NUMERICAL ANALYSIS OF THE YAGI-UDA ARRAY

A Thesis

Presented to

the Faculty of the

Department of Electrical Engineering

University of Houston

In Partial Fulfillment

of the Requirements for the Degree

Master of Science

in

Electrical Engineering

---

by

Nabil Kamel Takla

April 1974

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ABSTRACT

An important performance index of a Yagi-Uda array is the directivity. Many studies have been made on the subject of optimization for Yagi-Uda antenna arrays.

Shen [1971] obtained an optimum design for a Yagi array under constraints on bandwidth, directivity or the size of the array. It was shown that the array configuration is determined by any two of these three being specified. However, due to the nature of the approximation used in Shen's analysis, no information on the input impedance of the Yagi antenna is given.

Recently, Cheng and Chen [1973] developed a method for the maximization of the forward directivity of a Yagi-Uda array by adjustment of the inter-element spacing. To what extent the bandwidth is affected when the dimensions of the array are adjusted for maximum directivity remains unanswered.

In this study the work of Shen and of Chen and Cheng is extended. Namely, the behavior of the input impedance of the Yagi-Uda array is investigated as the operating frequency is varied. The effect on the bandwidth of a Yagi-Uda array when its gain is optimized by adjusting the spacing of the directors as proposed by Cheng and Chen is also investigated. The analysis makes use of King's three term theory, which converts an integral equation into a complex matrix equation, which is solved using a digital computer.

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CHAPTER I

Introduction

A physical description of the Yagi-Uda array is presented in Section 1-1. Previous work is discussed in Section 1-2, and the present work in Section 1-3.

1-1 Physical Description of the Yagi-Uda Array

The Yagi-Uda array studied in this investigation consists of N thin linear dipole elements, of which No. 1 is parasitic and adjusted in length to function as a reflector, No. 2 is driven by a voltage V_{02}, and Nos. 3 through N are parasitic and adjusted in length to act as directors. The ith element has half-length h_i and radius a, and the spacing between the ith element and the jth element is $b_{i,j}$, where

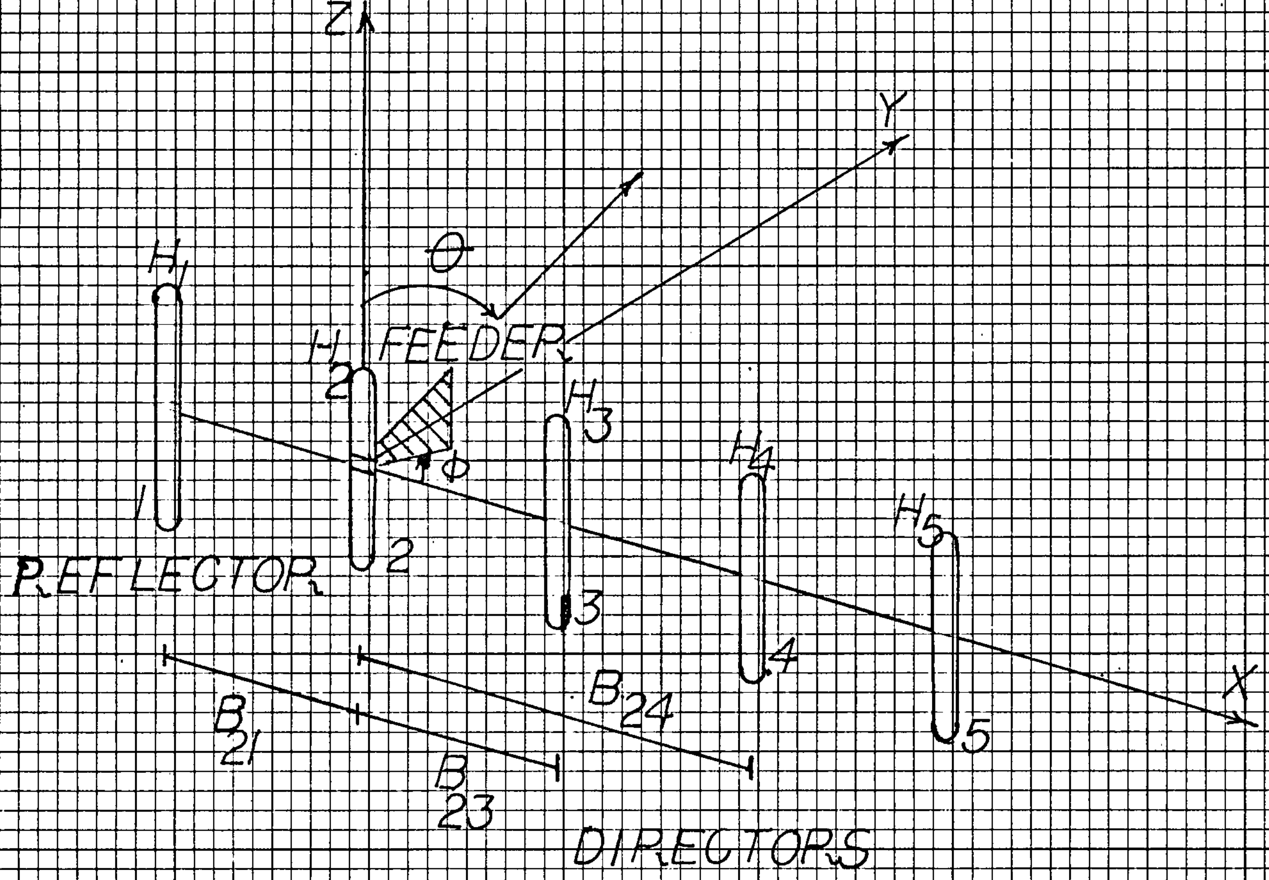
$i=1,2,\ldots,N$

$j=1,2,\ldots,N$

and $b_{k,k}=a$, where

$k=1,2,\ldots,N$.

Such an array is shown in Fig. 1.1. The driven element is normally tuned to resonance. Element No. 1 is usually longer than the driven element and elements Nos. 3 through N are usually shorter than the driven element.


Z
Y
X
θ
φ
H1
H2
H3
H4
H5
FEEDER
REFLECTOR
DIRECTORS
1
2
3
4
5
B21
B23
B24

2

1-2 Previous Work

An importance performance index of a Yagi-Uda array is the directivity. Many studies have been made on the subject of optimization for Yagi-Uda antenna arrays. Hansen and Woodyard [1938] calculated the optimum phase delays of the currents in the elements to give a maximum directivity for a Yagi array. However, their analysis did not provide any information as to how the prescribed phase shift could be realized. In a study by Ehrenspeck and Poeller [1950], correct dimensions for maximum directivity in a Yagi-Uda array with equally spaced directors of equal length were determined experimentally.

Shen [1971] obtained an optimum design for a Yagi array under constraints on bandwidth, directivity or the size of the array. It was shown that the array configuration is determined by any two of these three being specified. It was also shown that a properly designed Yagi array can be operated in two frequency bands, with the frequency ratio approximately equal to 3.5. However, due to the nature of the approximation used in Shen's analysis, no information on the input impedance of the Yagi antenna is given.

Recently, Cheng and Chen [1973] developed a method for the maximization of the forward directivity of a Yagi-Uda array by adjustment of the inter-element spacing. They made use of the three-term theory developed by King and his associates to approximate the current in the dipoles. Use was also made of a theory in matrix analysis. By an iteration process, the optimum spacing for maximum directivity is determined. To what extent the bandwidth is affected when the dimensions of the array are adjusted for maximum directivity remains unanswered.

1-3 Present Study

In this study, the work of Shen and the work of Cheng and Chen are extended. Namely, the behavior of the input impedance of the Yagi-Uda array as the operating frequency is varied is investigated. Thus, when a Yagi-Uda array is optimized under any two of the three constraints - bandwidth, directivity, size of array - using the method developed by Shen, information on the input impedance would be available. The effect on the bandwidth of a Yagi-Uda array when its gain is optimized by adjusting the spacing of the directors as proposed by Cheng and Chen is also investigated. The analysis makes use of King's three-term theory. It converts an integral equation into a complex matrix equation, which is solved using a digital computer.

CHAPTER II

Integral Equation and Three-Term Theory

The integral equation for the current on a thin cylindrical perfect conductor is described in Section 2-1. In Section 2-2, the three-term theory is described. In Section 2-3, the far field pattern is calculated.

2-1 Integral Equation

The theory developed is concerned exclusively with thin cylindrical conductors all aligned in the z direction in air, so that it suffices to use only the axial component of the vector potential. Element No. 2 is center driven by a delta-function generator.

The interaction of charges and currents on conductors in space is governed by Maxwell's equations. A convenient way of solving these vector partial differential equations is through the use of scalar and vector potentials, Φ and $\overline{A}$, respectively.

Since $\nabla \cdot \overline{B} = 0$, it follows from an important theory in vector analysis that the vector $\overline{B}$ be the curl of some other vector [Sokolnikoff, 1962, p.423]. So the magnetic field can be expressed in the form

$$\overline{B} = \nabla \times \overline{A} \qquad \text{---(2.1a)}$$

Using the above equation and substituting into the Maxwell equations, one can show that $\nabla\times(\overline{E}+j\omega\overline{A})=0$. Again using another important theory in vector analysis [Sokolnikoff, 1962, p.422], it follows from the above equation that the vector $\overline{E}+j\omega\overline{A}$ be the gradient of some function, so the expression of the electric field is given by

$$\overline{E} = -\nabla\Phi-j\omega\overline{A} \qquad \text{---(2.1b)}$$

where Φ and $\overline{A}$ are the scalar and vector potentials, respectively. The vector $\overline{A}$ is not defined completely by (2.1a); in order to define a vector, both its curl and it divergence must be defined [Mason and Weaver, 1922, p.353] and the normal component must be known over a closed surface or the vector must vanish as $1/r^2$ at infinity [loc. cit.]. The following condition relating $\overline{A}$ and Φ is imposed:

$$\nabla\cdot\overline{A} = -j(\beta_0^2/\omega)\Phi \qquad \text{---(2.2)}$$

which is known as the Lorentz condition. The quantity β_0 is $\omega\sqrt{\mu\varepsilon}$, where μ and ε are the permeability and permittivity of the medium.

Instead of simply assuming a convenient current along the antenna, a more scientific, albeit more difficult procedure is to determine the actual distribution of current by setting up and solving the appropriate integral equation. With (2.1) and (2.2), and from the boundary condition $E_z(z)=0$ on the surface of a perfectly conducting antenna, the vector potential is seen to satisfy the equation

$$(d^2/dz^2 + \beta_0^2)A_z(z)=0 \qquad \text{---(2.3)}$$

which has the general solution

$$A_z(z) = (-j/c)(C_1\cos\beta_0 z + C_2\sin\beta_0|z|) \qquad \text{---(2.4)}$$

if the symmetry conditions $I_z(-z)=I_z(z)$, $A_z(-z)=A_z(z)$ are imposed. C_1 and C_2 are arbitrary constants of integration and $c=1/\sqrt{\mu\varepsilon}$. The integral equation for the current is

$$(4\pi/\mu_0)A_z = \int_{-h}^{h} I_z(z')(e^{-j\beta_0 R})/R\, dz$$

$$= (-j4\pi/\zeta_0)(C_1\cos\beta_0 z + C_2\sin\beta_0|z|) \qquad \text{---(2.5)}$$

We have used the free space Green's function $G(\bar{r},\bar{r}')=(1/4\pi)(e^{-j\beta R})/R$ where $R=|\bar{r}-\bar{r}'|$. The problem of solving (2.5) for the current is very complicated. This is a linear integral equation of the first kind. It has been carried out approximately in a variety of ways [King, 1956]. The procedure to be followed in obtaining a useful approximate solution of (2.5) is the three-term theory developed by King [1968].

2-2 Three-Term Theory

It was shown by King [1968,p.149] that from the properties of the integral equation that an approximation of the current consists of three terms, of which each represents a different distribution; specifically, let the current distribution (2.5)

$$I_{zk}(z_k) = A_k M_{0zk} + B_k F_{0zk} + D_k H_{0zk} \qquad \text{---(2.6)}$$

where

$$M_{0zk} = \sin\beta_0(h_k - |z_k|) \qquad \text{---(2.7a)}$$

$$F_{0zk} = \cos\beta_0 z_k - \cos\beta_0 h_k \qquad \text{---(2.7b)}$$

$$H_{0zk} = \cos(\beta_0 z_k/2) - \cos(\beta_0 h_k/2) \qquad \text{---(2.7c)}$$

The currents induced by the interaction between charges moving in the more or less widely separated sections of the antenna appear in two parts. One of these, the shifted cosine, is maintained by that part of the interaction which is equivalent to a constant field acting in phase at all points along the antenna. The other part, the shifted cosine with half-angle arguments, is the correction that takes account of the phase lag introduced by the retarded instead of instantaneous interaction.

In a parasitic element, the coefficient A_k is zero, but the other two terms remain. When the several antennas in an array are not all equal in length, so that the h_i differ, the N simultaneous equations exist:

$$\sum_{i=1}^{N}\int_{-h_i}^{h_i} I_{zi}(z_i')K_{kid}(z_k,z_i')dz_i' = \{j4\pi/[\zeta_0\cos(\beta_0h_k)]\}$$

$$(1/2V_{0k}M_{0zk} + U_kF_{0zk}) \quad ---(2.8)$$

with $k=1,2,...,N$. The kernel has the form

$$K_{kid}(z_k,z_i') = K_{ki}(z_k,z_i')-K_{ki}(h_k,z_i') = e^{-j\beta_0R_{ki}}/R_{ki}-e^{-j\beta_0R_{kih}}/R_{kih}$$

where $R_{ki}=\sqrt{(z_k-z_i')^2+b_{ki}^2}$ $\quad R_{kih}=\sqrt{(h_h-z_i')^2+b_{ki}^2}$

The function U_k is

$$U_k=[-j\zeta_0/(4\pi)]\sum_{i=1}^{N}\int_{h_i}^{h_i} I_{zi}(z_i')K_{ki}(h_k,z_i')dz_i' \quad ---(2.9)$$

The integral equation for the driven element is

$$A_2\int_{-h_2}^{h_2} M_{0z'2}K_{22d}(z_2,z_2')dz_2' + \sum_{i=1}^{N}\int B_iF_{0z'i}K_{2id}(z_2,z_i')dz_i'$$
$$+ \sum_{i=1}^{N}\int D_iH_{0z'i}K_{2id}(z_2,z_i')dz_i'$$

$$=\{j4\pi/[\zeta_0\cos(\beta_0h_2)]\}(1/2V_{02}M_{0z2} + U_2F_{0z2}) \quad ---(2.10)$$

and the remaining N-1 integral equations are

$$A_2\int_{-h_2}^{h_2} M_{0z'2}K_{k2d}(z_k,z_2')dz_2' + \sum_{i=1}^{N}\int B_iF_{0z'i}K_{kid}(z_k,z_i')dz_i'$$
$$+ \sum_{i=1}^{N}\int D_iH_{0z'i}K_{kid}(z_k,z_i')dz_i'$$

$$=\{j4\pi/[\zeta_0\cos(\beta_0h_k)]\}(U_kF_{0zk}) \quad ---(2.11)$$

Use is made of the properties of the real and imaginary parts of the kernel as follows:

$$\int_{-h_k}^{h_k} G_{0z'k}K_{kkdR}(z_k,z_k')dz_k' \approx G_{0zk}$$

where $G_{0z'k}$ stands for $M_{0z'k}$, $F_{0z'k}$, or $H_{0z'k}$ and $K_{kkdR}(z_k,z_k')$ is the real part of the kernel. On the other hand,

$$\int_{-h_k}^{h_k} G_{0z_k'} K_{kkdI}(z_k, z_k') dz_k' \simeq H_{0z_k} \quad .$$

It follows that:

$$W_{kkV}(z_k) \equiv \int_{-h_k}^{h_k} M_{0z_k'} K_{kkd}(z_k, z_k') dz_k' \doteq \Psi^m_{kkdV} M_{0z_k} + \Psi^h_{kkdV} H_{0z_k} \qquad \text{---(2.12a)}$$

$$W_{kkU}(z_k) \equiv \int_{-h_k}^{h_k} F_{0z_k'} K_{kkd}(z_k, z_k') dz_k' \doteq \Psi^f_{kkdU} F_{0z_k} + \Psi^h_{kkdU} H_{0z_k} \qquad \text{---(2.12b)}$$

$$W_{kkD}(z_k) \equiv \int_{-h_k}^{h_k} H_{0z_k'} K_{kkd}(z_k, z_k') dz_k' \doteq \Psi^f_{kkdD} F_{0z_k} + \Psi^h_{kkdD} H_{0z_k} \qquad \text{---(2.12c)}$$

where the Ψ's are complex coefficients yet to be determined. $\Psi^f_{kkdD} F_{0z_k}$ is added to provide symmetry.

When $i \neq k$ and $\beta_0 b >= 1$, it can be shown by direct comparison that:

$$\int_{-h_i}^{h_i} G_{0z_i'} K_{kidR}(z_k, z_i') dz_i' \simeq F_{0z_k}$$

$$\int_{-h_i}^{h_i} G_{0z_i'} K_{kidI}(z_k, z_i') dz_i' \simeq H_{0z_k} \quad ,$$

where $G_{0z_i'}$ stands for $M_{0z_i'}$, $F_{0z_i'}$ or $H_{0z_i'}$. It follows that, with $i \neq k$:

$$W_{kiV}(z_k) \equiv \int_{-h_i}^{h_i} M_{0z_i'} K_{kid}(z_k, z_i') dz_i' \doteq \Psi^f_{kidV} F_{0z_k} + \Psi^h_{kidV} H_{0z_k} \qquad \text{---(2.13a)}$$

$$W_{kiU}(z_k) \equiv \int_{-h_i}^{h_i} F_{0z_i'} K_{kid}(z_k, z_i') dz_i' \doteq \Psi^f_{kidU} F_{0z_k} + \Psi^h_{kidU} H_{0z_k} \qquad \text{---(2.13b)}$$

$$W_{kiD}(z_k) \equiv \int_{-h_i}^{h_i} H_{0z_i'} K_{kid}(z_k, z_i') dz_i' \doteq \Psi^f_{kidD} F_{0z_k} + \Psi^h_{kidD} H_{0z_k} \qquad \text{---(2.13c)}$$

where the Ψ's are complex coefficients yet to be determined.

With (2.12) and (2.13), (2.10) becomes:

$$A_2(\Psi^m_{22dV} M_{0z2} + \Psi^h_{22dV} H_{0z2}) + \sum_{i=1}^{N} B_i(\Psi^f_{2idU} F_{0z2} + \Psi^h_{2idU} H_{0z2})$$

$$+ \sum_{i=1}^{N} D_i(\Psi^f_{2idD} F_{0z2} + \Psi^h_{2idD} H_{0z2})$$

$$= \frac{4j\pi}{\zeta_0 \cos(\beta_0 h_2)} (1/2\ V_{02} M_{0z2} + U_2 F_{0z2}) \qquad \text{---(2.14)}$$

For (2.11), the N-1 equations are:

$$A_2(\Psi^f_{k2dV} F_{0zk} + \Psi^h_{k2dV} H_{0zk}) + \sum_{i=1}^{N} B_i(\Psi^f_{kidU} F_{0zk} + \Psi^h_{kidU} H_{0zk})$$
$$+ \sum_{i=1}^{N} D_i(\Psi^f_{kidD} F_{0zk} + \Psi^h_{kidD} H_{0zk})$$

$$= \frac{4j\pi}{\zeta_0 \cos(\beta_0 h_k)} U_k F_{0zk} \quad k=1,3,4,\ldots,N \qquad \text{---(2.15)}$$

These equations will be satisfied if the coefficients of each of the distribution functions is individually required to vanish; i.e., in (2.11):

$$A_2 = \frac{j2\pi V_{02}}{\zeta_0 \Psi^m_{22dV} \cos(\beta_0 h_2)} \qquad \text{---(2.16)}$$

and

$$\sum_{i=1}^{N} (B_i \Psi^f_{2idU} + D_i \Psi^f_{2idD}) \cos\beta_0 h_2 - \frac{4\pi j}{\zeta_0} U_2 = 0 \qquad \text{---(2.17a)}$$

$$A_2 \Psi^h_{22dV} + \sum_{i=1}^{N} (B_i \Psi^h_{2idU} + D_i \Psi^h_{2idD}) = 0 \qquad \text{---(2.17b)}$$

Similarly, in (2.15), with k=1,3,4,...,N:

$$A_2 \Psi^f_{k2dV} + \sum_{i=1}^{N} (B_i \Psi^f_{kidU} + D_i \Psi^f_{kidD}) \cos\beta_0 h_k - \frac{j4\pi U_k}{\zeta_0} = 0 \qquad \text{---(2.18a)}$$

$$A_2 \Psi^h_{k2dV} + \sum_{i=1}^{N} (B_i \Psi^h_{kidU} + D_i \Psi^h_{kidD}) = 0 \qquad \text{---(2.18b)}$$

Equations (2.16), (2.17) and (2.18) are (2N+1), and they determine the 2N+1 constants A_2, B_i and D_i, i=1,2,...,N. To evaluate the functions U_k, define the following integrals:

$$\Psi_{kiV}(h_k)=\int_{-h_i}^{h_i} M_{0z_i'}K_{ki}(h_k,z_i')dz_i' \qquad \text{---(2.19a)}$$

$$\Psi_{kiU}(h_k)=\int_{-h_i}^{h_i} F_{0z_i'}K_{ki}(h_k,z_i')dz_i' \qquad \text{---(2.19b)}$$

$$\Psi_{kiD}(h_k)=\int_{-h_i}^{h_i} H_{0z_i'}K_{ki}(h_k,z_i')dz_i' \qquad \text{---(2.19c)}$$

where $K_{ki}(h_k,z_i') = \dfrac{e^{-j\beta_0 R_{kih}}}{R_{kih}}$ and $R_{kih}=\sqrt{(h_k-z_i')^2+b_{ik}^2}$.

