$(V,\mathcal{M})$-SEMI-REFLEXIVITY, $(V,\mathcal{M},\mathfrak{S})$-REFLEXIVITY AND $\mathcal{M}$-QUASI-REFLEXIVITY IN LOCALLY CONVEX SPACES

A Dissertation

Presented to

The Faculty of the Department of Mathematics

College of Arts and Sciences

University of Houston

In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

---

by

David Fling Norwood

August 1971

ACKNOWLEDGMENTS

I wish to express my gratitude to Dr. Paul J. Knopp for his guidance and encouragement throughout my career as a graduate student.

My graduate studies and the research for this dissertation were partially supported by NASA Grant NsG(T)-52 #4.

$(V,\mathcal{M})$-SEMI-REFLEXIVITY, $(V,\mathcal{M},\mathfrak{S})$-REFLEXIVITY AND $\mathcal{M}$-QUASI-REFLEXIVITY IN LOCALLY CONVEX SPACES

An Abstract of a Dissertation

Presented to

The Faculty of the Department of Mathematics

College of Arts and Sciences

University of Houston

In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

---

by

David Fling Norwood

August 1971

ABSTRACT

In this paper, "a locally convex space," or simply "an l.c. space," means "a locally convex, Hausdorff, topological vector space," either real or complex. For an arbitrary l.c. space $E[\mathcal{T}]$, necessary and sufficient conditions are found in order for there to exist an l.c. space $F[\mathcal{U}]$, a covering $\mathfrak{S}'$ of F with bounded subsets of F, and a linear homeomorphism $\Phi:E[\mathcal{T}] \to F'[\mathcal{T}_{\mathfrak{S}'}]$, where $\mathcal{T}_{\mathfrak{S}'}$ is the $\mathfrak{S}'$-topology on F' for the pairing $\langle F,F'\rangle$.

These conditions are given in terms of the properties of $(V,\mathcal{M})$-semi-reflexivity and $(V,\mathcal{M},\mathfrak{S})$-reflexivity of E, where V is a $\sigma(E',E)$-dense subspace of E', $\mathcal{M}$ is a covering of E with bounded subsets of E, and $\mathfrak{S}$ is a covering of V with $\sigma(V,V')$-bounded subsets of V, where V' denotes $(V[\mathcal{T}_{\mathcal{M}}|V])'$. When $\Omega:E\to V'$ denotes the canonical linear map, E is said to be $(V,\mathcal{M})$-semi-reflexive if Ω is a one-to-one map of E onto V'; in case $\mathcal{T}_{\mathcal{M}} = \beta(E',E)$, E is said to be V-semi-reflexive. The l.c. space $E[\mathcal{T}]$ is said to be $(V,\mathcal{M},\mathfrak{S})$-reflexive if $\Omega:E[\mathcal{T}] \to V'[\mathcal{T}_{\mathfrak{S}}]$ is a linear homeomorphism; if $\mathcal{T}_{\mathfrak{S}} = \beta(V',V)$, $E[\mathcal{T}]$ is said to be $(V,\mathcal{M})$-reflexive; if $\mathcal{T}_{\mathcal{M}} = \beta(E',E)$ and $\mathcal{T}_{\mathfrak{S}} = \beta(V',V)$, $E[\mathcal{T}]$ is said to be V-reflexive. If a linear homeomorphism $\Phi:E[\mathcal{T}] \to F'[\mathcal{T}_{\mathfrak{S}'}]$ exists, there exist V, $\mathcal{M}$, and $\mathfrak{S}$ such that (a) $F[\mathcal{U}]$ is linearly homeomorphic to $V[\mathcal{T}_{\mathcal{M}}|V]$, and (b) $E[\mathcal{T}]$ is $(V,\mathcal{M},\mathfrak{S})$-reflexive.

For an arbitrary l.c. space $E[\mathcal{T}]$, $(V,\mathcal{M})$-semi-reflexivity of E is characterized in terms of $\mathcal{T}_{\mathcal{M}}$-minimality of the $\mathcal{T}_{\mathcal{M}}$-closure $\overline{V}$ of V in E', relative $\sigma(E,V)$-compactness of the sets in $\mathcal{M}$, and $\sigma(E,V)$-

convergence in E of $\sigma(E,V)$-Cauchy filters in sets in $\mathcal{M}$. The $(V,\mathcal{M})$-semi-reflexivity of E is further characterized under the hypotheses that V be (1) $\sigma(V,E)$-separable; (2) $\mathcal{T}_{\mathcal{M}}|V$-separable; (3) $\sigma(V,E)$-separable and $\mathcal{T}_{\mathcal{M}}|V$-barrelled; (4) $\mathcal{T}_{\mathcal{M}}|V$-barrelled. These characterizations involve (1) relative $\sigma(E,V)$-sequential compactness and relative $\sigma(E,V)$-countable compactness of the sets in $\mathcal{M}$; (2) $\sigma(E,V)$-convergence in E of $\sigma(E,V)$-Cauchy sequences in the sets in $\mathcal{M}$; (3) intersections of decreasing sequences of convex, $\sigma(E,V)$-closed and bounded subsets of E; (4) an intersection property similar to (3). For a Banach space $E[\mathcal{T}]$, V-semi-reflexivity of E is equivalent to V-reflexivity of $E[\mathcal{T}]$ as defined herein, and is characterized in terms of conditions on the norm-closed unit ball of E corresponding to conditions on the sets in $\mathcal{M}$ for the general case.

For an arbitrary l.c. space $E[\mathcal{T}]$, necessary and sufficient conditions are found for $\Omega: E[\mathcal{T}] \to V'[\mathcal{T}_{\mathfrak{S}}]$ to be (1) continuous or (2) relatively open, and $(V,\mathcal{M},\mathfrak{S})$-reflexivity of $E[\mathcal{T}]$ is characterized. For the cases in which (a) $E[\mathcal{T}]$ is barrelled, or (b) $E[\mathcal{T}]$ is infra-barrelled and $\mathcal{T}_{\mathcal{M}} = \beta(E',E)$, or (c) $E[\mathcal{T}]$ is bornological and $V[\mathcal{T}_{\mathcal{M}}|V]$ is barrelled, or (d) $E[\mathcal{T}]$ is bornological and $\overline{V}[\mathcal{T}_{\mathcal{M}}|\overline{V}]$ is quasi-complete, it is shown that $(V,\mathcal{M})$-reflexivity of $E[\mathcal{T}]$ implies $(\overline{V},\mathcal{M})$-reflexivity of $E[\mathcal{T}]$.

Examples are given for the following cases:
(1) $\tau(E',E) \subsetneqq \mathcal{T}_{\mathcal{M}} \subsetneqq \beta(E',E)$ and E is $(V,\mathcal{M})$-semi-reflexive but <u>not</u> V-semi-reflexive; (2) $E[\mathcal{T}]$ is V-reflexive but <u>not</u> $\overline{V}$-reflexive; (3) $E[\mathcal{T}]$ is $\overline{V}$-reflexive but <u>not</u> V-reflexive.

If G denotes $(E'[\mathcal{T}_M])'$, E is said to be M-quasi-reflexive (of order n) if E is of finite codimension (n) in G. Under the hypotheses that E is $\beta(G,E')$-closed in G and either (a) $E[\mathcal{T}]$ is barrelled or (b) $E[\mathcal{T}]$ is infrabarrelled and $\mathcal{T}_M = \beta(E',E)$, the following are shown to be equivalent: (1) E is M-quasi-reflexive of order n ; (2) there is a $\sigma(E',E)$-dense, $\mathcal{T}_M$-closed subspace V of codimension n in E' such that $E[\mathcal{T}]$ is (V,M)-reflexive.

TABLE OF CONTENTS

INTRODUCTION

This paper is concerned with results relating to the following questions: When is an l.c. space $E[\mathcal{T}]$ linearly homeomorphic to the strong dual $F'[\beta(F',F)]$ of an l.c. space $F[\mathcal{U}]$? More generally, when is $E[\mathcal{T}]$ linearly homeomorphic to $F'[\mathcal{T}_{\mathfrak{S}}]$ where $\mathcal{T}_{\mathfrak{S}}$ is an $\mathfrak{S}$-topology on F' for a collection $\mathfrak{S}$ of bounded subsets of F covering F?

Dixmier [5] and Ruston [13] found necessary and sufficient conditions for a Banach space E to be (a) linearly homeomorphic or (b) isometrically isomorphic to the strong dual of a Banach space. Civin and Yood [4] and Singer [14 through 18] studied the question further, although still in the setting of Banach spaces, introducing the concepts of (a) V-pseudo-reflexivity and (b) V-reflexivity of E with respect to a subspace V of E'. Krishnamurthy [9] generalized the results of Dixmier, Ruston and Singer to a Mackey space E which is V-reflexive with respect to a strongly closed subspace V of E', where the concept of V-reflexivity is defined in terms of strong topologies. Of course, in this setting only linear homeomorphisms are considered, since the notion of an isometry is defined only for normed spaces. Lohman [10,11] extended certain results of Singer and Krishnamurthy to l.c. spaces E, still using the concepts of V-semi-reflexivity and V-reflexivity as defined in terms of strong topologies, and often assuming V to be strongly closed in E'.

In this paper we define $(V,\mathcal{M})$-semi-reflexivity and $(V,\mathcal{M},\mathfrak{S})$-reflexivity in terms of topologies perhaps different from strong ones, and show that these properties provide an answer to our general question. We also extend certain results of Krishnamurthy and Lohman in terms of these properties, and give examples to which our results apply while those of Krishnamurthy and Lohman do not. Moreover, we give characterizations of these properties in the general case, in certain cases involving conditions of separability, and in cases involving barrelled l.c. spaces.

The concepts of $(V,\mathcal{M})$-semi-reflexivity and $(V,\mathcal{M},\mathfrak{S})$-reflexivity of E provide answers to the questions of the opening paragraph by identifying the space F with a suitable subspace V of E'. The case in which E itself is identified with a subspace of $(E'[\mathcal{T}_{\mathcal{M}}])'$, for an appropriate topology $\mathcal{T}_{\mathcal{M}}$ on E', is of particular interest. Certain results are obtained when E is of finite codimension (n) in $(E'[\mathcal{T}_{\mathcal{M}}])'$, in which case we say that E is $\mathcal{M}$-quasi-reflexive (of order n). The principal result is Theorem 5.5, which states that if E is strongly closed in $(E'[\mathcal{T}_{\mathcal{M}}])'$ and $E[\mathcal{T}]$ is a barrelled l.c. space (or only infra-barrelled in case $\mathcal{T}_{\mathcal{M}} = \beta(E',E)$ on E'), then E is $\mathcal{M}$-quasi-reflexive of order n if and only if there is a $\sigma(E',E)$-dense, $\mathcal{T}_{\mathcal{M}}$-closed subspace V of codimension n in E' such that $E[\mathcal{T}]$ is $(V,\mathcal{M})$-reflexive.

In this paper we restrict our attention to vector spaces over the field of real numbers and to vector spaces over the field of complex numbers. "A locally convex space" (or simply "an l.c. space") means "a locally convex, Hausdorff, topological vector space" as defined by Horváth [6]. If a vector space E is equipped with a topology $\mathcal{T}$ for which it is an l.c. space, we refer to it as the l.c. space $E[\mathcal{T}]$, or simply as the l.c. space E if it is clear from the context that $\mathcal{T}$ is the topology on E. The vector space of all continuous linear functionals on E is called the dual of $E[\mathcal{T}]$ (or simply the dual of E), and is denoted by $(E[\mathcal{T}])'$ (or simply by E').

We use the definitions and the notation of Horváth [6] for the following:

1) A pairing $\langle F,G\rangle$ of F and G with respect to a bilinear form on $F \times G$, where F and G are vector spaces over the same field.
2) Separation of points of F (respectively G) with respect to the pairing $\langle F,G\rangle$.
3) A pairing $\langle F,G\rangle$ which is a dual system.
4) The polar in G of a subset A of F for $\langle F,G\rangle$.
5) The subspace $M^{\perp}$ of G which is orthogonal to the subset M of F for $\langle F,G\rangle$.
6) A collection $\mathfrak{S}$ of $\sigma(F,G)$-bounded subsets of F.
7) The $\mathfrak{S}$-topology on G (which we denote by $\mathcal{T}_{\mathfrak{S}}$).
8) Standard $\mathfrak{S}$-topologies on G for a pairing $\langle F,G\rangle$, such as $\sigma(G,F)$, $\tau(G,F)$, and $\beta(G,F)$.
9) Locally convex topologies on G which are compatible with the pairing $\langle F,G\rangle$.
10) The standard pairing $\langle E,E'\rangle$ (or $\langle E',E\rangle$) for an l.c. space E and its dual E'.

11) The equicontinuous subsets of E'. (Note: If $\mathcal{T}_1$ and $\mathcal{T}_2$ are locally convex topologies on E which are compatible with the pairing $\langle E,E'\rangle$, the subsets of E' which are equicontinuous for $\mathcal{T}_1$ are called the $\mathcal{T}_1$-equicontinuous subsets of E'.)

Also in accordance with Horváth [6, Ex. 2, pp. 202-03], we say that a collection $\mathfrak{S}$ of $\sigma(F,G)$-bounded subsets of F is saturated if all of the following are true:

1) Every subset of a set $A\varepsilon\mathfrak{S}$ belongs to $\mathfrak{S}$.

2) The union of a finite number of sets in $\mathfrak{S}$ belongs to $\mathfrak{S}$.

3) If $A\varepsilon\mathfrak{S}$, then $\lambda A\varepsilon\mathfrak{S}$ for all $\lambda \neq 0$.

4) The balanced, convex, $\sigma(F,G)$-closed hull of every set in $\mathfrak{S}$ belongs to $\mathfrak{S}$.

Note that if K is a linear subspace of G, then a collection $\mathfrak{S}$ of $\sigma(F,G)$-bounded subsets of F is also a collection of $\sigma(F,K)$-bounded subsets of F; but $\mathfrak{S}$ may be saturated for the pairing $\langle F,G\rangle$, but not saturated for the pairing $\langle F,K\rangle$. Thus we say that $\mathfrak{S}$ is $\langle F,G\rangle$-saturated but not $\langle F,K\rangle$-saturated. However, if the specific pairing is clearly indicated from the context, we say simply that $\mathfrak{S}$ is a saturated collection of bounded subsets of F. The use of saturated collections $\mathfrak{S}$ facilitates comparisons between $\mathfrak{S}$-topologies; indeed, if $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are saturated collections, then $\mathcal{T}_{\mathfrak{S}_1} = \mathcal{T}_{\mathfrak{S}_2}$ if and only if $\mathfrak{S}_1 = \mathfrak{S}_2$. Moreover, for a saturated collection $\mathfrak{S}$, a fundamental system of neighborhoods of 0 for $\mathcal{T}_{\mathfrak{S}}$ is formed by the polars of members of $\mathfrak{S}$, and it is not necessary to consider finite intersections of scalar multiples of polars of members of $\mathfrak{S}$.

If $\langle F,G\rangle$ and $\langle H,K\rangle$ are two pairings, and there are vector-space isomorphisms $\Psi:F\to H$ and $\Phi:G\to K$ such that $\langle f,g\rangle = \langle\Psi(f),\Phi(g)\rangle$ whenever $f\in F$ and $g\in G$, we say the pairings are algebraically isomorphic with respect to Ψ and Φ. In this case, the correspondence between subsets of F and G and their images under Ψ and Φ implies obvious relationships between the operations of taking polars and orthogonal subspaces with respect to the two pairings, and between saturated collections $\mathfrak{S}$ of bounded sets and corresponding $\mathfrak{S}$-topologies with respect to the two pairings.

In the interest of clarity or brevity, we use the following notation which is not used by Horváth [6].

1) If $\langle F,G\rangle$ is a pairing, and $A\subset F$ and $K\subset G$, the polar in K of A is denoted by A^{oK}; this notation facilitates distinction between A^{oK} and A^{oG}.

2) Similarly, when K is a linear subspace of G, the subspace of K which is orthogonal to A is denoted by $A^{\perp K}$; thus $A^{\perp K}$ is distinguished from $A^{\perp G}$.

3) If E is equipped with the topology $\mathcal{T}$ and $M\subset E$, we denote by $\mathcal{T}|M$ (or simply by $\mathcal{T}_i$) the topology induced on M by $\mathcal{T}$. Similarly if $V\subset E'$, $\beta(E',E)|V$ (or simply $\beta_i(E',E)$) denotes the topology induced on V by $\beta(E',E)$.

4) We refer to the closure of M in E for $\mathcal{T}$ as the $\mathcal{T}$-closure of M, and write $\overline{M}(\mathcal{T})$, or simply $\overline{M}$. Similarly if $A\subset M\subset E$, we refer to the $\mathcal{T}|M$-closure of A in M as the $\mathcal{T}|M$-closure of A, and write $\overline{A}(\mathcal{T}|M)$, or simply $\overline{A}$. (Of course, $\overline{A}(\mathcal{T}|M) = \overline{A}(\mathcal{T})\cap M$.)

Throughout this paper, $\mathcal{M}$ is always an $\langle E,E'\rangle$-saturated collection of bounded subsets of E covering E, $\mathcal{T}_{\mathcal{M}}$ is always the $\mathcal{M}$-topology on E', V is always a linear subspace of E', $\overline{V}$ always denotes $\overline{V}(\mathcal{T}_{\mathcal{M}})$, V' always denotes $(V[\mathcal{T}_{\mathcal{M}}|V])'$, and Ω always denotes the canonical linear map $\Omega:E\to V'$ where $\langle\Omega(e),v\rangle = \langle e,v\rangle$ whenever $e\in E$ and $v\in V$.

Sometimes we let G denote $(E'[\mathcal{T}_{\mathcal{M}}])'$, and Ω_G denote the canonical linear map $\Omega_G:E\to G$ where $\langle\Omega_G(e),e'\rangle = \langle e,e'\rangle$ whenever $e\in E$ and $e'\in E'$. (Of course, $\Omega(e)$ is the restriction of $\Omega_G(e)$ to V).

From the definition of Ω it is clear that Ω is one-to-one if and only if V separates points of E.

