HIGHER DEGREE EQUATIONS

A Thesis

Presented to

the Faculty of the College of Arts and Sciences

The University of Houston

In partial Fulfillment

of the Requirements for the Degree

Master of Science in Mathematics

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by

Hortensia Vargas

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This thesis is a summary of the history and solutions of algebraic equations whose degree is equal to or greater than three, together with a proof by the author that all three solutions of the quartic may be obtained by considering two different linear functions of a certain expression in the roots of the quartic.

OUTLINE

Continued

CHAPTER I

Brief History of the Solution of Equations

Babylonians.

The algebra of the Babylonians was empirical as can be shown in their astonishing solutions of cubic equations with numerical coefficients.[1] The equations solved by Babylonians, expressed in modern notation, were of the type

$$x^3 + px^2 + q = 0$$

which could be reduced to the form

$$y^3 + y^2 = r$$

by multiplying the original equation by $\frac{1}{q^3}$

and letting

$$y = \frac{x}{p} \text{ and } r = \frac{-q}{p^3}$$

If the resulting r was positive, the value of y, and so that of x, was obtained from tabulated values of $n^3 + n^2$, provided that the r was in the table. It is possible that the scribe proceeded from certain tabulated r's to construct the equations, so that they could be solvable.

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1 E. T. Bell, The Development of Mathematics (second edition; New York: McGraw-Hill Book Company, Inc., 1945), p. 36.

The Babylonian reduction of cubics appears to be the first recorded instance of this method which was again used in the sixteenth century by the Italian algebraists and latter on by Vieta.[2]

Greeks

Among the Greeks the oldest type of cubic equation of the form $x^3 = k$ was possibly due to Menaechmus (350 B. C.).[3] He solved this cubic by finding the intersection of two conics. Next, Archimedes tried to solve the problem by cutting the sphere by a plane so that the two segments shall have a given ratio which reduced to the proportion

$$\frac{c-x}{b} = \frac{c^2}{x^2}$$

and to the equation

$$x^3 + c^2 b = cx^2 .$$

According to Eutocius, Archimedes solved the problem by finding the intersection of two conics, namely the parabola

$$x^2 = \frac{a^2 y}{c}$$

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2 E. T. Bell, The Development of Mathematics (second edition; New York: McGraw-Hill Book Company, Inc., 1945), p. 37.

3 D. E. Smith, History of Mathematics (Vol. 2; New York: Ginn and Company, 1925), p. 454.

and the hyperbola

$$y(c-x) = bc.$$

Diophantus solved a single cubic equation $x^3 + x = 4x^2 + 4$ in connection with the problem of finding a right - angled triangle such that the area added to the hypotenuse gives a square while the perimeter is a cube. His method is not given, but possibly he saw that

$$x(x^2+1) = 4(x^2+1) \qquad \text{for } x = 4.$$

Arabs and Persians

The problem of Archimedes was taken up by the Arabs and Persians in the ninth century. The equations were solved by geometric methods. Alkayami was noted for his geometrical treatment of cubic equations by which he obtained a root as the abscissa of a point of intersection of a conic and a circle.[4] He considered equations of the following form in which a and c stand for positive integers.

a) $x^3 + b^2x = b^2c$ whose root he said is the abscissa of a point of intersection of $x^2 = by$ and $y^2 = x(c-x)$

4 W. W. R. Ball, <u>A Short Account of the History of Mathematics</u> (New York: The MacMillan Company, 1927), p. 159.

b) $x^3 + ax^2 = c^3$ whose root is the abscissa of a point of intersection of

$$xy = c^2 \text{ and } y^2 = c(x+a)$$

c) $x^3 \pm ax^2 + b^2x = b^2c$ whose root is the abscissa of a point of intersection of

$$y^2 = (x \pm a)\ (c - x) \text{ and } x\ (b \pm y) = bc.$$

Chinese and Hindus

The Chinese algebraists did not pay too much attention to the cubic equation. Their interest was in applied problems which led to numerical equations. The numerical cubic first appeared in a work by Wang H's iao-t'ung about 625 in relation to a problem with a right-angled triangle.[5] He used an equation of the form

$$x^3 + ax^2 - b = 0.$$

Other Chinese algebraists treated cubics, but it was not until the thirteenth century when the European influence was powerful, that any attempt was made by Chinese to classify third degree equations. Between 1662 and 1722 nine types of cubics were given

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5 D. E. Smith, History of Mathematics (Vol. 2; New York: The Ginn and Company, 1925), p. 456.

$$x^3 \pm bx = c \qquad x^3 \pm ax^2 = c$$

$$x^3 \pm ax^2 \pm bx = c \qquad -x^3 + ax^2 = c.$$

But in every case the solution was numerical and only a single positive root was given.

The Hindus were not interested in cubics. Bhāskara (1150) gave the following example

$$x^3 + 12x = 6x^2 + 35$$

in which the root 5 was found by trial.

Medieval Interest

European scholars of the Middle Ages attempted to solve cubics. Fibonacci, for example, attacked the problem in his Flos of 1225. He said that a scholar of Palermo, proposed to him the problem of finding a cube which, with two squares and ten roots, should be equal to 20. The problem was solved by this equation

$$x^3 + 2x^2 + 10x = 20.$$

Another attempt was made by an anonymous writer of the thirteenth century whose work has been described by Libri.[6] He took two cubics, one of the type $ax^3 = cx + k$ and another of the type $ax^3 = bx^2 + k$. His method in the

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6 D. E. Smith, History of Mathematics (Vol. 2; New York: The Ginn and Company, 1925,) p. 457.

first case was

$$ax^3 = cx + k$$

$$x^3 = \frac{cx}{a} + \frac{k}{a}$$

$$x = \frac{c}{2a} + \sqrt{\left(\frac{c}{2a}\right)^2 + \frac{k}{a}}$$ which is the

root of equation $ax^2 = cx + k$, but not of the given equation. His method in the second equation was equally fallacious.

Pacioli in 1494 stated that the general solution of a cubic was impossible. Rudolff (German) in 1525 suggested three numerical equations each with one integral root and each being easily solved by factoring. His method in modern symbols is as follows.

Given $x^3 = 10x^2 + 20x + 48$

Add 8 to both sides and divide by $(x + 2)$

$$(x^3 + 8) = 10x^2 + 20x + 56$$

$$x^2 - 2x + 4 = 10x + \frac{56}{x+2}.$$

Split the two members of the equation

$$x^2 - 2x = 10x$$

$$4 = \frac{56}{x+2}$$ which are satisfied by $x = 12$.

Similar solutions of special cases were found in several works of the sixteenth century, notably in a work by Nicolas Petri published at Amsterdam in 1567.

Cubic equations were considered by Diophantus about 300 A. D., but the first European mathematicians to give a complete solution of them belonged to the Italian school of Bologna at the time of the Renaissance. They were Scipio Ferro, Nicolas Fontana, surnamed Tartaglia, and Cardan. The solution which usually bears the name of Cardan, was really due to Tartaglia.[7] Ferro solved cubic equations of the form $x^3 + ax = b$ possibly basing his work on Arab sources. He did not reveal his method to the scholars; but he told the secret to his pupil Antonio Maria Fior. Some time later Tartaglia and Fior proposed to meet in a mathematical contest. Tartaglia devoted most of his time in devising a method for solving cubics in which the first degree term was missing. Tartaglia succeeded in the contest. Cardan asked Tartaglia to show him the discovery, but he refused. So, Cardan informed Tartaglia that a wealthy nobleman was interested

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7 H. W. Turnbull, <u>Theory of Equations</u> (New York: Interscience Publishers, Inc., 1947), p. 117.

in it. Cardan arranged a meeting at Milan between the mathematician and his would - be patron. On reaching Milan, Tartaglia found that it was all a hoax, but he was persuaded to give Cardan the information he desired, pledging him to secrecy. Tartaglia claimed that he divulged the entire theory; but Cardan published a treatment of the cubic covering Tartaglia's contributions as well as other points in his Ars Magna (1545). When Tartaglia protested, Ferrari, Cardan's most capable pupil, claimed that Cardan had received his information from Ferro. Cardan was the first to exhibit three real roots for any cubic. He advanced beyond the mere formal solution in recognizing the irreducible case (all roots irrational) when the radicals appearing are cube roots of complex numbers. The first to recognize the reality of the roots in the irreducible case was R. Bombelli in 1572.[8]

Fibonacci was a true mathematician far ahead of his time.[9] Being unable to give an algebraic solution of the

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8 E. T. Bell, The Development of Mathematics (second edition; New York: McGraw-Hill Book Company, Inc., 1945), p. 118.

9 Ibid., p. 116.

equation

$$x^3 + 2x^2 + 10x = 20,$$

he attempted to prove that a geometrical construction of a root by straightedge and compass alone was impossible; but he could not have succeeded with what was known at his time. So he proceeded to find a numerical approximation to a root.

Although Cardan reduced his particular equation to forms lacking the term in x^2, it was Vieta who began with the general form of the cubic[10] $x^3 + px^2 + qx + r = 0$.

Concerning equations of fourth degree, Abûl-Faradsh referred in his Fihrist to a problem which involved an equation of the type

$$x^4 + px^3 = q$$

which could be solved by the intersection of the hyperbola $y^2 + axy + b = 0$ and the parabola $x^2 - y = 0$; but the work in which the problem appeared was lost and no one knows what he did for a solution.

Woepcke, a French orientalist, has called attention to an anonymous manuscript of an Arab or Persian algebraist in which is given the biquadratic equation

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10 D. E. Smith, History of Mathematics (Vol. 2; New York: The Ginn and Company, 1925), p. 467.

$$(100-x^2)\cdot(10-x)^2 = 8100.$$

It was solved by taking the intersection of $(10-x)\,y = 90$ and $x^2 + y^2 = 100$. But there is no evidence that the author was concerned with the algebraic theory.[11]

The problem of a biquadratic equation was laid prominently before Italian mathematicians by Zuanne de Tonini da Coi in 1540 when he proposed this problem. Divide ten into three parts such that they shall be in continued proportion and that the product of the first two shall be six. He gave this problem to Cardan with the statement that it could not be solved, but Cardan denied the assertion, although he did not solve it. He gave it to Ferrari, who, though a mere youth, solved the problem.

Vieta (1590) was the first algebraist after Ferrari to make any noteworthy advance in the solution of the biquadratic.[11] He began with an equation of the type

$$x^4 + 2qx^2 + bx = c \quad \text{wrote it as}$$

$$x^4 + 2qx^2 = c - bx$$

He added $q^2 + \frac{y^2}{2} + yx^2 + qy$ to both sides, made the right side a perfect square, and from there on he followed Ferrari's method.

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11 D. E. Smith, History of Mathematics (Vol. 2; New York: The Ginn and Company, 1925), p. 467.

Next, Descartes (1637) took up the question of the biquadratic equation and succeeded in effecting a simple solution of problems of this type $x^4 + px^2 + qx + r = 0$.

Euler (1770) solved the general quartic by a method differing from that of Ferrari. This unexpected success led him to believe that the general quintic equation was solvable by radicals.

Concerning the quintic equation, the German E. W. Tschirnhausen (1651-1708) applied a rational substitution to remove certain terms from a given equation generalizing the removal of the second term from cubics and quartics used by Cardan, Vieta and others. A century later, E. S. Bring (Swedish - 1736 - 1798) reduced the general quintic to one of its trinomial forms $x^5 + ax + b = 0$ by a Tschirnhausen transformations with coefficients involving one cube root and three square roots, a result of capital importance in the trascendental solution of the quintic.[12]

Lagrange (1770-1771), instead of trying to solve the general quintic by ingenious tricks, examined the

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12 E. T. Bell, The Development of Mathematics (second edition; New York: McGraw-Hill Book Company, Inc., 1945), p. 232.

extant solutions of the equations of second, third and fourth degrees in an attempt to discover why the particular devices used by his predecessors had succeeded. He found that in each instance the solution was reducible to that of an equation of lower degree, whose roots were linear functions of the roots of the given equation, and of the roots of unity. He thought that it was a general method; but on applying to the general quintic, Lagrange obtained a sextic which meant that the degree of the equation, instead of being reduced as before, was raised.

The first noteworthy attempt to prove that an equation of fifth degree could not be solved by algebraic methods was due to Ruffini (1803-1805) although it had already been considered by Gauss.[13]

The modern theory of equations is commonly said to date from Abel and Galois. The latter's posthumous memoir on the subject established the theory in a satisfactory manner. The Norwegian mathematician Abel, as a student in Christiana, thought for a while that he had

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13 D. E. Smith, History of Mathematics (Vol. 2; New York: The Ginn and Company, 1925), p. 469.

solved the quintic equation; but he corrected himself in a pamphlet published in 1824.[14] This was the famous paper in which Abel proved the impossibility of solving an equation of fifth degree by radicals, a problem which had puzzled the mathematicians for many years.

In 1884 Klein (1849-1925, German) reviewed all the labors of his predecessors and unified them with respect to the group of rotations of a regular icosahedron about its axis of symmetry. In other words, Klein handled the subject of the quintic equation in a simple manner by introducing the icosahedron equation as the normal form, and showed that the method could be generalized so as to embrace the whole theory of higher degree equations.[15]

Other contributions to this subject were a transcendental solution of the quintic given by M. Hermite and Sylvester's transformation of the fifth degree equation.