From (2.9), it follows that:

$$U_k=\frac{-j\zeta_0}{4\pi}\sum_{i=1}^{N} A_i\Psi_{kiV}(h_k) + B_i\Psi_{kiU}(h_k) + D_i\Psi_{kiD}(h_k) \qquad \text{---(2.20a)}$$

$A_i=0$ for i=1,3,4,...,N

$$U_k=\frac{-j\zeta_0}{4\pi}\{A_2\Psi_{k2V}(h_k) + \sum_{i=1}^{N} B_i\Psi_{kiU}(h_k) + D_i\Psi_{kiD}(h_k) \qquad \text{---(2.20b)}$$

Equations (2.17) and (2.18) can be combined with the aid of the Kronecker δ defined by:

$$\delta_{ik} = \begin{cases} 0 & i\neq k \\ 1 & i=k \end{cases}$$

The equations are:

$$A_2(1-\delta_{k2})\Psi^f_{k2dV} + \sum_{i=1}^{N} (B_i\Psi^f_{kidU} + D_i\Psi^f_{kidD})\cos\beta_0 h_k - \frac{j4\pi U_k}{\zeta_0} = 0 \qquad \text{--(2.21}$$

$$A_2\Psi^h_{k2dV} + \sum_{i=1}^{N} (B_i\Psi^h_{kidU} + D_i\Psi^h_{kidD}) = 0 \qquad \text{---(2.21b)}$$

Substituting (2.20b) in (2.21a), we get:

$$A_2(\Psi_{k2V}(h_k) - (1-\delta_{k2})\Psi^f_{k2dV}\cos\beta_0 h_k) + \sum_{i=1}^{N} B_i(\Psi_{kiU}(h_k) - \Psi^f_{kidU}\cos\beta_0 h_k)$$
$$+ \sum_{i=1}^{N} D_i(\Psi_{kiD}(h_k) - \Psi^f_{kidD}\cos\beta_0 h_k) = 0 \qquad \text{---(2.22)}$$

A simplification seems possible by defining:

$$\Phi_{k2V} = \Psi_{k2V}(h_k) - (1-\delta_{k2})\Psi^f_{k2dV}\cos(\beta_0 h_k) \qquad \text{---(2.23a)}$$

$$\Phi_{kiU} = \Psi_{kiU}(h_k) - \Psi^f_{kidU}\cos(\beta_0 h_k) \qquad \text{---(2.23b)}$$

$$\Phi_{kiD} = \Psi_{kiD}(h_k) - \Psi^f_{kidD}\cos(\beta_0 h_k) \qquad \text{---(2.23c)}$$

With this notion, (2.22) and (2.21b) give the following set of equations:

$$\sum_{i=1}^{N} \Phi_{kiU}B_i + \Phi_{kiD}D_i = -\Phi_{k2V}A_2 \qquad \text{---(2.24a)}$$

$$\sum_{i=1}^{N} \Psi^h_{kidU}B_i + \Psi^h_{kidD}D_i = -\Psi^h_{k2dV}A_2 \qquad \text{---(2.24b)}$$

These equations may be expressed in matrix form after the introduction of the following notation:

$$\Phi_U = \begin{bmatrix} \Phi_{11U} & \Phi_{12U} & \cdots & \Phi_{1NU} \\ \vdots & \vdots & & \vdots \\ \Phi_{N1U} & \Phi_{N2U} & & \Phi_{NNU} \end{bmatrix} \qquad \text{---(2.25a)}$$

$$\Phi_D = \begin{bmatrix} \Phi_{11D} & \Phi_{12D} & \cdots & \Phi_{1ND} \\ \vdots & \vdots & & \vdots \\ \Phi_{N1D} & \Phi_{N2D} & & \Phi_{NND} \end{bmatrix} \qquad \text{---(2.25b)}$$

$$\Psi^h_{dU} = \begin{bmatrix} \Psi^h_{11dU} & \Psi^h_{12dU} & \cdots & \Psi^h_{1NdU} \\ \vdots & \vdots & & \vdots \\ \Psi^h_{N1dU} & \Psi^h_{N2dU} & & \Psi^h_{NNdU} \end{bmatrix} \qquad \text{---(2.25c)}$$

$$\Psi^h_{dD} = \begin{bmatrix} \Psi^h_{11dD} & \Psi^h_{12dD} & \cdots & \Psi^h_{1NdD} \\ \vdots & \vdots & & \vdots \\ \Psi^h_{N1dD} & \Psi^h_{N2dD} & & \Psi^h_{NNdD} \end{bmatrix} \qquad \text{---(2.25d)}$$

$$\{\Phi_{2V}\} = \begin{bmatrix} \Phi_{12V} \\ \vdots \\ \Phi_{N2V} \end{bmatrix} \qquad \text{---(2.25e)} \qquad \{\Psi^h_{2dV}\} = \begin{bmatrix} \Psi^h_{12dV} \\ \vdots \\ \Psi^h_{N2dV} \end{bmatrix} \qquad \text{---(2.25f)}$$

$$\{B\} = \begin{bmatrix} B_1 \\ \vdots \\ B_N \end{bmatrix} \qquad \text{---(2.25g)} \qquad \{D\} = \begin{bmatrix} D_1 \\ \vdots \\ D_N \end{bmatrix} \qquad \text{---(2.25h)}$$

The matrix form of (2.24) is:

$$[\Phi_U]\{B\} + [\Phi_D]\{D\} = -\{\Phi_{2V}\}A_2 \qquad \text{---(2.26a)}$$

$$[\Psi^h_{dU}]\{B\} + [\Psi^h_{dD}]\{D\} = -\{\Psi^h_{2dV}\}A_2 \qquad \text{---(2.26b)}$$

It remains to evaluate the Ψ's which occur in the Φ's in (2.23).

Equations (2.12) and (2.13) approximate each integral by a linear combination of two terms with arbitrary coefficients; these can be evaluated by equating both sides at two values of z. The values chosen are z=0 and $z=h_k/2$. Define the following:

$$W_{kiV}(0) \equiv A_i^{-1}\int_{-h_i}^{h_i} I_{Vi}(z_i')K_{kid}(0,z_i')dz_i'$$

$$\doteq \int_{-h_i}^{h_i} M_{0z'i}K_{kid}(0,z_i')dz_i' \qquad \text{---(2.27a)}$$

$$W_{kiV}(h_k/2) \equiv A_i^{-1}\int_{-h_i}^{h_i} I_{Vi}(z_i')K_{kid}(h_k/2,z_i')dz_i'$$

$$\doteq \int_{-h_i}^{h_i} M_{0z'i}K_{kid}(h_k/2,z_i')dz_i' \qquad \text{---(2.27b)}$$

$$W_{kiU}(0) \equiv B_i^{-1}\int_{-h_i}^{h_i} I_{Ui}(z_i')K_{kid}(0,z_i')dz_i'$$

$$\doteq \int_{-h_i}^{h_i} F_{0z'i}K_{kid}(0,z_i')dz_i' \qquad \text{---(2.27c)}$$

$$W_{kiU}(h_k/2) \equiv B_i^{-1}\int_{-h_i}^{h_i} I_{Ui}(z_i')K_{kid}(h_k/2,z_i')dz_i'$$

$$\doteq \int_{-h_i}^{h_i} F_{0z'i}K_{kid}(h_k/2,z_i')dz_i' \qquad \text{---(2.27d)}$$

$$W_{kiD}(0) \equiv D_i^{-1}\int_{-h_i}^{h_i} I_{Di}(z_i')K_{kid}(0,z_i')dz_i'$$

$$\doteq \int_{-h_i}^{h_i} H_{0z'i}K_{kid}(0,z_i')dz_i' \qquad \text{---(2.27e)}$$

$$W_{kiD}(h_k/2) \equiv D_i^{-1}\int_{-h_i}^{h_i} I_{Di}(z_i')K_{kid}(h_k/2,z_i')dz_i'$$

$$\doteq \int_{-h_i}^{h_i} H_{0z'i}K_{kid}(h_k/2,z_i')dz_i' \qquad \text{---(2.27f)}$$

Once the W's in (2.27) have been determined for all values of i and k, the coefficients Ψ may be determined from the equations (2.12) and (2.13). At z=0 these become:

$$\Psi^m_{kkdV}\sin\beta_0 h_k + \Psi^h_{kkdV}[1-\cos(\beta_0 h_k/2)] = W_{kkV}(0) \quad \text{---(2.28a)}$$

$$\Psi^f_{kidV}(1-\cos\beta_0 h_k) + \Psi^h_{kidV}[1-\cos(\beta_0 h_k/2)] = W_{kiV}(0) \quad i\neq k \quad \text{---(2.28b)}$$

$$\Psi^f_{kidU}(1-\cos\beta_0 h_k) + \Psi^h_{kidU}[1-\cos(\beta_0 h_k/2)] = W_{kiU}(0) \quad \text{---(2.28c)}$$

$$\Psi^f_{kidD}(1-\cos\beta_0 h_k) + \Psi^h_{kidD}[1-\cos(\beta_0 h_k/2)] = W_{kiD}(0) \quad \text{---(2.28d)}$$

At $z=h_k/2$, they are

$$\Psi^m_{kkdV}\sin(\beta_0 h_k/2) + \Psi^h_{kkdV}[\cos(\beta_0 h_k/4)-\cos(\beta_0 h_k/2)] = W_{kkV}(h_k/2) \quad \text{---(2.29a)}$$

$$\Psi^f_{kidV}[\cos(\beta_0 h_k/2)-\cos\beta_0 h_k] + \Psi^h_{kidV}[\cos(\beta_0 h_k/4)-\cos(\beta_0 h_k/2)]$$
$$= W_{kiV}(h_k/2) \quad i\neq k \quad \text{---(2.29b)}$$

$$\Psi^f_{kidU}[\cos(\beta_0 h_k/2)-\cos\beta_0 h_k] + \Psi^h_{kidU}[\cos(\beta_0 h_k/4)-\cos(\beta_0 h_k/2)]$$
$$= W_{kiU}(h_k/2) \quad \text{---(2.29c)}$$

$$\Psi^f_{kidD}[\cos(\beta_0 h_k/2)-\cos\beta_0 h_k] + \Psi^h_{kidD}[\cos(\beta_0 h_k/4)-\cos(\beta_0 h_k/2)]$$
$$= W_{kiD}(h_k/2) \quad \text{---(2.29d)}$$

The solutions of these equations for the Ψ's are obtained directly. They are:

$$\Psi^m_{kkdV} = \Delta_1^{-1}\{W_{kkV}(0)[\cos(\beta_0 h_k/4)-\cos(\beta_0 h_k/2)]$$
$$-W_{kkV}(h_k/2)[1-\cos(\beta_0 h_k/2)]\} \quad \text{---(2.30)}$$

$$\Psi^h_{kkdV} = \Delta_1^{-1}\{W_{kkV}(h_k/2)\sin\beta_0 h_k - W_{kkV}(0)\sin(\beta_0 h_k/2)\} \qquad \text{---(2.31)}$$

$$\Psi^f_{kidV} = \Delta_2^{-1}\{W_{kiV}(0)[\cos(\beta_0 h_k/4)-\cos(\beta_0 h_k/2)]$$

$$-W_{kiV}(h_k/2)[1-\cos(\beta_0 h_k/2)]\} \qquad i\neq k \qquad \text{---(2.32)}$$

$$\Psi^h_{kidV} = \Delta_2^{-1}\{W_{kiV}(h_k/2)[1-\cos\beta_0 h_k]$$

$$-W_{kiV}(0)[\cos(\beta_0 h_k/2)-\cos\beta_0 h_k]\} \qquad i\neq k \qquad \text{---(2.33)}$$

$$\Psi^f_{kidU} = \Delta_2^{-1}\{W_{kiU}(0)[\cos(\beta_0 h_k/4)-\cos(\beta_0 h_k/2)]$$

$$-W_{kiU}(h_k/2)[1-\cos(\beta_0 h_k/2)]\} \qquad \text{---(2.34)}$$

$$\Psi^h_{kidU} = \Delta_2^{-1}\{W_{kiU}(h_k/2)[1-\cos\beta_0 h_k]$$

$$-W_{kiU}(0)[\cos(\beta_0 h_k/2)-\cos(\beta_0 h_k)]\} \qquad \text{---(2.35)}$$

$$\Psi^f_{kidD} = \Delta_2^{-1}\{W_{kiD}(0)[\cos(\beta_0 h_k/4)-\cos(\beta_0 h_k/2)]$$

$$-W_{kiD}(h_k/2)[1-\cos(\beta_0 h_k/2)]\} \qquad \text{---(2.36)}$$

$$\Psi^h_{kidD} = \Delta_2^{-1}\{W_{kiD}(h_k/2)[1-\cos\beta_0 h_k]$$

$$-W_{kiD}(0)[\cos(\beta_0 h_k/2)-\cos\beta_0 h_k]\} \qquad \text{---(2.37)}$$

where $\Delta_1 = \sin\beta_0 h_k[\cos(\beta_0 h_k/4)-\cos(\beta_0 h_k/2)]$

$$-\sin(\beta_0 h_k/2)[1-\cos(\beta_0 h_k/2)] \qquad \text{---(2.38)}$$

and $\Delta_2 = [1-\cos\beta_0 h_k][\cos(\beta_0 h_k/4)-\cos(\beta_0 h_k/2)]$

$$-[\cos(\beta_0 h_k/2)-\cos\beta_0 h_k][1-\cos(\beta_0 h_k/2)] \qquad \text{---(2.39)}$$

All of the Ψ's have been determined. The $\Psi(h)$ coefficients are given in (2.19). The elements of the Φ matrices are obtained from (2.23). This completes the solution for all of the currents in the elements of the Yagi-Uda array.

When the driven element (No. 2) in a Yagi-Uda array is a half-wave dipole, as it often is, $\beta_0 h_2=\pi/2$ and $\cos\beta_0 h_2=0$. Some of the quantities will become indeterminate. Although they will yield definite values in the limiting process, an alternative formulation is preferred in order to avoid computational difficulties. The equations needed are described by King[1968,p.198].

2-3 Radiation Pattern

Once the current distribution on the elements of the Yagi-Uda array is known, the far field pattern can be calculated. The configuration of Fig. (1.1) is used to derive the far fields of the Yagi-Uda array.

The electromagnetic field is

$$E_\theta(R_2,\theta,\phi)=(j\zeta_0/2\pi)\{A_2(e^{-j\beta_0 R_2}/R_2)F_m(\theta,\beta_0 h_2)$$
$$+\sum_{i=1}^{N}(e^{-j\beta_0 R_i}/R_i)[B_i G_m(\theta,\beta_0 h_i)+D_i D_m(\theta,\beta_0 h_i)]\} \quad ---(2.40)$$

where R_i is the distance from the point of calculation to the center of the element i and F_m, G_m, and D_m are defined as follows:

$$F_m(x,y)=1/2\int_{-y}^{y}\sin(y-|x'|)e^{jx'\cos x}\sin x\, dx'$$

$$=[\cos(y\cos x)-\cos y]/\sin x \quad ---(2.41)$$

$$G_m(x,y)=1/2\int_{-y}^{y}(\cos x'-\cos y)e^{jx'\cos x}\sin x\, dx'$$
$$=[\sin y\cos(y\cos x)\cos x - \cos y\sin(y\cos x)]/$$
$$(\sin x\cos x) \quad ---(2.42)$$

$$D_m(x,y)=1/2\int_{-y}^{y}(\cos(x'/2)-\cos(y/2))e^{jx'\cos x}\sin x\, dx'$$
$$=\{[2\cos(y\cos x)\sin(y/2)-4\sin(y\cos x)\cos(y/2)\cos x]/$$
$$(1-4\cos^2 x)-[\sin(y\cos x)\cos(y/2)]/\cos x\}\sin x \quad ---(2.43)$$

Equation (2.40) may be arranged as follows:

$$E_{\theta N}(R_2,\theta,\phi) = (-V_{02}/\Psi)(e^{-j\beta_0 R_2})/R_2(f_{VN}(\theta,\phi)) \quad ---(2.44)$$

Since no ambiguity can arise, the symbol Ψ without subscripts and superscripts is used for Ψ^m_{22dV} as defined in (2.30). The field factor in (2.44) for the N element array is given by:

$$f_{VN}(\theta,\phi)=\{F_m(\theta,\beta_0 h_2) + \sum_{i=1}^{N} e^{-j\beta_0(R_i-R_2)}[T_{Ui}G_m(\theta,\beta_0 h_i)$$
$$+ T_{Di}D_m(\theta,\beta_0 h_i)]\}\sec\beta_0 h_2 \quad ---(2.45)$$

In obtaining (2.44) and (2.45) the far field approximation $R_i \doteq R_2$ has been made. The following set of parameters has been introduced:

$$T_{Ui}=B_i/A_2 \qquad T_{Di}=D_i/A_2 \quad \text{where } A_2 \text{ is defined in (2.16)} \quad ---(2.46)$$

The field pattern in the equatorial plane is given by $|f_{VN}(\pi/2,\phi)|/|f_{VN}(\pi/2,0)|$ as a function of ϕ.

The ratio of the field in the forward direction ($\phi=0$) to the field in the backward direction ($\phi=\pi$) in the equatorial plane ($\theta=\pi/2$) is known as the front-to-back ratio. It is given by:

$$R_{FB}=|f_{VN}(\pi/2,0)|/|f_{VN}(\pi/2,\pi)| \quad ---(2.47)$$

The front-to-back ratio in decibels is:

$$r_{FB}=20\log_{10}(R_{FB}) \quad ---(2.48)$$

Since the total power radiated by an array is given by the integral over a great sphere of the normal component of the Poynting vector

$$S_R=1/2(E\times H^*) \qquad ---(2.49)$$

$$|S_R(R,\theta,\phi)|=|E_\theta(R,\theta,\phi)|^2/2\zeta_0 \qquad ---(2.50)$$

the distribution of S_R as a function of θ and ϕ is of interest. The total power supplied to the N element array is:

$$P_{2N}=1/2|V_{02}|^2G_{2N} \qquad ---(2.51)$$

where G_{2N} is the driving point inductance of element 2 when driving the N element parasitic array; but substituting (2.44) and (2.51) in (2.50)

$$|S_R(R_2,\theta,\phi)|=(P_{2N}/G_{2N})(1/|\Psi|^2)(1/\zeta_0R_2^2)|f_{VN}(\theta,\phi)|^2 \qquad ---(2.52)$$

If the ohmic losses in the conductors of the antennas and in the surrounding dielectric medium (air) are neglected, the total power radiated by an array outside a great sphere of radius R_2 is the same as the total power supplied at the terminals of the driven element 2; that is:

$$P_{2N}=\int_0^{2\pi}\int_0^{\pi}|S_R(R_2,\theta,\phi)|R_2^2\sin\theta d\theta d\phi \qquad ---(2.53)$$

With (2.52) and (2.53), formulas are obtained for G_{2N}. Actually, G_{2N} is already known from:

$$I_{22}/V_{02}=G_{2N}+jB_{2N}$$

when the medium in which the array is immersed is lossless.

The absolute directivity of the N element Yagi-Uda array is defined in terms of the power radiated by a fictitious isotropic antenna which maintains the same field in all directions as the Yagi-Uda array does in the direction of its maximum ($\theta=\pi/2$, $\phi=0$).

$$P_{Niso} = 4\pi R_2^2 |S_R(R_2,\pi/2,0)|$$

$$= 4\pi R_2^2 (P_{2N}/G_{2N})(1/|\Psi|^2)(1/\zeta_0 R_2^2)|f_{VN}(\pi/2,0)|^2$$

The ratio P_{Niso}/P_{2N} is the absolute directivity. Thus

$$D_N = (4\pi/\zeta_0)(1/|\Psi|^2)(1/G_{2N})|f_{VN}(\pi/2,0)|^2 \qquad \text{---(2.54)}$$

but from (2.16), (2.54) becomes

$$D_N(\pi/2,0)=(4\pi/\zeta_0)\{[|A_2|^2\zeta_0^2 * \cos^2(\beta_0 h_2)]/4\pi^2\}(1/G_{2N})|f_{VN}(\pi/2,0)|^2$$

$$=(\zeta_0/\pi)\{[|A_2|^2 * \cos^2(\beta_0 h_2)]/G_{2N}\}|f_{VN}(\pi/2,0)|^2 \qquad \text{---(2.55)}$$

The quantity

$$G_N(\pi/2,0) = 10\log_{10} D_N(\pi/2,0) \qquad \text{---(2.56)}$$

is the absolute gain in decibels.

If the driven element is or is near a half wavelength long, the more convenient alternative form of the gain is obtained if $F_m(\theta,\beta_0 h_2)$ is replaced by $H_m(\theta,\beta_0 h_2)$ in equations (2.40) and (2.45). Also, $\cos\beta_0 h_2$ is omitted from equation (2.55). $H_m(x,y)$ is defined as follows:

$$H_m(x,y) = 1/2\int_{-h}^{h} (\sin(|x'|)-\sin y)e^{j\beta_0 x'\cos x}\sin x\, dx$$

$$=\frac{[1-\cos y\cos(y\cos x)]\cos x - \sin y\sin(y\cos x)}{\sin x\cos x} \quad ---(2.57)$$

CHAPTER III

Input Impedance

The general description of a computer program for the analysis of the Yagi-Uda array is presented in Section 3-1. In Section 3-2, a comparison of the present results with previous results is discussed. Section 3-3 deals with the relationship of input impedance and frequency. Conclusions are drawn in Section 3-4.

3-1 General Description of Computer Program

The main purpose of this study is to develop a numerical technique for calculating the characteristics of a Yagi-Uda array. In this regard, a general computer program has been written in the FORTRAN IV language to obtain a solution for the matrix equation (2.26).

Equations (2.26a) and (2.26b) involve elements of the N×N matrices $[\Phi_U]$, $[\Phi_D]$, $[\Psi^h_{dU}]$ and $[\Psi^h_{dD}]$. These in turn depend on the parameters Ψ introduced in (2.12) and (2.13), and the parameters $\Psi(h)$ defined in (2.19). Since each integral is approximated by a linear combination of two terms with arbitrary coefficients, it can be evaluated by equating both sides in (2.12) and (2.13) at two values of z. The values chosen are $z=0$ and $z=h_k/2$, in addition to $z=h_k$, where both sides must vanish. The integration subroutine used for this purpose is a subprogram for numerical integration of an arbitrary function by Romberg's method of numerical integration. The required degree of accuracy is achieved by adjusting a convergence test constant. Throughout this study, a convergence test constant of 0.01 and in interval limit of 40 are employed.

