INFINITE DIMENSIONAL MICROBUNDLES

A Dissertation

Presented to

the Faculty of the Department of Mathematics

University of Houston

Houston, Texas

In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

---

by

William Glenn Whitley

August 1973

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ABSTRACT

This dissertation considers microbundles with no restriction on their fiber space. The first part of this dissertation extends some of the basic theorems about finite dimensional microbundles. Induced microbundles are defined and shown to behave like pull backs. Induced microbundles are then shown to have a homotopy invariance property.

In the later chapters attention is restricted to microbundles with normed spaces as fibers. "Fredholm" type structures are defined and a microbundle k-theory is introduced. Inductive limits of microbundles are discussed and the k-theory of simplicial complexes is investigated.

TABLE OF CONTENTS

Chapter I Basic Definitions 1

Chapter II Bundle Map Germs and the Homotopy Theorem 14

Chapter III Microbundle K-Theory 27

Chapter IV k_{top} on Complexes 30

References

Chapter I: Basic Definitions

Let F denote a topological space with base point 0.

Definition 1.1 A microbundle χ with fiber F is a diagram

$$B \xrightarrow{i} E \xrightarrow{j} B$$

consisting of the following:

a) a topological space B called the base space;

b) a topological space E or $E(\chi)$ called the total space, and

c) continuous maps i and j called the injection and projection maps respectively. The composition ji is required to be the identity map of B. Furthermore one requires:

Local Triviality: For each $b \in B$ there exists an open neighborhood U of b and an open neighborhood V of $i(b)$ with $i(U) \subseteq V$ and $j(V) \subseteq U$ so that V is homeomorphic to $U \times F$ under a homeomorphism h which makes the following diagram commute:

$$\begin{tikzcd} V \arrow[r, "j|_V"] & U \\ U \arrow[u, "i|_U"] \arrow[r, "\times 0"] & U \times F \arrow[u, "p_1"'] \end{tikzcd} .$$

Here $\times 0$ denotes the injection $x \longmapsto (x,0)$ and p_1 denotes the projection map $(x,y) \longmapsto x$.

Examples

1) $$B \xrightarrow{\times 0} B \times F \xrightarrow{p_1} B .$$

This bundle is called the standard F bundle over B and is denoted by e_B^F .

2) Let ξ be a vector bundle over B with fiber F . Let E denote the total space of ξ , j the projection and $i:B \longrightarrow E$ the 0 cross section. Then

$$B \xrightarrow{i} E \xrightarrow{p} B$$

is a microbundle. This microbundle is called the underlying microbundle of ξ and is denoted by $|\xi|$.

3) Let M be a topological manifold and let $\Delta:M \longrightarrow M \times M$ denote the diagonal map. The diagram

$$M \xrightarrow{\Delta} M \times M \xrightarrow{p_1} M ,$$

denoted by t_M , is a microbundle and is called the tangent microbundle of M.

Lemma 1.2 The diagram t_M is a microbundle with fiber the model space of M.

Proof Let F denote the model space of M. Clearly $p_1 \circ \Delta$ is the identity map of M . We must verify the local triviality condition. Given $p \in M$, let U be an open neighborhood of p which is homeomorphic to F under the homeomorphism f .

Define $h\colon U \times U \longrightarrow U \times F$ by $h(a,b) = (a, f(b) - f(a))$.
Clearly h is a homeomorphism since f is a homeomorphism. Also,

$$h \circ \Delta_{|U}(a) = h(a,a) = (a,0) = \times 0(a), \text{ and } p_1 \circ h(a,b) = p_1(a,b),$$

and one has commutivity of the diagram

$$\begin{tikzcd} U \times U \arrow[r, "p_1"] \arrow[dr, "h"] & U \\ U \arrow[u, "\Delta_{|U}"] \arrow[r, "\times 0"] & U \times F \arrow[u, "p_1"'] \end{tikzcd}$$

<u>Definition 1.3</u> Suppose χ_1 and χ_2 are microbundles with diagrams

$$B \xrightarrow{i_1} E_1 \xrightarrow{j_1} B \quad \text{and} \quad B \xrightarrow{i_2} E_2 \xrightarrow{j_2} B .$$

We say χ_1 is isomorphic to χ_2 if there are open neighborhoods V_1 of $i_1(B)$ in E_1 and V_2 of $i_2(B)$ in E_2, and a homeomorphism h from V_1 to V_2 such that the following diagram commutes:

$$\begin{tikzcd} V_1 \arrow[r, "j_1"] \arrow[dr, "h"] & B \\ B \arrow[u, "i_1"] \arrow[r, "i_2"'] & V_2 \arrow[u, "j_2"'] \end{tikzcd}$$

The notation $\chi_1 \cong \chi_2$ will be used for this relation of isomorphism.

It is easily seen that this relation of isomorphism is reflexive, symmetric, and transitive.

Definition 1.4 A microbundle over a space B is called trivial if it is isomorphic to some standard microbundle. A manifold is topologically parallelizable if its tangent microbundle t_M is trivial.

If M is a smooth (C^∞) manifold we have two concepts of tangent bundle, t_M and TM. These structures appear to be different since $|TM|$ is a vector bundle and t_M, in general, is not. However, as microbundles they are isomorphic if M admits smooth partitions of unity. Let us recall some facts from differential topology which will be used in proving this statement. See Lange [8, Chapter 4] for details.

Let X be a manifold of class C^p with $p \geq 2$, and let $\pi : TX \to X$ be its tangent bundle. A vector field on X is a map of class C^{p-1}

$$\xi : X \longrightarrow TX$$

such that $\pi\xi = id_X$. Suppose x_0 is a point of X and ξ is a vector field on X. An integral curve for the vector field ξ with initial condition x_0 is a function of class C^{p-1}

$$\alpha : J \longrightarrow X$$

mapping an open interval J of R which contains 0 into X, such that

$$\alpha(0) = x_0 \qquad \text{and}$$
$$\alpha'(t) = \xi(\alpha(t))$$

for all $t \in J$.

Now, assume that X is a manifold of class C^p with $p \geq 3$. A second order differential equation over X is a vector field ξ on TX (of class C^{p-2}) such that, if π denotes the canonical projection of TX onto X, then

$$\pi_* \xi(v) = v$$

for all v in TX [8, pp. 67-68]. If ξ is a second order differential equation over X and $v \in TX$ there is a unique integral curve of ξ with initial condition v, [8, Theorem 2, pg. 64]. Denote this curve by β_v. Let D be the set of vectors v on TX such that β_v is defined at least on $[0,1]$. D is open in TX and the map

$$v \longrightarrow \beta_v(1)$$

is a C^{p-1} map of D into TX. Define the exponential map of ξ

$$\exp \xi : D \longrightarrow X$$

to be

$$\exp \xi(v) = \pi\beta_v(1) .$$

Exp ξ is a C^{p-2} function [8, pg. 69].

This exponential map is giving us a preferred set of paths in X, one through each point of X. However, for an arbitrary second order differential equation we have very little control over the lengths of these paths or how they range from point to point of X. Hence, we will restrict ourselves to a special class of second order differential equations. This class will give us a 'nice' exponential map.