SECTION 1

$(V,\mathcal{M},\mathfrak{S})$-REFLEXIVITY AND LINEAR HOMEOMORPHISMS OF LOCALLY CONVEX DUAL SPACES

In this section we define $(V,\mathcal{M})$-semi-reflexivity of E and $(V,\mathcal{M},\mathfrak{S})$-reflexivity of $E[\mathcal{T}]$, and we show that these properties provide an answer to our general question.

We have already noted in the introduction that Ω is a one-to-one map if and only if V separates points of E. Before stating or definitions, which are concerned with Ω, we give another necessary and sufficient condition for Ω to be one-to-one. It is well known, but a proof is given for completeness.

Proposition 1.1 Ω is one-to-one if and only if V is $\sigma(E',E)$-dense in E'.

Proof: According to [6, p. 192, Thm. 1], $\overline{V}(\sigma(E',E)) = V^{oE\ oE'}$. The following are equivalent: V is $\sigma(E',E)$-dense in E'; $E' \subset V^{oE\ oE'}$; $V^{oE} \subset (E')^{oE}$; for every $\delta > 0$, $\delta V^{oE} \subset \delta (E')^{oE}$.

Suppose Ω is not one-to-one. Then $\ker \Omega \neq \{0\}$, and there is an $e \in E$, $e \neq 0$, such that $\langle \Omega(e), v\rangle = \langle e,v\rangle = 0$ for all $v \in V$. Since E is an l.c. space, there is an $e' \in E'$ such that $\langle e,e'\rangle \neq 0$. Let $\delta = \frac{1}{2} |\langle e,e'\rangle|$. Then $e \in \delta V^{oE}$, but $e \notin \delta(E')^{oE}$ since $|\langle e,e'\rangle| = 2\delta > \delta$. Thus $\delta > 0$, but $\delta V^{oE} \not\subset \delta(E')^{oE}$; V is not $\sigma(E',E)$-dense in E'.

Conversely, suppose V is not $\sigma(E',E)$-dense in E'. Then $V^{oE} \not\subset (E')^{oE}$, and there is an $e \in E$ such that $|\langle e,v\rangle| \leq 1$ for all $v \in V$, but there is an $e'\varepsilon E'$ such that $|\langle e,e'\rangle| > 1$. Choose a $v_o \varepsilon V$; for all n, $nv_o \varepsilon V$, and $n|\langle e,v_o\rangle| = |\langle e,nv_o\rangle| \leq 1$. Then $|\langle e,v_o\rangle| = 0$. Since $|\langle e,e'\rangle| > 1$, $e \neq 0$; but $\langle e,v_o\rangle = 0$ for all $v_o \varepsilon V$. Thus $\ker \Omega \neq \{0\}$, and Ω is not one-to-one.

<u>Definition 1.2</u> Suppose V is $\sigma(E',E)$-dense in E'. Then <u>E is $(V,\mathcal{M})$-semi-reflexive</u> if and only if Ω maps E <u>onto</u> V'. If $\mathcal{T}_{\mathcal{M}} = \beta(E',E)$, then we say <u>$E$ is V-semi-reflexive</u>; if $V = E'$, then we say <u>E is $\mathcal{M}$-semi-reflexive</u>; if $\mathcal{T}_{\mathcal{M}} = \beta(E',E)$ <u>and</u> $V = E'$, then we say <u>E is semi-reflexive</u>.

<u>Remark 1.3</u> In the above definition, the case where $\mathcal{T}_{\mathcal{M}} = \beta(E',E)$ is the V-semi-reflexivity of Lohman [10,11], with the additional requirement that V be $\sigma(E',E)$-dense in E'. The case where $\mathcal{T}_{\mathcal{M}} = \beta(E',E)$ <u>and</u> $V = E'$ is simply the usual definition of semi-reflexivity. In every case, the definition is equivalent to the statement that the topology $\mathcal{T}_{\mathcal{M}}|V$ is compatible with the pairing $\langle V,E\rangle$ [6, p. 198, Def. 1].

<u>Definition 1.4</u> Suppose V is $\sigma(E',E)$-dense in E', and $\mathfrak{S}$ is a saturated collection of $\sigma(V,V')$-bounded subsets of V covering V. Then <u>$E[\mathcal{T}]$ is $(V,\mathcal{M},\mathfrak{S})$-reflexive</u> if and only if $\Omega: E[\mathcal{T}] \to V'[\mathcal{T}_{\mathfrak{S}}]$ is a linear homeomorphism. If $V = E'$, or $\mathcal{T}_{\mathcal{M}} = \beta(E',E)$, or $\mathcal{T}_{\mathfrak{S}} = \beta(V',V)$, then reference to V, or $\mathcal{M}$, or $\mathfrak{S}$, respectively, is deleted from the statement that $E[\mathcal{T}]$ is $(V,\mathcal{M},\mathfrak{S})$-reflexive. For example, if $\mathcal{T}_{\mathfrak{S}} = \beta(V',V)$, we say <u>$E[\mathcal{T}]$ is $(V,\mathcal{M})$-reflexive</u>; if $\mathcal{T}_{\mathcal{M}} = \beta(E',E)$ <u>and</u> $\mathcal{T}_{\mathfrak{S}} = \beta(V',V)$, we say <u>$E[\mathcal{T}]$ is V-reflexive</u>; if $V = E'$ <u>and</u> $\mathcal{T}_{\mathcal{M}} = \beta(E',E)$ <u>and</u> $\mathcal{T}_{\mathfrak{S}} = \beta(V',V)$, we say <u>$E[\mathcal{T}]$ is reflexive</u>.

Remark 1.5 In the above definition, V-reflexivity of $E[\mathcal{T}]$ is the same as that defined by Lohman [11, Def. 2] and Krishnamurthy [9, Def. 5]; our definition of reflexivity is equivalent to the usual definition of reflexivity of $E[\mathcal{T}]$.

Theorem 1.6 Suppose $E[\mathcal{T}]$ and $F[\mathcal{U}]$ are l.c. spaces, and $\mathfrak{S}'$ is a saturated collection of bounded subsets of F covering F. If there is a linear homeomorphism $\Phi:E[\mathcal{T}]\to F'[\mathcal{T}_{\mathfrak{S}'}]$, then there are a $\sigma(E',E)$-dense subspace V of E', a saturated collection $\mathcal{M}$ of bounded subsets of E covering E, and a saturated collection $\mathfrak{S}$ of $\sigma(V,V')$-bounded subsets of V covering V, such that (a) $F[\mathcal{U}]$ is linearly homeomorphic to $V[\mathcal{T}_{\mathcal{M}}|V]$, and (b) $E[\mathcal{T}]$ is $(V,\mathcal{M},\mathfrak{S})$-reflexive.

Proof: Let $\Psi:F\to(F'[\mathcal{T}_{\mathfrak{S}'}])'$ denote the canonical linear injection, and let $\Phi^T:(F'[\mathcal{T}_{\mathfrak{S}'}])'\to E'$ denote the linear map defined so that $\Phi^T(f'') \equiv f''\cdot\Phi$ for all $f''\varepsilon(F'[\mathcal{T}_{\mathfrak{S}'}])'$. Clearly $\Phi^T(f'')\varepsilon E'$. We show that Φ^T is one-to-one, and onto E'. Indeed, if $e'\varepsilon E'$ then $(e'\cdot\Phi^{-1})\varepsilon(F'[\mathcal{T}_{\mathfrak{S}'}])'$ such that $\Phi^T(e'\cdot\Phi^{-1}) \equiv (e'\cdot\Phi^{-1})\cdot\Phi \equiv e'$; also, if $f'' \not\equiv 0$ there is an $f'\varepsilon F'$ such that $\langle f'',f'\rangle \neq 0$, and there is an $e\varepsilon E$ such that $\Phi(e) = f'$, and it follows that $\Phi^T(f'')(e) = f''\ (\Phi(e)) = f''(f') \neq 0$, so that $\Phi^T(f'') \not\equiv 0$. We observe that Φ^T is an algebraic isomorphism.

Let $V = \Phi^T\cdot\Psi(F)$. We show V is $\sigma(E',E)$-dense in E'. Indeed, since $\langle f,f'\rangle = \langle\Phi^T\cdot\Psi(f),\Phi^{-1}(f')\rangle$ whenever $f\varepsilon F$ and $f'\varepsilon F'$, the pairings $\langle F,F'\rangle$ and $\langle V,E\rangle$ are algebraically isomorphic with respect to $\Phi^T\cdot\Psi:F\to V$ and $\Phi^{-1}:F'\to E$. Since F separates points of F', then V separates points of E.

Let $\mathcal{M} = \{M \subset E \mid \Phi(M) \text{ is } \mathcal{U}\text{-equicontinuous}\}$. If $M \in \mathcal{M}$, $\Phi(M)$ is $\beta(F',F)$-bounded [6, p. 217], hence $\mathcal{T}_{\mathfrak{S}'}$-bounded since $\beta(F',F)$ is finer than $\mathcal{T}_{\mathfrak{S}'}$. Thus M is a bounded subset of E. Since the $\mathcal{U}$-equicontinuous subsets of F' form a <F,F'>-saturated covering of F', and $\mathcal{M}$ is the collection of their images under Φ^{-1}, then $\mathcal{M}$ is a <V,E>-saturated covering of E. Thus $\mathcal{M}$ has the first three properties of a saturated collection described on page 4, and in order to show that $\mathcal{M}$ is <E',E>-saturated it remains only to show that $\mathcal{M}$ satisfies the fourth for <E'E>. Let $M \in \mathcal{M}$; then $\Phi(M)$ is $\mathcal{U}$-equicontinuous, and so is $(\Phi(M))^{oF\ oF'} = \Phi(M^{oV\ oE})$, and it follows that $M^{oE'\ oE} \subset M^{oV\ oE} \in \mathcal{M}$.

We prove (a): If $\mathcal{N}$ is the collection of all $\mathcal{U}$-equicontinuous subsets of F', then $\mathcal{U}$ is the $\mathcal{N}$-topology on F for the pairing <F,F'> [6, p. 200]. Since $\mathcal{T}_{\mathcal{M}}|V$ is defined by polars in V of members of $\mathcal{M}$ for the pairing <V,E>, and the members of $\mathcal{M}$ are the images under Φ^{-1} of the members of $\mathcal{N}$, then the $\mathcal{T}_{\mathcal{M}}|V$-neighborhoods of 0 in V are precisely the images under $\Phi^T \cdot \Psi$ of the $\mathcal{U}$-neighborhoods of 0 in F. Therefore $\Phi^T \cdot \Psi : F[\mathcal{U}] \to V[\mathcal{T}_{\mathcal{M}}|V]$ is a linear homeomorphism.

We show that Ω maps E <u>onto</u> V' (i.e., that E is $(V,\mathcal{M})$-<u>semi</u>-reflexive), and then define $\mathfrak{S}$ in order to prove (b). Denote by $(\Phi^T \cdot \Psi)^T : V' \to F'$ the linear map defined so that $(\Phi^T \cdot \Psi)^T(v') \equiv v' \cdot (\Phi^T \cdot \Psi)$ for all $v' \in V'$. Then the result of the previous paragraph implies that $(\Phi^T \cdot \Psi)^T$ is an algebraic isomorphism, and we already know that Φ is an algebraic isomorphism. If $e \in E$ and $f \in F$, then $[(\Phi^T \cdot \Psi)^T \cdot \Omega(e)](f) = [\Omega(e) \cdot (\Phi^T \cdot \Psi)](f) = [\Omega(e)](\Phi^T \cdot \Psi(f)) = [\Phi^T \cdot \Psi(f)](e) = [\Psi(f) \cdot \Phi](e) = [\Psi(f)]\Phi(e) = [\Phi(e)](f)$; thus

$(\Phi^T\cdot\Psi)^T\cdot\Omega(e) \equiv \Phi(e)$ for all $e\varepsilon E$, and $(\Phi^T\cdot\Psi)^T\cdot\Omega \equiv \Phi$. Therefore $\Omega \equiv ((\Phi^T\cdot\Psi)^T)^{-1}\cdot\Phi$ is an algebraic isomorphism, and Ω maps E onto V'.

Let $\mathfrak{S} = \{\Phi^T\cdot\Psi(A) \mid A\varepsilon\mathfrak{S}'\}$. Note that the pairings $\langle F,F'\rangle$ and $\langle V,V'\rangle$ are algebraically isomorphic with respect to $\Phi^T\cdot\Psi:F\to V$ and $\Omega\cdot\Phi^{-1}:F'\to V'$, because $\langle F,F'\rangle$ and $\langle V,E\rangle$ are so with respect to $\Phi^T\cdot\Psi:F\to V$ and $\Phi^{-1}:F'\to E$, and $\langle V,E\rangle$ and $\langle V,V'\rangle$ are so with respect to id: $V\to V$ and $\Omega:E\to V'$. Note also that $\Omega\cdot\Phi^{-1} \equiv ((\Phi^T\cdot\Psi)^T)^{-1}$. Since $\mathfrak{S}'$ is a $\langle F,F'\rangle$-saturated collection of $\sigma(F,F')$-bounded subsets of F covering F, $\mathfrak{S}$ is a $\langle V,V'\rangle$-saturated collection of $\sigma(V,V')$-bounded subsets of V covering V; moreover, since the members of $\mathfrak{S}$ are the images under $\Phi^T\cdot\Psi$ of the members of $\mathfrak{S}'$, the $\mathcal{T}_{\mathfrak{S}}$-neighborhoods of 0 in V' are precisely the images under $((\Phi^T\cdot\Psi)^T)^{-1}$ of the $\mathcal{T}_{\mathfrak{S}'}$-neighborhoods of 0 in F'. Thus $((\Phi^T\cdot\Psi)^T)^{-1}:F'[\mathcal{T}_{\mathfrak{S}'}]\to V'[\mathcal{T}_{\mathfrak{S}}]$ is a linear homeomorphism. Of course, so is $\Phi:E[\mathcal{T}]\to F'[\mathcal{T}_{\mathfrak{S}'}]$, and $\Omega \equiv ((\Phi^T\cdot\Psi)^T)^{-1}\cdot\Phi$. Therefore $\Omega:E[\mathcal{T}]\to V'[\mathcal{T}_{\mathfrak{S}}]$ is a linear homeomorphism, and (b) is proved.

SECTION 2

$(V,\mathcal{M})$-SEMI-REFLEXIVITY OF LOCALLY CONVEX SPACES, AND V-PSEUDO-REFLEXIVITY OF BANACH SPACES

In this section we study conditions relating to the algebraic properties of Ω. In order for Ω to be one-to-one, we already have two equivalent conditions: (1) that V separate points of E; (2) that V be $\sigma(E',E)$-dense in E'. Throughout this section we assume that V is $\sigma(E',E)$-dense in E', and we seek necessary and sufficient conditions in order for Ω to map E onto V'; that is, we seek characterizations of $(V,\mathcal{M})$-semi-reflexivity of E.

We take special notice, however, of the following: (1) that for certain l.c. spaces E (cf. Corollaries 3.7 and 3.9), Ω is always continuous; (2) that for certain of these l.c. spaces E and certain choices of V and $\mathcal{M}$, $(V,\mathcal{M})$-semi-reflexivity is equivalent to $(V,\mathcal{M})$-reflexivity--i.e., a certain kind of open-mapping theorem holds for Ω.

For example, if E is a Banach space, Ω is always continuous, regardless of the choices for $V,\mathcal{M}$, and $\mathfrak{S}$, whether or not Ω maps E onto V' (cf. Corollary 3.7). Moreover, if $\mathcal{T}_{\mathcal{M}} = \beta(E',E)$, $\mathcal{T}_{\mathfrak{S}} = \beta(V',V)$, and Ω does map E onto V', then by the open-mapping theorem for Banach spaces Ω is a linear homeomorphism. Thus for a Banach space E, V-semi-reflexivity of Definition 1.2 is equivalent to V-reflexivity of Definition 1.4. This is the condition which Civin and Yood [4] and Singer [14 through 18] have called the V-pseudo-reflexivity of E. (Those authors have reserved V-reflexivity of E for the case in which Ω is not only a linear homeomorphism but also an isometry.) Moreover, they have called a Banach space E quasi-reflexive (of order n) when E has finite codimension (n) upon being embedded into its bidual E", and have shown that E has this property if and only if there is a $\sigma(E',E)$-dense subspace V of finite codimension (n) in E' such that E is V-pseudo-reflexive (cf. Singer [18]). Thus for a Banach space a characterization of V-semi-reflexivity (i.e., of V-pseudo-reflexivity) gives a characterization not only of V-reflexivity of Definition 1.4 but also of quasi-reflexivity. (Of course, quasi-reflexivity of order zero is just semi-reflexivity, and therefore reflexivity of Definition 1.4.)

In Section 5 we show (cf. Theorem 5.3) that for an arbitrary l.c. space E, a characterization of $(V,\mathcal{M})$-semi-reflexivity gives a characterization of $\mathcal{M}$-quasi-reflexivity (Definition 5.1). Moreover, we show (cf. Corollary 3.7) that for a barrelled l.c. space and any choice of $\mathcal{M}$, or even for an infrabarrelled l.c. space where $\mathcal{T}_{\mathcal{M}} = \beta(E',E)$, Ω is always continuous, and if E is $\beta(G,E')$-closed in $G = (E'[\mathcal{T}_{\mathcal{M}}])'$ and V is of finite codimension in E', then (cf. Theorem 5.5) E is $(V,\mathcal{M})$-semi-reflexive if and only if E is $(V,\mathcal{M})$-reflexive. For such l.c. spaces E, we therefore have a kind of open-mapping theorem for Ω when V is of finite codimension in E'.

We now proceed with our characterization of $(V,\mathcal{M})$-semi-reflexivity.

<u>Proposition 2.1</u> If $M \in \mathcal{M}$, then $\overline{M}(\sigma(E,V)) \subset \overline{M}(\sigma(E,\overline{V}))$; since $\sigma(E,\overline{V})$ is finer than $\sigma(E,V)$, the two closures are equal.

