STATISTICAL PROPERTIES OF HIGH DIMENSIONAL NONSTATIONARY DYNAMICAL SYSTEMS

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Doctor of Philosophy

By
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To thank a village...

My village is big

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Abstract

Classical dynamical systems involves the study of the long-time behavior of a fixed map or vector field. When dynamical instabilities are present, it is advantageous to study the dynamical system from a statistical perspective. It is important to move beyond the classical setup in order to model a more diverse array of physical and biochemical phenomena. The recent theory of nonstationary dynamical systems endeavors to do exactly that. In this theory, the dynamical model itself varies in time. This allows modelers to handle dynamical processes that evolve in time-varying environments, as well as systems with time-varying parameters. This dissertation formulates and solves two novel problems in nonstationary dynamical systems.

The first project concerns what we call the quasistatic limit, an idea inspired by quasistatic processes in thermodynamics. We address the following question: If one assumes that the dynamical model itself varies sufficiently slowly, is it possible to recover a quasistatic ergodic theorem? We answer this question affirmatively for a class of quasistatic dynamical systems built from piecewise-smooth expanding maps in higher dimensions.

The second project moves the theory of coupled map lattices (CMLs) into the nonstationary realm. CMLs have been used extensively to model phenomena in biology and physics. Classically, a CML consists of a lattice (or graph), a local dynamical system at each lattice site, and interactions between different lattice sites. Here, we allow the local dynamical model at each lattice site to vary in time, thereby producing nonstationary CMLs, a novel construct. For a certain class of nonstationary CMLs, we define a notion of statistical memory loss, an analog of decay of correlations. We then prove that memory is lost at an exponential rate.

A common theme links the two parts of the dissertation: Dimension is high.

Contents

Chapter 1

Preliminaries

1.1 Introduction

The field of dynamical systems focuses on analyzing phenomena that evolve in time. Dynamical systems have been used as models in various fields such as biology, physics, weather modeling, economics, etc. In mathematical terms, a dynamical system is characterized by a phase space (or state space) X and a transformation T from X to itself. The pair (X, T) is used to denote a dynamical system. Given a state x in X, the collection $\{x, Tx, T^2x, \dots\}$ describes the orbit/trajectory of state x under iterates of T. In the case of a continuous-time system, the dynamic is called a *flow* and is described by an indexed family of maps $\{T_t : t \in \mathbb{R}\}$ that satisfies $T_{t_1} \circ T_{t_2} = T_{t_1+t_2}$.

Dynamical systems theories are interested in the asymptotic behaviors, which means the properties related to the behaviors as time goes to infinity [3]. Most of the systems of interest have some special structures. In general, we can divide dynamical systems into three main streams:

1. Ergodic Theory: (X, μ) is a probability space, and the transformation $T : X \to X$ preserves μ (μ is T-invariant), i.e., $\int \varphi d\mu = \int \varphi \circ T d\mu$ for all $\varphi \in L^1(\mu)$.

2. Topological dynamics: the phase space X is a topological space, and transformation $T : X \to X$ is continuous. Usually, we are more interested in compact metric space.

3. Smooth dynamics: the phase space X is a smooth manifold, and transformation $T : X \to X$ is a C^1-diffeomorphism. Again, we usually consider compact space.

1.2 Ergodic Theory

Ergodic theory stems from statistical physics, wherein Boltzmann introduced a hypothesis: trajectory of an isolated mechanical system runs through all states compatible with the total energy of the system [23]. The term "ergodic theory" can be used to describe the quantitative study of action of groups on measure space [3]. We summarize the basics of ergodic theory when the acting semigroup is the set of natural numbers. The classical case involves a probability space $(X, \mathcal{B}, \mu)$ and a measure-preserving transformation $T : X \to X$.

Definition 1.2.1. Given a probability space $(X, \mathcal{B}, \mu)$, a transformation $T : X \to X$ is said to be *measure-preserving* if:

1. T is measurable,

2. $\mu(T^{-1}(B)) = \mu(B)$ for all $B \in \mathcal{B}$.

If in addition to having a measure-preserving transformation, we have $0 < \mu(B) < 1$ and $T^{-1}(B) = B$ for $B \in \mathcal{B}$, then instead of studying the transformation T, we can study two simpler transformations $T|_B$ and $T|_{X\setminus B}$. On the other hand, if $T^{-1}(B) = B$ implies $\mu(B) = 0$ (or $\mu(X \setminus B) = 0$), the study of the transformation T is not reduced much since null sets are negligible from the point of view of measure theory. Such indecomposable transformations are called *ergodic*.

Definition 1.2.2. [24] Let $(X, \mathcal{B}, \mu)$ be a probability space. A measure-preserving transformation $T : X \to X$ is said to be *ergodic* if for any $B \in \mathcal{B}$ satisfying $T^{-1}(B) = B$, $\mu(B) = 0$ or $\mu(B) = 1$.

One of the most important results in ergodic theory is the Birkhoff Ergodic Theorem.

Theorem 1.2.3. *(Birkhoff Ergodic Theorem) [24] Let $(X, \mathcal{B}, \mu)$ be a probability space and $T : X \to X$ be a measure-preserving transformation. Fix $f \in L^1(\mu)$, then:*

1. $\lim\limits_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} f \circ T^k$ *exists almost everywhere $x \in X$.*
2. *The limit mentioned above is equal (almost everywhere) to a function $f^* \in L^1(\mu)$, where $f^* \circ T = f^*$ almost everywhere.*
3. *If T is ergodic, then $\lim\limits_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} f \circ T^k = \int_X f d\mu$ almost everywhere.*

Remark 1.2.4. 1. The Birkhoff Ergodic Theorem exists for the one-parameter flow $\{T_t\}_{t\in\mathbb{R}}$ of measure-preserving transformations. In this case, the statement of the result is that $\lim\limits_{N\to\infty} \frac{1}{N} \int_0^N f(T_t(x))dt$ exists almost everywhere for $f \in L^1(\mu)$ and equals to $\int_X f d\mu$ if the flow $\{T_t\}$ is ergodic.

2. The expression $\lim\limits_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} f \circ T^k$ is interpreted as the time average of the system, and $\int_X f d\mu$ is the space average. Hence, Birkhoff Ergodic Theorem implies that when T is ergodic, time average equals space average almost everywhere.

Another important notion in ergodic theory is *mixing*.

Definition 1.2.5. [24] Let $(X, \mathcal{B}, \mu)$ be a probability space and $T : X \to X$ be a measure-preserving transformation. T is said to be *mixing* if for all E and F in $\mathcal{B}$,

$$\lim_{k\to\infty} \mu(E \cap T^{-k}(F)) = \mu(E)\mu(F). \tag{1.1}$$

Remark 1.2.6. 1. The above notion of mixing sometimes is referred to as the *strong mixing*.

2. T is said to be *weak mixing* if for all E and F in $\mathcal{B}$,

$$\lim_{k\to\infty} \frac{1}{k} \sum_{i=0}^{k-1} |\mu(T^{-i}E \cap F) - \mu(E)\mu(F)| = 0. \tag{1.2}$$

3. A strong mixing transformation is weak mixing.

4. A weak mixing transformation is ergodic.

From the definition above, if T is *strong mixing*, then $T^{-1}E$ becomes asymptotically independent of any set F. On the other hand, ergodicity implies that $T^{-1}(E)$ becomes independent of F on the average.

One can also define mixing in terms of *observables*. Let $(X, \mathcal{B}, \mu)$ be a probability space. An observable is a real-valued function $\varphi : X \to \mathbb{R}$. A measure μ on X is said to be (strongly) mixing if for all $\varphi \in C(X)$ and $\psi \in L^1(\mu)$:

$$\lim_{k\to\infty} \int \varphi(T^k x)\psi(x)d\mu(x) = \int \varphi d\mu \int \psi d\mu \tag{1.3}$$

The above formulation gives an easier way to calculate the rate of mixing. If we define the correlation function $C_n(\varphi, \psi)$ as follows:

$$C_k(\varphi, \psi) := \int (\varphi \circ T^k)\psi d\mu - \int \varphi d\mu \int \psi d\mu \tag{1.4}$$

Then μ is mixing is equivalent to saying that C_k converges to 0 as $k \to \infty$.

Determining the rate of mixing is of great interest of dynamicists, and transfer operators are great tools to find the rate of mixing.

1.3 Transfer Operators

The *transfer operator* (or sometimes referred to as *Perron-Frobenius operator*) is a popular tool in ergodic theory. The transfer operator describes the evolution of probability densities under the system's dynamics [2].

Definition 1.3.1. Let $(X, \mathcal{B}, \mu)$ be a probability space, and $T : X \to X$ a non-singular transformation. The *transfer operator* associated with T, $P_T : L^1 \to L^1$ is defined as below:

For an $f \in L^1$, $P_T f$ is the unique (up to almost everywhere equivalence) function in L^1 such that:

$$\int_A P_T f d\mu = \int_{T^{-1}A} f d\mu \tag{1.5}$$

for any set $A \in \mathcal{B}$.

Proposition 1.3.2. *[2] (Properties of transfer operators)*

1. *(Linearity)* $P_T : L^1 \to L^1$ *is a linear operator.*

2. *(Positivity) If* $f \in L^1$ *and* $f \geq 0$*, then* $P_T f \geq 0$.

3. $\int_X P_T f d\mu = \int_X f d\mu$.

4. P_T *is a contraction, i.e.,* $\|P_T f\|_{L^1} \leq \|f\|_{L^1}$ *for any* $f \in L^1$.

5. *If* $T, \hat{T} : X \to X$ *are both non-singular transformations, then:*

$$P_{T \circ \hat{T}} f = P_T \circ P_{\hat{T}} f \tag{1.6}$$

$$P_{T^n} f = P_T^n f \tag{1.7}$$

6. *(Adjoint property) If* $f \in L^1$ *and* $g \in L^\infty$*, then:*

$$\int (P_T f) g d\mu = \int f (g \circ T) d\mu. \tag{1.8}$$

Transfer Operators are a major tool to prove the existence of absolutely continuous invariant measure. Existence of such measures corresponds to the existence of fixed point of the transfer operator P_T. Moreover, using property 1.8, the correlation function can be characterized in terms of $P_T^k(\psi)$; hence, decay of correlation can be studied via the spectrum of P_T.

Chapter 2

Introduction to Nonstationary Dynamical Systems

Nonstationary systems are those that have the underlying dynamics depending on time. Compared to the classical systems where only one map gets iterated, nonstationary systems appeal to a wider range of physical models and applications where we can let the environments vary with time. In the discrete-time situations, the nonstationary dynamics is determined by a composition of the form $T_n \circ \cdots \circ T_1$, where $T_j : X \to X$. If the maps are chosen randomly according to a distribution, we usually refer to such systems as *random dynamical systems*. In the subsequent sections, we do not assume any distribution on the collection of maps.

2.1 Ergodicity and Mixing

Nonstationary systems, sometimes, are referred to as sequential dynamical systems. The term was coined by Berend and Bergelson in [1]. In this dissertation, they extended the notions of ergodicity and mixing to nonstationary systems, under the condition that all transformations are assumed to be measure preserving.

Definition 2.1.1. (Ergodicity) Let $(X, \mathcal{B}, \mu)$ be a probability space and $\tilde{T} = \{T_n\}_n$ be a sequence of measure-preserving transformations, $T_j : X \to X$. $\tilde{T}$ is said to be *ergodic* if for all A, $B \in \mathcal{B}$:

$$\lim_{N\to\infty} \frac{1}{N^2} \sum_{m,n=1}^{N} \mu(T_m^{-1}A \cap T_n^{-1}B) = \mu(A)\mu(B). \tag{2.1}$$

Definition 2.1.2. (Bounded fibre) Let A be any set, and $B \subset A \times A$. We say that B is of *bounded fibre* if there exists some number c such that for every $a_1 \in A$, the set B contains at most c elements of the form (a_1, a_2) with $a_2 \in A$.

Definition 2.1.3. (Strongly mixing) Given a probability space $(X, \mathcal{B}, \mu)$, and $\tilde{T}$ a sequence of measure-preserving transformations as in 2.1.1. $\tilde{T}$ is said to be stronly mixing if for any $A, B \in \mathcal{B}$, and $\epsilon > 0$, the set of solutions (m, n) of:

$$|\mu(T_m^{-1}A \cap T_n^{-1}B) - \mu(A)\mu(B)| \geq \epsilon \tag{2.2}$$

is of bounded fibre.

In the above context, ergodicity and mixing were shown for a sequence of affine transformations on a compact Hausdorff group [1].