Let ξ be a second order differential equation on the C^p manifold X $(p \geq 3)$. ξ is a spray over X if for each $v \in TX$, a number t is in the domain of β_v if and only if 1 is in the domain of β_{tv}, and then

$$\pi\beta_v(t) = \pi\beta_{tv}(1) .$$

Now, suppose that X also admits C^p partitions of unity. Then there is a spray ξ over X [8, Theorem 7, pg. 70] and the exponential map, $\exp \xi$, for this spray is a local isomorphism on the fibers in TX [8, Theorem 8, pg. 72]. Hence, we can use this exponential map to construct tubular neighborhoods [8, Theorem 9, pg. 73].

<u>Theorem 1.5</u> Let M be a C^k manifold $(k \geq 3)$ which admits C^k partitions of unity. Then the underlying microbundle $|TM|$ is C^{k-2} isomorphic to the tangent microbundle t_M.

<u>Proof</u> Since M admits C^k partitions of unity there is a spray γ over M. Thus there is an open neighborhood U of the 0 cross section in TM such that the exponential map for γ, $\exp \gamma$, maps U C^{k-2} diffeomorphicly onto an open neighborhood V of the diagonal in $M \times M$. Recall also that $\exp \gamma$ maps the 0 cross section onto the diagonal. Thus, if α denotes the 0 section, the diagram

$$\begin{tikzcd} U \arrow[r, "j"] \arrow[dr, "\exp\gamma"] & M \\ M \arrow[u, "\alpha"] \arrow[r, "\Delta"] & V \arrow[u, "p_1"'] \end{tikzcd}$$

commutes, and $\exp^{\gamma}$ is a C^{k-2} isomorphism. Hence $|TM|$ and t_M are C^{k-2} isomorphic as microbundles.

<u>Definition 1.6</u> Suppose χ is a microbundle with diagram

$$B \xrightarrow{i} E \xrightarrow{j} B ,$$

and f is a continuous map from the space A to B. Then the induced microbundle $f^{*}\chi$ is the diagram

$$A \xrightarrow{i'} E' \xrightarrow{p'} A ,$$

where

$$E' = \{(a,e) \in A x E \mid f(a) = j(e)\}, \; i'(a) = (a, if(a)) \text{ and } p_1(a,e) = a .$$

<u>Theorem 1.7</u> The diagram $f^{*}\chi$ is a microbundle.

<u>Proof</u> Let E' have the relative topology of the product topology on $A \times E$. Clearly i' and p_1 are continuous. Moreover,

$$p_1 i'(a) = p_1(a, if(a)) = a .$$

We must now verify the local triviality condition. Choose an $a \in A$. Since χ is locally trivial, there are open sets U and V about $f(a)$ and $if(a)$, respectively, and a homeomorphism h such that the following diagram commutes:

$$\begin{tikzcd}
V \arrow[r] \arrow[dr, "h"] & U \\
U \arrow[u, "i|_U"] \arrow[r, "x0"'] & U \times F \arrow[u, "p_1"']
\end{tikzcd} .$$

Let S be the inverse image of U under f and let T be the intersection of E' and $S \times V$. The set S is open in A and $a \in S$; therefore T is open in E' and $i'(a) = (a, if(a)) \in T$. Define

$$k: T \longrightarrow S \times F$$

by

$$k(a',e) = (a',p_2h(e)) \quad .$$

Since id_A, p_2, and h are continuous and open maps, k is an open and continuous map. One must show that k is bijective and that the following diagram commutes:

$$\begin{tikzcd} T \arrow[r, "j'|t"] \arrow[dr, "k"] & S \\ S \arrow[u, "i'|S"] \arrow[r, "x0"'] & S \times F \arrow[u, "p_1"'] \end{tikzcd} \quad .$$

Suppose $k(a',e) = k(a'',e')$. Then $a' = a''$ and $p_2h(e) = p_2h(e')$. Since (a',e) and (a',e') are elements of E' we have

$$p_1h(e) = j(e) = f(a') = j(e') = p_1h(e) \quad .$$

Hence $h(e) = h(e')$, and $e = e'$.

Choose an (a',x) in $S \times F$. Then $(f(a'),x) \in U \times F$; hence, there is a $v \in V$ such that $h(v) = (f(a'),x)$. Now $(a',v) \in f^{-1}(U) \times V$, and $j(v) = p_1(h(v)) = f(a')$, and $(a',v) \in T$. Also

$$k(a'v) = (a',p_2h(v)) = (a',x) \quad .$$

Thus k is bijective.

The diagram commutes since

$$ki'(a') = k(a',if(a')) = (a',p_2hif(a')) = (a',p_2(f(a'),0)) = (a',0)$$

and

$$p_1k(a',e) = p_1(a',p_2h(e)) = a' = p_1(a',e) \ .$$

<u>Theorem 1.8</u> If χ is a trivial microbundle and f is a continuous function, then $f^*\chi$ is a trivial microbundle.

<u>Proof</u> Suppose χ has fiber F and diagram

$$B \xrightarrow{i} E \xrightarrow{j} B \ .$$

Since χ is trivial, there are open sets V_1 and V_2 in E and $B \times F$ respectively, and a homeomorphism h such that the diagram

$$\begin{tikzcd} V_1 \arrow[r, "j"] \arrow[dr, "h"] & B \\ B \arrow[u, "i"] \arrow[r, "\times 0"] & V_2 \arrow[u, "p_1"'] \end{tikzcd}$$

commutes. Let E' denote the total space of $f^*\chi$. Let $V_3 = E' \cap p_2^{-1}(V_1)$, where p_2 is the projection of $A \times E$ onto the second factor. The set V_3 is open in E' and $i'(A) \subseteq V_3$. This is the case, since if $a \in A$, then

$$p_2i'(a) = p_2(a,if(a)) = if(a) \in V_1 \ .$$

Define the function ϕ from V_3 to $A \times F$ by

$$\phi = (id_A \times p_2 h)\big|_{V_3} ,$$

where p_2 is the second projection of $B \times F$ onto F.

We will show that ϕ is an embedding of V_3 onto an open set in $A \times F$. Since id_A, p_2 and h are continuous open maps ϕ is continuous and open. Now suppose $\phi(a,e) = \phi(x,y)$. Then $a = x$ and $p_2h(e) = p_2h(y)$. But

$$p_1h(e) = j(e) = f(a) = j(y) = p_1h(y) .$$

Thus ϕ is injective.