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14 D. J. Struik, A Concise History of Mathematics (Vol. 2; New York: Dover Publication Inc., 1948), p. 226.

15 D. E. Smith, History of Modern Mathematics (New York: John Wiley and Sons, 1906), p. 21.

CHAPTER II

Solution of the Cubic Equation

Cardan's Solution

In solving the cubic, Cardan first eliminated the term of second degree by substituting

$$x = y - \frac{p}{3} \quad \text{in the general cubic}$$

$$x^3 + px^2 + qx + r = 0$$

which gives $y^3 + my + n = 0$.

Let $$y = z - \frac{m}{3z} \qquad (1)$$

obtaining $$z^6 + nz^3 - \frac{m^3}{27} = 0$$

where $$z^3 = -\frac{n}{2} \pm \sqrt{\frac{n^2}{4} + \frac{m^3}{27}}$$

and $$z = \sqrt[3]{-\frac{n}{2} \pm \sqrt{\frac{n^2}{4} + \frac{m^3}{27}}} \;.$$

Substituting the z for its value in (1)

$$y = \sqrt[3]{-\frac{n}{2} \pm \sqrt{\frac{n^2}{4} + \frac{m^3}{27}}} - \frac{m}{3\sqrt[3]{-\frac{n}{2} \pm \sqrt{\frac{n^2}{4} + \frac{m^3}{27}}}}$$

Eliminating the denominator in the second fraction and omitting double signs, since they give no more values than do single ones.

Then $$y = \sqrt[3]{-\frac{n}{2} + \sqrt{\frac{n^2}{4} + \frac{m^3}{27}}} + \sqrt[3]{-\frac{n}{2} - \sqrt{\frac{n^2}{4} + \frac{m^3}{27}}} \;.$$

Cardan's formula fails when all the roots are real and unequal.

Vieta's Solution

Given the general equation of third degree

$$x^3 + px^2 + qx + r = 0.$$

Vieta reduced it to

$$y^3 + 3by = 2c$$

by substituting

$$x = y - \frac{p}{3} \; .$$

Now, let

$$b = z^3 + yz \quad \text{and} \quad y = \frac{b - z^2}{z}$$

obtaining

$$z^6 + 2cz^3 = b^3$$

a sextic which he solved as a quadratic.

Hudde's Contribution

Hudde simplified Vieta's work (1658) by taking advantage of Descartes' symbolism. He brought the theory of the cubic to its present status. He was also the first algebraist who unquestionably recognized that a letter stands for either a positive or negative number.[16]

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16 D. E. Smith, History of Mathematics (Vol. 2; New York: The Ginn and Company, 1925), p. 466.

In the equation

$$x^3 = qx + r$$

he let $\qquad x = y + z$

which results in

$$y^3 + 3y^2z + 3yz^2 + z^3 = qx + r.$$

If

(1) $$y^3 + z^3 = r,$$

then $3zy^2 + 3z^2y = qx,$

from where $\quad y = \frac{q}{3z}.$

Substituting in (1)

$$y^3 = r - z^3 = \frac{1}{27}\,\frac{q^3}{3}$$

$$z^3 = \frac{r}{2} \pm \sqrt{\frac{r^2}{4} - \frac{q^3}{27}} = A$$

$$y^3 = \frac{r}{2} \mp \sqrt{\frac{r^2}{4} - \frac{q^3}{27}} = B$$

$$x = \sqrt[3]{A} + \sqrt[3]{B}\,.$$

Trigonometric Solution

In the sixteenth century Vieta suggested the treatment of the numerical cubic equation by trigonometry. Girald (1629) was one of the first, however to attack the problem.

He solved the equation

$x^3 = 13x + 12$ by the help of the identity

$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$. This is the so - called irreducible case which is of interest because it is the case in which all the roots are real.

Given the cubic $x^3 + px + q = 0$ (1)

let $x = \frac{z}{n}$. Then (1) becomes $z^3 + pn^2z + n^3q = 0$. (2)

Now the trigonometric identity

$$\cos^3\theta - \frac{3}{4}\cos\theta - \frac{1}{4}\cos 3\theta = 0 \quad (3)$$

is identical to (2) if

$$z = \cos\theta \qquad pn^2 = -\frac{3}{4} \qquad n^3q = -\frac{1}{4}\cos 3\theta.$$

Whence $n = \sqrt{\frac{-3}{4p}}$.

Substituting n in the expression

$$\cos 3\theta = -4n^3q,$$

results in

$$\cos 3\theta = -4q\left(\frac{-3}{4p}\right)^{3/2} = \frac{q}{2}\left(\frac{-27}{p^3}\right)^{1/2}. \quad (4)$$

These equations can always be solved if p is negative, and

$$\left|\frac{q}{2}\left(\frac{-27}{p^3}\right)^{1/2}\right| < 1 .$$

This last condition reduces to $-4p^3-27q^2 = \Delta > 0$, and so is satisfied in the cases under consideration.[17]

If θ is the smallest angle satisfying equation (4), then the values $\theta + 120°$ and $\theta + 240°$ also satisfy it, so the roots of the equation $x^3 + px + q = 0$ are

$$\frac{1}{n}\cos\theta \qquad \frac{1}{n}\cos(\theta + 120°) \qquad \frac{1}{n}\cos(\theta + 240°),$$

correct to a number of decimal places depending on the tables used.

Solving the Irreducible Case by Cardan's Method.

The equation having the three commensurable roots a, b, c is $x^3-(a+b+c)x^2+(ab+ac+bc)x-abc=0$.

Reduce the roots of this equation by $\frac{1}{3}(a+b+c)$, we have

$$y^3-\frac{1}{3}(a^2+b^2+c^2-ab-ac-bc)y-\frac{1}{27}(2a^3+2b^3+2c^3-3a^2b$$
$$-3a^2c-3ab^2-3ac^2-3b^2c-3bc^2+12abc)=0.$$

This being of the form $y^3+my+n=0$.

Substituting in Cardan's formula and reducing results in

$$y=\frac{1}{3}\sqrt[3]{-\frac{27n}{2}+\frac{3}{2}(a-b)(a-c)(b-c)\sqrt{-3}}.$$

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17 W. V. Lovitt, Elementary Theory of Equations (New York: Prentice-Hall, Inc., 1939), p. 99.

Set the right-hand side equal to $u+v$. "Since u is a binomial imaginary, its cube root will be of the type $\alpha + \sqrt{-\beta}$ and $\sqrt{\beta/3}$ will be rational."[18] Hence

$$\alpha(\alpha^2 - 3\beta) = 1/2(2a^3 + 2b^3 + 2c^3 - 3a^2b - 3a^2c - 3ab^2 - 3ac^2 - 3b^2c - 3bc^2 + 12abc), \quad (1)$$

$$\sqrt{-\beta}(3\alpha^2 - \beta) = 3/2(a-b)(a-c)(b-c)\sqrt{-3} = r. \quad (2)$$

Since α must be rational, $\sqrt{-\beta}$ must be of the first degree with reference to a, b, c, and the only factors of r of that degree are of the form

$\dfrac{(a-b)\sqrt{-3}}{p}$ where p is some integer. However, p must be 2, for substituting

$\sqrt{-\beta} = \dfrac{(b-c)\sqrt{-3}}{p}$ in (2) and reducing we have $\alpha^2 = p/2(a^2 - ab - ac + bc) + \dfrac{b^2 - 2bc + c^3}{p^2}$ which will not give a rational value to α unless $p = 2$.

Assume $\sqrt{-\beta} = \dfrac{(b-c)\sqrt{-3}}{2}$.

Substitute in (2) and reduced. We find $\alpha = \pm(a - \frac{b+c}{2})$ and by substitution in (1) $\alpha = a - \frac{b+c}{2}$ and similarly for other factors of r.

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18 O. H. Kendall, "Solving the Irreducible Case of Cardan's Method," American Journal of Mathematics, 1:285-87, 1878.

Hence $$u = \tfrac{1}{3}\left(a - \frac{b+c}{2} + \frac{b-c}{2}\sqrt{-3}\right)$$

$$= \tfrac{1}{3}\left(b - \frac{a+c}{2} + \frac{a-c}{2}\sqrt{-3}\right)$$

$$= \tfrac{1}{3}\left(c - \frac{a+b}{2} + \frac{a-b}{2}\sqrt{-3}\right).$$

Similarly
$$v = \tfrac{1}{3}\left(a - \frac{b+c}{2} - \frac{b-c}{2}\sqrt{-3}\right)$$

$$= \tfrac{1}{3}\left(b - \frac{a+c}{2} - \frac{a-c}{2}\sqrt{-3}\right)$$

$$= \tfrac{1}{3}\left(c - \frac{a+b}{2} - \frac{a-b}{2}\sqrt{-3}\right).$$

and $$y = \tfrac{1}{3}(2a - b - c)$$

$$= \tfrac{1}{3}(2b - a - c)$$

$$= \tfrac{1}{3}(2c - a - b)$$

and $x = a,\ b,\ c.$

If two roots are equal, then $r = 0$ and $-n/_2 = (a - b)^3$ and if two are imaginary $(\gamma \pm \sigma\sqrt{-3})$, then r becomes rational and $-\frac{n}{2} + r \quad (a - \gamma - 3\delta)^3$.

Solution of a Cubic by Symmetric Functions of the Roots.[19]

Assume equation of the cubic in the form

$ax^3 + 3bx^2 + 3cx + d = 0$.

Put this general equation under the form

$z^3 + 3Hz + G = 0$ where

$z \equiv ax + b \qquad H \equiv ac - b^2 \qquad G \equiv a^2d - 3abc + 2b^3$.

Since the three values of the expression

$$1/3\left[\alpha + \beta + \gamma + (\alpha + \omega\beta + \omega^2\gamma)\theta + (\alpha + \omega^2\beta + \omega\gamma)\theta^2\right]$$

when θ takes the values 1, w, w^2 are α, β, γ it is plain that if the functions $(\alpha + \omega\beta + \omega^2\gamma)\theta$ and $(\alpha + \omega^2\beta + \omega\gamma)\theta^2$ were expressed in terms of the coefficients of the cubic, we could by substituting their values in the formula given above, arrive at an algebraical solution of the cubic equation. This cannot be done by solving directly a quadratic because the sum of the two functions above written is not a rational symmetric function of α, β, γ. Take the cubes of the two functions in question which can be expressed in terms of the coefficients.

For convenience $\qquad L \equiv (\alpha + \omega\beta + \omega^2\gamma)$

$\qquad M \equiv (\alpha + \omega^2\beta + \omega\gamma)$.

19 W. S. Burnside and A. W. Panton, Theory of Equations (Vol. 1; seventh edition; Dublin: Hodges, Figgis, and Company, Ltd., 1912), p. 113.

Then

$(\theta L)^3 = A + B\omega + C\omega^2$

$(\theta^2 M)^3 = A + B\omega^2 + C\omega$, where

$A = (\alpha^3 + \beta^3 + \gamma^3 + 6\alpha\beta\gamma)$,

$B = 3(\alpha^2\beta + \beta^2\gamma + \alpha\gamma^2)$,

$C \quad 3(\alpha\beta^2 + \beta\gamma^2 + \alpha^2\gamma)$. From which we obtain

$$L^3 + M^3 = 2\sum\alpha^3 - 3\sum\alpha^2\beta + 12\,\alpha\beta\gamma - \frac{27G}{a^3} ,$$

where the symbol $\sum$ signifies that one is to take the sum of all terms, like the one following the symbol, that can be formed from the given variables by permutations of those variables.

Again

$$(\theta L)(\theta^2 M) = LM = \alpha^2 + \beta^2 + \gamma^2 - \beta\gamma - \alpha\gamma - \alpha\beta = -\frac{9H}{a^2}$$

whence

$(\alpha + \omega\beta + \omega^2\gamma)^3$ and $(\alpha + \omega^2\beta + \omega\gamma)^3$ are the roots of the quadratic equation

$$t^2 + \frac{3^3G}{a^3}\,t - \frac{3^6H^2}{a^6} = 0 .$$

Denoting the roots of this equation by t_1 and t_2

then $\dfrac{3^3}{2a^3}\left(-G \pm \sqrt{G^2 + 4H^3}\right)$.