Proof: Recall from the introduction that $\overline{V} = \overline{V}(T_M)$. Suppose $x \in \overline{M}(\sigma(E,V))$, and suppose that $w_1, . . . , w_n$ are in $\overline{V}$. Since $M \in \mathcal{M}$, and since $\mathcal{M}$ covers E, whenever $1 \leq i \leq n$, $(w_i + \frac{1}{3}\{x\}^{oE'}) \cap (w_i + \frac{1}{3}M^{oE'})$ is a T_M-neighborhood of w_i, and since $w_i \in \overline{V}$ there is a $v_i \in V \cap (w_i + \frac{1}{3}\{x\}^{oE'}) \cap (w_i + \frac{1}{3}M^{oE'})$. Now $(x + \frac{1}{3}\{v_1, . . . , v_n\}^{oE})$ is a $\sigma(E,V)$-neighborhood of x, and since $x \in \overline{M}(\sigma(E,V))$ there is an $m \in M \cap (x + \frac{1}{3}\{v_1, . . . , v_n\}^{oE})$. Then for $1 \leq i \leq n$, $|\langle m,w_i\rangle - \langle x,w_i\rangle| \leq |\langle m,w_i\rangle - \langle m,v_i\rangle| + |\langle m,v_i\rangle - \langle x,v_i\rangle| + |\langle x,v_i\rangle - \langle x,w_i\rangle| \leq \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1$; thus $m \in M \cap (x + \{w_1, . . . , w_n\}^{oE})$. We have shown that every $\sigma(E,\overline{V})$-neighborhood of x contains an $m \in M$. Thus $x \in \overline{M}(\sigma(E,\overline{V}))$.

Proposition 2.2 Suppose $M \in \mathcal{M}$. Then $\sigma(E,V)$ and $\sigma(E,\overline{V})$ induce the same topology on $\overline{M}$, where $\overline{M}$ denotes $\overline{M}(\sigma(E,V)) = \overline{M}(\sigma(E,\overline{V}))$.

Proof: Since $\sigma(E,\overline{V})$ is finer than $\sigma(E,V)$, the topology $\sigma(E,\overline{V})|\overline{M}$ is finer than the topology $\sigma(E,V)|\overline{M}$. To show that $\sigma(E,V)|\overline{M}$ is finer than $\sigma(E,\overline{V})|\overline{M}$, suppose $x_o \in \overline{M}$ and $w \in \overline{V}$. There is a $v \in V \cap (w + \frac{1}{5}\{x_o\}^{oE'}) \cap (w + \frac{1}{5}M^{oE'})$. Suppose $x \in \overline{M} \cap (x_o + \frac{1}{5}\{v\}^{oE})$; since $x \in \overline{M}$ there is an $m \in M \cap (x + \frac{1}{5}\{v,w\}^{oE})$, and it follows that $|\langle x_o,w\rangle - \langle x,w\rangle| \leq |\langle x_o,w\rangle - \langle x_o,v\rangle| + |\langle x_o,v\rangle - \langle x,v\rangle| + |\langle x,v\rangle - \langle m,v\rangle| + |\langle m,v\rangle - \langle m,w\rangle| + |\langle m,w\rangle - \langle x,w\rangle| \leq \frac{1}{5} + \frac{1}{5} + \frac{1}{5} + \frac{1}{5} + \frac{1}{5} = 1$. Thus $\overline{M} \cap (x_o + \frac{1}{5}\{v\}^{oE}) \subset \overline{M} \cap (x_o + \{w\}^{oE})$. We have shown that each member of a fundamental system of $\sigma(E,\overline{V})|\overline{M}$-neighborhoods of x_o contains a $\sigma(E,V)|\overline{M}$-neighborhood of x_o; thus the proof is complete.

We state in our notation a theorem of Luxemburg [12, p. 310]: Suppose V_1 and V_2 are linear subspaces of E'. Then $\overline{V}_1(T_M) = \overline{V}_2(T_M)$

if and only if the topologies $\sigma(E,V_1)|M$ and $\sigma(E,V_2)|M$ coincide for each $M\varepsilon\mathcal{M}$.

<u>Definition 2.3</u> Suppose Y is a $\sigma(E',E)$-dense, $\mathcal{T}_{\mathcal{M}}$-closed subspace of E'. Then Y is $\mathcal{T}_{\mathcal{M}}$-minimal in E' if and only if no proper subspace of Y is both $\sigma(E',E)$-dense and $\mathcal{T}_{\mathcal{M}}$-closed in E'.

The following is a generalization of a theorem of Krishnamurthy [9, Theorem 1]. Its proof is a straightforward adaptation of Krishnamurthy's proof. We provide it for completeness.

<u>Proposition 2.4</u> The following are equivalent:

(a) $\overline{V}$ is $\mathcal{T}_{\mathcal{M}}$-minimal in E'.

(b) $G = \Omega_G(E) \oplus (\overline{V})^{\perp G}$ (recall notation from introduction).

(c) Each $M \varepsilon \mathcal{M}$ is relatively $\sigma(E,\overline{V})$-compact.

<u>Proof</u>:

(a)$\Longrightarrow$(b): Since V is $\sigma(E',E)$-dense in E', so is $\overline{V}$. Suppose $e\varepsilon E$ and $\Omega_G(e)\varepsilon(\overline{V})^{\perp G}$. For all $w\varepsilon\overline{V}$, $<\Omega_G(e),w> = <e,w> = 0$; thus $e = 0$, and $\Omega_G(e) = 0$. Therefore $\Omega_G(E)\cap(\overline{V})^{\perp G} = \{0\}$. To show that (b) holds, we show that each $z\varepsilon G$ has a representation $z = z' + z''$ where $z'\varepsilon\,\Omega_G(E)$ and $z'' \;\varepsilon\; (\overline{V})^{\perp G}$; since $\Omega_G(E)\cap(\overline{V})^{\perp G} = \{0\}$, we already know that such a representation for z will be unique. Suppose $z\varepsilon G$, $z \notin (\overline{V})^{\perp G}$. Let $W = (\ker z)\cap\overline{V}$. Then W is a $\mathcal{T}_{\mathcal{M}}$-closed subspace of E', and is properly contained in $\overline{V}$ since $z \notin (\overline{V})^{\perp G}$. Then (a) implies that W is not $\sigma(E',E)$-dense in E'. There is an $e\varepsilon E$, $e \neq 0$, such that for all $w\varepsilon W$, $<e,w> = 0$. The restriction $\Omega_G(e)|\overline{V}$ of $\Omega_G(e)$ to $\overline{V}$ is a linear functional on $\overline{V}$; so is the restriction $z|\overline{V}$ of z to $\overline{V}$, and $W = \ker z|\overline{V}$. Thus W and $(\ker \Omega_G(e)|\overline{V})$ are hyperplanes in $\overline{V}$ such that $W \subset (\ker \Omega_G(e)|\overline{V})$. Therefore $W = (\ker \Omega_G(e)|\overline{V})$; i.e., $\ker(z|\overline{V}) = \ker(\Omega_G(e)|\overline{V})$. The equations of these identical hyperplanes are $z|\overline{V} = 0$ and $\Omega_G(e)|\overline{V} = 0$, respectively. There is a non-zero scalar λ (where $\lambda = \frac{<z,v>}{<e,v>}$ for some $v\varepsilon V$, $v\notin W$) such that $z|\overline{V} \equiv \lambda\cdot\Omega_G(e)|\overline{V}$. Since $\Omega_G(e) \in \Omega_G(E)$, then $\lambda\cdot\Omega_G(e) \in \Omega_G(e)$. Let $z' = \lambda\cdot\Omega_G(e)$, and let $z'' = z - z'$. Then $z'' = (z - \lambda\cdot\Omega_G(e)) \in (\overline{V})^{\perp G}$, and $z' \in \Omega_G(E)$, and $z = z' + z''$. Thus (b) is proved.

(b)$\Longrightarrow$(c): Suppose (b) holds. Denote $(\overline{V}[\mathcal{T}_M|\overline{V}])'$ by $\overline{V}'$. Define the linear map $\overline{\Omega}:E\to\overline{V}'$ so that $\overline{\Omega}(e) = \Omega_G(e)|\overline{V}$ for $e\varepsilon E$. Then $\overline{\Omega}$ is one-to-one, for if $e\varepsilon E$, $e \neq 0$, then for some $v\varepsilon\overline{V}$, $0 \neq <e,v> = <\Omega_G(e),v> = <\overline{\Omega}(e),v>$, and thus $\overline{\Omega}(e) \neq 0$. Moreover, $\overline{\Omega}$ maps E <u>onto</u> $\overline{V}'$, for if $f\varepsilon\overline{V}'$ there is an $e'' \in G$ such that $e''|\overline{V} \equiv f$, and (b) implies there are an $e\varepsilon E$ and a $z''\varepsilon(\overline{V})^{\perp G}$ such that $e'' \equiv \Omega_G(e) + z''$; if $x\varepsilon\overline{V}$ then $<\overline{\Omega}(e),x> = <\Omega_G(e),x> = <\Omega_G(e),x> + <z'',x> = <\Omega_G(e) + z'',x> = <e'',x>$; thus $\overline{\Omega}(e) \equiv e''|\overline{V} \equiv f$. Therefore $\overline{\Omega}:E\to\overline{V}'$ is a vector-space isomorphism, and $<\overline{\Omega}(e),f> = <e,f>$ whenever $e\varepsilon E$ and $f\varepsilon\overline{V}$. Therefore the pairings $<E,\overline{V}>$ and $<\overline{V}',\overline{V}>$ are algebraically isomorphic with respect to $\overline{\Omega}:E\to\overline{V}'$ and id: $\overline{V}\to\overline{V}$, and it follows that $\overline{\Omega}:E[\sigma(E,\overline{V})] \to\overline{V}'[\sigma(\overline{V}',\overline{V})]$ is a linear homeomorphism. To show that (c) holds, suppose $M\varepsilon\mathcal{M}$; then $M^{oE'\ oE} \in \mathcal{M}$. Denote $M^{oE'\ oE}$ by M_1. Then $M_1^{o\overline{V}}$ is a $\mathcal{T}_M|\overline{V}$-neighborhood of 0 in $\overline{V}$; the Alaoglu-Bourbaki theorem [6, p. 201] implies that $M_1^{o\overline{V}\ o\overline{V}'}$ is $\sigma(V',V)$-compact. But $M_1^{o\overline{V}\ oE} = \overline{\Omega}^{-1}(M_1^{o\overline{V}\ o\overline{V}'})$; thus $M_1^{o\overline{V}\ oE}$ is $\sigma(E,\overline{V})$-compact. Now

$\overline{M}(\sigma(E,\overline{V})) \subset \overline{M_1}(\sigma(E,\overline{V})) = M_1^{\circ\overline{V}\ \circ E}$, so that M is relatively $\sigma(E,\overline{V})$-compact.

(c)$\Longrightarrow$(a): Assume (c) holds. Suppose W is a $\sigma(E',E)$-dense, $\mathcal{T}_{\mathcal{M}}$-closed subspace of E', and $W \subset \overline{V}$. Then W separates points of E, and $\sigma(E,W)$ is a Hausdorff topology [6, p. 196, Prop. 1] on E which is coarser than $\sigma(E,\overline{V})$. For each $M \in \mathcal{M}$, denote $\overline{M}(\sigma(E,\overline{V}))$ by $\overline{M}$. Then for each $M \in \mathcal{M}$, $\sigma(E,W)|\overline{M}$ is a Hausdorff topology on $\overline{M}$ and is coarser than $\sigma(E,\overline{V})|\overline{M}$, for which $\overline{M}$ is compact. Therefore $\sigma(E,W)|\overline{M} = \sigma(E,\overline{V})|\overline{M}$, and $\sigma(E,W)|M = \sigma(E,\overline{V})|M$ for all $M \in \mathcal{M}$. By Luxemburg's theorem, as stated immediately preceding Definition 2.3, $\overline{W}(\mathcal{T}_{\mathcal{M}}) = \overline{V}(\mathcal{T}_{\mathcal{M}})$. But since W and $\overline{V}$ are already $\mathcal{T}_{\mathcal{M}}$-closed, $W = \overline{V}$. This proves that (a) holds.

<u>Theorem 2.5</u> Suppose E is an l.c. space, $\mathcal{M}$ is a saturated collection of bounded subsets of E covering E, V is a $\sigma(E',E)$-dense subspace of E', and $\overline{V}$ denotes $\overline{V}(\mathcal{T}_{\mathcal{M}})$. Then the following are equivalent:

(a) $\overline{V}$ is $\mathcal{T}_{\mathcal{M}}$-minimal in E'.

(b) If $M \in \mathcal{M}$, then $\overline{M}(\sigma(E,V))$ is complete for $\sigma(E,V)$.

(b') If $M \in \mathcal{M}$, then every $\sigma(E,V)$-Cauchy filter (or net) in M converges for $\sigma(E,V)$ to a point of E.

(c) Each $M \in \mathcal{M}$ is relatively $\sigma(E,\overline{V})$-compact.

(c') Each $M \in \mathcal{M}$ is relatively $\sigma(E,V)$-compact.

(d) E is $(\overline{V},\mathcal{M})$-semi-reflexive.

(d') E is $(V,\mathcal{M})$-semi-reflexive.

<u>Proof</u>: Since $\sigma(E,V)$ is coarser than $\sigma(E,\overline{V})$, which in turn is coarser than $\sigma(E,E')$, and since each $M \in \mathcal{M}$ is $\sigma(E,E')$-bounded, each

$M \in \mathcal{M}$ is also $\sigma(E,V)$-bounded and $\sigma(E,\overline{V})$-bounded. As we noted in Remark 1.3, (d') holds if and only if $\mathcal{T}_{\mathcal{M}}$ is compatible with the pairing $\langle E,V\rangle$ [6, p. 198, Def. 1]; similarly, (d) holds if and only if $\mathcal{T}_{\mathcal{M}}$ is compatible with the pairing $\langle E,\overline{V}\rangle$. By the corollary on page 205 of [6] which precedes Horváth's statement and proof of the Mackey-Arens theorem, (c') and (d') are equivalent; similarly, (c) and (d) are equivalent. Propositions 2.1 and 2.2 imply that (c) and (c') are equivalent. Proposition 2.4 implies that (a) and (c) are equivalent. We have shown that (a), (c), (c'), (d), and (d') are all equivalent. If suffices now to show that (b), (b') and (c') are equivalent. Suppose $M \in \mathcal{M}$; since M is $\sigma(E,V)$-bounded, so is $\overline{M}(\sigma(E,V))$; thus $\overline{M}(\sigma(E,V))$ is $\sigma(E,V)$-precompact [8, p. 248, (3)]. Thus $\overline{M}(\sigma(E,V))$ is complete for $\sigma(E,V)$ if and only if $\overline{M}(\sigma(E,V))$ is $\sigma(E,V)$-compact [6, p. 145, Def. 1 and remarks preceding and following]. Therefore (b) and (c') are equivalent. Now clearly (b) imples (b'). If (b') holds, then each such filter (or net) as described in (b') converges to a point in $\overline{M}(\sigma(E,V))$; by Proposition 9 on page 186 of [2], (b) holds.

<u>Remark 2.6</u> In view of the equivalence of (d) and (d') shown in Theorem 2.5, the following remarks about (d) apply equally to (d'). For topologies $\mathcal{T}$ and $\mathcal{U}$ on the same set, we write $\mathcal{T} \leqq \mathcal{U}$ to denote that $\mathcal{T}$ is coarser than $\mathcal{U}$. In the setting of Theorem 2.5, $\sigma(E,V) \leqq \sigma(E,E')$; thus, if $A \subset E$ and A is $\sigma(E,E')$-compact, then A is $\sigma(E,V)$-compact, since a $\sigma(E,V)$-open cover of A is a $\sigma(E,E')$-open cover of A (cf. [6, p. 143, (C4)]). Therefore $\tau(E',E)|V \leqq \tau(V,E)$ (cf. [6, p. 206, Def. 1]).

Also, $\sigma(E',E)|V = \sigma(V,E)$. Thus if $\sigma(E',E) \leqq \mathcal{T}_M \leqq \tau(E',E)$, then $\sigma(V,E) \leqq \mathcal{T}_M|V \leqq \tau(E',E)|V \leqq \tau(V,E)$, and it follows [6, p. 206, Prop. 4] that E is $(V,\mathcal{M})$-semi-reflexive. Of course, there are many examples of l.c. spaces E which are semi-reflexive; i.e., $(V,\mathcal{M})$-semi-reflexive where $V = E'$ and $\mathcal{T}_M = \beta(E',E)$. In Section 4, however, we give an example of an l.c. space E, a $\sigma(E',E)$-dense subspace V of E', and a collection $\mathcal{M}$ such that $\tau(E',E) \leqq \mathcal{T}_M \leqq \beta(E',E)$ and $\tau(E',E) \neq \mathcal{T}_M \neq \beta(E',E)$, and E is $(V,\mathcal{M})$-semi-reflexive but <u>not</u> V-semi-reflexive.

<u>Remark 2.7</u> Although Theorem 2.5 shows that $(V,\mathcal{M})$-semi-reflexivity is equivalent to $(\overline{V},\mathcal{M})$-semi-reflexivity, we give an example in Section 4 which shows that $(V,\mathcal{M})$-reflexivity is <u>not</u> equivalent to $(\overline{V},\mathcal{M})$-reflexivity; in fact the example shows an l.c. space E and a $\sigma(E',E)$-dense subspace V of E' such that E is V-reflexive but <u>not</u> $\overline{V}$-reflexive. Further observations about $(V,\mathcal{M},\mathfrak{S})$-reflexivity are made in Section 3.

<u>Corollary 2.8</u> Suppose $\mathcal{M}_1$ and $\mathcal{M}_2$ are saturated collections of bounded subsets of E covering E, and $\mathcal{M}_1 \subset \mathcal{M}_2$. If E is $(V,\mathcal{M}_2)$-semi-reflexive, then E is $(V,\mathcal{M}_1)$-semi-reflexive.

<u>Proof</u>: Each $M \in \mathcal{M}_2$ is relatively $\sigma(E,V)$-compact. Since $\mathcal{M}_1 \subset \mathcal{M}_2$, each $M \in \mathcal{M}_1$ is relatively $\sigma(E,V)$-compact.

<u>Corollary 2.9</u> Suppose V_1 and V_2 are $\sigma(E',E)$-dense subspaces of E', and $V_1 \subset V_2$. If E is $(V_2,\mathcal{M})$-semi-reflexive, E is $(V_1,\mathcal{M})$-semi-reflexive.