Let V_4 be $\phi(V_3)$. Then V_4 is open in $A \times F$ and the following diagram commutes:

$$\begin{tikzcd} V_3 \arrow[r, "p_1"] \arrow[dr, "\phi"] & A \\ A \arrow[u, "i'"] \arrow[r, "\times 0"'] & V_4 \arrow[u, "p_1"'] \end{tikzcd} .$$

<u>Definition 1.9</u> Suppose χ_1 and χ_2 are microbundles with diagrams

$$B \xrightarrow{i} E \xrightarrow{j} B \quad \text{and} \quad A \xrightarrow{g} D \xrightarrow{h} A ,$$

respectively. Suppose f is a continuous function from the space A to B. An f microbundle homomorphism from χ_2 to χ_1 is a continuous map from D to E such that the following diagram commutes:

$$\begin{tikzcd} B \arrow[r, "i"] & E \arrow[r, "j"] & B \\ A \arrow[u, "f"] \arrow[r, "g"] & D \arrow[u, "\phi"] \arrow[r, "h"] & A \arrow[u, "f"'] \end{tikzcd} .$$

We will say that ϕ is a lift of f, or that ϕ covers f.

Note that we have been using the language and notation $f^*\chi$ of pullbacks. The following collection of theorems will justify this terminology. We will show that the induced bundles satisfy a "universal mapping property" which is very similar to that of a pullback.

Lemma 1.10 Suppose χ is a microbundle with diagram

$$B \xrightarrow{i} E \xrightarrow{j} B ,$$

and f is a continuous map from the space A to B. Let E' denote the total space of $f^*\chi$. The map f^* from E' to E defined by $f^*(a,e) = e$ is an f microbundle homomorphism from $f^*\chi$ to χ .

Proof Clearly f^* is continuous since E' has the product topology. Since,

$$f^*i'(a) = f^*(a,if(a)) = if(a)$$

and

$$jf^*(a,e) = j(e) = f(a) = f(p_1(a,e))$$

for (a,e) in E', we have commutativity in the diagram

$$\begin{tikzcd} A \arrow[r, "i'"] \arrow[d, "f"'] & E' \arrow[r, "p_1"] \arrow[d, "f^*"] & A \arrow[d, "f"] \\ B \arrow[r, "i"] & E \arrow[r, "j"] & B \end{tikzcd} .$$

<u>Theorem 1.11</u> Suppose χ_1 and χ are microbundles with diagrams

$$A \xrightarrow{h} K \xrightarrow{g} A \quad \text{and} \quad B \xrightarrow{i} E \xrightarrow{j} B ,$$

and f is a continuous function from A to B. Suppose k is an f microbundle homomorphism from χ_1 to χ. Then there is a unique id_A microbundle homomorphism ϕ from χ_1 to $f^*\chi$ such that $f^*\phi = k$.

<u>Proof</u> Define the function ϕ from K to E' by

$$\phi(x) = (g(x), k(x)) .$$

a) $\phi(x) \in E'$ for each $x \in K$, since $fg(x) = jk(x)$.

b) Since g and k are continuous, ϕ is continuous.

c)

$$\begin{tikzcd} A \arrow[r, "i'"] & E' \arrow[r, "p_1"] & A \\ A \arrow[u, "id_A"] \arrow[r, "h"] & K \arrow[u, "\phi"'] \arrow[ru, "g"'] & \end{tikzcd} \quad \text{commutes.}$$

d) $f^*\phi(x) = f^*(g(x),k(x)) = k(x)$.

e) ϕ is unique with these properties.

Suppose θ is an id_A microbundle homomorphism from χ_1 to $f^*\chi$ and $f^*\phi = k = f^*\theta$. Choose an element x of K. Since f^* is the restriction of the second projection of $A \times E$ onto E and $f^*\phi(x) = f^*\theta(x)$ the second components of $\theta(x)$ and $\phi(x)$ are the same. Since θ and ϕ are id_A homomorphisms, $p_1\phi = p_1\theta$. Hence the first components of $\phi(x)$ and $\theta(x)$ are the same. Thus $\phi = \theta$.

<u>Definition 1.12</u> A microbundle χ with diagram

$$B \xrightarrow{i} E \xrightarrow{j} B$$

and fiber F is strongly trivial if there is an open set V containing $i(B)$ and a homeomorphism $h:V \longrightarrow B \times F$ which is compatible with the injections and projections.

Chapter II: Bundle Map Germs and the Homotopy Theorem

This chapter will be devoted to proving the following homotopy invariance theorem for induced microbundles.

Theorem 2.1 Suppose B is a paracompact space, f_0 and f_1 are homotopic maps from B to B', and χ is a microbundle with base space B'. Then $f_0{}^*\chi$ and $f_1{}^*\chi$ are isomorphic.

First let us recall the definition of a map germ and some of their basic properties. Suppose (E,B) and (E',B') are pairs of spaces. A map germ G from (E,B) to (E',B'), denoted by $G'(E,B) \Longrightarrow (E',B')$ is an equivalence class of continuous mappings g, each defined on some open neighborhood U_g of B in E, and mapping the pair (U_g,B) into (E',B'). Two such maps, g and g', are equivalent if and only if $g|V = g'|V$ for some sufficiently small open neighborhood V of B.

The composition HG of two map germs

$$(E,B) \overset{G}{\Longrightarrow} (E',B') \overset{H}{\Longrightarrow} (E'',B'')$$

if defined in the following manner. Choose $g \in G$ and $h \in H$. Suppose g and h have domains U_g and U_h respectively. Let $V = g^{-1}(U_h)$. HG is the map germ determined by $hg|V$.

Now consider a microbundle χ over B. We adopt the following notation from J. Milnor [9]. The projection map $j:E \to B$ determines a map germ $(E,iB) \Longrightarrow (B,B)$ denoted by J and called the projection germ of χ . Two convenient notation simplifications will be used:

1) The pair (B,B) will be denoted by B.

2) The space B will be identified with its image iB.

Using these conventions we may simply write

$$J:(E,B) \Longrightarrow B \ .$$

Suppose χ and χ' are microbundles with diagrams

$$B \xrightarrow{i} E \xrightarrow{j} B \quad \text{and} \quad B' \xrightarrow{i'} E' \xrightarrow{j'} B'$$

respectively. Suppose $G:(E,B) \Longrightarrow (E',B')$ is a map germ with representative $g:U_g \longrightarrow E'$.

<u>Definition 2.2</u> G is a bundle map germ if it satisfies the following conditions:

a) There is an open neighborhood T of $i(B)$ in U_g such that g maps each fiber $j^{-1}(b)$ T in a one to one open fashion into some fiber $j'^{-1}(b')$ of χ' .

b) For each $b_0 \in B$ there are trivializing maps h and h'

$$\begin{tikzcd} V \arrow[r, "j"] \arrow[dr, "h"] & U \\ U \arrow[u, "i"] \arrow[r, "\times 0"] & U \times F \arrow[u, "p_1"'] \end{tikzcd} \qquad \begin{tikzcd} V' \arrow[r, "j'"] \arrow[dr, "h'"] & U' \\ U' \arrow[u, "i'"] \arrow[r, "\times 0"] & U' \times F' \arrow[u, "p_1"'] \end{tikzcd}$$

at b_0 and $j'gi(b_0)$ such that there exist open sets S about 0 in F and Y about b_0 in B such that if (b,x_0) is an element of $Y \times S$

and R is a sufficiently small open set about x_0, there are open sets Q about $p_2h'gh^{-1}(b,x_0)$ and P about b such that

$$h'gh^{-1}(b,x_0) \in j'gi(P) \times Q \subseteq h'gh^{-1}(P \times R) .$$

Sufficiently small, as used here, means that there is an open set R_0 about x_0 such that if R is an open set about x_0 and $R \subseteq R_0$ then R meets the required conditions.