Evaluating the Ψ and Φ coefficients, the 2N coefficients B_i and D_i can be determined form equations (2.26); substituting in equations (2.7), the currents in the N elements are determined. These computations are straightforward matrix manipulation. The subroutine for solution of systems of complex simultaneous equations makes use of the Gauss elimination method. Rows are arranged to improve the accuracy of the results.

Finally, the input impedance of the driven elements is obtained from the current at the center of the driven elements in terms of the driving voltage V_{02}. The far field radiation patterns are obtained form numerically computed currents on all elements of the Yagi-Uda antenna array, and the absolute directivity of the array is calculated.

3-2 Comparison With Previous Results

In order to confirm the adequacy of the method of solution described in Section 3-1, a comparison between numerical results from this analysis and the existing data is required; such results are presented in this chapter. The distribution of the current in an array of two full-wave elements in which element 1 is center driven is element 2 is parasitic is studied by King [1968, p.212]. The agreement between the results of this study and those obtained by King is good. The coefficients of the trigonometric components of the current, the admittance, the impedance, the current distribution, the horizontal field pattern, the forward gain, the backward gain, and the front-to-back ratio are all in agreement within 1%.

The input impedance, the far field pattern and the gain for a three element array [King, 1968, p.215] and for an array with four identical directors [King, 1968, p.220] are studied. The results obtained from this analysis are in agreement with those obtained by King within 5%. The difference is due to the fact that a somewhat different procedure was used. The results presented in [King, 1968] are based on the work of Morris[1965], in which the entire procedure carried out in Section 2-3 for arrays with half-wave elements was repeated with the distribution function M_{0z2} replaced by S_{0z2}. This also involved a simple rearrangement of the integral equation so that when k=2, the right-hand member is $(j4\pi/\zeta_0)(1/2V_{02}S_{0z_2} + C_2F_{0z_2})$, where $S_{0z_2}=\sin\beta_0|z_2|-\sin\beta_0 h_2$.

3-3 Input Impedance

It was shown by Shen [1971] that a Yagi-Uda array of an infinite number of equally spaced elements can support a traveling wave along the direction of the array when the frequency is within one of the passbands. The existence of the traveling wave on such a structure has been confirmed by recent experiments [Shen et al., 1971]. The phase velocity was found to be smaller than the velocity of light. Shen [1971] assumed that currents on the elements are constant in amplitude with equally progressive phase shifts; i.e., if $I(z)$ denotes the current at $y=0$ (Fig. 1.1), then the current at $y=b$ is $e^{j\phi}I(z)$, $e^{j2\phi}I(z)$ at $y=2b$, and so on. The quantity ϕ is the phase shift between currents in adjacent elements. It is seen that the phase angle along each element is assumed constant. The present analysis confirms the adequacy of this assumption. Typical examples are shown in Table I for comparison.

TABLE I
Phase Delay Versus Director Length

Kh	b/h	ϕ(Shen)	ϕ(Present Study)	N
1.37	1.5	2.18	2.26	16
1.33	1.0	1.45	1.48	20

Note: a/h=0.01, first passband ϕ=average phase delay in radians
K=free space wave number

Figures (3.1)-(3.4) show the phase angle as a function of the distance from the reflector. The lines drawn through the points have no physical significance, but serve merely to interrelate the discrete points and thus reveal how nearly constant the phase change from director to director actually is.

Also, Figures (3.5)-(3.8) show that for closely spaced directors (e.g., b/h=1.0), the magnitude of $I_k(0)$ is almost equal in the directors and much smaller than the current in the driven element, but for larger spacings (e.g., b/h=1.5), the currents are comparable in magnitude with the current in the driven element. Now, since Shen [1971] assumed constant current amplitudes on all elements, the discrepancy between the measured directivity and the theoretical results obtained by him would be expected to be less for longer arrays (b/h=1.5). This was already described by Shen in the same paper. Also, as was explained by Shen [1971], the theoretical results obtained by him for the directivity are not necessarily equal to the maximum value and so, for longer arrays, the bandwidth is narrower and therefore the discrepancy between the calculated directivity and the maximum value gradually disappears.

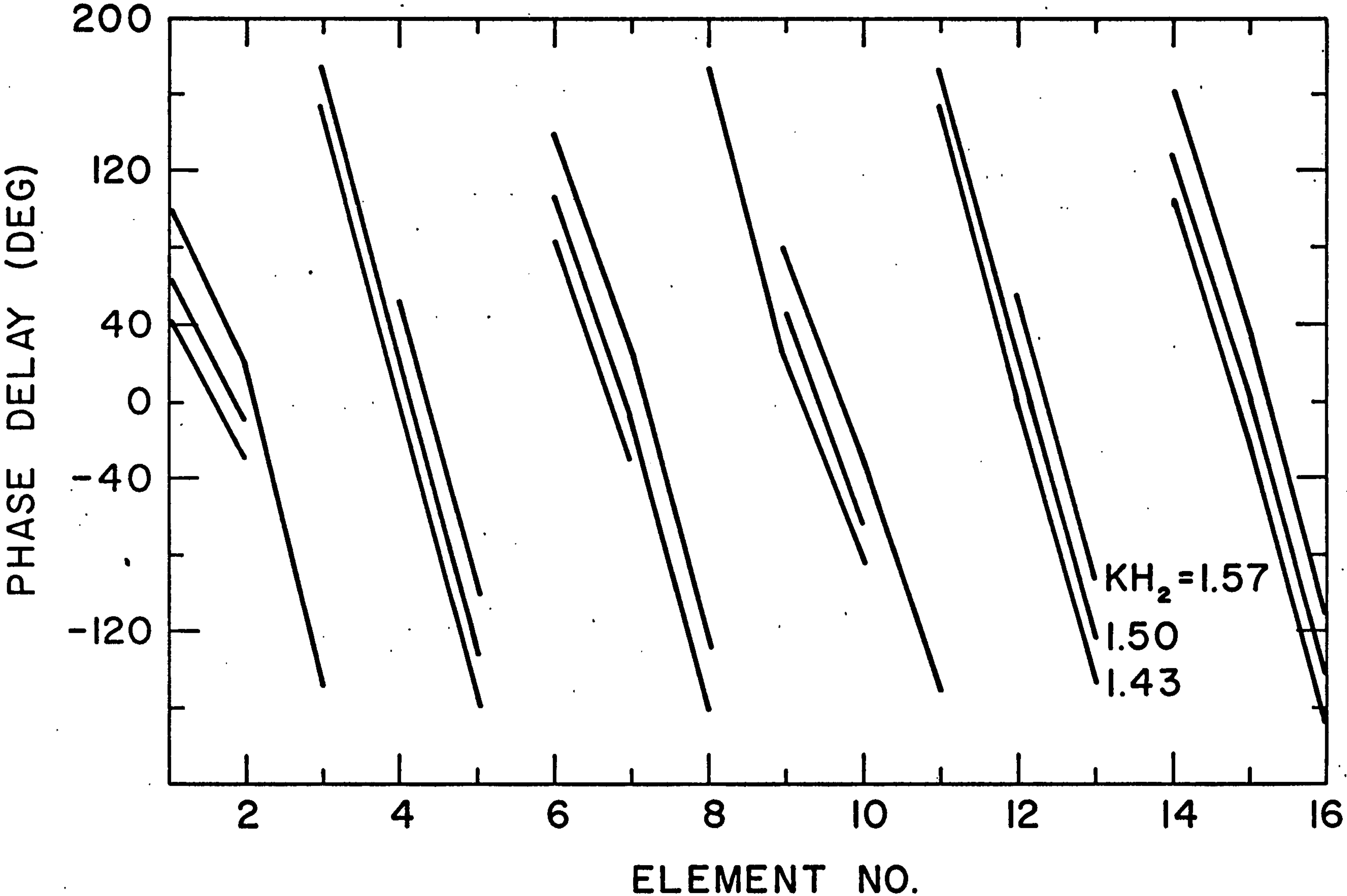


FIG. 3.1

($N=16$, $kh=1.37$, $b/h=1.5$, $a/h=.01$, $kh_1=1.67$)

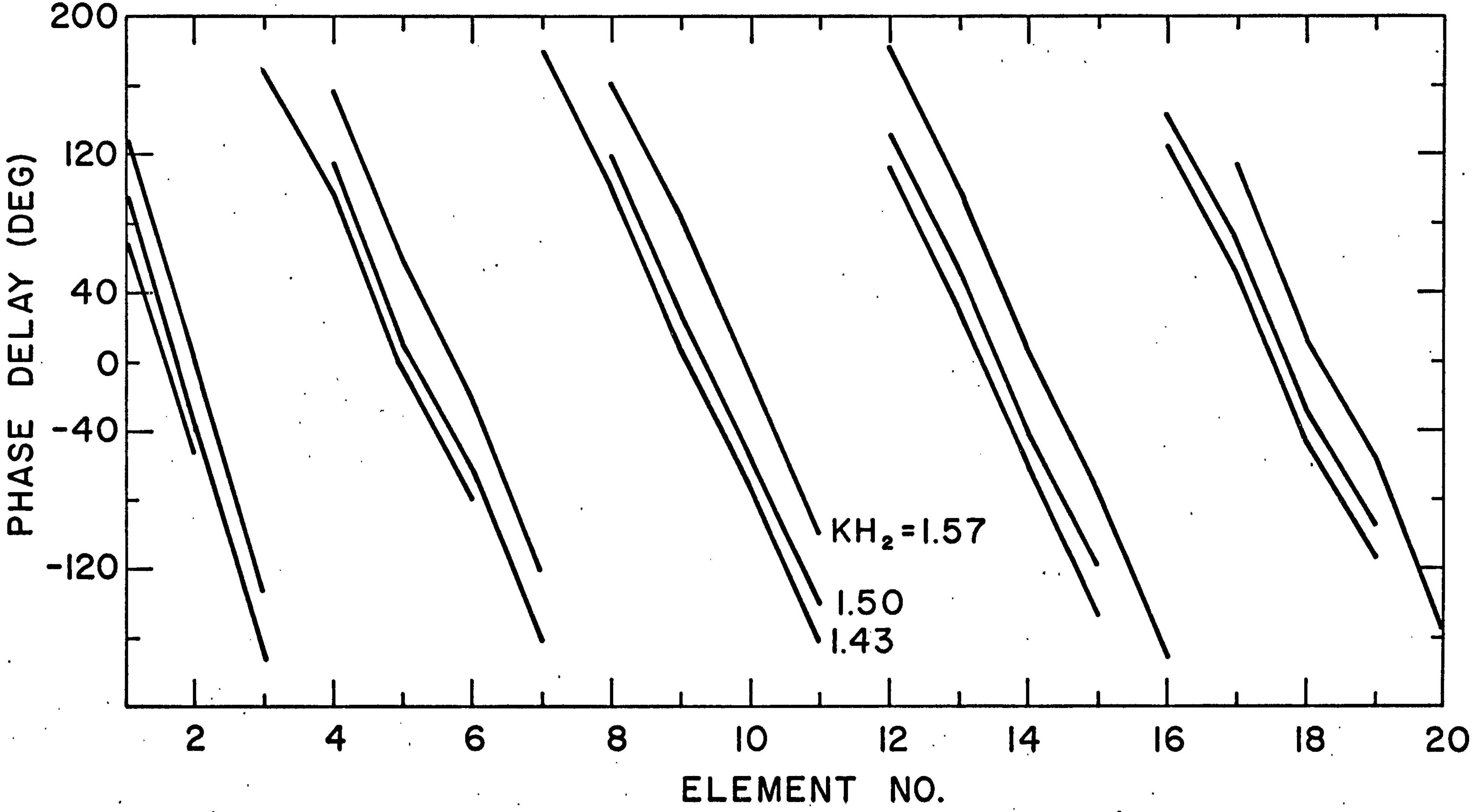


FIG. 3.2

(N=20, kh=1.33, b/h=1.0, a/b=.01, kh_1=1.67)

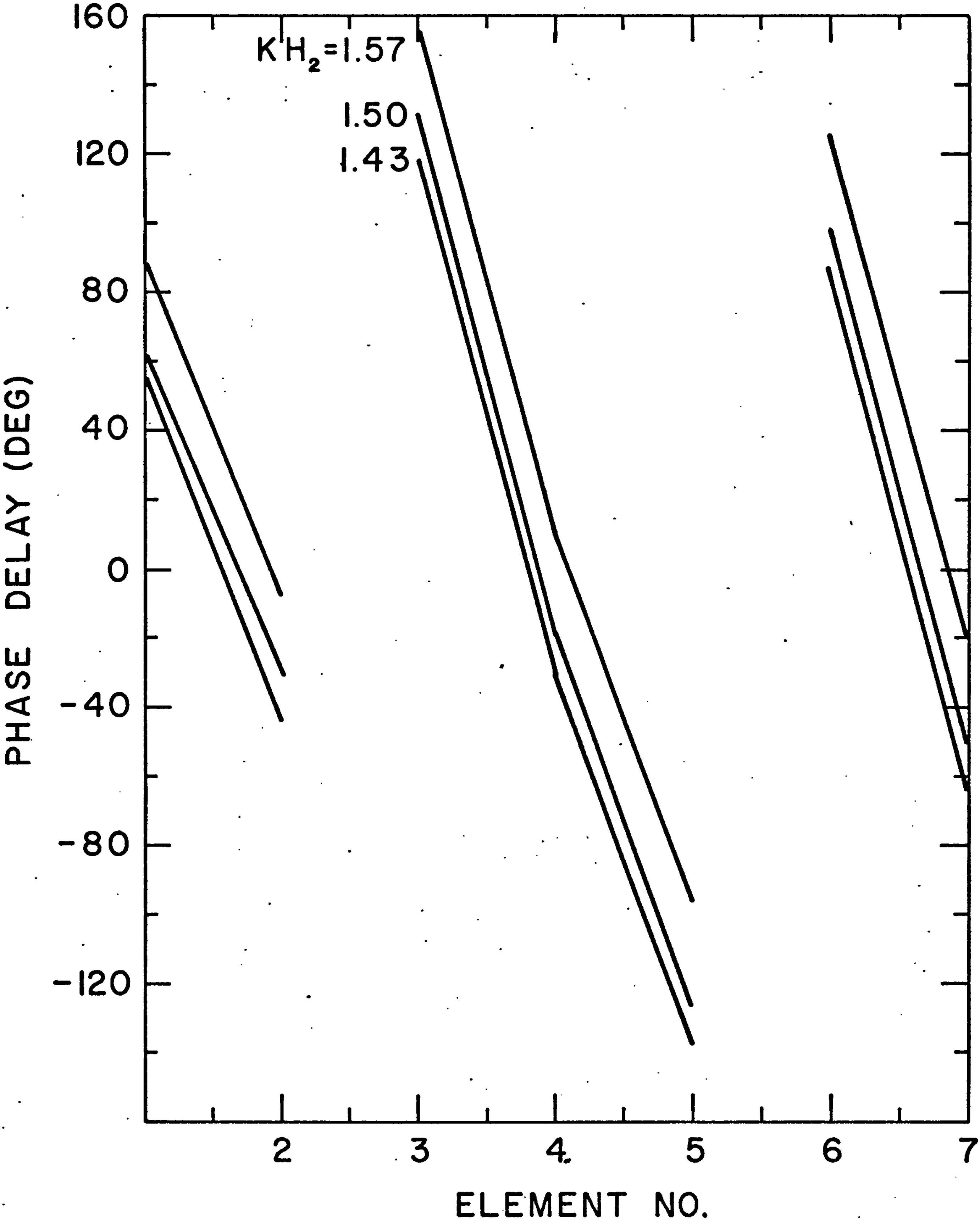


FIG. 3.3

(kh=1.37, b/h=1.5, ka=0.0136, kh_1=1.67)

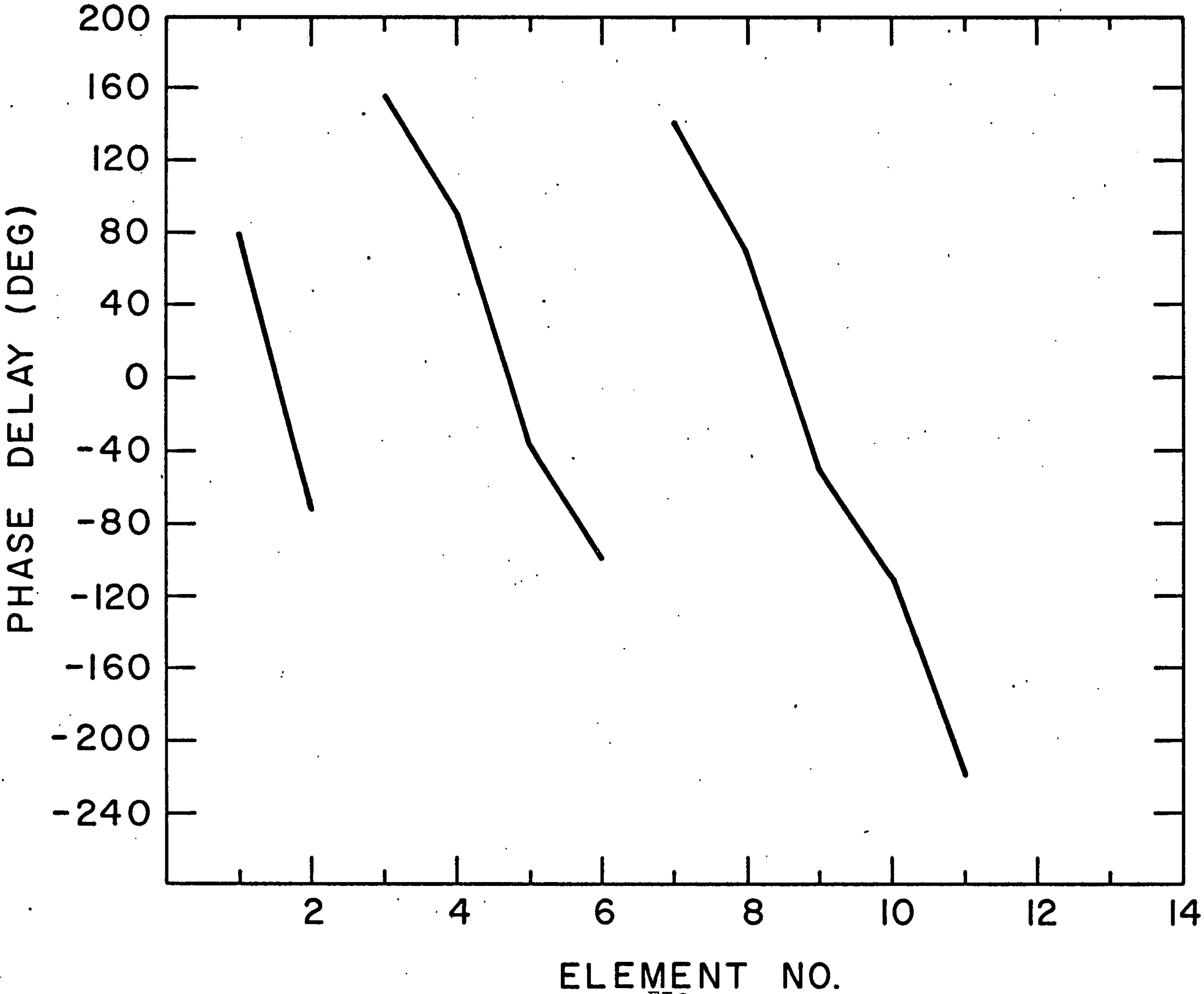


FIG. 3.4
(N=10, b/h=1.0, a/b=0.01, kh=1.37)

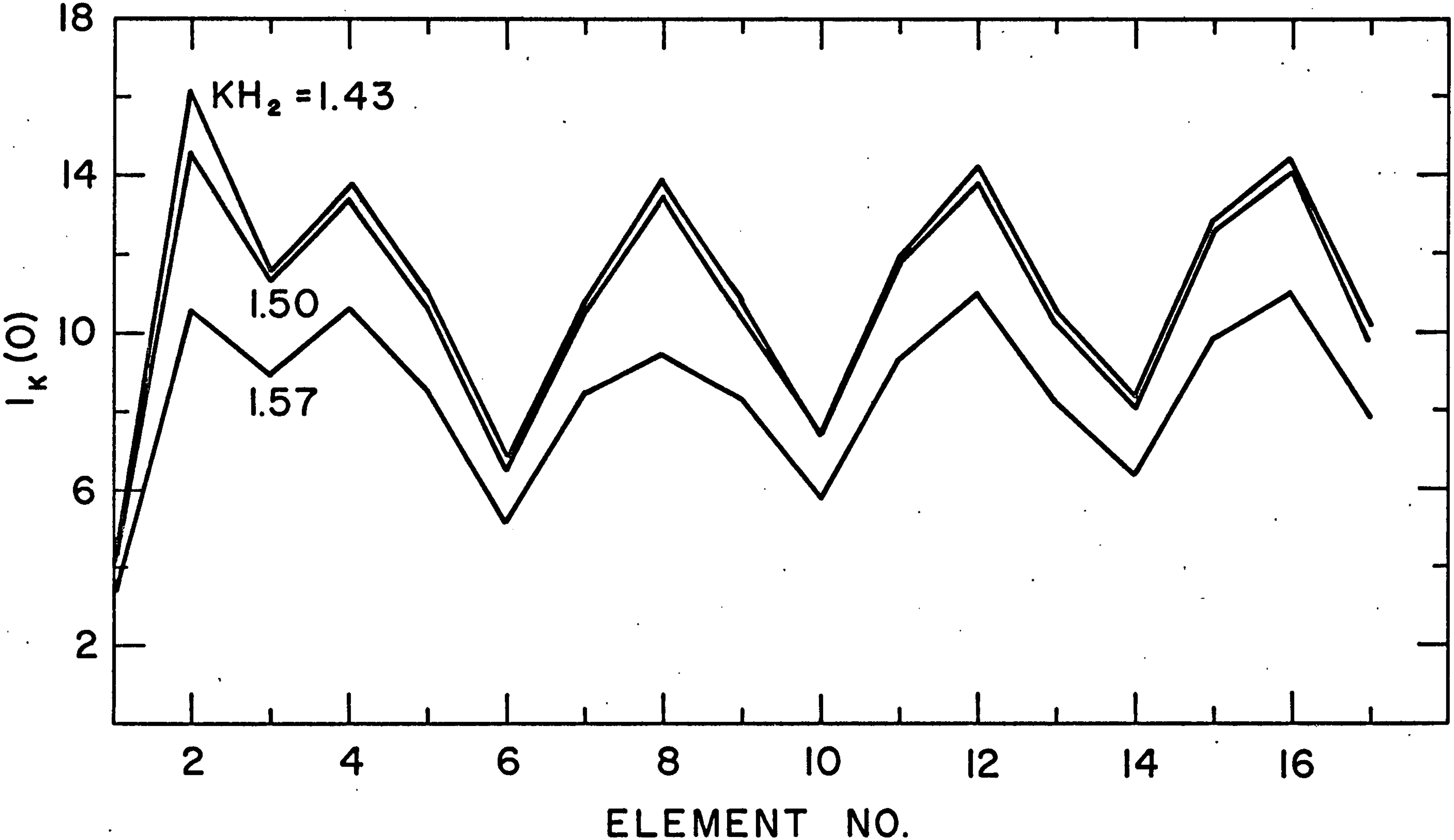


FIG. 3.5

(N=16, kh=1.37, a/h=0.01, kh_1=1.67, b/h=1.5)