<u>Proof</u>: Suppose W is a $\mathcal{T}_M$-closed, proper subspace of $\overline{V}_1(\mathcal{T}_M)$; W is a $\mathcal{T}_M$-closed, proper subspace of $\overline{V}_2(\mathcal{T}_M)$. Since $\overline{V}_2$ is $\mathcal{T}_M$-minimal in E', W is not $\sigma(E',E)$ dense in E'. Therefore $\overline{V}_1$ is $\mathcal{T}_M$-minimal in E'.

In a Banach space E, if S_E denotes the norm-closed unit ball, then the positive multiples of S_E are all bounded, and every bounded subset of E is contained in a positive multiple of S_E. Thus, the collection $\mathcal{M}$ of <u>all</u> bounded subsets of E is the collection of all subsets of positive multiples of S_E. By Theorem 2.5, in order for E to be V-semi-reflexive (i.e., V-pseudo-reflexive), it is therefore necessary and sufficient that one of the following be true:

(a) $\overline{V}(\beta(E',E))$ is $\beta(E',E)$-minimal in E'; i.e., the norm-closure of V is norm-minimal in E'.

(b) $\overline{S_E}(\sigma(E,V))$ is complete for $\sigma(E,V)$ (using the fact that a closed subset of a complete set is complete).

(c) S_E is relatively $\sigma(E,V)$-compact (using the fact that a closed subset of a compact space is compact).

These are precisely the results of Dixmier [5], Ruston [13], Civin and Yood [4], and Singer [15] regarding characterizations of V-pseudo-reflexivity of E in terms of norm-closed, norm-minimal subspaces V of E', and in terms of conditional (i.e., relative) $\sigma(E,V)$-compactness and conditional $\sigma(E,V)$-completeness of S_E. Their results further characterized what they called V-reflexivity of E (i.e., V-pseudo-reflexivity where Ω is an isometry) in terms of such subspaces V of E' which are also of characteristic 1 (cf. Dixmier [5]) or duxial (cf. Ruston [13]), and in terms of strict $\sigma(E,V)$-compactness (cf. Dixmier [5], and Singer [15] and [14]) and strict $\sigma(E,V)$-completeness (cf. Singer [17]) of S_E.

In [16], [17], and [18], Singer gave characterizations of V-pseudo-reflexivity of a separable Banach space E in terms of conditions of $\sigma(E,V)$-sequential compactness and $\sigma(E,V)$-sequential completeness of S_E. Our next results on the $(V,\mathcal{M})$-semi-reflexivity of an l.c. space E are concerned with cases in which V is $\sigma(V,E)$-separable. The next two remarks pertain to the applicability of results obtained under this assumption.

Remark 2.10 If $E[\mathcal{T}]$ and $F[\mathcal{U}]$ are l.c. spaces, and F is separable, and there is a linear homeomorphism Φ as in the hypothesis of Theorem 1.6, then by the conclusion of Theorem 1.6, $V[\mathcal{T}_{\mathcal{M}}|V]$ is separable. Since $\sigma(E',E) \leqq \mathcal{T}_{\mathcal{M}}$ and $\sigma(E',E)|V = \sigma(V,E)$, it follows that $\sigma(V,E) \leqq \mathcal{T}_{\mathcal{M}}|V$, and therefore V is $\sigma(V,E)$-separable.

Remark 2.11 If E is a separable, metrizable l.c. space and V is an arbitrary linear subspace of E', then V is $\sigma(V,E)$-separable. To see this, suppose $U_1, U_2, U_3, \ldots$ is a fundamental sequence of neighborhoods of 0 in E. For each n, $U_n^{oE'}$ is a $\sigma(E',E)$-closed, equicontinuous subset of E'; since E is separable, $U_n^{oE'}[\sigma_i(E',E)]$ is a compact metric space [3, Ch. IV, p. 66], and is therefore separable. Moreover, $U_n^{oV} = U_n^{oE'} \cap V$, and $\sigma(E',E)|V = \sigma(V,E)$; thus $U_n^{oV}[\sigma_i(V,E)]$ has the topology induced on U_n^{oV} by the topology of $U_n^{oE'}[\sigma_i(E',E)]$. Therefore U_n^{oV}, as a subset of the separable metric space $U_n^{oE'}[\sigma_i(E',E)]$, is separable for $\sigma_i(V,E)$. Now $E' = \bigcup_{n=1}^{\infty} U_n^{oE'}$, and so $V = \bigcup_{n=1}^{\infty} U_n^{oV}$. Since V is thus a countable union of subsets which are separable for $\sigma_i(V,E)$, it follows that V is separable for $\sigma(V,E)$.

The following theorem and its proof are straightforward adaptations of those of Lohman [10, Thm. 2.15] and [11, Thm. 1].

Theorem 2.12 Suppose E is an l.c. space, $\mathcal{M}$ is a saturated collection of bounded subsets of E covering E, and V is a $\sigma(E',E)$-dense subspace of E' which is $\sigma(V,E)$-separable. Then the following are equivalent:

(a) $\overline{V}$ is $\mathcal{T}_{\mathcal{M}}$-minimal in E'.

(b) If $M \in \mathcal{M}$, then each $\sigma(E,V)$-Cauchy filter (or net) in M converges for $\sigma(E,V)$ to a point of E.

(c) Each $M \in \mathcal{M}$ is relatively $\sigma(E,V)$-compact.

(d) Each $M \in \mathcal{M}$ is relatively $\sigma(E,V)$-sequentially compact.

(e) Each $M \in \mathcal{M}$ is relatively $\sigma(E,V)$-countably compact.

(f) E is $(V,\mathcal{M})$-semi-reflexive.

Proof: By Theorem 2.5, (a), (b), (c), and (f) are all equivalent. By definition, (d) implies (e). By Šmulian's theorem [8, p. 311], (e) implies (d); indeed, if V is considered as the dual of $E[\sigma(E,V)]$, our hypothesis is that this dual is weakly separable, and by Šmulian's theorem every relatively $\sigma(E,V)$-countably compact subset of E is relatively $\sigma(E,V)$-sequentially compact. It suffices now to show that (a) and (e) are equivalent. To show that (e) implies (a): Suppose $f \in V'$; then $\{f\}^{oV}$ is a $\sigma(V,V')$-neighborhood of 0, hence a $\mathcal{T}_{\mathcal{M}}|V$-neighborhood of 0 in V. There is a balanced, convex $M \in \mathcal{M}$ such that $M^{oV} \subset \{f\}^{oV}$; since $M^{oV} = (\Omega(M))^{oV}$, it follows that $f \in M^{oV\ oV'} = (\Omega(M))^{oV\ oV'} = \overline{\Omega(M)}(\sigma(V',V))$. There is a net $\{x_\alpha\}_{\alpha \in A}$ in M such that the net $\{\Omega(x_\alpha)\}$

converges to f for $\sigma(V',V)$. Thus each $y \varepsilon V$, considered as a $\sigma(V',V)$-continuous linear functional on V', has the property that $\langle f,y\rangle = \langle \lim_\alpha \Omega(x_\alpha),y\rangle = \lim_\alpha \langle\Omega(x_\alpha),y\rangle = \lim_\alpha \langle x_\alpha,y\rangle$. Let $\{y_n\}$ be a $\sigma(V,E)$-dense sequence in V. Choose α_1 so that $\alpha \geqq \alpha_1$ implies $|\langle f,y_1\rangle - \langle x_\alpha,y_1\rangle| \leqq 1$. Inductively, for each n choose $\alpha_{n+1} \geqq \alpha_n$ so that $\alpha \geqq \alpha_{n+1}$ implies $|\langle f,y_i\rangle - \langle x_\alpha,y_i\rangle| \leqq \frac{1}{n+1}$ for $1 \leqq i \leqq n+1$. It follows that for each n, $\langle f,y_n\rangle = \lim_{m\to\infty} \langle x_{\alpha_m},y_n\rangle$. Since M is relatively countably compact, the sequence $\{x_{\alpha_m}\}$ has a $\sigma(E,V)$-cluster point $z \varepsilon E$. For each n, if $\varepsilon>0$ there is an integer j such that $|\langle f,y_n\rangle - \langle x_{\alpha_m},y_n\rangle| \leqq \frac{\varepsilon}{2}$ whenever $m \geqq j$, and there is an $m \geqq j$ such that $x_{\alpha_m} \varepsilon z + \frac{\varepsilon}{2} \{y_n\}^{OE}$. Hence $|\langle f,y_n\rangle - \langle z,y_n\rangle| \leqq |\langle f,y_n\rangle - \langle x_{\alpha_m},y_n\rangle| + |\langle x_{\alpha_m},y_n\rangle - \langle z,y_n\rangle| \leqq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. Thus for each n, $\langle f,y_n\rangle = \langle z,y_n\rangle$. Now for each $y \varepsilon V$ we repeat the process to obtain for the sequence $y,y_1,y_2,y_3,\ldots$ another sequence $\{x'_{\alpha_m}\}$ in M and a $\sigma(E,V)$-cluster point z' of $\{x'_{\alpha_m}\}$ in E such that $\langle f,y\rangle = \langle z',y\rangle$ and $\langle f,y_n\rangle = \langle z',y_n\rangle$ for each n. Then $\langle z,y_n\rangle = \langle z',y_n\rangle$ for each n; since V separates points of E and $\{y_n\}$ is $\sigma(V,E)$-dense in V, then $\{y_n\}$ separates points of E; therefore $z = z'$. Hence, $\langle f,y\rangle = \langle z,y\rangle$ for all $y \varepsilon V$; that is, $\Omega(z) \equiv f$. We have shown that Ω maps E <u>onto</u> V'; thus (a) holds. To show that (a) implies (e): Let $M \varepsilon \mathcal{M}$; since (a) implies (c), $\overline{M}(\sigma(E,V))$ is $\sigma(E,V)$-compact. Since $\{y_n\}$ separates points of $\overline{M}(\sigma(E,V))$, by Theorem 16.7 on page 143 of [7], $\overline{M}[\sigma i(E,V)]$ is metrizable. Since $\overline{M}[\sigma_i(E,V)]$ is a compact metric space, every infinite sequence in M has a $\sigma(E,V)$-cluster point in E. Thus E is relatively $\sigma(E,V)$-countably compact; i.e., (e) holds.

Remark 2.13 Theorem 2.12 fails without the assumption that V is $\sigma(V,E)$-separable. Lohman has given an example [11, Ex. 1] of a V-semi-reflexive Banach space E in which S_E is not relatively $\sigma(E,V)$-sequentially compact. However, Singer [18, Thm. 3] has proved that if E is a Banach space, and V is of finite codimension in E', and $\mathcal{T}_M = \beta(E',E)$, then (a) through (f) are equivalent even without the $\sigma(V,E)$-separability of V.

Corollary 2.14 If V is $\mathcal{T}_M|V$-separable, then (a) through (f) of Theorem 2.12 are all equivalent, and each is equivalent to:
(g) If $M \in \mathcal{M}$, then each $\sigma(E,V)$-Cauchy sequence in M converges to a point of E for $\sigma(E,V)$.

Proof: Since V is $\mathcal{T}_M|V$-separable, and $\sigma(V,E) \leq \mathcal{T}_M|V$, then V is $\sigma(V,E)$-separable, and Theorem 2.12 applies. We identify E with $\Omega(E)$ as a subspace of V' in the pairing $\langle V',V \rangle$. Then $\sigma(V',V)$ induces $\sigma(E,V)$ on E. Let $M \in \mathcal{M}$; denote $\overline{M}(\sigma(E,V))$ by $\overline{M}$, and $\overline{M}(\sigma(V',V))$ by $\overline{M}'$. Then $\overline{M} = \overline{M}' \cap E$. Now M^{oV} is a $\mathcal{T}_M|V$-neighborhood of 0 in V, so $M^{oV\ oV'}$ is a $\sigma(V',V)$-closed, $\mathcal{T}_M|V$-equicontinuous subset of V'. Thus $M^{oV\ oV'}[\sigma_i(V',V)]$ is a compact metric space [2, Ch. IV, p. 66]. Now $\overline{M} \subset \overline{M}' \subset M^{oV\ oV'}$, and $\overline{M}[\sigma_i(E,V)] = \overline{M}[\sigma_i(V',V)]$. If (g) holds, then each $\sigma(E,V)$-Cauchy sequence in M converges to a point of $\overline{M}$ for $\sigma(E,V)$, and $\overline{M}$ is a complete subset of $M^{oV\ oV'}[\sigma_i(V',V)]$. Thus $\overline{M}[\sigma_i(E,V)]$ is a compact metric space, and (c) holds. We have shown that (g) implies (c). Clearly (b) implies (g). The corollary is proved.

Proposition 2.15 With the same hypothesis, notation, and identifications as in Corollary 2.14, if $M \varepsilon \mathcal{M}$, then for each $v' \varepsilon \overline{M}'$, there is a sequence $\{m_i\}$ in M such that for all $v \varepsilon V$, $\langle v',v\rangle = \lim_{i\to\infty} \langle m_i,v\rangle$.

Proof: As we saw in the proof of Corollary 2.14, $\overline{M}'[\sigma_i(V',V)]$ is a metric space. There is a sequence $\{m_i\}$ in M which converges to v' for $\sigma(V',V)$; i.e., for all $v \varepsilon V$, $\langle v',v\rangle = \lim_{i\to\infty} \langle m_i,v\rangle$.

The following is a proposition proved by Lohman [11, Lemma 4]: Suppose V is a $\sigma(E',E)$-dense, $\sigma(V,E)$-separable subspace of E. If each decreasing sequence of nonempty, $\sigma(E,V)$-bounded, $\sigma(E,V)$-closed, convex subsets of E has a nonempty intersection, then E is V-semi-reflexive.

Proposition 2.16 If V is a $\sigma(E',E)$ dense, $\sigma(V,E)$-separable subspace of E', $\mathcal{M}$ is a saturated collection of bounded subsets of E covering E, and each decreasing sequence of nonempty, $\sigma(E,V)$-bounded, $\sigma(E,V)$-closed, convex subsets of E has a nonempty intersection, then E is $(V,\mathcal{M})$-semi-reflexive.

Proof: By Lohman's Lemma 4 as stated above, E is V-semi-reflexive; i.e., $(V,\mathcal{M}_2)$-semi-reflexive where $\mathcal{M}_2$ is the collection of all bounded subsets of E. Therefore, by Corollary 2.8, E is $(V,\mathcal{M})$-semi-reflexive.

Proposition 2.17 Suppose V is a $\sigma(E',E)$-dense subspace of E', and $\mathcal{M}$ is a saturated collection of bounded subsets of E covering E, and $V[\mathcal{T}_{\mathcal{M}}|V]$ is barrelled. If E is $(V,\mathcal{M})$-semi-reflexive, then each $\sigma(E,V)$-bounded subset of E is relatively $\sigma(E,V)$-compact.

Proof: Since E is $(V,\mathcal{M})$-semi-reflexive, the two pairings $\langle E,V\rangle$ and $\langle V',V\rangle$ are algebraically isomorphic with respect to $\Omega:E\to V'$ and $id:V\to V$. If B is a $\sigma(E,V)$-bounded subset of E, then $\Omega(B)$ is a $\sigma(V',V)$-bounded subset of V'. Since $V[\mathcal{T}_{\mathcal{M}}|V]$ is barrelled, $\Omega(B)$ is therefore relatively $\sigma(V',V)$-compact. Since $\Omega:E[\sigma(E,V)] \to V'[\sigma(V',V)]$ is a linear homeomorphism, B is relatively $\sigma(E,V)$-compact.

Theorem 2.18 Suppose V is $\sigma(E',E)$-dense subspace of E', and $\mathcal{M}$ is a saturated collection of bounded subsets of E covering E, and V is $\sigma(V,E)$-separable and $\mathcal{T}_{\mathcal{M}}|V$-barrelled. Then E is $(V,\mathcal{M})$-semi-reflexive if and only if each decreasing sequence of nonempty, $\sigma(E,V)$-bounded, $\sigma(E,V)$-closed, convex subsets of E has a nonempty intersection.

Proof: Sufficiency of the condition is proved by Proposition 2.16. Necessity is proved by Proposition 2.17, and the fact that a decreasing sequence of nonempty, $\sigma(E,V)$-bounded, $\sigma(E,V)$-closed, convex subsets of E is therefore a collection of $\sigma(E,V)$-closed subsets of a $\sigma(E,V)$-compact set, such that the collection has the finite intersection property.

Remark 2.19 Theorem 2.18 fails without the separability hypothesis. Lohman has given an example [11, Ex. 3] of a Banach space E and a $\sigma(E',E)$-dense subspace V of E' such that the intersection property holds but E is not V-semi-reflexive. However, see Singer [18, Thm. 4] for a result similar to Theorem 2.18.

In the absence of the $\sigma(V,E)$-separability of V, the following proposition holds. It is adapted from [10, Thm. 2.20] of Lohman.

<u>Proposition 2.20</u> Suppose V is a $\sigma(E',E)$-dense subspace of E such that each family of non-empty, convex, $\sigma(E,V)$-closed and bounded subsets of E which is directed by inclusion (i.e., the intersection of any two contains a third) has non-empty intersection. Then for any saturated collection $\mathcal{M}$ of bounded subsets of E covering E, E is $(V,\mathcal{M})$-semi-reflexive.