Alessandro, Melzic and Dahleh took a step further to show the existence of the time average under the condition that not only are the transformations $\{T\}_n$ invertible and measure-preserving but also the sequence of transformations $\{T_n\}_n$ converges to an ergodic transformation in the weak topology on the space of invertible, measure-preserving transformation [7]. In particular, let $(X, \mathcal{B}, \mu)$ be a probability space and $\{T_n\}_n$ be a sequence of invertible, measure-preserving transformations. Let $\mathcal{T}_n = T_n \circ \cdots \circ T_1$. For any observable $f : X \to \mathbb{R}$ so that $f \in L^1(\mu)$, the Birkhoff time average for the nonstationary systems $(X, \mathcal{B}, \mu, (T_n)_n)$ is defined to be:

$$S_N \coloneqq \frac{1}{N} \sum_{n=0}^{N-1} f(\mathcal{T}_n(x)) \tag{2.3}$$

Before stating the theorem, let us recall the weak topology on the space of invertible, measure-preserving transformations from [7]. Denote this space as $\mathcal{F}$. $\mathcal{F}$ is endowed with a metric:

$$\rho_1(T_1, T_2) = \mu\{x \in X : T_1(x) \neq T_2(x)\} \tag{2.4}$$

The metric (2.4) induces a strong topology on $\mathcal{F}$. The weak topology on $\mathcal{F}$ is defined as follows: given any two sequences $(T_n^{(1)})$ and $(T_n^{(2)})$ on $\mathcal{F}$, we say that $(T_n^{(1)})$ tends to $(T_n^{(2)})$ if:

$$\lim_{n\to\infty} \mu(T_n^{(1)} A \triangle T_n^{(2)} A) = 0 \tag{2.5}$$

for each $A \in \mathcal{B}$.

Theorem 2.1.4. *[7] Let $\{T_n\}_n$ be a sequence of invertible, measure-preserving transformation on the probability space $(X, \mathcal{B}, \mu)$. Assume that $\{T_n\}_n$ converges to an ergodic transformation $T : X \to X$ in the weak topology (2.5). Then, for every $f : X \to \mathbb{R}$, $f \in L^2(\mu)$, the Birkhoff time average S_N converges to $\int_X f d\mu$ in the L^2 sense.*

2.2 Almost Sure Invariance Principle

While it is sensible to ask for the existence and uniqueness of absolutely continuous invariant measures for stationary systems, this is not the case for general nonstationary systems. The reason is because the dynamics in the nonstationary systems vary with time, and it is not always that all transformations in the sequence $\{T_n\}_n$ are measure-preserving. For that reason, dynamicists have been pursuing other statistical properties of nonstationary dynamical systems. These properties include, most notably, the *almost sure invariance principle* [9, 12]. Roughly speaking, the almost sure invariance principle (ASIP) is the property that allows matching of trajectories of the dynamical systems of interest with a Brownian motion so that the error is negligible in comparison with the Birkhoff sum. In mathematical terms, we can define the almost sure invariance principle as follows:

Definition 2.2.1. (Almost sure invariance principle) [12] Let $\{U_j\}_j$ be a sequence of random variables on the probability space (X, μ) with $\mu(U_j) = 0$ for all j. We say that $\{U_j\}_j$ satisfies ASIP if there is a sequence of independent centered Gaussian random variables $\{Z_j\}_j$ such that on possibly an extended space:

$$\sum_{j=1}^{n} U_j = \sum_{j=1}^{n} Z_j + \mathcal{O}(\sigma_n^{1-\gamma}) \tag{2.6}$$

almost surely for some $\gamma > 0$, where:

$$\sum_{j-1}^{n} \mathbb{E}[Z_j^2] = \sigma_n^2 \to \infty. \tag{2.7}$$

($\mathbb{E}$ denotes the expectation).

Haydn, Nicol, Török, and Vaienti showed that on a large class of expanding nonstationary dynamical systems, ASIP is satisfied for Hölder and bounded observations [12]. Similarly to Conze and Raugi in [5], they consider sequences of non-invertible, non-singular transformations with respect to Lebesgue or Haar measure on a compact subset X of $\mathbb{R}^d$ or torus $\mathbb{T}^d$. We note that a transformation $T : (X, \mathcal{B}, \mathfrak{m}) \to (X, \mathcal{B}, \mathfrak{m})$ is non-singular if given $A \in \mathcal{B}$ that satisfies $\mathfrak{m}(T(A)) = 0$, then $\mathfrak{m}(A) = 0$. In order to obtain ASIP, Haydn et al. impose some additional conditions:

> Given a sequence of non-invertible, non-singular transformations $\{T_n\}_n$ on a compact set X, let $\mathcal{V} \subset L^1(\mathfrak{m})$ be a Banach space of functions over X with norms $\|\cdot\|_\alpha$ such that $\|\varphi\|_\infty \leq \text{const}\|\varphi\|_\alpha$. Denote $\mathcal{T}_n = T_n \circ \cdots \circ T_1$ as before. Assume further that:
>
> 1. Suppose that $P_1, \ldots, P_n$ are Perron-Frobenius transfer operators corresponding to the maps $T_1, \ldots, T_n$. There exist constants $C_1, C_2 > 0$, and $\hat{\gamma} \in (0, 1)$ such that for any n, and $f \in \mathcal{V}$ of zero Lebesgue mean:
>
> $$\|P_n \circ \cdots \circ P_1 f\|_\alpha \leq C_1 \hat{\gamma}^n \|f\|_\alpha + C_2 \|f\|_1. \tag{2.8}$$
>
> 2. There exists $\delta > 0$ such that for any sequence $P_n, \ldots, P_1$ of transfer operators:
>
> $$\inf_{x \in X} P_n \circ \cdots \circ P_1 \mathbb{1}(x) \geq \delta. \tag{2.9}$$

Definition 2.2.2. (Smooth expanding map) Given X a compact, connected Riemannian manifold without boundary, a smooth map $T : X \to X$ is called expanding if there exists $\lambda > 1$ such that:

$$|DT(x)v| \geq \lambda |v| \tag{2.10}$$

for every $x \in X$ and every tangent vector v at x.

The theorem below assumes that $X = [0,1]$ and the sequence of transformations $\{T_n\}_n$ is of expanding maps. Given a sequence $\{\varphi_n\}_n$ in $\mathcal{V}$, we define $\sigma_n^2 = \mathbb{E}(\sum_{i=1}^n \tilde{\varphi}_i(T_i \dots T_1))^2$, where $\tilde{\varphi}_n = \varphi_n - \mathfrak{m}(\varphi(T_n \dots T_1))$.

Theorem 2.2.3. *(ASIP for nonstationary expanding maps on $[0,1]$) [12]*

Let $\{\varphi_n\}_n$ be a sequence in $\mathcal{V}$ such that $\sup_n \|\varphi_n\|_\alpha < \infty$ (hence, $\sup_n \mathbb{E}(\varphi_n)^4 < \infty$). Assume (2.8) and (2.9) and $\sigma_n \geq n^{\frac{1}{4}+\delta}$ for some $0 < \delta < \frac{1}{4}$. Then, the sequence $\{\varphi_n \circ \mathcal{T}_n\}_n$ satisfies the ASIP. In other words, there exists a sequence $\{Z_n\}_n$ of independent centered Gaussian variables (on possibly an extended space) such that for any $\beta < \delta$:

$$\sup_{1\leq k\leq n} |\sum_{i=1}^{k} \tilde{\varphi}_n(T_i \dots T_1) - \sum_{i=1}^{k} Z_i| = o(\sigma_n^{1-\beta}) \qquad \mathfrak{m}\text{-}\textit{almost surely} \tag{2.11}$$

Moreover,

$$\sum_{j=1}^{n} \mathbb{E}[Z_i^2] = \sigma_n^2 + \mathcal{O}(\sigma_n). \tag{2.12}$$

Furthermore, ASIP can be shown for systems of 1-dimensional covering maps. In particular, assume that the transformation $T : X \to X$, where $X = [0,1]$, and T is a piecewise uniformly expanding transformation, i.e.:

1. There exists a partition $\mathcal{A} = \{A_k : 1 \leq k \leq m\}$ of the unit interval into intervals so that T is locally injective on the open intervals A_k,

2. T is C^2 on each A_k and it has a C^2 extension to the boundaries,

3. There exist constants $\lambda > 1$, $C > 0$ so that:

$$\inf_{x\in X} |DT(x)| > \lambda \qquad \text{and} \qquad \sup_{x\in X} \left|\frac{D^2T(x)}{DT(x)}\right| \leq C \tag{2.13}$$

Haydn et al. introduce some additional local noise to the dynamics T by defining maps T_ε on each interval A_k in the following form: $T_\varepsilon(x) = T(x) + \varepsilon$, where $|\varepsilon| < 1$ and extend the continuity to the boundaries. The sign of ε (may) change with A_k so that the image $T_\varepsilon(A_k)$ stays within the unit interval. In addition, there exists a set $J \subset [0, 1]$ so that:

- $J \subset T_\varepsilon(A_k)$ for all T_ε, for $k = 1, \ldots, m$,
- There exists a number $L' \in (0, 1]$ such that $|T(J) \cap A_k| > L'$, for all $k = 1, \ldots, m$.

Theorem 2.2.4. *[12] Let $\mathcal{F}$ be the family of T_ε described as above. $\mathcal{F}$ consists of the sequence $\{T_{\varepsilon_k}\}_k$ where $\{\varepsilon_k\}_{k\geq 1}$ satisfies $|\varepsilon_k| \leq k^{-\theta}, \theta \geq \frac{1}{2}$. Let $\mathcal{T}_n = T_{\varepsilon_n} \circ \cdots \circ T_{\varepsilon_1}$. Suppose that φ (in the Banach space $\mathcal{V}$) is not a coboundary for T, then the sequence*

$$U_n \coloneqq \sum_{j=0}^{n-1} \varphi \circ \mathcal{T}_j \tag{2.14}$$

satisfies a standard ASIP with variance σ^2.

2.3 Statistical Memory Loss

As opposed to Conze and Raugi in [5] or Haydn and collaborators in [12], Ott, Stenlund, and Young derived an analogous notion of decay of correlation for nonstationary systems called *statistical memory loss* [19].

Definition 2.3.1. Let $\rho_0, \hat{\rho}_0$ be initial distributions (with respect to the reference measure $\mathfrak{m}$) of a dynamical system and $\int \rho_0 \, d\mathfrak{m} = \int \hat{\rho}_0 \, d\mathfrak{m}$. Denote by $\rho_t, \hat{\rho}_t$ the time evolution of ρ_0 and $\hat{\rho}_0$, respectively. The system loses its memory in the statistical sense if:

$$\lim_{t\to\infty} \int |\rho_t - \hat{\rho}_t| d\mathfrak{m} = 0. \tag{2.15}$$

Definition 2.3.2. Let $\rho_0, \hat{\rho}_0, \rho_t, \hat{\rho}_t$ be as in the previous definition. We say that the system has *exponential memory loss* if there exist constants $C > 0$ and $\alpha > 0$ so that for $t \geq 0$:

$$\int |\rho_t - \hat{\rho}_t| < Ce^{-\alpha t}. \tag{2.16}$$

Remark 2.3.3. The exponential statistical memory loss can happen over a finite time in which the Definition 2.3.2 holds true for $t \leq t_0$, for some time t_0.

2.3.1 Smooth Expanding Nonstationary System

Exponential statistical memory loss was successfully proved for nonstationary systems of smooth expanding maps on a compact Riemannian manifold and C^2- piecewise expanding circle maps on using the method of "matching" (coupling) densities [19]. For the smooth expanding maps case, in order to obtain statistical memory loss, Ott, Stenlund and Young impose a bound on the set of transformations as follows: let X be a compact, connected Riemannian manifold without boundary. For constants $\lambda \geq 0$ and $\Gamma \geq 0$, define:

$$\mathcal{E}(\lambda, \Gamma) := \{T : X \to X : \|T\|_{C^2} \leq \Gamma, |DT(x)v| \geq \lambda |v| \quad \forall (x, v)\} \tag{2.17}$$

where v is the tangent vector at x.

Analogous to the Banach space $\mathcal{V}$ in the context of the almost sure invariance principle is the set of densities:

$$\mathcal{D} := \{\varphi > 0 : \int \varphi \; d\mathfrak{m} = 1, \quad \varphi \text{ is Lipschitz}\} \tag{2.18}$$

As before, we denote the Perron-Frobenius transfer operator corresponding to a map T by P_T, and the nonstationary dynamics is $\mathcal{T}_n = T_n \circ \cdots \circ T_1$ for all $T_j \in \mathcal{E}$.

Theorem 2.3.4. *[19] (Statistical memory loss for nonstationary smooth expanding maps)*

Given $\lambda > 1$ and Γ as above (see (2.17)), there exists a constant $\Lambda \in (0,1)$, Λ depends on λ and Γ, such that for any sequence $\{T_n\}_n \in \mathcal{E}(\lambda, \Gamma)$, the following is true: for any $\varphi, \psi \in \mathcal{D}$, there exists a constant $C_{(\varphi,\psi)}$ such that for all $n \geq 0$:

$$\int |P_{\mathcal{T}_n}(\varphi) - P_{\mathcal{T}_n}(\psi)| \, d\mathfrak{m} \leq C_{(\varphi,\psi)} \Lambda^n. \tag{2.19}$$

2.3.2 Nonstationary Piecewise Expanding Systems

Similarly to Haydn et al. in [12], Ott, Stenlund, and Young also consider a nonstationary system with a piecewise uniformly expanding maps on the circle (see Section 2.2 for the definition of a piecewise uniformly expanding transformation). In order to obtain statistical memory loss, they use the *enveloping* property for the underlying dynamics.