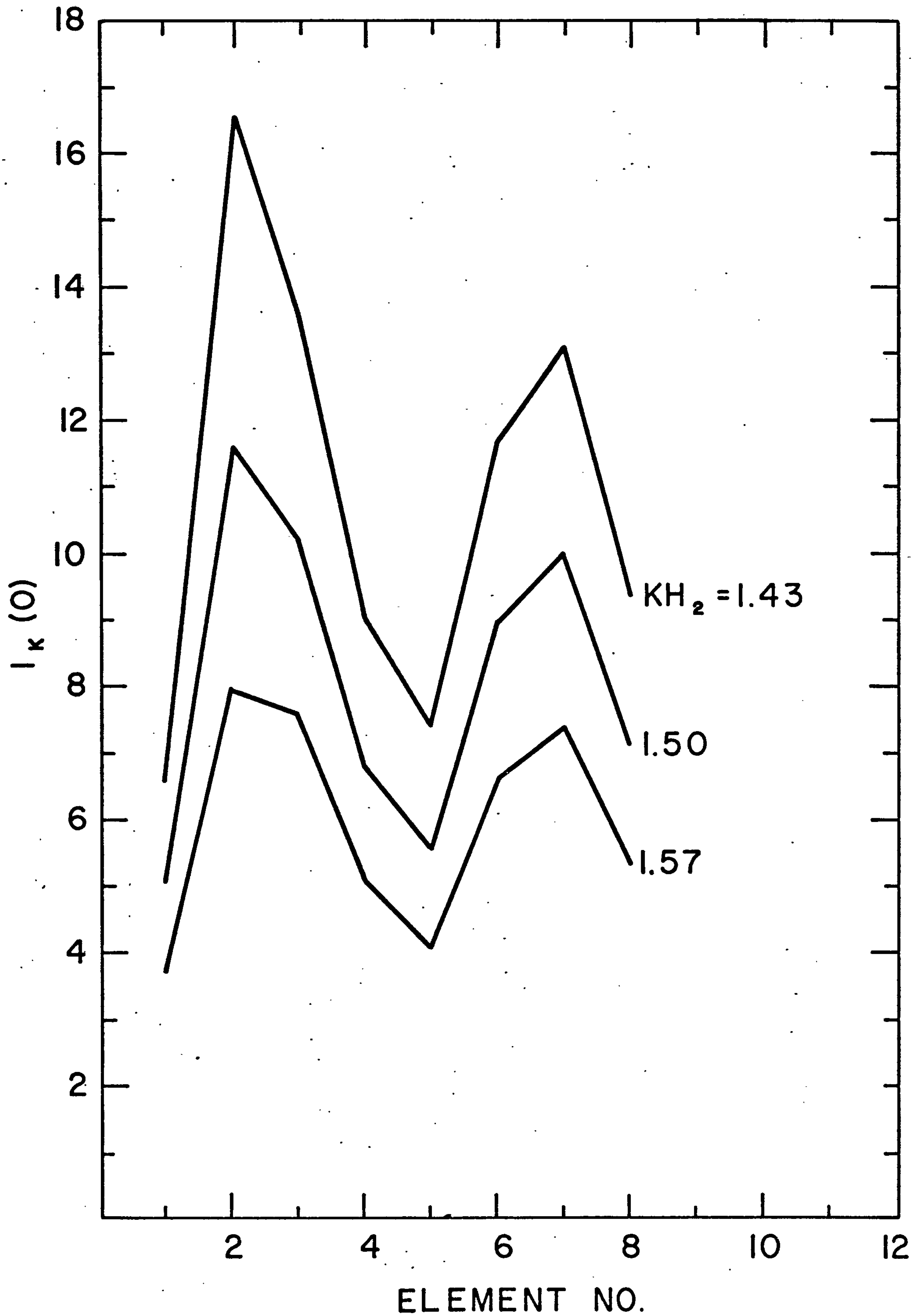


FIG. 3.6
(kh=1.37, b/h=1.5, ka=.0136, kh_1=1.67)

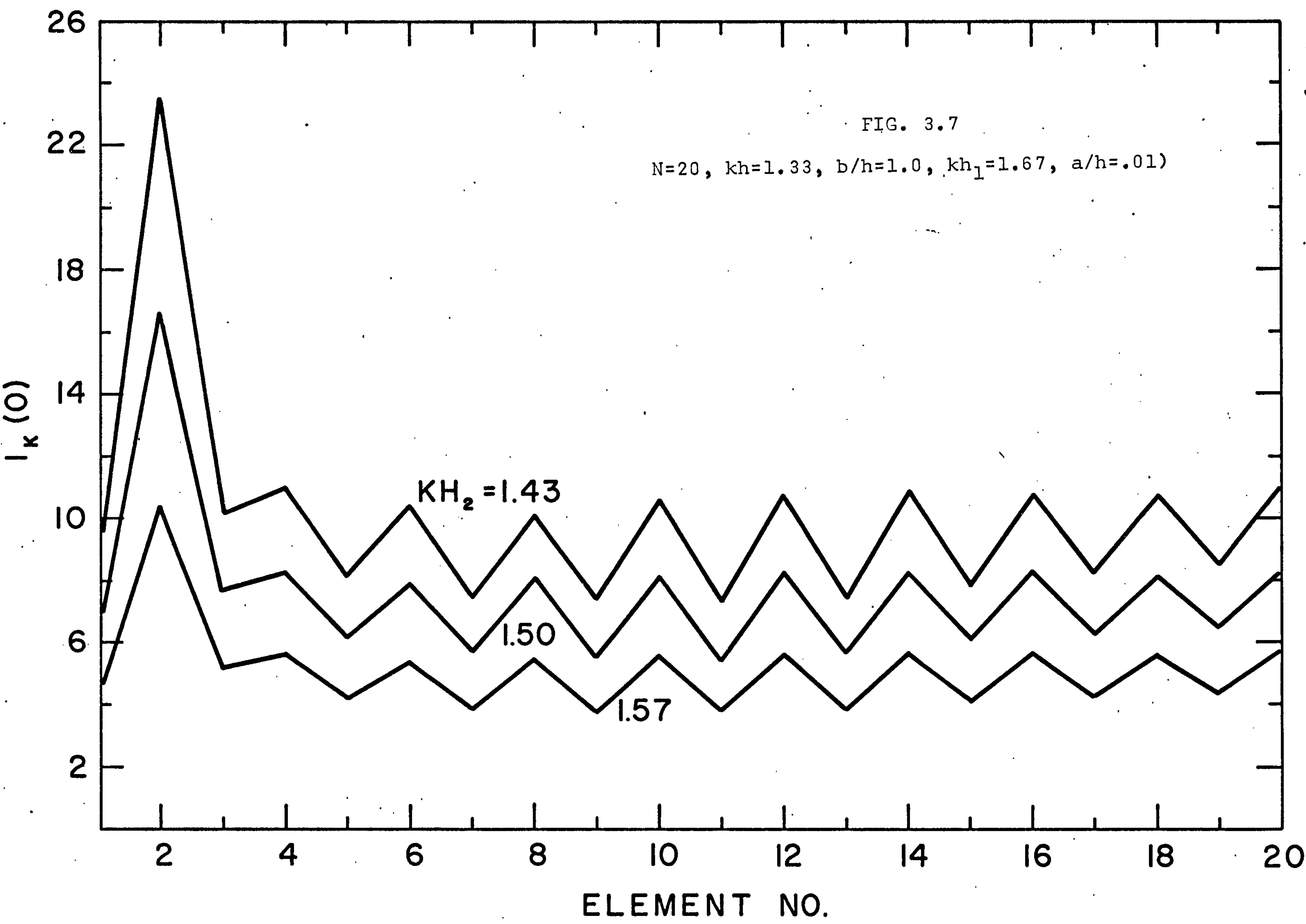


FIG. 3.7
N=20, kh=1.33, b/h=1.0, kh1=1.67, a/h=.01)
26
22
18
14
10
6
2
Ik (0)
KH2 =1.43
1.50
1.57
2
4
6
8
10
12
14
16
18
20
ELEMENT NO.

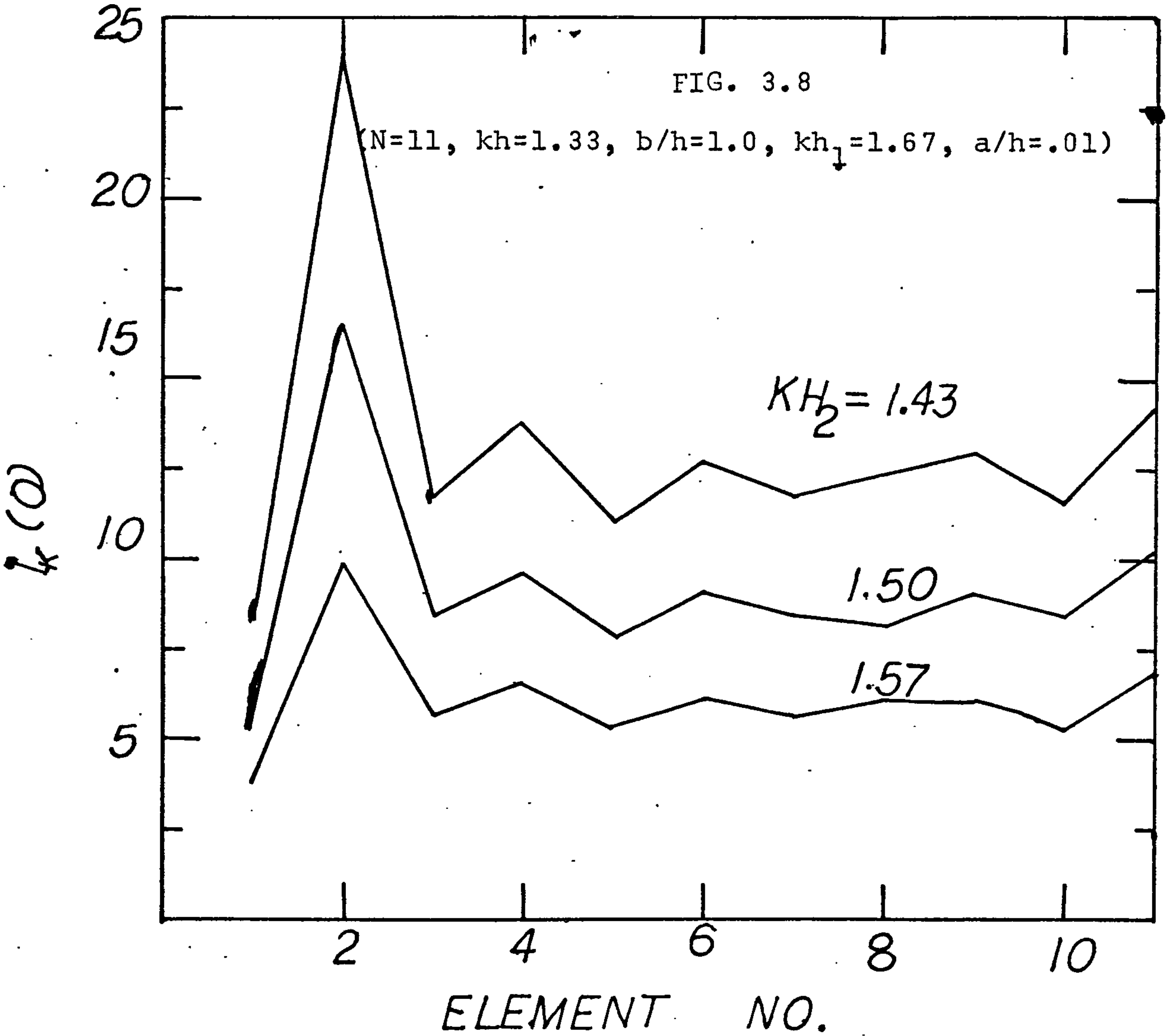


FIG. 3.8
(N=11, kh=1.33, b/h=1.0, kh$_1$=1.67, a/h=.01)
KH$_2$= 1.43
1.50
1.57
25
20
15
10
5
2
4
6
8
10
ELEMENT NO.

3-4 Conclusions

The input impedance is calculated for different lengths of the director and the driven element. For the four examples shown in Figs. (3.9)-(3.12) it was found that the resistive part of the input impedance is more sensitive to changes in the reflector length and the reactive component of the input impedance is more sensitive to changes in the driven element length.

For the cases shown in Figs. (3.1), (3.4) and (3.13), the phase shift ϕ between currents in adjacent elements is apparently independent of the length of the driven element and the director.

The magnitude of $I_k(0)$ takes the form of a standing wave, the pattern of which is not affected by the length of the reflector as shown in Fig. (3.14), or by the length of the driven element as shown in Figs. (3.5)-(3.8). The magnitude is more affected by changes in the length of the driven element than by the length of the reflector.

In summary, to obtain a desirable input impedance for a Yagi-Uda array, the length of the driven element and the length of the reflector can be changed. The resistive part can be changed effectively by adjusting the reflector length, and the reactive part can be changed by adjusting the length of the driven element. It is observed that for a 10% change of these lengths, the phase shift between currents on adjacent elements

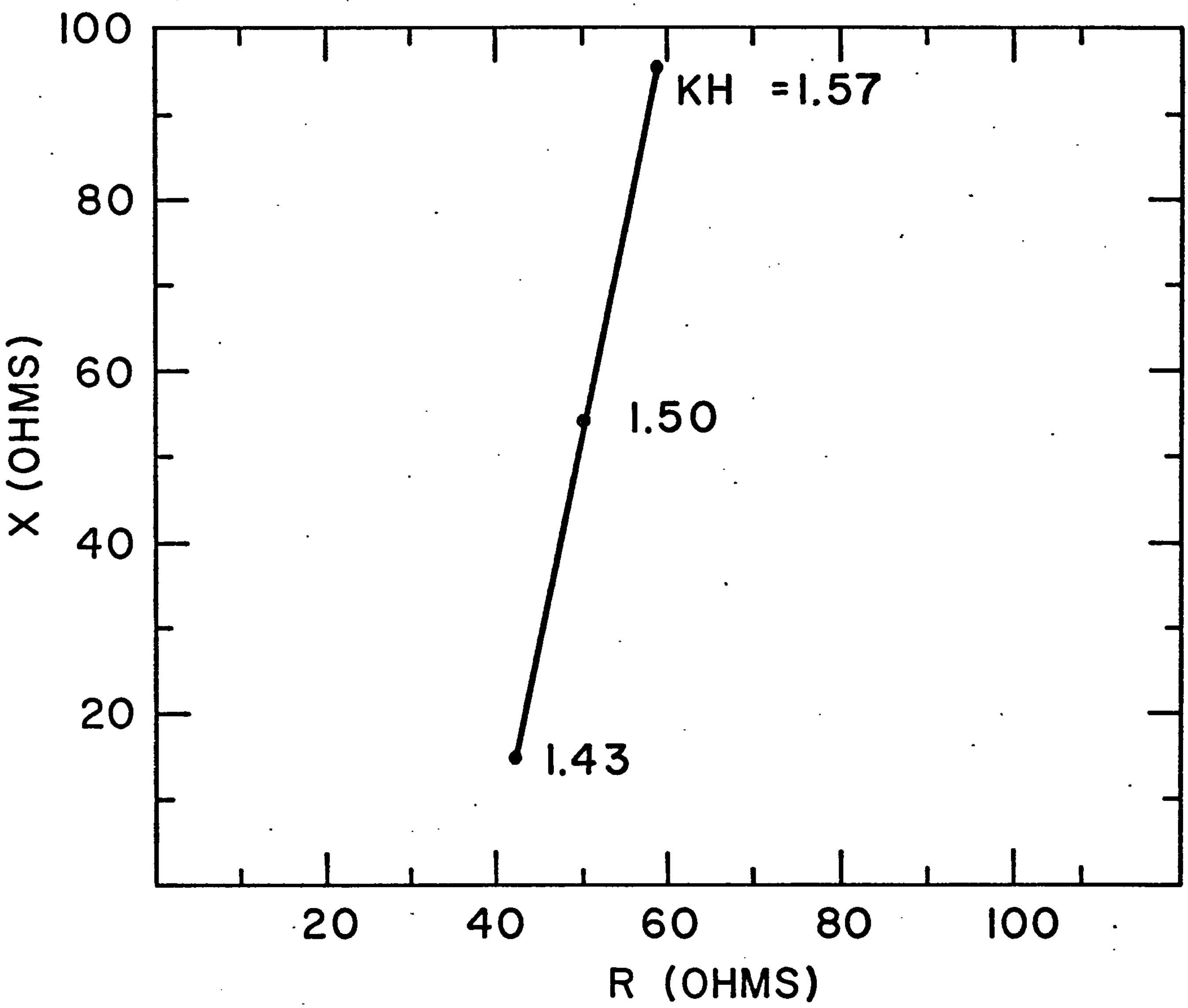


FIG. 3.9

(N=19, kh=1.33, b/h=1, a/b=.01, kh_1=1.67)

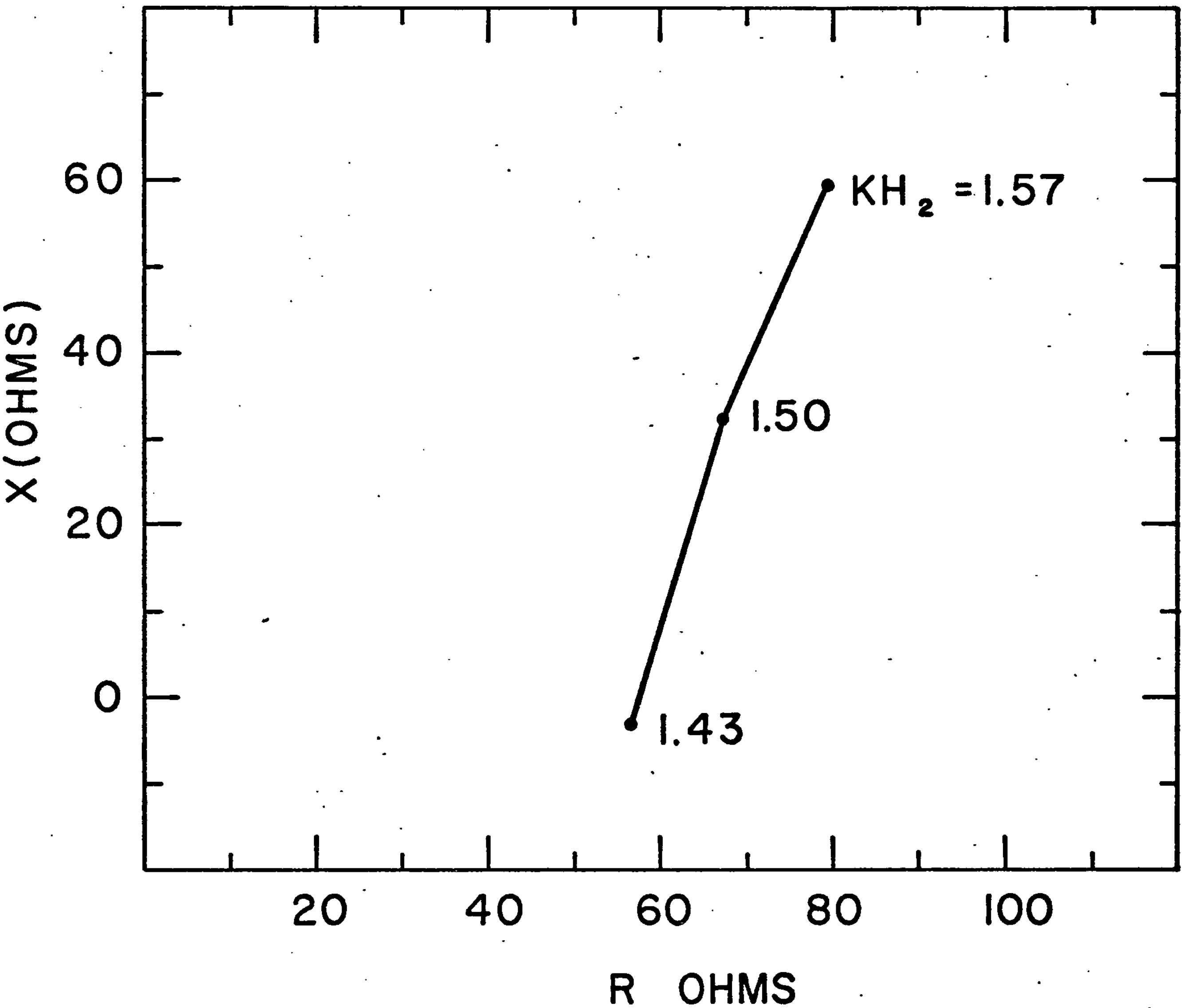


FIG. 3.10

(N=16, kh=1.37, b/h=1.5, a/h=.01, kh_1=1.67)