The pair of trivializations h and h' described in b) will be called an allowable trivializing pair for G. Statement b) also implies that if g is an open map on iB to $i'B$, then on a small enough neighborhood V of iB, g is an open map from V to E', as can be seen by slightly modifying Lemma 2.3.

Lemma 2.3 Suppose $B = B'$ and G is a bundle map germ from χ to χ' which covers the identity map of B. Then G is an isomorphism germ.

Proof First consider the special case that $g \in G$ and $g: B \times F \longrightarrow B \times F'$. Clearly g is continuous and 1-1 on a sufficiently small open neighborhood of iB. All one need show is that g is open on an open set about iB. For $b \in B$ define $g_b: F \longrightarrow F'$ by $g_b(x) = g(b,x)$. Choose a $b_0 \in B$ and S and Y as in Definition 2.2(b).

Claim The map g is open on $Y \times S$.

Choose a $(b_1,x_1) \varepsilon Y \times S$ and an open set 0 about (b_1,x_1). Choose open sets R, Q, and P as in Definition 2.2 part (b) such that

$$(b_1,x_1) \varepsilon R \subseteq \{x \mid (\bar{b},x) \varepsilon 0\} ,$$

P is a subset of Y and $P \times R$ is a subset of 0. Thus

$$g(b_1,x_1) \varepsilon j'gi(P) \times Q \subseteq g(P \times R) \subseteq g(0) .$$

Since $jgi(P) = P$, g is open at (b_1,x_1). Hence, g is open on $Y \times S$.

Let V be the union over $b_0 \varepsilon B$ of the $Y \times S$. Then V is an open neighborhood of iB and g is open on V.

Now consider the general case. Let $g:U_g \longrightarrow E'$ be a representative of G. Assume that U_g is sufficiently small so that g is fiber preserving, 1-1, and open on the fibers. We need only find an open set W such that g is open on W and iB is contained in W. We have shown that for each $b \varepsilon B$ there is an open set W_b in a trivializing neighborhood of $i(b)$ in U_g such that g maps W_b onto an open set $g(w_b) \subseteq E'$. Taking W to be the union over the $b \varepsilon B$ of the W_b, it follows immediately that g maps W homeomorphicly onto an open set in E'. Therefore, G is an isomorphism germ.

<u>Corollary 2.4</u> If $f:B \longrightarrow B'$ is covered by a bundle map germ $G: \chi \Longrightarrow \chi'$, then χ is isomorphic to the induced bundle $f^*\chi'$.

<u>Proof</u> Let $g:U_g \longrightarrow E'$ be a representitive for G. Then

$$\begin{tikzcd}
B \arrow[r, "i"] \arrow[d, "f"'] & U_g \arrow[r, "j'"] \arrow[d, "g"] & B \arrow[d, "f"] \\
B' \arrow[r, "i'"] & E' \arrow[r, "j'"] & B'
\end{tikzcd}$$

commutes and there is a map ϕ such that the following diagram commutes:

$$\begin{tikzcd}
B \arrow[r] \arrow[dd, bend right=60, dash, "f"'] & f^*(E') \arrow[r] \arrow[dd, bend left=80, dash, "f^*"] & B \arrow[dd, bend left=60, dash, "f"] \\
B \arrow[u, "id"'] \arrow[r] \arrow[d, "f"'] & U_g \arrow[u, "\phi"'] \arrow[r] \arrow[d, "g"] & B \arrow[u, "id"'] \arrow[d, "f"] \\
B' \arrow[r] & E' \arrow[r] & B'
\end{tikzcd}$$

Recall that ϕ is defined by $\phi(x) = (j(x),g(x))$. Clearly ϕ covers the identity map. Suppose ϕ is an allowable representative map for a bundle map germ H. Then H is an isomorphism germ and f^*X is isomorphic to X.

We now show that ϕ is an allowable map for a bundle map germ. Since g is a representative for a bundle map germ, there is an open set T in E such that T contains iB, and g is 1-1 and open on the fibers in T. ϕ is also 1-1 and open on fibers in T. We will show that ϕ maps fibers in a trivialization just as g maps them. This will verify part b of Definition 2.2. Let h and h' be allowable trivializations for g. Let k be the trivialization of X' associated with h' as defined in Theorem 1.7:

$$k(b,e) = (b,p_2h'(e)) .$$

Then we have

$$\begin{aligned} k\phi h^{-1}(b,x) &= k(jh^{-1}(x),gh^{-1}(x)) = k(b,gh^{-1}(x)) \\ &= (b,p_2h'gh^{-1}(x)) = k'gh^{-1}(b,x) . \end{aligned}$$

Lemma 2.5 (Henderson, [4, Proposition 3]) Let χ be a microbundle with base space Y. Suppose Y is the union of two closed sets A and B. Suppose there is a retraction $r:B \longrightarrow A \cap B$, χ is strongly trivial over B, and χ is trivial over A. Then χ is trivial.

Corollary 2.6 Let χ be a microbundle with base space $B \times [0,1]$ such that χ is trivial over $B \times [0,\frac{1}{2}]$ and strongly trivial over $B \times [\frac{1}{2},1]$. Then χ is trivial.

Lemma 2.7 Suppose h and h' are allowable trivializations for a bundle map germ $G:\chi \Longrightarrow \chi'$ at b_0 and $b_1 = G_B(b_0)$. If k is any trivialization of χ' at b_1, then h and k are allowable trivializations of G. Here G_B denotes the restriction of any element of G to B.

Proof Choose a representative map $g:U_g \longrightarrow E'$ of G and assume that h' and k have been restricted so that they have the same range

$U' \times F'$. Thus we have

$$U \times F \xrightarrow{h^{-1}} V \xrightarrow{g} V' \xrightarrow{h'} U' \times F' \xrightarrow{kh'^{-1}} U' \times F$$

(A) and

$$(b,x) \longrightarrow (b',x') \longrightarrow (b',x'') .$$

Note that g may not be defined on all of V, but it maps an open subset of V into V'. Assume S and Y have been chosen as in Definition 2.2(b). Choose $(b,x) \in Y \times S$. Choose R, Q, and P as in 2.2,b. We will use the notation of diagram (A). Let $X = kh'^{-1}(U' \times Q)$. Then X is open and $(b',x'') \in X$ since $(b',x') \in U' \times Q$. Hence there are open sets W about b' and Q'' about x'' such that $(b',x'') \in W \times Q' \subseteq X$.
Let $P' = ((j'gi)^{-1}W) \cap P$. P' is an open set and $b \in P'$. Also $j'gi(P') \times Q \subseteq h'gh^{-1}(P' \times R)$. Thus we have our desired result:
$j'gi(P') \times Q' \subseteq kh'^{-1}(jgi(P') \times Q) \subseteq kh'^{-1}(h'gh^{-1}(P' \times R)) = kgh^{-1}(P' \times R)$.