<u>Proof</u>: Suppose $M_o \in \mathcal{M}$. There is a balanced, convex $M \in \mathcal{M}$ such that $M_o \subset M$. Denote $\overline{M}(\sigma(E,V))$ by $\overline{M}$. Since M is $\sigma(E,V)$-bounded, so is $\overline{M}$. Thus $\overline{M}$ is $\sigma(E,V)$-precompact [8, p. 248]. We show that $\overline{M}$ is also $\sigma(E,V)$-complete, hence $\sigma(E,V)$-compact. Let $\{x_\alpha\}\alpha \in A$ be a $\sigma(E,V)$-Cauchy net in $\overline{M}$. For each $\alpha_o \varepsilon A$, let $B_{\alpha_o} = <\{x_\alpha | \alpha \geqq \alpha_o\}>$ denote the convex hull of $\{x_\alpha | \alpha \geqq \alpha_o\}$, and $\overline{B_{\alpha_o}} = \overline{<\{x_\alpha | \alpha \geqq \alpha_o\}>}$ the $\sigma(E,V)$-closure of B_{α_o} in E. Then $\mathcal{F} = \{\overline{B_{\alpha_o}} | \alpha_o \varepsilon A\}$ is a family of non-empty, convex, $\sigma(E,V)$-closed and bounded subsets of E. Suppose $\alpha_o, \alpha'_o \varepsilon A$. There is an $\alpha''_o \varepsilon A$ such that $\alpha_o'' \geqq \alpha_o'$ and $\alpha_o'' \geqq \alpha_o$. Thus $B_{\alpha_o''} \subset B_{\alpha_o'} \cap B_{\alpha_o}$, and $\overline{B_{\alpha_o''}} \subset \overline{B_{\alpha_o'} \cap B_{\alpha_o}} \subset \overline{B_{\alpha_o'}} \cap \overline{B_{\alpha_o}}$. Hence $\mathcal{F}$ is directed by inclusion. By hypothesis, there is an $x \in \bigcap_{\alpha_o \varepsilon A} \overline{B_{\alpha_o}}$. Note that, since $B_{\alpha_o} \subset \overline{M}$ and $\overline{B_{\alpha_o}} \subset \overline{M}$, we have $x \in \overline{M}$. We show that $\{x_\alpha\}$ converges to x with respect to $\sigma(E,V)$. Let U be a balanced, convex, $\sigma(E,V)$-neighborhood of 0 in E. There is an $\alpha_o \in A$ such that if $\alpha_1, \alpha_2 \geqq \alpha_o$ then $x_{\alpha_1} - x_{\alpha_2} \in \frac{1}{2} U$. Let $\alpha \geqq \alpha_o$. We show that $x - x_\alpha \in U$; indeed, there is a $y \in B_\alpha \cap (x + \frac{1}{2}U)$, and there are elements $x_{\alpha_1}, \ldots, x_{\alpha_n} \varepsilon B_\alpha$ and $a_1, \ldots, a_n > 0$ such that $\sum_{i=1}^n a_i = 1$ and $y = \sum_{i=1}^n a_i x_{\alpha_i}$. Therefore $x - x_\alpha = (x-y) + (y-x_\alpha) = (x-y) + \sum_{i=1}^n a_i(x_{\alpha_i} - x_\alpha)$.

Now $(x-y) \in \frac{1}{2}U$; also, for $1 \leq i \leq n$, $a_i(x_{\alpha_i}-x_\alpha) \in \frac{a_i}{2}U$, so that $\sum_{i=1}^{n} a_i(x_{\alpha_i}-x_{\alpha_o}) \in \frac{1}{2}U$. Thus $x - x_\alpha \in \frac{1}{2}U + \frac{1}{2}U = U$. Thus $\overline{M}$ is $\sigma(E,V)$-compact, and so is $\overline{M}_o \subseteq \overline{M}$. By Theorem 2.5, E is $(V,\mathcal{M})$-semi-reflexive.

Theorem 2.21 Suppose V is $\sigma(E',E)$-dense subspace of V, $\mathcal{M}$ is a saturated collection of bounded subsets of E covering E, and $V[\mathcal{T}_{\mathcal{M}}|V]$ is barrelled. Then E is $(V,\mathcal{M})$-semi-reflexive if and only if each family of non-empty, convex, $\sigma(E,V)$-closed and bounded subsets of E which is directed by inclusion has non-empty intersection.

Proof: Proposition 2.20 proves the sufficiency of the intersection condition. Conversely, if E is $(V,\mathcal{M})$-semi-reflexive, then by Proposition 2.17 each set in such a family is $\sigma(E,V)$-compact, and by the finite intersection property the family has non-empty intersection.

Since the hypotheses of Proposition 2.17 and Theorem 2.21 include the assumption that $V[\mathcal{T}_{\mathcal{M}}|V]$ is barrelled, and each is concerned with the $(V,\mathcal{M})$-semi-reflexivity of E, we give the following general result concerning the two properties.

Proposition 2.22 Suppose $E[\mathcal{T}]$ is an l.c. space, $\mathcal{M}$ is a saturated collection of bounded subsets of E covering E, and V is a $\sigma(E',E)$-dense subspace of E'. The following are equivalent:

(a) $E[\mathcal{T}]$ is $(V,\mathcal{M})$-semi-reflexive, and $V[\mathcal{T}_{\mathcal{M}}|V]$ is barrelled.

(b) $E[\sigma(E,V)]$ is semi-reflexive, and for each $\sigma(E,V)$-bounded $B \subset E$ there is an $M \in \mathcal{M}$ such that $B \subset \overline{M}\ (\sigma(E,V))$.

(c) $E[\sigma(E,V)]$ is quasi-complete, and for each $\sigma(E,V)$-bounded $B \subset E$ there is an $M \in \mathcal{M}$ such that $B \subset \overline{M}(\sigma(E,V))$.

<u>Proof</u>: Since V is the dual of $E[\sigma(E,V)]$, it follows [6, p. 228, Prop. 2] that $E[\sigma(E,V)]$ is semi-reflexive if and only if $E[\sigma(E,V)]$ is quasi-complete. Thus (b) and (c) are equivalent. We prove that (a) and (b) are equivalent.

First note that, if $\mathcal{W} = \{A \subset V' \mid \text{for some } M \in \mathcal{M},\ A \subset M^{oV\ oV'}\}$ then $\mathcal{W}$ is the collection of all $\mathcal{T}_{\mathcal{M}}|V$-equicontinuous subsets of V' [6, p. 200, Prop. 6]. Note also that $V[\mathcal{T}_{\mathcal{M}}|V]$ is barrelled if and only if every $\sigma(V',V)$-bounded subset of V' is in $\mathcal{W}$ [6, p. 212, Prop. 2].

Suppose (a) holds. Then the pairings $\langle E,V\rangle$ and $\langle V',V\rangle$ are algebraically isomorphic with respect to $\Omega:E\to V'$ and $id:V\to V$, and $\Omega:E[\sigma(E,V)] \to V'[\sigma(V',V)]$ is a linear homeomorphism. Suppose $B \subset E$ is $\sigma(E,V)$-bounded. Then $\Omega(B)$ is $\sigma(V',V)$-bounded. Since $V[\mathcal{T}_{\mathcal{M}}|V]$ is barrelled, $\Omega(B)$ is relatively $\sigma(V',V)$-compact [6, p. 212, Cor.]. Thus B is relatively $\sigma(E,V)$-compact. We have shown that every $\sigma(E,V)$-bounded subset of E is relatively $\sigma(E,V)$-compact. Thus $E[\sigma(E,V)]$ is semi-reflexive. Moreover, since $\Omega(B)$ is $\sigma(V',V)$-bounded, and $V[\mathcal{T}_{\mathcal{M}}|V]$ is barrelled, $\Omega(B) \in \mathcal{W}$. There is an $M \in \mathcal{M}$ such that $\Omega(B) \subset M^{oV\ oV'}$; since $\mathcal{M}$ is saturated, M may be assumed to be balanced and convex. But $(M^{oV})^{oV'} = \Omega((M^{oV})^{oE})$, so that $\Omega(B) \subset \Omega(M^{oV\ oE})$. Thus $B \subset M^{oV\ oE} = \overline{M}(\sigma(E,V))$. This proves (b).

Suppose (b) holds. Then every $\sigma(E,V)$-bounded subset of E is relatively $\sigma(E,V)$-compact. If $M \in \mathcal{M}$, then M, being $\sigma(E,E')$-bounded, is $\sigma(E,V)$-bounded and thus relatively $\sigma(E,V)$-compact. By Theorem 2.5, $E[\mathcal{T}]$ is $(V,\mathcal{M})$-semi-reflexive. Thus $\Omega:E[\sigma(E,V)] \to V'[\sigma(V',V)]$ is a linear homeomorphism as above. For each $\sigma(V',V)$-bounded $C \subset V'$, $\Omega^{-1}(C)$ is $\sigma(E,V)$-bounded, so there is an $M \in \mathcal{M}$ such that $\Omega^{-1}(C) \subset \overline{M}(\sigma(E,V)) \subset M^{oV\ oE}$. Thus $C \subset \Omega(M^{oV\ oE}) = M^{oV\ oV'}$, and $C \in \mathcal{W}$. Therefore $V[\mathcal{T}_{\mathcal{M}}|V]$ is barrelled. This proves (a).

SECTION 3

CHARACTERIZATIONS OF $(V,M,\mathfrak{S})$-REFLEXIVITY: TOPOLOGICAL PROPERTIES OF Ω

Here we determine necessary and sufficient conditions for $\Omega:E[T] \to V'[T_{\mathfrak{S}}]$ to be (a) continuous or (b) relatively open. Then we characterize the property of $(V,M,\mathfrak{S})$-reflexivity of $E[T]$ (Definition 1.4). We also show that for certain l.c. spaces E, Ω is always continuous for any choice of $T_{\mathfrak{S}}$, and we determine, for such cases, certain relationships between the properties of (V,M)-reflexivity and $(\overline{V},M)$-reflexivity (Definition 1.4) of E.

Proposition 3.1 Suppose $E[T]$ is an l.c. space, M is a saturated collection of bounded subsets of E covering E, and V is a $\sigma(E',E)$-dense subspace of E'. Let V' and Ω be defined as in the introduction to this paper. Suppose $\mathfrak{S}$ is a saturated collection of $\sigma(V,V')$-bounded subsets of V covering V, and let $T_{\mathfrak{S}}$ denote the $\mathfrak{S}$-topology on V'. Then the map $\Omega:E[T] \to V'[T_{\mathfrak{S}}]$ is continuous if and only if each set $B \in \mathfrak{S}$ is T-equicontinuous.

Proof: Note that if $B \subset V$ then $\Omega^{-1}(B^{oV'}) = B^{oE}$. Thus the following are all equivalent:

(a) $\Omega:E[T] \to V'[T_{\mathfrak{S}}]$ is continuous.

(b) For each $B \in \mathfrak{S}$, $\Omega^{-1}(B^{oV'}) = B^{oE}$ is a T-neighborhood of 0 in E.

(c) For each $B \in \mathfrak{S}$, there is a T-equicontinuous $C \subset E'$ such that $C^{oE} \subset B^{oE}$.

(d) For each $B \in \mathfrak{S}$, there is a T-equicontinuous $D \subset E'$ such that $B \subset D$.

(e) Each set $B \in \mathfrak{S}$ is T-equicontinuous.

<u>Definition 3.2</u> Suppose $E[\mathcal{T}]$ is an l.c. space, V is a linear subspace of E', and $\mathfrak{S}$ is a collection of $\sigma(V,E)$-bounded subsets of V. Then V is $(\mathcal{T},\mathfrak{S})$-p.c. in E' if and only if, for each $\mathcal{T}$-equicontinuous $B \subset E'$, there is a $C \in \mathfrak{S}$ such that $B \subset \overline{C}(\sigma(E',E))$.

<u>Remark 3.3</u> Note that in Definition 3.2 no mention is made of a collection M, of a topology $\mathcal{T}_M|V$ on V, or of V'. The definition depends only on the l.c. space $E[\mathcal{T}]$, the subspace V of E', and the collection $\mathfrak{S}$ of $\sigma(V,E)$-bounded subsets of V.

<u>Remark 3.4</u> In Definition 3.2, if $\mathfrak{S}$ is the collection of all $\beta(E',E)$-bounded subsets of V, then V is $(\mathcal{T},\mathfrak{S})$-p.c. in E' if and only if V is $\mathcal{T}$-p.c. in E', in the sense of Lohman [11, Defn. 2]; if, in addition, $\mathcal{T} = \tau(E,E')$, then V is $(\mathcal{T},\mathfrak{S})$-p.c. in E' if and only if V is a "duxial" subspace of E', in the sense of Krishnamurthy [9, Defn. 2], that is, if and only if V is $\tau(E,E')$-p.c. in E' in the sense of Lohman; if $E[\mathcal{T}]$ is a Banach space, then V is "of positive characteristic" in E' in the sense of Dixmier [5] if and only if V is "duxial" in the sense of Krishnamurthy. Dixmier [5] and Ruston [13] defined a subspace V of a Banach space as being "duxial" if and only if V "has characteristic 1" in the sense of Dixmier.

<u>Proposition 3.5</u> With the same hypothesis as in Proposition 3.1, $\Omega: E[\mathcal{T}] \to V'[\mathcal{T}_{\mathfrak{S}}]$ is relatively open if and only if V is $(\mathcal{T},\mathfrak{S})$-p.c. in E'.

<u>Proof</u>: Since Ω maps E into V', each set in $\mathfrak{S}$, being $\sigma(V,V')$-bounded, is also $\sigma(V,E)$-bounded, since $\sigma(V,E) \leq \sigma(V,V')$. Thus

Definition 3.2 applies to V with respect to $E[\mathcal{T}]$ and $\mathfrak{S}$. Denote the subspace $\Omega(E)$ of V' by H. Define the pairing $\langle H,E'\rangle$ so that $\langle\Omega(e),e'\rangle = \langle e,e'\rangle$ for $e\in E$ and $e'\in E'$. Note that if $B\subset E'$ then $\Omega(B^{oE}) = B^{oH}$. Also, if $D\subset V$ then $D^{oH\ oE'} = D^{oE\ oE'}$. The following are all equivalent:

(a) $\Omega{:}E[\mathcal{T}] \to V'[\mathcal{T}_{\mathfrak{S}}]$ is relatively open.

(b) For each $\mathcal{T}$-neighborhood U of 0 in E, $\Omega(U)$ is a neighborhood of 0 in $H[\mathcal{T}_{\mathfrak{S}}|H]$.

(c) For each $\mathcal{T}$-equicontinuous $B\subset E'$, $\Omega(B^{oE}) = B^{oH}$ is a neighborhood of 0 in $H[\mathcal{T}_{\mathfrak{S}}|H]$.

(d) For each $\mathcal{T}$-equicontinuous $B\subset E'$, there is a balanced, convex set $D\in\mathfrak{S}$ such that $D^{oH}\subset B^{oH}$.

(e) For each $\mathcal{T}$-equicontinuous $B\subset E'$, there is a balanced, convex set $D\in\mathfrak{S}$ such that $B\subset D^{oH\ oE'} = D^{oE\ oE'}$.

(f) For each $\mathcal{T}$-equicontinuous $B\subset E'$, there is a set $C\in\mathfrak{S}$ such that $B\subset \overline{C}(\sigma(E',E))$.

(g) V is $(\mathcal{T},\mathfrak{S})$-p.c. in E'.

All of the adjacent statements are clearly equivalent to each other, except perhaps (e) and (f), whose equivalence we show. If (e) holds, then let $C = D^{oV'\ oV}$; since $\mathfrak{S}$ is $\langle V,V'\rangle$-saturated, $C\in\mathfrak{S}$; since $B\subset D^{oE\ oE'}$ and $D^{oE\ oE'} = \overline{D}(\sigma(E',E))$, $B\subset\overline{D}(\sigma(E',E))\subset\overline{(D^{oV'\ oV})}(\sigma(E',E) = \overline{C}(\sigma(E',E))$. Thus (f) holds. Conversely, if (f) holds, then let $D = C^{oV'\ oV}$; then D is balanced and convex, and $D\in\mathfrak{S}$; since $B\subset\overline{C}(\sigma(E',E))$, $B\subset\overline{(C^{oV'\ oV})}(\sigma(E',E)) = \overline{D}(\sigma(E',E))\subset D^{oE\ oE'}$. Thus (e) holds.

Theorem 3.6 With the same hypothesis as in Proposition 3.1, $E[\mathcal{T}]$ is $(V,\mathcal{M},\mathfrak{S})$-reflexive if and only if all of the following are true:

(a) E is $(V,\mathcal{M})$-semi-reflexive (see Theorem 2.5).

(b) Each set in $\mathfrak{S}$ is $\mathcal{T}$-equicontinuous.

(c) V is $(\mathcal{T},\mathfrak{S})$-p.c. in E'.

Proof: Definitions 1.4 and 1.2, and Propositions 3.1 and 3.5.

The remaining propositions of this section are concerned with cases in which Ω is always continuous, and with relationships between $(V,\mathcal{M})$-reflexivity and $(\overline{V},\mathcal{M})$-reflexivity (Definition 1.4) for these cases.

Corollary 3.7 If $E[\mathcal{T}]$ is barrelled, or if $\mathcal{M}$ is the collection of all bounded subsets of E(i.e., $\mathcal{T}_{\mathcal{M}} = \beta(E',E)$) and $E[\mathcal{T}]$ is infra-barrelled [6, p. 217, Defn. 2], then $\Omega:E[\mathcal{T}] \to V'[\mathcal{T}_{\mathfrak{S}}]$ is continuous, whatever the choice of $\mathfrak{S}$.

Proof: The sets in $\mathfrak{S}$ are assumed to be $\sigma(V,V')$-bounded; i.e., $\mathcal{T}_{\mathcal{M}}|V$-bounded. Thus each $B \in \mathfrak{S}$ is a $\mathcal{T}_{\mathcal{M}}$-bounded subset of E'. If $E[\mathcal{T}]$ is barrelled, each $B \in \mathfrak{S}$, being $\mathcal{T}_{\mathcal{M}}$-bounded and thus $\sigma(E',E)$-bounded, is also $\mathcal{T}$-equicontinuous [6, p. 212, Prop. 2]. If $\mathcal{T}_{\mathcal{M}} = \beta(E',E)$ and $E[\mathcal{T}]$ is infrabarrelled, each $B \in \mathfrak{S}$, being $\mathcal{T}_{\mathcal{M}}$-bounded (i.e., $\beta(E',E)$-bounded), is also $\mathcal{T}$-equicontinuous [6, p. 217, Prop. 6]. By Proposition 3.1, $\Omega:E[\mathcal{T}] \to V'[\mathcal{T}_{\mathfrak{S}}]$ is continuous.

<u>Proposition 3.8</u> With the same hypothesis as Proposition 3.1, if $V[\mathcal{T}_M|V]$ is barrelled or quasi-complete, then Ω maps $\sigma(E,V)$-bounded subsets of E onto $\mathcal{T}_{\mathfrak{S}}$-bounded subsets of V', and consequently Ω does the same to $\mathcal{T}$-bounded subsets of E.

<u>Proof</u>: The last part of the conclusion follows from the first part, since the $\mathcal{T}$-bounded subsets of E are $\sigma(E,V)$-bounded. To prove the first part, suppose B is a $\sigma(E,V)$-bounded subset of E; then $\Omega(B)$ is $\sigma(V',V)$-bounded. The hypotheses on $V[\mathcal{T}_M|V]$ imply that each $\sigma(V',V)$-bounded subset of V' is $\beta(V',V)$-bounded [6, p. 210, Thm. 4, and p. 212, Cor.]; so $\Omega(B)$ is $\beta(V',V)$-bounded, and therefore $\mathcal{T}_{\mathfrak{S}}$-bounded.