The original formula expressed in terms of the coefficients of the cubic gives the three roots.

$$\alpha = \frac{-b}{a} + \frac{1}{3} \left(\sqrt[3]{t_1} + \sqrt[3]{t_2} \right)$$

$$\beta = \frac{-b}{a} + \frac{1}{3} \left(\omega \sqrt[3]{t_1} + \omega^2 \sqrt[3]{t_2} \right)$$

$$\gamma = \frac{-b}{a} + \frac{1}{3} \left(\omega^2 \sqrt[3]{t_1} + \omega \sqrt[3]{t_2} \right) .$$

CHAPTER III

Solution of the Quartic Equation

Ferrari's Solution.[20]

Let the equation of the quartic be of the form

(1) $$x^4 + bx^3 + cx^2 + dx + e = 0.$$

(2) Transposing terms, we have $x^4 + bx^3 = -cx^2 - dx - e.$

Completing the square in the left member results in

$$(x^2 + \tfrac{1}{2}bx)^2 = (\tfrac{1}{4}b^2 - c)x^2 - dx - e.$$

Adding $(x^2 + \frac{1}{2}bx)y + \frac{1}{4}y^2$ to each member leads to

(3) $$(x^2 + \tfrac{1}{2}bx + \tfrac{1}{2}y)^2 = (\tfrac{1}{4}b^2 - c + y)x^2 + (\tfrac{1}{2}by - d)x + \tfrac{1}{4}y^2 - e.$$

The second member of (3) is a perfect square of a linear function of x, if and only if, its discriminant is zero,

$$(\tfrac{1}{2}by - d)^2 - 4(\tfrac{1}{4}b^2 - c + y)(\tfrac{1}{4}y^2 - e) = 0,$$

which may be written in the form

(4) $$y^3 - cy^2 + (bd - 4e)y - b^2e + 4ce - d^2 = 0.$$

Choose any root y of the resolvent cubic (4), then the right member of (3) is the square of a linear function, say $mx + n$.

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20 L. E. Dickson, First Course in Theory of Equations (New York: John Wiley and Sons, Inc., 1922), p. 50.

Thus

(5) $$x^2 + \tfrac{1}{2}bx + \tfrac{1}{2}y = mx + n \text{ or}$$

$$x^2 + \tfrac{1}{2}bx + \tfrac{1}{2}y = -mx - n.$$

The roots of these quadratic equations are the four roots of (3) and hence of the equation (1).

Descartes' Solution[21]

The general quartic of the form

$$ax^4 + bx^3 + cx^2 + dx + e = 0$$

can be reduced to

(1) $$z^4 + qz^2 + rz + s = 0$$

The left member of (1) can be expressed as the product of two quadratic factors

$$(z^2 + 2kz + \ell)(z^2 + 2kz + m) =$$
$$z^4 + (\ell + m - 4k^2)z^2 + 2k(m - \ell)z + \ell m,$$

where
$$q = \ell + m - 4k^2,$$
$$r = 2k(m - \ell),$$
$$s = \ell m.$$

If $k \neq 0$, the first two give

$$2\ell = q + 4k^2 - \frac{r}{2k},$$

$$2m = q + 4k^2 + \frac{r}{2k}.$$

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21 L. E. Dickson, <u>First Course in Theory of Equations</u> (New York: John Wiley and Sons, Inc., 1922), p. 52.

Substituting the values of ℓ and m in $(2\ell)(2m) = 4s$, results in

(2) $$64k^6 + 32qk^4 + 4(q^2 - 4s)\,k^2 - r^2 = 0,$$

which can be solved as a cubic. Any root $k^2 \neq 0$ gives a pair of quadratic factors of equation (1)

$$z^2 \pm 2kz + \tfrac{1}{2}q + 2k^2 \mp \frac{r}{4k}\,.$$

The four roots of these two quadratic functions are the four roots of equation (1).

Euler's Solution[22]

Assume the quartic in the form

(1) $$ax^4 + 4bx^3 + 6cx^2 + 4dx + e = 0.$$

(2) Let $z = ax + b$.

By eliminating x between (1) and (2) we obtain the equation

(3) $$z^4 + 6Hz^2 + 6Gz + a^2I + 3H^2 = 0.$$

Euler assumed a root of the form

$$z = \sqrt{p} + \sqrt{q} + \sqrt{r}.$$

Squaring twice and reducing by means of the relation above, we obtain

$$z^4 - 2(p+q+r)z^2 - 8z(\sqrt{p}\,\sqrt{q}\,\sqrt{r}) + (p+q+r)^2 - 4(qr+pr+pq) = 0.$$

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22 J. D. Hutchinson, "An Analysis of Several of the Methods of Solution of the General Quartic Equation from the point of view of Resolvent Functions, Including an Illustration of Lagrange's Theorem," (unpublished Master's thesis, The University of Illinois, Urbana, 1930), p. 11.

Comparing this equation with (3) we obtain

$$p+q+r = -3H,$$

(4) $$qr + pr + pq = 3H^2 - \frac{a^2I}{4},$$
$$\sqrt{p}\sqrt{q}\sqrt{r} = -\frac{G}{2}.$$

From (4) we see that p, q, r are the roots of the equation

(5) $$\theta^3 + 3H\theta^2 + \left(3H^2 - \frac{a^2I}{4}\right)\theta - \frac{G^2}{4} = 0$$

where $G^2 = 4H^3 - a^2HI + a^3J$.

The three values of θ from (5), together with equations (2) and (4) and the relation

$$z = \sqrt{p} + \sqrt{q} + \sqrt{r}$$

determine the four roots of the given equation.

Solution of the Quartic by symmetric Functions of the Roots.[23]

Let $ax^4 + 4bx^3 + 6cx^2 + 4dx + e = 0$ be the quartic.

The solution of a quartic equation can be reduced to that of a cubic by forming a function of the four roots of the quartic $\alpha, \beta, \gamma, \delta$ which admits only three values under the twenty-four permutations of $\alpha, \beta, \gamma, \delta$.

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23 W. S. Burnside, A. W. Panton, The Theory of Equations (seventh edition; Vol. 2; Dublin: Hodges, Figgis, and Company, Ltd., 1912), p. 139.

We proceed to form the equation whose roots are the three values of

$$t \equiv \frac{(\alpha-\beta+\gamma-\delta)^2}{4}$$

$$t_1 \equiv \frac{(\beta+\gamma-\alpha-\delta)^2}{4} \qquad t_2 \equiv \frac{(\gamma+\alpha-\beta-\delta)^2}{4}$$

$$t_3 \equiv \frac{(\alpha+\beta-\gamma-\delta)^2}{4}$$

and since

$$(\beta+\gamma-\alpha-\delta)^2 \equiv \sum\alpha^2 + 2\lambda - 2u - 2v$$

$$\sum(\alpha-\beta)^2 \equiv 3\sum\alpha^2 - 2\lambda - 2u - 2v - \frac{48H}{a^2}$$

where $H = ac - b^2$,

$\lambda = (\beta\gamma + \alpha\delta)$,

$U = -(\alpha\beta + \gamma\delta)$,

$V = -(\beta\delta + \alpha\gamma)$.

We find the following values of t_1, t_2, t_3

$$t_1 \equiv \frac{2\lambda - u - v}{12} - \frac{H}{a^2}, \qquad t_2 \equiv \frac{2u - v - \lambda}{12} - \frac{H}{a^2}$$

$$t_3 \equiv \frac{2v - \lambda - u}{12} - \frac{H}{a^2}.$$

So

$$t_1 + t_2 + t_3 = -\frac{3H}{a^2},$$

$$t_1t_2 + t_1t_3 + t_2t_3 = \frac{3H^2}{a^4} - \frac{1}{96}\sum(u-v)^2 = \frac{3H^2}{a^4} - \frac{I}{4a^2},$$

$$t_1t_2t_3 = \frac{G^2}{4a^6}.$$ Where $I \equiv ae - 4bd + 3c^2$,

$G \equiv a^2d - 3abc + 2b^3$,

$J \equiv ace + 2bcd - ad^2 - eb^2 - c^3$.

Then the equation whose roots are t_1, t_2, t_3 is

$$(a^2t)^3 + 3H(a^2t)^2 + \left(3H^2 - \frac{a^2I}{4}\right)(a^2t) - \frac{G^2}{4} = 0.$$

We substitute G for its value

$$4(a^2t + H)^3 - a^2I(a^2t + H) + a^3J = 0$$

which can be transformed into the standard reducing cubic by the substitution $(a^2t + H) = -a^2$.

To determine α, β, γ, δ we have the following equations

$$\begin{aligned} -\alpha + \beta + \gamma - \delta &= 4\sqrt{t_1}\,, \\ \alpha - \beta + \gamma - \delta &= 4\sqrt{t_2}\,, \\ \alpha + \beta - \gamma - \delta &= 4\sqrt{t_3}\,, \\ \alpha + \beta + \gamma + \delta &= -\frac{4b}{a}\,. \end{aligned}$$

From which we find

$$\alpha = -\frac{b}{a} - \sqrt{t_1} + \sqrt{t_2} + \sqrt{t_3}, \qquad \gamma = -\frac{b}{a} + \sqrt{t_1} + \sqrt{t_2} - \sqrt{t_3},$$

$$\beta = -\frac{b}{a} + \sqrt{t_1} - \sqrt{t_2} + \sqrt{t_3}, \qquad \delta = -\frac{b}{a} - \sqrt{t_1} - \sqrt{t_2} - \sqrt{t_3}.$$

We also find that $\sqrt{t_1}\,\sqrt{t_2}\,\sqrt{t_3} = \dfrac{G}{2a^3}$ by means of which one radical can be expressed in terms of the other two.

A Solution of the Quartic Equation by Means of a Twenty-four Valued Function of the Roots

We assume the quartic equation in the form

(1) $x^4 + ax^3 + bx^2 + cx + d = 0.$

We call its roots α, β, γ and δ and consider the twenty-four valued function

$$v = v_1 = (\alpha - \beta + i\gamma - i\delta) \text{ where } i^2 = -1$$

"Under the subgroup E_4 of G_{24}^4, v, takes the values."[24]

$$V_1 = \alpha - \beta + i\gamma - i\delta \qquad V_3 = \gamma - \delta + i\delta - i\beta$$
$$V_2 = \beta + \alpha - i\gamma + i\delta \qquad V_4 = \delta - \gamma + i\beta - i\alpha .$$

These four functions are the roots of the equation

(2) $$z^4 - (v_1^2 + v_3^2)\, z^2 + v_1^2 v_3^2 = 0.$$

The coefficients of equation (2) may be determined in terms of the coefficients of (1)

Then $$v_1^2 + v_3^2 = 4iy_1.$$

$$v_1^2 \quad v_3^2 = -\tfrac{1}{9}\left[\, 3a^2 - 8b - 2(y_2 - y_3)\right]^2$$

where
$$y_1 = (\alpha - \beta)(\gamma - \delta)$$
$$y_2 = (\beta - \gamma)(\alpha - \delta)$$
$$y_3 = (\gamma - \alpha)(\beta - \delta) .$$

However

$(y_2 - y_3)$ can be expressed in terms of y_1 for y_1, y_2, y_3 are roots of a certain cubic equation of the form $y^3 + c_2 y + c_3 = 0$.

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24 J. D. Hutchinson, "An Analysis of Several of the Methods of Solution of the General Quartic Equation from the point of View of Resolvent Functions, Including an Illustration of Lagrange's Theorem," (Unpublished Master's thesis, The University of Illinois, Urbana, 1930), p. 15.

We know that

(3) $$y_2 - y_3 = \frac{-432J}{(y_1 - y_2)(y_1 - y_3)} = \frac{-432J}{y_1^2 - y_1(y_2+y_3) + y_2 y_3}.$$

Let

$$\psi(y) = \frac{y^3 + c_2 y + c_3}{y - y_1} = y^2 - y_1 y + c_2 + y_1^2.$$

Then

$\psi(y)$ has roots y_2 and y_3, therefore

$-y_1 = y_2 + y_3$ and $y_2 y_3 = y_1^2 + c_2$.

Then expression (3) becomes

$$y_2 - y_3 = \frac{-432J}{3y_1^2 + c_2} \equiv P$$

where

$$c_2 \sum y_1 y_2 = 3ac - 12d - b^2.$$

The quartic (2) becomes

(4) $$z^4 - 4iy_1 z^2 + 1/9\left[3a^2 - 8b + \frac{2(432J)}{3y_1^2 + c_2}\right]^2.$$

If in equation (4) we set $Z^2 = \theta$ we have a quadratic, whose roots we may call θ_1 and θ_2. We have then the equations

(5) $$\begin{aligned} V_1 &= \theta_1 \\ V_3 &= \theta_2 \\ \alpha+\beta+\gamma+\delta &= -a. \end{aligned}$$

By using the other twenty values for v one can obtain another independent relation of α, β, γ and δ. This sixth equation, together with equations (5), form a system of equations which can be solved for the roots of the quartic, α, β, γ and δ.

CHAPTER IV

Higher Degree Equations

The theory of substitutions and groups of substitutions grew out of the investigation by Lagrange, Ruffini and Abel concerning the question of solvability by radicals of the general algebraic equation of degree n.

Galois"Theory, which is applicable to any algebraic equation, whether its coefficients are constants or depend upon one or more variables, establishes the modern theory of equations in a satisfactory manner.[25] To Galois, the solvability of any equation of n^{th} degree by radicals depends on the discovery that to each equation there corresponds a group of substitutions, which leaves the function unchanged, known as the group of the equation or Galois group. According to Galois' theory, given an equation we shall associate a group of substitutions on its roots. Then the algebraic equation is solvable by radicals if, and only if, the group is solvable.

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25 G. A. Miller, H. F. Blichfeldt, and L. E. Dickson, Theory and Applications of Finite Groups (New York: John Wiley and Sons, Inc., 1916), p. 279.