<u>Lemma 2.8</u> Suppose χ_0, χ_1 and χ_2 are microbundles with diagrams

$B_0 \xrightarrow{i_0} E_0 \xrightarrow{j_0} B_0$, $B_1 \xrightarrow{i_1} E_1 \xrightarrow{j_1} B_1$,

$B_2 \xrightarrow{i_2} E_2 \xrightarrow{j_2} B_2$, respectively. If $D: \chi_0 \Rightarrow \chi_1$ and $G: \chi_1 \Rightarrow \chi_2$ are bundle map germs, then GD is a bundle map germ.

<u>Proof</u> Choose $b_0 \in B_0$, $d \in D$, and $g \in G$. Let h_0 and k_1 be allowable trivializations for d at b_0 and $b_1 = j_1di_0(b_0)$. Let h_1 and h_2 be allowable trivializations for g at b_1 and $b_2 = j_2gi_1(b_1)$. By Lemma 2.7 we may assume that $k_1 = h_1$.

Choose Y_1 and S_1 about b_1 and 0 as in Definition 2.2,b. Choose Y_0 and S_0 for d as in Definition 2.2,b, such that

$$Y_0 \subseteq (j_1dij)^{-1}(Y_1) \quad \text{and} \quad h_1dhj^{-1}(Y_0 \times S_0) \subseteq Y_1 \times S_1 .$$

Let (b,x) be an element of $Y_0 \times S_0$. Choose a sufficiently small open set R about x, and open sets P and Q about b and $p_2h_1dh_0^{-1}(b,x)$ such that

$$(b',x') = h_1dh_0^{-1}(b,x) \in j_1di_0(P) \times Q \subseteq h_1dh_0^{-1}(P \times R) .$$

One may also assume that Q was chosen small enough so that there will be open sets P_1 and Q_1 about b' and $p_2h_2gh_1^{-1}(b',x')$, respectively, such that

$$h_2gh_1^{-1}(b',x') \in j_2gi_1(P_1) \times Q_1 \subseteq h_2gh_1^{-1}(P_1 \times Q) .$$

Let $P' = (j_1di_0)^{-1}(P_1) \cap P$. The set P' is open and $b \in P'$. Clearly $h_2gdh_0^{-1}(b,x) \in j_2gdi_0(P') \times Q_1$. All one needs to show is that $j_2gdi_0(P') \times Q_1 \subseteq h_2gdh_0^{-1}(P' \times R)$.

Choose $(b,x) \in P' \times Q_1$. Since $b \in P'$, $jdi_0(b) \in P_1$. Combining this fact with x being an element of Q_1, we see that

$$(j_2 g i_1 j_1 d i_0(b), x) \in h_2 g h_1^{-1}((j d i_0(b)) \times Q).$$

Hence, we have that

$$\begin{aligned}(j_2 g d i_0(b), x) &\in h_2 g h_1^{-1}((j_1 d i_0(b) \times Q) \\ &\subseteq h_2 g h_1^{-1}(h_1 d h_0^{-1}(b \times R)) \\ &= h_2 g d h_0^{-1}(b \times R) \\ &\subseteq h_2 g d h_0^{-1}(P' \times R)\ .\end{aligned}$$

Lemma 2.9 Suppose χ and χ' are microbundles with diagrams

$$B \xrightarrow{i} E \xrightarrow{j} B \quad \text{and} \quad B' \xrightarrow{i'} E' \xrightarrow{j'} B'\ ,$$

respectively. Suppose $G\colon \chi \Longrightarrow \chi'$ is a bundle map germ, $b \in B$, and h and h' are allowable trivializations for G at b and $j'G_B i(b)$. If k is any trivialization of χ at b, then k and h' are allowable trivializations for G.

Proof Consider the identity map germ from χ to χ. The map pair k and k is an allowable trivializing pair for the identity at b and b. The proof of Lemma 2.8 shows that k and h' are an allowable trivializing pair for $G(Id) = G$.

Lemma 2.7 and Lemma 2.9 together imply that if $G\colon \chi \Longrightarrow \chi'$ is a bundle map germ and $b \in B$, then any trivializations h of χ at b and h' of χ' at $j'G_B i(b)$, h and h' are allowable for G.

Thus we can choose trivializations arbitrarily for a bundle map germ without having to require them to be allowable.

Lemma 2.10 Let χ be a microbundle over B and let B_α be a locally finite collection of closed sets covering B. Suppose $G_\alpha : \chi|_{B_\alpha} \Longrightarrow \eta$ is a collection of bundle map germs such that for each (α,β) G_α coincides with G_β on $\chi|_{B_\alpha \cap B_\beta}$. Then there is a bundle map germ $G: \chi \Longrightarrow \eta$ which extends each G_α.

Proof Let $g_\alpha : U_\alpha \longrightarrow E'$ be a representitive for G_α. Suppose g_α coincides with g_β on a set $U_{\alpha\beta}$ which is an open neighborhood of $B_\alpha \cap B_\beta$ in $U_\alpha \cap U_\beta$. Let U be the set of all $e \in E$ such that

i) if $j(e) \in B_\alpha \cap B_\beta$, then $e \in U_{\alpha\beta}$

ii) if $j(e) \in B_\alpha$, then $e \in U_\alpha$.

Since the B_α's are locally finite U is open. Define the function $g: U \longrightarrow E'$ by $g(e) = g_\alpha(e)$ if $e \in U_\alpha \cap U$.

We now show that g is allowable map for a bundle map germ. Choose $b_0 \in B$ and an open set U' about b_0 so that U' intersects only finitely many B_α's, say $B_1, B_2, \ldots, B_n$. For simplicity we will consider only the case $n = 2$. Let h and h' be trivializations of χ and η at b_0 and $j'gi(b_0)$. If there is an open set about b_0 that is contained entirely in B_1 or B_2, then Definition 2.2(b) is satisfied at b_0. Suppose that any open set about b_0 intersects

B_1 and B_2. There are open sets S_1 and S_2 about 0 in F and Y_1 and Y_2 about b_0 such that $(Y_1 \cap B_1) \times S_1$ and $(Y_2 \cap B_2) \times S_2$ satisfy Definition 2.2(b) for g_1 and g_2, respectively. Choose open sets Y and S about b_0 and 0, respectively, so that $Y \subseteq Y_1 \cap Y_2 \cap j(\text{domain } h) \cap U'$, $S \subseteq S_1 \cap S_2$ and $Y \times S \subseteq h(U)$. Choose $(b,x) \in Y \times S$. Again, if b is in only one of B_1 or B_2 then, Difinition 2.2(b) is satisfied at (b,x). Suppose $b \in B_1 \cap B_2$ and $i \in \{1,2\}$. If $R \subseteq S$ is a sufficiently small open set about x, there are open sets Q_i and P_i about $p_2h'g_ih^{-1}(b,x)$ and b, respectively, such that

$$h'g_ih^{-1}(b,x) \in j^{-1}g_ii(P_i \cap B_i) \times Q_i \subseteq h'g_ih^{-1}(P_i \cap B_i) \times R) .$$