Let $\mathcal{A}$ be a partition of the circle S^1 into intervals. In analogy with the previous section, we think of S^1 as $[0,1]$ with endpoints identified.

Notation 2.3.5.

- $\mathcal{A}_n \coloneqq \bigvee_{i=1}^{n} T^{-(i-1)}(\mathcal{A})$ is the dynamical partition of S^1 by T after n iterations.
- Given a set $J \subset S^1$, $int(J)$ denotes the interior of J.
- $\mathcal{A}_n|I$ denotes the restriction of $\mathcal{A}_n$ to the set I.

Definition 2.3.6. [19] The transformation $T : S^1 \to S^1$ is said to be *enveloping* if there exists a natural number N such that for every $I \in \mathcal{A}$,

$$\bigcup_{J \in \mathcal{A}_N | I} T^N(int(J)) = S^1. \tag{2.20}$$

The smallest of such number N is called the *enveloping time.*

Furthermore, exponential memory loss property for this setting relies on the notion of *overcovering.*

Definition 2.3.7. Let $I \in \mathcal{A}$ be given. Suppose that the enveloping time of T is N. We say that $T^N|I$ *overcovers* S^1 if for every $z \in S^1$, the $z \in T^N(J)$ for some $J \in \mathcal{A}_N|I$. Moreover, the element z stays away from the boundary $T^N(\partial J)$.

Instead of adding some local noise to the transformation T, Ott, Stenlund, and Young take maps that are in a "good" neighborhood of an enveloping map T to produce a local result. In particular,

Definition 2.3.8. [19] Given a C^2-piecewise, enveloping transformation T with discontinuities $x_1 = x_{k+1}, x_2, \ldots, x_k$ labeled clockwise. Let $d_\Omega(T) := |x_{i+1} - x_i|$. Assume $\varepsilon < \frac{1}{4} d_\Omega(T)$. We say that a map $\hat{T}$ is ε-near T if:

1. If $\{y_1 = y_{k+1}, y_2, \ldots, y_k\}$ is the set of discontinuities of $\hat{T}$, then $|y_i - x_i| < \varepsilon$
2. Let ξ (dependent on $T, \hat{T}$) map $[x_i, x_{i+1}]$ affinely onto $[y_i, y_{i+1}]$. Then, ξ satisfies:

$$\|T \circ \xi - g\|_{C^2} < \varepsilon. \tag{2.21}$$

Unlike the smooth expanding maps case earlier, the space of densities that is employed for piecewise expanding maps is the space of bounded variation functions.

Let $\text{Var}(\varphi)$ denote the total variation of φ, then the space of bounded variation functions $BV(S^1, \mathbb{R})$ is defined as:

$$BV(S^1, \mathbb{R}) := \{\varphi : S^1 \to \mathbb{R} : \text{Var}(\varphi) < \infty\}. \tag{2.22}$$

Hence, the space of densities for C^2-piecewise expanding maps is

$$\mathcal{D} := \{\varphi \in BV(S^1, \mathbb{R}) : \varphi \geq 0, \int_{S^1} \varphi(x)dx = 1\}. \tag{2.23}$$

The local result is stated as below:

Theorem 2.3.9. *[19] Let T be the a C^2-piecewise expanding, enveloping transformation. Then, there exist constants $\Lambda > 1$ and $\varepsilon > 0$ small enough (ε depends on T) such that for all $\hat{T}_i$ that are ε-near T, the following holds true: Given $\varphi, \psi \in \mathcal{D}$, there exists a constant C (depending on φ and ψ) such that for all $n \in \mathbb{Z}$:*

$$\int_{S^1} |P_{\mathcal{T}_n}(\varphi) - P_{\mathcal{T}_n}(\psi)|dx \leq C\Lambda^n. \tag{2.24}$$

Exponential statistical memory loss from the one dimensional piecewise expanding case can be extended to higher dimensions. However, when we get to dimensions greater than 1, the enveloping condition no longer exists, and we have to deal with the complexity of the dynamical partitions, especially at the boundaries. Moreover, we need a space of densities that is analogous to the space of bounded variation functions for multidimensional systems. There are several papers in the literature that address the problem with the space of densities. Two spaces that are commonly used are: generalized bounded variation functions [6, 14] and quasi-Hölder functions [21].

Gupta, Ott and Török [11] gave a result of statistical memory loss for higher dimensional nonstationary systems in a similar form as the 1-D case given that the complexity around boundaries and the expanding properties are balanced. The details about the maps and the dynamical partitions are in the next section as we are taking advantage of the setting in [11]. We also note that in [11], the authors chose to use quasi-Höder functions for their space of densities instead of the generalized bounded variation functions.

Most recently, Geiger and Ott showed that a multidimensional nonstationary system with holes (a different term for it is *nonstationary open dynamical system*) also exhibits exponential statistical memory loss [10].

The next two chapters fit under the umbrella of non-stationary dynamical systems. However, the underlying space structures are different: Chapter 3 offers a generalization of non-stationary dynamical systems by using a *triangular array* and a *curve of maps* while Chapter 4 considers the non-stationary dynamical systems on a lattice. We also note that notations are used differently in Chapters 3 and 4.

Chapter 3

Quasistatic Dynamical Systems

3.1 Introduction

Quasistatic dynamical systems (QDS) were first introduced by Mikko Stenlund in [22]. The phenomenon resembles the notion of quasistatic in thermodynamics, where the system's thermodynamic process changes infinitesimally little so that at any point, the system remains at equilibrium. Mathematically, QDS is represented by a triangular array, and a curve of maps. Precisely,

Definition 3.1.1. [8] Let X be a set, and $\mathcal{M} = \{T : X \to X\}$ be a collection of self-maps, endowed with a topology. Define the triangular array $\mathcal{T}$ by:

$$\mathcal{T} = \{T_{n,k} \in \mathcal{M} : 0 \leq k \leq n, n \geq 1\}.$$

If there exists a piecewise-continuous curve $\gamma : [0, 1] \to \mathcal{M}$ such that for all

$t \in [0,1]$:

$$\lim_{n\to\infty} T_{n,\lfloor nt \rfloor} = \gamma_t, \tag{3.1}$$

then we say that the pair $(\mathcal{T}, \gamma)$ is a quasistatic dynamical system.

X is the state space, and $\mathcal{M}$ is the system space.

A simple example of QDS is when the curve is discretized. That means, all of the maps of the triangular array are on the curve γ, and as $n \to \infty$, the mesh becomes finer, the speed of traversing the curve is sent to 0. Another example is when $T_{n,k} = T$ for all $0 \le k \le n, n \ge 1$, in which case, we have an ordinary dynamical system. In other cases, we can think of n as the rows in the triangular arrays and k is the position of the transformation on the row. Therefore, the time t in the limit (3.1) can be thought as the "angle" of convergence when n approaches infinity.

To derive the statistical properties of this systems, the scheme is to first derive the properties on each row n, then see how these properties evolve when taking $n \to \infty$. Several results on statistical properties such as ergodicity and convergence in distribution have been shown for QDS when the transformations T are smooth one-dimensional maps [8, 22] and intermittent maps [17, 18]. Some results on the second moments were also present for the above maps in QDS [8, 18].

3.1.1 QDS Almost-sure Convergence for Circle Expanding Maps

Fix $\lambda > 1$ and $A > 0$. Let $T : \mathbb{S}^1 \to \mathbb{S}^1$ be the map from the circle to itself, and $\mathcal{M} = \{T : \mathbb{S}^1 \to \mathbb{S}^1\}$ so that all maps $T \in \mathcal{M}$ satisfy:

$$\inf_{x\in\mathbb{S}^1} T'(x) > \lambda \quad \text{and} \quad \|T''\|_\infty \le A. \tag{3.2}$$

$\mathcal{M}$ is endowed with a metric d_* defined by:

$$d_*(T_1, T_2) = \sup_{x \in \mathbb{S}^1} d(T_1(x), T_2(x)) + \|T_1' - T_2'\|_\infty, \tag{3.3}$$

where d is the natural metric on the circle $\mathbb{S}^1$.

Let $\gamma : [0,1] \to \mathcal{M}$ be a Hölder continuous curve with exponent $\eta \in (0,1)$ and $\mathcal{T} = \{T_{n,k} \in \mathcal{M}\}$ be a triangular array that satisfies:

$$\sup_{n \geq 1} \sup_{t \in (0,1)} d_*(T_{n,\lfloor nt \rfloor}, \gamma_t) < \infty. \tag{3.4}$$

Before defining QDS time average and space average, we introduce some shorthand notations that will be used throughout.

Notation 3.1.2.

1. Given an observable $f : X \to \mathbb{R}$ and $T_{n,j} \in \mathcal{T}$, $1 \leq j \leq k$, we denote $f_{n,k} = f \circ T_{n,k} \circ \cdots \circ T_{n,1}$.
2. If u is a real number, $\{u\}$ denotes the fractional part of u.

Definition 3.1.3. On the n-th row, the QDS's Birkhoff-like sum $S_n : X \times [0,1] \to \mathbb{R}$ is defined to be:

$$S_n(x,t) = \int_0^{nt} f_{n,\lfloor s \rfloor}(x) ds.$$

and the QDS time average is:

$$\xi_n(x,t) = \frac{1}{n} S_n(x,t) = \int_0^t f_{n,\lfloor ns \rfloor}(x) ds. \tag{3.5}$$

Remark 3.1.4.

1. For any given x and n, the map $t \mapsto S_n(x,t)$ is a piecewise linear interpolation of Birkhoff sums $\sum\limits_{j=0}^{nt-1} f_{n,j}(x) + \{nt\} f_{n,\lfloor nt \rfloor}(x)$.
2. Fix the row n, and if μ is an initial distribution of x, ξ_n can be considered as a random element of $C^0([0,1], \mathbb{R})$. We denote its distribution by $\mathbf{P}_n^\mu$.

Let $\mathfrak{m}$ denote the Lebesgue measure on $\mathbb{S}^1$. It is well-known that each $T \in \mathcal{M}$ has a unique invariant measure $\hat{\mu}_T$ that is absolutely continuous with respect to $\mathfrak{m}$.

Notation 3.1.5. Throughout the chapter, we will use $\hat{\mu}_t$ to denote the SRB measure $\hat{\mu}_{\gamma_t}$, and $\hat{\mu}_{n,k}$ to denote $\hat{\mu}_{T_{n,k}}$.

Theorem 3.1.6. *[22] Fix an initial distribution μ for x. Assume that μ is absolutely continuous with respect to the Lebesgue measure $\mathfrak{m}$ and the observable $f : \mathbb{S}^1 \to \mathbb{R}$ is Lipschitz, then*

$$\lim_{n\to\infty} \sup_{t\in[0,1]} |\xi_n - \xi| = 0 \tag{3.6}$$

almost μ-everywhere x and $\xi : [0, 1] \to \mathbb{R}$ is defined to be:

$$\xi(t) = \int_0^t \int_{\mathbb{S}^1} f \; d\hat{\mu}_s ds. \tag{3.7}$$

Theorem 3.1.6 is analogous to the Birkhoff theorem for the stationary dynamical systems.

An interesting question is as follows: if we use high-dimensional piecewise expanding maps instead of using expanding circle maps for the system space, how would the QDS time averages behave as n approaches infinity? This is what we will discuss in the next section.

3.2 QDS with Multidimensional Maps

For the setting, we will consider the same piecewise $C^{1+\alpha}$ expanding maps on N-dimensional torus $\mathbb{T}^N$ that were mentioned in [10,11]. As mentioned earlier in Chapter 2, the complexity of the dynamical partitions play an important role. We begin by some notations used throughout this chapter.

Notation 3.2.1. 1. We denote the Lebesgue measure on $\mathbb{T}^N$ by $\mathfrak{m}$

2. $\mathfrak{m}(f)$ is the same as $\int f d\mathfrak{m}$ for any $f \in L^1(\mathfrak{m})$.

3. $\mathcal{L}_T$ denotes the transfer operator associated to the map T.

Definition 3.2.2. (Partitions) Fix $K > 0$. Let $\mathcal{A} = \{U_i : 1 \leq i \leq m\}$ be a partition of $\mathbb{T}^N$ where U_i's are pairwise disjoint open sets. We say that $\mathcal{A} \in \mathcal{R}(K)$ if the following hold:

1. $\mathfrak{m}(\mathbb{T}^N \setminus \cup_{i=1}^m U_i) = 0$;

2. For each U_i, there exist finitely many compact C^2-embedded codimension-1 submanifolds $\{\Gamma_{ij}\}_j$. the boundary ∂U_i is contained in $\cup_j \Gamma_{ij}$;

3. For each Γ_{ij}, there are finitely many C^2-charts $\Phi_{l;ij} : B^N \subset \mathbb{R}^N \to W_{l;ij} \subset \mathbb{T}^N$, where B^N is the N-dim unit ball of $\mathbb{R}^N$, . C^2-norm of $\Phi_{l;ij}$ and $\Phi_{l;ij}^{-1}$ is less than K, and $\Gamma_{ij} \subset \cup_l \Phi_{l,ij}(B^N \cap (\mathbb{R}^{N-1} \oplus 0))$.