<u>Corollary 3.9</u> With the same hypothesis as in Proposition 3.1, if $V[\mathcal{T}_M|V]$ is barrelled or quasi-complete, and $E[\mathcal{T}]$ is bornological [6, p. 220, Defn. 1], then $\Omega:E[\mathcal{T}] \to V'[\mathcal{T}_{\mathfrak{S}}]$ is continuous.

<u>Proof</u>: Proposition 3.8 applies, so Ω maps $\mathcal{T}$-bounded subsets of E onto $\mathcal{T}_{\mathfrak{S}}$-bounded subsets of V'. By Proposition 1 on page 220 of [6], $\Omega:E[\mathcal{T}] \to V'[\mathcal{T}_{\mathfrak{S}}]$ is continuous, since $E[\mathcal{T}]$ is bornological.

<u>Proposition 3.10</u> Suppose E is (V,M)-semi-reflexive, and one of the following is true:

(a) $E[\mathcal{T}]$ is barrelled;

(b) $E[\mathcal{T}]$ is infrabarrelled, and $\mathcal{T}_M = \beta(E',E)$;

(c) $E[\mathcal{T}]$ is bornological, and $V[\mathcal{T}_M|V]$ is barrelled.

(d) $E[\mathcal{T}]$ is bornological, and $\overline{V}[\mathcal{T}_M|\overline{V}]$ is quasi-complete.

Then both of the following are true:

(1) If $\mathfrak{S}$ is a saturated collection of $\sigma(V,V')$-bounded subsets of V covering V, and $\overline{\mathfrak{S}}$ is a saturated collection of $\sigma(\overline{V},\overline{V}')$-bounded subsets of $\overline{V}$ covering $\overline{V}$ and containing $\mathfrak{S}$, and $E[\mathcal{T}]$ is $(V,\mathcal{M},\mathfrak{S})$-reflexive, then $E[\mathcal{T}]$ is $(\overline{V},\mathcal{M},\overline{\mathfrak{S}})$-reflexive.

(2) If $E[\mathcal{T}]$ is $(V,\mathcal{M})$-reflexive, then $E[\mathcal{T}]$ is $(\overline{V},\mathcal{M})$-reflexive.

<u>Proof</u>: Since E is $(V,\mathcal{M})$-semi-reflexive, E is $(\overline{V},\mathcal{M})$-semi-reflexive by Theorem 2.5; $\Omega:E \to V'$ and $\overline{\Omega}:E \to \overline{V}'$ are vector space isomorphisms. Each element $v'\varepsilon V'$ has a unique extension to an element $\overline{v}'\varepsilon\overline{V}'$; define $\Lambda:V' \to \overline{V}'$ so that $\Lambda(v') = \overline{v}'$. Then Λ is a vector space isomorphism, and if $B\subset V$ then $B^{o\overline{V}'} = \Lambda(B^{oV'})$. Since (1) requires that $\overline{\mathfrak{S}}$ contain $\mathfrak{S}$, $\Lambda:V'[\mathcal{T}_{\mathfrak{S}}] \to \overline{V}'[\mathcal{T}_{\overline{\mathfrak{S}}}]$ is an open map, because whenever $B\ \varepsilon\mathfrak{S}$, Λ maps the $\mathcal{T}_{\mathfrak{S}}$-neighborhood $B^{oV'}$ of 0 onto the $\mathcal{T}_{\overline{\mathfrak{S}}}$-neighborhood $B^{o\overline{V}'}$ of 0. Moreover, $\Lambda\cdot\Omega \equiv \overline{\Omega}$, and since (1) requires that $\Omega:E[\mathcal{T}] \to V'[\mathcal{T}_{\mathfrak{S}}]$ be a linear homeomorphism, it follows that $(\Lambda\cdot\Omega \equiv \overline{\Omega}):E[\mathcal{T}] \to \overline{V}'[\mathcal{T}_{\overline{\mathfrak{S}}}]$ is an open map; in order to prove the conclusion of (1) it suffices to show that $\overline{\Omega}:E[\mathcal{T}] \to \overline{V}'[\mathcal{T}_{\overline{\mathfrak{S}}}]$ is continuous. By Corollary 3.7, either (a) or (b) implies that $\overline{\Omega}:E[\mathcal{T}] \to \overline{V}'[\mathcal{T}_{\overline{\mathfrak{S}}}]$ is continuous; by Corollary 3.9, (d) implies that $\overline{\Omega}:E[\mathcal{T}] \to \overline{V}'[\mathcal{T}_{\overline{\mathfrak{S}}}]$ is continuous. To complete the proof of (1) we show that (c) implies $\overline{\Omega}:E[\mathcal{T}] \to \overline{V}'[\mathcal{T}_{\overline{\mathfrak{S}}}]$ is continuous; we show that (c) implies $\overline{V}[\mathcal{T}_{\mathcal{M}}|\overline{V}]$ is barrelled, so that the conclusion follows from Corollary 3.9. Suppose B is a $\sigma(\overline{V}',\overline{V})$-bounded subset of $\overline{V}'$; then B is also $\sigma(\overline{V}',V)$-bounded, and $\Lambda^{-1}(B)$ is a $\sigma(V',V)$-bounded subset of V'. Since $V[\mathcal{T}_{\mathcal{M}}|V]$ is barrelled, $\Lambda^{-1}(B)$ is $\mathcal{T}_{\mathcal{M}}|V$-equicontinuous; there is a

balanced, convex $M \in \mathcal{M}$ such that $\Lambda^{-1}(B) \subset (M^{oV})^{oV'}$. Thus $B \subset \Lambda((M^{oV})^{oV'}) = (M^{oV})^{o\overline{V}'} = \overline{\Omega}((M^{oV})^{oE}) = \overline{\Omega}(\overline{M}(\sigma(E,V)))$. By Proposition 2.1, $\overline{M}(\sigma(E,V)) = \overline{M}(\sigma(E,\overline{V}))$; thus $B \subset \overline{\Omega}(\overline{M}(\sigma(E,\overline{V}))) = \overline{\Omega}((M^{o\overline{V}})^{oE}) = (M^{o\overline{V}})^{o\overline{V}'}$, and B is $\mathcal{T}_{\mathcal{M}}|\overline{V}$-equicontinuous. We have shown that $\overline{V}[\mathcal{T}_{\mathcal{M}}|\overline{V}]$ is barrelled [6, p. 212, Prop. 2]; the proof of (1) is complete. To prove (2), we note that it is just the particular case of (1) in which $\mathfrak{S}$ is the collection of <u>all</u> $\sigma(V,V')$-bounded subsets of V, and $\overline{\mathfrak{S}}$ is the collection of <u>all</u> $\sigma(\overline{V},\overline{V}')$-bounded subsets of $\overline{V}$.

<u>Remark 3.11</u> In general, neither of the following implies the other: $(V,\mathcal{M})$-reflexivity of $E[\mathcal{T}]$; $(\overline{V},\mathcal{M})$-reflexivity of $E[\mathcal{T}]$. We give counter-examples in Section 4.

SECTION 4

EXAMPLES

Example 4.1 The following is an example of an l.c. space E, a $\sigma(E',E)$-dense subspace V of E', and a saturated collection $\mathcal{M}$ of bounded subsets of E covering E, such that E is not V-semi-reflexive but E is $(V,\mathcal{M})$-semi-reflexive, and $\tau(E',E) \leqq \mathcal{T}_{\mathcal{M}}$ but $\tau(E',E) \neq \mathcal{T}_{\mathcal{M}}$.

Let c_o and ℓ^1 [6, p. 11] have their usual normed topologies, and let $E[\mathcal{T}] = c_o \oplus \ell^1$, the locally convex direct sum of c_o and ℓ^1. Then $(c_o)' = \ell^1$, where the pairing $\langle c_o,\ell^1\rangle$ is defined so that if $\{\mu_i\} \in c_o$ and $\{\xi_i\} \in \ell^1$ then $\langle\{\mu_i\},\{\xi_i\}\rangle = \sum_{i=1}^{\infty} \mu_i\xi_i$ [6, p. 55, Ex. 2]. Also, $(\ell^1)' = m$, where the pairing $\langle\ell^1,m\rangle$ is defined so that if $\{\lambda_i\} \in \ell^1$ and $\{\zeta_i\} \in m$ then $\langle\{\lambda_i\},\{\zeta_i\}\rangle = \sum_{i=1}^{\infty} \lambda_i\zeta_i$ [6, p. 56, Ex. 3]. Then $E' = \ell^1 \oplus m$, where the pairing $\langle E,E'\rangle$ is defined so that $\langle\{\mu_i\} + \{\lambda_i\},\{\xi_i\} + \{\zeta_i\}\rangle = \langle\{\mu_i\},\{\xi_i\}\rangle + \langle\{\lambda_i\},\{\zeta_i\}\rangle$ [6, p. 267, Prop. 2].

Let $V = \ell^1 \oplus c_o \subset E'$, where $c_o \subset m$. Since ℓ^1 separates points of c_o, and c_o separates points of ℓ^1, V separates points of E; i.e., V is $\sigma(E',E)$-dense in E'.

The topology $\sigma(E,V)$ is the locally convex direct sum (i.e., the product) of $\sigma(c_o,\ell^1)$ with $\sigma(\ell^1,c_o)$, and the topology $\sigma(E,E')$ is the locally convex direct sum (i.e., the product) of $\sigma(c_o,\ell^1)$ with $\sigma(\ell^1,m)$ [6, p. 268, Prop. 3].

We show that there is a balanced, convex, closed, bounded $B \subset E$ such that B is not relatively $\sigma(E,V)$-compact. Let S_{c_o} denote the norm-closed unit ball in c_o. Then S_{c_o} is balanced, convex, closed, and bounded in c_o; S_{c_o} is also $\sigma(c_o,\ell^1)$-closed, but not $\sigma(c_o,\ell^1)$-compact according to Theorem 2.5 since c_o is not semi-reflexive [6, p. 59, Ex. 5]. Let $B = S_{c_o} + \{0\} \subset E$. Then B is balanced, convex, closed, and bounded in E. Now B is also $\sigma(E,V)$-closed since S_{c_o} is $\sigma(c_o,\ell^1)$-closed and $\{0\}$ is $\sigma(\ell^1,c_o)$-closed; but B is not $\sigma(E,V)$-compact, since S_{c_o} is not $\sigma(c_o,\ell^1)$-compact. Therefore B has all the properties required at the beginning of this paragraph.

According to Theorem 2.5, the previous paragraph implies that E is not V-semi-reflexive.

We show there is a balanced, convex, closed, bounded $D \subset E$ such that D is $\sigma(E,V)$-compact, but not $\sigma(E,E')$-compact. Let $S_{\ell 1}$ denote the norm-closed unit ball in ℓ^1. Then $S_{\ell 1}$ is balanced, convex, closed, and bounded in ℓ^1; moreover, $S_{\ell 1}$ is $\sigma(\ell^1,c_o)$-compact [6, p. 201, Cor. to Thm. 1]. Let $D = \{0\} + S_{\ell 1} \subset E$. Then D is balanced, convex, closed, and bounded in E, and D is also $\sigma(E,V)$-compact. But since ℓ^1 is not semi-reflexive [8, pp. 424-26], Theorem 2.5 implies that $S_{\ell 1}$, although $\sigma(\ell^1,m)$-closed, is not $\sigma(\ell^1,m)$-compact. Therefore D is not $\sigma(E,E')$-compact.

Define M to be the collection of all subsets of sets of the form $(\bigcup_{i=1}^{n} B_i)^{oE'\ oE}$, where for $1 \leqq i \leqq n$ B_i is a balanced, convex, $\sigma(E,E')$-bounded, $\sigma(E,V)$-compact subset of E. Then M covers E, because if $x \in E$ then $\{\lambda x|\ |\lambda| \leqq 1\}$ contains x and is in M, since $\{\lambda x|\ |\lambda| \leqq 1\}$ is balanced, convex, $\sigma(E,E')$-bounded, and $\sigma(E,V)$-compact. Each $M \in M$ is $\sigma(E,E')$-bounded, for if $M \subset (\bigcup_{i=1}^{n} B_i)^{oE'\ oE}$ then $(\bigcup_{i=1}^{n} B_i)^{oE'\ oE}$

is the balanced, convex, $\sigma(E,E')$-closed hull of the $\sigma(E,E')$-bounded set $(\bigcup_{i=1}^{n} B_i)$, and is thus also $\sigma(E,E')$-bounded. We show that $\mathcal{M}$ is $\langle E,E'\rangle$-saturated. Clearly all subsets of members of $\mathcal{M}$ are members of $\mathcal{M}$. If $\lambda \neq 0$, and $M \in \mathcal{M}$ where $M \subset (\bigcup_{n=1}^{n} B_i)^{\circ E' \circ E}$, then $\lambda M \subset \lambda[(\bigcup_{i=1}^{n} B_i)^{\circ E' \circ E}] = [\bigcup_{i=1}^{n} (\lambda B_i)]^{\circ E' \circ E}$, where each (λB_i) is balanced, convex, $\sigma(E,E')$-bounded, and $\sigma(E,V)$-compact; thus $\lambda M \in \mathcal{M}$. The balanced, convex, $\sigma(E,E')$-closed hull of M is $M^{\circ E' \circ E}$, which is contained in $(\bigcup_{i=1}^{n} B_i)^{\circ E' \circ E}$, and so is also in $\mathcal{M}$. Suppose that $M_1, M_2 \in \mathcal{M}$ where $M_1 \subset (\bigcup_{i=1}^{n_1} B_i^{(1)})^{\circ E' \circ E}$ and $M_2 \subset (\bigcup_{i=1}^{n_2} B_i^{(2)})^{\circ E' \circ E}$; then

$$M_1 \cup M_2 \subset [(\bigcup_{i=1}^{n_1} B_i^{(1)})^{\circ E' \circ E} \cup (\bigcup_{i=1}^{n_2} B_i^{(2)})^{\circ E' \circ E}]^{\circ E' \circ E} =$$

$$[(\bigcap_{i=1}^{n_1} (B_i^{(1)})^{\circ E'})^{\circ E} \cup (\bigcap_{i=1}^{n_2} (B_i^{(2)})^{\circ E'})^{\circ E}]^{\circ E' \circ E} =$$

$$[(\bigcap_{i=1}^{n_1} (B_i^{(1)})^{\circ E'})^{\circ E \circ E'} \cap (\bigcap_{i=1}^{n_2} (B_i^{(2)})^{\circ E'})^{\circ E \circ E'}]^{\circ E} =$$

$$[(\bigcap_{i=1}^{n_1} (B_i^{(1)})^{\circ E'}) \cap (\bigcap_{i=1}^{n_2} (B_i^{(2)})^{\circ E'})]^{\circ E} =$$

$$[(\bigcup_{i=1}^{n_1} B_i^{(1)})^{\circ E'} \cap (\bigcup_{i=1}^{n_2} B_i^{(2)})^{\circ E'}]^{\circ E} = [(\bigcup_{i=1}^{n_1} B_i^{(1)}) \cup (\bigcup_{i=1}^{n_2} B_i^{(2)})]^{\circ E' \circ E};$$

hence $M_1 \cup M_2 \in \mathcal{M}$. Thus finite unions of members of $\mathcal{M}$ are members of $\mathcal{M}$. We have shown that $\mathcal{M}$ is $\langle E,E'\rangle$-saturated.

We show that each $M \in \mathcal{M}$ is relatively $\sigma(E,V)$-compact. If $M \subset (\bigcup_{i=1}^{n} B_i)^{\circ E' \circ E}$ then $\overline{M}(\sigma(E,V)) \subset M^{\circ V \circ E} \subset [(\bigcup_{i=1}^{n} B_i)^{\circ E' \circ E}]^{\circ V \circ E} \subset [(\bigcup_{i=1}^{n} B_i)^{\circ V \circ E}]^{\circ V \circ E} = (\bigcup_{i=1}^{n} B_i)^{\circ V \circ E}$; but since each B_i is convex and

$\sigma(E,V)$-compact, so is $(\bigcup_{i=1}^{n} B_i)^{oV\ oE}$ [8, p. 243, (8)]. Thus $\overline{M}(\sigma(E,V))$ is $\sigma(E,V)$-compact. Therefore, by Theorem 2.5, E is (V,M)-semi-reflexive.

Since M contains the set D, we have $\tau(E',E) \leqq T_M$ but $\tau(E',E) \neq T_M$.

The next two examples show that neither of the following implies the other: V-reflexivity of $E[T]$; $\overline{V}$-reflexivity of $E[T]$. Both examples are based upon the fact that there exists a metrizable l.c. space F such that there is a bounded subset of the completion $\hat{F}$ of F which is not contained in the completion of any bounded subset of F. This fact is shown by an example of Amemiya [1], which is also cited by Köthe [8, p. 404].

First we prove the following lemma.

Lemma 4.2 Suppose $\langle K,J \rangle$ is a pairing of vector spaces, and H is a subspace of J. The following are equivalent:

(a) $\beta(K,J)$ is coarser than $\beta(K,H)$; since $\beta(K,J)$ is always finer than $\beta(K,H)$, the topologies are equal.

(b) For each $\sigma(J,K)$-bounded $C \subset J$, there is a $\sigma(H,K)$-bounded $B \subset H$ such that $C \subset \overline{B}(\sigma(J,K))$.

Proof: Suppose (b) holds. If U is a $\beta(K,J)$-neighborhood of 0 in K, there is a $\sigma(J,K)$-bounded $C \subset J$ such that $C^{oK} \subset U$, and (b) implies there is a $\sigma(H,K)$-bounded $B \subset H$ such that $C \subset \overline{B}(\sigma(J,K))$. Thus

$B^{oK} = (B^{oK\ oJ})^{oK} \subset (\overline{B}(\sigma(J,K)))^{oK} \subset C^{oK} \subset U$, and U is a $\beta(K,H)$-neighborhood of 0 in K. This proves (a).

