APPLICATIONS OF THE THEORY OF MATRICES

TO

ELECTRIC NETWORKS

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The objective of this thesis is the investigation of the application of the theory of matrices to analysis of electric networks. Various theorems from the theory of matrices which are applicable to matrix analysis of electric networks are stated and proved in Part I. Part II, Chapters III and IV, consists of problems of electric network theory whose solutions are facilitated by use of the theory of matrices. Using matrix methods, the mesh and node pair equations are developed and solved in Chapter III. Applications of matrix methods to transmission networks and transmission lines are considered in Chapter IV.

It is the opinion of the author that the method of developement of equation (3.2.13), Theorems 4.1.1 and 4.2.1, and the proofs of Theorems 4.4.1 and 4.4.2 are original contributions. Equation (3.2.13) is one of the principal equations used in node pair analysis of electric networks. Theorems 4.1.1 and 4.2.1 are concerned with the matrix associated with a transmission network. Some interesting properties of a particular matrix are stated in Theorems 4.4.1 and 4.4.2. This matrix is useful in the analysis of fully transposed transmission lines.

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PART I

SOME THEOREMS ON MATRICES

CHAPTER I

BASIC THEORY OF MATRICES

1.1. Introduction. This thesis is divided into two parts. In Part I the theorems of matrix theory, which are to be applied to electric networks in Part II, will be stated and proved.

The word matrix was first used by J. J. Sylvester to describe a rectangular array of numbers "out of which determinants can be formed."[1] The concept of matrix was used explicitly by Arthur Cayley in 1858.[2] He defined the matrix in a similar manner to Sylvester but was insistent that "the idea of matrix precedes that of determinant."[3] A matrix has been defined as a rectangular array of mn quantities, arranged in m rows and n columns.[4] According to C. C. MacDuffee, a matrix is an element of a total matric algebra.[5] We will use the following definition: A matrix is a rectangular array of mn quantities arranged in m rows and n columns

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1 C. C. MacDuffee, "What is a Matrix?", American Mathematical Monthly, 1943, L, 360.
2 E. T. Bell, Developement of Mathematics (New York, 1945), p. 205.
3 MacDuffee, American Mathematical Monthly, p. 360.
4 John M. H. Olmsted, Solid Analytic Geometry (New York, 1947)
5 MacDuffee, American Mathematical Monthly, p. 361.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = (a_{ij})$$

with elements a_{ij} in a field F.

In order to clarify this definition somewhat, we will define an <u>equals relation</u> and <u>abelian group</u>. Then using these definitions, a <u>field</u> will be defined.

An equals relation a=b is characterized by the following properties:[1]

i. Either a = b or a ≠ b. (The relation is determinative.)

ii. a = a. (The relation is reflexive.)

iii. If a = b, then b = a. (The relation is symmetric.)

iv. If a = b and b = c, then a = c. (The relation is transitive.)

An <u>abelian group</u> is a mathematical system composed of elements, an equals relation, and one operation X subject to the following postulates:[2]

i. The system is closed under the operation X, which is well defined.

ii. The operation X is associative.

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1 C. C. MacDuffee, <u>Introduction to Abstract Algebra</u> (New York, 1940), p. 47.
2 <u>Ibid</u>., p. 47.

iii. There exists an identity element I such that

$$a \times I = I \times a = a$$

for every element a of the group.

iv. Every element a has an inverse a^{-1} such that

$$a^{-1} \times a = a \times a^{-1} = I.$$

v. The operation X is commutative.

A field F is a mathematical system composed of elements, an equals relation, and two well defined operations, addition and multiplication defined by the postulates:[1]

i. The elements constitute an abelian group relative to the operation of addition, the identity element being denoted by z, and the inverse of a by - a.

ii. The elements with z omitted constitute an abelian group relative to multiplication, the identity element being denoted by I and the inverse of a by a^{-1}.

iii. Multiplication is distributive with respect to addition.

We will be interested in two fields, the field of real numbers and the complex field. Unless otherwise specified, it will be understood that these are the fields under consideration.

1.2. Addition and multiplication of matrices.[2] Two matrices $A = (a_{ij})$ and $B = (b_{ij})$ are equal if and only if

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1 MacDuffee, Abstract Algebra, p. 79.
2 Sam Perlis, Theory of Matrices (Cambridge, 1952), pp. 2 - 17.

$a_{ij} = b_{ij}$ for every i and j.

The addition of two matrices A and B is defined to be a third matrix whose elements are equal to the sums of corresponding elements of A and B, i.e.,

(1.2.1) $$A + B = (a_{ij} + b_{ij}) = (c_{ij}) = C.$$

We define the product of two matrices $A = (a_{ij})$ and $B = (b_{jk})$ to be the array

(1.2.2) $$AB = (\sum_{j=1}^{n} a_{ij} b_{jk}) = (c_{ij}) = C,$$

or row by column multiplication of the matrices. From this definition, we see that the number of columns of A must equal the number of rows of B. The resulting matrix will have the same number of rows as A and the same number of columns as B. Multiplication of matrices is in general not commutative.

Multiplication of a matrix by a scalar number k is accomplished by multiplying each element of the matrix by k, $kA = (ka_{ij})$.

Theorem 1.2.1. Multiplication of matrices is associative.

Proof. Let $A = (a_{ij})$, $B = (b_{jk})$, and $C = (c_{kl})$ where $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$; $k = 1, 2, \ldots, s$; $l = 1, 2, \ldots, t$. By definition

(1.2.3) $$AB = \sum_{j=1}^{n} a_{ij} b_{jk} = (d_{ik}).$$

Then

(1.2.4) $$(AB)C = \sum_{k=1}^{s} d_{ik} c_{kl} = \sum_{k=1}^{s} (\sum_{j=1}^{n} a_{ij} b_{jk}) c_{kl}.$$

Multiplication in a field is associative and distributive with respect to addition so that

$$(1.2.5) \qquad \sum_{k=1}^{s}\Big(\sum_{j=1}^{n} a_{ij}b_{jk}\Big)c_{kl} = \sum_{k=1}^{s}\sum_{j=1}^{n}(a_{ij}b_{jk})c_{kl}$$

$$= \sum_{k=1}^{s}\sum_{j=1}^{n} a_{ij}(b_{jk}c_{kl}).$$

Since any finite double sum is independent of the order of summation

$$(1.2.6) \qquad (AB)C = \sum_{j=1}^{n}\sum_{k=1}^{s} a_{ij}(b_{jk}c_{kl})$$

$$= \sum_{j=1}^{n} a_{ij}\Big(\sum_{k=1}^{s} b_{jk}c_{kl}\Big) = A(BC).$$

Theorem 1.2.2. Multiplication of matrices is distributive with addition.

Proof. Let A and B be m by n matrices and C be an n by s matrix. Then

$$(1.2.7) \qquad (A + B)C = \sum_{j=1}^{n}(a_{ij} + b_{ij})c_{jk} = \sum_{j=1}^{n}(a_{ij}c_{jk} + b_{ij}c_{jk})$$

$$= \Big(\sum_{j=1}^{n} a_{ij}c_{jk}\Big) + \Big(\sum_{j=1}^{n} b_{ij}c_{jk}\Big) = AC + BC.$$

Similarily we see that

$$C(A + B) = CA + CB.$$

The matrix $A' = (a_{ji})$ obtained from $A = (a_{ij})$ by changing rows to columns is called the transpose of A. A matrix $S' = S$ is called symmetric, and a matrix Q such that $Q' = -Q$ is said to be skew or skew-symmetric.

Theorem 1.2.3. The transpose of a product of two matrices is the product of their transposes in reverse order.

Proof. If $A = (a_{ij})$ and $B = (b_{jk})$, then

$$AB = \sum_{j=1}^{n} a_{ij}b_{jk} = (c_{ik}) = C. \tag{1.2.8}$$

By definition

$$C' = (\sum_{j=1}^{n} a_{ij}b_{jk})' = \sum_{j=1}^{n} b_{kj}a_{ji} = B'A'. \tag{1.2.9}$$

If a matrix B is obtained from the matrix A by striking out certain rows or columns of A, then we refer to B as a <u>submatrix</u> of the matrix A. At times it may be desirable to refer to a given matrix as being made up of its submatrices. If A is an m by n matrix and a_{ij}, $(i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n)$, is in one and only one of the submatrices of A, then A may be written $A = (A_{ij})$, $(i = 1, 2, \ldots, s;\ j = 1, 2, \ldots, t)$, where each A_{ij} is a matrix. If for a fixed i, A_{i1}, A_{i2}, A_{i3}, $\ldots$, A_{it} all have the same number of rows and for fixed j, A_{1j}, A_{2j}, $\ldots$, A_{sj} all have the same number of columns, then $A = (A_{ij})$ is a <u>partitioning</u> of A. The partitioning of a matrix may also be thought of as drawing lines parallel to the rows and columns of A, and between them, and representing the submatrices thus formed by A_{ij}.

If a partitioned matrix B is to be multiplied on the left by a partitioned matrix A, then the partitioning of the columns of A must be the same as the partitioning of the rows of B.

1.3. <u>Diagonal and scalar matrices</u>. A square matrix D, whose non-zero elements occupy the <u>principal diagonal</u>, of the type

$$D = \begin{bmatrix} k_1 & 0 & 0 & 0 & 0 \\ 0 & k_2 & 0 & 0 & 0 \\ 0 & 0 & k_3 & 0 & 0 \\ 0 & 0 & 0 & k_4 & 0 \\ 0 & 0 & 0 & 0 & k_5 \end{bmatrix} = (k_1, k_2, k_3, k_4, k_5)$$

is said to be of diagonal form. If D has n rows and columns and $k_i = k$, $(i = 1, 2, \ldots, n)$, and k is a constant, the matrix is said to be a scalar matrix.

From the definition of multiplication, it is evident that the multiplication of a matrix A by a scalar matrix is equivalent to multiplying A by a scalar number.

A scalar matrix whose non-zero elements are the identity of the field F are called identity matrices. The I is commonly used to denote such a matrix. A matrix with elements all zero is the zero or null matrix and is designated by O.

1.4. Elementary operations on matrices.[1] The elementary operations are:

i. Interchange of two rows or of two columns.

ii. Addition to a row of a multiple of another row, or addition to a column of a multiple of another column.

iii. Multiplication of a row or of a column by a non-vanishing constant.

Theorem 1.4.1 Each elementary operation on the rows (columns) of a matrix A can be accomplished by multiplying A

1 R. A. Frazer, W. J. Duncan, A. R. Collar, Elementary Matrices (New York, 1946), pp. 87-89.

on the left(right) by the matrix J which is obtained by performing the given elementary operation upon the unit matrix I. The matrix J is sometimes called an <u>elementary matrix</u>.[1]

Proof. Let J be a matrix obtained by interchanging the i-th and j-th rows(or columns) of I. Then the only non-zero element in the i-th row of J is a 1 in the (i,j) position, and the only non-zero element in the j-th row of J is a 1 in the (j,i) position. Hence in the product JA, when the i-th row of J is multiplied by the k-th column of A, the product will be the element of A in the (j,k) position. Similarly, the product of the j-th row of J times the k-th column of A will be the element of A in the (i,k) position. Therefore the result will be the interchanging of rows i and j. In the product AJ, when the k-th row of A is multiplied by the i-th column of J, the result will be the element in the (k,j) position of A. The product of the k-th row of A and the j-th column of J will be the element in the (k,i) position of A. Consequently, the product AJ will simply interchange the i-th and j-th columns of A.

By examining the product AJ, where J is obtained from I by adding a multiple of the i-th column of I to the j-th column of I, in a similar manner we see that the result is addition of the same multiple of the i-th column of A to the

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1 C. C. MacDuffee, <u>Theory of Matrices</u> (New York, 1946), p. 32.

j-th column of A. Similarly for the product JA, where J is obtained from I by adding a multiple of the i-th row of I to the j-th row of I, the result is addition of the same multiple of the i-th row of A to the j-th row of A.

If J is the matrix obtained by multiplying the i-th row (or column) of I by a non-vanishing constant, then it follows from the definition of multiplication, the product JA results in the multiplying of the i-th row of A by the same non-vanishing constant. The product AJ results in the multiplying of the i-th column of A by the same non-vanishing constant.

1.5. The determinant. If A is an n by n (sometimes said to be of order n and written n^2) matrix, there is associated with A a number $d(A)$ which serves as an absolute value of A. This number $d(A)$ is characterized by

i. For every A, $d(A)$ is a non-constant rational integral function of the elements of A of lowest degree such that

ii. $d(AB) = d(A)d(B)$.[1]

In general

$$d(A) = \sum e_{h_1h_2\ldots h_n} a_{1h_1} a_{2h_2} \cdots a_{nh_n}, \tag{1.5.1}$$

where the summation is over all permutations $(h_1, h_2, \ldots, h_n)$ of $(1, 2, \ldots, n)$ and $e_{h_1h_2\ldots h_n}$ is 1 or -1 according to the permutation being even or odd.

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1 MacDuffee, Theory of Matrices, p. 6.

The determinant of the $(n-1)^2$ matrix derived from $A = (a_{ij})$ by deleting the i-th row and the j-th column of A and assigning to it the sign of $(-1)^{i+j}$ is the <u>cofactor</u> of a_{ij}. We will designate this determinant by $\underline{A}_{ij}$.

A few well known theorems of determinant theory will be stated here without proof.

Theorem 1.5.1. The sum of the products of the elements of a row (or column) of a matrix by their respective cofactors is equal to the determinant, while the sum of the products of the elements of a row (or column) by the cofactors of the elements of a different row (or column) is zero. Notationally this is equivalent to

$$(1.5.2) \qquad \sum_{i=1}^{n} A_{ir}a_{is} = \sum_{i=1}^{n} a_{ri}A_{si} = \delta_{rs}d(A),$$

where δ_{rs} is an operator such that $\delta_{rs} = 0$ if $r \neq s$, $\delta_{rs} = 1$ if $r = s$. δ_{rs} is <u>Kronecker's delta</u>.

Theorem 1.5.2. If B is obtained from A by multiplying any row or any column of A by k, then

$$(1.5.3) \qquad d(B) = kd(A).$$

Theorem 1.5.3. If B is obtained from A by the interchange of two rows or of two columns

$$(1.5.4) \qquad d(B) = -d(A).$$

Theorem 1.5.4. If A is a square matrix each element of whose k-th row is a sum

$$d_{ks1} + d_{ks2} + \dots + d_{ksm} \qquad (s = 1, 2, \dots, n)$$

then

(1.5.5) $$d(A) = d(A_1) + d(A_2) + \dots + d(A_m),$$

where A_h is the array obtained by replacing the elements of the k-th row by $d_{k1h}, d_{k2h}, \dots, d_{knh}$ respectively. Similarly for columns.

Theorem 1.5.5. If B is obtained from A by adding to any row(or column) a linear combination of the other rows(or columns), then

(1.5.6) $$d(A) = d(B).$$

Theorem 1.5.6. The determinant of a partioned matrix of the form

$$A = \begin{bmatrix} A_{11} & 0 & 0 & \dots & 0 \\ 0 & A_{22} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & A_{nn} \end{bmatrix}$$

is equal to the product of the determinants of the matrices making up the diagonal,

$$d(A) = d(A_{11})d(A_{22})d(A_{33}) \dots d(A_{nn}).$$[1]

A matrix A is said to be <u>singular</u> if $d(A) = 0$, otherwise it is <u>non-singular</u>. If A is a rectangular matrix then we may add a sufficient number of rows of zeros to A on the bottom or a sufficient number of columns of zeros to A on the right to make A a square matrix without changing the effectiveness of A. The new matrix will be singular. We will define

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1 Sam Perlis, p. 72.

a rectangular matrix to be a singular matrix.

1.6. Adjoint and Inverse. The transpose of the matrix obtained from $A = (a_{ij})$ by replacing each element a_{ij} by its cofactor is the adjoint of A, written A^A or adj. A. It is evident from equation (1.5.2) that

Theorem 1.6.1. If A is an n by n matrix

$$A^A A = AA^A = Id(A). \tag{1.6.1}$$

For obvious reasons the matrix defined by $A^A/d(A)$ is called the inverse of A written A^{-1}.

Theorem 1.6.2. The inverse of a product of two matrices is the product of their inverses in reverse order.

Proof.[1] Consider the product of the two n by n matrices A and B. Let $AB = C$. Multiplying this equation on the left by A^{-1} and then B^{-1}, we have $I = B^{-1}A^{-1}C$. Then multiplying on the right by C^{-1}, $C^{-1} = B^{-1}A^{-1}$.

Theorem 1.6.3. Let S be a non-singular, symmetric matrix. If K is any skew matrix such that $(S + K)(S - K)$ is non-singular and $P = (S + K)^{-1}(S - K)$, then[2]

$$P'SP = S.$$

Proof. Let $T = S^{-1}K$, then

$$P = [S(I + T)]^{-1}S(I - T) = (I + T)^{-1}(I - T).$$

$$P' = (S' - K')(S' + K')^{-1} = (S + K)(S - K)^{-1}$$

$$= S(I + T)(I - T)^{-1}S^{-1}.$$

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1 Frazer, Duncan, and Collar, p. 25.
2 Perlis, p. 104.