Denote $\mathcal{R} = \cup_{K>0}\mathcal{R}(K)$, and $\kappa(\mathcal{A}) = \sup_{x\in\mathbb{T}^N}\{\Gamma_{ij} : x \in \Gamma_{ij}\}$.

Definition 3.2.3. (Piecewise Expanding Maps) Fix $0 \leq \lambda \leq 1, K > 0, \kappa > 0, \alpha > 0$.:

$$\lambda^\alpha + \left(\frac{4\lambda\kappa}{1-\lambda}\right)\left(\frac{\Pi_{N-1}}{\Pi_N}\right) < 1,$$

where Π_N is the volume of the unit ball in $\mathbb{R}^N$.

$T : \mathbb{T}^N \to \mathbb{T}^N$ is said to be $C^{1+\alpha}$ piecewise expanding map if:

1. $T \in C(\mathcal{A})$ for some partition $\mathcal{A} \in \mathcal{R}(K)$, i.e., there exists a partition $\mathcal{A} = \{U_i\}_i$. T is continuous on each U_i. We often refer $\mathcal{A}$ as $\mathcal{A}(T)$.

2. For each U_i, there exists an open set $V_i \supset \overline{U_i}$ s.t. $T_{(i)} : V_i \to \mathbb{T}^N$ is a C^1 diffeomorphism onto its image, and for some small $\epsilon_0, T_{(i)}(V_i) \supset \overline{B_{\epsilon_0}(T(U_i))}$;

3. For each i, $T|U_i$ and all its partial derivaties extend to continuous functions on the closure of U_i and $\|D(T_{(i)}^{-1})\| < \lambda$ on $T_{(i)}(V_i)$;

4. $\|T_{(i)}\|_{1+\alpha} < K$ on V_i;

5. $\kappa(\mathcal{A}) < \kappa$.

Notation 3.2.4. We denote the space of piecewise expanding maps in Definition 3.2.3 by $\mathcal{M}$.

Definition 3.2.5. (Nearby maps) Let $T, \tilde{T}$ be piecewise expanding maps (Definition 3.2.3). We say that $\tilde{T}$ is a δ-perturbation of T if:

1. There exist partitions $\mathcal{A}$, and $\tilde{\mathcal{A}} \in \mathfrak{R}(K)$ s.t. $T \in C(\mathcal{A})$, and $\tilde{T} \in C(\tilde{\mathcal{A}})$. Moreover,

   (a) $\mathcal{A}$ and $\tilde{\mathcal{A}}$ have the same number of partition elements;

   (b) There is a correspondence between the boundary components Γ_{ij} and $\tilde{\Gamma}_{ij}$

   (c) For each i, Hausdorff distance between U_i and $\tilde{U}_i$ is less than δ, where Hausdorff distance is defined as:

   $$d_{haus}(U_i, \tilde{U}_i) = \max\{\sup\{dist(x, U_i) : x \in \tilde{U}_i\}, \sup\{dist(x, \tilde{U}_i) : x \in U_i\}\}.$$

2. Outside δ-neighborhood of the boundaries, the maps $T, \tilde{T}$ are δ-close in $C^{1+\alpha}$, i.e.:

   $$\|T|_{W_i} - \tilde{T}|_{W_i}\|_{1+\alpha} < \delta$$

   where $W_i = \{x \in U_i \cap \tilde{U}_i \mid dist(x, U_i^c) > \delta, dist(x, \tilde{U}_i^c) > \delta\}$

Remark 3.2.6. The topology on $\mathcal{M}$ is created by varying T and δ.

Definition 3.2.7. (Mixing maps) Let $\zeta_1 \in (0,1)$ and $\zeta_2 \in (1,\infty)$. We say that $T \in \mathcal{E}(\zeta_1,\zeta_2)$ if for every finite partition $\mathcal{A}$ of X, there exists $J(\mathcal{A},\zeta_1,\zeta_2)$ such that for all $A_1, A_2 \in \mathcal{A}$, we have:

$$\zeta_1 < \frac{\mathfrak{m}(A_1 \cap T^i(A_2))}{\mathfrak{m}(A_1)\mathfrak{m}(A_2)} < \zeta_2, \tag{3.8}$$

for all $i \geq J(\mathcal{A},\zeta_1,\zeta_2)$.

3.2.1 Curve of Maps and Triangular Array

Our QDS setup will be a special case of the general QDS in a sense that each map of the triangular array will live on the curve, hence discretize the curve on each level n. In particular,

Fix $T \in \mathcal{E}(\zeta_1,\zeta_2)$ and let $\mathcal{N}(T,\delta)$ be the collection of maps $\tilde{T}$ where $\tilde{T}$ is a δ-perturbation of T. Let the curve $\gamma : [0,1] \to \mathcal{N}(T,\delta)$ be Hölder continuous with exponent $\eta \in (0,1)$, and define the triangular array:

$$\mathcal{T} = \{T_{n,k} = \gamma_{\frac{k}{n}} : 0 \leq k \leq n, n \geq 1; k,n \in \mathbb{N}\}. \tag{3.9}$$

Before stating the main theorem, we note that each map $T \in \mathcal{E} \cap \mathcal{M}(\zeta_1,\zeta_2)$ has a unique absolutely continuous invariant measure $\hat{\mu}_T$. This is a consequence of the Hilbert cone argument from the local result in [11].

3.2.2 Statement of the Result

Theorem 3.2.8. *Suppose that the observable f is Lipschitz continuous, and the initial distribution μ is absolutely continuous with respect to $\mathfrak{m}$. Assume that the* *function $t \mapsto \hat{\mu}_t(f)$ is continuous. Then, the distributions $\{\mathbf{P}_n^\mu\}_n$ of QDS time averages ξ_n converge to the point mass at $\xi \in C^0([0,1],\mathbb{R})$, where $\xi : [0,1] \to \mathbb{R}$ is defined to be:*

$$\xi(t) = \int_0^t \hat{\mu}_s(f)ds.$$

3.3 Proof of Theorem 3.2.8

3.3.1 Probabilistic Argument

To prove Theorem 3.2.8, we follow a standard probabilistic argument by proving that given any absolutely continuous initial distribution μ, the sequence $\{\mathbf{P}_n^\mu\}_n$ is tight, hence relatively compact, i.e. there exists a subsequence that has a weak limit. We will use Dynkin's formula [20] to prove that the limit is unique.

Lemma 3.3.1. *Given an arbitrary initial distribution μ, the sequence of distribution $(\mathbf{P}_n^\mu)_{n\geq 1}$ is tight.*

Proof. For any $t_1, t_2 \in [0,1]$, we have:

$$\xi_n(t_2) - \xi_n(t_1) = \int_{t_1}^{t_2} f_{n,\lfloor ns \rfloor}(x)ds.$$

Since f is Lipschitz, $|\xi_n(t_2) - \xi_n(t_1)| \leq (t_2 - t_1)\|f\|_\infty$. Hence, ξ_n is Lipschitz continuous and bounded for all n. Therefore, the sequence of distribution $(\mathbf{P}_n^\mu)_{n\geq 1}$ is tight.

□

Notation 3.3.2. Let us denote the expectation of ξ_n by $\mathbf{E}_n^{\mu}$.

For Dynkin's formula, we define the evaluation functional $\pi_t : C^0([0,1],\mathbb{R}) \to \mathbb{R}$

$$\pi_t(\omega) = \omega(t),$$

$t \in [0,1]$

Lemma 3.3.3. *Suppose that* $\mathbf{P}$ *is the weak limit of a subsequence* $\{\mathbf{P}_{n_k}^{\mu}\}_{(k\geq 1)}$. *Let* $\mathbf{E}$ *be the corresponding expectation of* $\mathbf{P}$. *Then, for any* $A \in C^{\infty}(\mathbb{R})$*:*

$$\frac{d}{dt}\mathbf{E}[A \circ \pi_t] = E[A' \circ \pi_t] \cdot \hat{\mu}_t(f). \tag{3.10}$$

Proof. Given $A \in C^{\infty}(\mathbb{R})$, as proved above $(\xi_n)_{n\geq 1}$ is Lipschitz and bounded, we have:

$$A(\xi_n(t+h)) - A(\xi_n(t)) = A'(\xi_n(t)).(\xi_n(t+h) - \xi(t)) + O(h^2).$$

Integrate the above with respect to μ, and take $n \to \infty$ along the subsequence $(n_k)_{k\geq 1}$. We will calculate the left-hand side and the right-hand side separately.

The left-hand side becomes

$$\lim_{k\to\infty} \mu[A(\xi_{n_k}(t+h)) - A(\xi_{n_k}(t))] = \mathbf{E}[A \circ \pi_{t+h} - A \circ \pi_t]$$

by definition. For the right-hand side, we make the following claims:

Claim 3.3.4. *As* $n \to \infty$*,*

$$\mu[A'(\xi_n(t)).(\xi_n(t+h) - \xi_n(t))] - \mu[(A'(\xi_n(t))].\mu[\xi_n(t+h) - \xi_n(t)] = o(1)$$

Proof. See Section 3.6. $\square$

Note that:

$$\lim_{k\to\infty} \mu[A'(\xi_n(t))] = \lim_{k\to\infty} \mathbf{E}^{\mu}_{n_k}(A' \circ \pi_t) = \mathbf{E}[A' \circ \pi_t]$$

We can easily show that $\mu(f_{n,\lfloor ns \rfloor}) = \mu_{n,\lfloor ns \rfloor}(f)$. Therefore,

$$\begin{aligned}
\mu(f_{n,\lfloor ns \rfloor}) - \hat{\mu}_s(f) &= \int f(\rho_{n,\lfloor ns \rfloor} - \hat{\rho}_s) d\mathfrak{m} \\
&\leq \|f\|_\infty \|\rho_{n,\lfloor ns \rfloor} - \hat{\rho}_s\|_{L^1}.
\end{aligned}$$

Claim 3.3.5. *For any $0 < \eta' < \eta\alpha < 1$, there exists a constant C such that:*

$$\|\rho_{n,\lfloor ns \rfloor} - \hat{\rho}_s\|_{L^1} \leq C n^{-\eta'},$$

for all n, and $s > \frac{-\eta' \log n}{n \log \Lambda}$.

Proof. See Section 3.5 □

Now,

$$\begin{aligned}
\mu(\xi_n(t+h) - \xi_n(t)) &= \int \xi_n(t+h) - \xi_n(t) d\mu \\
&= \int \Big(\int_0^{t+h} f_{n,\lfloor ns \rfloor} ds - \int_0^t f_{n,\lfloor ns \rfloor} ds \Big) d\mu \\
&= \int \int_t^{t+h} f_{n,\lfloor ns \rfloor} ds \; d\mu \\
&= \int_t^{t+h} \mu(f_{n,\lfloor ns \rfloor}) ds \\
&= \int_t^{t+h} \hat{\mu}_s(f) + \|f\|_\infty . C n^{-\eta'} ds
\end{aligned}$$

Take $n \to \infty$, we have that $\lim_{n\to\infty} \mu(\xi_n(t+h) - \xi_n(t)) = \int_t^{t+h} \hat{\mu}_s(f) ds$.

Lastly, $\int_t^{t+h} \hat{\mu}_s(f)ds = \hat{\mu}_t(f).h + o(h)$.

Hence, we finish the proof of Dynkin's formula (3.10). $\square$

Proposition 3.3.6. *Limit* $\mathbf{P}$ *is the point mass at* $\xi \in C^0([0,1], \mathbb{R})$*, where*

$$\xi(t) = \int_0^t \hat{\mu}_s(f)ds.$$

Proof.

$$\frac{d}{dt}\mathbf{E}[(\pi_t - \xi(t))^2] = \frac{d}{dt}\mathbf{E}[\pi_t^2] - 2\xi(t)\frac{d}{dt}\mathbf{E}[\pi_t] - 2\mathbf{E}[\pi_t]\frac{d}{dt}\xi(t) + \frac{d}{dt}(\xi(t))^2.$$

Applying 3.10 to the first two terms, we have that

$$\frac{d}{dt}\mathbf{E}[(\pi_t - \xi(t))^2] = 2\mathbf{E}[\pi_t]\hat{\mu}_t(f) - 2\xi(t)\hat{\mu}_t(f) - 2\mathbf{E}[\pi_t]\hat{\mu}_t(f) + 2\xi(t)\hat{\mu}_t(f) = 0.$$

Since $\pi_0 = 0$ almost surely with respect to $\mathbf{P}$, and $\xi(0) = 0$, $\mathbf{E}[(\pi_t - \xi(t))^2] = 0$ for $t \in [0,1]$. Hence,

$$\mathbf{E}(\int_0^1 [(\pi_t - \xi(t))^2]dt) = \int_0^1 \mathbf{E}[(\pi_t - \xi(t))^2]dt = 0.$$

We can conclude that the limit $\mathbf{P}$ is the point mass at $\xi \in C^0([0,1], \mathbb{R})$. $\square$

3.4 Statistical Memory Loss

Both Claims 3.3.4 and 3.3.5 rely heavily on statistical memory loss properties. Hence, as a preliminary to the proofs, we will make use of Quasi-Hölder space (see also [11] and [21]) and statistical memory loss results from [11]

Definition 3.4.1. (Quasi-Hölder space) Given $\varphi \in L^1(\mathfrak{m})$ and a Borel set $S \subset \mathbb{T}^N$, define the oscillation of φ on S by:

$$\operatorname{osc}(\varphi, S) \coloneqq \operatorname{Esup}(\varphi, S) - \operatorname{Einf}(\varphi, S).$$

where $\operatorname{Esup}(\varphi, S)$ and $\operatorname{Einf}(\varphi, S)$ are the essential supremum and essential infimum of the function φ in the set S Fix $\varepsilon_0 > 0$, define the seminorm:

$$|\varphi|_{\alpha,\varepsilon_0} \coloneqq \sup_{0<\varepsilon\leq\epsilon_0} \varepsilon^{-\alpha} \int_{\mathbb{T}^N} \operatorname{osc}(\varphi, B_\varepsilon(x)) d\mathfrak{m}(x).$$

Define:

$$\mathrm{OSC}_\alpha \coloneqq \{\varphi \in L^1(\mathfrak{m}) : |\varphi|_{\alpha,\varepsilon_0} < \infty\}.$$

Note: Although the seminorm $|\varphi|_{\alpha,\varepsilon_0}$ depends on ε_0, the space OSC does not. Moreover, OSC contains all compactly supported α-Hölder functions. We can also define the norm $\|\cdot\|_{\alpha,\varepsilon_0}$ on OSC_α by:

$$\|\varphi\|_{\alpha,\varepsilon_0} \coloneqq \|\varphi\|_{L^1(\mathfrak{m})} + |\varphi|_{\alpha,\varepsilon_0}.$$

$(\mathrm{OSC}_\alpha, \|\cdot\|_{\alpha,\varepsilon_0})$ is a Banach space, and the unit ball of $(\mathrm{OSC}_\alpha, \|\cdot\|_{\alpha,\varepsilon_0})$ is precompact in $L^1(\mathfrak{m})$.