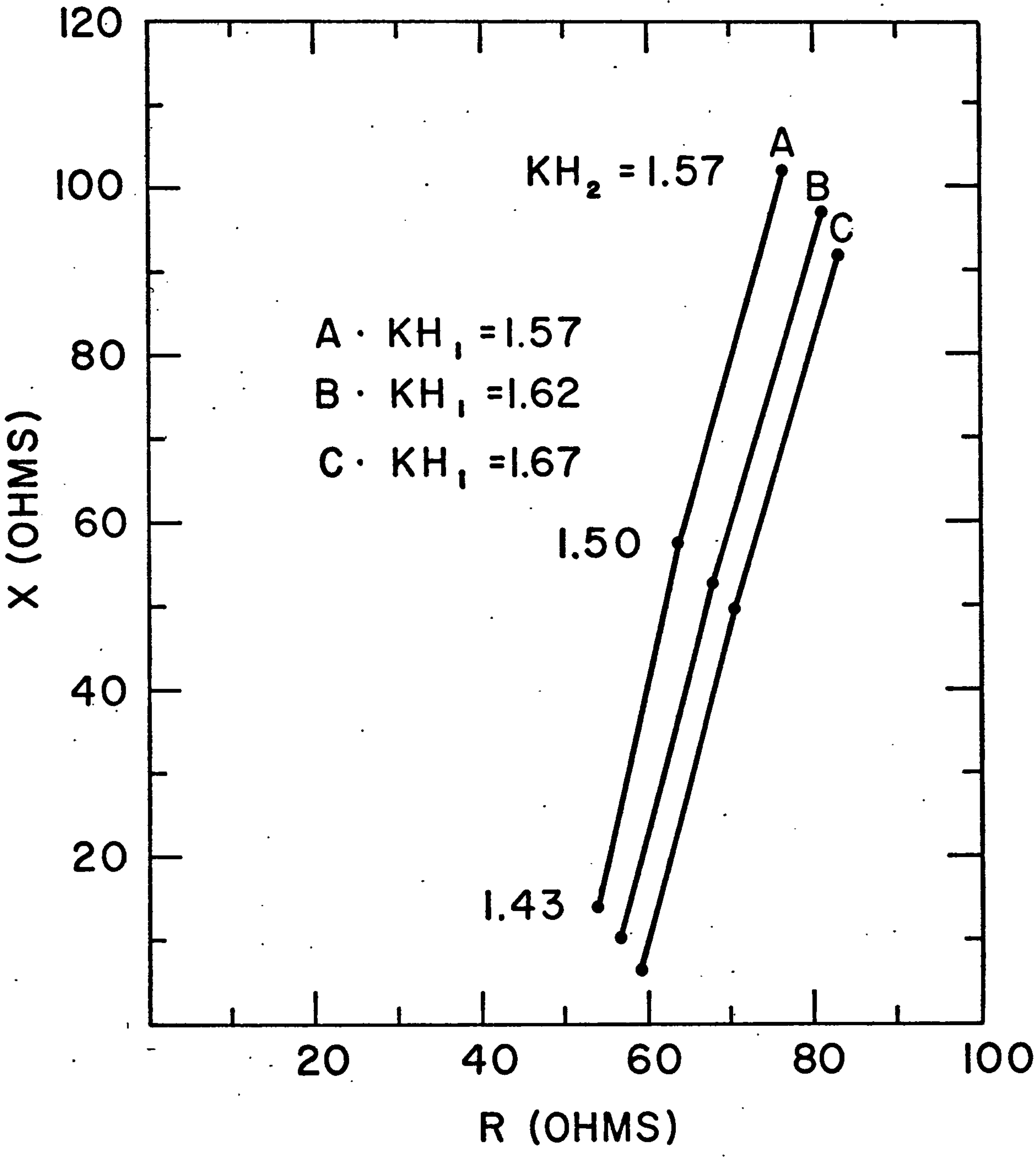


FIG. 3.11

(N=7, kh=1.37, b/h=1.5, ka=.0136)

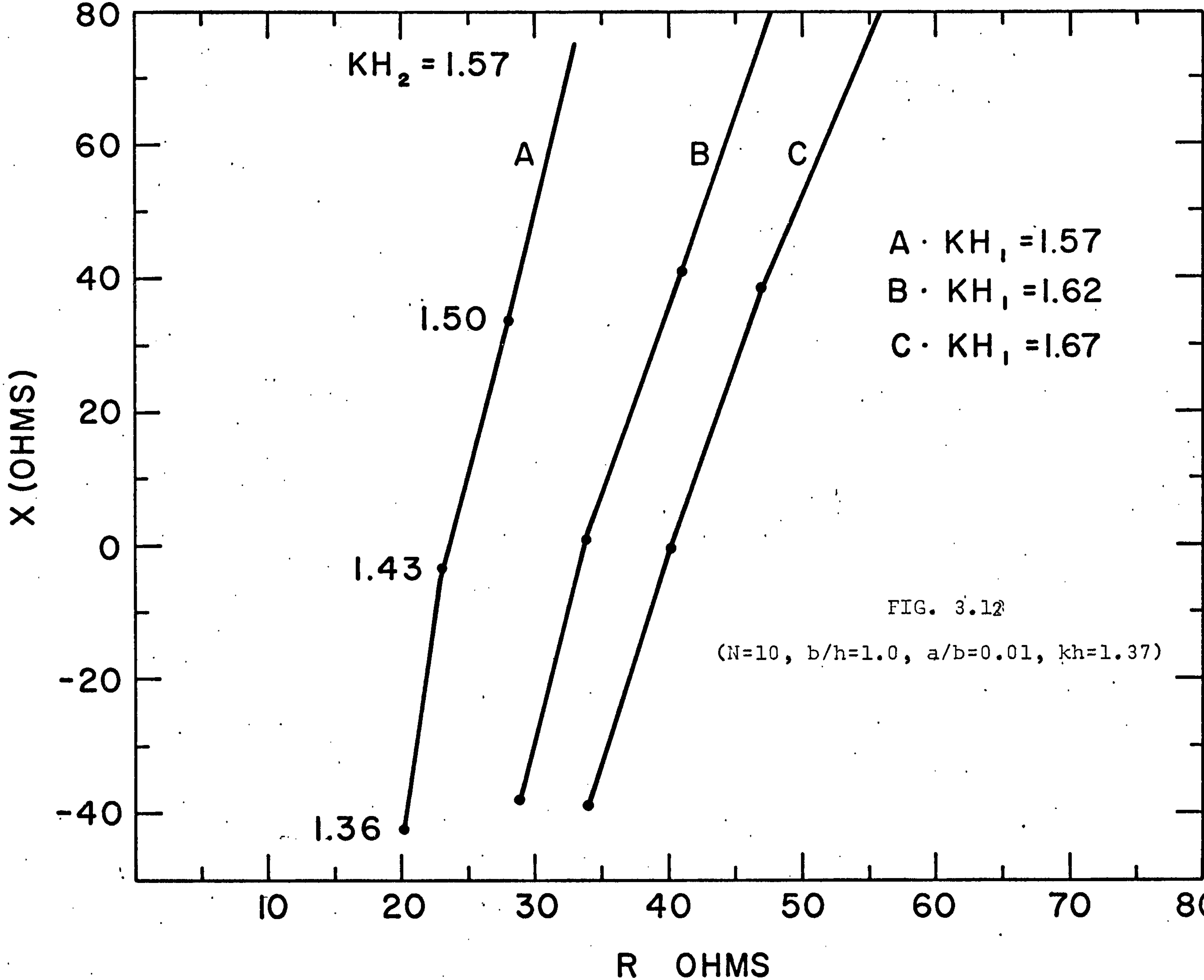


FIG. 3.12

(N=10, b/h=1.0, a/b=0.01, kh=1.37)

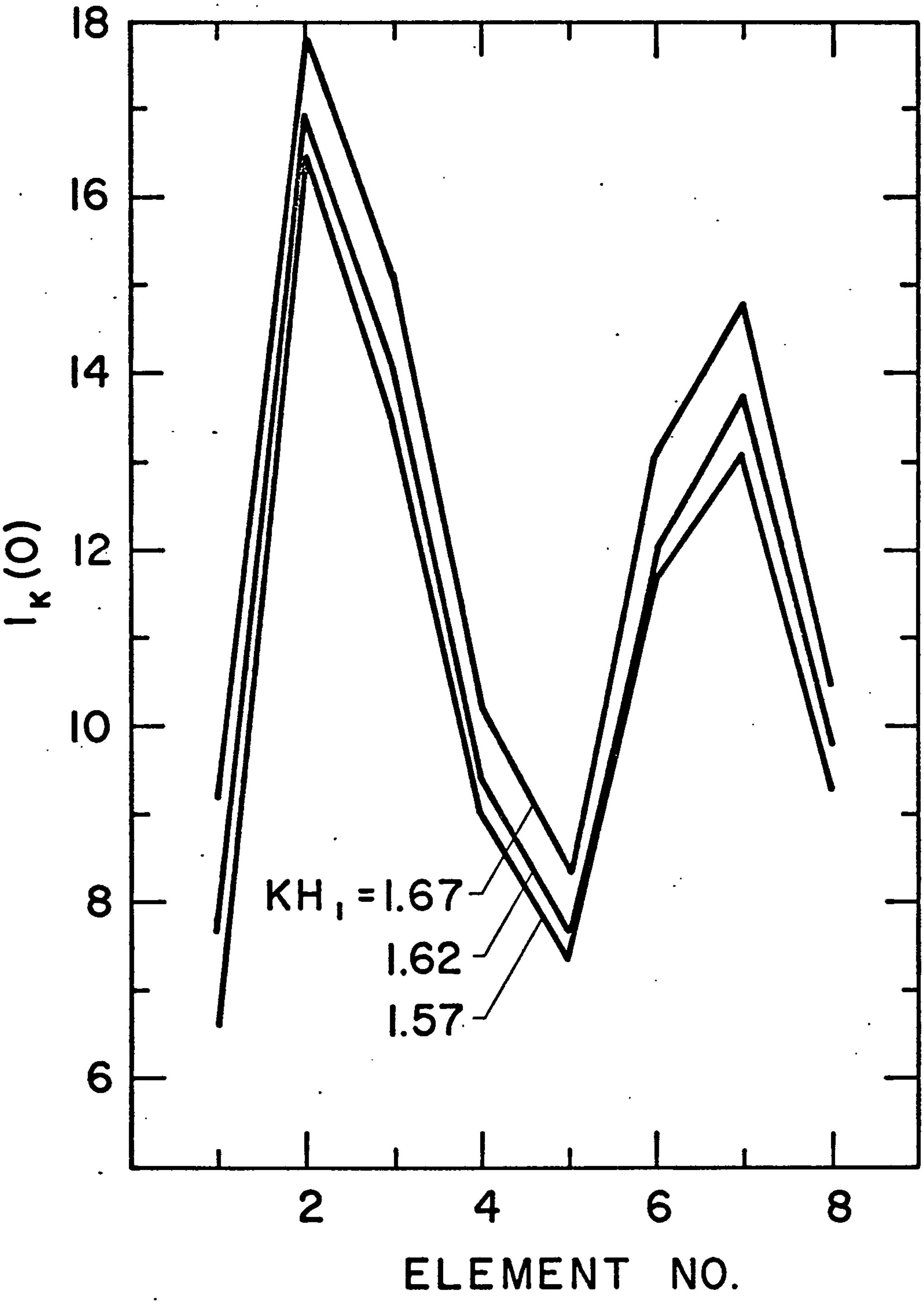


FIG. 3.13

(kh=1.37, b/h=1.5, ka=.0136, kh_2=1.43)

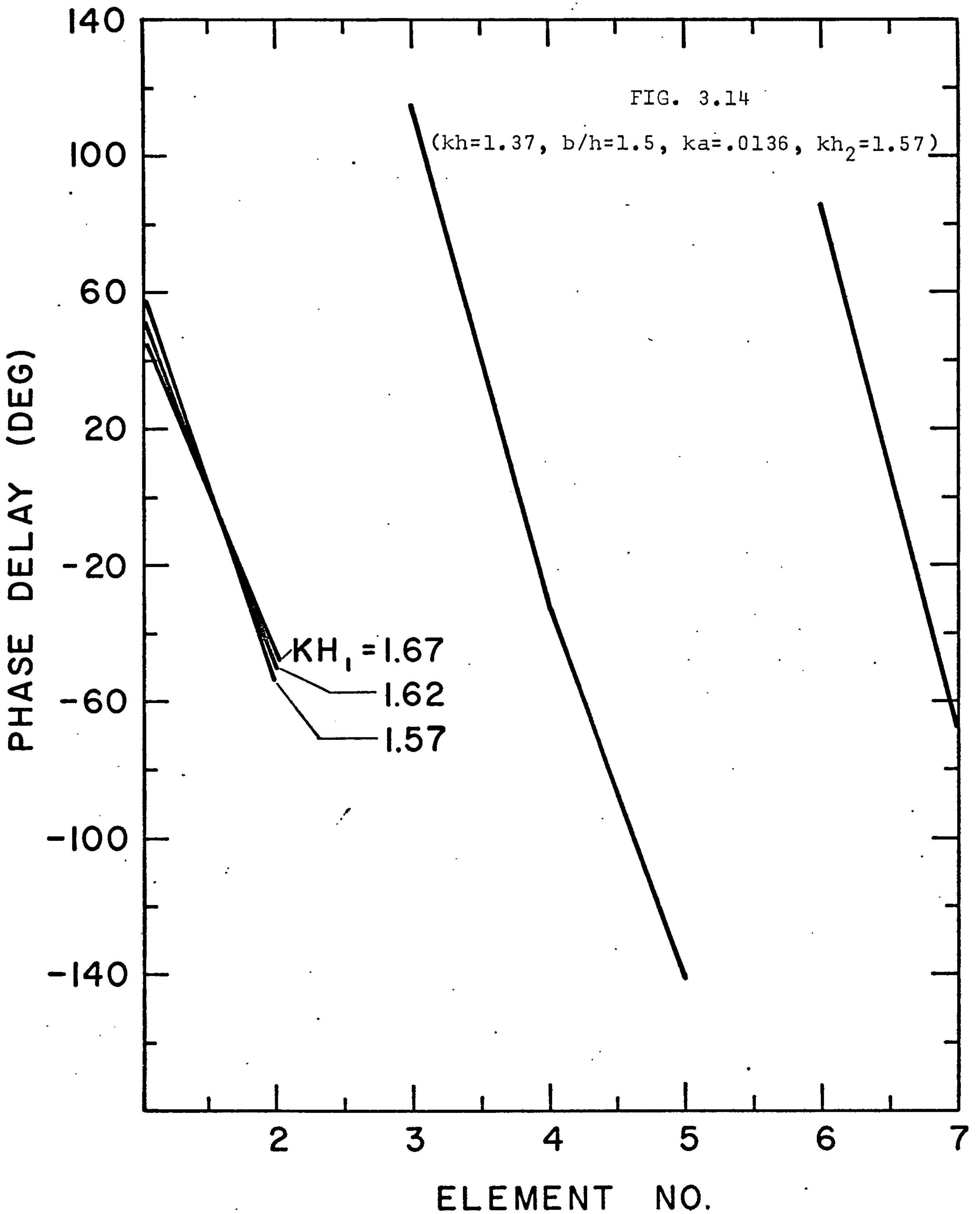


FIG. 3.14
(kh=1.37, b/h=1.5, ka=.0136, kh$_2$=1.57)
140
100
60
20
-20
-60
-100
-140
PHASE DELAY (DEG)
KH$_1$=1.67
1.62
1.57
2
3
4
5
6
7
ELEMENT NO.

and the relative current amplitudes are changed very little($\approx$2%). Hence, the field pattern and directivity are also insensitive to the change of director and reflector lengths. This is shown in Fig. (3.15).

This observation is useful in application, since in the design of a Yagi array, it is now possible to separate the design of the radiation pattern from the design of the input admittance. When the array is optimized in its radiation pattern using Shen's method, the input impedance can be optimized using the present method, or simply by trial and error in measurement.

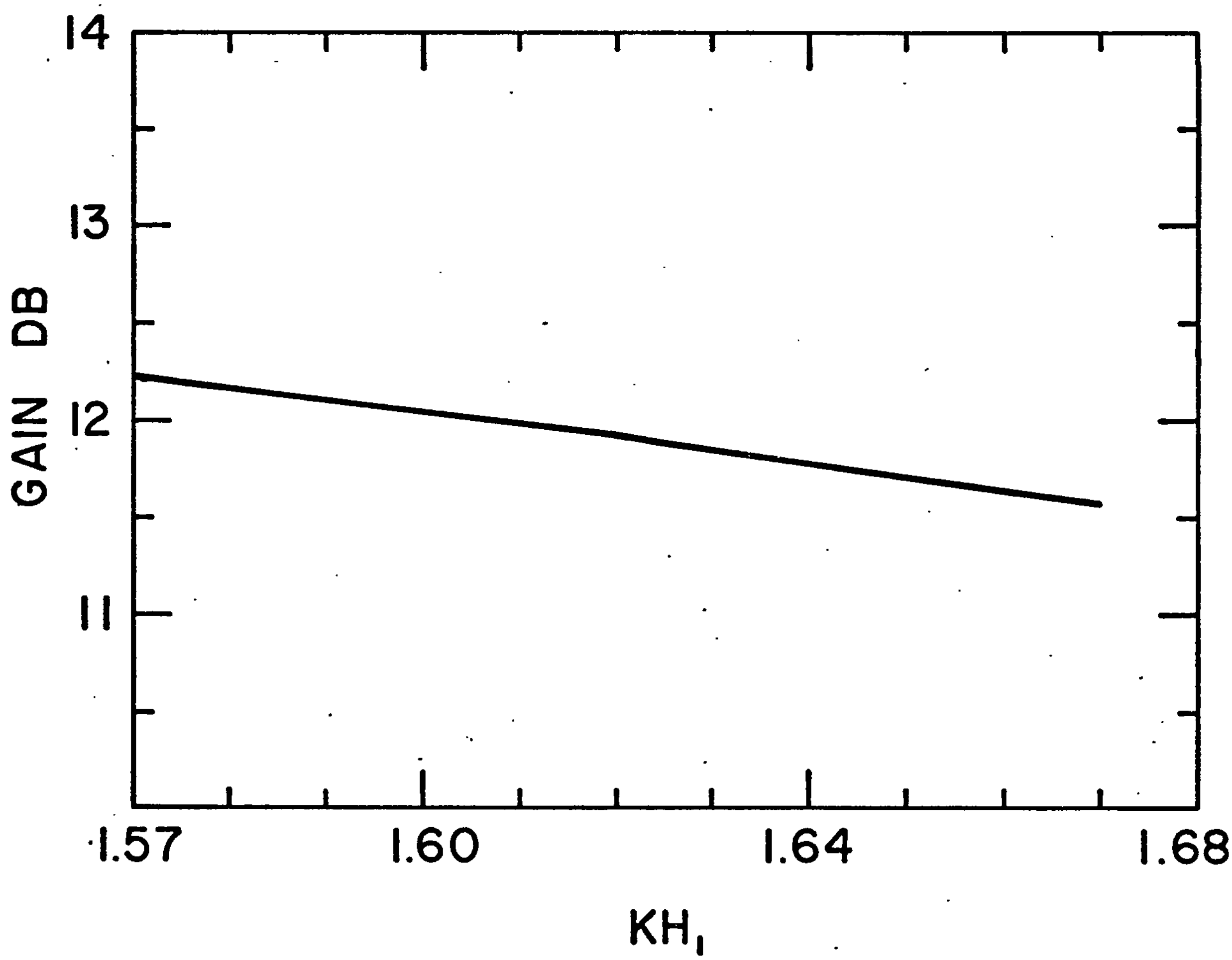


FIG. 3.15

(N=7, kh=1.37, b/h=1.5, ka=.0136, kh_2=1.43, 1.50 & 1.57

CHAPTER IV

Optimization of the Yagi-Uda Array

A method was developed by Cheng and Chen [1973] for the optimization of the forward gain by adjustment of the inter-element spacings. This method is described in Section 4-1. In Section 4-2, some numerical results obtained from the present study are discussed. Also discussed in this section is the extent to which the bandwidth is affected when the inter-element spacings are adjusted for maximum directivity. In Section 4-3, conclusions are drawn.

4-1 Spacing Perturbation

With a view to adjusting the element spacings in a Yagi-Uda array, we assume that the positions of the kth and the ith elements be displaced by small amounts Δd_k and Δd_i, respectively.

$$R_{ki}=[(z_k-z_i')^2 + b_{ki}^2]^{1/2}$$

If $(\Delta d_k-\Delta d_i)<<b_{ki}$, then

$$\Delta R_{ki}=b_{ki}/R_{ki}(\Delta d_k-\Delta d_i) \qquad ---(4.1)$$

We write the new perturbed matrices:

$$[\Phi_U]^P = [\Phi_U]+[\Delta\Phi_U] \qquad ---(4.2a)$$

$$[\Phi_D]^P = [\Phi_D]+[\Delta\Phi_D] \qquad ---(4.2b)$$

$$[\Psi_{dU}^h]^P = [\Psi_{dU}^h]+[\Delta\Psi_{dU}^h] \qquad ---(4.2c)$$

$$[\Psi_{dD}^h]^P = [\Psi_{dD}^h]+[\Delta\Psi_{dD}^h] \qquad ---(4.2d)$$

$$\{\Phi_{2V}\}^P = \{\Phi_{2V}\}+\{\Delta\Phi_{2V}\} \qquad ---(4.2e)$$

$$\{\Psi_{2dV}^h\}^P = \{\Psi_{2dV}^h\}+\{\Delta\Psi_{2dV}^h\} \qquad ---(4.2f)$$

The coefficients for the current terms will also be changed. We write:

$$\{B\}^P = \{B\}+\{\Delta B\} \qquad ---(4.2g)$$

$$\{D\}^P = \{D\}+\{\Delta D\} \qquad ---(4.2h)$$

With this change in the inter-element spacings, a typical term in the integrals contained in (2.10) and (2.11) can be written as

$$\int_{-h_i}^{h_i} G_{0z_i'}K_{ki}(z_k,z_i')dz_i' \qquad ---(4.3)$$

where $G_{0z_i'}$ stands for $M_{0z_i'}$, $F_{0z_i'}$, or $H_{0z_i'}$, and $M_{0z_i'}$, $F_{0z_i'}$, and $H_{0z_i'}$ have been defined in (2.7).