Suppose (a) holds. If C is a $\sigma(J,K)$-bounded subset of J, then C^{oK} is a $\beta(K,J)$-neighborhood of 0 in K, and (a) implies that C^{oK} is also a $\beta(K,H)$-neighborhood of 0 in K. There is a balanced, convex, $\sigma(H,K)$-bounded $B \subset H$ such that $B^{oK} \subset C^{oK}$. Thus $C \subset C^{oK\ oJ} \subset B^{oK\ oJ} = \overline{B}(\sigma(J,K))$. This proves (b).

Example 4.3 Let F be a metrizable l.c. space, and $\hat{F}$ the completion of F, and let C be a bounded subset of $\hat{F}$ which is not contained in the completion of any bounded subset of F.

Let $E = F'$, $T = \beta(F',F)$, $\Psi: F \to F''$ the canonical linear injection, and $V = \Psi(F) \subset E'$. Since F separates points of F', V separates points of $(F' = E)$; i.e., V is $\sigma(E',E)$-dense in E'.

Since we are interested in V-reflexivity and $\overline{V}$-reflexivity of $E[T]$, we let $T_M = \beta(E',E) = \beta(F'',F')$. Thus $\overline{V} = \overline{V}(\beta(F'',F'))$, and $V' = (V[\beta_i(F'',F')])'$, and $\overline{V}' = (\overline{V}[\beta_i(F'',F')])'$.

Since F is metrizable and thus infrabarrelled [6, p. 222, Prop. 3, and p. 220, Defn. 1], we have the following [6, p. 229, Prop. 5]:

(a) $\Psi: F \to V[\beta_i(F'',F')]$ is a linear homeomorphism.

Moreover, $F''[\beta(F'',F')]$ is a Fréchet space, i.e., a complete, metrizable l.c. space [8, §23,4(4) and §29,2(3)]. Hence $\overline{V}[\beta_i(F'',F')]$ is a complete, metrizable l.c. space such that V is dense in $\overline{V}[\beta_i(F'',F')]$ and (a) above holds. Therefore we consider that $\hat{F} = \overline{V}[\beta_i(F'',F')]$.

Define the linear map $\Psi^T:V' \to (E=F')$ so that $\Psi^T(v') \equiv v'\cdot\Psi$ whenever $v'\varepsilon V'$. By (a) above, Ψ^T is an algebraic isomorphism. Moreover, since $\langle\Psi^T(v'),f\rangle = \langle v'\cdot\Psi,f\rangle = \langle v',\Psi(f)\rangle$ whenever $v'\varepsilon V'$ and $f\varepsilon F$, it follows that

(b) $\langle E,F\rangle$ and $\langle V',V\rangle$ are algebraically isomorphic with respect to $(\Psi^T)^{-1}:E \to V'$ and $\Psi:F \to V$.

But $\Psi^T\cdot\Omega$ is the identity map on $(F' = E)$; indeed, if $f'\varepsilon E$ and $f\varepsilon F$, then $(\Psi^T\cdot\Omega(f'))(f) = [\Omega(f')\cdot\Psi](f) = \Omega(f')[\Psi(f)] = [\Psi(f)](f') = f'(f)$, so that $\Psi^T\cdot\Omega(f') \equiv f'$. Therefore, $\Omega \equiv (\Psi^T)^{-1}$, and it follows from (b) above that

(c) $\Omega:E[\beta(E,F)] \to V'[\beta(V',V)]$ is a linear homeomorphism.

But since $\mathcal{T} = \beta(F',F) = \beta(E,F)$, we have from (c) that <u>$E[\mathcal{T}]$ is V-reflexive</u>.

For the set C of the hypothesis, since $\hat{F} = \overline{V}[\beta_i(F'',F')]$ and $(\hat{F})' = \overline{V}'$, we may assume that C is a $\sigma(\overline{V},\overline{V}')$-bounded subset of $\overline{V}$ such that, for any $\sigma(F,F')$-bounded $B\subset F$, $\Psi(C)$ is not contained in the completion $\hat{B}$ of B. But if $B\subset F$, then $\hat{B}$ is the closure of $\Psi(B)$ in $(\hat{F} = \overline{V})$. Moreover, if B is $\sigma(F,F')$-bounded, then $B^{oF'\ oF}$ is balanced, convex, and $\sigma(F,F')$-bounded, and the closure of $\Psi(B^{oF'\ oF})$ in $\overline{V}$ is $\overline{\Psi(B^{oF'\ oF})}(\sigma(\overline{V},\overline{V}'))$. Thus, if $B\subset F$ is $\sigma(F,F')$-bounded, then $\Psi(C)\not\subset \overline{\Psi(B)}(\sigma(\overline{V},\overline{V}'))$, for otherwise there would be the $\sigma(F,F')$-bounded $(B^{oF'\ oF})\subset F$ such that $\Psi(C)\subset \overline{\Psi(B)}(\sigma(\overline{V},\overline{V}'))\subset \overline{\Psi(B^{oF'\ oF})}(\sigma(\overline{V},\overline{V}'))$, the completion of $B^{oF'\ oF}$.

Since E is V-reflexive, E is V-semi-reflexive, and thus E is also $\overline{V}$-semi-reflexive by Theorem 2.5. Hence the canonical linear injection $\overline{\Omega}:E \to \overline{V}'$ maps E onto $\overline{V}'$. Moreover, if $f'\epsilon E$ and $f\epsilon F$, then $\langle f',f\rangle = \langle f',\Psi(f)\rangle = \langle\overline{\Omega}(f'),\Psi(f)\rangle$. Therefore the pairings $\langle E,F\rangle$ and $\langle\overline{V}',V\rangle$ are algebraically isomorphic with respect to $\overline{\Omega}:E \to \overline{V}'$ and $\Psi:F \to V$. It follows that

(d) $\overline{\Omega}:E[\beta(E,F)] \to \overline{V}'[\beta(\overline{V}',V)]$ is a linear homeomorphism.

But by the previous paragraph and the isomorphic pairings $\langle E,F\rangle$ and $\langle\overline{V}',V\rangle$ just described, $\Psi(C)$ is a $\sigma(V,\overline{V}')$-bounded subset of V, and thus a $\sigma(\overline{V},\overline{V}')$-bounded subset of $\overline{V}$, such that for any $\sigma(V,\overline{V}')$-bounded $\Psi(B) \subset V$, $\Psi(C) \not\subset \overline{\Psi(B)}(\sigma(\overline{V},\overline{V}'))$. We rephrase and emphasize the last statement:

(e) There is a $\sigma(\overline{V},\overline{V}')$-bounded $C \subset \overline{V}$ such that, for any $\sigma(V,\overline{V}')$-bounded $B \subset V$, $C \not\subset \overline{B}(\sigma(\overline{V},\overline{V}'))$.

By Lemma 4.2, $\beta(\overline{V}',\overline{V})$ is strictly finer than $\beta(\overline{V}',V)$; thus

(f) id: $\overline{V}'[\beta(\overline{V}',V)] \to \overline{V}'[\beta(\overline{V}',\overline{V})]$ is not continuous.

From (d) and (f) above, we conclude that $\overline{\Omega}:E[\beta(E,F)] \to \overline{V}'[\beta(\overline{V}',\overline{V})]$ is not a linear homeomorphism, for otherwise we would have

(g) (id. $= \overline{\Omega}\cdot\overline{\Omega}^{-1}$)$:\overline{V}'[\beta(\overline{V}',V)] \to \overline{V}'[\beta(\overline{V}',\overline{V})]$ a linear homeomorphism,

a contradiction in view of (f). Therefore E[$\mathcal{T}$] is not $\overline{V}$-reflexive.

Example 4.4 Let F, $\hat{F}$, and C be as in Example 4.3. Let $E = \hat{F}'$, and $\mathcal{T} = \beta(\hat{F}',\hat{F})$. We consider F to be a dense subspace of $\hat{F}$.

Let $\Psi:\hat{F} \to (E' = \hat{F}'')$ denote the canonical linear injection, and let $V = \Psi(F)$. We show that V is $\sigma(E',E)$-dense in E'; indeed, if $\hat{f}'\varepsilon E$ and $\hat{f}'\neq 0$, there is an $\hat{f}\varepsilon\hat{F}$ such that $<\hat{f}',\hat{f}>\neq 0$, and there is an $f\varepsilon F$ such that $0 \neq <\hat{f}',f> = <\hat{f}',\Psi(f)>$; thus V separates points of E. Since we are again interested in V-reflexivity and $\overline{V}$-reflexivity, we let $\mathcal{T}_M = \beta(E',E) = \beta(\hat{F}'',\hat{F}')$. Thus $\overline{V} = \overline{V}(\beta(\hat{F}'',\hat{F}'))$, and $V' = (V[\beta_i(\hat{F}'',\hat{F}')])'$, and $\overline{V}' = (\overline{V}[\beta_i(\hat{F}'',\hat{F}')])'$.

By an argument similar to the one used in the paragraph of Example 4.3 which contains the statement (a), it can be shown that $\hat{F}''[\beta(\hat{F}'',\hat{F}')]$ is a complete, metrizable l.c. space, and that

(a) $\quad\Psi:\hat{F} \to \Psi(\hat{F})[\beta_i(\hat{F}'',\hat{F}')]$ is a linear homeomorphism.

Since $\hat{F}$ is complete, so is $\Psi(\hat{F})$; thus $\overline{\Psi(\hat{F})}(\beta(\hat{F}'',\hat{F}')) = \Psi(\hat{F})$. Since $V = \Psi(F) \subset \Psi(\hat{F})$, $\overline{V} \subset \overline{\Psi(\hat{F})} = \Psi(\hat{F})$. If $f\varepsilon\hat{F}$, there is a sequence $f_1,f_2,f_3,\ldots$ in F such that $\hat{f} = \lim_n f_n$. Since Ψ is continuous, it follows that $\Psi(\hat{f}) = \lim_n \Psi(f_n)$, where $\Psi(f_n)\varepsilon V$ for each n; thus $\Psi(\hat{f})\varepsilon\overline{V}$, and $\Psi(\hat{F}) \subset \overline{V}$. We have shown that $\overline{V} = \Psi(\hat{F})$. It follows that

(b) $\quad\Psi:\hat{F} \to \overline{V}[\beta_i(\hat{F}'',\hat{F}')]$ is a linear homeomorphism;

moreover, if $\Psi^T:\overline{V}' \to (\hat{F}'=E)$ is defined as usual, then

(c) $\quad$ the pairings $<E,\hat{F}>$ and $<\overline{V}',\overline{V}>$ are algebraically isomorphic with respect to $(\Psi^T)^{-1}:E \to \overline{V}'$ and $\Psi:\hat{F} \to \overline{V}$.

But it is easily verified that $\Psi^T\cdot\overline{\Omega}$ is the identity map on E; thus $(\Psi^T)^{-1} \equiv \overline{\Omega}$. Moreover, since $\mathcal{T} = \beta(E,\hat{F})$, (c) implies that

(d) $\overline{\Omega}:E[T] \to \overline{V}'[\beta(\overline{V}',\overline{V})]$ is a linear homeomorphism.

Therefore E[T] is $\overline{V}$-reflexive.

Since E is $\overline{V}$-semi-reflexive, E is also V-semi-reflexive, and $\Omega:E \to V'$ is an algebraic isomorphism. Every $v'\varepsilon V'$ has a unique extension to an element $\overline{v}'\varepsilon\overline{V}'$; if the map $\Gamma:\overline{V}' \to V'$ is defined so that $\Gamma(\overline{v}')$ is the restriction $(\overline{v}'|V)$ of $\overline{v}'$ to V whenever $\overline{v}'\varepsilon\overline{V}'$, then the pairings $\langle\overline{V}',V\rangle$ and $\langle V',V\rangle$ are algebraically isomorphic with respect to $\Gamma:\overline{V}' \to V'$ and id.: $V \to V$; it follows that

(e) $\Gamma:\overline{V}'[\beta(\overline{V}',V)] \to V'[\beta(V',V)]$ is a linear homeomorphism.

Similarly, every $f'\varepsilon F'$ has a unique extension to an element $\hat{f}'\varepsilon E$; if the map $\Theta:F' \to E$ is defined so that $\Theta(f')$ is the unique extension of f' to an element of E whenever $f'\varepsilon F'$, then

(f) the pairings $\langle F,F'\rangle$ and $\langle V,\overline{V}'\rangle$ are algebraically isomorphic with respect to $\Psi:F \to V$ and $\overline{\Omega}\cdot\Theta:F' \to \overline{V}'$.

There is the $\sigma(\hat{F},\hat{F}')$-bounded $C \subset \hat{F}$ which is not contained in the completion of any $\sigma(F,F')$-bounded $B \subset F$. By an argument similar to the one in Example 4.3, it can be shown that for any $\sigma(F,F')$-bounded $B \subset F$, $C \not\subset \overline{B}(\sigma(\hat{F},\hat{F}'))$. By (c) and (f) above, there is the $\sigma(\overline{V},\overline{V}')$-bounded $\Psi(C) \subset \overline{V}$ such that for any $\sigma(V,\overline{V}')$-bounded $\Psi(B) \subset V$, $\Psi(C) \not\subset \overline{\Psi(B)}(\sigma(\overline{V},\overline{V}'))$. We rephrase and emphasize the last statement:

(g) There is a $\sigma(\overline{V},\overline{V}')$-bounded $C \subset \overline{V}$ such that, for any $\sigma(V,\overline{V}')$-bounded $B \subset V$, $C \not\subset \overline{B}(\sigma(\overline{V},\overline{V}'))$.

By Lemma 4.2, (g) implies that $\beta(\overline{V}',\overline{V})$ is strictly finer than $\beta(\overline{V}',V)$; it follows that

(h) $\quad \text{id}: \overline{V}'[\beta(\overline{V}',V)] \to \overline{V}'[\beta(\overline{V}',\overline{V})]$ is not continuous.

Now we show that $\overline{\Omega}\cdot\Omega^{-1}\cdot\Gamma \equiv$ id. on $\overline{V}'$; indeed, if $\overline{v}\varepsilon\overline{V}$ and $\overline{v}'\varepsilon\overline{V}'$, then there is a sequence $v_1,v_2,v_3,\ldots$ in V such that $\overline{v} = \lim_n v_n$, and if we denote $\Omega^{-1}\cdot\Gamma(\overline{v}')$ by $\hat{f}'$, it follows that $<\overline{\Omega}\cdot\Omega^{-1}\cdot\Gamma(\overline{v}'), \overline{v}> = <\overline{\Omega}(\hat{f}'), \lim_n v_n> = \lim_n <\overline{\Omega}(\hat{f}'),v_n> = \lim_n <\hat{f}',v_n> = \lim_n <\Omega^{-1}\cdot\Gamma(\overline{v}'),v_n> = \lim_n <\Omega^{-1}(\overline{v}'|V),v_n> = \lim_n <(\overline{v}'|V),v_n> = \lim_n <\overline{v}',v_n> = <\overline{v}', \lim_n v_n> = <\overline{v}',\overline{v}>$; thus, for all $\overline{v}'\varepsilon\overline{V}'$, $\overline{\Omega}\cdot\Omega^{-1}\cdot\Gamma(\overline{v}') \equiv \overline{v}'$; therefore $\overline{\Omega}\cdot\Omega^{-1}\cdot\Gamma \equiv$ id. on $\overline{V}'$.

We conclude that $\Omega:E[\beta(E,\hat{F})] \to V'[\beta(V',V)]$ is not a linear homeomorphism, for otherwise it would follow from (d) and (e) that $(\overline{\Omega}\cdot\Omega^{-1}\cdot\Gamma \equiv \text{id.}):\overline{V}'[\beta(\overline{V}',V)] \to \overline{V}'[\beta(\overline{V}',\overline{V})]$ would be continuous, in contradiction to (h) above. We have shown that $\Omega:E[T] \to V'[\beta(V',V)]$ is not a linear homeomorphism. Therefore E[T] is not V-reflexive.

SECTION 5

M-QUASI-REFLEXIVITY IN LOCALLY CONVEX SPACES

Definition 5.1 Suppose E is an l.c. space and M is a saturated collection of bounded subsets of E covering E. Let $G = (E'[T_M])'$. Then E is M-quasi-reflexive (of order n) if and only if, for the canonical linear injection $\Omega_G : E \to G$, $\Omega_G(E)$ is of finite codimension (n) in G.

Remark 5.2 If, in Definition 5.1, we consider the case in which $T_M = \beta(E',E)$, we find that in this case E is M-quasi-reflexive (of order n) if and only if E is quasi-reflexive (of order n) in the sense of Lohman [10,11]. This concept is related to results obtained by Civin and Yood [4] and Singer [18] for Banach spaces. Those authors have shown (cf. [18, pp. 77,78]) that a Banach space E is quasi-reflexive (of order n) if and only if there is a $\sigma(E',E)$-dense, $\beta(E',E)$-closed subspace V of finite codimension (n) in E' such that E is V-pseudo-reflexive (i.e., V-semi-reflexive, which for Banach spaces means V-reflexive according to Definition 1.4). Our remarks on pages 12, 13, and 20, regarding the equivalence of V-semi-reflexivity and V-reflexivity for Banach spaces, are therefore applicable to the concept of quasi-reflexivity of Banach spaces. In Theorem 5.3 we show that we can extend to M-quasi-reflexivity a result of Lohman [10, Thm. 2.40, and 11, Lemma 7] which establishes the equivalence of quasi-reflexivity (of order n) in an l.c. space E to the V-semi-reflexivity of E for a $\sigma(E',E)$-dense, $\beta(E',E)$-closed subspace V of finite codimension (n) in E'. In [11, Thm. 3], Lohman showed that the equivalence between quasi-reflexivity (of order n) and V-reflexivity of a Banach space $E[\mathcal{T}]$ for a $\sigma(E',E)$-dense, $\beta(E',E)$-closed subspace V of finite codimension (n) in E' is still valid for a Fréchet space $E[\mathcal{T}]$. In Theorem 5.5 we prove a similar but much stronger result.

Theorem 5.3 Suppose E is an l.c. space, and $\mathcal{M}$ is a saturated collection of bounded subsets of E covering E. The following are equivalent:

(a) E is $\mathcal{M}$-quasi-reflexive of order n.

(b) There is a $\sigma(E',E)$-dense, $\mathcal{T}_{\mathcal{M}}$-closed subspace V of codimension n in E' such that E is $(V,\mathcal{M})$-semi-reflexive.