We denote the space of densities as:

$$\mathcal{D} = \{\varphi \in \mathrm{OSC}_\alpha : \varphi \geq 0, \|\varphi\|_{L^1(\mathfrak{m})} = 1\}$$

Theorem 3.4.2. *There exists a constant $0 < \Lambda < 1$ such that the following holds for any $\mathfrak{T} = T_m \circ \ldots . \circ T_1$, where $T_1, \ldots, T_m \in \mathcal{T}$: given any densities $\phi, \psi \in \mathcal{D}$, there is a constant $C_{(\phi,\psi)} > 0$ such that*

$$\int_X |\mathcal{L}_{\mathcal{T}}(\phi) - \mathcal{L}_{\mathcal{T}}(\psi)| d\mathfrak{m} \leq C_{(\phi,\psi)} \Lambda^m.$$

for all $m \in \mathbb{N}$.

Proof. Proof of statistical memory loss was detailed in [11] by Gupta, Ott and Török . □

3.5 Proof of Claim 3.3.5

In order to prove the Claim 3.3.5, we need the following pertubation lemma:

Lemma 3.5.1. *Fix $n \in \mathbb{N}$. Given piecewise expanding maps $T = T_{n,k}, \tilde{T} = T_{n,l}$. then there exists a contant $C > 0$ such that:*

$$\int |\mathcal{L}_T(\varphi) - \mathcal{L}_{\tilde{T}}(\varphi)| \; d\mathfrak{m} \leq Cn^{-\eta\alpha} \|\varphi\|_{\alpha,\epsilon_0}$$

for all $\varphi \in \mathrm{OSC}_\alpha$.

Proof.

$$\begin{aligned}
\int |\mathcal{L}_T(\varphi) - \mathcal{L}_{\tilde{T}}(\varphi)| \; d\mathfrak{m} &= \sum_i \int_{T_{(i)}(U_i)\cap\tilde{T}_{(i)}(\tilde{U}_i)} \left| \frac{\varphi \circ T_{(i)}^{-1}}{|\det DT_{(i)}|} - \frac{\varphi \circ \tilde{T}_{(i)}^{-1}}{|\det D\tilde{T}_{(i)}|} \right| \; d\mathfrak{m} \\
&+ \int_{T_{(i)}(U_i)\setminus\tilde{T}_{(i)}(\tilde{U}_i)} \left| \frac{\varphi \circ T_{(i)}^{-1}}{|\det DT_{(i)}|} \right| \; d\mathfrak{m} \\
&+ \int_{\tilde{T}_{(i)}(\tilde{U}_i)\setminus T_{(i)}(U_i)} \left| \frac{\varphi \circ \tilde{T}_{(i)}^{-1}}{|\det D\tilde{T}_{(i)}|} \right| \; d\mathfrak{m} \\
&= \sum_i A_{(i)} + B_{(i)} + C_{(i)}
\end{aligned}$$

We will estimate $A_{(i)}$, $B_{(i)}$, and $C_{(i)}$ separately. For $A_{(i)}$:

$$\left| \frac{\varphi\circ T_{(i)}^{-1}}{|\det DT_{(i)}|} - \frac{\varphi\circ \tilde{T}_{(i)}^{-1}}{|\det D\tilde{T}_{(i)}|} \right| \leq \frac{\left|\varphi\circ T_{(i)}^{-1} - \varphi\circ \tilde{T}_{(i)}^{-1}\right|}{|\det DT_{(i)}|} + \left|\varphi \circ \tilde{T}_{(i)}^{-1}\right| \left[\frac{1}{|\det DT_{(i)}} - \frac{1}{|\det D\tilde{T}_{(i)}|} \right].$$

Since T and $\tilde{T}$ are δ-close and the curve is Hölder, for any $x \in T_{(i)}(U_i) \cap \tilde{T}_{(i)}(\tilde{U}_i)$,

$$|T_{(i)}^{-1} - \tilde{T}_{(i)}^{-1}| < n^{-\eta}, \tag{3.11}$$

so $\int_{T_{(i)}(U_i)\cap\tilde{T}_{(i)}(\tilde{U_i})} \left|\varphi\circ T_{(i)}^{-1} - \varphi\circ\tilde{T}_{(i)}^{-1}\right| \, d\mathfrak{m} \leq \int_{T_{(i)}(U_i)} \operatorname{osc}(\varphi, B_{n^{-\eta}}(T_{(i)}^{-1}(x))) \, d\mathfrak{m}$
Thus,

$$\begin{aligned}
&\int_{T_{(i)}(U_i)\cap\tilde{T}_{(i)}(\tilde{U_i})} \left|\varphi\circ T_{(i)}^{-1} - \varphi\circ\tilde{T}_{(i)}^{-1}\right| \cdot |\det DT_{(i)}|^{-1} \, d\mathfrak{m} \\
&\leq \int_{T_{(i)}(U_i)} \operatorname{osc}(\varphi, B_{n^{-\eta}}(T_(i)^{-1}(x))) \cdot |\det DT_{(i)}|^{-1} \, d\mathfrak{m} \\
&\leq \int_{U_i} \operatorname{osc}(\varphi, B_{n^{-\eta}}(y)) \, d\mathfrak{m}(y)
\end{aligned}$$

Thus,

$$\begin{aligned}
\sum_i \int_{T_{(i)}(U_i)\cap\tilde{T}_{(i)}(\tilde{U_i})} \left|\varphi\circ T_{(i)}^{-1} - \varphi\circ\tilde{T}_{(i)}^{-1}\right| \cdot |\det DT_{(i)}|^{-1} \, d\mathfrak{m} &\leq \sum_i \int_{U_i} \operatorname{osc}(\varphi, B_{n^{-\eta}}(y)) \, d\mathfrak{m}(y) \\
&\leq \int_{\mathbb{T}^N} \operatorname{osc}(\varphi, B_{n^{-\eta}}(y)) \, d\mathfrak{m} \\
&\leq n^{-\eta\alpha}|\varphi|_{\alpha,\varepsilon_0}.
\end{aligned}$$

Moreover, since $\|DT_{(i)}\|, \|D\tilde{T}_{(i)}\| \leq K$,

$$\begin{aligned}
\left|\frac{1}{|\det DT_{(i)}|} - \frac{1}{|\det D\tilde{T}_{(i)}|}\right| &= \frac{|\det DT_{(i)} - \det D\tilde{T}_{(i)}|}{|\det DT_{(i)}| \cdot |\det D\tilde{T}_{(i)}|} \\
&\leq C_{\det}(N, K, \lambda)
\end{aligned}$$

where

$$\begin{aligned}
C_{\det}(N, K, \lambda) = \sup\{|\det A - \det B| : A, B \in \mathrm{Mat}_{N\times N}(\mathbb{R}), A \neq B, \|A\|, \|B\| \leq K\} \\
\cdot \Big(\sup\{|\det(A^{-1}) : A \in \mathrm{Mat}_{N\times N}(\mathbb{R}), \|A^{-1}\| \leq \lambda \leq 1\}\Big)^2
\end{aligned}$$

so

$$\sum_i \int_{T_{(i)}(U_i)\cap\tilde{T}_{(i)}(\tilde{U_i})} \left|\varphi\circ\tilde{T}_{(i)}^{-1}\right| \left[\frac{1}{|\det DT_{(i)}} - \frac{1}{|\det D\tilde{T}_{(i)}|}\right] \tag{3.12}$$

$$\leq C_{\det}(N, K, \lambda)\|\varphi\|_{\alpha,\varepsilon_0}(\#\text{partition elements}) \tag{3.13}$$

Hence,

$$\sum_i A_{(i)} \leq n^{-\eta\alpha}|\varphi|_{\alpha,\varepsilon_0} + C_{\det}(N,K,\alpha)\|\varphi\|_{\alpha,\varepsilon_0}(\#\text{partition elements}) \tag{3.14}$$

For $B_{(i)}$:

$$\sum_i B_{(i)} = \sum_i \int_{T_{(i)}(U_i)\setminus\tilde{T}_{(i)}(\tilde{u_i})} \frac{\varphi \circ T_{(i)}^{-1}}{|\det DT_{(i)}|} d\mathfrak{m} \leq \|\varphi\|_{\alpha,\varepsilon_0} \cdot \lambda \cdot (\#\text{partition elements}). \tag{3.15}$$

The argument for $C_{(i)}$ is analogous. □

Corollary 3.5.2. *Fix $n \in \mathbb{N}$, and $T_{n,1}, \ \ldots, \ T_{n,k}$, then there exists a constant $\tilde{C}$ such that:*

$$\|\mathcal{L}_{T_{n,k}\circ\cdots\circ T_{n,1}}(\varphi) - \mathcal{L}^k_{T_{n,k}}(\varphi)\|_{L^1(\mathfrak{m})} \leq k\tilde{C}n^{-\eta\alpha}\|\varphi\|_{\alpha,\varepsilon_0}. \tag{3.16}$$

for all $1 \leq k \leq n$, and $\varphi \in \mathrm{OSC}_\alpha$.

Proof. Write $T_{n,j} = T_j$ for $1 \leq j \leq n$. We note that: $\mathcal{L}_{T_k\circ\cdots\circ T_1} = \mathcal{L}_{T_k}\ldots\mathcal{L}_{T_1}$

$$\mathcal{L}_k \ldots \mathcal{L}_1 - \mathcal{L}_k^k = \sum_{j=1}^k \mathcal{L}_k \ldots \mathcal{L}_{j+1}(\mathcal{L}_j - \mathcal{L}_k)\mathcal{L}_k^{j-1} \tag{3.17}$$

By the Lasota-York inequality [11, 21], for any map T piecewise expanding, $\mathcal{L}_T$ maps OSC_α into itself. Combined with the fact that $\mathcal{L}_T$ does not increase L^1- norm, and Lemma 4.6, we have the desired result. □

Completion of proof of claim 3.3.5

First, we recall the claim's statement:

For any $\eta' < \eta\alpha$, there exists a constant C such that:

$$\|\rho_{n,\lfloor ns\rfloor} - \hat{\rho}_s\|_{L^1} \leq Cn^{-\eta'}, \tag{3.18}$$

for all n, and $0 < s < \frac{-\eta' \log n}{n \log \Lambda}$.