Now we can write the kith element of the square deviation matrices and the kith element of the column deviation matrices as follows:

$$[\Delta\Phi_U]_{ki} = (\Delta d_k-\Delta d_i)[\Phi_U']_{ki}(1-\delta_{ki}) \qquad \text{---(4.9a)}$$

$$[\Delta\Phi_D]_{ki} = (\Delta d_k-\Delta d_i)[\Phi_D']_{ki}(1-\delta_{ki}) \qquad \text{---(4.9b)}$$

$$[\Delta\Psi_{dU}^h]_{ki} = (\Delta d_k-\Delta d_i)[\Psi_{dU}^{h\prime}]_{ki}(1-\delta_{ki}) \qquad \text{---(4.9c)}$$

$$[\Delta\Psi_{dD}^h]_{ki} = (\Delta d_k-\Delta d_i)[\Psi_{dD}^{h\prime}]_{ki}(1-\delta_{ki}) \qquad \text{---(4.9d)}$$

$$\{\Delta\Phi_{2V}\}_k = (\Delta d_k-\Delta d_2)\{\Phi_{2V}'\}_k(1-\delta_{k2}) \qquad \text{---(4.9e)}$$

$$\{\Delta\Psi_{2dV}^h\}_k = (\Delta d_k-\Delta d_2)\{\Psi_{2dV}^{h\prime}\}_k(1-\delta_{k2}) \qquad \text{---(4.9f)}$$

If second order derivatives are neglected, substituting (4.2) into (2.26) yields:

$$[\Phi_U]\{\Delta B\}+[\Phi_D]\{\Delta D\} = -\{\Delta\Phi_{2V}\}A_2-[\Delta\Phi_U]\{B\}-[\Delta\Phi_D]\{D\} \qquad \text{---(4.10)}$$

and

$$[\Psi_{dU}^h]\{\Delta B\}+[\Psi_{dD}^h]\{\Delta D\} = -\{\Delta\Psi_{2dV}^h\}A_2-[\Delta\Psi_{dU}^h]\{B\}-[\Delta\Psi_{dD}^h]\{D\} \qquad \text{---(4.11)}$$

where A_2 is defined in (2.16).

In view of (4.9), the kth element of the right hand side of (4.10) can be written as:

$$\sum_{i=1}^{N} [P_2]_{ki}\Delta d_i \qquad \text{---(4.12)}$$

The proof is as follows:

$$\Delta\int=\int\Delta=\int_{-h_i}^{h_i} G_{0z'i}\Delta K_{ki}(z_k,z_i')dz_i' \qquad \text{---(4.4)}$$

but $\Delta K_{ki} = \Delta e^{-j\beta_0 R_{ki}}/R_{ki}$

$$=[R_{ki}(-j\beta_0)e^{-j\beta_0 R_{ki}} - e^{-j\beta_0 R_{ki}}]/R_{ki}^2\ \Delta R_{ki}$$

$$=[-e^{-j\beta_0 R_{ki}}(j\beta_0 R_{ki}+1)]/R_{ki}^2\ \Delta R_{ki} \qquad \text{---(4.5)}$$

If $\beta_0 R_{ki}>>1$

$$\Delta K_{ki} = [-j\beta_0 e^{-j\beta_0 R_{ki}}]/R_{ki}\ \Delta R_{ki} \qquad \text{---(4.6)}$$

Substitution of (4.1) and (4.6) in (4.4) yields

$$\Delta\int=\int_{h_i}^{h_i} G_{0z'i}[-j\beta_0 b_{ki}(\Delta d_k-\Delta d_i)e^{-j\beta_0 R_{ki}}]/R_{ki}^2\ dz_i'$$

$$= -j\beta_0 b_{ki}(\Delta d_k-\Delta d_i)\int_{-h_i}^{h_i} G_{0z'i}[e^{-j\beta_0 R_{ki}}/R_{ki}]dz_i' \qquad \text{---(4.7)}$$

Now, if $k \neq i$, using equations (2.10) and (2.11) would imply that they would still apply with K_{kid}^p substituted for K_{kid}, where

$$K_{kid}^p = e^{-j\beta_0 R_{ki}}/R_{ki}^2 - e^{-j\beta_0 R_{kih}}/R_{kih}^2 \qquad \text{---(4.8)}$$

Although the kernel is now a different function, use is made of the properties shown in (2.12) and (2.13). Complex matrices $[\Phi_U']$, $[\Phi_D']$, $[\Psi_{dU}^{h'}]$, and $[\Psi_{dD}^{h'}]$ are defined as $[\Phi_U]$, $[\Phi_D]$, $[\Psi_{dU}^h]$, and $[\Psi_{dD}^h]$ were defined in Chapter II, except that the integrals W defined in (2.27) and the integrals $\Psi(h_k)$ defined in (2.19) are multiplied by $-j\beta_0(d_k-d_i)$ and K_{kid}^p is substituted for K_{kid}, as was shown in (4.7).

The kth element of $\{\Delta\Phi_{2V}\}A_2$ is

$$(\Delta d_k-\Delta d_2)\{\Phi'_{2V}\}_k A_2\ (1-\delta_{k2})$$

The kth element of $[\Delta\Phi_U]\{B\}$ is

$$\sum_{i=1}^{N}(\Delta d_k-\Delta d_i)[\Phi'_U]_{ki}(1-\delta_{ki})B_i$$
$$=\Delta d_k\sum_{i=1}^{N}[\Phi'_U]_{ki}(1-\delta_{ki})B_i - \sum_{i=1}^{N}[\Phi'_U]_{ki}(1-\delta_{ki})B_k\Delta d_i$$

The kth element of $[\Delta\Phi_D]\{D\}$ is

$$\sum_{i=1}^{N}(\Delta d_k-\Delta d_i)[\Phi'_D]_{ki}(1-\delta_{ki})D_i$$
$$=\Delta d_k\sum_{i=1}^{N}[\Phi'_D]_{ki}(1-\delta_{ki})D_i - \sum_{i=1}^{N}[\Phi'_D]_{ki}(1-\delta_{ki})D_i\Delta d_i$$

The sum of the above three terms yields:

$$\Delta d_k[\{\Phi'_{2V}\}_k A_2(1-\delta_{k2}) + \sum_{i=1}^{N}[\Phi'_U]_{ki}(1-\delta_{ki})B_i + \sum_{i=1}^{N}[\Phi'_D]_{ki}(1-\delta_{ki})D_i]$$
$$- [\{\Phi'_{2V}\}_k A_2(1-\delta_{k2})\Delta d_2] - \sum_{i=1}^{N}[\Phi'_U]_{ki}(1-\delta_{ki})B_i\Delta d_i$$
$$+ \sum_{i=1}^{N}[\Phi'_D]_{ki}(1-\delta_{ki})D_i\Delta d_i \qquad ---(4.13)$$

The following expressions for the elements of $[P_2]$ are derived:

$$[P_2]_{ki} = [\Phi'_U]_{ki}\{B\}_i + [\Phi'_D]_{ki}\{D\}_i \qquad i\neq k \quad i\neq 2 \qquad ---(4.14)$$

$$[P_2]_{k2} = [\Phi'_U]_{k2}\{B\}_2 + [\Phi'_D]_{k2}\{D\}_2 + \{\Phi'_{2V}\}_k A_2(1-\delta_{k2}) \quad k\neq 2 \qquad ---(4.15)$$

$$[P_2]_{kk} = -\sum_{i=1}^{N}[\Phi'_U]_{ki}(1-\delta_{ki})B_i - \sum_{i=1}^{N}[\Phi'_D]_{ki}(1-\delta_{ki})D_i$$
$$- \{\Phi'_{2V}\}_k A_2(1-\delta_{k2}) \qquad ---(4.16)$$

And similarly, another N×N matrix can be defined and equations (4.10) and (4.11) become:

$$[\Phi_U]\{\Delta B\}+[\Phi_D]\{\Delta D\} = [P_2]\{\Delta d\} \qquad \text{---(4.17)}$$

$$[\Psi^h_{dU}]\{\Delta B\}+[\Psi^h_{dD}]\{\Delta D\} = [P_3]\{\Delta d\}$$

The perturbed current coefficients $\{B\}^P$ and $\{D\}^P$ can then be determined from (4.2g) and (4.2h).

The preceding formulation is not suitable when $\beta_0 h_i=\pi/2$ as long as it is less than $5\pi/4$. The preceding expressions must be modified in accordance with the procedure in Chapter II. (4.17) and (4.18) become:

$$\begin{bmatrix} [\Phi_U] & [\Phi_D] \\ [\Psi^h_{dU}] & [\Psi^h_{dD}] \end{bmatrix} \begin{Bmatrix} \{\Delta B\} \\ \{\Delta D\} \end{Bmatrix} = \begin{Bmatrix} [P_2] \\ [P_3] \end{Bmatrix} \{\Delta d\} \qquad \text{---(4.19)}$$

From (4.19), $\{\Delta B\}$ and $\{\Delta D\}$ can be found by matrix inversion:

$$\begin{Bmatrix} \{\Delta B\} \\ \{\Delta D\} \end{Bmatrix} = \begin{bmatrix} [\Phi_U] & [\Phi_D] \\ [\Psi^h_{dU}] & [\Psi^h_{dD}] \end{bmatrix}^{-1} \begin{Bmatrix} [P_2] \\ [P_3] \end{Bmatrix} \{\Delta d\} = \begin{Bmatrix} [Q_2] \\ [Q_3] \end{Bmatrix} \{\Delta d\} \qquad \text{---(4.20)}$$

The radiation field of a linear array at a distance R_0 from a reference origin is:

$$E(\theta,\phi) = \frac{j\omega\beta_0}{4\pi R_0} \sum_{i=1}^{N} e^{j\beta_0 d_i \sin\theta\cos\phi} \sin\theta \int_{-h_i}^{h_i} I_i(z_i')e^{j\beta_0 z_i'\cos\theta}\,dz_i' \qquad \text{---(4.21)}$$

Let us consider the term $e^{j\beta_0 d_i \sin\theta\cos\phi} \int_{-h_i}^{h_i} I_i(z_i')e^{j\beta_0 z_i'\cos\theta}\,dz_i'$

For small Δd_i; i.e., for $\Delta d_i/d_i \ll 1$

$$\Delta\left(e^{j\beta_0 d_i \sin\theta\cos\phi}\int_{-h_i}^{h_i} I_i(z_i')e^{j\beta_0 z_i'\cos\theta}dz_i'\right)$$

$$=\Delta\left(e^{j\beta_0 d_i \sin\theta\cos\phi}\right)\times I + e^{j\beta_0 d_i \sin\theta\cos\phi}\times\Delta I \qquad \text{---(4.22)}$$

where $I=\int_{-h_i}^{h_i} I_i(z_i')e^{j\beta_0 z_i'\cos\theta}dz_i'$

but $\Delta\left(e^{j\beta_0 d_i \sin\theta\cos\phi}\right) = j\beta_0\sin\theta\cos\phi e^{j\beta_0 d_i \sin\theta\cos\phi}\Delta d_i \qquad \text{---(4.23)}$

and $\Delta I = \int_{-h_i}^{h_i}(\Delta B_i F_{0z_i'} + \Delta D_i H_{0z_i'})e^{j\beta_0 z_i'\cos\theta}dz_i'$

$$\Delta I = (\Delta B_i)\int_{-h_i}^{h_i} F_{0z_i'}e^{j\beta_0 z_i'\cos\theta}dz_i'$$

$$+ (\Delta D_i)\int_{-h_i}^{h_i} H_{0z_i'}e^{j\beta_0 z_i'\cos\theta}dz_i' \qquad \text{---(4.24)}$$

Substituting (4.23) and (4.24) into (4.22), the right hand side of (4.22) can be written as:

$$(\Delta d_i)e^{j\beta_0 d_i\sin\theta\cos\phi}(j\beta_0\sin\theta\cos\phi)\left[B_i\int_{-h_i}^{h_i} F_{0z_i'}e^{j\beta_0 z_i'\cos\theta}dz_i'\right.$$

$$\left.+ D_i\int_{-h_i}^{h_i} H_{0z_i'}e^{j\beta_0 z_i'\cos\theta}dz_i'\right] + e^{j\beta_0 d_i\sin\theta\cos\phi}\left[\vphantom{\int}\right.$$

$$\left.\Delta B_i\int_{-h_i}^{h_i} F_{0z'}e^{j\beta_0 z_i'\cos\theta}dz_i' + \Delta D_i\int_{-h_i}^{h_i} H_{0z'}e^{j\beta_0 z_i'\cos\theta}dz_i'\right] \qquad \text{---(4.25)}$$

A simplification seems possible by defining:

$$M_i^{(2)}(\theta) = \frac{\beta_0}{2}\int_{-h_i}^{h_i} F_{0z_i'} e^{j\beta_0 z_i' \cos\theta} \sin\theta dz_i' \qquad \text{---(4.26)}$$

$$M_i^{(3)}(\theta) = \frac{\beta_0}{2}\int_{-h_i}^{h_i} H_{0z_i'} e^{j\beta_0 z_i' \cos\theta} \sin\theta dz_i' \qquad \text{---(4.27)}$$

Thus, the radiation field of a spacing-perturbed linear array at a distance R_0 from a reference origin is:

$$E'(\theta,\phi) \simeq E(\theta,\phi) + \frac{j60}{R_0}\sum_{i=1}^{N} e^{j\beta_0 d_i \sin\theta\cos\phi}\{j\beta_0 \sin\theta\cos\phi(\Delta d_i)\times [M_i^{(2)}(\theta)B_i + M_i^{(3)}(\theta)D_i]+[M_i^{(2)}(\theta)\Delta B_i + M_i^{(3)}(\theta)\Delta D_i]\} \qquad \text{---(4.28)}$$

where $E(\theta,\phi)$ is the radiation field of the unperturbed array.

It is convenient to define an N element column matrix $\{D'\}$ with the kth element:

$$D_k' = \frac{j60}{R_0}\{j\beta_0 \sin\theta\cos\phi e^{j\beta_0 d_k \sin\theta\cos\phi}[M_k^{(2)}(0)B_k + M_k^{(3)}(0)D_k] + \sum_{i=1}^{N} e^{j\beta_0 d_i \sin\theta\cos\phi}(M_i^{(2)}[Q_2]_{ik} + M_i^{(3)}[Q_3]_{ik})\} \qquad \text{---(4.29)}$$

where $[Q_2]_{ik}$ and $[Q_3]_{ik}$ denote the ikth elements of $[Q_2]$ and $[Q_3]$, respectively.

With (4.29), we can write (4.28) as:

$$E'(\theta,\phi) = E(\theta,\phi)+\{D'\}^T\{\Delta d\} = E(\theta,\phi)+\{\Delta d\}^T\{D'\} \qquad \text{---(4.30)}$$

where the superscript T denotes transposition.

Now we consider the problem of gain optimization by spacing perturbation. The gain of an array in the direction (θ_0,ϕ_0) is:

$$G(\theta_0,\phi_0) = \frac{E(\theta_0,\phi_0)E^*(\theta_0,\phi_0)}{60P_{in}} \qquad \text{---(4.31)}$$

where P_{in} is the time averaged input power. With spacing perturbation, E becomes E', P_{in} becomes P'_{in} and the perturbed gain becomes:

$$G'(\theta_0,\phi_0) = \frac{E'(\theta_0,\phi_0)E'^*(\theta_0,\phi_0)}{60P'_{in}}$$

$$\begin{aligned} E'E'^* &= (E+\Delta E)(E+\Delta E)^* \\ &= (E+\Delta E)(E^*+\Delta E^*) \\ &= EE^*+E^*\Delta E+E\Delta E^*+\Delta E\Delta E^* \\ &\doteq EE^*+2\mathrm{Re}(E\Delta E^*)+\Delta E\Delta E^* \end{aligned}$$

From (4.30),

$$E'E'^* = EE^* + 2\mathrm{Re}(E\{\Delta d\}^T\{D'^*\}) + \{\Delta d\}^T\{D'^*\}\{D'\}^T\{\Delta d\} \quad ---(4.33)$$

We define:

$$\{B_1\} = \mathrm{Re}E\{D^*\} \quad ---(4.34)$$

and $[C_1] = \{D'^*\}\{D'\}^T$ ---(4.35)

Equation (4.33) now becomes:

$$E'E'^* = EE^* + 2\{\Delta d\}^T\{B_1\} + \{\Delta d\}^T[C_1]\{\Delta d\} \quad ---(4.36)$$

The N×N square matrix $[C_1]$ is positive semi-definite, and since $\{\Delta d\}$ is a real matrix, $[C_1]$ in the last term of of (4.33) can be replaced by $[\mathrm{Re}C_1]$. P'_{in} in the equation above is

$$P'_{in}=1/2\mathrm{Re}[V^*_{02}I^P_2(0)] = P_{in} + \{\Delta d\}^T\{B_2\} \quad ---(4.37)$$

where

$$P_{in} = 1/2V_{02}\mathrm{Re}[A_2M_{0z_2}(0) + B_2F_{0z_2}(0) + D_2H_{0z_2}(0)] \quad \text{---(4.38)}$$

and the kth element of the column matrix $\{B_2\}$ is

$$\{B_2\}_k = 1/2V_{02}\mathrm{Re}\{[Q_2]_{2k}F_{0z_2}(0) \pm [Q_3]_{2k}H_{0z_2}(0)\} \quad \text{---(4.39)}$$

For a lossless array, the input power equals the total power radiated, and P'_{in} can be written in an alternative form:

$$P'_{in} = 1/60\{1/(4\pi)\int_0^{2\pi}d\phi\int_0^{\pi}|E'(\theta,\phi)|^2\sin\theta d\theta d\phi \quad \text{---(4.40)}$$

Using (4.30), (4.40) can be expressed as

$$P'_{in} = P_{in}+2\{\Delta d\}^T\{B_3\}+\{\Delta d\}^T[\mathrm{Re}C_2]\{\Delta d\} \quad \text{---(4.41)}$$

where the radiated power of the unperturbed array

$$P_{in} = 1/(240\pi)\int_0^{2\pi}d\phi\int_0^{\pi}|E(\theta,\phi)|^2\sin\theta d\theta d\phi \quad \text{---(4.42)}$$

$$\{B_3\} = 1/(240\pi)\int_0^{2\pi}d\phi\int_0^{\pi}\{B_1\}\sin\theta d\theta d\phi \quad \text{---(4.43)}$$

and

$$[C_2] = 1/(240\pi)\int_0^{2\pi}d\phi\int_0^{\pi}[C_1]\sin\theta d\theta d\phi \quad \text{---(4.44)}$$

$\{B_1\}$ and $[C_1]$ have been previously defined, respectively, in (4.34) and (4.35), and $[C_2]$ is a positive definite Hermitian matrix.

The objective of gain optimization by spacing perturbation is to find the small changes in the element spacings such that the array gain in a given direction is increased and to repeat the process until further increases in gain are negligible. Hence, it is essential that

$$\Delta G(\theta_0,\phi_0) = G'(\theta_0,\phi_0) - G(\theta_0,\phi_0) \qquad \text{---(4.45)}$$

be positive. Substitution of (4.31)--(4.41) in (4.45) yields

$$\Delta G(\theta_0,\phi_0) = \frac{1}{60}\,\frac{\{\Delta d\}^T\{B\}+\{\Delta d\}^T[ReC_1]\{\Delta d\}}{P_{in}+2\{\Delta d\}^T\{B_3\}+\{\Delta d\}^T[ReC_2]\{\Delta d\}} \qquad \text{---(4.46)}$$

where

$$\{B\} = 2\{B_1\}-60G(\theta_0,\phi_0)\{B_2\} \qquad \text{---(4.47)}$$

Note that the negative sign in (4.47) for {B} in the numerator of $G(\theta_0,\phi_0)$ in (4.46) implies that the array gain will decrease for an improper choice of $\{\Delta d\}$.

In order to be certain that $G(\theta_0,\phi_0)$ will be positive, we make use of a known relation in the theory of matrices. Applied to the present problem, the relation asserts that if $[ReC_2]$ is positive definite, then

$$(\{B\}^T[ReC_2]^{-1}\{B\})(\{\Delta d\}^T[ReC_2]\{\Delta d\}) \geq (\{\Delta d\}^T\{B\}) \qquad \text{---(4.48)}$$

In (4.48), the equality sign holds when

$$\{\Delta d\} = \alpha[ReC_2]^{-1}\{B\} \qquad \text{---(4.49)}$$

where α is a positive constant. Hence, if the spacing changes in $\{\Delta d\}$ are chosen such that

$$\{\Delta d\} = \alpha[\mathrm{Re}C_2]^{-1}(2\{B_1\}-60G\{B_2\}) \qquad \text{---(4.50)}$$

then

$$\Delta G = \frac{1}{60}[\ \alpha(2\{B_1\}-60G\{B_2\})^T[\mathrm{Re}C_2]^{-1}(2\{B_1\}-60G\{B_2\})$$
$$+\{\Delta d\}^T[\mathrm{Re}C_1]\{\Delta d\}\ ]/[\ P_{in}+2\{\Delta d\}^T\{B_3\}$$
$$+\{\Delta d\}^T[\mathrm{Re}C_2]\{\Delta d\}\ ] > 0$$

4-2 Numerical Results

In this section we present some of the examples that were used by Cheng [1973] to illustrate the effectiveness of increasing the directivity of the Yagi-Uda array by spacing perturbation.

Example 1

Six element Yagi array with a half-wave feeder ($2h_2=.5\lambda$), one reflector ($2h_1=.51\lambda$), four directors ($2h_3=2h_4=2h_5=2h_6=.43\lambda$), $a=0.003369\lambda$. In the initial array, $b_{21}=0.25\lambda$, $b_{32}=b_{43}=b_{54}=b_{65}=0.31\lambda$. The director spacings are to be adjusted for gain maximization with the reflector spacing being fixed. The initial array is found to have a 9.06 dB gain [D.K. Cheng, 1973]. The present numerical calculation yields a 10.93 dB gain. The normalized radiation patterns are shown in Fig. 4.1, which agrees well with the patterns of Fig. 2 of [Cheng, 1973]. It is seen that the pattern for the optimum array has not only narrower main beam but also lower sidelobes. The optimized array is unequally spaced and was found to have 10.72 dB gain by Cheng [1973] and 12.49 dB gain by the present study. The increase of the array gain is 1.66 dB according to Cheng and 1.66 dB according to the present study. As was mentioned in Section 3-2, the difference is due to a somewhat different procedure. The results are summarized in Table I.