Proof: The proof is a straightforward adaptation of Lohman's. proof for the case in which $\mathcal{T}_{\mathcal{M}} = \beta(E',E)$[10, Thm. 2.40, and 11, Lemma 7], which in turn is patterned after the proof of Civin and Yood [4, Thm. 3.3], using the criteria for $\mathcal{T}_{\mathcal{M}}$-minimality of V in E' as stated in Proposition 2.4. We indicate the essential steps.

Suppose (a) holds. Let $G = (E'[\mathcal{T}_{\mathcal{M}}])'$. Let $G = \Omega_G(E) \oplus L$, and let $\{\ell_1, \ldots, \ell_n\}$ be a basis for L. Let $V = L^{\perp E'}$. Then $V = \bigcap_{\ell \in L} \ker \ell$, and V is $\mathcal{T}_{\mathcal{M}}$-closed. Since L is finite-dimensional, L is $\sigma(G,E')$-closed, and $L^{\perp E' \perp G} = L$; thus $V^{\perp G} = L$. Since $\{0\} = \Omega_G(E) \cap L = \Omega_G(E) \cap V^{\perp G}$, V separates points of E, and V is $\sigma(E',E)$-dense in E'; also, since $G = \Omega_G(E) \oplus L = \Omega_G \oplus V^{\perp G}$, Proposition 2.4 implies that V is $\mathcal{T}_{\mathcal{M}}$-minimal in E', and Theorem 2.5 implies that E is $(V,\mathcal{M})$-semi-reflexive. Now there are linearly independent $\{e_1',\ldots,e_n'\}$ in E' such that $\langle e_i',\ell_j\rangle = \delta_{ij}$ for $1 \leq i,j \leq n$. Let H be the linear span of $\{e_1',\ldots,e_n'\}$ in E'. To show that $E' = V + H$, suppose $e' \in E'$ and $e' \notin V$; then for $e_o' = \sum_{i=1}^{n} \langle e',\ell_i\rangle e_i'$ and for $e_{oo}' = e' - e_o'$, we have $e_o' \in H$ and $e_{oo}' \in (L^{\perp E'} = V)$ and $e' = e_{oo}' + e_o'$. To show that $V \cap H = \{0\}$, suppose $e_o' \in H$ and $e_o' \neq 0$; then there are scalars $\lambda_1,\ldots,\lambda_n$ such that $e_o' = \sum_{i=1}^{n} \lambda_i e_i'$ where $\lambda_{i_o} \neq 0$ for some i_o; thus $\langle e_o',\ell_{i_o}\rangle = \lambda_{i_o} \neq 0$, and $e_o' \notin (L^{\perp E'} = V)$. Therefore (b) holds.

Suppose (b) holds. Let $E' = V \oplus H$, and let $\{e_1',\ldots,e_n'\}$ be a basis for H. Let $L = V^{\perp G}$. For $1 \leq i_o \leq n$, let H_{i_o} denote the linear span in E' of the set $\{e_i' \mid 1 \leq i \leq n \text{ and } i \neq i_o\}$; since V is $\mathcal{T}_{\mathcal{M}}$-closed and H_{i_o} is finite-dimensional, $V + H_{i_o}$ is $\mathcal{T}_{\mathcal{M}}$-closed, and there is an $e_{i_o}'' \in (V^{\perp G} = L)$ such that $\langle e_i',e_{i_o}''\rangle = \delta_{ii_o}$ for $1 \leq i \leq n$. It is easily verified that $\{e_1'',\ldots,e_n''\}$ are linearly independent in G. Let $e'' \in (V^{\perp G} = L)$; then $e'' = \sum_{i=1}^{n} \langle e_i',e''\rangle e_i''$; thus L is the linear span of $\{e_1'',\ldots,e_n''\}$; therefore L has dimension n. By Theorem 2.5 and Proposition 2.4, (b) implies that $G = \Omega_G(E) \oplus V^{\perp G} = \Omega_G(E) \oplus L$. Thus (a) holds.

<u>Proposition 5.4</u> Suppose $E[\mathcal{T}]$ is an l.c. space and $\mathcal{M}$ is a saturated collection of bounded subsets of E covering E. Let $\mathcal{E}_{\mathcal{T}}$ be the collection of all $\mathcal{T}$-equicontinuous subsets of E', and let $\mathcal{B}_{\mathcal{T}_{\mathcal{M}}}$ be

the collection of all $\mathcal{T}_M$-bounded subsets of E'. Denote $(E'[\mathcal{T}_M])'$ by G, and let $\Omega_G : E \to G$ be the canonical linear injection. Suppose that $\mathcal{E}_\mathcal{T} = \mathcal{B}_{\mathcal{T}_M}$. Then $\Omega_G : E[\mathcal{T}] \to \Omega_G(E)[\beta_i(G,E')]$ is a linear homeomorphism.

<u>Proof</u>: By the definition of Ω_G, the pairings $\langle E,E'\rangle$ and $\langle \Omega_G(E),E'\rangle$ are algebraically isomorphic with respect to $\Omega_G : E \to \Omega_G(E)$ and $id : E' \to E'$. Now since both $\mathcal{T}$ and $\beta(G,E')|\Omega_G(E)$ are defined by polars of the sets in $(\mathcal{E}_\mathcal{T} = \mathcal{B}_{\mathcal{T}_M})$, the conclusion is obvious.

<u>Theorem 5.5</u> Suppose $E[\mathcal{T}]$ is an l.c. space, and $\mathcal{M}$ is a saturated collection of bounded subsets of E covering E. Let $(E'[\mathcal{T}_M])' = G$, and let Ω_G denote the canonical linear injection of E into G. Assume $\Omega_G(E)$ is $\beta(G,E')$-closed in G, and one of the following is true:

(a) $E[\mathcal{T}]$ is barrelled;

(b) $E[\mathcal{T}]$ is infrabarrelled, and $\mathcal{T}_M = \beta(E',E)$.

Then the following are equivalent:

(1) E is $\mathcal{M}$-quasi-reflexive of order n.

(2) There is a $\sigma(E',E)$-dense, $\mathcal{T}_M$-closed subspace V of codimension n in E' such that $E[\mathcal{T}]$ is $(V,\mathcal{M})$-reflexive.

<u>Proof</u>: By Theorem 5.3, (2) implies (1). Conversely, if (1) holds, then by Theorem 5.3 and its proof we have the following:

(i) There is an n-dimensional subspace L of G such that $G = \Omega_G(E) \oplus L$.

(ii) The subspace $(V = L^{\perp E'})$ of E' is $\sigma(E',E)$-dense and $\mathcal{T}_M$-closed in E', and $V^{\perp G} = L$.

(iii) There is an n-dimensional subspace H of E' such that $E' = V \oplus H$.

(iv) E is $(V,\mathcal{M})$-semi-reflexive; i.e., the canonical linear injection $\Omega: E \to V'$ is an algebraic isomorphism.

It remains only to show that $\Omega: E[\mathcal{T}] \to V'[\beta(V',V)]$ is a homeomorphism.

Since $\Omega_G(E)$ is $\beta(G,E')$-closed and of codimension n in G, $G[\beta(G,E')]$ is the topological direct sum of $\Omega_G(E)$ and L [6, p. 142, Prop. 4], and the projection maps of G onto $\Omega_G(E)$ and L are continuous [6, p. 121, Prop. 1]. Consider the following diagram:

$$\begin{tikzcd}
G = (\Omega_G(E) \oplus L)[\beta(G,E')] \arrow[r, "P"] \arrow[d, "Q"'] & \Omega_G(E)[\beta_i(G,E')] \\
G/L = (\Omega_G(E) \oplus L)/L[\beta_Q(G,E')] \arrow[ur, "\overline{P}"'] &
\end{tikzcd}$$

where Q is the canonical surjection, P is the projection, $\overline{P}$ is the associated bijection, and $\beta_Q(G,E')$ is the quotient topology for G/L induced by $\beta(G,E')$. Now Q is always both continuous and open, and P and $\overline{P}$ are always open. We have observed that P is continuous, and thus $\overline{P}$ is continuous as well as open.

Now if we define the map $\Psi: G/L \to V' = (V[\mathcal{T}_{\mathcal{M}}|V])'$ so that $\Psi(e'' + L) = (e''|V)$ for all $e'' \varepsilon G$, since $L = V^{\perp}$ it follows that Ψ is an algebraic isomorphism. We shall show that for the topologies $\beta_Q(G,E')$ and $\beta(V',V)$, Ψ is in fact a linear homeomorphism.

First we note that the pairings $\langle G/L, V\rangle$ and $\langle V', V\rangle$ are algebraically isomorphic with respect to $\Psi: G/L \to V'$ and id: $V \to V$; indeed,

$\langle\Psi(e'' + L),v\rangle = \langle(e''|V),v\rangle$ whenever $e''\varepsilon G$ and $v\varepsilon V$. Therefore,

(I) $\Psi:G/L[\beta(G/L,V)] \to V'[\beta(V',V)]$ is a linear homeomorphism.

We claim that the topologies $\beta(G/L,V)$ and $\beta_Q(G,E')$ are identical.

We show that $\beta(G/L,V) \leqq \beta_Q(G,E')$. Indeed, suppose $A \subset V$ and A is $\sigma(V,G/L)$-bounded. If $\{e_1'',\ldots,e_n''\} \subset G$, then $\{(e_1'' + L),\ldots,(e_n'' + L)\} \subset G/L$, so there is an $r > 0$ such that, for $v\varepsilon A$ and for $1 \leqq i \leqq n$, $|\langle(e_i'' + L),v\rangle| = |\langle e_i'',v\rangle| \leqq r$. Thus A is $\sigma(V,G)$-bounded, and hence $\sigma(E',G)$-bounded. Now $A^{\circ\, G/L} = \{(e'' + L)\ \varepsilon\ G/L|\ |\langle e'',v\rangle| \leqq 1 \text{ for all } v\varepsilon A\} = Q(\{e''\varepsilon G|\ |\langle e'',v\rangle| \leqq 1 \text{ for all } v\varepsilon A\}) = Q(A^{\circ\, G})$. Since $(A^{\circ\, G})$ is a $\beta(G,E')$-neighborhood of 0 in G, and since $Q:G[\beta(G,E')] \to G/L[\beta_Q(G,E')]$ is an open map, $Q(A^{\circ\, G})$ is a $\beta_Q(G,E')$-neighborhood of 0 in G/L; hence, so is $A^{\circ\, G/L}$.

Since we now know that the identity map $G/L[\beta_Q(G,E)] \to G/L[\beta(G/L,V)]$ is continuous, then in view of (I) above we know that

(II) $\Psi:G/L[\beta_Q(G,E')] \to V'[\beta(V',V)]$ is continuous.

We shall show that this map is also open. Note that if $D \subset G/L$, then D is a $\beta_Q(G,E')$-neighborhood of 0 if and only if there is a $\beta(G,E')$-neighborhood U of 0 in G such that $D = Q(U)$. But U is a $\beta(G,E')$-neighborhood of 0 in G if and only if there is a $\sigma(E',G)$-bounded, i.e., T_M-bounded, subset B of E' such that $B^{\circ G} \subset U$. Suppose B is T_M-bounded in E'. In order for $\Psi(Q(B^{\circ\, G})) = \{(e''|V)\varepsilon V'|\ e''\varepsilon G \text{ and } |\langle e'',b\rangle| \leqq 1 \text{ for all } b\varepsilon B\}$ to be a $\beta(V',V)$-neighborhood of 0 in V', it is sufficient to show there is a $\sigma(V,V')$-bounded, i.e., $\mathcal{T}_M$-bounded, $A \subset V$ such that $A^{\circ\, V'} \subset \Psi(Q(B^{\circ\, G}))$. Thus it is sufficient to show that there is a $\mathcal{T}_M$-bounded $A \subset V$ such that if $v' \varepsilon A^{\circ\, V'}$, there is a $\mathcal{T}_M$-continuous linear extension e'' of v' to E' such that $e'' \varepsilon B^{\circ\, G}$. Since V is $\mathcal{T}_M$-closed in E' and the dimension of E'/V is n, then again using Proposition 4 on p. 142 and Proposition 1 on p. 121 of [6], we have that the projection P_V of E' onto V is continuous. Since B is $\mathcal{T}_M$-bounded, then $P_V(B)$ is $\mathcal{T}_M$-bounded in V. Let $A = P_V(B)$. If $v' \varepsilon A^{\circ\, V'}$, then define e'' on $E' = V \oplus H$ so that, if $e' \varepsilon E'$ and $v+h$ is the unique expression of e' where $v \varepsilon V$ and $h \varepsilon H$, then $\langle e'', e' \rangle = \langle v', v \rangle$. Then e'' is a linear functional on E', and $(\ker e'') = \{v+h \mid v \varepsilon \ker v' \text{ and } h \varepsilon H\} = (\ker v') + H$. Since $(\ker v')$ is $\mathcal{T}_M$-closed in V, and V is $\mathcal{T}_M$-closed in E', $(\ker v')$ is $\mathcal{T}_M$-closed in E'. Since H is an n-dimensional subspace of E', it follows from Proposition 3 on p. 142 of [6] that $(\ker v') + H$ is $\mathcal{T}_M$-closed in E'. Therefore $e'' \varepsilon G$, and $e'' | V \equiv v'$. Moreover, if $b \varepsilon B$ then $b = P_V(b) + P_H(b)$, where $P_V(b) \varepsilon A$ and $P_H(b) \varepsilon H$, and $|\langle e'', b \rangle| = |\langle e'', P_V(b) \rangle + \langle e'', P_H(b) \rangle| = |\langle v', P_V(b) \rangle + 0| \leqq 1$; thus $e'' \varepsilon B^{\circ\, G}$.

We have now proved that the map (II) is a linear homeomorphism. Either hypothesis (a) or (b) implies that $\mathcal{E}_{\mathcal{T}} = \mathcal{B}_{\mathcal{T}_M}$ as in Propsition 5.4. Indeed, (b) holds if and only if $\mathcal{T}_M = \beta(E',E)$ and $\mathcal{E}_{\mathcal{T}} = \mathcal{B}_{\mathcal{T}_M}$ and if (a) holds each set W in $\mathcal{E}_{\mathcal{T}}$, being $\beta(E',E)$-bounded, is also in $\mathcal{B}_{\mathcal{T}_M}$, and each set X in $\mathcal{B}_{\mathcal{T}_M}$, being $\phi(E',E)$-bounded, is also in $\mathcal{E}_{\mathcal{T}}$. Thus by Proposition 5.4, $\Omega_G : E[\mathcal{T}] \to \Omega_G(E)[\beta_i(G,E')]$ is a linear homeomorphism. Combining this fact with our previous results, we have each map in the following diagram a linear homeomorphism:

$$\begin{tikzcd}
& \Omega_G(E)[\beta_i(G,E')] & E[\mathcal{T}] \arrow[l, "\Omega_G"'] \\
G/L\,[\beta_Q(G,E')] \arrow[ur, "\overline{P}"] \arrow[r, "\Psi"] & V'[\beta(V',V)] &
\end{tikzcd}$$

Thus the composition map $(\Psi\cdot\overline{P}^{\,-1}\cdot\Omega_G):E[\mathcal{T}] \to V'[\beta(V',V)]$ is a linear homeomorphism. But this map is precisely the canonical linear bijection $\Omega:E \to V'$; indeed, if $e\epsilon E$ then $\Psi\cdot\overline{P}^{\,-1}\cdot\Omega_G(e) = \Psi(\Omega_G(e) + L) = (\Omega_G(e)|V) = \Omega(e)$.

LIST OF REFERENCES

1. Amemiya, I. "Some Examples of (F)-and (DF)-Spaces." Proc. Japan Acad., 33 (1957), pp. 169-171.

2. Bourbaki, N. Elements of Mathematics: General Topology, Part I. Paris: Hermann, 1966.

3. Bourbaki, N. Éléments de Mathématique, Livre V: Espaces Vectoriels Topologiques, 2 Vols. Paris: Hermann, 1953, 1955.

4. Civin, P., and B. Yood. "Quasi-reflexive Spaces." Proc. Am. Math. Soc., 8 (1957), pp. 906-911.

5. Dixmier, J. "Sur un Théorème de Banach." Duke Math. J., 15 (1948), pp. 1057-1071.

6. Horváth, J. Topological Vector Spaces and Distributions, Vol. I. Reading, Mass: Addison-Wesley, 1966.

7. Kelley, J. L., et. al. Linear Topological Spaces. Princeton, N.J.: Van Nostrand, 1963.

8. Köthe, G. Topological Vector Spaces I. New York: Springer-Verlag, 1969.

9. Krishnamurthy, V. "Conjugate Locally Convex Spaces." Math. Zeitschr, 87 (1965), pp. 334-344.

10. Lohman, R. H. "V-Semi-reflexivity, V-Reflexivity and Quasi-reflexivity in Locally Convex Spaces." Ph.D. dissertation, University of Iowa, 1968.

11. Lohman, R. H. "V-Semi-reflexivity, V-Reflexivity and Quasi-reflexivity in Locally Convex Spaces." Math. Ann., 187 (1970), pp. 85-94.

12. Luxemburg, W. A. J. "On Closed Linear Subspaces and Dense Linear Subspaces of Locally Convex Topological Linear Spaces." Proc. International Symp. on Linear Spaces Jerusalem, (1961), pp. 307-318.

13. Ruston, A. F. "Conjugate Banach Spaces." Proc. Cambridge Philos. Soc., 53 (1957), pp. 576-580.

14. Singer, I. "On a Theorem of J. D. Weston." Journal of London Math. Soc., 34 (1959), pp. 320-324.

15. Singer, I. "On Banach Spaces Reflexive with respect to a Linear Subspace of their Conjugate Space." Bull. Math. de la Soc. Sci. Math. Phys. de la R.P.R., 2 (1958), pp. 449-462.

16. Singer, I. "On Banach Spaces Reflexive with respect to a Linear Subspace of their Conjugate Space. II." Math. Ann., 145 (1962), pp. 64-76.

17. Singer, I. "On Banach Spaces Reflexive with respect to a Linear Subspace of their Conjugate Space. III." Rev. Math. Pures et Appl., 8 (1963), pp. 139-150.

18. Singer, I. "Weak Compactness, Pseudo-Reflexivity and Quasi-Reflexivity." Math. Ann., 154 (1964), pp. 77-87.