Then

$$P'SP = S(I + T)(I - T)^{-1}S^{-1}S(I + T)^{-1}(I - T)$$
$$= S(I + T)\left[(I + T)(I - T)\right]^{-1}(I - T)$$
$$= S(I + T)\left[(I - T)(I + T)\right]^{-1}(I - T).$$

Using Theorem 1.6.2,

$$P'SP = S(I + T)(I + T)^{-1}(I - T)^{-1}(I - T) = S.$$

Theorem 1.6.4. The inverse, A^{-1}, of a symmetric matrix, A, is symmetric.

Proof. Consider the cofactor of a_{ij}. It is obtained from A by deleting the i-th row and j-th column, taking the determinant of the resulting matrix and affixing the sign $(-1)^{i+j}$ to the determinant. Since the i-th row of A is identical with the i-th column, the cofactor of a_{ij} is the same as the cofactor of a_{ji}. Hence A^{A} is symmetric and the theorem follows from the definition of A^{-1}.

If $A^{-1} = A'$, then A is an <u>orthogonal</u> matrix.

If p is any positive integer and A any matrix we understand by A^p the product $AAA\ldots A$ to p factors. If A is a non-singular matrix, we define its negative and zero powers by the formulae

$$(1.6.2) \qquad A^{-p} = (A^{-1})^p, \quad A^0 = I.$$ [1]

From this definition we have

Theorem 1.6.5. The laws of exponents

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1 Maxime Bôcher, Introduction to Higher Algebra, (New York, 1931), p. 75.

$$A^pA^q = A^{p+q}, \quad (A^p)^q = A^{pq}$$

hold for all matrices when the exponents p and q are positive integers, and for all non-singular matrices when p and q are any integers.[1]

1.7. Rank of a matrix. A matrix is said to be of rank r if it contains at least one r-rowed determinant which is not zero, while all its determinants of order higher than r are zero.[2] If A is square matrix of order n, then n - r is called the nullity of A.[3] If A is a m by n matrix there are two different nullities, a row-nullity and a column nullity.[4]

Theorem 1.7.1. Let A be an m by n matrix partitioned into m_i by n_j submatrices and let B be an n by p matrix partitioned into n_i by p_j submatrices such that AB = C, then if C has the same row partitioning as A and the same column partitioning as B,

$$C_{ij} = \sum_{k=1}^{t} A_{ik}B_{kj}, \tag{1.7.1}$$

where t is the number of columns in A and the number of rows in B after they have been partitioned.

Proof. Since the row partitioning of C is the same as that of A, C_{ij} will have the same number of rows as A_{ik}. Similarly, C_{ij} has the same number of columns as B_{kj}. Let $C_{ij} = (c_{uv})$, where (u,v) is the index of the element in C. Since

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1 Bôcher, p. 75.
2 MacDuffee, The Theory of Matrices, p. 10.
3 Ibid., p. 10.
4 A. C. Aitken, Determinants and Matrices (New York, 1946), p. 60.

AB = C,

(1.7.2) $$c_{uv} = \sum_{r=1}^{n} a_{ur}b_{rv}.$$

Now the (u,v)-th element of the product on the right of (1.7.1) is

(1.7.3) $$d_{uv} = \sum_{r=1}^{n_1'} a_{ur}b_{rv} + \sum_{r=n_1'+1}^{n_1'+n_2'} a_{ur}b_{rv} + \cdots + \sum_{r=(\sum_1^{t-1} n_1)+1}^{\sum_1^{t} n_1} a_{ur}b_{rv}.$$

However,

$$n_1' + n_2' + \ldots + n_t' = n.$$

Hence, combining the summations in (1.7.3),

(1.7.4) $$d_{uv} = \sum_{r=1}^{n} a_{ur}b_{rv}.$$

Then from (1.7.2) and (1.7.4), it follows that

$$c_{uv} = d_{uv},$$

and the theorem is true.

Corollary 1.7.1. If A is any matrix with n columns and B is any matrix with n rows, any t-rowed determinant D of matrix AB is equal to a sum of terms each a product of a t-rowed determinant of A by a t-rowed determinant of B.[1]

Proof. The corollary follows from the application of property (ii) of a determinant and Theorem 1.5.4 to equation (1.7.1).

Corollary 1.7.2. The rank of the product of two matrices cannot exceed the rank of either factor.[2]

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1 L. E. Dickson, Modern Algebraic Theories (Chicago, 1926), p. 49.
2 Ibid., p. 51.

Proof. If all t-rowed determinants of A(or of B) are zero, the same is true of all t-rowed determinants of AB.

Theorem 1.7.2. If A is any matrix with m rows and n columns and B is any non-singular n-rowed square matrix, then A and AB have the same rank. If C is any non-singular m-rowed square matrix, then A and CA have the same rank.[1]

Proof. If r is the rank of A and p is the rank of $P = AB$, then according to the Corollary 1.7.2, $p \leq r$. Applying the same idea to $A = PB^{-1}$, $r \leq p$; consequently, $r = p$. The same reasoning shows that A and CA have the same rank.

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1 Dickson, p. 51.

CHAPTER II

ALGEBRAIC FORMS, EQUALS RELATIONSHIPS AND THE CHARACTERISTIC MATRIX

2.1. Equivalence of two matrices. Two matrices are said to be equivalent if one can be derived from the other by any finite number of elementary operations.[1] The relation of equivalence is an equals relation.[2]

Theorem 2.1.1. Every matrix A of rank r is equivalent to a matrix C whose elements are all zero with the exception of r ones occupying the first r places in the principal diagonal.

Proof. If the matrix is a null matrix the theorem is obviously true. If A is not null, then at most two elementary operations of the type (i) will be required to bring a non-zero element to the (1,1) position of A. Then by an operation of the type (iii), we can reduce this element to unity. Next we reduce all other elements of the first row and first column to zero by operations of the type (ii). If elements lying below the first row are not all zero, we bring a non-zero element to the second place in the principal dia-

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1 Bôcher, p. 55.
2 MacDuffee, The Theory of Matrices, p. 41.

gonal. This can be done without altering the first row or first column. We can now reduce the element in the (2,2) position to unity and all other elements in the second row and column to zero. If not all the elements below the second row are zero, we bring a non-zero element to the third place in the principal diagonal. In this way, after a finite number of elementary operations on A, we have a diagonal matrix C with units occupying the first t places in the principal diagonal and all other elements of C are zeros. By Corollary 1.7.2, $t \leq r$.

According to Theorem 1.4.1, we may perform any of the elementary operations on A by multiplying A on the left by a matrix U if the operation is on the rows of A, and on the right by a matrix V if the operation is on the columns of A. U and V are obtained by performing the given operation on I. Suppose n operations are required on the rows of A and m operations on the columns of A in order to obtain C. Then

(2.1.1) $$U_n U_{n-1} \cdots U_2 U_1 A V_1 V_2 \cdots V_m = C.$$

By Theorems 1.5.2, 1.5.3, and 1.5.5 we know that each U_i and V_j, $(i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, m)$, is non-singular. Then, since n and m are finite,

(2.1.2) $$B = U_n U_{n-1} \cdots U_2 U_1, \quad D = V_1 V_2 \cdots V_{m-1} V_m$$

are non-singular. Since

(2.1.3) $$BAD = C,$$

from Theorem 1.7.2, C has rank r. Hence $t = r$.

Theorem 2.1.2. Two matrices A and B are equivalent if and only if they have the same rank.

Proof. If A is equivalent to B, then from Theorem 1.7.2, A and B have the same rank.

Suppose A and B have the same rank r. Both A and B can be reduced to the diagonal form described in Theorem 2.1.1. Hence there exists a P_1, P_2, Q_1 and Q_2, each having the same form as B and D in equation (2.1.2), such that

$$P_1 A P_2 = Q_1 B Q_2. \tag{2.1.4}$$

And since P_1, P_2, Q_1, and Q_2 are non-singular

$$A = P_1^{-1} Q_1 B Q_2 P_2^{-1}. \tag{2.1.5}$$

Since the inverse of an elementary matrix is an elementary matrix, A is equivalent to B.

2.2. <u>Linear forms and linear transformations</u>. A linear homogeneous function, such as $5y - 6z$ is called a <u>linear form</u>.[1] The set of m equations

$$\begin{array}{l} y_1 = a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ y_m = a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n \end{array} \tag{2.2.1}$$

expressing the m variables $y_1, y_2, y_3, \dots, y_m$ in terms of the n variables $x_1, x_2, x_3, \dots, x_n$ can be written in the matrix form as

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1 Dickson, p. 39.

$$(2.2.2) \qquad \begin{bmatrix} y_1 \\ y_2 \\ \cdot\cdot \\ y_m \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot\cdot \\ x_n \end{bmatrix}$$

or more precisely

$$Y = AX.$$

If $m = n$, then (2.2.1) is said to be a linear transformation.[1]

The set of linear equations (2.2.1) may also be written in the compact notations

$$(2.2.3) \qquad y_i = \sum_{j=1}^{n} a_{ij} x_j \qquad (i = 1, 2, \ldots, m).$$

Suppose the n variables x_j are expressible linearly in terms of new variables $z_1, z_2, \ldots, z_n$. Then

$$(2.2.4) \qquad x_j = \sum_{k=1}^{n} b_{jk} z_k \qquad (j = 1, 2, \ldots, n).$$

Substituting (2.2.4) into (2.2.3), we have

$$(2.2.5) \qquad y_i = \sum_{j=1}^{n} \sum_{k=1}^{n} a_{ij} b_{jk} z_k \qquad (i = 1, 2, \ldots, m).$$

Since a finite sum is independent of the order of summation

$$(2.2.6) \qquad y_i = \sum_{k=1}^{n} \sum_{j=1}^{n} a_{ij} b_{jk} z_k \qquad (i = 1, 2, \ldots, m).$$

But

$$(2.2.7) \qquad AB = \sum_{j=1}^{n} a_{ij} b_{jk}.$$

Hence we have

Theorem 2.2.1. A linear transformation with the matrix B replaces a system of linear forms with the matrix A by a system of linear forms with the matrix AB.

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1 Dickson, p. 41.

Consider a set of m equations in n unknown such as (2.2.1) such that $y_1 = y_2 = \ldots = y_m = 0$. Such a set of equations are <u>linear</u> <u>homogeneous</u> <u>equations</u> in the n unknown x_1, x_2, x_3, ..., x_n.

Theorem 2.2.2. A necessary and sufficient condition that m linear homogeneous equations in n unknowns have solutions not all zero is that the matrix of the coefficients of the unknowns have rank $r < n$.

Proof. Suppose $r = n$. Then $m \geq n$. By rearrangement of the equations or by elementary operations, if necessary, we can obtain an n by n submatrix A_1 of the coefficient matrix A, such that A_1 is non-singular and consists of the first n rows of A. Since solutions of the m equations must also be solutions of the first n, the solutions must satisfy

(2.2.8) $$A_1X = 0.$$

But since A_1 is non-singular

(2.2.9) $$A_1^{-1}A_1X = X = 0.$$

Hence the assumption that $r = n$ has led to the identically zero solution of the m equations. Hence $r < n$ is a necessary condition that $X \neq 0$.

If $r < n$, then we can arrange the equations and the unknowns so that the r by r submatrix of A consisting of the first r elements of the first r rows of A is non-singular. Now we partition A so that

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \tag{2.2.10}$$

where A_{11} is an r by r matrix, A_{12} is r by $n - r$, A_{21} is $m - r$ by r and A_{22} is $m - r$ by $n - r$. Multiplying A on the left by the elementary matrix

$$J = \begin{bmatrix} I & 0 \\ -A_{21}A_{11}^{-1} & I \end{bmatrix}, \tag{2.2.11}$$

we have

$$JA = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{bmatrix}. \tag{2.2.12}$$

$A_{22} - A_{21}A_{11}^{-1}A_{12}$ is a null matrix. For if it contained a non-zero element, this element could be moved to the $(r + 1, r + 1)$ position of JA, making JA of rank $r + 1$. By Theorem 1.7.2, this cannot be true.

Writing equations (2.2.1) in the form

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = 0 \tag{2.2.13}$$

and multiplying (2.2.13) on the left by J, we have

$$\begin{bmatrix} A_{11} & A_{12} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = 0.$$

Hence

$$A_{11}X_1 + A_{12}X_2 = 0.$$

Therefore

$$X_1 = - A_{11}^{-1}A_{12}X_2. \tag{2.2.13'}$$

Then $r < n$ is sufficient to insure solutions to a set of m

homogeneous equations in n unknowns.

Corollary 2.2.2. If the coefficient matrix of m homogeneous equations in n unknowns is of rank $r < n$, then $n - r$ unknowns can be arbitrarily assigned. These $n - r$ unknowns are parameters in terms of which the other r unknowns can be linearly and uniquely expressed.

Corollary 2.2.21. A system of m homogeneous equations in n unknowns, where $m < n$, always has solutions not all zero.

A system of equations (2.2.1) such that the y_i are not all zero is a system of non-homogeneous equations. The matrix $[A, Y]$ associated with such a set of equations is called the augmented matrix of the set.

A set of m equations in n unknowns is said to be consistent if there exists values of the unknowns which satisfy all the equations. Otherwise the equations are inconsistent.

Theorem 2.2.3. Any m non-homogeneous equations in n unknowns with coefficient matrix A of rank r are consistent and solvable in terms of $n - r$ parameters if and only if the rank of the augmented matrix $[A, Y]$ is equal to the rank of A.

Proof. Let A be the coefficient matrix of m non-homogeneous equations in n unknowns. Suppose A and $[A, Y]$ are each of rank r. Partition A in the same manner as in the proof of Theorem 2.2.2. Then we have

$$(2.2.14) \qquad \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}.$$

where X_1 and Y_1 is an r by 1 matrix, X_2 is an n - r by one matrix and Y_2 is m - r by one. Multiplying equation (2.2.14) on the left by the matrix J of equation (2.2.11),

$$(2.2.15)\qquad \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} Y_1 \\ Y_2 - A_{21}A_{11}^{-1}Y_1 \end{bmatrix}.$$

Since the coefficient matrix is of rank r, $A_{22} - A_{21}A_{11}^{-1}A_{12} = 0$, and, since the augmented matrix is of rank r, $Y_2 - A_{21}A_{11}Y_1 = 0$. Then (2.2.15) can be written

$$(2.2.16)\qquad A_{11}X_1 + A_{12}X_2 = Y_1.$$

From which we have the solution

$$(2.2.17)\qquad X_1 = A_{11}^{-1}Y_1 - A_{11}^{-1}A_{12}X_2$$

of (2.2.14). The elements of X_2 are arbitrarily assigned, which uniquely determines the r elements of X_1. Hence, if the rank of the augmented matrix is the same as the rank of the coefficient matrix, the equations are consistent.

Suppose the equations are consistent. Equation (2.2.17) satisfies the first r of the equations. If the equations are consistent, then (2.2.17) must satisfy all of the other equations also. Now the rank of $J[A, Y]$ is the same as $[A, Y]$.

$$(2.2.18)\qquad J[A, Y] = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 - A_{21}A_{11}^{-1}Y_1 \end{bmatrix}.$$

Since A is of rank r, $A_{22} - A_{21}A_{11}^{-1}A_{12} = 0$. Consider $Y_2 - A_{21}A_{11}^{-1}Y_1$.

From equation (2.2.14)

$$(2.2.19)\qquad Y_2 = A_{21}X_1 + A_{22}X_2.$$

Therefore,

$$(2.2.20)\qquad Y_2 - A_{21}A_{11}^{-1}Y_1 = A_{21}X_1 + A_{22}X_2 - A_{21}A_{11}^{-1}Y_1.$$

Then using (2.2.17)

$$\begin{aligned} Y_2 - A_{21}A_{11}^{-1}Y_1 &= A_{21}(A_{11}^{-1}Y_1 - A_{11}^{-1}A_{12}X_2) + A_{22}X_2 - A_{21}A_{11}^{-1}Y_1 \\ &= A_{21}A_{11}^{-1}Y_1 + (A_{22} - A_{21}A_{11}^{-1}A_{12})X_2 - A_{21}A_{11}^{-1}Y_1. \\ &= 0 \end{aligned}$$

Hence, $[A\ Y]$ has rank r.

2.3. <u>Bilinear and quadratic forms</u>. A polynomial in the m + n variables $x_1, \ldots, x_m, y_1, \ldots y_n$ is called a <u>bilinear form</u> if each of its terms is of the first degree in the x's and also of the first degree in the y's.[1] In compact notation

$$(2.3.1)\qquad A(x,y) = \sum_{i=1}^{m}\ \sum_{j=1}^{n} a_{ij}x_i y_j.$$

In matrix notation

$$(2.3.2)\qquad A(x,y) = \begin{bmatrix} x_1 & x_2 & \cdots & x_m \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \cdots \\ y_n \end{bmatrix},$$

or more precisely

$$(2.3.3)\qquad A(x,y) = XAY,$$

in which X is an one by m matrix and Y is an n by one matrix. If m = n and $a_{ij} = a_{ji}$, then (2.3.1) is a symmetric <u>bilinear</u>

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1 Dickson, p. 51.