Let $P = P_1 \cap P_2 \cap Y$ and $Q = Q_1 \cap Q_2$. Since $P \subseteq Y \subseteq U'$, $P \subseteq B_1 \cup B_2$. If $(a,y) \in P \times R$, then $(a,y) \in Y \times S$ and $h^{-1}(a,y) \in U$. Thus

$$h'g_1h^{-1}(a,y) = h'g_2h^{-1}(a,y) = h'gh^{-1}(a,y)$$

whenever they are defined. We now have

$$\begin{aligned} h'gh^{-1}(b,x) \in j'gi(P) \times Q &= (jg_1i(P \cap B_1) \times Q) \cup (jg_2i(P \cap B_2) \times Q) \\ &\subseteq h'g_1h^{-1}((P \cap B_1) \times R) \cup h'g_2h^{-1}((P \cap B_2) \times R) \\ &= hgh^{-1}((P \cap B_1) \times R) \cup h'gh^{-1}((P \cap B_2) \times R) \\ &= h'gh^{-1}(P \times R) . \end{aligned}$$

Lemma 2.11 Let χ be any microbundle over $B \times [0,1]$. Then each $b \in B$ has an open neighborhood V such that $\chi|_{V\times[0,1]}$ is trivial.

Proof Choose $b \in B$. For each $t \in [0,1]$ choose an open set $V_t \times (t-\epsilon, t+\epsilon)$ of (b,t) such that χ is strongly trivial over this neighborhood. The compact set $\{b\} \times [0,1]$ is covered by finitely many such neighborhoods. Let V be the interesection of the corresponding neighborhoods V_t for some such finite subcover of $\{b\} \times [0,1]$. Then there exists a partition $0 = t_0 < t_1 < \ldots < t_n = 1$ of $[0,1]$ so that $\chi|_{V\times[t_{i-1},t_i]}$ is strongly trivial. Applying Lemma 2.6 inductively, we see that $\chi|_{V\times[0,1]}$ is trivial.

Lemma 2.12 Let χ be a microbundle over $B \times [0,1]$, where B is paracompact. Then the standard retraction r of $B \times [0,1]$ to $B \times \{1\}$ is covered by a bundle map germ $R: \chi \Longrightarrow \chi|_{B\times\{1\}}$.

Proof Let V_α be a locally finite covering of B by open sets V_α such that $\chi|_{V_\alpha\times[0,1]}$ is trivial. Choose $\lambda_\alpha: B \longrightarrow [0,1]$ such that the support of λ_α is contained in V_α, and so that for each $b \in B$, $\lambda_\alpha(b) = 1$ for some α. Define

$$r_\alpha: B \times [0,1] \longrightarrow B \times [0,1]$$

$$r_\alpha(b,t) = (b, \mathrm{Max}\,(t,\lambda_\alpha(b))) .$$

Cover r_α by a bundle map germ $R_\alpha : \chi \Longrightarrow \chi$ as follows. Define

$$A_\alpha = (\text{Support } \lambda_\alpha)$$

$$A'_\alpha = \{(b,t) | t \geq \lambda_\alpha(b)\} \quad .$$

A_α and A'_α are closed sets and their union is $B \times [0,1]$. Since $\chi|A_\alpha$ and $\chi|A_\alpha \cap A'_\alpha$ are trivial, there is a bundle map germ $G_\alpha : \chi|A_\alpha \Longrightarrow \chi|A_\alpha \cap A'_\alpha$ which covers $r_\alpha|A_\alpha : A_\alpha \longrightarrow A_\alpha \cap A'_\alpha$ and the identity on $\chi|A_\alpha \cap A'_\alpha$. Using 2.10 we can piece G together with the identity map germ $\chi|A'_\alpha \Longrightarrow \chi|A'_\alpha$ to obtain the required map germ R_α.

Choose some ordering of the index set α. $R : \chi \Rightarrow \chi|B \times \{1\}$ will be defined as the composition of the R_α's in the prescribed order. R is a bundle map germ as locally it is the composition of finitely many bundle map germs.

Proof of Theorem 2.1 Suppose $H : B \times [0,1] \longrightarrow B'$ is a homotopy from f_0 to f_1. By Lemma 2.12 there is a bundle map germ $R : H^*\chi \Longrightarrow H^*\chi|B \times \{1\}$ which covers the standard retraction of $B \times [0,1]$ to $B \times \{1\}$. Forming the composition

$$f_0^*\chi \xrightarrow{\subset \text{inc}} H^*\chi \xRightarrow{R} H^*\chi|_{B \times \{1\}} = f_1^*\chi \quad ,$$

we obtain a bundle map germ $G : f_0^*\chi \Longrightarrow f_1^*\chi$ which covers the identity on B. Hence G is an isomorphism germ and $f_0^*\chi \cong f_1^*\chi$.

Chapter III: Microbundle K-Theory

In this chapter χ will always denote a microbundle with diagram

$$B \xrightarrow{t} E \xrightarrow{j} B \; .$$

Moreover, the fiber F of χ is a normed space.

<u>Definition 3.1</u> Two chart structures $(U_\alpha, h_\alpha, V_\alpha)_{\alpha \varepsilon A}$ and $(U'_\beta, h'_\beta, V'_\beta)_{\beta \varepsilon B}$ of χ are GC isomorphic if there is a homeomorphism $g\colon {}_{\alpha} V_\alpha \longrightarrow {}_{\beta} V'_\beta$ such that the following is true:

> For each $(\alpha, \beta) \varepsilon A \times B$ there is a collection $(V_s^{\alpha\beta}, A_s^{\alpha\beta})_{s \varepsilon S_{\alpha\beta}}$ where $(V_s^{\alpha\beta})_{s \varepsilon S_{\alpha\beta}}$ is an open cover of $U_\alpha \cap U'_\beta$ by pairwise disjoint sets, and for each $s \varepsilon S_{\alpha\beta}$, $A_s^{\alpha\beta} \varepsilon \mathrm{Homeo}(F,F)$ such that for each $(\alpha,\beta) \varepsilon A \times B$, $s \varepsilon S_{\alpha\beta}$ and $x \varepsilon V_s^{\alpha\beta}$ the map $\phi'_x g_x \phi_x^{-1} - A_s^{\alpha\beta}$ is the restriction of a compact operator. Here ϕ_x and ϕ'_x are the maps induced on the fibers by h_α and h'_β.

<u>Definition 3.2</u> A GC microbundle is a microbundle equiped with a maximal class of GC isomorphic chart structures. Two GC microbundles χ and χ' are GC isomorphic if there are chart structures in the GC structures on χ and χ' which are isomorphic in the sense of 3.1.

Definition 3.3 Two GC microbundles χ and χ' are stably equivalent if there are trivial GC bundles τ and τ' such that $\chi \oplus \tau$ and $\chi' \oplus \tau'$ are GC isomorphic.