Proof. Note: $\mathcal{L}_{T_{n,k}}(\hat{\rho}_{n,k}) = \hat{\rho}_{n,k}$, and the pushforward density $\rho_{n,k} = \mathcal{L}_{T_{n,k}} \dots \mathcal{L}_{T_{n,1}} \rho$

$$\begin{aligned}
\|\rho_{n,k} - \hat{\rho}_{n,k}\|_{L^1(\mathfrak{m})} &= \|\mathcal{L}_{T_{n,k}} \dots \mathcal{L}_{T_{n,1}} \rho - \mathcal{L}^k_{T_{n,k}}(\hat{\rho}_{n,k})\|_{L^1(\mathfrak{m})} \\
&\le \|\mathcal{L}_{T_{n,k}} \dots \mathcal{L}_{T_{n,k-i+1}}(\mathcal{L}_{T_{n,k-i}} \dots \mathcal{L}_{T_{n,1}} \rho) - \mathcal{L}_{T_{n,k}} \dots \mathcal{L}_{T_{n,k-i+1}} \hat{\rho}_{n,k}\|_{L^1(\mathfrak{m})} \\
&+ \|\mathcal{L}_{T_{n,k}} \dots \mathcal{L}_{T_{n,k-i+1}} \hat{\rho}_{n,k} - \mathcal{L}^i_{T_{n,k}} \hat{\rho}_{n,k}\|_{L^1(\mathfrak{m})} \\
&= \|\mathcal{L}_{T_{n,k}} \dots \mathcal{L}_{T_{n,k-i+1}}(\rho_{n,k-i} - \hat{\rho}_{n,k})\|_{L^1(\mathfrak{m})} \\
&+ \|(\mathcal{L}_{T_{n,k}} \dots \mathcal{L}_{T_{n,k-i+1}} - \mathcal{L}^i_{T_{n,k}})\hat{\rho}_{n,k}\|_{L^1(\mathfrak{m})} \\
&\le C\Lambda^i + iCn^{-\eta\alpha}\|\hat{\rho}_{n,k}\|_{\alpha,\varepsilon_0}
\end{aligned}$$

Choose $i = \left\lceil -\eta \frac{\log n}{\log \Lambda} \right\rceil$, so for any $0 < \eta' < \eta\alpha < 1$:

$$C\Lambda^i = Cn^{-\eta} < Cn^{-\eta'}$$

and

$$\begin{aligned}
Cin^{-\eta\alpha} &= Ci\frac{1}{n^{\eta\alpha}} \\
&= C(-\eta\frac{\log n}{\log \Lambda})\frac{1}{n^{\eta\alpha}} \\
&< Cn^{\eta\alpha - \eta'}\frac{1}{n^{\eta\alpha}} \\
&= Cn^{-\eta'}.
\end{aligned}$$

Since $\|\hat{\rho}_{n,k}\|_{\alpha,\varepsilon_0}$ is uniformly bounded, for $k > \left\lceil -\eta \frac{\log n}{\log \Lambda} \right\rceil$:

$$\|\rho_{n,k} - \hat{\rho}_{n,k}\|_{L^1(\mathfrak{m})} \le Cn^{-\eta'}. \tag{3.19}$$

Hence, $\|\rho_{n,\lfloor ns \rfloor} - \hat{\rho}_s\|_{L^1} \le Cn^{-\eta'}$, for $s > \frac{-\eta' \log n}{n \log \Lambda}$. □

3.6 Proof of Claim 3.3.4

First, we make the following definition of dynamical partition.

Definition 3.6.1. Let $\mathcal{A}_1 = \mathcal{A}(T_1)$. The dynamical partition $\mathcal{A}(\mathcal{T})$ of $\mathcal{T} = T_k \circ \cdots \circ T_1$ is defined by:

$$\mathcal{A}(\mathcal{T}) = \mathcal{A}_1 \vee \bigvee_{j=1}^{k} (T_j \circ \cdots \circ T_2)^{-1}(\mathcal{A}_1). \tag{3.20}$$

Fix $n \in \mathbb{N}$, $t \in [0,1]$, and let $\mathcal{A} = \{U_i\}_{i=1}^m$ be the dynamical partition of $T_{n,\lfloor nt \rfloor} \circ \cdots \circ T_{n,1}$

For x, y in the same partition element of $\mathcal{A}$, due to expansion:

$$dist(T_{n,\lfloor ns \rfloor} \circ \cdots \circ T_{n,1}(x), T_{n,\lfloor ns \rfloor} \circ \cdots \circ T_{n,1}(y)) \le C\lambda^{n(s-t)} \tag{3.21}$$

for any $0 < s \le t < 1$.

Therefore, given any observable $f : \mathbb{T}^N \to \mathbb{R}$ that is Lipschitz,

$$|\xi_n(x,s) - \xi_n(y,s)| \le Cn^{-1}. \tag{3.22}$$

Hence, if $A : \mathbb{R} \to \mathbb{R}$ is Lipschitz, there exists a constant C(dependent on A) such that for $x \in U_i$:

$$\int_{U_i} A(\xi_n(x,s)) d\mu(y) - \int_{U_i} A(\xi_n(y,s))\ d\mu(y) \le C.n^{-1} \tag{3.23}$$

$1 \le i \le m$. So

$$A(\xi(x,s)) - \frac{1}{\mu(U_i)} \int_{U_i} A(\xi_n(y,s))\ d\mu(y) \le Cn^{-1} \tag{3.24}$$

We denote the integral $\frac{1}{\mu(U_i)} \int_{U_i} A(\xi_n(y,s))\ d\mu(y)$ by $\mu_i(A(\xi_n(y,s)))$. From here on, we will drop the x-dependence in $\xi_n(x,s)$, and simply write $\xi_n(s)$ instead. Now,

$$\begin{aligned}\mu[A(\xi_n(s))[\xi_n(t)-\xi_n(s)]] &= \sum_i \mu[\mathbb{1}_{U_i}A(\xi(s))(\xi_n(t)-\xi_n(s))]\\ &= \sum_i \mu_i(A(\xi_n(s)))\mu[\mathbb{1}_{U_i}(\xi_n(t)-\xi_n(s))]+O(n^{-1})\\ &= \sum_i \mu(A(\xi_n(s)))\mu_i[\xi_n(t)-\xi_n(s)]+O(n^{-1}).\end{aligned}$$

To finish off the proof, we want to show that:

$$\max_i |\mu_i(\xi_n(t)-\xi_n(s))-\mu(\xi_n(t)-\xi_n(s))| = o(1) \tag{3.25}$$

as $n\to\infty$.

To this end,

$$\begin{aligned}&\mu_i\left[\textstyle\int_s^t f_{n,\lfloor ns\rfloor}dr\right]-\mu\left[\textstyle\int_s^t f_{n,\lfloor ns\rfloor}dr\right]\\ &=\int_s^t \frac{1}{\mu(U_i)}\int_{U_i} f_{n,\lfloor nr\rfloor}d\mu dr-\int_s^t\int f_{n,\lfloor nr\rfloor}d\mu\\ &=\int_s^t\int \frac{1}{\mu(U_i)}\mathbb{1}_{U_i}f\circ T_{n,\lfloor nr\rfloor}\circ\cdots\circ T_{n,1}(x)d\mu(x)-\int f\circ T_{n,\lfloor nr\rfloor}\circ\cdots\circ T_{n,1}(x)d\mu(x)\\ &=\int_s^t\int f\cdot\mathcal{L}_{T_{n,\lfloor nr\rfloor}\circ\cdots\circ T_{n,1}}(\frac{1}{\mu(U_i)}\mathbb{1}_{U_i})-f\cdot\mathcal{L}_{T_{n,\lfloor nr\rfloor}\circ\cdots\circ T_{n,1}}(\mathbb{1})d\mu\end{aligned}$$

The memory loss result from Theorem 3.4.2 gives us that :

$$\max_i |\mu_i(\xi_n(t)-\xi_n(s))-\mu(\xi_n(t)-\xi_n(s))| = o(1) \tag{3.26}$$

as $n\to\infty$.

Chapter 4

Nonstationary Coupled Map Lattices

4.1 Introduction

Coupled map lattices (CML) were first proposed by Kunihiko Kaneko in 1984 [13] as a model to study spatiotemporal chaos in biology. Since they were first introduced, coupled map lattices have gained popularity for its convenience in computer simulations and their practicality in extended dynamical systems.

A coupled map lattice system includes:

1. A lattice Ω (a discrete structure) that acts as the underlying physical space. An example is the integers $\mathbb{Z}$. Each point $\omega \in \Omega$ on the lattice is called a "site" or a "node". Ω can be finite or countable.

2. At each site ω is a local phase space X_ω, and a local dynamics $T_\omega : X_\omega \to X_\omega$.

3. The global phase space is a direct product of the local phase spaces on the lattice, denoted as $M = \prod_{\omega \in \Omega} X_\omega$. Hence, each point $\mathbf{x}$ on the (global) phase space can be represented as $\mathbf{x} = (x_\omega)_{\omega \in \Omega}$

4. The global uncoupled dynamics $T : M \to M$ preserves the product structure. In other words, at each site ω, T acts as the local dynamics T_ω:

$$(T(\mathbf{x}))_\omega = T_\omega(x_\omega) \tag{4.1}$$

5. Spatial interaction between sites $\Phi : M \to X$. The most common type of spatial interaction in CML is *nearest-neighbor diffusive coupling* where each site only interacts with its closest neighbors. We note that from now on, we may use spatial interaction and coupling interchangeably. The formulation nearest-neighbor diffusive coupling $\Phi_\epsilon : M \to M$ is given by:

$$(\Phi_\epsilon \mathbf{x})_\omega = \frac{\epsilon}{2} x_{\omega-1} + (1-\epsilon) x_\omega + \frac{\epsilon}{2} x_{\omega+1} \tag{4.2}$$

   where ω represents the strength of coupling.

6. The coupled dynamics on M is the composition $\hat{T} = \Phi \circ T$.

Bunimovich and Sinai [4] introduced the coupled map lattices to the dynamical systems community in 1988 in a joint paper where they studied the integer coupled lattice system generated by expanding maps on a unit interval analytically using tools from ergodic theory. In particular,

- $\Omega = \mathbb{Z}$, $X_\omega = [0, 1]$
- The local dynamics T_ω's are expanding maps

- Sites interact with each other according to the nearest-neighbor diffusive coupling.

Theorem 4.1.1. *[4] When ϵ is sufficiently small, the following hold true:*

1. *There exists a $\hat{T}$-invariant measure μ on M.*
2. *For any integers N_1, N_2, the induced measure on the space of finite sequence $\{x_\omega\}$, $N_1 \leq \omega \leq N_2$ is absolutely continuous with respect to the Lebesgue measure.*
3. *$\hat{T}$ is mixing*

In 2005, Keller and Liverani [16] considered the system with the same piecewise C^2 expanding maps on $[0, 1]$ at each lattice site. They also derived an invariant measure and decay of correlation for the infinite coupled system. Their coupled systems consist of:

1. An integer lattice
2. Local dynamics are piecewise expanding maps that are mixing
3. Spatial interaction strength is small and with a "short" range, i.e., one can only influence finitely many sites around it.

Here, we denote the global coupled map as T_ϵ. Keller and Liverani's results are stated as below:

Theorem 4.1.2. *[?] Given a coupled map lattice systems described as above. Then,*

1. *The coupled system T_ϵ has an invariant probability measure μ_ϵ whose finite dimensional marginals are absolutely continuous with respect to Lebesgue measures and have densities of bounded variation.*

2. *There are constants $\gamma, \gamma', \theta \in (0,1)$ and $C > 0$ such that for bounded observables $\phi, \psi : M \to \mathbb{R}$ which depend only on coordinates $x_{a+1}, \dots, x_b$*

$$\left| \int \phi \cdot (\psi \circ T_\epsilon^n) d\mu_\epsilon - \int \phi d\mu_\epsilon \int \psi d\mu_\epsilon \right| \leq C\theta^{-(b-a)}\gamma^n \|\phi\|_{C^1} \|\psi\|_{C^0}. \tag{4.3}$$

Remark 4.1.3. In the results for expanding maps and piecewise expanding maps, both groups of authors utilized the space of measures whose finite-dimensional marginals are absolute continuous with respect to Lebesgue measure when they evaluate the infinite lattice systems.

When considering nonstationary systems in the setting of CML, one can put finite compositions of maps at local sites then couple the system, or let the coupling happen at every step before the next transformation at local site takes place. The latter is what we consider for the remaining sections. Our goal is to prove the statistical memory loss of a nonstationary dynamical system on a finite lattice.

4.2 Nonstationary Dynamics in CML

4.2.1 Uncoupled Dynamics At a Single Site

We recall a few definitions about piecewise expanding maps on the circle.

Let $\mathcal{S}^1$ be the interval $[0, 1]$ with endpoints identified. The uncoupled dynamics at each individual site is defined by a composition of $\mathcal{C}^2$-piecewise expanding enveloping maps, defined as below:

Definition 4.2.1. (Piecewise Expanding Map) $\tau : \mathcal{S}^1 \to \mathcal{S}^1$ is said to be C^2-piecewise expanding if there exists a finite partition $\mathcal{A} = \mathcal{A}(\tau)$ into intervals such that for each interval $I \in \mathcal{A}$:

1. $\tau|I$ is monotone and $\mathcal{C}^2$
2. There exists $\lambda > 1$ such that $\|\tau'(x)\| \geq \lambda$ for each $x \in I$

In this context, we will assume that $\lambda > 2$.

Notation 4.2.2. 1. $\mathcal{A}_n \coloneqq \bigvee_{i=1}^{n} f^{-(i-1)}(\mathcal{A})$ is the join of pullbacks of $\mathcal{A}$.

2. $\mathcal{A}_n|I$ is the restriction of $\mathcal{A}_n$ to the set I
3. For any $J \subset \mathcal{S}^1, \mathrm{int}(J)$ denotes the interior of J.

Definition 4.2.3. (Enveloping Map) τ is said to be *enveloping* if there exists $N \in \mathbb{Z}^+$ such that for every $I \in \mathcal{A}$, we have:

$$\bigcup_{J \in \mathcal{A}_N|I} \tau^N(\mathrm{int}(J)) = \mathcal{S}^1. \tag{4.4}$$

The smallest such N is called enveloping time.