<u>form</u>.[1] If the sets of variables are identical such that $X' = Y$, then the bilinear form becomes a <u>quadratic form</u>.[2]

2.4. <u>Congruence of matrices</u>. Any two n-rowed matrices A and B are congruent if there exists a non-singular n-rowed square matrix P such that $A = P'BP$.[3]

Theorem 2.4.1. Every real square symmetric matrix A of order n and of rank r is congruent to a diagonal matrix whose diagonal elements are either 1, -1, or 0. The number of 1's plus the number of -1's equals r.[4]

Proof. If $r = 0$, the theorem is true. We assume $r > 0$.

If $a_{11} = 0$, then some $a_{1j} \neq 0$. We consider first the elements of the first row and first column. If $a_{1j} \neq 0$ and $2a_{1j} + a_{jj} \neq 0$, we add the j-th row of A to the first row and the j-th column of A to the first column. The result is a matrix B congruent to A with $b_{11} \neq 0$. If $a_{1j} \neq 0$, but $2a_{1j} + a_{jj} = 0$, we multiply the first row and column of A by -1. Then if the j-th row is added to the first row and the j-th column is added to the first column, we obtain a matrix B congruent to A with $b_{11} \neq 0$.

If every element of the first row and first column is zero then there is some $a_{ij} \neq 0$. By interchanging the first row and i-th row and the first column and i-th column, we ar-

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1 Dickson, p. 64.
2 Frazer, Duncan, and Collar, p. 28.
3 Dickson, p. 65.
4 Saunders MacLane, <u>Notes on Higher Algebra</u> (Ann Arbor 1939), p. 153.

rive at a matrix C congruent to A with some $c_{1j} \neq 0$. We then proceed as before to obtain a matrix B with $b_{11} \neq 0$.

If we add $-b_{1k}/b_{11}$ times the first row to the k-th row of B and similarly for the columns, we reduce all elements except b_{11} of the first row and first column to 0.

Consider b_{22}. If $b_{22} = 0$, we operate on B in a manner analogous to the operations described on A to obtain a matrix whose second principal diagonal element is not zero. These operations leave b_{11} unchanged. Then we have a matrix whose first two diagonal elements are not zero, but all other elements in the first and second rows and columns are zero. Continuing this process until a diagonal form is obtained, we have a matrix E of rank r. Hence its first r diagonal elements are not zero, but the remaining diagonal elements are zero.

Next we multiply E on the right and on the left by a diagonal matrix D whose first r diagonal elements are determined as follows: If e_{ii} is positive, then $d_{ii} = 1/e_{ii}^{\frac{1}{2}}$. If e_{ii} is negative, then $d_{ii} = 1/-e_{ii}^{\frac{1}{2}}$. The remaining n - r diagonal elements of D are one.

The resulting matrix is the desired diagonal form.

Corollary 2.4.1. Every square complex symmetric matrix A of order n and of rank r is congruent to a diagonal matrix whose first r diagonal elements are one and the remaining elements are zero.

If p is the number of ones in the diagonal form of Theorem 2.4.1 and r is its rank, then the number $2p - r = s$ is called the <u>signature</u> of A. An n by n symmetric matrix A is called <u>positive definite</u> if $r = s = n$, <u>negative definite</u> if $r = -s = n$. It is <u>semi-definite</u> if $r = s$ or $r = -s$.[1]

If we have several sets of variables $(x_1, x_2, \ldots)$, $(y_1, y_2, \ldots)$ and $(z_1, z_2, \ldots)$ and agree that whenever one of these sets is subjected to a transformation every other set shall be subjected to the same transformation, then we say that we have sets of <u>cogredient variables</u>.[2] Such a transformation is a <u>cogredient transformation</u>.

Theorem 2.4.2. Two symmetric bilinear forms are equivalent under non-singular cogredient transformations if and only if their matrices are congruent.[3]

Proof. Consider the cogredient transformation

$$(2.4.1)\qquad x_i = \sum_{j=1}^{n} b_{ij}X_j \qquad (i = 1, 2, \ldots, n),$$

$$y_i = \sum_{j=1}^{n} b_{ij}Y_j \qquad (i = 1, 2, \ldots, n),$$

with non-singular matrix $B = (b_{ij})$.

We may write the symmetric bilinear form

$$(2.4.2)\qquad A(x,y) = \sum_{i,j=1}^{n} a_{ij}x_i y_j$$

as

$$(2.4.3)\qquad A(x,y) = \sum_{j=1}^{n} c_j y_j, \quad c_j = \sum_{i=1}^{n} a_{ij}x_i \quad (j = 1, \ldots, n).$$

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1 MacDuffee, <u>The Theory of Matrices</u>, p. 57.
2 Bôcher, p. 90.
3 Dickson, p. 65.

The matrix of the linear forms $c_1, c_2, \ldots, c_n$ is the transpose A' of A. Since A is symmetric, $A' = A$. Applying Theorem 2.2.1, the transformation (2.4.1) with B matrix replaces $c_1, c_2, \ldots, c_n$ by a set of linear functions whose coefficient matrix is $A'B$. Hence this transformation replaces the bilinear form $A(x,y)$ with a bilinear form with the matrix $(A'B)'$. By Theorem 1.2.3, $(A'B)' = B'A$.

Let

$$(2.4.4) \qquad B'A = D.$$

Then we have the bilinear form

$$(2.4.5) \qquad D(x,y) = \sum_{i,j=1}^{n} d_{ij}x_i y_j$$

which may be written

$$(2.4.6) \qquad D(x,y) = \sum_{i=1}^{n} x_i l_i, \qquad l_i = \sum_{j=1}^{n} d_{ij}y_j \quad (i = 1, \ldots, n).$$

By Theorem 2.2.1, the linear transformation (2.4.1) with matrix B replaces l_i with a linear form with matrix DB. Hence the cogredient transformation defined by equations (2.4.1) replaces a symmetric bilinear form with matrix A with a symmetric bilinear form with matrix $A_1 = B'AB$.

By Theorem 1.7.2, A_1 and A have the same rank, hence are equivalent. From the definitions it follows that if the two matrices A_1 and A are congruent, they are equivalent.

From the definition of a quadratic form it follows that

Theorem 2.4.3. Two quadratic forms are equivalent under a non-singular transformation if and only if their matrices are congruent.[1]

Corollary 2.4.3. Two symmetric bilinear forms with matrices A and A_1 are equivalent under cogredient transformation if and only if the quadratic froms with matrices A and A_1 are equivalent.

2.5. M-affine congruence.[2] Consider the quadratic form

$$F = \sum_{i,j=1}^{n} a_{ij} x_i x_j \tag{2.5.1}$$

whose matrix is $A = (a_{ij})$. If the x's are subjected to the transformation

$$x_i = x_i \qquad (i = 1, 2, \ldots, m),$$
$$x_i = \sum_{j=1}^{n} b_{ij} x_j \qquad (i = m+1, \ldots, n)$$

with matrix

$$T = \begin{bmatrix} 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 1 & 0 & \cdots & 0 \\ b_{m+1,1} & \cdots & b_{m+1,m} & b_{m+1,m+1} & \cdots & b_{m+1,n} \\ \vdots & & \vdots & \vdots & & \\ b_{n1} & \cdots & b_{nm} & b_{n,m+1} & \cdots & b_{nn} \end{bmatrix} \tag{2.5.2}$$

Then

$$T'AT = \bar{A} \tag{2.5.3}$$

is the matrix of the transformed quadratic form.

Two matrices A and B are said to be m-affine congruent if and only if there exists a non-singular matrix T of the

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1 Dickson, p. 65.

2 R. S. Burington, "On the Equivalence of Quadrics in m-affine n-space and its relation to the equivalence of 2m-pole networks", Transactions of the American Mathematics Society, 1935, XXXVIII, pp. 163-176.

form (2.5.2) such that $A = T'BT$.

If we have a system of polynomials in the set of variables $(x_1, x_2, \ldots)$ and a set of transformations of these variables, then any function of the coefficients is called an <u>invariant</u>(or absolute invariant) with regard to these transformations if it is unchanged when the polynomials are subjected to the transformation of the set.[1]

A rational function of the coefficients of a form or system of forms which, when these forms are subjected to any non-singular linear transformation, is merely multiplied by the j-th power of the determinant of the transformation is called a <u>relative invariant</u> of weight j of the form or system.[2]

If $\bar{A}_{r_1 \ldots r_s}$ is $\bar{A}$ with the $r_1, \ldots, r_s$ rows and columns deleted, the r_i's being all distinct and such that $r_i \leq m$ for all i, then

(2.5.4) $\quad \bar{A}_{r_1 \ldots r_s} = T'_{r_1 \ldots r_s} A_{r_1 \ldots r_s} T_{r_1 \ldots r_s}.$

Thus $A_{r_1 \ldots r_s}$ is an invariant matrix of A under T in the sense that $\bar{A}_{r_1 \ldots r_s}$ can be formed either by transforming and then deleting the rows and columns, or by deleting the rows and columns of A and T and then transforming.

Let the ranks of $A_{r_1 \ldots r_s}$ be denoted by $\rho_{r_1 \ldots r_s}$.

Theorem 2.5.1. The $\rho, \rho_1, \ldots, \rho_{r_1 \ldots r_s}$ are integer invariants of $A, A_1, \ldots, A_{r_1 \ldots r_s}$, respectively and hence of

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1 Bôcher, p. 89.
2 <u>Ibid</u>., p. 96.

matrix A.

Proof. This follows from Theorems 2.1.2 and 2.4.3, equation (2.5.4) and the definition of m-affine congruence.

By taking determinants of equation (2.5.4), it follows that

$$(2.5.5) \qquad d(\bar{A}_{r_1 \ldots r_s}) = (d(T_{r_1 \ldots r_s}))^2 d(A_{r_1 \ldots r_s}).$$

Then

Theorem 2.5.2. The $d(A)$, $d(A_1)$, ... , $d(A_{r_1 \ldots r_s})$ are relative invariants of A under T.

We denote by

$$A^{s_1 \ldots s_t}_{r_1 \ldots r_t}$$

the submatrix of A obtained from A by striking out the rows numbered r_1, r_2, ..., r_t and all the columns numbered s_1, s_2, ..., s_t. It follows from the definitions that, for $r_i \leq m$, $s_i \leq m$, $(i = 1, 2, \ldots, t)$,

Theorem 2.5.3.

$$(2.5.6) \qquad \bar{A}^{s_1 \ldots s_t}_{r_1 \ldots r_t} = T'_{r_1 \ldots r_t} A^{s_1 \ldots s_t}_{r_1 \ldots r_t} T_{s_1 \ldots s_t}$$

is an invariant matrix, and $d(A^{s_1 \ldots s_t}_{r_1 \ldots r_t})$ is a relative invariant of A under T.

Since $d(T_{r_1 \ldots r_t}) = d(T_{s_1 \ldots s_t}) = d(T)$, for $r_i = m$, $s_i = m$, $(i = 1, 2, \ldots, t)$,

Theorem 2.5.4. If R_1 and R_2 are any two of the above relative invariants, then

$$(2.5.7) \qquad I_{1,2} = R_1/R_2$$

is an absolute invariant of A under T.

2.6. R-matrices.[1] A real square matrix $B = (b_{ij})$ of order n is said to be an R-matrix if and only if $\sum_{i=1}^{n} b_{ij} \leq 2b_{jj}$, for each $j = 1, 2, \ldots, n$.

If $(x) = (x_1, x_2, \ldots, x_n)$ is an one by n matrix, then so is $(x)B$, where B is an n by n matrix. Let γ_i be an operator such that

$$\gamma_i b_{ij} = -\left|b_{ij}\right| \qquad i \neq j,$$
$$\gamma_i b_{jj} = b_{jj}.$$

Letting $x_i = \gamma_i$ and $(x) = (\gamma)$, the statement $(\gamma)B = 0$ will be used to mean that each element of this one by n matrix is greater than or equal 0.

Theorem 2.6.1. A necessary and sufficient condition that a real matrix B be an R-matrix is $(\gamma)B \geq 0$.

Proof. From the definition, we see that if B is an R-matrix then $(\gamma)B \geq 0$.

Each element of $(\gamma)B$ is less than or equal to each element of $(b_{11} - b_{21} - \ldots - b_{n1}, \ldots, -b_{1k} - b_{2k} - \ldots + b_{kk} - b_{k+1,k} - \ldots - b_{nk}, \ldots, - b_{1n} - \ldots + b_{nn})$. Then if $(\gamma)B \geq 0$

$$0 \leq b_{jj} - \sum_{\substack{i \neq j \\ i=1}}^{n} b_{ij} \tag{2.6.1}$$

or

$$b_{jj} \geq \sum_{\substack{i \neq j \\ i=1}}^{n} b_{ij}.$$

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1 R. S. Burington, "R-Matrices and Equivalent Networks", *Journal of Mathematics and Physics*, M.I.T., 1937, XVI, pp. 85-102.

Adding b_{jj} to both sides

$$2b_{jj} \geq \sum_{i=1}^{n} b_{ij},$$

which is the definition of an R-matrix.

Consider the set of all possible distinct one by n matrices $(x_q) = (x_{q1}, x_{q2}, \ldots, x_{qn})$, in which the x_{qi} are one or minus one, $i = 1, 2, \ldots, n$. There are 2^n such matrices, $(x_1), (x_2), \ldots, (x_{2^n})$.

Let $X_q = (x_q)A$, $(q = 1, 2, \ldots, 2^n)$, be the set of functions generated from the real matrix A and the matrices (x_q). Denote by $(x_q^{(k)}) = (x_{q1}, \ldots, x_{q,k-1}, 1, x_{q,k+1}, \ldots, x_{qn})$, $(k = 1, \ldots, n)$, i.e., $x_{qk} = 1$. Since there are 2^{n-1} such matrices, $q = 1, 2, \ldots, 2^{n-1}$. Then let $X_q^{(k)} = (x_q^{(k)})A$, $(k = 1, 2, \ldots, n)$.

The k-th element of each $X_q^{(k)}$ is $\sum_{i=1}^{n} x_{qi}a_{ik}$ and

(2.6.2) $$\sum_{i=1}^{n} x_{qi}a_{ik} \geq \sum_{i=1}^{n} {}_ia_{ik}, \quad (k = 1, 2, \ldots, n;\ q = 1, 2, \ldots, 2^{n-1}).$$

Since all possible signs will be assigned to a_{ik} in (2.6.2), equality will hold once for each k. Then if

(2.6.3) $$\sum_{i=1}^{n} x_{qi}a_{ik} \geq 0$$

for all q and k, $(\gamma)A = 0$. Therefore, it follows from Theorem (2.6.1) that

Theorem 2.6.2. A necessary and sufficient condition that $A = T'BT$, T non-singular, be an R-matrix is

$$\sum_{i=1}^{n} x_{qi}a_{ik} \geq 0, \quad (k=1, 2, \ldots, n;\ q = 1, 2, \ldots, 2^{n-1}).$$

Let γ be an n^2 space consisting of the totality of real points $(t_{11}, \ldots, t_{n1}, \ldots, t_{1n}, \ldots, t_{nn})$, that is, the space of the matrix T of Theorem 2.6.2. In general, from the nature of the product T'BT, each member of X_q is a quadric in 2n variables imbedded in $\mathcal{T}$.

Let $\gamma^{(k)} = \sum_{i=1}^{n} x_{qi} a_{ik}$, $(k = 1, 2, \ldots, n;\ q = 1, 2, \ldots, 2^{n-1})$.

The various surfaces $\gamma^{(k)} = 0$, $q = 1, \ldots, 2^{n-1}$, k fixed, divide $\mathcal{T}$ into regions $\pi^{(k)}$ and $\eta^{(k)}$, where $\pi^{(k)}$ contains all points of $\mathcal{T}$ such that $\gamma^{(k)} \geq 0$, $q = 1, 2, \ldots, 2^{n-1}$, and $\eta^{(k)}$ contains all other points of $\mathcal{T}$ not in $\pi^{(k)}$.

Let Π denote that portion of $\mathcal{T}$ composed of all points common to $\pi^{(1)}, \pi^{(2)}, \ldots, \pi^{(n)}$, and no other points. Delete from Π all points for which T is singular. Denote this region by W. Then

Theorem 2.6.3. A necessary and sufficient condition that the matrix A = T'BT be an R-matrix, where B is real and T is real and non-singular, is that the elements of T belong to the region W in $\mathcal{T}$.

If we restrict the space $\mathcal{T}$ to be an n by n-m space, then Theorem 2.6.3 holds for m-affine congruence as well as congruence.

From Theorem 2.4.1, we know that any real square symmetric positive semi-definite matrix B of order n and rank r is congruent to a diagonal matrix whose diagonal consists of r ones in the first r places and n - r zeros in the remaining places. Let J be such a diagonal matrix. Evidently J is an R-matrix. Consider the matrix A congruent to J, $A = T_2'JT_2$, where

$$T_2 = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & t & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 0 & 1 \end{bmatrix}.$$

In other words, T_2 is a unit matrix which has been altered so that the element in the (r,r) position is t. Then A is a diagonal matrix whose diagonal consists of ones in the first $r - 1$ positions, t^2 in the r-th position and the remainder of the positions occupied by zeros. Obviously A is an R-matrix for all real values of t. If $J = T_1'BT_1$, then $A = T_2'T_1'BT_1T_2$. Hence there is a region W containing an infinite number of points for which A is an R-matrix.