Example 2

Six element Yagi-Uda array with a half-wave feeder ($2h_2=.5\lambda$), one reflector ($2h_1=.51\lambda$), four directors ($2h_3=2h_4=2h_5=2h_6=.43\lambda$), $a=0.003369\lambda$. In the initial array, $b_{21}=.28\lambda$, $b_{32}=b_{43}=b_{54}=b_{65}=.31\lambda$. All element spacings are to be adjusted for gain optimization.

The reflector spacing b_{21} in the initial array is arbitrarily chosen to be 0.28λ and all other element spacings are given as 0.31λ. The gain of the initial array was found to be 8.77 dB by Cheng and Chen and was found to be 10.88 dB by the present study. The gain of the optimized array was found to be 10.74 dB by Cheng and Chen and 12.51 dB by the present study. The increase of the array gain is 1.97 dB according to Cheng and Chen and 1.63 dB according to the present study. Again the difference is somewhat due to a different procedure, as was explained in Section 3-2. The results are summarized in Table II.

Example 3

Ten element Yagi-Uda array with a half-wave feeder ($2h_2=.5\lambda$), one reflector ($2h_1=.51\lambda$), eight directors ($2h_i=.43\lambda, i=3,4,\ldots,10$), $a=0.003369\lambda$. In the initial array, $b_{21}=.25\lambda$, $b_{32}=b_{43}=\ldots=b_{109}=.31\lambda$. The director spacings are to be adjusted for gain maximization. The gain of the initial array was found to be 10.92 dB by Cheng and Chen and 12.3 dB by the present study. The gain of the optimized array was found as 12.1 dB by Cheng and Chen and 13.9 dB by the present study. The increase in the array gain is 1.18 dB according to Cheng and 1.6 dB according to the present study. The results are summarized in Table III.

The passband of the array was defined by Shen [1972] as the frequency band in which the directivity varies within 3 dB of the maximum value. Using this definition, a comparison of the bandwidth as well as the array gain is described in Table IV, from which it can be seen that the bandwidth is sacrificed when the array gain is optimized. A more interesting result is shown in Figures 4.2 and 4.3, where the array gain is plotted versus frequency for both the initial and optimized arrays of Cheng [1973]. It was found that if the array's forward gain is maximized by frequency perturbation before it is maximized by changing the inter-element spacings, then the gain increase described by Cheng [1973] is not as much. This is shown in Table V.

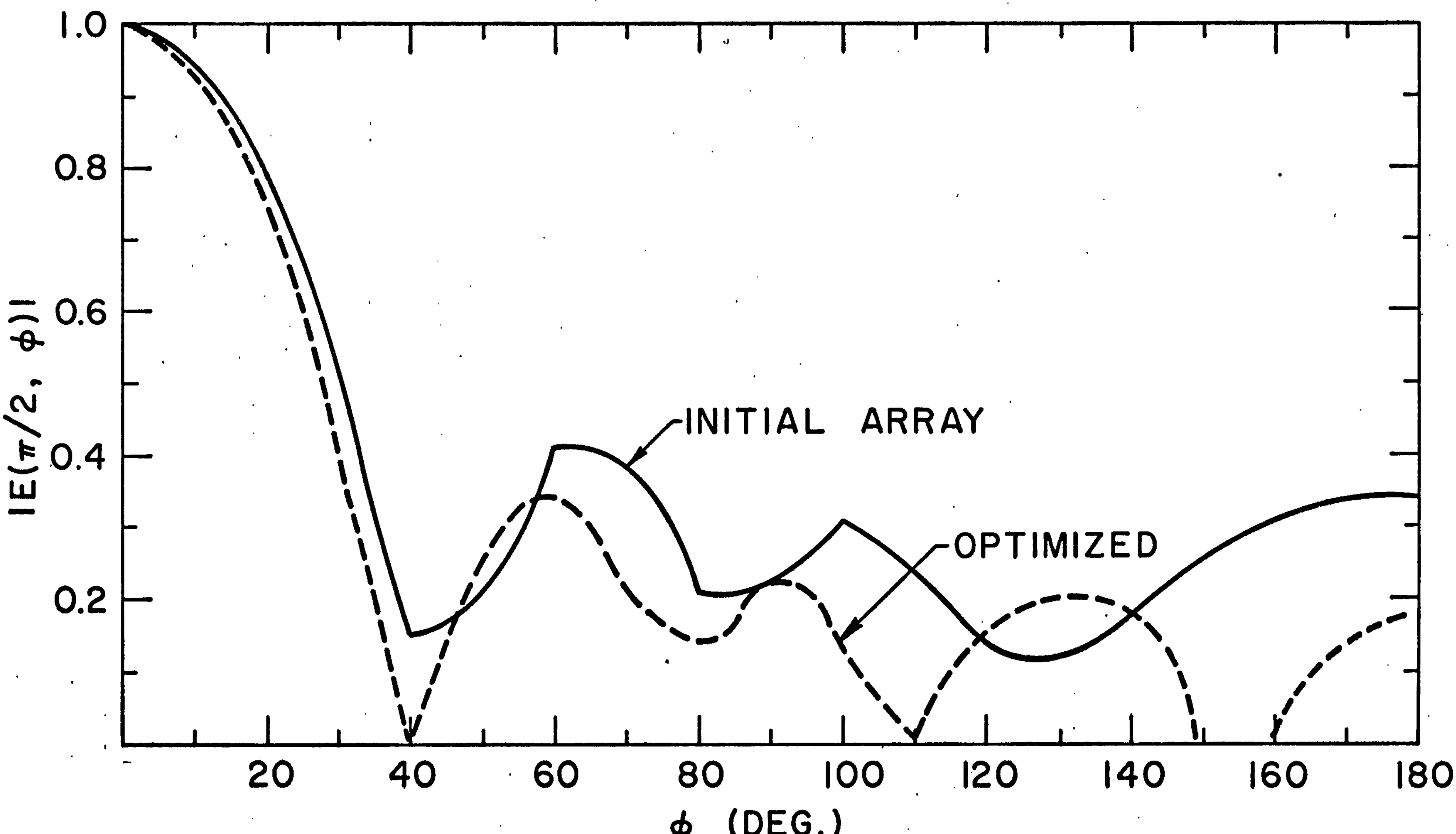


FIG. 4.1

(Cheng and Chen)
Ex. 1

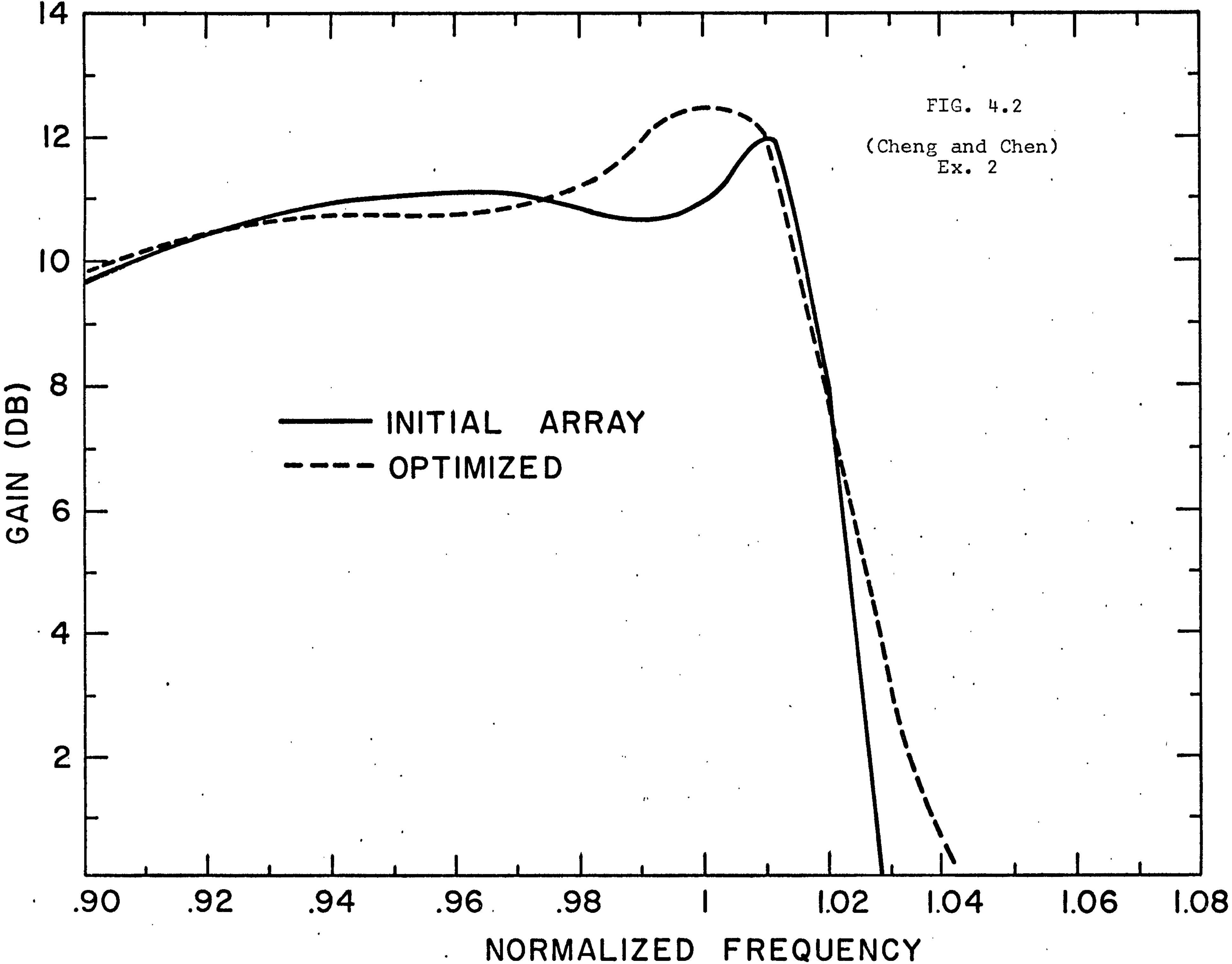


FIG. 4.2
(Cheng and Chen)
Ex. 2
INITIAL ARRAY
OPTIMIZED
GAIN (DB)
NORMALIZED FREQUENCY
14
12
10
8
6
4
2
.90
.92
.94
.96
.98
1
1.02
1.04
1.06
1.08

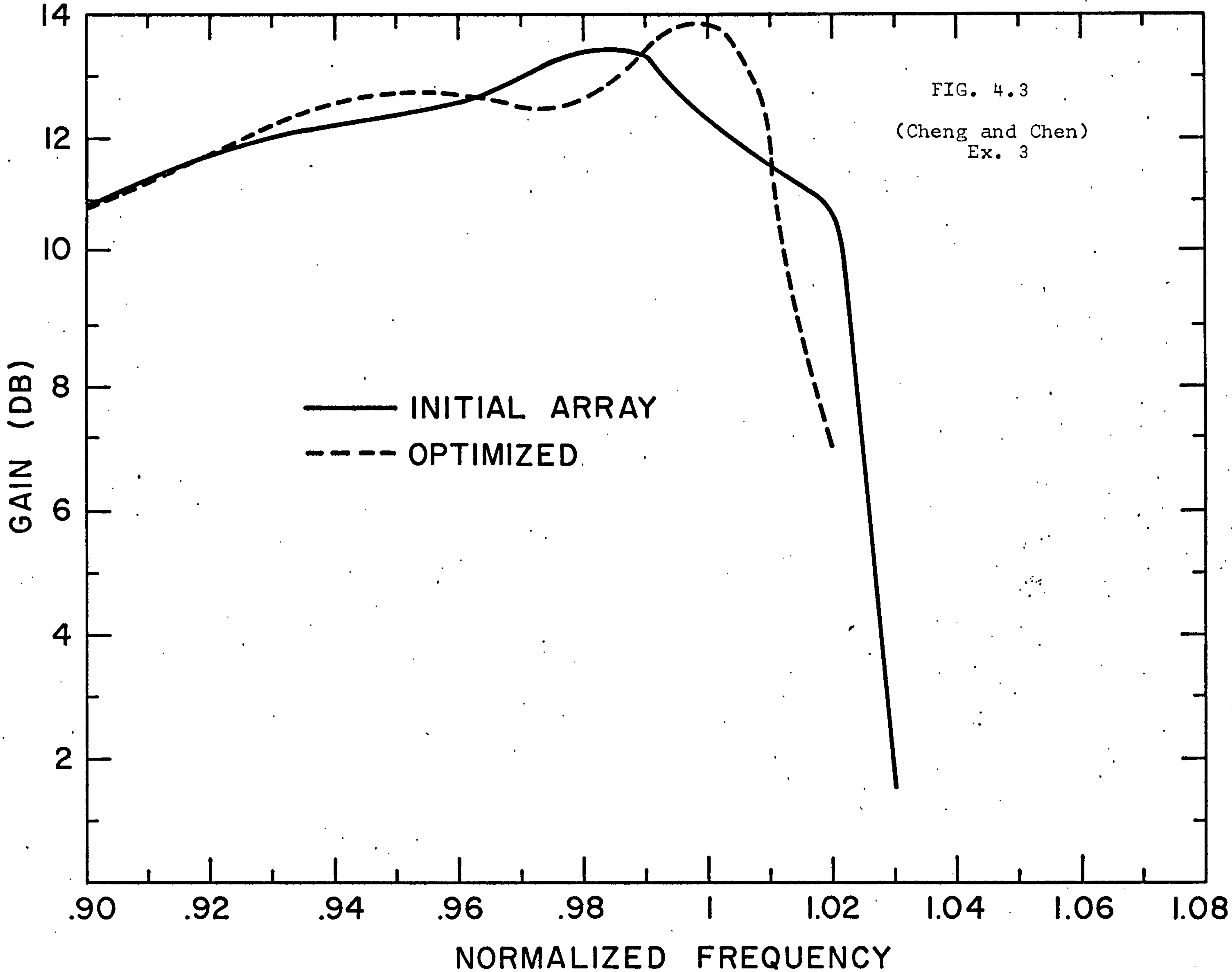


FIG. 4.3

(Cheng and Chen)
Ex. 3

TABLE I
Gain Optimization for Six-Element Yagi-Uda Array
(Perturbation of Director Spacings)

Array	b_{21}/λ	b_{32}/λ	b_{43}/λ	b_{54}/λ	b_{65}/λ	GAIN (dB) Cheng	GAIN (dB) Present Study
Initial	0.250	0.310	0.310	0.310	0.310	9.06	10.93
Optimized	0.250	0.336	0.398	0.310	0.407	10.72	12.49

TABLE II
Gain Optimization for Six-Element Yagi-Uda Array
(Perturbation of All Element Spacings)

Array	b_{21}/λ	b_{32}/λ	b_{43}/λ	b_{54}/λ	b_{65}/λ	GAIN (dB) Cheng	GAIN (dB) Present Study
Initial	0.280	0.310	0.310	0.310	0.310	8.77	10.88
Optimized	0.250	0.352	0.355	0.354	0.373	10.74	12.51

TABLE III
Gain Optimization for Ten-Element Yagi-Uda Array
(Perturbation of Director Spacings)

Array	b_{21}/λ	b_{32}/λ	b_{43}/λ	b_{54}/λ	b_{65}/λ	b_{76}/λ	b_{87}/λ	b_{98}/λ	b_{109}/λ	GAIN (dB) Cheng	GAIN (dB) Present Study
Initial	0.250	0.330	0.330	0.330	0.330	0.330	0.330	0.330	0.330	10.92	12.3
Optimized	0.250	0.319	0.357	0.326	0.400	0.343	0.320	0.355	0.397	12.1	13.9

TABLE IV

	Gain	Bandwidth %
Example 2		
Initial array	10.88	12.9
Optimized array	12.51	11.6
Example 3		
Initial array	12.3	13.0
Optimized array	13.90	10.8

TABLE V

Example	GAIN INCREASE		
	Cheng	Present Study	If Maximized by Frequency Adjustment
2	1.97dB	1.63dB	.55dB
3	1.18dB	1.6 dB	.50dB

4-3 Conclusions

The procuedure used by Cheng and Chen to calculate the forward gain of the Yagi-Uda array is somewhat different than the procedure used in the present study. This was described in Chapter III. These differences are small, and either procedure gave satisfactory results. In both, the effects of a finite element radius and the mutual coupling between array elements are taken into consideration; also, in both the three-term theory with complex coefficients is used to approximate the current distribution in the elements and to convert the integral equations into simultaneous algebraic equations. The examples used by Cheng and Chen to illustrate the gain increase by adjusting interelement spacings were verified using the method developed in the present study, and they are in agreement within 1/2 dB (comparing the absolute gain increase of the initial Yagi-Uda array).

The convergent iterative technique developed by Cheng and Chen yields the optimum spacings for maximum array gain without the need for a haphazard trial and error approach or for interpreting a vast data collection.

On the other hand, it was found out that should the array gain be optimized by adjusting the frequency, the net gain increase is less than the typical gain increases which are attainable with Cheng and Chen's techniques as described in their paper.