Before describing the group of the equation we define the domain of rationality R. If we denote the constants or variables of a given problem by R^{1}, R^{11}, . . ., R^{u} together with all quantities derived from them by a finite number of additions, substractions, multiplications and divisions (except by zero), the resulting system of quantities is called the domain of rationality.

The set of all substitutions (on the roots $x_1, x_2, \ldots, x_n$ of the equation) satisfying properties a and b, listed below, form a unique group G of order N. This is called the group of the equation.

a Every rational function $(x_1, x_2, \ldots, x_n)$ of the roots which remains unaltered by all substitutions of G lies in the domain of rationality R.

b Every rational function $(x_1, x_2, \ldots, x_n)$ if the roots which equals a quantity in R remains unaltered by all the substitutions of G.

An integral rational function $f(x)$ of degree n of a variable x whose coefficients belong to the domain R is said to be reducible in R if it can be expressed as a product of integral rational functions of x, each of degree less than n, with coefficients in R; irreducible if no such factorization is possible.

For better understanding let

(1) $f(x) \equiv x^n - c_1x^{n-1} + c_2x^{n-2} - \dots + (-1)^n c_n = 0$

whose coefficients belong to the domain R. We assume that the roots $x_1, x_2, \dots, x_n$ are all distinct. It is possible to construct a rational function v_1 of the roots with coefficients in R such that v_1 takes $n!$ distinct values under the $n!$ substitutions on $x_1 \dots, x_n$. Such a function is

(2) $v_1 \equiv m_1x_1 + m_2x_2 + \dots + m_n \ x_n$

where $m_1, m_2, \dots, m_n$ are properly chosen in R. Then the $n!$ values of the function v_1 are the roots of an equation

$$F(v) \equiv (v-v_1)(v-v_2) \dots (v-v_{n!}) = 0,$$

whose coefficients are integral rational functions of the m's in (2) and the c's in (1), with integral coefficients. Hence the v's belong to R.

If $F(v)$ is reducible in R, we let $F_0(v)$ be that irreducible factor for which $F_0(v_1) = 0$, if $F(v)$ is irreducible in R, let $F_0(v)$ be $F(v)$ itself. Then $F_0(v) = 0$ is an irreducible equation called the Galois resolvent of equation (1).

Next, we let the roots of this resolvent be v_1, $v_a, v_b, \dots, v_k$

The substitutions by which they are derived from v, are

1. a, b, ..., l

which form a group G which is the group of the given equation (1) with respect to the domain R.
The group of a given equation for a given domain is unique. In particular the group of an equation is independent of the special $n!$ - valued function v_1 chosen.[26]
The group of the general equation of degree n whose coefficients and roots are independent variables is the symmetric group $Gn!$. The group of the equation is solvable if it has a composition series in which the indices are all prime numbers.

Numerical Methods of Approximation of the Roots of an Equation.

Finding roots of numbers, the solution of equations and even the approximation methods go as far back as the early Egyptian and Babilonian civilization. Babylonians tried to find the solution of equations by the method of False Position (Regula Falsi) which is the oldest one.[27]

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26 L. Dickson, Introduction to the Theory of Algebraic Equations (first edition; New York: John Wiley and Sons, Inc., 1903), p. 55.

27 J. B. Scarborough, Numerical Mathematical Analysis (Baltimore: The John Hopkins Press, 1930), p. 174.

To solve the equation $x + \frac{x}{7} = 19$, the unknown number x was assumed to be seven. The sum of the number and its seventh part was eight and the number solution of the equation is the same multiple of seven that nineteen is of the guessed number eight.[28]

Chuquet used the rule of mean numbers which is illustrated in the following example

$$x^2 + x = 39\frac{13}{81}$$

Let $x = 5$ and substitute in the equation. It is too small. Let $x = 6$. It is too big. We write these two numbers in rational form to obtain the first mean.

First mean $\frac{5 + 6}{1 + 1} = \frac{11}{2}$ By substitution we see it is too small.

New bounds $\frac{11}{2}$ and $\frac{6}{1}$.

Second mean $\frac{11 + 6}{2 + 1} = \frac{17}{3}$. It is too small.

New bounds $\frac{17}{3}$ and $\frac{6}{1}$.

Third mean $\frac{17 + 6}{3 + 1} = \frac{23}{4}$. It is too small.

New bounds $\frac{23}{4}$ and $\frac{6}{1}$.

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28 V. Sanford, A Short History of Mathematics (New York: Houghton Mifflin Company, 1930), p. 160.

Fourth mean $\frac{23+6}{4+1} = \frac{29}{5}$. It is large.

New bounds $\frac{23}{4}$ and $\frac{29}{5}$.

Fifth mean $\frac{23+29}{4+5} = \frac{52}{9}$ which is the exact root.[29]

The Regula Aurea (Golden Rule) of Cardan published in his Ars Magna (1545) was built on the basis of two false positions and a particular mode of interpolation. He used it for equations of third and fourth degree, but it is applicable to equations of every degree.[30] To solve an equation by this method of double false position we let the equation be

(1) $$f(x) = v.$$

We assume for the moment two values, say a and b. Then we determine the errors by substituting a and b in (1) and we write $f(a) = A$ and $f(b) = B$.

Next we compute the error for a and b,

$$E_a = V - A \quad \text{and } E_b = V - B.$$

Then an approximation to the value of x is x $\frac{bE_a - aE_b}{E_a - E_b}$. But it is accurate whenever $f(x) = V$ is a linear function of x.

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29 M. A. Nordgaard, A Historical Survey of Algebraic Methods of Approximation the Roots of Numerical Higher Degree Equations up to the Year 1819 (New York: Columbia University 1922), p. 6.

30 F. Cajori, A History of Mathematics (New York: The MacMillan Company, 1919), p. 103.

Newton's method is applicable to any equation $f(x) = 0$, whether $f(x)$ is a polynomial or not as long as $f(x)$ is differentiable.

First determine two numbers a and b $(a < b)$ such that there is one and only one root of $f(x) = 0$ between them, We find a closer approximation $a + h$ to the root by neglecting the powers h^2, h^3 . . . of the small number h in Taylor's formula.

$$f(a+h) = f(a) + f'(a)h + \frac{f''(a)h^2}{2!} + \frac{f'''(a)h^3}{3!} \ldots \text{ and}$$

hence by taking $f(a) + f'(a)h = 0$

$$h = -\frac{f(a)}{f'(a)} \text{ .}$$

We repeat the process with $a_1 = a + h$ in place of the former a.

Numerical example.[31]

$$f(x) \equiv x^3 - 2x - 5 = 0.$$

For $a = 2$ $\qquad h = -\frac{(2)}{f'(2)} = \frac{1}{10}$.

$$a_1 = a + h$$
$$a_1 = 2 + .1 \text{ .}$$

For $a_1 = 2.1$ $\qquad h = -\frac{f(2.1)}{f'(2.1)} = -.0054$

$$a_2 = a_1 + h_1$$
$$a_2 = 2.1 - .0054$$

For $a_2 = 2.0946$ $\qquad h_2 = -\frac{f(2.0946)}{f'(2.0946)} = -.00004852$

31 L. D. Dickson, First Course in Theory of Equations (New York: John Wiley and Sons, 1922), p. 91.

$$a_3 = a_2 + h_2$$

$$a_3 = 2.0946 - .00004852$$

$a_3 = 2.09455148$ in which seven decimal places are correct.

A Modification of Newton's Method.[32]

Newton's formula for approximating the roots of an equation $f(x) = 0$, namely,

(1) $$x_{p+1} = x_p - \frac{f(x_p)}{f'(x_p)} \qquad (p = 0, 1, 2, \ldots)$$

may be modified in the following manner. The equation of the parabola through the point $[x_p, f(x_p)]$ having the same first and second derivatives at x x_p as y $f(x)$ is $y = f(x_p) + (x - x_p)\, f'(x_p) + \frac{1}{2}(x - x_p)^2\, f''(x_p)$.
Let x_{p+1} be a solution of the equation which results if we put $y = 0$. Then $x_{p+1} = \dfrac{-f(x_p)}{f'(x_p) + \frac{1}{2}(x_{p+1} - x_p) f''(x_p)}$

If we take in this formula

$$x_{p+1} - x_p = \frac{-f(x_p)}{f'(x_p)}$$ we obtain

(2) $$x_{p+1} = \frac{-f(x_p)}{f'(x_p) - \dfrac{f(x_p)\, f''(x_p)}{2f'(x_p)}} \qquad (p = 0, 1, 2 \ldots)$$

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32 H. S. Wall, "A Modification of Newton's Method," The American Mathematical Monthly, 55:90-94, February, 1948.

which is the desired modification of Newton's formula. The convergence of the sequence $\{x_p\}$ is more rapid in the case of formula (2) than in the case of formula(1).

Newton's method has a serious defect; it is inoperative if two roots are close together, for then the series would not converge, and might after a while actually diverge. In 1767 Lagrange announced a new method. It consists of three parts.

1 A method of finding the integral part of the root.

2 A rule for separating roots.

3 Technique of approximation by the use of continued fractions.

Suppose that the equation $f(x)$ 0 has a real positive root between p and $(p+1)$. Then $x = p + \frac{1}{y}$ and $y > 1$. Substituting the value of x in $f(x) = 0$ gives the equation $\phi(y) = 0$. Since $y > 1$ we find its integral value by the prescribed method. Suppose it lies between integers q and $(q+1)$. Then $y = q + \frac{1}{z}$ and so on.

$$x = p + \cfrac{1}{q + \cfrac{1}{r + \cdots}}$$

Numerical example

$$x^3 - 2x - 5 = 0$$

One root lies between 2 and 3.

Let $x = 2 + \frac{1}{y}$; then the equation becomes $y^3 - 10y^2 - 6y - 1 = 0$. We find where y lies $10 < y < 11$. Then $y = 10 + \frac{1}{z}$ and we get

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an equation in z

$$61z^3 - 92z^2 - 20z - 1 = 0.$$

We find where z lies $1 < z < 2.$

Then $z = 1 + \frac{1}{u}$ gives the equation $54u^3 + 25u^2 - 89u - 61 = 0.$ We find where u lies, $1 < u < 2$. A continuation of this process gives the series 2, 10, 1, 1, 2, 1, 3, 1, 1, 12. Hence

$$x = 2 + \cfrac{1}{10 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \cdots}}}}$$

Evaluating we have

$$x = \frac{16415}{7837} = 2.09455149.$$[33]

In Horner's method the first step in finding the numerical value of a real root of a rational integral algebraic equation is to isolate the root. The root is obtained digit by digit, in the successive order of decimal places; that means, first the coefficient of the highest power of ten and then the other coefficients, till any desired number of places of accuracy. His method is based on the next two theorems.

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33 M. A. Nordgaard, A Historical Survey of Algebraic Methods of Approximating the Roots of Numerical Higher Degree Equations up to the Year 1819 (New York: Columbia University, 1922), p. 50.

1 If the first member of an equation of the form $f(x) = 0$ be divided by $(x - a)$, then the quotient be again divided by $(x - a)$, and so on, the successive remainders will be, in reverse order, the coefficients of an equation whose roots are less by a than those of the given equation.[34]

2 When the root of an equation is small, with respect to all the coefficients, it is approximately equal to the absolute term divided by the coefficient of the first power of x.

Numerical Example

$f(x) \equiv x^3 \quad x^2 - 6x - 1 \quad 0$

$f(0) = -$

$f(1) = -$

$f(2) = -$

$f(3) = +$. So there is a root between 2 and 3. We diminish the roots of the equation by 2

1	1	-6	-1	2
	2	6	0	
1	3	0	-1	
	2	10		
1	5	10		
	2			
1	7			

The first transformed equation is

34 J. Downey, Higher Algebra (New York: American Book Co., 1901), p. 331.

$f_1(x_1) = x_1^3 + 7\,x_1^2 + 10x_1 - 1 = 0$ which has a root between 0 and 1. We obtain, from $10x_1 = 1$, the approximate value $x_1 = .1$.

This value of x_1 makes the first two terms positive and $f_1(.1) > 0$ hence the constant term in the second transformed equation would be positive which shows that the value of x_1 is too large. The constant term in each transformed equation must retain the same sign as the constant term in the original equation.

For $x_1 = .09$, $f_1(x_1) < 0$. Now we diminish the roots of $f_1(x_1) = 0$ by .09 which gives us the second transformed equation

$$x_2^3 + 7.27x_2^2 + 11.2843x_2 - .042571 = 0.$$

From the two last terms we get $x_2 = .003$. We diminish the roots of $f_2(x_2) = 0$ by .003 which gives the third transformed equation

$$x_3^3 + 7.279x_3^2 + 11.327947x_3 - .008652643 = 0.$$

From the two last terms we find

$$.0007 < x_3 < .0008 .$$

Whence
$.000003570 < (x_3^3 \quad 7.279x_3^2) < .000004664$. We ignore the first two terms provided the constant term is reduced by an amount between these limits.

$.008652643 - .000003570 = .008649073$

$.008652643 - .000004664 = .008647979$.

From
$11.327947x_3 - .008647979 = 0$ we obtain

$$x_3 = .0007634 +$$.

From
$11.327947x_3$ - $.008649073$ 0 we obtain

$$x_3 = .0007635 +$$.