Theorem 2.6.4. If B is a real square symmetric matrix, there exists an infinite number of points in $\mathcal{T}$ for which $A = T'BT$ is an R-matrix. B need not be an R-matrix.

If W_1 is the region of $\mathcal{T}$ for which $A_1 = T'B_1T$, and W_2 is the region of $\mathcal{T}$ for which $A_2 = T'B_2T$, where A_1 and A_2 are R-matrices, then if W_{12} is the intersection of W_1 and W_2, we have

Theorem 2.6.5. If B_1 and B_2 are real square symmetric positive semi-definite matrices, there exists a region W_{12} in $\mathcal{T}$ for which $A_1 = T'B_1T$, $A_2 = T'B_2T$ are both R-matrices. The region W_{12} contains infinitely many points.

In a similar manner Theorem 2.6.4 can be extended to any finite number of matrices.

2.7. λ-matrices and the characteristic equation. The matrix $A - \lambda I = f(\lambda)$ is called the characteristic matrix of A; it may be obtained by subtracting λ from each element in the principal diagonal of A.[1] The determinant $d(A - \lambda I)$ is the characteristic determinant of A and is a polynomial of degree n.

(2.7.1) $d(f(\lambda)) = a_n\lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_1\lambda + a_0,\ a_n = (-1)^n.$

The equation $d(f(\lambda)) = 0$ is called the characteristic equation of A. The roots of the characteristic equation of A are the characteristic roots, latent roots or eigen-values of A.

Theorem 2.7.1.(Hamilton-Cayley) Any square matrix satisfies its characteristic equation.

Proof. Let (2.7.1) be the characteristic determinant of A. Since the elements of $A - \lambda I$ are linear functions of λ, and the elements of its adjoint C are $(n - 1)$-rowed determinants, they are polynomials in λ of degree less than or equal to $n - 1$. If the element in the i-th row and j-th column of C is $\sum_{k=0}^{n-1} c_{ijk}\lambda^k$, then

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1 Dickson, p. 65.

$$(2.7.2) \qquad C = \sum_{k=1}^{n-1} C_k \lambda^k, \; C_k = (c_{ijk}) \quad (i, j = 1, 2, \ldots, n).$$

From Theorem 1.6.1, we have

$$(2.7.3) \qquad (A - \lambda I)C = d(f(\lambda))I.$$

Therefore,

$$(2.7.4) \qquad A\sum_{k=0}^{n-1} C_k \lambda^k - \sum_{k=0}^{n-1} C_k \lambda^{k+1} = \sum_{k=0}^{n} a_k \lambda^k I.$$

Equating like coefficients in equation (2.7.4), we get

$$\begin{aligned} AC_0 \quad\quad &= a_0 I, \\ AC_1 - C_0 &= a_1 I, \\ AC_2 - C_1 &= a_2 I, \\ \ldots\ldots&\ldots\ldots \\ AC_{n-1} - C_{n-2} &= a_{n-1} I, \\ - C_{n-1} &= a_n I. \end{aligned}$$

Multiplying these equations on the left by I, A, A^2, $\ldots$, A^{n-1}, A^n respectively and adding, the result is

$$0 = a_0 I + a_1 A + a_2 A^2 + \ldots + a_{n-1} A^{n-1} + a_n A^n = d(f(A)).$$

Suppose the elements of a matrix A are functions of a variable, say t. Then if t receives an increment Δt, the elements of A receive a corresponding increment. The matrix of increments assigned to the elements of A may be denoted by ΔA. Then

$$\frac{dA}{dt} = D^{(1)}A = \lim_{\Delta t \to 0} \frac{\Delta A}{\Delta t}.$$ [1]

Theorem 2.7.2. If λ_s is a simple root of the characteristic equation of the n by n matrix $f(\lambda)$, then $f(\lambda_s)$ is of

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1 Frazer, Duncan, and Collar, p. 43.

rank n - 1.[1]

Proof. Let $d(f(\lambda)) = D(\lambda)$. $D(\lambda_s) = 0$; therefore, by definition $f(\lambda_s)$ has rank less than n. If λ_s is a simple root of $D(\lambda)$, then $D^{(1)}(\lambda_s) \neq 0$, where $D^{(1)}(\lambda_s)$ is the first derivative of $D(\lambda)$ with respect to λ evaluated at $\lambda = \lambda_s$. Since $D^{(1)}(\lambda)$ is a linear homogeneous function of the first minors of $f(\lambda)$, it then follows that $f(\lambda_s)$ has at least one n - 1 by n - 1 minor whose determinant is not zero. Hence $f(\lambda_s)$ is of rank n - 1.

Theorem 2.7.3. If $F(\lambda)$ is the adjoint of $f(\lambda)$, and $f(\lambda_s)$ is of rank n - 1, then $F(\lambda_s)$ is of rank one and

$$F(\lambda_s) = k_s h_s,$$

where the elements of the n by one matrix k_s and one by n matrix h_s are appropriate to the root λ_s.[2]

Proof. $f(\lambda_s)$ is of rank n - 1; therefore, $F(\lambda_s)$ cannot be a null matrix. From Theorem 1.6.1, $f(\lambda_s)F(\lambda_s) = D(\lambda_s)I$. Since $f(\lambda_s)$ is of rank n - 1, this leads to $f(\lambda_s)F(\lambda_s) = 0$. The p-th column of this product can be written as

$$(2.7.5) \qquad f(\lambda_s)\begin{bmatrix} F_{1p} \\ F_{2p} \\ \cdots \\ F_{np} \end{bmatrix} = 0.$$

Applying Corollary 2.2.2 and equation (2.2.13') to equation (2.7.5), we can arrive at an expression of every other row

1 Frazer, Duncan, and Collar, p. 61.
2 *Ibid.*, p. 61.

of $F(\lambda_s)$ as a multiple of some given row of $F(\lambda_s)$. Hence $F(\lambda_s)$ must be of rank 1.

Now if we let h_s be any row of $F(\lambda_s)$, we have shown that we can express any other row of $F(\lambda_s)$ as a multiple of h_s. Hence

(2.7.6) $$F(\lambda_s) = k_s h_s.$$

Theorem 2.7.4. If the latent roots of an n by n matrix A are all distinct, then

$$A = K \Lambda K^{-1},$$

where K is a matrix whose s-th column is k_s and Λ is a diagonal matrix whose j-th diagonal element is λ_j.[1]

Proof. Consider the linear transformation $Y = AX$ which is to be satisfied by $Y = \lambda X$, where λ is a scalar factor of proportionality. This then leads to

(2.7.7) $$(\lambda I - A)X = f(\lambda)X = 0.$$

Hence the roots λ are the characteristic roots of A.

We have seen that

(2.7.8) $$f(\lambda_s)F(\lambda_s) = F(\lambda_s)f(\lambda_s) = 0.$$

Then from the preceding theorem,

(2.7.9) $$f(\lambda_s)k_s h_s = k_s h_s f(\lambda_s) = 0.$$

If λ_s is a simple root, $F(\lambda_s)$ is of rank one, and at least one of the elements of h_s is not zero. Then (2.7.9) requires that

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1 Frazer, Duncan, and Collar, p. 66.

(2.7.10) $$f(\lambda_s)k_s = 0.$$

Comparing (2.7.10) and (2.7.7), $X(\lambda_s)$ can be taken proportional to any non-zero column of $F(\lambda_s)$. Now when all of the characteristic roots of A are distinct, there is a column k_s corresponding to each root. The columns are known as <u>modal columns</u>. Then

(2.7.11) $$Ak_1 = \lambda_1 k_1, \; Ak_2 = \lambda_2 k_2, \; \ldots, \; Ak_n = \lambda_n k_n.$$

This is equivalent to

(2.7.12) $$AK = K\Lambda,$$

where K is a matrix whose j-th column is the j-th modal column, and Λ is the diagonal matrix whose j-th diagonal element is λ_j. K is called the <u>modal matrix</u> of A.

Suppose K is singular of rank r. Then at least one column, say the j-th, of K is expressible as a linear combination of r others. Then

(2.7.13) $$c_1k_1 + c_2k_2 + \ldots + c_rk_r = k_j.$$

Substituting this value for k_j in the j-th equation of (2.7.11) leads to

(2.7.14) $$c_1Ak_1 + c_2Ak_2 + \ldots + c_rAk_r = c_1\lambda_j k_1 + \ldots + c_r\lambda_j k_r,$$

and, since $Ak_i = \lambda_i k_i$, ($i = 1, 2, \ldots, n$),

(2.7.15) $$c_1\lambda_1k_1 + c_2\lambda_2k_2 + \ldots + c_r\lambda_rk_r = c_1\lambda_jk_1 + \ldots + c_r\lambda_jk_r.$$

Hence $\lambda_i = \lambda_j$, ($i = 1, 2, \ldots, r$), which contradicts λ_j being distinct. Therefore, K is non-singular.

It then follows from (2.7.12) that $A = K\Lambda K^{-1}$.

PART II

MATRICES AND ELECTRIC NETWORKS

CHAPTER III

NETWORK DIFFERENTIAL EQUATIONS AND EQUIVALENT NETWORKS

3.1. Definitions and basic laws. In order to present ideas concisely, it is often advisable to use words which have a particular meaning when applied to the topic under consideration. Network topology and network analysis make use of many such terms. We will not attempt to define a full "network" vocabulary[1], however, it is felt that certain terms should be explained.

A branch is one or several passive elements such as inductance, L, resistance, R, and elastance, D, connected in series between two terminals. Sometimes branch is used more generally to denote any system of elements, active or passive, connected to the remainder of the network by way of two terminals. The word node is used to denote a terminal. A mesh is a closed contour arbitrarily drawn on a network diagram. Node pair means two nodes arbitrarily chosen from one network.

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1 For a more complete set of definitions see "Standards on Circuits: Definition of Terms in Network Topology, 1950". Proceedings of the Institute of Radio Engineers, XXXIX, 27-29; also, Ernst A Guillemin, Communication Networks(New York, 1935), Vol. I, Chapt. IV.

Two statements which are usually given as axioms in network analysis are Kirchhoff's laws. Kirchhoff's first law states that at any point in a circuit there is as much current flowing away from the point as there is current flowing into it, or the algebraic sum of the currents at any one point in a circuit is zero. The second law of Kirchhoff states that the sum of the products of the current by the resistance taken around any closed path in a network of conductors is equal to the sum of the electromotive forces[1] which one passes in going around the closed circuit. The latter of these two laws is usually generalized by changing the word resistance to impedance. Instead of using these two laws as axioms, we will postulate the existence of circuital currents and the following equation and develop Kirchhoff's laws.

(3.1.1) $\underline{E}_k + V_k - V_{-k} = \underline{L}_k \frac{d\underline{I}}{dt} + \underline{R}_k\underline{I} + \underline{D}_k\int_0^t \underline{I}\, dt, \quad (k = 1, \ldots, B),$[2]

where V_k is the force of constraint associated with the source node for the current $\underline{I}_k$, V_{-k} is the force of constraint associated with the sink node for $\underline{I}_k$, $\underline{E}_k$ is the external e.m.f. acting in the k-th branch and B is the number of branches of the network. $\underline{L}_k$, $\underline{R}_k$, and $\underline{D}_k$ are one by B matrices whose k-th elements are the total inductance, resistance and elastance in the k-th branch respectively and whose j-th elements are

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1 Electromotive force will be abbreviated hereafter by e.m.f.
2 W. H. Ingram and C. M. Cramlet, "Foundations of Electrical Network Theory", Journal of Mathematics and Physics, M. I. T., 1944, XXIII, p. 141.

the inductance, resistance and elastance due to coupling, symmetric or non-symmetric, between the k-th and the j-th branches of the network. I is a B by one matrix consisting of the currents in the B different branches.

To facilitate matrix analysis of networks we will make use of two relationships which have been developed from topological considerations.[1] The first of these states that the number of independent node pairs in a network, and the number of independent node equations obtainable by application of Kirchhoff's first law to a network, is N - S, where N is the number of nodes and S is the number of subnetworks in the network. The second relationship is M = B - N + S, where M is the number of independent meshes and B is the number of branches in the network. The latter relationship apparently originated with Kirchhoff.[2]

3.2. Developement of the mesh and node pair equations. Consider a network consisting of B branches, N nodes and S separable parts. In the present discussion the values of e.-m.f. and current will refer to instantaneous values. Instantaneous values are used in order to be able to refer to positive and negative voltages and directed currents. Since we

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1 Ingram and Cramlet, pp. 135-140.
2 G. Kirchhoff, "Über die Auflösung der Gleichungen, auf welche man bei der Untersuchung der linearen Vertheilung galvanisher Ströme geführt wird.", Annalen der Physik und Chemie, von J. C. Poggendorff, 1847, LXXII, Series 2, p. 497.

will assume that all of the generators in our network are of the same frequency, this does not detract from the generality of the results. Associated with each branch k of the network are a set of active or passive elements, a directed current $\underline{I}_k$ and a generated e.m.f. $\underline{E}_k$. When a particular branch contains no external generator then its e.m.f. $\underline{E}_k = 0$. According to Thevenin's Theorem, a linear generator can be represented by a d.c. or a.c. source of constant amplitude, E_o, and no internal resistance in series with a resistance $R_g = E_o/I_o$, where I_o is the short circuit current of the generator and E_o is its open circuit e.m.f. Using this theorem $\underline{E}_k$ will be the d.c. or a.c. source associated with the branch k and the R_g will be included in the $\underline{R}_k$ component of the passive elements of the branch.

From the network diagram we may select M independent meshes. With each such mesh there is associated a circuital current. The direction in which these currents circulate may be arbitrarily chosen; however, it is customary to assume a clockwise direction of flow. The branch k may be common to one or more of the M meshes. Since the value of the current in the branch k is $\underline{I}_k$, $\underline{I}_k$ must be the algebraic sum of all the mesh currents flowing through branch k. Expressing this statement in matrix form we have

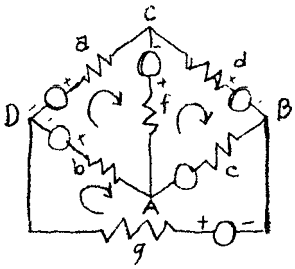

Figure 1. A network diagram with B = 6, N = 4. The meshes are indicated.

(3.2.1) $\underline{I} = PI$,

where $\underline{I}$ is a B by one matrix consisting of B branch currents, P is a B by M matrix whose elements are 0, 1, or -1, and I is a M by one matrix made up of the M mesh currents. The element of P in the (j,k)-th position is 0 if the branch j is not in mesh k, 1 if $\underline{I}_j$ is directed in the same direction as I_k and -1 if $\underline{I}_j$ is directed in the opposite direction to I_k.

The <u>incidence</u> <u>matrix</u>, Q, of a network diagram, and for any directed quantities associated one-to-one with the branches of the diagram, is an N by B matrix in which the element in the j-th row and k-th column is 1 if the k-th directed quantity leaves the j-th node, -1 if the k-th quantity enters the j-th node and zero if the k-th quantity avoids the j-th node.[1]

Theorem 3.2.1. The N by M matrix QP is one all of whose elements are zero.

Proof. By definition of Q, $Q\underline{I}$ is a N by one matrix each of whose elements is the algebraic sum of the branch currents leaving the node corresponding to the position of the element in the column of $Q\underline{I}$. Then it follows from equation (3.2.1) that QPI is a N by one matrix each of whose elements is the algebraic sum of the circuital currents leaving the node in question. Therefore, QP is an incidence matrix for the mesh currents. Now the incidence of any mesh current at

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1 Ingram and Cramlet, p. 138.

a node is zero if the mesh avoids the node. If the mesh does not avoid the node, the current both enters and leaves the node, and the incidence is zero. Hence each element of QP is zero.

Theorem 3.2.2. The existence of circuital currents implies Kirchhoff's first law.

Proof. Since QP = 0, from equation (3.2.1), it follows that

(3.2.2) $$Q\underline{I} = 0.$$

But from the definition of Q and $\underline{I}$, the j-th element of the product $Q\underline{I}$ is the algebraic sum of the currents leaving the j-th node of our network. Kirchhoff's first law then follows from equation (3.2.2).

The preceding theorem is the converse of Theorem 4 of Ingram.[1]

Theorem 3.2.3. If V is the N by one matrix whose elements are the forces of constraint associated with the nodes of the network, then Q'V is a B by one matrix whose j-th element is the algebraic sum of the forces of constraint on the current in the j-th branch of the network.

Proof. Q is an N by B matrix whose j-th column consists of one 1, one -1, and zeros. The 1 is in the k-th row of Q, which means that the k-th node of the network is the source

1 Ingram and Cramlet, p. 137.

node of $\underline{I}_j$. The -1 is in the m-th row of Q, showing that the m-th node of the network is the sink node for $\underline{I}_j$. Then the result of multiplying the j-th row of Q' by the matrix V would be an element consisting of the net forces of constraint on the j-th branch of the network.