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APPENDIX

```
      SUBROUTINE YAGI(N,NO,A,H1,SX,AS2,X,SOUM,SM,ZN2,GAIN)
     EXTERNAL WVKN1,WVKNI1,WVKN2,WVKNI2,WVKN3,WVKNI3
     EXTERNAL WUKN1,WUKNI1,WUKN2,WUKNI2,WUKN3,WUKNI3
     EXTERNAL WDKN1,WDKNI1,WDKN2,WDKNI2,WDKN3,WDKNI3
     COMPLEX WV1(20,20),WV2(20,20),WV3(20,20)
     COMPLEX WU1(20,20),WU2(20,20),WU3(20,20)
     COMPLEX WD1(20,20),WD2(20,20),WD3(20,20)
     COMPLEX EPSHDD(20,20),EPSHDU(20,20),EPSFDU(20,20),EPSFDD(20,20)
      COMPLEX PHIU(20,20),PHID(20,20)
     COMPLEX PHIV(20)
     COMPLEX EPSHV(10)
     COMPLEX SOUM(NO,NO),SM(NO),X(NO)
     COMPLEX AS2,Y
     COMPLEX TD(20),TU(20)
     COMPLEX YN2,CURNT,FLD,XNORM
     COMPLEX ZN2
     DIMENSION DELTA1(20),DELTA2(20)
     DIMENSION A(N),H1(N)
     DIMENSION PHIVR(20),PHIVI(20),EPSHVR(20),EPSHVI(20)
     DIMENSION WVR1(20,20),WVI1(20,20),WVR2(20,20),WVI2(20,20)
     DIMENSION WUR1(20,20),WUI1(20,20),WUR2(20,20),WUI2(20,20)
     DIMENSION WDR1(20,20),WDI1(20,20),WDR2(20,20),WDI2(20,20)
     DIMENSION WVR3(20,20),WVI3(20,20)
     DIMENSION WUR3(20,20),WUI3(20,20)
     DIMENSION WDR3(20,20),WDI3(20,20)
     COMMON/BS/B(20,20)/HS/H(20)
     FM(R,Q)=(COS(Q*COS(R))-COS(Q))/SIN(R)
     GM(R,Q)=(SIN(Q)*COS(Q*COS(R))*COS(R)-COS(Q)*SIN(Q*COS(R)))
    1/SIN(R)/COS(R)
     DM(R,Q)=((2.0*COS(Q*COS(R))*SIN(Q/2.0)-4.0*SIN(Q*COS(R))*COS(Q/2.0
    2)*
    1COS(R))/(1.0-4.0*COS(R)*COS(R))-SIN(Q*COS(R))*COS(Q/2.0)/COS(R))*
    2SIN(R)
     TPI=6.283185307156
     DO 699 K=1,N
     H(K)=H1(K)
 699 CONTINUE
     DO 355 K=1,N
     DO 355 I=1,N
     IF (I.EQ.K) GO TO 350
     B(K,I)=A(K)-A(I)
     B(K,I)=ABS(B(K,I))
     GO TO 355
 350 B(K,I)=SX
 355 CONTINUE
     DO 20 K=1,N
     DO 20 I=1,N
     WVR1(K,I)=SIQD(-H(I),H(I),WVKN1,0.0,0.01,K,I)
     WVI1(K,I)=SIQD(-H(I),H(I),WVKNI1,0.0,0.01,K,I)
     WVR2(K,I)=SIQD(-H(I),H(I),WVKN2,0.0,0.01,K,I)
     WVI2(K,I)=SIQD(-H(I),H(I),WVKNI2,0.0,0.01,K,I)
     WVR3(K,I)=SIQD(-H(I),H(I),WVKN3,0.0,0.01,K,I)
     WVI3(K,I)=SIQD(-H(I),H(I),WVKNI3,0.0,0.01,K,I)
     WUR1(K,I)=SIQD(-H(I),H(I),WUKN1,0.0,0.01,K,I)
```

```
      WUI1(K,I)=SIQD(-H(I),H(I),WUKNI1,0.0,0.01,K,I)
      WUR2(K,I)=SIQD(-H(I),H(I),WUKN2,0.0,0.01,K,I)
      WUI2(K,I)=SIQD(-H(I),H(I),WUKNI2,0.0,0.01,K,I)
      WUR3(K,I)=SIQD(-H(I),H(I),WUKN3,0.0,0.01,K,I)
      WUI3(K,I)=SIQD(-H(I),H(I),WUKNI3,0.0,0.01,K,I)
      WDR1(K,I)=SIQD(-H(I),H(I),WDKN1,0.0,0.01,K,I)
      WDI1(K,I)=SIQD(-H(I),H(I),WDKNI1,0.0,0.01,K,I)
      WDR2(K,I)=SIQD(-H(I),H(I),WDKN2,0.0,0.01,K,I)
      WDI2(K,I)=SIQD(-H(I),H(I),WDKNI2,0.0,0.01,K,I)
      WDR3(K,I)=SIQD(-H(I),H(I),WDKN3,0.0,0.01,K,I)
      WDI3(K,I)=SIQD(-H(I),H(I),WDKNI3,0.0,0.01,K,I)
      WV1(K,I)=CMPLX(WVR1(K,I),WVI1(K,I))
      WV2(K,I)=CMPLX(WVR2(K,I),WVI2(K,I))
      WV3(K,I)=CMPLX(WVR3(K,I),WVI3(K,I))
      WU1(K,I)=CMPLX(WUR1(K,I),WUI1(K,I))
      WU2(K,I)=CMPLX(WUR2(K,I),WUI2(K,I))
      WU3(K,I)=CMPLX(WUR3(K,I),WUI3(K,I))
      WD1(K,I)=CMPLX(WDR1(K,I),WDI1(K,I))
      WD2(K,I)=CMPLX(WDR2(K,I),WDI2(K,I))
      WD3(K,I)=CMPLX(WDR3(K,I),WDI3(K,I))
   20 CONTINUE
      DO 60 K=1,N
      DELTA1(K)=SIN(H(K))*(COS(H(K)/4.0)-COS(H(K)/2.0))
      DELTA1(K)=DELTA1(K)-SIN(H(K)/2.0)*(1.0-COS(H(K)/2.0))
      DELTA2(K)=(1.0-COS(H(K)))*(COS(H(K)/4.0)-COS(H(K)/2.0))
   60 DELTA2(K)=DELTA2(K)-(COS(H(K)/2.0)-COS(H(K)))*(1.0-COS(H(K)/2.0))
      DO 80 K=1,N
      DO 80 I=1,N
      EPSHDD(K,I)=WD2(K,I)*(1.0-COS(H(K)))
      EPSHDD(K,I)=EPSHDD(K,I)-WD1(K,I)*(COS(H(K)/2.0)-COS(H(K)))
      EPSHDD(K,I)=EPSHDD(K,I)/DELTA2(K)
      EPSHDU(K,I)=WU2(K,I)*(1.0-COS(H(K)))
      EPSHDU(K,I)=EPSHDU(K,I)-WU1(K,I)*(COS(H(K)/2.0)-COS(H(K)))
      EPSHDU(K,I)=EPSHDU(K,I)/DELTA2(K)
      EPSFDU(K,I)=WU1(K,I)*(COS(H(K)/4.0)-COS(H(K)/2.0))
      EPSFDU(K,I)=EPSFDU(K,I)-WU2(K,I)*(1.0-COS(H(K)/2.0))
      EPSFDU(K,I)=EPSFDU(K,I)/DELTA2(K)
      EPSFDD(K,I)=WD1(K,I)*(COS(H(K)/4.0)-COS(H(K)/2.0))
      EPSFDD(K,I)=EPSFDD(K,I)-WD2(K,I)*(1.0-COS(H(K)/2.0))
      EPSFDD(K,I)=EPSFDD(K,I)/DELTA2(K)
      PHIU(K,I)=WU3(K,I)-EPSFDU(K,I)*COS(H(K))
      PHID(K,I)=WD3(K,I)-EPSFDD(K,I)*COS(H(K))
   80 CONTINUE
      DO 210 K=1,N
      IF (K.EQ.2) GO TO 210
      PHIV(K)=WV1(K,2)*(COS(H(K)/4.0)-COS(H(K)/2.0))
      PHIV(K)=PHIV(K)-WV2(K,2)*(1.0-COS(H(K)/2.0))
      PHIV(K)=PHIV(K)*COS(H(K))/DELTA2(K)
      PHIV(K)=PHIV(K)-WV3(K,2)
      PHIV(K)=-PHIV(K)
  210 CONTINUE
       PHIV(2)=WV3(2,2)
      DO 400 K=1,N
      IF (K.EQ.2) GO TO 400
```

```
      EPSHV(K)=WV2(K,2)*(1.0-COS(H(K)))
      EPSHV(K)=EPSHV(K)-WV1(K,2)*(COS(H(K)/2.0)-COS(H(K)))
      EPSHV(K)=EPSHV(K)/DELTA2(K)
  400 CONTINUE
      EPSHV(2)=WV2(2,2)*SIN(H(2))-WV1(2,2)*SIN(H(2)/2.0)
      EPSHV(2)=EPSHV(2)/DELTA1(2)
      NO=2*N
      Y=CMPLX(0.0,1.0)
      AS2=DELTA1(2)*Y/60.0
      AS2=AS2/(WV1(2,2)*(COS(H(2)/4.0)-COS(H(2)/2.0))-WV2(2,2)*(1.0-COS(
     1H(2)/2.0)))
      AS2=AS2/COS(H(2))
      DO 69 K=1,N
      PHIV(K)=PHIV(K)*AS2
      PHIV(K)=-PHIV(K)
      EPSHV(K)=EPSHV(K)*AS2
      EPSHV(K)=-EPSHV(K)
   69 CONTINUE
      DO 78 K=1,NO
      DO 78 I=1,N
      IF (K.GT.N) GO TO 115
      SOUM(I,K)=PHIU(I,K)
      GO TO 78
  115 SOUM(I,K)=PHID(I,K-N)
   78 CONTINUE
      DO 39 K=1,NO
      KL=N+1
      DO 39 I=KL,NO
      IF (K.GT.N) GO TO 130
      SOUM(I,K)=EPSHDU(I-N,K)
      GO TO 39
  130 SOUM(I,K)=EPSHDD(I-N,K-N)
   39 CONTINUE
      DO 52 K=1,N
   52 SM(K)=PHIV(K)
      DO 65 K=KL,NO
   65 SM(K)=EPSHV(K-N)
      CALL EQSOL(SOUM,SM,X,NO)
      YN2=AS2*SIN(H(2))+X(2)*(1.0-COS(H(2)))+X(N+2)*(1.0-COS(H(2)/2.0))
       ZN2=1.0/YN2
      WRITE (6,30) ZN2
   30 FORMAT (5X18H  INPUT IMPEDENCE=F10.4,3X,F10.4)
      DO 126 K=1,N
      IF (K.EQ.2) GO TO 126
      DO 113 L=1,10
      RATIO=(FLOAT(L)-1.0)/10.0
      CURNT=X(K)*(COS(H(K)*RATIO)-COS(H(K)))+X(N+K)*(COS(H(K)*RATIO/2.0)
     1-COS(H(K)/2.0))
      ANGLE=360.0/TPI*ATAN2(AIMAG(CURNT),REAL(CURNT))
      AMAG=CABS(CURNT)
  113 WRITE (6,111) K,CURNT,AMAG,ANGLE
  111 FORMAT (I11,2X,E12.6,3XE12.6,3X,E12.6,3X,E12.6)
  126 CONTINUE
      DO 139 L=1,10
```

```
      RATIO=(FLOAT(L)-1.0)/10.0
      CURNT=AS2*(SIN(H(2))-SIN(ABS(RATIO*H(2))))
      CURNT=CURNT+X(2)*(COS(H(2)*RATIO)-COS(H(2)))
      CURNT=CURNT+X(N+2)*(COS(H(2)*RATIO/2.0)-COS(H(2)/2.0))
      ANGLE=360.0/TPI*ATAN2(AIMAG(CURNT),REAL(CURNT))
      AMAG=CABS(CURNT)
  139 WRITE (6,111) L,CURNT,AMAG,ANGLE
      DO 152 K=1,N
      TU(K)=X(K)/AS2
      TD(K)=X(N+K)/AS2
  152 CONTINUE
      DO 626 KHI=1,19
      FLD=FM(TPI/4.0,H(2))
      DO 639 I=1,N
      PHI=(FLOAT(KHI)-1.0)*TPI/36.0
      IF (I.EQ.2) GO TO 713
      FLD=FLD+(COS((2.0-FLOAT(I))*B(2,I)*COS(PHI)/ABS(2.0-FLOAT(I)))
     1       -Y*SIN((2.0-FLOAT(I))*B(2,I)*COS(PHI)/ABS(2.0-FLOAT(I))))*
     1(TU(I)*GM(TPI/4.0,H(I))+TD(I)*DM(TPI/4.0,H(I)))
  726 GO TO 739
  713 FLD=FLD+TU(2)*GM(TPI/4.0,H(2))+TD(2)*DM(TPI/4.0,H(2))
  739 CONTINUE
  639 CONTINUE
      FLD=FLD/COS(H(2))
      IF (KHI.NE.1) GO TO 652
      XNORM=FLD
      RESIS=REAL(YN2)
      RESIS=ABS(RESIS)
      GAIN=60.0*COS(H(2))*CABS(AS2)
      GAIN=GAIN*GAIN/RESIS
      GAIN=GAIN/30.0
      GAIN=GAIN*CABS(XNORM)*CABS(XNORM)
      GAIN=10.0*ALOG10(GAIN)
  652 FLD=FLD/XNORM
      E=CABS(FLD)
      FLDDB=20.0*ALOG10(E)
       ANGLE=360.0/TPI*ATAN2(AIMAG(FLD),REAL(FLD))
  626 WRITE (6,181) KHI,FLDDB
  181 FORMAT (I14,5X,F10.4)
      WRITE (6,187) GAIN
  187 FORMAT (11H      GAIN= E13.4)
  559 CONTINUE
      STOP
      END
```

```
          FUNCTION SIQD(A,B,FCN,EPS,ETA,KM,IM)
    C     ROMBERG INTEGRATION FOR REAL    FUNCTION WITH REAL ARGUMENT
           DIMENSION Q(41)
          H=(B-A)/2.
          T=H*(FCN(A,KM,IM)+FCN(B,KM,IM))
          NX=2
          DO 12 N=1,40
01        SUM=0.0
01        DO 2 I=1,NX,2
02        FCNXI=FCN(A+FLOAT(I)*H,KM,IM)
02      2 SUM=SUM+FCNXI
01        T=T/2.+H*SUM
01        Q(N)=(T+H*SUM)/1.5
01        IF(N-2) 10,3,3
01      3 F=4.
01        I=N
01        DO 4 J=2,N
02        I=I-1
02        F=F*4.
02      4 Q(I)=Q(I+1)+(Q(I+1)-Q(I))/(F-1.)
01        IF(N-3)9,6,6
01      6 X2=ABS(Q(I)-QX2)+ABS(QX2-QX1)
01         TABS2=ABS(Q(1))
01     20 IF(TABS2)71,81,71
01     71 IF(X2/TABS2-ETA)11,11,81
01     81 IF(X2-EPS)11,11,9
01      9 QX1=QX2
01     10 QX2=Q(I)
01        H=H/2.
01     12 NX=NX*2
          WRITE(6,100)A,B
      100 FORMAT(47H ACCURACY LESS THAN SPECIFIED VALUES--SIQD--A,B 2
       11 SIQD=Q(I)
          RETURN
          END

ING 002537 .... THE SYMBOL 'J' OCCURS ONLY ONCE IN THE PROGRAM.

ING 002537 .... THE SYMBOL '#00020' OCCURS ONLY ONCE IN THE PROGRAM.
```

```
          SUBROUTINE EQSOL(A,B,X,N)
          COMPLEX A(N,N),B(N),X(N),TEMP,FACTOR,SUM
       30 FORMAT (2F10.6)
       25 NM1=N-1
          DO 610 K=1,NM1
01        KP1=K+1
01        L=K
01        DO 400 I=KP1,N
02    400 IF (CABS(A(I,K)).GT.CABS(A(L,K))) L=I
01        IF (L.EQ.K) GO TO 500
01        DO 410 J=K,N
02        TEMP=A(K,J)
02        A(K,J)=A(L,J)
02    410 A(L,J)=TEMP
01        TEMP=B(K)
01        B(K)=B(L)
01        B(L)= TEMP
01    500 DO 610 I=KP1,N
02        FACTOR=A(I,K)/A(K,K)
02        DO 600 J=KP1,N
03    600 A(I,J)=A(I,J)-FACTOR*A(K,J)
02    610 B(I)=B(I)-FACTOR*B(K)
          X(N)=B(N)/A(N,N)
          I=NM1
      710 IP1=I+1
          SUM=(0.0,0.0)
          DO 700 J=IP1,N
01    700 SUM=SUM+A(I,J)*X(J)
          X(I)=(B(I)-SUM)/A(I,I)
          I=I-1
          IF (I.GE.1) GO TO 710
          DO 901 I=1,N
01        U=CABS(X(I))
01        P=180.0/3.14159265*ATAN2(AIMAG(X(I)),REAL(X(I)))
01    901 WRITE (6,900) I,X(I),U,P
      900 FORMAT (1H0,I5,4E12.6)
          RETURN
          END

ING 002534 .... THE SYMBOL '#00030' OCCURS ONLY IN A DATA DECLARATION

ING 002537 .... THE SYMBOL '#00025' OCCURS ONLY ONCE IN THE PROGRAM.
```

V G LEVEL 19 WVKN1 DATE = 73319 16/27/48

```
      FUNCTION WVKN1(Z,K,I)
      COMMON/BS/B(2,2)/HS/H(2)
      REAL KERNDR
      FWV1=SIN(H(I)-ABS(Z))
      D1=SQRT(Z*Z+B(K,I)*B(K,I))
      D2=SQRT((Z-H(K))*(Z-H(K))+B(K,I)*B(K,I))
      KERNDR=COS(D1)/D1-COS(D2)/D2
      WVKN1=FWV1*KERNDR
      RETURN
      END
```

V G LEVEL 19 WVKNI1 DATE = 73319 16/27/48

```
      FUNCTION WVKNI1(Z,K,I)
      REAL KERNDI
      COMMON/BS/B(2,2)/HS/H(2)
      D1=SQRT(Z*Z+B(K,I)*B(K,I))
      D2=SQRT((Z-H(K))*(Z-H(K))+B(K,I)*B(K,I))
      FWV1=SIN(H(I)-ABS(Z))
      KERNDI=SIN(D2)/D2-SIN(D1)/D1
      WVKNI1=FWV1*KERNDI
      RETURN
      END
```

V G LEVEL 19 WVKN2 DATE = 73319 16/27/48

```
      FUNCTION WVKN2(Z,K,I)
      REAL KERNDR
      COMMON/BS/B(2,2)/HS/H(2)
      FWV1=SIN(H(I)-ABS(Z))
      D1=SQRT((H(K)/2.0-Z)*(H(K)/2.0-Z)+B(K,I)*B(K,I))
      D2=SQRT((Z-H(K))*(Z-H(K))+B(K,I)*B(K,I))
      KERNDR=COS(D1)/D1-COS(D2)/D2
      WVKN2=FWV1*KERNDR
      RETURN
      END
```

```
      FUNCTION WVKNI2(Z,K,I)
      REAL KERNDI
      COMMON/BS/B(2,2)/HS/H(2)
      FWV1=SIN(H(I)-ABS(Z))
      D1=SQRT((H(K)/2.0-Z)*(H(K)/2.0-Z)+B(K,I)*B(K,I))
      D2=SQRT((Z-H(K))*(Z-H(K))+B(K,I)*B(K,I))
      KERNDI=SIN(D2)/D2-SIN(D1)/D1
      WVKNI2=FWV1*KERNDI
      RETURN
      END
```

```
      FUNCTION WVKN3(Z,K,I)
      COMMON/BS/B(2,2)/HS/H(2)
      REAL KERNDR
      FWV1=SIN(H(I)-ABS(Z))
      D1=SQRT((H(K)-Z)*(H(K)-Z)+B(K,I)*B(K,I))
      KERNDR=COS(D1)/D1
      WVKN3=FWV1*KERNDR
      RETURN
      END
```



```
      FUNCTION WVKNI3(Z,K,I)
      COMMON/BS/B(2,2)/HS/H(2)
      REAL KERNDI
      FWV1=SIN(H(I)-ABS(Z))
      D1=SQRT((H(K)-Z)*(H(K)-Z)+B(K,I)*B(K,I))
      KERNDI=SIN(D1)/D1
      WVKNI3=KERNDI*FWV1
      RETURN
      END
```

```
      FUNCTION WUKN2(Z,K,I)
      COMMON/BS/B(2,2)/HS/H(2)
      REAL KERNDR
      FWV1=COS(Z)-COS(H(K))
      D1=SQRT((H(K)/2.0-Z)*(H(K)/2.0-Z)+B(K,I)*B(K,I))
      D2=SQRT((Z-H(K))*(Z-H(K))+B(K,I)*B(K,I))
      KERNDR=COS(D1)/D1-COS(D2)/D2
      WUKN2=FWV1*KERNDR
      RETURN
      END
```



```
      FUNCTION WUKNI1(Z,K,I)
      COMMON/BS/B(2,2)/HS/H(2)
      REAL KERNDI
      D1=SQRT(Z*Z+B(K,I)*B(K,I))
      D2=SQRT((Z-H(K))*(Z-H(K))+B(K,I)*B(K,I))
      KERNDI=SIN(D2)/D2-SIN(D1)/D1
      FWV1=COS(Z)-COS(H(K))
      WUKNI1=FWV1*KERNDI
      RETURN
      END
```



```
      FUNCTION WUKN1(Z,K,I)
      COMMON/BS/B(2,2)/HS/H(2)
      REAL KERNDR
      FWV1=COS(Z)-COS(H(K))
      D1=SQRT(Z*Z+B(K,I)*B(K,I))
      D2=SQRT((Z-H(K))*(Z-H(K))+B(K,I)*B(K,I))
      KERNDR=COS(D1)/D1-COS(D2)/D2
      WUKN1=FWV1*KERNDR
      RETURN
      END
```

V G LEVEL 19 WUKNI3 DATE = 73319 16/27/48

```
      FUNCTION WUKNI3(Z,K,I)
      COMMON/BS/B(2,2)/HS/H(2)
      REAL KERNDI
      FWV1=COS(Z)-COS(H(K))
      D1=SQRT((H(K)-Z)*(H(K)-Z)+B(K,I)*B(K,I))
      KERNDI=SIN(D1)/D1
      WUKNI3=FWV1*KERNDI
      RETURN
      END
```

V G LEVEL 19 WUKN3 DATE = 73319 16/27/48

```
       FUNCTION WUKN3(Z,K,I)
      REAL KERNDR
      COMMON/BS/B(2,2)/HS/H(2)
      FWV1=COS(Z)-COS(H(K))
      D1=SQRT((H(K)-Z)*(H(K)-Z)+B(K,I)*B(K,I))
      KERNDR=COS(D1)/D1
      WUKN3=FWV1*KERNDR
      RETURN
      END
```

V G LEVEL 19 WUKNI2 DATE = 73319 16/27/48

```
      FUNCTION WUKNI2(Z,K,I)
      COMMON/BS/B(2,2)/HS/H(2)
      REAL KERNDI
      FWV1=COS(Z)-COS(H(K))
      D1=SQRT((H(K)/2.0-Z)*(H(K)/2.0-Z)+B(K,I)*B(K,I))
      D2=SQRT((Z-H(K))*(Z-H(K))+B(K,I)*B(K,I))
      KERNDI=SIN(D2)/D2-SIN(D1)/D1
      WUKNI2=FWV1*KERNDI
      RETURN
      END
```

V G LEVEL 19 WDKN1 DATE = 73319 16/27/48

```
      FUNCTION WDKN1(Z,K,I)
      COMMON/BS/B(2,2)/HS/H(2)
      REAL KERNDR
      FWV1=COS(Z/2.0)-COS(H(K)/2.0)
      D1=SQRT(Z*Z+B(K,I)*B(K,I))
      D2=SQRT((Z-H(K))*(Z-H(K))+B(K,I)*B(K,I))
      KERNDR=COS(D1)/D1-COS(D2)/C2
      WDKN1=FWV1*KERNDR
      RETURN
      END
```

V G LEVEL 19 WDKNI1 DATE = 73319 16/27/48

```
      FUNCTION WDKNI1(Z,K,I)
      COMMCN/BS/B(2,2)/HS/H(2)
      REAL KERNDI
      D1=SQRT(Z*Z+B(K,I)*B(K,I))
      D2=SQRT((Z-H(K))*(Z-H(K))+B(K,I)*B(K,I))
      FWV1=COS(Z/2.0)-COS(H(K)/2.0)
      KERNDI=SIN(D2)/D2-SIN(D1)/D1
      WDKNI1=FWV1*KERNDI
      RETURN
      END
```

V G LEVEL 19 WDKN2 DATE = 73319 16/27/48

```
      FUNCTION WDKN2(Z,K,I)
      COMMCN/BS/B(2,2)/HS/H(2)
      REAL KERNDR
      FWV1=COS(Z/2.C)-COS(H(K)/2.0)
      D1=SCRT((H(K)/2.0-Z)*(H(K)/2.0-Z)+B(K,I)*B(K,I))
      D2=SCRT((Z-H(K))*(Z-H(K))+B(K,I)*B(K,I))
      KERNCR=COS(C1)/D1-COS(D2)/D2
      WDKN2=FWV1*KERNDR
      RETURN
      END
```

G LEVEL 19　　　WDKNI2　　　DATE = 73319　　　16/27/48

```
      FUNCTION WDKNI2(Z,K,I)
      COMMON/BS/B(2,2)/HS/H(2)
      REAL KERNDI
      FWV1=COS(Z/2.0)-COS(H(K)/2.0)
      D1=SQRT((H(K)/2.0-Z)*(H(K)/2.0-Z)+B(K,I)*B(K,I))
      D2=SQRT((Z-H(K))*(Z-H(K))+B(K,I)*B(K,I))
      KERNDI=SIN(D2)/D2-SIN(D1)/D1
      WDKNI2=FWV1*KERNDI
      RETURN
      END
```

G LEVEL 19　　　WDKN3　　　DATE = 73319　　　16/27/48

```
      FUNCTION WDKN3(Z,K,I)
      REAL KERNDR
      COMMON/BS/B(2,2)/HS/H(2)
      FWV1=COS(Z/2.0)-COS(H(K)/2.0)
      D1=SQRT((H(K)-Z)*(H(K)-Z)+B(K,I)*B(K,I))
      KERNDR=COS(D1)/D1
      WDKN3=FWV1*KERNDR
      RETURN
      END
```

V G LEVEL 19　　　WDKNI3　　　DATE = 73319　　　16/27/48

```
      FUNCTION WDKNI3(Z,K,I)
      COMMON/BS/B(2,2)/HS/H(2)
      REAL KERNDI
      FWV1=COS(Z/2.0)-COS(H(K)/2.0)
      D1=SQRT((H(K)-Z)*(H(K)-Z)+B(K,I)*B(K,I))
      KERNDI=SIN(D1)/D1
      WDKNI3=FWV1*KERNDI
      RETURN
      END
```