Let $K_0(X)$ denote the standard K-theory of finite dimensional vector bundles over X. Let $k_{top}(X)$ denote the stable classes of microbundles with real normed spaces as fibers. Let B_n denote the n^{th} Bernoulli number and let $num(B_n/n)$ denote the numerator of B_n/n when expressed in reduced form. Let $\phi: K_0(X) \longrightarrow k_{top}(X)$ be the map that takes a vector bundle to its corresponding microbundle.

Theorem 3.4 $\phi: K(X) \longrightarrow k_{top}(X)$ is not in general an isomorphism. Specifically, if π is the generator of $K_0(S^{4n})$, then $\phi(\pi)$ is divisible by $(2^{2n-1} - 1)num(B_n/n)$.

Proof Kervaire and Milnor [6, Lemma 7.4] have shown that for $n > 1$ there is a manifold W with boundary such that W is smooth and parallelizable, ∂W is isomorphic with S^{4n-1}, and the intersection number pairing $H_{2n}(W) \otimes H_{2n}(W) \longrightarrow Z$ is positive definite. Let $M = W \cup C(\partial W)$ be the topological manifold obtained from W by adjoining a cone over ∂W. Let $f: M \longrightarrow S^{4n}$ have degree 1. Milnor [9, Lemma 8.2] has shown that there is a finite dimensional microbundle χ over S^{4n} so that $f^*\chi$ is isomorphic to the tangent microbundle t_M .

Let j_n denote the order of the image

$$J(\pi_{4n-1}(SO_e)) \subseteq \pi_{4n-1+e}(S^e)$$

of the stable J-homomorphism $(e > 4n)$; let a_n denote 1 or 2 as n is even or odd. Denote

$$2^{2n-4}(2^{2n-1} - 1)B_n j_n a_{n/n}$$

by b_n. Thus b_n is an integer. Since j_n is relatively prime to $(2^{2n-1} - 1)\mathrm{num}(B_{n/n})$, there are integers r and s so that

$$\text{(B)} \qquad r j_n + s(2^{2n-1} - 1)\mathrm{num}(B_{n/n}) = 1 .$$

Furthermore,

$$\phi(\pi) = (1 - r j_n)(\phi(\pi)) + r b_n(\chi)$$

since $b_n(\chi) = j_n(\phi(\pi))$. From (B) we see that $1 - r j_n$ is divisible by $(2^{2n-1} - 1)\mathrm{num}(B_{n/n})$. A theorem of J. F. Adams [2, Theorem 3.7] shows that j_n is either 4 or 8 times the denominator of B_n/n. Hence $b_n = 2^m(2^{2n-1} - 1)(B_n/n)$ for some integer m, and $r b_n$ is divisible by $(2^{2n-1} - 1)\mathrm{num}(B_n/n)$. Thus $\phi(\pi)$ is divisible by $(2^{2n-1}-1)\mathrm{num}(B_n/n)$, and we can condlude that the "natural" map of $K_0(X)$ into $k_{top}(X)$ is not always an isomorphism.

Chapter IV: k_{top} on Complexes

In this chapter we will look at $k_{top}(B)$ where B is a locally finite not necessarily finite dimensional simplicial complex with the weak topology. It will be shown that $k_{top}(B)$ is a commutative monoid.

Lemma 4.1 If B is a paracompact contractible space then any microbundle over B is trivial.

Proof Since B is contractible the identity map is null homotopic. Applying the homotopy theorem we get our desired result.

For a topological space A let $CA = A \times [0,1]/A \times \{0\}$ denote the cone over A. Suppose $f:A \longrightarrow B$ is a continuous map. The space $B\cup_f CA$ is the one obtained from B by attaching the cone over A and identifying each $(a,1)$ with $f(a)$.

Lemma 4.2 If A is paracompact and $f:A \longrightarrow B$ is a continuous map then a microbundle χ over B can be extended to a microbundle over $B\cup_f CA$ if and only if $f^*\chi$ is trivial.

Proof The composition

$$A \xrightarrow{f} B \subseteq B\cup_f CA$$

if null homotopic. Hence if χ extends, it follows that $f^*\chi$ is trivial.

To prove the converse, consider the mapping cylinder $Z = B \cup_f (A \times [0,1])$ of f. Since B is a retract of Z, it follows that χ can be extended to Z. Call this extension χ_1. Now suppose $f^*\chi$ is trivial. Then $\chi_1|_{A\times\{0\}}$ is trivial. Moreover, $\chi_1|_{A\times[0,1/2]}$ is trivial. Thus some open set V in $E(\chi_1|_{A\times[0,1/2]})$ is homeomorphic to an open neighborhood U of $A \times [0,1/2] \times \{0\}$ in $A \times [0,1/2] \times F$ under a homeomorphism h which is compatable with the injections and projections.

Let $E(\chi_2)$ be obtained from $E(\chi_1)$ by collapsing $h^{-1}(A \times \{0\} \times \{x\})$ to a point for each $x \in F$. Since we obtain $B \cup_f CA$ from Z by collapsing $A \times \{0\}$ to a point, $E(\chi_2)$ is the total space of the required microbundle.

<u>Definition 4.3</u> Let χ and χ' be microbundles over the same space B. The Whitney sum of χ and χ', $\chi + \chi'$, is defined as follows. The base space is B; the total space is the subset of $E(\chi) \times E(\chi')$ consisting of all (x,y) such that $j(x) = j'(y)$; and the injection and projection maps are defined by $b \longmapsto (i(b),\ i'(b))$ and $(x,y) \longrightarrow j(x)$, respectively. Local triviality can be easily verified as seen below.

Suppose $C = \{U_\alpha, h_\alpha, V_\alpha\}_{\alpha\in A}$ and $C' = \{U'_\beta, h'_\beta, V'_\beta\}_{\beta\in B}$ are chart structures for χ and χ', respectively. Let $C \oplus C' = \{U_\alpha \cap U'_\beta,\ h_\alpha \oplus h'_\beta,\ V_\alpha \tilde{\times} V'_\beta\}_{(\alpha,\beta) \in A \times B}$ where

$V_\alpha \tilde{x} V'_\beta = E(\chi_1 \oplus \chi_2) \cap (V_\alpha \times V'_\beta)$ and

$h_\alpha \oplus h'_\beta = V_\alpha \tilde{x} V_\beta \longrightarrow (U_\alpha \cap U_\beta) \times (F \times F')$ is defined by:

$$h_\alpha \oplus h'_\beta(x,y) = (j(x),(P_2h_\alpha(x),P_2h'_\alpha(y))) \quad .$$

$C \oplus C'$ is a chart structure for $\chi \oplus \chi'$. Suppose that C and C' are representatives of GC structures on χ and χ', respectively. Then the GC structure defined by $C \oplus C'$ on $\chi \oplus \chi'$ is said to be the induced GC structure. This sum is commutative and associative up to isomorphism. Thus if B is any space, $k_{top}(B)$ is a commutative monoid.