Therefore correct to six decimal places we have

$$x_3 = .0007634$$.

This can be shortened in this way

$$11.3\dot{2}7947 \overline{\smash{)}\,.008647} \quad \begin{matrix} .0007634 \end{matrix}$$.

Since the quotient is .0007 we use only two decimals in the divisor, except by inspection, to see how much should be carried in making the first multiplication. Place a dot above the figure 2 in the divisor and use 11.32 as a divisor. Before multiplying by 6, the second significant figure in the quotient, place a dot over the figure 3 and use 11.3. For the root of the original equation we have $x = 2.0937634+$ where the six first decimal places are correct. There is doubt as to whether the last figure should be 4 or 5. If more decimals are required, it is not necessary to form a new transformed equation. We need to revise the constant term in

$f_3(x_3) = 0$ making use of our present better value of x_3. "This contracted method may be used after three or four decimals have been found."[35]

Horner's Method Shortened.[36]

When the real root of an equation above the second degree is wanted accurately, Horner's method is the old standby. But it is very laborious and the work increases with each digit. The following shows how to get an answer to two, and often three, more decimal places for a given number of transformations. After the original equation is transformed to a new equation one of whose roots lie between 0 and 1 it is of the type ending in

(1) $$. . . bx^2 + cx + k = 0.$$

Since $x < 1$, the square, cube and higher terms are small so that a rough value for the root can be obtained

(2) $$cx + k = 0 \quad \text{where } x = -\frac{k}{c}.$$

This is sometimes useful, but since it is a linear approximation it may be a poor fit to a curve with a distinct curvature. A curvilinear approximation would be obviously

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35 W. V. Lovitt, Elementary Theory of Equations (New York: Prentice Hall, Inc., 1939), p. 135.

36 H. D. Hatch, "Horner's Method Shortened," School, Science and Mathematics, 36:1007-8, December, 1936.

better and can be gotten as follows:

Consider the result of dividing

$$c^2x \text{ by } (c - bx).$$

The quotient $cx + bx^2 + \frac{b^2x^3}{c} + \ldots$ is a series which is convergent for values of x between 1 and -1. For such values the cube and higher powers are small and $\frac{c^2x}{c - bx}$ is a good approximation for $bx^2 + cx$.

Let us substitute then for the last two x terms of (1)

$$\frac{c^2x}{c - bx} + k = 0.$$

Then

$$c^2x + kc - kbx = 0$$

$$x(c^2 - kb) = - kc$$

$$x = \frac{kc}{kb - c^2}.$$ There is less to calculate with this than if we continue

$$\frac{1}{x} = \frac{kb - c^2}{kc} \text{ and finally}$$

(3) $\frac{1}{x} = \frac{b}{c} - \frac{c}{k}.$ This is only an approximation because the cube and higher terms have been omitted, but it becomes increasingly accurate as the omitted terms become small.

Note: In the thirteenth century a Chinese employed a method of approximation, virtually the same as Horner's method." The Chinese method did not pass into the living

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stream because neither in the Orient nor in Europe did it start a forward movement."[37]

In 1804 Paolo Ruffini invented a similar method in Italy which was soon forgotten and in 1819 the same procedure was reinvented by Horner.

Graeffe's Method

Of the many methods which have been proposed for solving algebraic equations the most practical one, where complex roots are concerned, is the one known root squaring method usually referred to as Graeffe's method even though the astronomer J. F. Encke was an early exponent of this process and did all he could to make it well known.[38] It was suggested independently by Dandelin in 1826, Lobachevsky in 1834 and Graeffe in 1837. But Dandelin's work was not widely circulated and the process went under the name of Carl Graeffe who published it as a prize paper.[39] This method has the advantage

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37 E. T. Bell, The Development of Mathematics (second edition; New York: McGraw-Hill Book Company, Inc., 1945), p. 116

38 D. H. Lehmer, "The Graeffe Process as Applied to Power Series," Mathematical Tables and Aids to Computation, 1:377-83, 1943-45.

39 C. A. Hutchinson, "On Graeffe's Method for the Numerical Solution of Algebraic equations," The American Mathematical Monthly, 42:149-61, March, 1935.

of finding all the roots at once and not requiring any preliminary determination of their approximate position. Its principle, for an equation with only real roots, is to form a new equation whose roots are some high power of the roots of the given equation. Suppose we say the 128^{th} power, so that if the roots of the given equation are $x_1, x_2, x_3, \ldots, x_n$ then, the roots of the new equation are $x_1^{128}, x_2^{128}, x_3^{128}, \ldots, x_n^{128}$. These numbers are widely separated; thus if x_1 were twice x_2, then x_1^{128} would be more than 10^{38} times x_2^{128}. The advantage of an equation whose roots are very widely separated is that it can be solved at once numerically.

Let the equation be

(1) $$x^n + a_1x^{n-1} + a_2^{n-2} + \cdots + a_n = 0$$

with real coefficients. We write all terms of even degree on one side of the equation and all terms of odd degree on the other side. Squaring both sides we have

$$(x^n + a_2x^{n-2} + a_4x^{n-4} + \cdots)^2 = (a_1x^{n-1} + a_3x^{n-3} + \cdots)^2 .$$

If $x^2 = -y$.

(2) Then $$y^n + b_1y^{n-1} + b_2y^{n-2} + \cdots + b_{n-1}y + b_n = 0$$

where

$$b1 = a_1^2 - 2a_2$$
$$b_2 = a_2^2 - 2a_1a_3 + 2a_4$$
$$b_3 = a_3^2 - 2a_2a_4 + 2a_1a_5 - 2a_6$$
$$\vdots$$

$\vdots$

$b_k \quad a^2k - 2(a_{k-1})(a_{k+1}) + 2(a_{k-2})(a_{k+2}) + \dots + (-1)^k$

$2a_{2k}$

$\vdots$

$b_n = a_n^2$.

The following rule will give the coefficients of equation (2).

The coefficient of any power of y is formed by squaring the coefficient of the corresponding power of x in the original equation and adding twice the product of every pair of coefficients which are equally distant on either side, these products being taken with signs alternately positive and negative, missing powers of x being supplied with zero coefficients.

We let the roots of the original equation be a, b, c,... Then, the Encke roots of (2) are a^2, b^2, c^2,... The process may be repeated m times giving an equation whose Encke roots are the 2^{mth} powers of the Encke roots of the original equation. The equation whose Encke roots are a^m, b^m, c^m,.. is

$$(x + a^m)(x + b^m)(x + c^m) \dots = 0 \quad \text{or}$$

$$x^n + [a^m]\, x^{n-1} + [a^m b^m]\, x^{n-2} + [a^m b^m c^m]\, x^{n-3} \dots = 0$$

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where

$$[a^m] = a^m + b^m + c^m + \ldots$$

$$[a^m b^m] = a^m b^m + a^m c^m + \ldots + b^m c^m \ldots \text{ etc.}$$

We continue the process until the doubled products bring no change in the digits we wish to retain. Under the assumption that all the roots are real and unequal $|a| > |b| > |c| > \ldots$ if m is sufficiently large the ratio of a^m to $[a^m]$ is approximately one. Likewise the ratio of $a^m b^m$ to $[a^m b^m]$ is approximately one. Then the equation becomes $x^n + a^m x^{n-1} + a^m b^m x^{n-2} + \cdots + (a^m b^m \ldots) = 0$.

The numbrical value of a can be determined from the second coefficient; $|b|$ from the third coefficient and so on. The sign of the actual roots can be checked by Descarte's rule of signs.

Numerical example.

$$x^3 - 2x^2 - 5x + 6 = 0.$$

1	1	-2	-5	6
	1	4	25	36
		10	24	
2	1	14	49	36
	1	196	2401	1296
		-98	-1008	
4	1	98	1393	1296
	1	9.604^3	1.940^6	1.680^6
		-2.786^3	-0.254^6	
8	1	6.818^3	1.686^6	1.6[illegible]^6
	1	4.649^7	2.843^{12}	2.822^{12}
		-0.337^7	-0.023^{12}	
16	1	4.312^7	2.820^{12}	2.822^{12}
	1	1.859^{15}	7.952^{24}	7.964^{24}
		-0.006^{15}	-0.000	
32	1	1.853^{15}	7.952^{24}	7.964^{24}

Determination of absolute value of the roots using

Logarithms

$a^{32} = 1.853^{15}$ $\qquad$ $a^{32}b^{32} = 7.952^{24}$

$\log a^{32} = 15.2679$ $\qquad$ $\log b^{32} = 24.9005 - 15.2679 = 9.6326$

$\log |a| = .4771$ $\qquad$ $\log |b| = .3010$
$a = 3.000$ $\qquad$ $b = 2.000$

$a^{32}b^{32}c^{32} = 7.964^{24}$

$\log c^{32} = 24.9011 - 24.9005 = .0006$

$\log |c| = .0000$
$c = 1.000.$

The original equation must have two positive roots and one negative, so the roots are 3, -2, 1.

This method can be used in the case of complex roots. We take the case of a cubic equation with one real root and a pair of conjugate complex roots.

If the Encke roots are a, $re^{-i\theta}$ where $r > 0$ then the m^{th} power equation is

$(x + a^m)(x^2 + 2r^m \cos m\theta x + r^{2m}) = 0$ or

(1) $x^3 + (a^m + 2r^m \cos m\theta)x^2 + (r^{2m} + 2a^m r^m \cos m\theta)x + a^m r^{2m} = 0.$

If $|a| > r$, and m is large enough, a^m is large compared to $2^{rm} \cos m\theta$ and a can be computed by taking the m^{th} root of the coefficient of x^2, and r, from the constant term $a^m r^{2m}$. In this case the approximate equation is

$x^3 + a^m x^2 + 2a^m r^m + \cos m\theta x + a^m r^{2m} = 0.$

If $|a| < r$, then r^{2m} is large compared to $2a^m r^m \cos m\theta$, and r can be computed by taking the $2m^{th}$ root of the coefficient of x, and a from the constant term.
The approximate equation in this case is

$$x^3 + 2r^m \cos m\theta x^2 + r^{2m}x + a^m r^{2m} = 0.$$

Suppose the complex roots are $u \pm iv$, then u can be computed from the relation $-a_1 = 2u - a$
and u can be computed from the relation $r^2 = u^2 + v^2$.
In this particular case one column of the coefficients shows minus sign after the first row during the process which means that there is a pair of complex roots present.

In the case of a quartic equation two pairs of complex roots may occur. We let the Encke roots of the equation be $re^{\pm i\phi}$ and $se^{\pm i\theta}$ where r and $s > 0$.
The equation of the m^{th} powers of the roots is
$x^4 + 2(r^m \cos m\phi + s^m \cos m\theta)x^3 + (r^{2m} + 4r^m s^m \cos m\phi$
$\cos m\theta + s^{2m})x^2 + 2r^m s^m(r^m \cos m\theta\ s^m \cos m\phi)x + r^{2m} s^{2m} = 0.$
The approximate equations are:
If $r > s$ then
$x^4 + 2r^m \cos m\phi x^3 + r^{2m}x^2 + 2r^{2m}s^m \cos m\theta x + r^{2m}s^{2m} = 0$, and r can be determined from the coefficient of x^2 and then s from the constant term.

In either case we have two columns behaving irregularly with respect to signs; these two columns are separated by one regular column. The complex roots of the original equation can be represented by

$$u_1 \pm iv \quad \text{and} \quad u_2 \pm iv_2$$

u_1 and u_2 can be determined from the equations

$$2u_1 + 2u_2 = -a$$

$$2r_2^2 u_1 + 2r_1^2 u_2 = -a_3 .$$

Then v_1 and v_2 can be determined from these relations

$$r_1^2 = u_1^2 + v_1^2 \quad \text{and} \quad r_2^2 = u_2^2 + v_2^2 .$$

The equation which has three pairs of complex roots, say $u_j \pm iv_j \qquad j = 1, 2, 3$

is

$$x^6 - 2\left[u_1 + u_2 + u_3\right] x^5 + \left[r_1^2 + r_2^2 + r_3^2 + 4u_1u_2 + 4u_1u_3 + 4u_2u_3\right] x^4$$
$$-2\left[2u_1(r_2^2 + r_3^2) + 2u_2(r_1^2 + r_3^2) + 2u_3(r_1^2 + r_2^2) + 8u_1u_2u_3\right] x^3$$
$$+\left[r_1^2r_2^2 + r_1^2r_3^2 + r_2^2r_3^2 + 4(u_1u_2r_3^2 + u_1u_3r_2^2 + u_2u_3r_1^2\right] x^2$$
$$-2\ r_2^2r_3^2u_1 + r_1^2r_3^2u_2 + r_1^2r_2^2u^3\Big] x + r_1^2r_2^2r_3^2 = 0 .$$

We find r_1, r_2, r_3 by Graeffe's method and if we proceed as in the case of two complex roots we must determine u_1, u_2, u_3 from the equations

$$2u_1 + 2u_2 + 2u_3 = -a.$$

$$2(r_2^2 + r_3^2)u_1 + 2(r_1^2 + r_3^2)u_2 + 2(r_1^2 + r_2^2)u_3 + 8u_1u_2u_3 = -a_3$$
$$2r_2^2r_3^2u_1 + 2r_1^2r_3^2u_2 + 2r_1^2r_2^2u_3 = -a_5 .$$

Elimination of u_2, u_3 leads to a cubic in u_1 which we can solve.