Now we can write equation (3.1.1) as

$$(3.2.3) \qquad \underline{E} + Q'V = \underline{L}\frac{d\underline{I}}{dt} + \underline{R}\underline{I} + \underline{D}\int_0^t \underline{I}dt,$$

where $\underline{E}$ and $\underline{I}$ are B by one matrices whose elements are the branch e.m.f.'s and branch currents respectively, and $\underline{L}$, $\underline{R}$, and $\underline{D}$ are square matrices whose k-th rows are $\underline{L}_k$, $\underline{R}_k$, and $\underline{D}_k$ from (3.1.1). Since $P'Q' = (QP)'$ is a matrix of zeros, if we multiply (3.2.3) on the left by P' the resulting equation is

$$(3.2.4) \qquad P'\underline{E} = P'\underline{L}\frac{d\underline{I}}{dt} + P'\underline{R}\underline{I} + P'\underline{D}\int_0^t \underline{I}dt,$$

which is free of the forces of constraint. Equation (3.2.4) expresses mathematically

Theorem 3.2.4.(Kirchhoff's second law) The algebraic sum of the counter e.m.f.'s around a closed circuit in a network of conductors is equal to the algebraic sum of the impressed electromotive forces which one passes in going around the closed circuit.

Proof. From the definition of P, the product of the j-th row of P' and E will be the sum of the generated e.m.f.'s in the j-th mesh. All that remains to be shown is that the right side of (3.2.4) is the sum of the counter e.m.f.'s around the various meshes.

Substituting the expression for $\underline{I}$ from equation (3.2.1) into (3.2.4), we have

(3.2.5) $$P'\underline{E} = P'\underline{L}P\frac{dI}{dt} + P'\underline{R}PI + P'\underline{D}P\int_0^t I\,dt.$$

Consider the three quadratic forms

$$F = \underline{I}'\underline{R}\underline{I}/2,$$

(3.2.6) $$T = \underline{I}'\underline{L}\underline{I}/2$$

$$V = \underline{Q}'\underline{D}\underline{Q}/2, \text{ with } \frac{d\underline{Q}}{dt} = \underline{I},$$

which represent half the total instantaneous power loss, the total instantaneous magnetic energy, and total instantaneous electrostatic energy in the network. We have seen in the proof of Theorem 2.4.2 that a transformation such as (3.2.1) will replace $\underline{R}$ in the above quadratic forms by $P'\underline{R}P$, $\underline{L}$ by $P'\underline{L}P$, and $\underline{D}$ by $P'\underline{D}P$. Hence the matrices $P'\underline{R}P = R$, $P'\underline{L}P = L$, and $P'\underline{D}P = D$ must have the same relationship to the mesh currents as the matrices $\underline{R}$, $\underline{L}$, and $\underline{D}$ have to the branch currents. Then R, L, and D will be M by M matrices whose (j,k)-th elements are the total resistance, inductance, and elastance respectively of mesh j if j = k, and the resistance, inductance, and elastance due to coupling between mesh j and k if $j \neq k$. Therefore, the j-th row of the matrix on the right side of equation (3.2.4) or (3.2.5) is the sum of the counter e.m.f.'s in mesh j.

The equation (3.2.5) is the same equation that Kirchhoff expressed in 1847 without the use of matrices.[1]

In the second method of matric analysis of the electrical network to be presented, the role of the e.m.f and current will be interchanged. According to Norton's Theorem a linear generator can be represented by a source of constant direct or alternating current I_o and no internal conductance in parallel with a conductance $G_g = I_o/E_o$.[2] Representing all of the generators in this manner, we will assume the generated current, the capacitance, conductance and the reciprocal inductance of the branch to be known with the e.m.f. of the node pairs associated with the network to be determined.[3]

As stated in Section 3.1, there are N - S independent node pairs. The e.m.f.'s associated with the remaining node pairs can be written as combinations of sums and differences of the e.m.f.'s associated with the N - S independent node pairs. The resulting matrix equation is

(3.2.7) $$\underline{E}^* = UE^*,$$

where $\underline{E}^*$ and E^* are matrices whose elements are the B node pair e.m.f.'s and the N - S independent node pair e.m.f.'s respectively, and U is a B by N - S matrix whose elements are 0, 1, or -1. The element in the j-th row and k-th column of

1 G. Kirchhoff, p. 500.
2 P. Le Corbeiller, Matrix Analysis of Electric Networks (New York, 1950), p. 63.
3 This method of analysis exemplifies, in part, the <u>principle of duality</u>. For a complete explanation of this principle see Ernst A. Guillemin, Vol. II, pp. 246-252.

U is 0 if the k-th element of E^* is not a member of the independent e.m.f.'s whose combination is the j-th element of E^*, 1 if the k-th element of E^* has a plus sign affixed to it in the combination which is equal to E^*_j, and -1 if the sign of E^*_k is minus in the combination.

The circuital matrix, W, of a network and for any directed quantities associated one to one with the branches of the network diagram is a matrix such that the element in the j-th row and k-th column is 1 if the k-th quantity is directed in the same direction as the j-th circuit, -1 if the k-th quantity is directed in a direction opposite to the direction of the j-th circuit, and zero if the k-th quantity is not in the j-th circuit.

Theorem 3.2.5. The M by N - S matrix WU is a matrix all of whose elements are zeros.

Proof. We will adopt the convention that an e.m.f. is a quantity which is directed from positive to negative.

From the definition of W it follows that WE^* is an M by one matrix each of whose elements is the sum of the e.m.f.'s around a closed circuit. Then from equation (3.2.7), it follows that WUE^* is an M by one matrix each of whose elements is the sum of the independent e.m.f.'s around a closed circuit. Then WU is a circuital matrix for the independent node pair e.m.f.'s. By Theorem 3.2.4, or Kirchhoff's second law, the algebraic sum of the node pair e.m.f.'s around each circuit is zero. Hence,

(3.2.8) $$WUE^* = 0.$$

W and U are determined by the manner in which the network is connected, while E^* is determined by the circuit elements. Then by altering the circuit elements we may assign arbitrary values to E^* without altering W and U. Hence, $WU = 0$.

Consider the node pair equation

(3.2.9) $$\underline{C}_k \frac{d\underline{E}^*}{dt} + \underline{G}_k \underline{E}^* + \underline{\Gamma}_k \int_0^t \underline{E}^* dt = \underline{I}_k^* + \mathcal{I}_k,$$

where $\underline{C}_k$, $\underline{G}_k$, and $\underline{\Gamma}_k$ are one by B matrices whose k-th elements are the total capacitance, conductance, and reciprocal inductance in the k-th node pair and whose j-th elements are the capacitance, conductance, and reciprocal inductance respectively due to coupling, symmetric or non-symmetric, between the k-th and j-th node pairs. $\underline{E}^*$ is a B by one matrix whose k-th element is the k-th node pair e.m.f., $\underline{I}_k^*$ is the current generated by the generator in the k-th node pair and $\mathcal{I}_k$ is the incident current due to the k-th node pair.

Theorem 3.2.6. If I is a columnar matrix whose elements are the mesh currents of the given network with each node pair replaced by a single branch, then $W'I$ is a B by one matrix whose k-th element is the incident current associated with the k-th branch.

Proof. The element of W in the j-th row and k-th column is 0 if $\underline{E}_k^*$ is not in the j-th mesh, 1 if $\underline{E}_k^*$ is polarized so that the j-th mesh current flows from positive $\underline{E}_k^*$ to negative $\underline{E}_k^*$, and -1 if the j-th mesh current flows from negative $\underline{E}_k^*$ to positive $\underline{E}_k^*$. Comparing this with the definition of P, we see that P = W'. It then follows from equation (3.2.1) that

$$\mathcal{I} = W'I, \tag{3.2.10}$$

where $\mathcal{I}$ is a B by one matrix whose B elements are the incident currents associated with the B different node pairs of the network.

Then from equation (3.2.10) and (3.2.9), we have

$$\frac{\underline{C}d\underline{E}^*}{dt} + \underline{G}\underline{E}^* + \underline{\Gamma}\int_0^t \underline{E}^* dt = \underline{I}^* + W'I, \tag{3.2.11}$$

where $\underline{C}$, $\underline{G}$, and $\underline{\Gamma}$ are the capacitance, conductance, and reciprocal inductance matrices whose k-th rows are $\underline{C}_k$, $\underline{G}_k$, and $\underline{\Gamma}_k$ and $\underline{I}^*$ is a B by one matrix consisting of the B generated currents. Multiplying (3.2.11) on the left by U',

$$U'\frac{\underline{C}d\underline{E}^*}{dt} + U'\underline{G}\underline{E}^* + U'\underline{\Gamma}\int_0^t \underline{E}^* dt = U'\underline{I}^*, \tag{3.2.12}$$

for U'W' = (WU)' = 0. In general (3.2.12) has more unknowns than equations, but making use of (3.2.7),

$$U'\underline{C}U\frac{dE^*}{dt} + U'\underline{G}UE^* + U'\underline{\Gamma}U\int_0^t E^* dt = U'\underline{I}^*. \tag{3.2.13}$$

Now consider the quadratic forms in terms of the node pair e.m.f.'s analogous to equation (3.2.6)

$$\begin{aligned} F &= \underline{E}^{*\prime}\underline{G}\underline{E}^*/2, \\ V &= \underline{\varphi}^{*\prime}\underline{C}\underline{\varphi}^*/2, \text{ with } \frac{d\underline{\varphi}^*}{dt} = \underline{E}^*, \\ T &= \underline{E}^*\underline{\Gamma}\underline{E}^*, \end{aligned} \tag{3.2.14}$$

which represent half the total instantaneous power loss, the total instantaneous electrostatic energy, and the total instantaneous magnetic energy in the network. Again referring to the proof of Theorem 2.4.2, we see that the transformation (3.2.7) replaces $\underline{C}$ by $U'\underline{C}U$, $\underline{G}$ by $U'\underline{G}U$, and $\underline{\Gamma}$ by $U'\underline{\Gamma}U$. Then $C = U'\underline{C}U$, $G = U'\underline{G}U$, and $\Gamma = U'\underline{\Gamma}U$, where C, G, and Γ are the matrices of the above quadratic forms for the e.m.f.'s associated with the independent node pairs. Hence we have

(3.2.15) $$C\frac{dE^*}{dt} + GE^* + \Gamma\int_0^t E^*dt = U'I.$$

By applying Kirchhoff's first law to the network we have

(3.2.16) $$C\frac{dE^*}{dt} + GE^* + \Gamma\int_0^t E^*dt = I^*,$$

where the j-th element of I^* is the sum of the generated currents incident on the nodes of the j-th independent node pair. Hence

(3.2.17) $$U'\underline{I}^* = I^*.$$

Le Corbeiller arrives at the relationship (3.2.17) through the consideration of some rather strange manipulations of the node pairs.[1]

3.3. <u>Solution of the network differential equations and the natural modes of the network</u>. Equation (3.2.5) and (3.2.13) are differential-integral equations whose solutions give us the equations of the currents and e.m.f.'s respectively in a given network. The general solution of such a set of equations is the sum of the particular or steady state and the complementary or transient solutions. Since each of these

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1 Le Corbeiller, pp. 68-73.

equations are of the same type, the same method will yield a solution of either of them. Consequently, we will solve only one of the matrix equations, equation (3.2.5).

In the steady state solution of the equation the frequency, ω, of the impressed voltage is known and equation (3.2.5) becomes

$$(3.3.1) \qquad P'\underline{E} = P'\underline{Z}PI, \text{ where } \underline{Z} = \underline{L}\lambda + \underline{R} + \underline{D}\lambda^{-1} \text{ and } \lambda = j\omega \quad (j = \sqrt{-1}).$$

Since, in general, there are M independent mesh currents, P'ZP is non-singular and possesses an inverse.[1] Then the steady state solution of (3.2.5) is

$$(3.3.2) \qquad (P'\underline{Z}P)^{-1}P'\underline{E} = I,$$

or if the branch currents are desired

$$(3.3.3) \qquad P(P'\underline{Z}P)^{-1}P'\underline{E} = \underline{I}.$$

Heretofore we have assumed that all of the generators were of the same frequency. However, the e.m.f.'s generated may be of several different frequencies. Applying the Superposition Theorem[2], there exists a solution of the form (3.3.2) or (3.3.3) for each separate frequency present in the generated e.m.f.'s.

Theorem 3.3.1. If $\underline{Z}_s$ and $\underline{E}_s$ are the branch impedance and e.m.f. matrices corresponding to the steady state angular

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1 The impedance matrix may be singular if the frequency of one of the impressed e.m.f.'s coincides with a natural frequency of the network. This case is handled subsequently.
2 William Littell Everitt, Communication Engineering, (New York, 1937), p. 53.

frequency ω_s, the steady state solution of (3.2.5) is

(3.3.4) $$\underline{I} = \sum_{s=1}^{h} P(P'Z_sP)^{-1}P'E_s, \text{ or } I = \sum_{s=1}^{h} (P'Z_sP)^{-1}P'E_s,$$

in which h is the number of different frequencies being generated.

Le Corbeiller presents a detailed discussion of the various ways of using mixtures of the mesh and node pair methods of analysis of the network.[1]

An interesting application of equation (3.3.1) arises when the matrices of this equation are partitioned in the proper manner. Let $P'\underline{E} = E$, $P'\underline{Z}P = Z$. Now partition the matrices in a conformable manner

(3.3.5) $$\begin{bmatrix} E_1 \\ E_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}.$$

Then writing this as two matrix equations

(3.3.6) $$E_1 = Z_{11}I_1 + Z_{12}I_2,$$
$$E_2 = Z_{21}I_1 + Z_{22}I_2.$$

If Z_{22} is non-singular

(3.3.7) $$Z_{22}^{-1}E = Z_{22}^{-1}Z_{21}I_1 + I_2,$$
$$I_2 = Z_{22}^{-1}E_2 - Z_{22}^{-1}Z_{21}I_1.$$

Substituting this value for I_2 in the first equation of (3.3.6)

$$E_1 = Z_{11}I_1 + Z_{12}(Z_{22}^{-1}E_2 - Z_{22}^{-1}Z_{21}I_1),$$

or

(3.3.8) $$E_1 - Z_{12}Z_{22}^{-1}E_2 = (Z_{11} - Z_{12}Z_{22}^{-1}Z_{21})I1.$$

1 Le Corbeiller, Chap. IV.

From (3.3.8) we can determine the impedances and generated e.m.f.'s necessary to design a new network whose current distribution is the same as I_1. Other useful relationships can be derived in a similar manner by partitioning of (3.3.1) or its dual in the node pair analysis.[1]

If the frequency of one of the impressed e.m.f.'s coincides with the frequency of a transient solution of the network, then from Theorem 2.2.2, we know the impedance matrix is singular for that frequency and equation (3.3.3) gives no solution for that particular frequency. Instead of equation (3.2.5) we will consider the equivalent set of equations

$$(3.3.9) \qquad E = (Lp^2 + Rp + D)Q, \quad dQ/dt = I,$$

where $p = d/dt$, Q and I are M by one matrices whose elements are the charges and currents circulating in the meshes of the network. If the network is passive, the meshes can be chosen so that the coupling from mesh j to mesh k will be the same as the coupling from mesh k to mesh j, and the operational matrix

$$(3.3.10) \qquad f(p) = (Lp^2 + Rp + D)$$

will be symmetric. Let ω_g be the frequency for which the mesh impedance matrix Z is singular. Then assuming $E_g = e_g \exp(j\omega_g t)$, where e_g is a columnar matrix consisting of the amplitudes of

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1 Cyril B. and Georgia B. Reed, Mathematical Methods in Electrical Engineering (New York, 1951), p. 81.

the e.m.f.'s being generated in the various meshes, we may substitute $j\omega_g$ in (3.3.10) for p.

(3.3.11) $$f(j\omega_g) = (Lj\omega_g + R + D/j\omega_g)j\omega_g = j\omega_g Z_g.$$

The resulting solution of the first equation of (3.3.9) is[1]

(3.3.12) $$Q_g = \frac{(\exp j\omega_g t)\left[F^{(1)}(j\omega_g t) + tF(j\omega_g t)\right]}{f^{(1)}(j\omega_g t)} e_g,$$

where $F^{(n)}(j\omega_g t) = \left[d^n F(j\omega)/dj\omega^n\right]_{j\omega = j\omega_g}$, and $F(j\omega)$ is the adjoint of $f(j\omega)$. Then dQ_g/dt will be the g-th term of the second equation of (3.3.4). This completes the steady state solution of (3.2.5).

The transient solution is a solution of the system of equations

(3.3.13) $$\left[Lp^2 + Rp + D\right]Q = 0$$
$$dQ/dt = I.$$

It is customary to assume a constituent of the solution of the form $Q = K(\exp \lambda t)$, where $\exp \lambda t$ is a scalar multiplier and K is a column of constants to be determined. Then λ is substituted in (3.3.13) for p. If solutions of (3.3.13) exist, from Theorem 2.2.2 it follows that

(3.3.14) $$d(f(\lambda)) = d(L\lambda^2 + R\lambda + D) = 0.$$

The roots of this equation are known by various names such as <u>natural angular frequencies</u>, <u>natural angular velocities</u>, <u>natural modes</u>, etc. Equation (3.3.14) is of degree $n \leq 2M$.

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1 Frazer, Duncan, and Collar, p. 183.