We now turn our attention to the special case where B is a simplicial complex. Since B is the inductive limit of its skeletons, one can take a microubndle χ over B and restrict it to the skeletons and get microbundles over very nice spaces namely, finite dimensional complexes. Suppose on the other hand one has a collection $\{\chi_n\}_{n\in N}$ of microbundles over the skeletons of B. Can these microbundles be pieced together to give a microbundle over B? With this thought in mind we will define an inductive limit of a system of microbundles.

For each $n \in N$ let χ_n be a microbundle with diagram

$$B_n \xrightarrow{i_n} E_n \xrightarrow{j_n} B_n$$

and fiber F_n. Suppose, also, that the following matching conditions are satisfied:

(a) $B_1 \subseteq B_2 \subseteq B_3 \subseteq \cdots \subseteq B_n \subseteq \cdots$

(b) $E_1 \subseteq E_2 \subseteq E_3 \subseteq \cdots \subseteq E_n \subseteq \cdots$

(c) $F_1 \subseteq F_2 \subseteq F_3 \subseteq \cdots \subseteq F_n \subseteq \cdots$

(d) $i_n = i_{n+1}|B_n$ and $j_n = j_{n+1}|E_n$

(e) For each $n \in N$ there is a chart structure $C_n = \{U_{n_b}, h_{n_b}, V_{n_b}\}_{b \in B_n}$ on X_n such that if $b_n \in B_n$ and $m_1, m_2 \geq n$, then $b \in U_{m,b}$ and $h^{-1}_{m_1 b}(b,x) = h^{-1}_{m_2 b}(b,x)$.

Let B, E, and F be the inductive limits of the B_n, E_n, and F_n respectively, and equip each with the weak topology. Define $i:B \longrightarrow E$ and $j:E \longrightarrow B$ by $b_n \longrightarrow i_n(b_n)$ and $e_n \longrightarrow j_n(e_n)$ respectively. The maps i and j are well defined by (d) and are continuous. Also $ji = id_B$. Now we must verify the local triviality requirement. Choose a $b \in B$. Then there is an n such that $b \in B_n$. Let $U = \bigcup_{m \geq n} U_{mb}$ and let $V = \bigcup_{m \geq b} V_{mb}$. Define $k:U \times F \longrightarrow V$ by $k(x,y) = h_{mb}^{-1}(x,y)$ where $x \in U_{mb}$. Condition (e) guarantees that k is well defined. Also k is a homeomorphism. Thus the diagram

$$B \xrightarrow{i} E \xrightarrow{j} B$$

is the diagram for a microbundle X. The microbundle X will be called the inductive limit of the system of $\{X_n\}$.

Recall the following theorem of Milnor [9, Lemma 4.2].

Suppose that the CW-complex B is a 'bouquet' of finitely or infinitely many spheres, meeting at a point. Let $r: B \to B$ map each sphere into itself with degree -1. Then for any (finite dimensionsl) microbundle χ over B, the sum $\chi \oplus r^*\chi$ is trivial.

Since finite dimensional microbundles χ and χ' are GC isomorphic if they are isomorphic in the usual sense, it follows that $\chi \oplus r^*\chi$ is GC isomorphic to a trivial GC bundle. Moreover, slight modifications of Milnor's proof of this theorem shows that if χ is any microbundle over B with finite or infinite dimensional fiber, then the sum $\chi \oplus r^*\chi$ is isomorphic to a trivial bundle. However, for an infinite dimensional GC microbundle χ it is not known whether the GC sum is GC isomorphic to a trivial bundle.

We now look at $k_{top}(B)$ and ask if it is a group and if it is not a group, which, if any, of its elements have inverses. Milnor [9, Theorem 4.1] has shown that if χ is a finite dimensional microbundle over B and B is finite dimensional, then there is a microbundle χ' such that the sum $\chi \oplus \chi'$ is trivial. Let us assume that B is locally finite but necessarily finite dimensional and χ is a GC microbundle over B. Let us now view B as the direct limit of its (finite dimensional) skeletons. It will be shown that if B_n denotes the n^{th} skeleton of B, then there is a microbundle χ_n over B_n such that $\chi \oplus \chi_n$ is trivial.

If $n = 0$, then $\chi|_{B_0}$ is trivial and there is nothing to prove. Now let B' denote the n-1 skeleton of B and assume that there is a microbundle χ_{n-1} over B' such that $\chi|_{B'} \oplus \chi_{n-1}$ is trivial. Suppose F is the fiber of χ. Let e^F denote the trivial bundle with fiber F over B'. Let σ be an n-simplex of B. Also σ is the cone over its boundary. Thus $\chi_{n-1} \oplus e^F$ can be extended to σ if and only if its restriction to $\dot{\sigma}$ is trivial. However, χ is defined over σ; thus $\chi|_{\dot{\sigma}}$ is trivial. By assumption $\chi \oplus \chi_{n-1}$ is trivial. Thus $(\chi_{n-1} \oplus e^F)|_{\dot{\sigma}}$ is trivial and $(\chi_{n-1} \oplus e^F)|_{\dot{\sigma}}$ can be extended to σ.

Now let us extend $\chi_{n-1} \oplus e^F$ over all n-simplexes of B. Let B'' be obtained from B by removing a small open n-simplex from the interior of each n-simplex. B' is a retract of B'', and $\chi_{n-1} \oplus e^F$ extends over B''. Now extend $\chi_{n-1} \oplus e^F$ over each of the small n-simplexes that were removed. Since these simplexes are separated, one obtains an extension of $\chi_{n-1} \oplus e^F$. Call this extension ξ.

Consider the complex $B \cup CB'$. Since $(\chi \oplus \xi)|_{B'}$ is trivial, it extends to some microbundle ω over $B \cup CB'$. Now $B \cup CB'$ has the homotopy type of a bouquet of n-spheres. Hence $\omega \oplus r^*\omega$ is trivial over $B \cup CB'$. Now $\chi \oplus \xi \oplus (r^*\omega)|_{B_n}$ is trivial and $\xi \oplus (r^*\omega)|_{B_n}$ is the required microbundle.

Note that this construction procedure gives us an "increasing" sequence of microbundles which satisfy the five conditions on page 33. Thus, if χ' is the inductive limit of this system, the sum $\chi \oplus \chi'$ is trivial as each $\chi \oplus \chi_n$ is trivial. Moreover, if χ is finite dimensional, then at each stage we have a finite dimensional microbundle χ_n, so that $\chi|_{B_n} \oplus \chi_n$ is isomorphic, hence GC isomorphic, to a trivial bundle.

However the general case, where χ is an arbitrary GC microbundle over B, has not yet been settled.

<u>Conjecture</u> If B is a locally finite simplicial complex and χ is a GC microbundle over B, then there is a GC microbundle χ' over B such that $\chi \oplus \chi'$ is GC trivial.

If this conjecture is true then $k_{top}(B)$ is a group and we have a contravariant functor k_{top} from the category of locally finite simplicial complexes and continuous maps to the category of groups. Moreover, Theorem 3.4 shows that this functor would be different from the standard K functor.

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