For four pairs of complex roots the problem becomes more complicated. A system of simultaneous equations has to be solved for u_1, u_2, u_3, u_4. This can be solved as follows.[40]

$$f(z) = C_0 z^n + C_1 z^{n-1} + \cdots + C_n = 0$$

where $z = x + iy$. We expand by Taylor's series

$$f(x+iy) = f(x) + f^{I}(x)iy - f^{II}\frac{(x)y^2}{2!} - f^{III}\frac{(x)iy^3}{3!} + f^{IV}\frac{(x)y^4}{4!} + \cdots = 0$$

which can be written

(1) $f(x) - f^{II}\frac{(x)y^2}{2!} + f^{IV}\frac{(x)y^4}{4!} = 0$

(2) $f^{I}(x) - f^{III}\frac{(x)y^2}{3!} + f^{V}\frac{(x)y^4}{5!} = 0.$ We substitute

(3) $y^2 = r_3^2 - x^2$ in (2) and we solve for x the resulting equation by Graeffe's method. One of the real roots obtained will be u_3. If the equation has more than one real root, we compute the corresponding values of y for each real x by means of (3) and we substitute each pair in (1).

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40 B. A. Hausmann, "Graeffe's Method, and Complex Roots," The American Mathematical Monthly, 43:225-29, April, 1936.

The pair which satisfies (1) gives the values, namely u_3 and v_3. For more than three pairs of complex roots we repeat the process.

Let $y^2 = r_4^2 - x^2$ to get u_4 and v_4.

To find u_1 and u_2 the best procedure is to use symmetric functions of the roots employing the second and the next to the last terms of the original equation which are linear expressions in u_1 and $u_2 \ldots$ and substituting in them the values for $u_3, u_4 \ldots$ which have been found already.

Numerical example

$$z^6 - 12z^5 + 72z^4 - 262z^3 + 601z^2 - 850z + 650 = 0 \ .$$

Using Graeffe's method we find that all the roots are complex with the square of the absolute values

$$r_1^2 = 13 \qquad\qquad r_2^2 = 10 \qquad\qquad r_3^2 = 5.$$

We use equations (1) and (2)

$$f(x) \equiv x^6 - 12x^5 + 72x^4 - 262x^3 + 601x^2 - 850x + 650 = 0$$

$$f^{I}(x) = 6x^5 - 60x^4 + 288x^3 - 786x^2 + 1202x - 850$$

$$f^{II}(x) = 30x^4 - 240x^3 + 864x^2 - 1572x + 1202$$

$$f^{III}(x) = 120x^3 - 720x^2 + 1720x - 1572$$

$$f^{IV}(x) = 360x^2 - 1440x + 1720$$

$$f^{V}(x) = 720x - 1440$$

$$f^{VI}(x) = 720.$$

From $r^2_3 = 5$

$$y^2 = r_3^2 - x^2$$

$y^2 = 5 - x^2$. We substitute this value in equation (2) together with the values of $f^{I}(x)$, $f^{III}(x)$, etc... which gives the equation

$$E(x) = 32x^5 - 192x^4 + 416x^3 - 328x^2 - 88x + 160 = 0.$$

If we solve it by Graeffe's method we find that the only root which satisfies (1) is one. Hence $u_3 = 1$. We find u_1 and u_2 from these relations

$$2u_1 \quad + \quad 2u_3 \quad + \quad 2u_3 \quad = \quad 12$$

$$2r_2^{\,2} r_3^{\,2} u_1 \quad + \quad 2r_1^{\,2} r_3^{\,2} u_2 + 2r_1^{\,2} r_2^{\,2} u_3 = 850$$

which gives $u_1 = 2$ $\qquad u_2 = 3$.

From
$$r_1^{\,2} = u_1^{\,2} + v_1^{\,2}$$
$$r_2^{\,2} = u_2^{\,2} + v_2^{\,2}$$
$$r_3^{\,2} = u_3^{\,2} + v_3^{\,2}$$ we get

$v_1 = 3 \qquad v_2 = 1 \qquad v_3 = 2$.

Then the roots of $f(z) = 0$ are

$2 \pm 3i$

$3 \pm i$

$1 \pm 2i$.

If the equation has multiple roots, they can be detected and eliminated by finding the highest common factor of $f(x)$ and $f^{I}(x)$. But if this test and elimination have not been made before applying Graeffe's method the procedure is as follows.

Let the Encke roots of a cubic equation be

$a, \ -a, \ b \qquad |a| \neq b$.

Then the equation of the m^{th} powers of the roots is

$a^3 + (2a^m + b^m)x^2 + (a^{2m} + 2a^m b^m)x + a^{2m}b^m = 0$.

If $|a| > |b|$ the approximate equation is

$x^3 + 2a^m x^2 + a^{2m}x + a^{2m}b^m = 0$

and if $|a| < |b|$

$x^3 + b^m x^2 + 2a^m b^m x + a^{2m}b^m = 0$.

If m is so large that the doubled products of the coefficients are negligible one of the columns exhibits the pecularity that its coefficient is not squared by another root-squaring transformation, but becomes one half of the square of its former value.

$2a^{2m} = \frac{1}{2}(2a^m)^2$ and $2a^{2m}b^{2m} = \frac{1}{2}(2a^m b^m)^2$.

This change in the magnitudes of the coefficients of one column with no irregularity in sign, shows the presence of a pair of roots equal in magnitude, but the signs can either be equal or opposite.

Now, suppose $f(x) = 0$, a cubic, has the Encke roots a, a, $-a$, then, the m^{th} power of the equation is

$x^3 + 3a^m x^2 + 3a^{2m}x + a^{3m} = 0$.

In this case two adjacent columns will increase ultimately at one third of normal rate, since $3a^{2m} = 1/_3(3a^m)^2$.

If four roots are equal the corresponding equation will be

$$x^4 + 4a^m x^3 + 6a^{2m}x^2 + 4a^{3m}x + a^{4m} = 0.$$

Then three adjacent columns increase ultimately at one fourth, one sixth, one fourth of normal rate respectively. Notice that the fractions appearing here are the reciprocals of the binomial coefficients. This behavior extends to multiplicities of any order.

This behavior can be summarized in a set of rules of identification.[41]

First detect and eliminate equal roots.

1 All signs plus after the given equation and all columns increase at normal rate, all roots real and of unequal absolute values.

2 A single column irregular in sign, one pair of complex roots.

3 If two adjacent columns are irregular in sign, one pair of complex roots with modulus equal to that of the real root.

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41 C. A. Hutchinson, "On Graeffe's Method for the Numerical Solution of Algebraic Equations," The American Mathematical Monthly, 42:149-61, March, 1935.

4 One column increases eventually at one-half of normal rate, two equal roots in magnitude, but unequal sign.

5 Two adjacent columns increase eventually at one-third of normal rate, triplet.

6 Two non-adjacent columns increase at one-half of normal rate, two doublets, not a quadruplet.

7 Three adjacent columns increase at one-fourth, one-sixth and one-fourth of normal rate respectively, quadruplet.

8 One column increases at one-half of normal rate, and non-adjacent column is irregular in sign, doublet and a pair of complex roots.

9 Two non-adjacent columns irregular in sign: two pairs of complex roots with unequal moduli.

10 Three adjacent columns irregular in sign, two pairs of complex roots with equal moduli.

11 One column irregular in sign and one column adjacent on each side, regular in sign, but irregular in magnitude, doublet and a pair of complex roots with the same moduli as the doublet.

Location and Separation of the Roots

The real roots of an equation, $f(x) = 0$, are said to be isolated if one or more intervals have been found such that each real root is contained in one of these intervals and no other roots of $f(x)=0$ are known to lie in those intervals.

We may isolate the real roots of $f(x)=0$ by means of the graph of $y=f(x)$. But to obtain a reliable graph it is necessary to employ the critical points, whose abscissas occur among the roots of $f^{I}(x)=0$. Since the latter equation is of degree $(n-1)$ when $f(x)=0$ is of degree n, this method is usually impracticable when n exceeds three.

Rolle's theorem states that between two consecutive real roots of $f(x)=0$ there exists an odd number of real roots of $f^{I}(x)=0$, provided a root of multiplicity m is counted as m roots. A method based on this theorem is open to the same objection as the method described above.

Descartes' rule of signs says that the number of positive real roots of a rational integral algebraic equation $f(x)=0$, with real coefficients, is either equal to the number of variations of sign in its coefficients or less than that number by a positive even integer. The number of negative roots of the same equation is either equal to the number of variations of sign of $f(-x)=0$, or less than that number by a positive even integer. This rule of signs gives, in many cases, information regarding the total number of real roots.

Budan's theorem (1807) is another theorem concerning isolation of roots of an equation. In this case we let $f(x) = 0$ be an integral algebraic equation of degree n, with real coefficients, and a and b two real numbers $(a < b)$ neither a root of $f(x) = 0$, be substituted in the series formed by $f(x)$ and its successive derived functions $f(x)$, $f^{I}(x)$, $f^{II}(x), \ldots, f^{n}(x)$; then the excess of the number of variations of sign in the series when $x = a$, over the number of variations of sign when $x = b$, either equals the number of real roots of $f(x) = 0$, between a and b or exceeds the number of roots by a positive even integer. A root of multiplicity m is here counted as m roots. This method has one advantage over that of Sturm, in that Budan's functions are easily obtained.

Numerical Example.

Locate the roots of

$$x^5 + x^4 - 4x^3 - 3x^2 + 3x + 1 = 0$$

$f(x) \equiv x^5 + x^4 - 4x^3 - 3x^2 + 3x + 1 = 0$

$f^{I}(x) = 5x^4 + 4x^3 - 12x^2 - 6x + 3$

$f^{II}(x) = 20x^3 + 12x^2 - 24x - 6$

$f^{III}(x) = 60x^2 + 24x - 24$

$f^{IIII}(x) = 120x + 24$

$f^{5}(x) = 120$.

We form the following table

x	f	f^{1}	f^{11}	f^{3}	f^{4}	f^{5}	v
-2	-	+	-	+	-	+	5
-1	-	-	+	+	-	+	3
0	+	+	-	-	+	+	2
1	-	-	+	+	+	+	1
2	+	+	+	+	+	+	0

From the table we see that there may be two roots in the interval (-2,-1) and must be one root in each of the intervals (-1, 0); (0, 1); (1,2).

The first complete solution of the problem of isolating the real roots of an equation with real coefficients was furnished by Sturm in 1829.[42] His work was as follows Let $f_0(x)$, $f_1(x), \ldots, f_r(x)$ be an ordered set of polynomials in the field of real numbers Substituting $x = a$, (a is a real number) an ordered set of real numbers $fa(a)$, $f_1(a), \ldots, f_r(a)$ is obtained. All zeros present in this set are supressed, except $f(a)$ and $f(b)$. The number of variations in sign in passing from term to term is counted

42 L. Weisner, Introduction to the Theory of Equations (second edition; New York: The Macmillan Company, 1947), p. 80.

and denoted by Va. The basic idea of Sturm's method is to construct, for every polynomial with real coefficients, a sequence of polynomials of which it may be asserted that for any a and b $(a < b)$, the exact number of real roots of $f(x) = 0$, between a and b, is exactly equal to the number of variations of sign in the series when $x = a$, diminished by the number of variations of sign in the series when $x = b$.

The polynomials which Sturn proved to have the desired property were constructed in the following manner.

Let $f(x)$, the given polynomial, be identical equal to $f_0(x)$ and $f^I(x) \equiv f_1(x)$. On dividing $f_0(x)$ by $f_1(x)$ a remainder is obtained whose negative was $f_2(x)$. In the same manner $f_3(x)$ is the negative of the remainder obtained when $f_1(x)$ is divided by $f_2(x)$, etc.
The polynomials $f_0(x)$, $f_1(x)$, $f_2(x)$... are called Sturm's functions for the given polynomial $f(x)$. The calculations terminate naturally when a remainder is obtained which is a non-zero constant whose negative is the last Sturm function. During the passage of x from a to b, the only cases in which there can be any changes in the number of variations of sign of the series of Sturm's functions are the following:

1 When x passes through a value which causes one of the functions $f_0(x)$, $f_1(x)$... to vanish

2 When x passes through a root of $f(x) = 0$.

In the case of equal roots $f_0(x)$ and $f_1(x)$ have a common factor; hence $f_r(x)$, the last of Sturm's functions, is not a non-zero constant, but the greatest common divisor of $f_0(x)$ and $f_1(x)$.

The advantage of Sturm's method is that it gives always the exact number of real and distinct roots of $f(x) = 0$, between, a and b.

Numerical Example

Locate the roots of $x^4 - 4x^3 + 4x^2 + 4x - 3 = 0$.

We find first $f^1(x)$ of $f(x) = 0$ and divide $\frac{f(x)}{f^1(x)} = \frac{f_0(x)}{f^1(x)}$.