The n roots are denoted by $\lambda_1, \lambda_2, \ldots, \lambda_n$. From equation (2.7.10), we see that the constituent solution corresponding to the unrepeated root λ_r is

$$(3.3.15) \qquad Q_r = K_r(\exp \lambda_r t).$$

The column K_r may be chosen proportional to any non-vanishing column of $F(\lambda_r)$. However, if λ_s is a member of s multiple roots of (3.3.14), the constituent solution associated with λ_s can be written

$$(3.3.16) \qquad Q_s = K_s(t)(\exp \lambda_s t).$$

The s columns relevant to the complete set of roots equal to λ_s may be chosen proportional to any s linearly independent columns of the family of matrices[1]

$$(3.3.17) \qquad \begin{aligned} U_0(t,\lambda_s) &= F(\lambda_s), \\ U_1(t,\lambda_s) &= F^{(1)}(\lambda_s) + tF(\lambda_s), \\ U_2(t,\lambda_s) &= F^{(2)}(\lambda_s) + 2tF^{(1)}(\lambda_s) + t^2F(\lambda_s), \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ U_{s-1}(t,\lambda_s) &= F^{(s-1)}(\lambda_s) + (s-1)tF^{(s-2)}(\lambda_s) + \ldots \\ &\qquad + t^{(s-1)}F(\lambda_s). \end{aligned}$$

Then the elements $k_{is}(t)$ of the columnar matrices K_s will be polynomials in t of degree s - 1 at most.

The most general transient solution of (3.3.5) is a linear combination of the constituent solutions

$$(3.3.18) \qquad \begin{aligned} Q = c_1K_1(t)(\exp \lambda_1 t) + c_2K_2(t)(\exp \lambda_2 t) + \ldots \\ + c_nK_n(t)(\exp \lambda_n t), \end{aligned}$$

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1 Frazer, Duncan, and Collar, p. 288.

in which the c_j's are arbitrary constants, real or complex. This solution may be written more concisely as

(3.3.19) $$Q = K(t)M(t)c.$$

In this equation K(t) is an M by n matrix whose j-th column is K_j,

$$M(t) = \begin{bmatrix} \exp \lambda_1 t & 0 & 0 & \cdots\cdots & 0 \\ 0 & \exp \lambda_2 t & 0 & \cdots\cdots & 0 \\ 0 & 0 & \exp \lambda_3 t & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots\cdots & \exp \lambda_n t \end{bmatrix},$$

while c represents the column of arbitrary constants. Taking the derivative of (3.3.19) with respect to time and adding it to the right side of the second equation of (3.3.4), we have the general solution of (3.2.5).

Other methods of obtaining the steady state and transient solutions of equation (3.2.5) are given in the literature.[1]

Pipes proposes a method for determination of natural modes which is particularly applicable to networks involving only inductance and elastance.[2] In this case equation (3.3.13) becomes

(3.3.20) $$(Lp^2 + D)Q = 0,$$

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1 Frazer, Duncan, and Collar, Chap. V and VI.
2 L. A. Pipes, "Matrix Theory of Oscillatory Networks", Journal of Applied Physics, 1939, X, 851.

or assuming a solution of the form exp $j\omega t$

(3.3.21) $$(-L\omega^2 + D)Q = 0.$$

Then if both L and D are non-singular, determination of the largest and smallest natural mode of the given network can be found by determining the dominant latent root of $L^{-1}D$ and $D^{-1}L$ respectively. These dominant latent roots can be readily determined by iterative processes.[1] If λ_1 is the dominant root of $L^{-1}D$, $\omega_1^2 = \lambda_1$, and if λ_2 is the dominant root of $D^{-1}L$, $\omega_2^2 = 1/\lambda_2$, where ω_1 is the largest natural mode of the given network and ω_2 is the smallest natural mode of the given network.

The scalar quantity $Z_{mn} = d(Z)/d(Z_m^n)$ is the <u>generalized network impedance</u>. If $m = n$, it is the <u>driving point impedance</u> of the given network in mesh m. If $m \neq n$, it is the <u>transfer</u> impedance from mesh m to mesh n. If the network consists of passive elements the matrix Z is symmetric, and $d(Z_m^n) = d(Z_n^m)$. Therefore, $Z_{mn} = Z_{nm}$.

Theorem 3.3.2 (Reciprocity Theorem). In any network composed of passive elements, if an e.m.f. E is applied in any mesh and I is the current in any mesh as a result of the applied e.m.f., then if the positions of the e.m.f. and currents are

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1 Frazer, Duncan, Collar, pp. 133-145; also H. I. Flomenhoft, "<u>A Method for Determining Mode Shapes and Frequencies Above the Fundamental by Matrix Iteration</u>", <u>Journal of Applied Mechanics</u>, 1950, XVII, pp. 249-256.

reversed the transfer impedance remains unchanged.

Theorem 3.3.3(Generalized Reciprocity Theorem). If a set $\underline{E}_1$ of e.m.f.'s all acting on the B branches of a passive network produce a current distribution $\underline{I}_1$, and a second set of e.m.f.'s $\underline{E}_2$ produce a second current distribution $\underline{I}_2$, then

(3.3.22) $$\underline{E}_1'\underline{I}_2 = \underline{I}_1'\underline{E}_2.$$

Proof. From equation (3.3.3)

(3.3.23) $$P(P'\underline{Z}P)^{-1}P'\underline{E}_1 = \underline{I}_1$$

and

(3.3.24) $$P(P'\underline{Z}P)^{-1}P'\underline{E}_2 = \underline{I}_2.$$

Since our network is passive $\underline{Z}$ is symmetric. If Z is a symmetric matrix then so is $P'\underline{Z}P$ and $(P'\underline{z}P)^{-1}$. Hence, taking the transpose of both sides of (3.3.23)

(3.3.25) $$\underline{I}_1' = \underline{E}_1'P(P'\underline{Z}P)^{-1}P'.$$

Multiplying (3.3.24) on the left by E_1'

(3.3.26) $$\underline{E}_1'\underline{I}_2 = \underline{E}_1'P(P\underline{Z}P)^{-1}P'\underline{E}_2.$$

Using (3.3.25)

(3.3.27) $$\underline{E}_1'\underline{I}_2 = \underline{I}_1'\underline{E}_2.$$

4. <u>Energy relationships and equivalent networks.</u> Writing equations (3.2.6) in terms of the mesh parameters we have

(3.4.1) $$F = \frac{\underline{I'RI}}{2} = 1/2 \sum_{j,k=1}^{M} R_{jk}i_j i_k,$$

I is a columnar matrix whose j-th element is the j-th mesh current of the network.

(3.4.2) $$T = I'LI/2 = 1/2 \sum_{j,k=1}^{M} L_{jk}i_j i_k.$$

If Q is a columnar matrix whose j-th element is the charge present in the j-th mesh of the network,

(3.4.3) $$V = Q'DQ/2 = 1/2 \sum_{j,k=1}^{M} D_{jk} q_j q_k .$$

The matrices R, L and D are used here in the same sense as in equation (3.3.9).

In the above quadratic forms subject the currents to a cogredient transformation with matrix A. According to Theorem 2.4.2, the matrices for the new quadratic forms are

(3.4.5) $$\begin{aligned} \mathcal{R} &= A'RA \\ \mathcal{L} &= A'LA \\ \mathcal{D} &= A'DA. \end{aligned}$$

From equation (3.3.10) we know that the natural modes of the new network are given by

(3.4.6) $$d(\mathcal{L}\lambda^2 + \mathcal{R}\lambda + \mathcal{D}) = 0$$

But

$$\mathcal{L}\lambda^2 + \mathcal{R}\lambda + \mathcal{D} = A'\left[L\;^2 + R\; + D\right]A.$$

Since A is non-singular and the determinant of a product of matrices is equal to the product of the determinants, (3.4.6) is the same as

(3.4.7) $$d(L\lambda^2 + R\lambda + D) = 0.$$

Hence

Theorem 3.4.1. The natural modes of a network are unchanged by a cogredient transformation.

or, stated differently

Theorem 3.4.1'. The natural modes of a network are absolute invariants under congruence relationship.

According to a theorem due to Hermite, Theorem 1.6.3, if Q is any non-singular symmetric matrix whatever of order n and S is a skew matrix of order n such that $(Q + S)\ (Q - S)$ is non-singular the matrix

$$R = (Q + S)^{-1}(Q - S)$$

is such that

$$R'QR = Q.$$

It follows then that

Theorem 3.4.2. If $A = (L + S)^{-1}(L - S)$, S any skew matrix whatever as long as $(L + S)\ (L - S)$ is non-singular, then the inductance, total and due to a coupling, are absolute invariants under the cogredient transformation whose matrix is A.

Upon substituting R and D for L the above theorem also holds for the resistance and elastance matrices.

The transformations (3.4.5) may lead to negative circuit elements. Such negative circuit elements require the introduction of ideal transformers[1] into the network. Networks requiring several ideal transformers are of little practical use; consequently, transformation which result in negative circuit elements are to be avoided. If R, L and D of (3.4.5) are R-matrices no negative circuit elements are present. Then

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1 Guillemin, Vol. II, p. 151.

from Theorem 2.6.3, 2.6.4 and 2.6.5 with the obvious extension to three circuit parameters we have[1]

Theorem 3.4.3. If R(L or D) is the matrix associated with a network which involves only one type of circuit parameter and A is non-singular, then a necessary and sufficient condition that R(L or D) be realizable without one or more ideal transformers is that A belong to the region W.

Theorem 3.4.4. If the matrix of the network involves two(three) types of circuit parameters and A is non-singular, then a sufficient condition that two(three) types of circuit parameters be realizable under a cogredient transformation with matrix A without the use of one or more ideal transformers is that the elements of A belong to $W_{12}(W_{123})$.

If a given network has m terminal pairs which may be used as input or output terminals, it is sometimes desirable to keep the generalized network impedances associated with these terminals invariant under the transformation (3.4.5). From the definition of the generalized network impedance and Theorem 2.5.4,

Theorem 3.4.5. The generalized network impedances associated with the m-terminal pairs of the network are absolute invariants under a cogredient transformation whose

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1 Richard Stevens Burington, "R-matrices and Equivalent Networks", p. 101.

matrix is m-affine.

The conclusions of the last theorem have been reached by Parodi[1] and Howitt[2] by different methods than those employed above.

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1 Maurice Parodi, "Réseaux Électrique et Théorie des Transformations", Journal of Physics Radium (8), VII, 1946, pp. 94-96.
2 Nathan Howitt, "Group Theory and the Electrical Networks", Physical Review, XXXVII, 1931, pp. 1583-1595.

CHAPTER IV

TRANSMISSION NETWORKS

AND

TRANSMISSION LINES

4.1. The matrix associated with a transmission network. In the analysis which follows we shall be concerned with passive networks having two sets of terminals, equal in number, those at the sending end, the input terminals, and those at the receiving end, the output terminals. Suppose each of these sets consists of n terminal pairs. Let $P'\underline{E} = E$ and $(P'\underline{Z}P)^{-1} = Z^{-1} = Y$, then equation (3.3.2) reduces to

(4.1.1)
$$
\begin{aligned}
i_1 &= y_{11}e_1 + y_{12}e_2 + \cdots + y_{1,2n}e_{2n} \\
i_2 &= y_{21}e_1 + y_{22}e_2 + \cdots + y_{2,2n}e_{2n} \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
i_{2n} &= y_{2n,1}e_1 + y_{2n,2}e_2 + \cdots + y_{2n,2n}e_{2n} \\
i_k &= y_{k1}e_1 + y_{k2}e_2 + \cdots + y_{k,2n}e_{2n}, \quad (k = 2n, \ldots, M).
\end{aligned}
$$

Our attention will be focused on the first 2n of these equations. The coefficient matrix of these first 2n equations is assumed to be non-singular. Then these equations may be solved for the e.m.f.'s in terms of the first 2n currents.

(4.1.2)
$$
\begin{aligned}
e_1 &= a_{11}i_1 + a_{12}i_2 + \cdots + a_{1,2n}i_{2n} \\
e_2 &= a_{21}i_1 + a_{22}i_2 + \cdots + a_{2,2n}i_{2n} \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
e_{2n} &= a_{2n,1}i_1 + a_{2n,2}i_2 + \cdots + a_{2n,2n}i_{2n}.
\end{aligned}
$$

or

$$(4.1.3) \qquad \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix},$$

where E_1 and I_1 are n by one matrices whose n elements are the n input e.m.f.'s and currents of the transmission network, E_2 and I_2 are columnar matrices whose n elements are the output e.m.f.'s and currents of the network, and A_{ij} are n by n matrices whose elements are constants of the circuit. We will adopt the convention of taking I_2, the output currents as negative.[1] The resulting matrix equations are

$$(4.1.4) \qquad \begin{aligned} E_1 &= A_{11}I_1 - A_{12}I_2 \\ E_2 &= A_{21}I_1 - A_{22}I_2. \end{aligned}$$

Theorem 4.1.1. $A_{21} = A'_{12}$.

Proof. The network is passive, hence Z can be chosen symmetric. It follows then that A will also be symmetric, for the inverse of a symmetric matrix is symmetric. Now

$$A_{12} = \begin{bmatrix} a_{1,n+1} & a_{1,n+2} & \cdots & a_{1,2n} \\ \vdots & \vdots & & \vdots \\ a_{n,n+1} & a_{n,n+2} & \cdots & a_{n,2n} \end{bmatrix}$$

and

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1 Leon Brillouin, Wave Propagation in Periodic Structures (New York, 1946), p. 201; Maynard G. Arsove, "The Algebraic Theory of Linear Transmission Networks", Journal of the Franklin Institute, 1953, CCLV, p. 308.

$$A_{21} = \begin{bmatrix} a_{n+1,1} & a_{n+1,2} & \cdots & a_{n+1,n} \\ \vdots & \vdots & & \vdots \\ a_{2n,1} & a_{2n,2} & & a_{2n,n} \end{bmatrix}$$

From the symmetry of A we see that the j-th row of A_{12} is the same as the j-th column of A_{21}. The theorem is then evident from the definition of A_{12}'.

Generally in the study of transmission networks it is desirable to have the output(input) e.m.f.'s and currents expressed in terms of the input(output) e.m.f.'s and currents. This can be done by using (4.1.4) if A_{22} and A_{12} are non-singular.

(4.1.5) $$\begin{bmatrix} E_2 \\ I_2 \end{bmatrix} = \begin{bmatrix} B_{11}, & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} E_1 \\ I_1 \end{bmatrix} = B \begin{bmatrix} E_1 \\ I_1 \end{bmatrix},$$

where

$$B_{11} = A_{22}A_{12}^{-1}, \quad B_{12} = A_{21} - A_{22}A_{12}^{-1}A_{11},$$
$$B_{21} = -A_{12}^{-1}, \quad B_{22} = A_{12}^{-1}A_{11}.$$

Theorem 4.1.2. The determinant of the transformation matrix B is one.[1]

Proof. Factor B so that

(4.1.6) $$B = \begin{bmatrix} J_n & 0 \\ B_{21}B_{11}^{-1} & J_n \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ 0 & -B_{21}B_{11}^{-1}B_{12} + B_{22} \end{bmatrix},$$

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1 This theorem is proven in a slightly different manner by H. V. Lowry, "The Application of the Characteristic Equation of a Matrix to the Evaluation of the Range of Frequencies for Which Currents are Passed Through Networks With Four or More Terminals Without Attenuation", <u>Philosophical</u> <u>Magazine</u>, 1945, XXXVI, p. 258.

where J_n is the identity matrix of order n. Then

$$(4.1.7) \qquad d(B) = d\begin{bmatrix} J_n & 0 \\ B_{21}B_{11}^{-1} & J_n \end{bmatrix} d\begin{bmatrix} B_{11} & B_{12} \\ 0 & -B_{21}B_{11}^{-1}B_{12} + B_{22} \end{bmatrix}.$$

From an extension of Theorem 1.5.6,

$$(4.1.8) \qquad d\begin{bmatrix} J_n & 0 \\ B_{21}B_{11}^{-1} & J_n \end{bmatrix} = 1,$$

and

$$(4.1.9) \qquad d\begin{bmatrix} B_{11} & B_{12} \\ 0 & -B_{21}B_{11}^{-1}B_{12} + B_{22} \end{bmatrix} = d(-B_{11}B_{21}B_{11}^{-1}B_{12} + B_{11}B_{22})$$

$$= d(B_{11})d(B_{22} - B_{21}B_{11}^{-1}B_{12}).$$

Making use of the definitions of the elements of B

$$\begin{aligned} -B_{21}B_{11}^{-1}B_{12} + B_{22} &= A_{12}^{-1}A_{12}A_{22}^{-1}(A_{21} - A_{22}A_{12}^{-1}A_{11}) + A_{12}^{-1}A_{11} \\ &= A_{22}^{-1}A_{21} \\ &= (B_{11}^{-1})'. \end{aligned}$$

But $d(B_{11}^{-1})' = d(B_{11}^{-1})$. Therefore, from equations (4.1.8) and (4.1.9), we have

$$d(B) = d(B_{11})d(B_{11}^{-1}) = 1.$$

The input e.m.f.'s and currents may be expressed as linear combinations of the output e.m.f.'s and currents by again referring to equation (4.1.4) if A_{11} and A_{21} are non-singular.

$$(4.1.10) \qquad \begin{bmatrix} E_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \begin{bmatrix} E_2 \\ I_2 \end{bmatrix}$$

with

$$C_{11} = B_{22}', \; C_{12} = -B_{12}', \; C_{21} = -B_{21}', \; C_{22} = B_{11}'.$$

Since

$$B\begin{bmatrix}E_1\\I_1\end{bmatrix} = \begin{bmatrix}E_2\\I_2\end{bmatrix}, \tag{4.1.5'}$$

$$C^{-1} = B. \tag{4.1.11}$$

Theorem 4.1.3. The transformation matrix of n transmission networks each having the same number of terminals connected in cascade is the product of the n transformation matrices of the individual transmission networks. The ordering of the matrices from right to left in the product is the same as the ordering of the networks starting from the input terminals and proceeding to the output terminals.