From now on, we let $\mathcal{E}$ be the collection of all piecewise C^2-expanding, enveloping maps.

Definition 4.2.4. (Nearby Maps) Let $\tilde{\tau} \in \mathcal{E}$ be given. Let $\Omega(\tilde{\tau})$ contains the points of discontinuity $\zeta_1 \ = \zeta_k, \ldots, \zeta_{k-1}$ of g on $\mathbb{S}^1$, labeled counterclockwise, and $d_\Omega(\tilde{\tau}) \coloneqq \min_i |\zeta_{i+1} - \zeta_i|$.
Given $\varepsilon < \frac{1}{4} d_\Omega(g)$, $\tau \in \mathcal{E}$ is said to be ε-near $\tilde{\tau}$, written $\tau \in \mathcal{U}_\varepsilon(\tilde{\tau})$ if:

1. τ and $\tilde{\tau}$ have the same number of points of discontinuity, $y_1, \ldots, y_k = y_1$;

2. $|\zeta_i - y_i| < \varepsilon$, for each $i = 1, \ldots, k$;

3. if $\xi_{\tau\tilde{\tau}}$ maps each interval $[\zeta_i, \zeta_{i+1}]$ affinely onto $[y_i, y_{i+1}]$, then on each $[\zeta_i, \zeta_{i+1}]$,

$$\|\tau \circ \xi_{\tau\tilde{\tau}} - \tilde{\tau}\|_{C^2} < \varepsilon. \tag{4.5}$$

Given $\tau_1, \ldots, \tau_n \in \mathcal{E}$, the (uncoupled) dynamics at each site is $\tau = \tau_n \circ \cdots \circ \tau_1$.

4.2.2 Global Uncoupled Map F_0

Our phase space X is a d-fold direct product of $\mathcal{S}^1$, $X = (\mathcal{S}^1)^L$, where $L = \{1, \ldots, d\}$. At each site $l \in L$, the uncoupled map $F_n : X \to X$ acts like the dynamics at site l, i.e.,

$$(F_n\mathbf{x})_l = \tau(x_l). \tag{4.6}$$

4.2.3 Space Interaction Φ_ϵ

The simplest type of diffusive coupling is of the nearest neighbor, i.e., for ϵ small enough, and each $l \in L$, the coupling map $\Phi_\epsilon : X \to X$ is defined as:

$$(\Phi_\epsilon\mathbf{x})_l = \frac{\epsilon}{2}x_{l-1} + (1-\epsilon)x_l + \frac{\epsilon}{2}x_{l-1}. \tag{4.7}$$

For the setting, we wish to use the generalization of diffusive coupling which was defined on a finite system as in [**?**].

The map $\Phi_\epsilon : X \to X$:

$$\Phi_\epsilon(\mathbf{x}) = \mathbf{x} + A_\epsilon(\mathbf{x}) \tag{4.8}$$

is said to be (a_1, a_2)-coupling if there exist operators A_1, A_2 are $L \times L$-matrices with $a_1 = \|A_1\|_1$ and $a_2 = \|A_2\|_1$ (column sum norm) such that for all $i, j, k \in L$:

1. $|(A_\epsilon)_1| \leq 2|\epsilon|$

2. $|(DA_\epsilon)_{ij}| \leq 2|\epsilon|(A_1)_{ij}$

3. $|\partial_k (DA_\epsilon)_{ij}| \leq 2|\epsilon|(A_2)_{ij}$, where ∂_k denotes the partial derivative w.r.t x_k

4. Φ_ϵ has finite coupling range $\omega > 0$: $\partial_j \Phi_{\epsilon,i} = 0$ whenever $|j-i| > \omega$. Equivalently, $(A_1)_{ij} = (A_2)_{ij} = 0$ when $|j-i| > \omega$

Remark 4.2.5. The diffusive coupling of the nearest neighbor (4.7) is a $(1,0)$-coupling.

4.2.4 Global Coupled Map $F_{n,\epsilon}$

For $1 \leq j \leq n$, let $f_{j,\epsilon} \coloneqq \Phi_\epsilon \circ (\tau_j \times \cdots \times \tau_j)$

The coupled map $F_{n,\epsilon} : X \to X$ is defined as:

$$F_{n,\epsilon} = f_{n,\epsilon} \circ \cdots \circ f_{1,\epsilon}. \tag{4.9}$$

4.3 Functions of Bounded Variation

Given any $\varphi : X \to \mathbb{R}$.

Notation 4.3.1. For $i \in L$,:

1. $\mathbf{x}_{\neq i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_d)$.

2. $\mathbf{x}$ can be written as $(x_i, \mathbf{x}_{\neq \mathbf{i}})$.

3. $\mathbf{X}_{\neq \mathbf{i}} = \{\mathbf{x}_{\neq \mathbf{i}} : \mathbf{x} \in X\}$.

Given any $\varphi \in L^1(X)$, define the $\mathbf{x_{\neq i}}$-section of φ, namely $\varphi_{\mathbf{x_{\neq i}}} : \mathcal{S}^1 \to \mathbb{R}$ by:

$$\varphi_{\mathbf{x_{\neq i}}}(x) = \varphi(x, \mathbf{x_{\neq i}}). \tag{4.10}$$

Finally, the variation of φ at each coordinate i can be defined to be:

$$\mathrm{Var}^i_X(\varphi) = \int_{\mathbf{X_{\neq i}}} \mathrm{Var}_{\mathcal{S}^1}(\varphi_{\mathbf{x_{\neq i}}}) d\mathbf{x_{\neq i}}. \tag{4.11}$$

Hence, variation of φ on X is:

$$\mathrm{Var}_X(\varphi) = \max_{i=1,\ldots,d} \mathrm{Var}^i_X(\varphi). \tag{4.12}$$

Remark 4.3.2. By Fubini's theorem, $\varphi_{\mathbf{x_{\neq i}}} \in L^1(\mathcal{S}^1)$ almost everywhere $\mathbf{x_{\neq i}} \in X_{\neq i}$, so $\mathrm{Var}_{\mathcal{S}^1}(\varphi_{\mathbf{x_{\neq i}}})$ is well-defined almost everywhere $\mathbf{x_{\neq i}}$. The measurability of $\mathrm{Var}_{\mathcal{S}^1}(\varphi_{\mathbf{x_{\neq i}}})$ is shown in [**?**](see Lemma 3.1).

For technical calculation, we will use an alternative definition of $\mathrm{Var}^i_X(\varphi)$

Let $\mathcal{G}_X = \{\psi \in C^1(X) : |\psi| \leq 1\}$, and $\varphi \in L^1(\mathfrak{m})$ be given. The i^{th}-variation of φ is defined as:

$$\mathrm{Var}^i_X(\varphi) = \sup_{\psi \in \mathcal{G}_X} \int_X \varphi(\mathbf{x}) \partial_i \psi(\mathbf{x})\ d\mathbf{x} \tag{4.13}$$

It was shown in [**?**] that the definition (4.11) is equivalent to (4.13).

For the subsequent sections, we will use the following space of densities:

$$\mathcal{D} = \{\varphi \in BV(X) : \int_X \varphi\ d\mathfrak{m} = 1, \varphi > 0\}, \tag{4.14}$$

where

$$BV(X) \coloneqq \{\varphi \in L^1(\mathfrak{m}) : Var_X(\varphi) < \infty\}. \tag{4.15}$$

4.4 Statements of the Results

Theorem 4.4.1. *(Local result) Fix* $\tilde{\tau} \in \mathcal{E}$*. There exists* $\Lambda < 1$*,* ϵ *and* $\varepsilon > 0$ *small enough (*ε *depending on* $\tilde{\tau}$*) so that for all* τ_j *that are* ε*-near* $\tilde{\tau}$*, and* $\varphi, \psi \in \mathcal{D}$*, the following holds: there exists a constant* $K(\varphi, \psi) > 0$ *such that for all* $n \in \mathbb{N}$*,*

$$\int_X |P_{F_{n,\epsilon}}(\varphi) - P_{F_{n,\epsilon}}(\psi)| \leq K\Lambda^n. \tag{4.16}$$

Fix $a < b$. At each site $l \in L$, define a continuous path $\gamma : [a, b] \to \mathcal{E}$ and consider a finite or infinite partition $a \leq t_1 \leq t_2 \leq \cdots \leq b$. Set $\tau_i = \gamma(t_i)$ and $\Delta := \max_i(t_{i+1} - t_i)$. Δ can be thought as the velocity of traversing the curve γ: decreasing Δ is the same as reducing the speed of traversing the curve. The theorem below can be interpreted as follows: when the curve is traversed slowly enough, the local result can be applied.

Theorem 4.4.2. *(Global result) At each site* $l \in L$*, let* $\gamma : [a, b] \to \mathcal{E}$ *be a continuous map (we use the same curve* γ *for all lattice sites, so* γ *is independent of* $l \in L$*). Then there exist* δ_0 *and* $\Lambda < 1$ *(depending on* γ*) such that the following holds: for every* $\{t_i\}$ *with* $\Delta \leq \delta_0$ *and* $\varphi, \psi \in \mathcal{D}$*, there exists a constant* $C(\varphi, \psi) > 0$ *such that for all relevant* $n \in \mathbb{N}$*:*

$$\int_X |P_{F_{n,\epsilon}}(\varphi) - P_{F_{n,\epsilon}}(\psi)| \leq C\Lambda^n. \tag{4.17}$$

4.5 Lasota-Yorke Inequality

It is natural to consider the space of bounded variation functions $BV(X) := \{\varphi : X \to \mathbb{R} : \mathrm{Var}_X(\varphi) < \infty\}$.

If for $a > 0$, $\mathcal{D}_a = \{\varphi \in BV(X) : \int_X \varphi \; d\mathfrak{m} = 1, \varphi > 0, \mathrm{Var}_X(\varphi) \leq a\}$, then $\bigcup_a \mathcal{D}_a = \mathcal{D}$

Proposition 4.5.1. *Given $\varphi \in \mathcal{D}$, there exist constants $\alpha_1(\lambda) < 1$ and $C(a_1, a_2, \epsilon) > 0$ such that:*

$$\mathrm{Var}_X(P_{F_{n,\epsilon}}(\varphi)) \leq \alpha_1^n \mathrm{Var}_X(\varphi) + \frac{C}{1-\alpha_1^n} \int_X |\varphi| \; d\mathfrak{m}. \tag{4.18}$$

Proof. For each $f_{j,\epsilon} = \Phi_\epsilon \circ (\tau_j \times \cdots \times \tau_j)$, the Lasota-Yorke inequality is proved in [**?**]:

$$\mathrm{Var}_X(P_{f_{j,\epsilon}}(\varphi)) \leq \alpha_1 \mathrm{Var}_X(\varphi) + C \int_X |\varphi| \; d\mathfrak{m}. \tag{4.19}$$

As $F_{n,\epsilon} = f_{n,\epsilon} \circ \cdots \circ f_{1,\epsilon}$, by applying (4.19) repeatedly, we obtain the desired result.

$\square$

4.6 Absorbing Set $\mathcal{D}_{a^*}$

Proposition 4.6.1. *For the same constants α_1 and C as in 4.5.1, we fix $a^* > \frac{C}{1-\alpha_1}$. Then for every $a > 0$, there exists a time $t(a) \in \mathbb{N}$ such that for all $\varphi \in \mathcal{D}_a$, $P_{F_{n,\epsilon}}(\varphi) \in \mathcal{D}_{a^*}$.*

Proof. This is a consequence of the Lasota-Yorke inequatliy in 4.5.1. We can choose $t(a) \geq \frac{\ln\left[(a^* - \frac{C}{1-\alpha_1})a^{-1}\right]}{ln(\alpha_1)}$.

$\square$

4.7 Perturbation Lemmas

Fix $\tilde{\tau} \in \mathcal{E}$. Let $g = \tilde{\tau} \times \cdots \times \tilde{\tau}$ and $g_\epsilon = \Phi_\epsilon \circ g$ be the uncoupled and coupled map on X, respectively.

Let $\mathcal{Z}_1(g)$ denote the partition for g. Because of the product structure of g, $\mathcal{Z}_1(g)$ is a direct product of partitions from local nodes. Similarly, for $n \in \mathbb{N}$, the dynamical partition $\mathcal{Z}_n(g)$ of g_n is also a direct product of elements of $\mathcal{A}_n(\tilde{\tau})$ mentioned earlier in Subsection 4.2.1.

Proposition 4.7.1. *There exists $n_0 \in \mathbb{N}$ and $\kappa_0(g)$ so that $P_{g^{n_0}}(\varphi) \geq \kappa_0$ for all $\varphi \in \mathcal{D}_{a^*}$.*

Proof. Let $\mathcal{Z}_1$ be the partition for g, and $n_1 \in \mathbb{N}$ be such that each element $Z \in \mathcal{Z}_{n1}$ have $\mathfrak{m}(Z) < \frac{1}{2a^*}$. First, we will show that given $\varphi \in \mathcal{D}_{a^*}$, there exists a element $Z = Z(\varphi) \in \mathcal{Z}_{n_1}$ so that $\varphi | Z \geq \frac{1}{2}$. Suppose by way of contradiction that for each $Z \in \mathcal{Z}_{n_1}$, there exists y_Z with $\varphi(y_Z) < \frac{1}{2}$. Then:

$$\int_Z \varphi \leq \mathfrak{m}(Z)(\varphi(y_Z) + \mathrm{Var}_Z(\varphi)) < \frac{\mathfrak{m}(Z)}{2} + \frac{1}{2a^*} \mathrm{Var}_Z(\varphi) \tag{4.20}$$

Summing over all Z, we get $\int_X \varphi < 1$, which is a contradiction.