Then we proceed to find the rest of Sturm's functions according the rule already mentioned. To avoid fractions, we may multiply $f_0(x)$ by a positive constant before dividing by $f_1(x)$, and multiply any $f_j(x)$ by a positive constant before dividing by $f_{j+1}(x)$. Also, we can remove any constant positive factor from $f_j(x)$ before using it as a divisor.

$f_0(x) = x^4 - 4x^3 + 4x^2 + 4x - 3 = 0$

$f_1(x) = 4(x^3 - 3x^2 + 2x + 1$

$f_2(x) = 4(x^2 - 5x + 2)$

$f_3(x) = 4(-10x + 3)$

$f_4(x) = - 236$.

We give a table of signs for the indicated values of x of Sturm's functions

x	$f_a(x)$	$f_1(x)$	$f_2(x)$	$f_3(x)$	$f_4(x)$	v
-3	+	-	+	+	-	3
-2	+	-	+	+	-	3
-1	+	-	+	+	-	3
0	-	-	+	+	-	2
1	+	+	-	-	-	1
2	+	+	-	-	-	1
3	+	+	-	-	-	1
4	+	+	-	-	-	1
5	+	+	+	-	-	1

Accordingly, we have one real root between (-1, 0), another real root between (0, 1) and two imaginary roots.

Sometimes Fourier's theorem is useful in separating roots of an equation. Fourier's theorem states that if $f(x) = 0$, is a rational integral algebraic equation, which has one and only one real root between a and b $(a < b)$, and if $f^{I}(x) = 0$ has no real root between a and b, and also $f^{II}(x) = 0$ has no real root between a and b; then Newton's method of approximation will certainly be successful if it be begun and continued from that bound for which $f(x)$ and $f^{II}(x)$ have the same sign. Cases must occur in practice where the roots of an equation cannot be separated by any of the well - known easier methods and where Sturm's functions involve too much work. In such cases a combination of Fourier's theorem and Lagrange's method of approximation is very useful.

Numerical example.[43]

$$x^{17} - 35x^{15} + 11x^{14} - 1000x^{10} + 2500x^{5} - 151x^{3} + 1 = 0.$$

At first application of Fourier's theorem shows that there are

1 Two positive roots: one between (1, 2) and one between (5, 6).

2 Three negative roots: one between (0, -1), one between (-1, -2) and one between (-6, -7).

3 A doubtful interval (0, 1) in which four changes of sign are lost and which consequently include four more possible positive roots.

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43 L. R. Manlove, "An Example of the Usefulness of Fourier's Theorem in Separating the Roots of an Equation," The American Mathematical Monthly, 19: 8-9, January, 1912.

Now, to dispose of the doubtful interval we let $x = \frac{1}{u_1}$ and we obtain an equation in u_1

$$u_1^{17} - 151u_1^{14} + 2500u_1^{12} - 1000u_1^{7} + 11u_1^{3} - 35u_1^{2} + 1 = 0.$$

Call it $F_A(u_1)$.

Fourier's functions of $Fa(u_1)$ give for $u_1 = 1$ two changes of sign and for $u_1 = 2$ no changes of sign. Then $F_A(u_1) = 0$ may have two roots between 1 and 2, but has no other positive roots greater than unity. Now the doubtful roots are reduced to two. To dispose the remaining pair we set $u_1 = 1 + \frac{1}{u_2}$ and we get an equation in u_2 which we call $F_B(u_2)$. This equation can have no positive roots greater than unity. The four originally doubtful roots are all imaginary.

Remarks.

1 The auxiliary equations were concerned only with positive values of the variable greater than unity.

2 Negative roots of the original equation are more conveniently located by substituting -y for x and seeking for the positive roots of the resulting equation.

3 Failure of the method is due to equal roots.

Concerning complex roots Sturm's Theorem gave a method for determining the number of them, but not their values. This was revealed by a general theorem of a great French mathematician Augustin Louis Cauchy (1789-1857) giving the number of roots, real or complex, which lie within a given contour.[44]

If the roots of an equation $f(x) = 0$ are under consideration the theorem states that if N is the number of roots in the complex plane within a closed circuit A, not passing through any of the roots, then

$$N = \frac{1}{2\pi i} \int \frac{f^1(x)\ dx}{f(x)}, \quad \text{the integral being taken around A.}$$

When two real roots of an integral rational equation are nearly equal, it is often difficult to separate them. This difficulty is frequently due to the fact that we cannot readily approximate the roots by Newton's method. The scheme described below for isolating such roots is usually satisfactory.[45]

Illustration.

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44 R. C. Archibald, "Outline of the History of Mathematics," The American Mathematical Monthly, 56: 1-103, January, 1949.

45 E. C. Kennedy, "Concerning Nearly Equal Roots," The American Mathematical Monthly, 48: 42-43, January, 1941.

We consider the quartic (Lovitt p. 138)

(1) $f(x) = x^4 + 8x^3 - 70x^2 - 144x + 936 = 0$.

We readily find that

$f(3) = 171$

$f(4) = 8$

$f(5) = 91$.

By Descartes' rule of signs the equation $f(x) = 0$ has two or no real positive roots. Hence we conclude that if the equation has any positive roots they are near $x = 4$.

We shift the axes horizontally by setting $x = (y + 4)$.

We obtain

(2) $f(y) = y^4 + 24y^3 + 122y^2 - 64y + 8 = 0$.

We discard the two first terms and solve the quadratic equation obtaining

$y^{I} = .206$ $\qquad x_3 = 4.206$

$y^{II} = .319$ $\qquad x_4 = 4.319$.

If $f(y)$ has any positive roots, they lie inside the interval (y_1, y_2) since the first two terms are positive for every $y > 0$. Hence, $f(y)$ cannot be zero except possibly for values of y, between (y_1, y_2) that make the quadratic negative. We can get better results by translating the axes by letting $y = z + a$ where $y_1 < a < y_2$.

Usually one would take $a = \frac{(y_1 + y_2)}{2}$ approximately.
If we take $a = \frac{1}{4}$ we get

$$G(z) = z^4 + 25z^3 + \frac{1123z^2}{8} + \frac{25z}{16} + \frac{1}{256} = 0.$$

From this we obtain by discarding the first two terms

$z_1 = -.00734$ or $x_1 = 4.24266$

$z_2 = -.00379$ $x_2 = 4.24261.$

Hence if $G(z) = 0$ has any real roots near zero, they lie outside the interval (z_1, z_2). Thus if $f(x) = 0$ has any positive roots, they lie inside the interval (x_1, x_2) and outside the interval (x_3, x_4) where

$$x_1 < x_3 < x_4 < x_2.$$

To decide the question we find the sign of $G(p)$ where p is any value between z_1 and z_2. For example we find $G(-.005) < 0$ or $f(4.245) < 0$. This proves the existence of two positive real roots of $f(x) = 0$. As a matter of fact, it is evident that $G(z) = 0$ has a root between zero and z_2 because $G(0) > 0$ and $G(z_2) < 0$, since $z^3(z + 25) < 0$ for negative values of z near zero and the quadratic vanishes at z_2. Since z_1 and z_2 are very small, it follows that x_3 and x_4 are close approximation to the two roots in question.

These roots are

$$R_1 = 4.24264$$

$$R_2 = 4.24622 .$$

Had we taken some value for a, a little different from .25, we might have found that the roots of $G(z)=0$ rested inside the interval (z_1, z_2). However the values obtained for x_3 and x_4 would have been close approximations to the roots R_1 and R_2 of $f(x)=0$, and the point $x = \frac{(x_3+x_4)}{2}$ would very likely have separated R_1 and R_2.

Incidentally, this value of x should be a very close approximation to the abscissa of the minimum point in this neighborhood.

Graphical and Mechanical Methods of Solution

One of the principal uses of the rectangular Cartesian coordinate system is the graphical representation of an equation $y = f(x)$ where the function is a polynomial with real coefficients. To construct this graph, we assign to x a series of values and compute the corresponding y's. It is usually convenient to start by assigning integral values of x, to plot the resulting points and then to approximate for fractional values of x where the general shape of the curve does not seem

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already to be clearly indicated. Considerable information about the roots of an equation $f(x)=0$ can be obtained by inspecting the graph of $y=f(x)$. These roots are the x's of the points where $y=0$, that is, the intersections of the curve with the x - axis, $f^1(x)$ is the slope of the tangent of $y=f(x)$. In studying the shape of the curve, it is convenient to think of it as traced with x varying from $-\infty$ (numerically large negative numbers) to $+\infty$ (numerically large positive numbers). The curve rises or falls according as $f^1(x)$ is positive or negative. If $f^1(x)=0$, the tangent is parallel to the x - axis.

It is useful to plot the curve y $f^1(x)$ either on the same axes as $y=f(x)$ or with the same y - axis and different x - axis. Similarly for the higher derivatives. These derived curves are very useful in bringing out properties of the function because they are interrelated.

Graphical Method for an n^{th} Degree Equation.[46]

We take any numerical equation of the form

$$A_nx^n + B_nx^{n-1} + C_nx^{n-2} + \text{--------} T_nx + U_n = 0$$

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46 W. H. Bixby, "Graphical Solution of Numerical Equations." The American Mathematical Monthly, 29:344-46, October, 1922.

where A_n is any positive number either whole or fractional. On a blank sheet of paper, we start at any assumed point α, and using any convenient scale, we lay off in a downward direction a distance A_n, from α to a, through a, we draw a perpendicular, and lay off upon it, with the same scale as before, the value B_n; (to the right if B_n is positive, to the left if B_n is negative) through the end of B_n we draw a perpendicular to B_n upon which we lay off the value of C_n, upward if positive, downward if negative; and so on. For each new line the positive direction turns through an angle counter - clockwise. We designate w the end of the last line. Then we have a rectangular contour, that is a broken line all of whose angles are right angles, of $n+1$ sides connecting α and w.

Now starting again at α, draw at random any straight line cutting B_n in some point,

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say b, through b we draw a perpendicular to αb cutting C_n in some point as b^1, and so on. Now we have a new rectangular contour of n sides. If the n^{th} side passes through the point w, then $\frac{ab}{\alpha a}$ taken with its sign changed, is a root of the given equation. There will be as many such cantours of n sides as there are real roots to a given equation.

Suppose only one root is found by the above method, given as one new rectangular contour, $\alpha bb^1....w$, of n sides; call its first side A_{n-1}, its second B_{n-1} and so on; this new rectangular contour, $\alpha bb^1...w = A_{n-1}$, B_{n-1}.... etc. represents the equation of degree (n-1) obtained by dividing out by the root. We treat the new contour of n sides like the preceeding, obtaining a new rectangular contour of (n-1) sides whose first vertex is at some point C upon the line bb^1; then $\frac{bc}{\alpha b}$, taken with its sign changed, will be another root of the given equation, etc. This method is especially applicable to cases where the desired roots be between $\pm \frac{1}{5}$ and ± 5; if the roots of the given equation lie beyond these limits, the given equation may be transformed into another whose roots will be between

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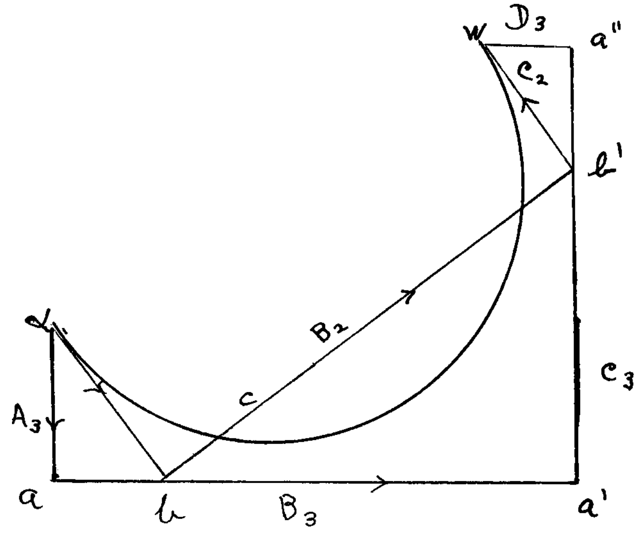
The Figure is for $n = 3$.



between the above limits. In the case of a quadratic, represented by the inner contour, the roots, if real, may be found by means of a circle on αw as diameter, as indicated in the figure.

Mechanical Solution of the Cubic Equation.[47]

We take the general cubic equation

(1) $u^3 + Au^2 + Bu + C = 0$ and by letting $u = v - \frac{A}{3}$ we reduce (1) to

(2) $v^2 + av + b = 0.$

By the following rational transformation $v = \frac{bx}{a}$ we get

(3) $x^3 - m(x+1) = 0$ where $m = -\frac{a^3}{b^2}$.