Proof. Consider two transmission networks whose matrices are A_1 and A_2. Suppose these two networks are connected in series. Since

$$\begin{bmatrix}E_2\\I_2\end{bmatrix} = A_1\begin{bmatrix}E_1\\I_1\end{bmatrix}, \quad \begin{bmatrix}E_3\\I_3\end{bmatrix} = A_2\begin{bmatrix}E_2\\I_2\end{bmatrix}, \tag{4.1.12}$$

it follows that

$$\begin{bmatrix}E_3\\I_3\end{bmatrix} = A_2A_1\begin{bmatrix}E_1\\I_1\end{bmatrix}. \tag{4.1.13}$$

Proceeding by induction, assume

$$\begin{bmatrix}E_n\\I_n\end{bmatrix} = A_{n-1}\cdots A_2A_1\begin{bmatrix}E_1\\I_1\end{bmatrix}, \tag{4.1.14}$$

and

$$\begin{bmatrix}E_{n+1}\\I_{n+1}\end{bmatrix} = A_n\begin{bmatrix}E_n\\I_n\end{bmatrix}. \tag{4.1.15}$$

Then

(4.1.16)
$$\begin{bmatrix} E_{n+1} \\ I_{n+1} \end{bmatrix} = A_n A_{n-1} \cdots A_2 A_1 \begin{bmatrix} E_1 \\ I_1 \end{bmatrix}.$$

Corollary 4.1.3. The transformation matrix for n equal transmission networks with matrix B connected in cascade is B^n.

4.2. <u>Reversible transmission networks</u>. A <u>reversed transmission network</u> is one in which the sense of the currents is changed and the input and output are interchanged. We call a transmission network <u>reversible</u> if it is identical with its reverse.[1]

Theorem 4.2.1. Necessary and sufficient conditions that a network with transformation matrix B be reversible is $B_{11} = B'_{22}$ and $B_{21} = B'_{21}$.

Proof. Consider the transmission network

(4.2.1)
$$\begin{bmatrix} E_2 \\ I_2 \end{bmatrix} = B \begin{bmatrix} E_1 \\ I_1 \end{bmatrix}.$$

If the input and output are interchanged

(4.2.2)
$$B^{-1} \begin{bmatrix} E_2 \\ I_2 \end{bmatrix} = C \begin{bmatrix} E_2 \\ I_2 \end{bmatrix} = \begin{bmatrix} E_1 \\ I_1 \end{bmatrix}.$$

Changing the sense of the currents gives us

(4.2.3)
$$\begin{bmatrix} E_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} C_{11} & -C_{12} \\ -C_{21} & C_{22} \end{bmatrix} \begin{bmatrix} E_2 \\ I_2 \end{bmatrix}.$$

From equation (4.1.10)

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1 Brillouin, p. 205.

$$(4.2.4) \qquad \begin{bmatrix} C_{11} & -C_{12} \\ -C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} B'_{22} & B'_{12} \\ B'_{21} & B'_{11} \end{bmatrix}.$$

By definition the network is reversible if and only if

$$(4.2.5) \qquad \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} B'_{22} & B'_{12} \\ B'_{21} & B'_{11} \end{bmatrix}.$$

It follows from equation (4.2.5) that $B_{11} = B'_{22}$ and $B_{21} = B'_{21}$ are necessary conditions for the network to be reversible.

To show that the conditions are sufficient all that remains is to show $B_{12} = B'_{12}$. Referring to the defining relations for the B_{ij}, $B_{21} = -A_{12}^{-1}$. Since $B_{21} = B'_{21}$, from Theorem 4.1.1, $A_{21} = A_{12}$. As a consequence of $B_{11} = B'_{22}$,

$$A_{22}A_{12}^{-1} = A_{11}A_{21}^{-1},$$

and

$$A_{12}^{-1}A_{11} = A_{21}^{-1}A_{22}.$$

Then

$$A_{22}A_{12}^{-1}A_{11} = A_{11}A_{21}^{-1}A_{11} = A_{11}A_{12}^{-1}A_{11} = A_{11}A_{21}^{-1}A_{22}.$$

But

$$B_{12} = A_{21} - A_{22}A_{12}^{-1}A_{11} = A_{12} - A_{11}A_{21}^{-1}A_{22} = B'_{12}.$$

The <u>four-terminal</u> network is a transmission network having only two pairs of terminals, one input and one output pair.

Corollary 4.2.1. A necessary and sufficient condition that a four terminal network with matrix B be reversible is[1]

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1 Brillouin, p. 206.

$$(4.2.6) \qquad B = \begin{bmatrix} \sqrt{1 + b_{12}b_{21}} & b_{12} \\ b_{21} & \sqrt{1 + b_{12}b_{21}} \end{bmatrix}.$$

Proof. Since the elements of our network matrix are no longer matrices but merely elements of the complex field, Theorem 4.2.1 shows that $b_{11} = b_{22}$ is necessary and sufficient that the network be reversible. Since $d(B) = 1$, $b_{11} = b_{22}$ leads to

$$(4.2.7) \qquad b_{11} = b_{22} = \sqrt{1 + b_{12}b_{21}}.$$

4.3. The impedances associated with a network and the propagation constant. A transmission network will be said to be terminated in a load impedance $Z(L)$ provided the output e.m.f.'s and currents satisfy $E_2 = Z(L)I_2$, where $Z(L)$ is an n by n matrix.[1]

If termination of a given transmission network in a load impedance $Z(L)$ implies that the input e.m.f.'s and currents satisfy $E_1 = Z(I)I_1$, where $Z(I)$ is an n by n matrix, then $Z(I)$ is called an input impedance of the transmission network.[2]

If a passive transmission network has input impedance Z_o when terminated in a load impedance Z_o, then Z_o is an iterative impedance of the network.[3]

Theorem 4.3.1. A necessary and sufficient condition

1 Arsove, Part II, p. 427.
2 Ibid., p. 427.
3 Ibid., p. 427.

that Z_o be an iterative impedance of a network with transformation matrix B is that Z_o be a solution of[1]

$$Z_oB_{21}Z_o + Z_oB_{22} - B_{11}Z_o - B_{12} = 0. \tag{4.3.1}$$

Proof. If Z_o is an iterative impedance, since $E_2 = Z_oI_2$ and $E_1 = Z_oI_1$, equation (4.1.5) becomes

$$\begin{bmatrix} Z_oI_2 \\ I_2 \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} Z_oI_1 \\ I_1 \end{bmatrix}, \tag{4.3.2}$$

or

$$\begin{aligned} Z_oI_2 &= (B_{11}Z_o + B_{12})I_1 \\ I_2 &= (B_{21}Z_o + B_{22})I_1. \end{aligned} \tag{4.3.3}$$

Then

$$(Z_oB_{21}Z_o + Z_oB_{22} - B_{11}Z_o - B_{12})I_1 = 0$$

for arbitrary I_1. Then by Corollary 2.2.2

$$Z_oB_{21}Z_o + Z_oB_{22} - B_{11}Z_o - B_{12} = 0.$$

Suppose Z_o is a solution of (4.3.1). It then follows that $B_{11} - Z_oB_{21}$ is non-singular. For if $B_{11} - Z_oB_{21}$ is singular there is a non-zero row matrix X such that $X(B_{11} - Z_oB_{21}) = 0$, and since $(B_{11} - Z_oB_{21})(-Z_o) = (B_{12} - Z_oB_{22})$, this would imply $X(B_{12} - Z_oB_{22}) = 0$. This leads to $X(J_n, -Z_o)B = 0$, from which the non-singularity of B forces $X = 0$, which is a contradiction.

Terminating the network in a load impedance Z_o yields $E_2 = Z_oI_2$. Then we have

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1 Arsove, Part II, p 427.

$$(4.3.4)\qquad \begin{bmatrix} Z_oI_2 \\ I_2 \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} E_1 \\ I_1 \end{bmatrix}.$$

From which it follows that

$$(4.3.5)\qquad (B_{11} - Z_oB_{21})E_1 + (B_{12} - Z_oB_{22})I_1 = 0.$$

But from (4.3.1), $B_{12} - Z_oB_{22} = -(B_{11} - Z_oB_{21})Z_o$. Hence (4.3.5) can be written

$$(4.3.6)\qquad (B_{11} - Z_oB_{21})(E_1 - Z_oI_1) = 0.$$

From which the non-singularity of $B_{11} - Z_oB_{21}$ leads to

$$(4.3.7)\qquad E_1 = Z_oI_1.$$

Theorem 4.3.2. A necessary and sufficient condition that Z_o be an iterative impedance of a reversible passive transmission network is that Z_o satisfy[1]

$$(4.3.8a)\qquad (Z_oB_{21})^2 = B_{11}^2 - J_n$$

and

$$(4.3.8b)\qquad (Z_oB_{21})B_{11} = B_{11}(Z_oB_{21}).$$

Proof. Upon multiplying (4.3.1) on the right by B_{21}, we have

$$(4.3.9)\qquad (Z_oB_{21})^2 + Z_oB_{22}B_{21} - B_{11}Z_oB_{21} - B_{12}B_{21} = 0.$$

Since $BC = J_{2n}$,

$$(4.3.10)\qquad \begin{aligned} B_{21}C_{11} + B_{22}C_{21} &= 0, \\ B_{11}C_{11} + B_{12}C_{21} &= J_n. \end{aligned}$$

However, for a reversible network, $C_{11} = B_{11}$ and $C_{21} = -B_{21}$. Then from (4.3.10)

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1 Arsove, Part II, p. 428.

$$(4.3.11) \qquad B_{22}B_{21} = B_{21}B_{11},$$
$$B_{12}B_{21} = B_{11}^2 - J_n.$$

Substituting from (4.3.11) into (4.3.9),

$$(4.3.12) \qquad (Z_oB_{21})^2 + (Z_oB_{21})B_{11} - B_{11}(Z_oB_{21}) - (B_{11}^2 - J_n) = 0.$$

Since B_{21} is non-singular (4.3.12) is equivalent to (4.3.1). Then

$$(Z_oB_{21} - B_{11})(Z_oB_{21} + B_{11}) = -J_n,$$

or

$$(B_{11} - Z_oB_{21})(Z_oB_{21} + B_{11}) = J_n.$$

In the proof of the preceding theorem we saw that $(B_{11} - Z_oB_{21})$ was non-singular if Z_o was a solution of (4.3.1), consequently

$$Z_oB_{21} + B_{11} = (B_{11} - Z_oB_{21})^{-1}.$$

Then

$$(4.3.13) \qquad B_{11} = 1/2([B_{11} - Z_oB_{21}] + [B_{11} - Z_oB_{21}]^{-1}).$$

Therefore, multiplying (4.3.13) on the left by $(B_{11} - Z_oB_{21})$,

$$(B_{11} - Z_oB_{21})B_{11} = 1/2([B_{11} - Z_oB_{21}]^2 + J_n),$$

and, multiplying (4.3.13) on the right by $(B_{11} - Z_oB_{21})$,

$$B_{11}(B_{11} - Z_oB_{21}) = 1/2([B_{11} - Z_oB_{21}]^2 + J_n).$$

Hence B_{11} is commutative with $B_{11} - Z_oB_{21}$ and with Z_oB_{21}. Equation (4.3.12) then reduces to

$$(4.3.14) \qquad (Z_oB_{21})^2 = B_{11}^2 - J_n.$$

Equation (4.3.14) and the commutativity of B_{11} with Z_oB_{21} imply equation (4.3.12), and the theorem is proven.

If m identical passive transmission networks are connected in cascade and the resulting network admits an input impedance $Z(m)$ for all sufficiently large integers m and if $\lim_{m\to\infty} Z(m)$ exists, $\lim_{m\to\infty} Z(m)$ is the <u>characteristic impedance</u> of the passive transmission network.[1]

Theorem 4.3.3. If a transmission network, B, admits a characteristic impedance Z_c, then Z_c is also an iterative impedance of the network.

Proof. If B is connected in cascade with m other networks identical with B, then B has output impedance $Z(m)$ and input impedance $Z(m+1)$. In the limit as $m\to\infty$, $Z(m) = Z(m+1)$, therefore, $E_2 = Z_c I_2$ and $E_1 = Z_c I_1$.

Suppose now that we cascade an infinite number of transmission networks. The propagation of a single wave along such a set of transmission networks is characterized by a complex <u>propagation constant</u> ξ such that as the wave passes from network m - 1 to network m

$$e_m^j = \xi_j e_{m-1}^j \qquad (j = 1, 2, \ldots, n) \tag{4.3.15}$$
$$i_m^j = \xi_j i_{m-1}^j$$

where the superscript on the e.m.f.'s and currents and the subscript on the propagation constant refers to the terminal pair with which the particular propagation constant is associated.[2]

Theorem 4.3.4. The propagation constants for a passive transmission network are the characteristic roots of the

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1 Arsove, Part II, p. 432.
2 Brillouin, p. 211.

network matrix.

Proof. From equation (4.1.5) and (4.3.15), we have

$$(4.3.16)\qquad \begin{bmatrix} \xi E_{n-1} \\ \xi I_{n-1} \end{bmatrix} = B \begin{bmatrix} E_{n-1} \\ I_{n-1} \end{bmatrix},$$

where the elements of the matrix on the left of the equation are the members of equations (4.3.15) on the right. In order that (4.3.16) have a solution $d(\xi J_{2n} - B) = 0$.

Theorem 4.3.5. A necessary and sufficient condition that a wave pass through a four terminal network without attenuation is $|b_{11} + b_{22}| \leq 2$ and $b_{11} + b_{22}$ be real.

Proof. From Theorem 4.3.4, it follows that

$$(b_{11} - \xi)(b_{22} - \xi) - b_{12}b_{21} = 0.$$

And since $d(B) = 1$,

$$(4.3.17)\qquad \xi^2 - (b_{11} + b_{22})\xi - 1 = 0.$$

Hence

$$(4.3.18)\qquad \xi = \frac{b_{11} + b_{22} \pm \sqrt{(b_{11} + b_{22})^2 - 4}}{2}$$

$$\xi = 1/2(b_{11} + b_{22}) \pm j\sqrt{1 - 1/4(b_{11} + b_{22})^2}. \quad (j^2 = -1)$$

The propagation function for a network may be written

$$(4.3.19)\qquad \xi = \exp(\alpha + j\beta) = (\exp\alpha)(\cos\beta + j\sin\beta),$$

with α being the attenuation constant and β the change in phase per network.[1] A necessary and sufficient condition that a wave pass through the network without attenuation is $\alpha = 0$. Then

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1 Brillouin, p. 212.

(4.3.20) $$\xi = \cos\beta + j\sin\beta.$$

Combining equations (4.3.18) and (4.3.20)

$$\cos\beta = 1/2(b_{11} + b_{22}).$$

Hence if $\alpha = 0$, $|b_{11} + b_{22}| \leq 2$ and $b_{11} + b_{22}$ is real.

Suppose $|b_{11} + b_{22}| \leq 2$ and $b_{11} + b_{22}$ is real, then from equation (4.3.18) and (4.3.19)

(4.3.21) $$1/2(b_{11} + b_{22}) = (\exp\alpha)\cos\beta,$$
$$\sqrt{1 - \left(\frac{b_{11} + b_{22}}{2}\right)^2} = (\exp\alpha)\sin\beta.$$

Squaring each of the equations in (4.3.21) and adding

$$1 = (\exp 2\alpha)(\cos^2\beta + \sin^2\beta).$$

Therefore,

$$\exp 2\alpha = 1,$$

which is satisfied only when $\alpha = 0$.

Theorem 4.3.6. If n four terminal networks are connected in cascade the network matrix associated with the resulting network is

$$B^n = \frac{1}{Z_{01} + Z_{02}}\begin{bmatrix} Z_{01}(\exp an) + Z_{02}(\exp -an), & 2Z_{01}Z_{02}(\sinh an) \\ 2(\sinh an), & Z_{02}(\exp an) + Z_{01}(\exp -an) \end{bmatrix},$$

where Z_{01} is the iterative impedance from left to right and Z_{02} is the iterative impedance from right to left, exp a is the propagation function from left to right and exp -a is the propagation function from right to left.[1]

Proof. The adjoint of the characteristic matrix B is

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1 L. A. Pipes, "The Transient Behavior of Four Terminal Networks", Philosophical Magazine, 1942, XXXIII, p. 190.