Due to the expanding property of τ and the the product structure of g, for each partition element $Z \in \mathcal{Z}_{n_1}$, we can choose a sub-partition Z_s and $s \in \mathbb{N}$, dependent on Z, so that $g^s(Z_s) \supset J$ for some $J \in \mathcal{Z}_1$. Let $n_0 = N + s_0$, where N is the enveloping time of τ and $s_0 = \max_{Z \in \mathcal{Z}_{n_1}} s$. Let $Z = Z(\varphi) \in \mathcal{Z}_{n_1}$ be such that $\varphi | Z \geq \frac{1}{2}$, and $J \in \mathcal{Z}_1$ be such that $g^s(Z_s) \supset J$. Then $P_{g^s}(\varphi)|J \geq (\frac{1}{2|\det Dg|})^s$. Since there is the same enveloping map τ at each site which has the same enveloping time N, $g^N(J) = X$. Therefore, $P_{g^{n_0}}(\varphi) \geq (\frac{1}{2|\det Dg|})^{n_0}$.

$\square$

Recall that $F_n = f_n \circ \cdots \circ f_1$, where $f_j = \tau_j \times \cdots \times \tau_j$, for $1 \leq j \leq n$. Since there is no coupling, the global dynamic of F_n can be described by taking the same composition $T_n = \tau_n \circ \ldots \tau_1$ at each site. The dynamical partition of T at a single site is defined as follows:

$$\mathcal{A}(T_n) = \mathcal{A}_1(\tau_1) \vee \bigvee_{k=2}^{n} \left[T_{k-1}^{-1}(\mathcal{A}_1(\tau_k)) \right]. \tag{4.21}$$

Proposition 4.7.2. *Let n_0 be the same as the in the previous proposition. There exist $\varepsilon > 0$ and $\kappa_\varepsilon > 0$ such that for all τ_j ε-close to $\tilde{\tau}$, $P_{F_{n_0}}(\varphi) > \kappa_\varepsilon$ for all $\varphi \in \mathcal{D}_{a^*}$.*

Proof. Fix $\varphi \in \mathcal{D}_{a^*}$.

As F_n preserves the lattice structure, its dynamical partition $\mathcal{Z}(F_n)$ is a direct product of the local partition for T at each coordinate. In [19], it was shown that when τ_j is ε-close to $\tilde{\tau}$ with ε small enough, there is a well-defined mapping between $\mathcal{A}_n(\tilde{\tau})$ and $\mathcal{A}(T_n)$ so that given an element $J \in \mathcal{A}_n(\tilde{\tau})$, its image in $\mathcal{A}(T_n)$ has the same itinerary. This is true for every coordinate $i \in L$. Therefore, if an interval $J \in \mathcal{A}_n(\tilde{\tau})$ is perturbed by a small amount of δ at both ends, by the enveloping property, X is still covered from every direction under the dynamics of F_N, where N is the enveloping time of $\tilde{\tau}$. And the result follows. $\square$

While the dynamical partition $\mathcal{Z}_n(g)$ is a direct product of the partitions for τ_n, $\mathcal{Z}_n(g_\epsilon)$ is not. The following lemma from [15] shows that when ϵ is small enough, the elements of dynamical partitions for uncoupled map g^n and coupled map g_ϵ^n are diffeomorphic.

Lemma 4.7.3. *[15] There exists $\epsilon_1 > 0$ so that for all $\epsilon \in (0, \epsilon_1)$ and $n \in \mathbb{N}$ the following hold:*

1. *The partitions $\mathcal{Z}_n(g)$ and $\mathcal{Z}_n(g_\epsilon)$ have the same cardinality,*

2. *The elements $U \in \mathcal{Z}_n(g)$ and $V \in \mathcal{Z}_n(g_\epsilon)$ can be relabelled so that there is a diffeomorphism $\Psi : V \to U$ that is C^1-close to the identity in the following sense:*

   - $\|\Psi(\mathbf{x}) - \mathbf{x}\|_\infty \leq const \cdot \epsilon,$
   - $\|D\Psi(\mathbf{x}) - I\|_\infty \leq const \cdot \epsilon$
   - $\|\partial_i D\Psi(\mathbf{x})\|_\infty \leq const \cdot \epsilon$

   for all $i \in L$ and $\mathbf{x} \in V$. The constants are independent of lattice size L.

Let $F_{n,\epsilon} = f_{n,\epsilon} \circ \cdots \circ f_{1,\epsilon}$, where $f_{j,\epsilon} = \Phi_\epsilon \circ (\tau_j \times \cdots \times \tau_j)$, for $1 \leq j \leq n$. The dynamical partition of $F_{n,\epsilon}$ is defined as follows:

$$\mathcal{Z}(F_{n,\epsilon}) = \mathcal{Z}_1(f_{1,\epsilon}) \vee \bigvee_{k=2}^{n} \left[F_{k-1,\epsilon}^{-1}(\mathcal{Z}_1(f_{k,\epsilon}))\right] \tag{4.22}$$

Lemma 4.7.4. *For $\epsilon_1 > 0$ as in lemma 4.7.3 and n_0 as before, there exists $\varepsilon > 0$ and $\kappa > 0$ (depending on ε, ϵ) such that for all τ_j that is ε-near $\tilde{\tau}$, $P_{F_{n_0,\epsilon}}(\varphi) > \kappa$.*

Proof. By Lemma 4.7.3, for each $1 \leq j \leq n$, when there is $\epsilon_1 > 0$ so that when $\epsilon \in (0, \epsilon_1)$, each element of $\mathcal{Z}_1(f_j)$ is diffeomorphic to some element of $\mathcal{Z}_1(f_{j,\epsilon})$. Therefore, given an element $U \in \mathcal{Z}(F_n)$ and ϵ sufficiently small there is a well-defined mapping Ψ_n between $\mathcal{Z}(F_n)$ and $\mathcal{Z}(F_{n,\epsilon})$. We can choose ε so that Proposition 4.7.2 holds and Ψ_n exists for $n = n_0$. Given $U \in \mathcal{Z}(F_{n_0})$ be such that $F_{n_0}(U)$ covers X, $F_{n_0}(U)$ and $F_{n_0,\epsilon}(\Psi_{n_0}(U))$ can be made arbitrarily close. As the enveloping dominates the coupling strength, $F_{n_0,\epsilon}(\Psi_{n_0}(U))$ will eventually cover the whole space, and the result follows.

□

4.8 Matching Densities and Proof of Theorem (4.4.1)

Given any φ and ψ in the set of densities $\mathcal{D}$, we carry out the following steps:

1. Iterate φ and ψ until $P_{F_{n,\epsilon}}(\varphi)$ and $P_{F_{n,\epsilon}}(\psi)$ are in $\mathcal{D}_{a^*}$ and they both become greater than κ. This step will account for the constant $K(\varphi, \psi)$ in the statement of the result.

2. Once the evolution of φ and ψ becomes greater than κ under the effect of the transfer operator, we can consider measures $(F_n)_*(\varphi)d\mathfrak{m}$ and $(F_n)_*(\psi)d\mathfrak{m}$ sharing a "common" part $\kappa d\mathfrak{m}$, and this common part will be matched.

3. We renormalized the "unmatched" part by letting:
$$\hat{\varphi} = \frac{\varphi - \kappa}{1 - \kappa} \quad \text{and} \quad \hat{\psi} = \frac{\psi - \kappa}{1 - \kappa} \tag{4.23}$$
It's clear that the new densities $\hat{\varphi}$ and $\hat{\psi}$ are in $\mathcal{D}_{a^*(1-\kappa)^{-1}}$

4. Let $N = t(a^*(1-\kappa)^{-1})$ be as in Proposition 4.6.1. Then, $\bar{\varphi}_N \coloneqq P_{F_{N,\epsilon}}(\hat{\varphi})$ and $\bar{\psi}_N \coloneqq P_{F_{N,\epsilon}}(\hat{\psi})$ are both in $\mathcal{D}_{a^*}$

5. Repeat Steps 2 and 3 above: wait for a some time (n_0 as in the perturbation lemmas) so that $\bar{\varphi}$ and $\bar{\psi} > \kappa$, match the common part, and renormalize the "unmatched" part to obtain
$$\hat{\varphi}_N = \frac{\bar{\varphi}_N - \kappa}{1 - \kappa} \quad \text{and} \quad \hat{\psi}_N = \frac{\bar{\psi}_N - \kappa}{1 - \kappa} \tag{4.24}$$
In general, for $k \in \mathbb{N}$, given $\hat{\varphi}_{(k-1)N}$, $\hat{\psi}_{(k-1)N}$ in $\mathcal{D}_{a^*(1-\kappa)^{-1}}$, we can set:
$$\bar{\varphi}_{kN} \coloneqq P_{F_{kN,\epsilon}}(\hat{\varphi}_{(k-1)N}) \quad \text{and} \quad \bar{\psi}_{kN} \coloneqq P_{F_{kN,\epsilon}}(\hat{\psi}_{(k-1)N}) \tag{4.25}$$
By Proposition 4.6.1, both $\bar{\varphi}_{kN}$ and $\bar{\psi}_{kN}$ are in $\mathcal{D}_{a^*}$. Repeating Steps 2 and 3, we can obtain $\hat{\varphi}_{kN}$ and $\hat{\psi}_{kN}$ that are in $\mathcal{D}_{a^*(1-\kappa)^{-1}}$. We observe that only a

fraction of κ is matched every $N + n_0$ steps (accounting for the time that the densities become uniformly positive). The induction will give us:

$$\int |P_{F_{n,\epsilon}}(\varphi) - P_{F_{n,\epsilon}}(\psi)| \leq ((1-\kappa)^{k/n})^n, \tag{4.26}$$

for $k(N + n_0) \leq n \leq (k+1)(N + n_0)$. We achieve memory loss with $\Lambda = (1-\kappa)^{(n_0 + t(a^*(1-\kappa)^{-1}))^{-1}}$.

4.9 Proof of Theorem 4.4.2

Let $s \in [a, b]$ be arbitrary. Then, $\gamma(s)$ is an element of $\mathcal{E}$, and Theorem 4.4.1 applies to $\gamma(s)$. For $s \in [a, b]$, there exists an open cover $(s - \alpha(s), s + \alpha(s)) \subset [a, b]$ such that $\gamma((s - \alpha(s), s + \alpha(S)) \cap [a, b])$ is contained in an ε-neighborhood of $\gamma(s)$ for which the local result (4.4.1) applies. We note some relevant constants associated with $\gamma(s)$ from the local result:

- ε: size of the neighborhood around $\gamma(s)$
- κ: the lower bound from the local result
- $n(\gamma(s)) = n_0 + t(a^*(1-\kappa)^{-1})$: from the perturbation lemmas and includes enveloping time of $\gamma(s)$

Since $\gamma[a, b]$ is compact, there exist a finite sequence $s_1 < s_2 < \cdots < s_D$ such that:

$$\bigcup_{j=1}^{D} (s_j - \frac{1}{2}\alpha(s_j), s_j + \frac{1}{2}\alpha(s_j)) \tag{4.27}$$

covers $[a, b]$.

Choose $\delta_0 = \min_j \frac{\alpha(s_j)}{2n(\gamma(s_j))}$.

If $\{t_i\}$ defines a partition on $[a, b]$, then each $t_i \in (s_j - \frac{1}{2}\alpha(s_j), s_j + \frac{1}{2}\alpha(s_j))$ for some j. The choice δ_0 ensures that the curves starting at t_i and ending at $t_{i+n(\gamma(s_j))-1}$ are all in the ε- neighborhood of $\gamma(s_j)$. Therefore, for each s_j, the local result (Theorem 4.2.1) applies to the collection $\{\gamma(t_{i+n(\gamma(s_j))-1}), \ldots, \gamma(t_i)\}$ in $\mathcal{E}$, and we get the desired statistical memory loss.

Chapter 5

Further Discussion

We conclude with open problems and future directions in both quasistatic dynamical systems and coupled map lattices settings.

For the quasistatic dynamical systems generated by the piecewise-smooth expanding maps we consider, it would be interesting to derive a limiting stochastic differential equation that describes the evolution of fluctuations with respect to the ergodic averages. Further, one could generalize the types of dynamical systems for which a quasistatic ergodic theorem holds.

For coupled map lattices, we have seen that exponential memory loss emerges in the case of nonstationary dynamical systems on a finite lattice in Chapter 4. Will a nonstationary infinite lattice system lose its statistical memory? If so, how do we prove it?

Another direction in coupled map lattices is to consider logistic maps at the local sites. Kaneko used this type of map for simulations of spatiotemporal chaos in nonlinear dynamics. However, tools from ergodic theory have not yet yielded any rigorous results in this setting, even when the lattice is finite.

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