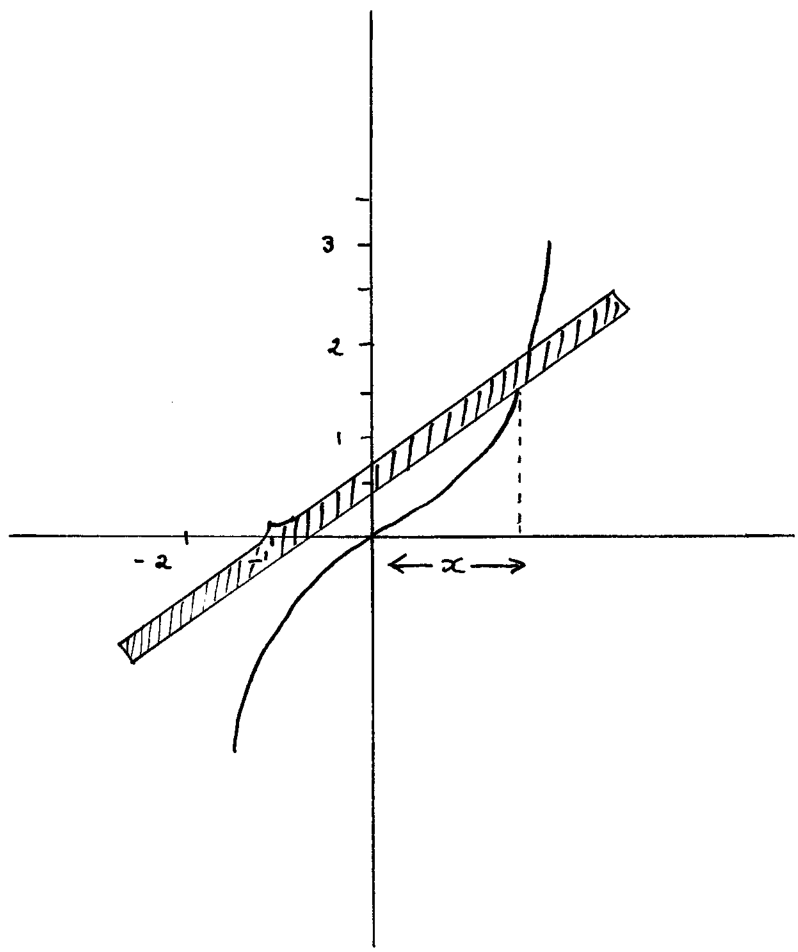
Now we solve (3) graphically by replacing (3) by the set

(4) $y = x^3$

$y = m(x+1)$ whose simultaneous values of x also belong to (3). So we have for all cubics, a fixed curve and a variable line. But this variable line has the distinct virtue of always passing through the point (-1, 0). This is the feature that prompts the mechanical arrangement shown. A cardboard strip is attached to the

47 R. C. Yates, "A Mechanical Solution of the Cubic Equation," Mathematics Teacher, 32:215.

A Mechanical Solution of a Cubic



plane on which the curve $y = x^3$ is drawn, so that its straight edge rotates about $(-1,0)$. Its position, of course is determined by the slope $m = \frac{-a^3}{b^2}$ measured directly upon the vertical scale. The root x is then determined by the perpendicular dropped from the intersection onto the horizontal axis.

Mechanical Solution of an Equation of n^{th} Degree[48]

The mechanism consists of a main bar thirty-two inches long, to which are hinged three arms each about eight inches long, the distance between the hinges being equal. A lighter connecting bar is attached to the free ends of the arms in such a manner that these arms always turn through the same angle. On the main bar, and also each of the arms, are beveled cleats along which grooved slides moves freely. Each of these slides on the main bar carries an eye headed screw placed so that when the instrument is closed and these slides are at their zero points, the eyes are in line with the pins of the hinges. Each slide on the arms carries a small drum that is held firm by means of a milled nut. To each of the drums is attached a small, flexible, inelastic cord, which passes through the eye carried by the adjacent slide on the main bar, and is fastened to the next slide below on the main bar, the lower end of the last cord being made fast to the main bar. The

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48 A. L. Candy, "A Mechanism for the Solution of an Equation of n^{th} Degree, "The American Mathematical Monthly, 27:195-99, May, 1920.

first slide is held in place by means of a small iron pin inserted in holes in the main bar. A graduated circular scale is placed under the first arm, from which the roots of the equation are read. The scale for reading the positions of the slides are marked off on the left side of the main bar. The instrument may be used in a vertical position, so that the lengthening of any string by unwinding will cause some of the slides to move downwards by their own weight, or lying on a table and operated with both hands.

Let us solve the equation

(1) $10x^3 + 24x^2 + 9x - 7 = 0.$

The process is as follows. First, close the instrument, wind up the drums until each slide comes to the zero point of its scale, and all the cords are taut. The arms will now move freely through an angle of 90°, with all the cords continuously taut. Now move the first slide ten units (the coefficient of x^3) downward, by moving the iron pin which always holds this slide in a fixed position; unwind twenty-four units (the coefficient of x^2) from the cord wound around the first drum; likewise, unwind nine units (coefficient of x) from the second drum;

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since the constant term is negative, wind up the last drum until the last cord is shortened by seven units. Now turn the arms through some angle until all the cords become taut, with the slides on the arms so adjusted that the cords attached to them shall be at right angles to the arms. The reading on the scale under the first arm now shows one root of the equation to be .366. The exact root is $\frac{(\sqrt{3} - 1)}{2}$.

The author has called the attention to the following limits of the mechanism

1 The mechanism will find only a root of the equation that lies between 0 and 1.

2 The equation to be solved must have the constant term negative, and all the coefficients positive.

3 An equation of first degree can be solved by using only the upper or lower arm. An equation of second degree can be solved by using two arms, either the upper two or the lower two. For a cubic equation three arms are needed, and so on.

CHAPTER V

Ferrari's, Descartes' and Euler's' Reducing Cubics Obtained from a Single Quadratic Function of the Roots

In chapter III, the solutions of the general quartic equation as discovered by Ferrari, Descartes and Euler were presented. In each case the solution depended on the solution of a certain reducing cubic. Also in chapter III it was shown that by using certain quadratic function of the roots of the quartic a solution by means of symmetric functions could be obtained. This latter method also gave rise to a reducing cubic that depended on the choice of the quadratic function of the roots. With a proper choice of this function the reducing cubics found by Ferrari, Descartes and Euler were obtained. In this part of the paper I wish to prove that all three of the functions chosen in chapter III, to produce the several reducing cubics, are linear functicus of t , (where t x_1x_2 x_3x_4, and and are arbitrary quantities in the domain of rationality) and that, by a proper choice of and all three solutions of the quartic can be obtained.

We assume the general quartic in the form

(1) $\quad ax^4 + 4bx^3 + 6cx^2 + 4dx + e = 0$

and let x_1, x_2, x_3 and x_4 be its roots.

We consider the given function $Q = \alpha t + \beta$ where $t = x_1x_2 + x_3x_4$. Under the twenty-four substitutions of G_{24}^4 Q takes only three values.

$$Q_1 = \alpha t_1 + \beta \qquad \text{where} \qquad t_1 = x_1x_2 + x_3x_4$$
$$Q_2 = \alpha t_2 + \beta \qquad\qquad\qquad t_2 = x_1x_3 + x_2x_4$$
$$Q_3 = \alpha t_3 + \beta \qquad\qquad\qquad t_3 = x_1x_4 + x_2x_3 \ .$$

We let Q_1, Q_2 and Q_3 be the roots of the resolvent cubic equation

$$y^3 - \sum Q_1 y^2 + \sum Q_1Q_2 y - Q_1Q_2Q_3 = 0 \ .$$

We compute $\sum Q_1$, $\sum Q_1Q_2$ and $Q_1Q_2Q_3$ which are elementary symmetric functions of the roots of the quartic. The roots of this cubic are expressed in terms of the coefficients of equation (1).

$$\sum Q_1 = \alpha(\sqrt{2}) + 3\beta$$

(2) $\quad \sum Q_1Q_2 = \alpha^2\left[\sqrt{1}\,\sqrt{3} - 4\sqrt{4}\right] \ 2\alpha\beta\left[\sqrt{2}\right] + 3\beta^2$

$$Q_1Q_2Q_3 = \alpha^3\left[\sqrt{1}^{\,2}\sqrt{4} + \sqrt{3}^{\,2} - 4\sqrt{2}\,\sqrt{4}\right] + \alpha^2\beta\left[\sqrt{1}\,\sqrt{3} - 4\sqrt{4}\right] +$$
$$\alpha\beta^2\left[\sqrt{2}\right] + \beta^3$$

where

$$\sqrt{1} = -\frac{4b}{a} = \sum x_1 \qquad\qquad \sqrt{3}\ -\frac{4d}{a} = \sum x_1x_2x_3$$

$$\sqrt{2} = \frac{6c}{a} = \sum x_1x_2 \qquad\qquad \sqrt{4} = \frac{e}{a} = x_1x_2x_3x_4 \ .$$

Ferrari's Case

His resolvent cubic is of the form

(3) $4a^3\theta^3 - (ae - 4bd \quad 3c^2)a\theta + ace + 2bcd - ad^2 - eb^2 - c^3 = 0$.

From (2) and (3) we have the following system of equations

(4) $\alpha\sigma_2 + 3\beta = 0$

(5) $\alpha^2\left[\sigma_1\sigma_2 - 4\sigma_4\right] + 2\alpha\beta(\sigma_2) + 3\beta^2 = -(ae - 4bd + 3c^2)a$

(6) $\alpha^3\left[\sigma_1^2\sigma_4 + \sigma_3^2 - 4\sigma_2\sigma_4\right] + \alpha^2\beta\left[\sigma_1\sigma_3 - 4\sigma_4\right] + \alpha\beta^2(\sigma_2) + \beta^3 =$

$$-\left[ace + 2\,bcd - ad^2 - eb^2 - c^3\right].$$

Solving (4) and (5) for α and β the roots are,

$$\beta = c\sqrt{a} \qquad \alpha = -\frac{a\sqrt{a}}{2}$$

and $$\beta = -c\sqrt{a} \qquad \alpha = \frac{a\sqrt{a}}{2}.$$

Next, we check the two pairs of roots in equation (6).

This yields the following expression

$$2a\sqrt{a}\left[ace + 2bcd - ad^2 - eb^2 - c^3\right] = \left[ace + 2bcd - ad^2 - eb^2 - c^3\right]$$

from which

(7) $2a\sqrt{a} = 1$. But the left - hand side of (7) is equal to 4α.

So $4\alpha = 1$

$\alpha = \frac{1}{4}$. Substituting this value in (4) we get

$$\beta = -\frac{\sigma_2}{12} \qquad \text{or} \qquad \beta = -\frac{c}{2a}.$$

Descartes' Case.

The values just found for α and β are the same for Descartes' case because his resolvent cubic is of the same form as that of Ferrari.

Euler's Case.

His resolvent is

$$(8) \qquad \theta^3 + 3H\theta^2 + \left(3H^2 - \frac{a^2I}{4}\right)\theta - \frac{G^2}{4} = 0$$

where

$$G = (a^2d - 3abc + 2b^3)$$

$$H = (ac - b^2)$$

$$I = (ae - 4bd + 3c^2) \,.$$

From (2) and (8) we can write

$$(9) \qquad \alpha\sqrt{2} + 3\beta = -3H$$

$$(10) \qquad \alpha^2\left[\sqrt{1}\sqrt{3} - 4\sqrt{4}\right] + 2\alpha\beta(\sqrt{2}) + 3\beta^2 = 3H^2 - \frac{a^2I}{4}$$

$$(11) \qquad \alpha^3\left[\sqrt{1}^2\sqrt{4} + \sqrt{3}^2 - 4\sqrt{2}\sqrt{4}\right] + \alpha^2\beta\left[\sqrt{1}\sqrt{3} - 4\sqrt{4}\right] + \alpha\beta^2\left[\sqrt{2}\right] + \beta^3 = -\left[-\frac{G^2}{4}\right].$$

We obtain two pairs of roots by solving equations (9) and (10)

First pair of roots $\alpha = \frac{a^2}{4}$ $\qquad \beta = -\frac{3ac}{2} + b^2.$

Second pair of roots $\alpha = -\frac{a^2}{4}$ $\qquad \beta = -\frac{ac}{2} + b^2.$

The first pair satisfies identically equation (11).

To complete the work we know that $\sum Q_1$; $\sum Q_1 Q_2$ and $Q_1 Q_2 Q_3$ can be expressed in terms of the coefficients of equation (1) and that the Q's are the roots of a cubic of the type

$Ay^3 + By^2 + Cy + D = 0$ which can be solved. Once we know the Q's, we can express the t's in terms of Q's, α's and β's, and write a quadratic equation of the form

$Rz^2 + Sz + T = 0$ whose roots are $z_1 = x_1 x_2$ and $z_2 = x_3 x_4$.

To find $(x_1 x_2)$, $(x_3 x_4)$, $(x_1 + x_2)$ and $(x_3 + x_4)$

We write the following equations.

(12) $$x_1 x_2 (x_3 + x_4) + x_3 x_4 (x_1 + x_2) = \sqrt{3} = -\frac{4d}{a}$$

or $$z_1 (x_3 + x_4) + z_2 (x_1 + x_2) = -\frac{4d}{a}$$

(13) $$(x_1 + x_2) + (x_3 + x_4) = \sqrt{1} = -\frac{4b}{a}\ .$$

Solving (12) and (13) for $(x_1 + x_2)$ and $(x_3 + x_4)$ and knowing $(x_1 x_2)$ and $(x_3 x_4)$ we can write two more quadratic equations whose roots are x_1 and x_2, for one of them, and x_3 and x_4 for the other one.

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