(4.3.22) $$F(\xi) = \begin{bmatrix} \xi - b_{22}, & b_{12} \\ b_{21} & \xi - b_{11} \end{bmatrix}.$$

Let $\xi_1 = (\exp a)$, $\xi_2 = (\exp -a)$. Now take k_1 of Theorem 2.7.4 to be $\begin{bmatrix} (\exp a) - b_{22} \\ b_{21} \end{bmatrix}$ and $k_2 = \begin{bmatrix} (\exp -a) - b_{22} \\ b_{21} \end{bmatrix}$ then

$$K = \begin{bmatrix} (\exp a) - b_{22} & (\exp -a) - b_{22} \\ b_{21} & b_{21} \end{bmatrix}.$$

Let

(4.3.23) $$s_1 = \frac{(\exp a) - b_{22}}{b_{21}}, \quad s_2 = \frac{(\exp -a) - b_{22}}{b_{21}}.$$

Then

$$K = b_{21}\begin{bmatrix} s_1 & s_2 \\ 1 & 1 \end{bmatrix} \text{ and } K^{-1} = \frac{1}{b_{21}(s_1 - s_2)}\begin{bmatrix} 1 & -s_2 \\ -1 & s_1 \end{bmatrix}.$$

From Corollary 4.1.3 the matrix of the network formed by cascading n equal four-terminal networks is B^n. Theorem 2.7.4 gives us

$$B^n = K\begin{bmatrix} (\exp na) & 0 \\ 0 & (\exp -na) \end{bmatrix}K^{-1}$$

or

(4.3.24) $$B^n = \frac{1}{(s_1 - s_2)}\begin{bmatrix} s_1(\exp na) - s_2(\exp -na), & 2s_1s_2(\sinh na) \\ 2(\sinh na), & s_1(\exp -na) - s_2(\exp na) \end{bmatrix}.$$

From equations (4.3.23) and (4.3.18), we can conclude that

(4.3.25) $$s_1 = \frac{b_{11} - b_{22} + \sqrt{(b_{11} + b_{22})^2 - 4}}{2b_{21}}$$

$$s_2 = \frac{b_{11} - b_{22} - \sqrt{(b_{11} + b_{22})^2 - 4}}{2b_{21}}.$$

Referring back to Theorem 4.3.1, equation (4.3.1) for a four-terminal network becomes

$$b_{21}Z_o^2 + (b_{22} - b_{11})Z_o - b_{12} = 0,$$

or

$$Z_o = \frac{b_{11} - b_{22} \pm \sqrt{(b_{11} + b_{22})^2 - 4}}{2b_{21}}. \tag{4.3.26}$$

An equation similar to (4.3.1) for the reversed four-terminal shows that the roots of (4.3.26) are Z_{01} and $-Z_{02}$. Comparing the solutions of (4.3.26) with (4.3.25), we see that

$$s_1 = Z_{01}, \quad s_2 = -Z_{02}.$$

The desired result follows upon substituting these values in (4.3.24).

Corollary 4.3.6. If n reversible four-terminal networks are connected in cascade, the network matrix associated with the resulting network is[1]

$$B^n = \begin{bmatrix} (\cosh na) & Z_{01}(\sinh na) \\ \dfrac{(\sinh na)}{Z_{01}} & (\cosh na) \end{bmatrix}.$$

Proof. The corollary follows from the fact that for a reversible network $Z_{01} = Z_{02}$.

4.4. Transmission lines. Much of the work of the preceding section can be extended to transmission lines by considering the transmission lines to consist of n transmission networks connected in cascade, letting n approach ∞ and the effect of the individual network matrices upon the individual e.m.f.'s and currents become infinitesimal, tending to zero

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1 Pipes, p. 190.

as n approaches ∞.[1] Since this would be mostly repetition of work already done, we will proceed instead with the problem of applying the theory of matrices to other problems concerning transmission lines.[2]

Consider n long, parallel, widely separated, cylindrical conductors over an equipotential ground. The differential equations governing the distribution of potentials and currents of the system are[3]

$$(4.4.1) \qquad -\frac{\partial V}{\partial x} = RI + L\frac{\partial I}{\partial t}$$

$$-\frac{\partial I}{\partial x} = GV + C\frac{\partial V}{\partial t},$$

where V and I are n by one matrices whose elements are the potentials and currents of the conductors. L, R, G, and C are n by n matrices whose elements are defined by

l_{rr} = self inductance of conductor r,

l_{rs} = mutual inductance between conductor r and s,

r_{rr} = series resistance of conductor r,

r_{rs} = 0, $(r \neq s)$,

g_{rr} = leakage conductance to ground of conductor r,

g_{rs} = leakage conductance between conductor r and s,

c_{rr} = self capacitance coefficient between conductor r and ground,

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1 Pipes, p. 200; Brillouin, Chapt. X.
2 This section is based on two papers by L. A. Pipes, "Transient analysis of Completely Transposed Multiconductor Transmission lines", Transactions of American Institute of Electrical Engineers, 1945, LX, pp. 346-351; and "Matrix Theory of Multiconductor Transmission Lines", Philosophical Magazine and Journal of Science, XXIV, 1937, pp. 97-115.
3 Guillemin, Vol. II, p. 33.

c_{rs} = mutual capacitance coefficient between conductors r and s.

Pipes gives formulas for the determination of the numerical values of these line parameters.[1]

Now let $Z = R + Lp$ and $Y = G + Cp$, where $p = \partial/\partial t$. Then using D for $\frac{\partial}{\partial x}$, (4.4.1) becomes

$$-DV = ZI$$

$$-DI = YV.$$

Since Z and Y are not functions of x,

(4.4.2) $$D^2V = -ZDI = ZYV$$

$$D^2I = YZI.$$

Let

$$V = (\exp j\omega t)\begin{bmatrix} E_1a_1 \\ \cdot \\ \cdot \\ \cdot \\ E_na_n \end{bmatrix},$$ where $a_r = (\exp j\theta_r)$, θ_r is the phase shift for the r-th potential and E_r is a function of x.

and

$$E = \begin{bmatrix} E_1a_1 \\ \cdot \\ \cdot \\ \cdot \\ E_na_n \end{bmatrix}.$$

Then

(4.4.3) $$D^2E = Z(j\omega)Y(j\omega)E, \text{ where } Z(j\omega) = Z(p)_{p = j\omega}.$$

Also

$$D^2I = Y(j\omega)Z(j\omega)I,$$

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1 Pipes, Transactions of A.I.E.E., p. 346.

with $I = \begin{bmatrix} I_1 b_1 \\ \cdot \\ \cdot \\ \cdot \\ I_n b_n \end{bmatrix}$, where b_r is of the same form as a_r and I_r is a function of x.

We will assume a solution of the form

(4.4.4) $E = A\cosh \alpha x + B\sinh \alpha x,$

$I = F\cosh \alpha x + H\sinh \alpha x,$

A, B, F, and H being matrices of constants which are to be determined. We now have

(4.4.5) $D^2 E = \alpha^2 A\cosh \alpha x + \alpha^2 B\sinh \alpha x$

$= \alpha^2 E.$

Equating the right side of (4.4.3) to the right side of (4.4.5)

(4.4.6) $Z(j\omega)Y(j\omega)E = \alpha^2 E.$

Letting $\alpha^2 = \lambda$, the solutions for (4.4.6) are the characteristic roots of $Z(j\omega)Y(j\omega)$. After these roots have been determined

(4.4.7) $\omega_r = \pm\sqrt{\lambda_r}$ $(r = 1, 2, \ldots, n).$

Since $Z(j\omega)$ and $Y(j\omega)$ are symmetrical and the characteristic roots of the transpose of a matrix are the same as the characteristic roots of the matrix, the roots of (4.4.6) are the same as the roots of the comparable equation for I.

From equation (2.7.8) and the statement which follows it, we know that for each characteristic root of $Z(j\omega)Y(j\omega)$ there is a solution of (4.4.4) which may be written

$E_r = k_r a_r \cosh \alpha_r x + k_r b_r \sinh \alpha_r x,$

$I_r = k_r f_r \cosh \alpha_r x + k_r h_r \sinh \alpha_r x,$ $(r = 1, \ldots, n)$

where a_r, b_r, f_r, and h_r are arbitrary constants and k_r is any non-zero column of the adjoint of the characteristic matrix fo $Z(j\omega)Y(j\omega)$. The general solution of (4.4.3) is then given by the sum of these particular solutions, i.e.,

$$V = K\begin{bmatrix} \cosh \alpha_1 x & 0 & \cdots & 0 \\ 0 & \cosh \alpha_2 x & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \cosh \alpha_n x \end{bmatrix} A + K\begin{bmatrix} \sinh \alpha_1 x & 0 & \cdots & 0 \\ 0 & \sinh \alpha_2 x & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \sinh \alpha_n x \end{bmatrix} H,$$

$$I = K\begin{bmatrix} \cosh \alpha_1 x & 0 & \cdots & 0 \\ 0 & \cosh \alpha_2 x & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \cosh \alpha_n x \end{bmatrix} F + K\begin{bmatrix} \sinh \alpha_1 x & 0 & \cdots & 0 \\ 0 & \sinh \alpha_2 x & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \sinh \alpha_n x \end{bmatrix} H, \tag{4.4.8}$$

where K is the modal matrix of $Z(j\omega)Y(j\omega)$ and the matrices A, B, F, and H are columnar matrices consisting of the $2n$ arbitrary constants of the solution of the differential equations. We have assumed the roots of (4.4.6) to be simple in the above solution. If this is not true the matrix K is determined in the same manner as it was determined for multiple roots in the solution of the mesh differential equations in Chapter 3, Section 3.

If the transmission lines are fully transposed the solution of the equations (4.4.1) is more easily obtained. In this case $l_{rs} = l$, $g_{rs} = g$, and $c_{rs} = c$, $(r \neq s;\ s,r = 1, 2, \ldots, n)$, $l_{rs} = l_o$, $g_{rs} = g_o$, $c_{rs} = c_o$, and $r_{rs} = r_o$, $(r = s;\ r,s = 1, 2, \ldots, n)$.

Consider the matrix $A = (a_{rs})$, $a_{rs} = (\exp j2\pi/n)^{-(r-1)(s-1)}$. The various rows of A are the sequence operators of Fortescue.[1]

Theorem 4.4.1. If $A = (a_{rs})$, then $A^{-1} = 1/n(\text{conjugate } A)$.

Proof. Let $B = 1/n(\text{conjugate } A)$, then

$$b_{rs} = (1/n) \exp(2\pi j/n)(r-1)(s-1).$$

If $AB = C$,

$$\begin{aligned} c_{rs} &= 1/n \sum_{k=1}^{n} \exp\left[(-2\pi j/n)(r-1)(k-1)\right] \exp\left[(2\pi j/n)(k-1)(s-1)\right], \\ &= 1/n \sum_{k=1}^{n} \exp\left[(-2\pi j/n)(k-1)(r-s)\right]. \end{aligned} \tag{4.4.9}$$

We see that for $r = s$, $c_{rs} = 1$. For $r \neq s$,

$$c_{rs} = 1/n\left[1 + \exp\left\{(-2\pi j/n)(r-s)\right\} + \exp\left\{(-4\pi j/n)(r-s)\right\} + \ldots + \exp\left\{(-2\pi j/n)(n-1)(r-s)\right\}\right]. \tag{4.4.10}$$

Using the formula

$$S = \frac{r^{k} - a}{r - 1}, \tag{4.4.11}$$

for the sum of a geometric series,

$$c_{rs} = \frac{\exp\left[(-2\pi j)(r-s)\right] - 1}{\exp\left[(-2\pi j/n)(r-s)\right] - 1}. \tag{4.4.12}$$

But $\exp(-2k\pi j) = 1$ for $k = 1, 2, \ldots$ and the numerator of (4.4.12) is zero for all values of r and s. Since $r - s$ is always less than n, the denominator of (4.4.12) is never zero.

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1 Pipes, Transactions of A.I.E.E., p. 347.

Therefore, $c_{rs} = 0$ when $r \neq s$. Consequently, $C = J_n$, and $B = A^{-1}$.

Theorem 4.4.2. If C is a matrix such that $c_{rs} = c_o$ for $r = s$, and $c_{rs} = c$ for $r \neq s$, then $A^{-1}CA = D$ is a matrix such that $d_{ss} = c_o - c$ for $s = 2, 3, \ldots, n$, $d_{11} = c_o + (n-1)c$, and $d_{rs} = 0$ for $r \neq s$.

Proof. From the definition if $A^{-1}C = F$,

$$f_{rs} = 1/n\Big\{c_o\exp\left[(2\pi j/n)(s-1)^2\right] + c\sum_{k=1}^{s-1}\exp\left[(2\pi j/n)(s-1)(k-1)\right] + c\sum_{k=s+1}^{n}\exp\left[(2\pi j/n)(s-1)(k-1)\right]\Big\}.$$

Then if $FA = D$,

$$d_{rs} = 1/n\sum_{m=1}^{n}\Big[c_o\exp\left[(2\pi j/n)(s-1)^2\right] + c\Big\{\sum_{k=1}^{s-1}\exp\left[(2\pi j/n)(s-1)(k-1)\right] + \sum_{k=s+1}^{n}\exp\left[(2\pi j/n)(s-1)(k-1)\right]\Big\}\exp\,(-2\pi j/n)(m-1)(s-1)\Big],$$

(4.4.13)

$$d_{rs} = 1/n\sum_{m=1}^{n}c_o\exp\left[(2\pi j/n)\left\{(s-1)^2 - (m-1)(s-1)\right\}\right] + 1/n\sum_{m=1}^{n}\sum_{k=1}^{n}c\,\exp\left[(2\pi j/n)\left\{(s-1)(k-1) - (m-1)(s-1)\right\}\right] - c/n\sum_{m=1}^{n}\exp\left[(2\pi j/n)\left\{(s-1)^2 - (m-1)(s-1)\right\}\right].$$

Now the first and third expressions on the right of (4.4.13) are of the same type as (4.4.9). Consequently, for $s \neq m$, they are both zero. For $s = 1$, the first expression on the right of (4.4.13) becomes c , and the third expression becomes $-c$. When $s = 1$, the second expression on the right of (4.4.13) is nc. Then $d_{11} = c_o + (n-1)c$.

If $s \neq 1$, again using (4.4.11), the second expression on the right is

$$c/n \sum_{m=1}^{n} \left[\frac{\exp\{(2\pi j)(s-1)\} - 1}{\exp\{(2\pi j/n)(s-1)\} - 1}\right] \exp\left[(-2\pi j/n)(m-1)\right].$$

But for $s > 1$, the numerator of the fraction in brackets is zero, but the denominator is not zero. Therefore, the above expression is zero for $s \neq 1$. Hence, D is of the desired diagonal form.

Multiplying (4.4.1) on the left by A

$$(4.4.14) \qquad -\frac{\partial A^{-1}V}{\partial x} = A^{-1}RAA^{-1}I + A^{-1}LAA^{-1}\frac{\partial I}{\partial t}$$

$$-\frac{\partial A^{-1}I}{\partial x} = A^{-1}GAA^{-1}V + A^{-1}CAA^{-1}\frac{\partial V}{\partial t}.$$

Define

$$V_s = A^{-1}V = (v_{rs}),$$

$$I_s = A^{-1}I = (i_{rs}), \qquad (r = 1, 2, \ldots, n)$$

$$R_s = A^{-1}RA = \begin{bmatrix} R_0 & 0 & 0 & \ldots & 0 \\ 0 & R_0 & 0 & \ldots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \ldots & R_0 \end{bmatrix},$$

$$L_s = A^{-1}LA = \begin{bmatrix} l_0 + (n-1)l & 0 & \ldots & 0 \\ 0 & l_0 - l & \ldots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \ldots & l_0 - l \end{bmatrix},$$

$$C_s = A^{-1}CA,$$

$$G_s = A^{-1}GA,$$

where the C_s and G_s take the same form as L_s.

Then

$$(4.4.15) \qquad -\frac{\partial V_s}{\partial x} = R_sI_s + L_s\frac{\partial I_s}{\partial t}$$

$$-\frac{\partial I_s}{\partial x} = G_s V_s + C_s \frac{\partial V_s}{\partial t}.$$

Since R_s, L_s, G_s, and C_s are all diagonal matrices (2.6.11) reduces to 2n equations

(4.4.16a) $$-\frac{\partial v_{1s}}{\partial x} = R_o i_{1s} + (l_o + (n-1)l)\frac{\partial i_{1s}}{\partial t},$$

$$-\frac{\partial v_{rs}}{\partial x} = R_o I_{rs} + (l_o - l)\frac{\partial i_{rs}}{\partial t}, \quad (r = 2, 3, \ldots, n),$$

and

(4.4.16b) $$-\frac{\partial i_{1s}}{\partial x} = (g_o + (n-1)g)v_{1s} + (c_o + (n-1)c)\frac{\partial v_{1s}}{\partial t},$$

$$-\frac{\partial i_{rs}}{\partial x} = (g_o - g)v_{rs} + (c_o - c)\frac{\partial v_{rs}}{\partial t},$$

$$(r = 2, 3, \ldots, n).$$

These differential equations can then be solved by La Place transformations.[1] The values of the actual forward waves in terms of the constant e.m.f. at $x = 0$, is given by

$$V^+ = 1/2ABA^{-1}V^o,$$

with B a diagonal matrix derived from (4.4.16a) by La Place transformations and integration. Similarly

$$I^+ = 1/2AHA^{-1}V^o,$$

where H is a diagonal matrix derived from (4.4.16b) by La Place transformations and integration.

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1 Pipes, Transactions of A.I.E.E., p. 348.

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