STOCHASTIC PROCESSES
NEW FOUNDATIONS
AND
REPRESENTATIONS IN HILBERT SPACE

A Dissertation

Presented to

the Faculty of the Department of Physics

University of Houston

In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

---

by

Forrest G. Hall

May 1970

ACKNOWLEDGEMENTS

I would like to thank my wife, Linda, for her warm support during my anxious moments; and my friend and advisor, Professor Collins for his support and technical assistance.

STOCHASTIC PROCESSES

NEW FOUNDATIONS

AND

REPRESENTATIONS IN HILBERT SPACE

An Abstract of a Dissertation

Presented to

the Faculty of the Department of Physics

University of Houston

In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

---

by

Forrest G. Hall

May 1970

ABSTRACT

In this thesis new foundations for the stochastic process are formulated which lead to the conventional stochastic formalism and in addition clearly define the notion of time reversibility for the stochastic process. The stochastic equations are shown to have representations in a separable Hilbert space $\mathcal{H}$, which have no counterpart in unitary quantum dynamics.

Beginning with an intuitive consideration of sequences of measurements, we define a time-ordered event space representing the collection of all imaginable outcomes for the measurement sequence.

We then postulate the generalized distributive relation on the event space and examine the class of measurements for which this relation can be experimentally validated. The generalized distributive relation is shown to lead to a σ-additive conditional probability on the event space and to a predictive and retrodictive formalism for stochastic processes.

We show that the dynamics of the stochastic formalism are distinct from unitary quantum dynamics in several major ways. We propose the basis for a mathematical structure in $\mathcal{H}$ which would include both the stochastic formalism and the quantum formalism as special cases.

TABLE OF CONTENTS

INTRODUCTION

This introduction will describe briefly, the contents of each of the thesis chapters.

Chapter I is concerned with background material and provides a matrix for an extensive bibliography documenting the various researches into stochastic theory and quantum theory. An examination of these references will point out the conflict between and the attempts to reconcile two highly successful theories: quantum theory and classical stochastic theory.

After an extensive study of these references the author concludes that these attempts at reconciliation have failed for three major reasons: (1) The author agrees with those who feel that the quantum axioms, historically and in more recent formulations, are not derived from a consideration of data, but are postulated and then are shown after the fact to agree with certain experiments. (2) Mathematicians have made little or no attempt to appeal to the nature of physical data in their axiomatic forumlation of stochastic theory; that such an appeal has not been made by physicists, can be explained by their loss of interest in stochastic theory with the advent of quantum theory. (3) Stochastic theory has never been formulated in a way which would allow its dynamical structure to be compared with the dynamical structure of quantum theory.

The author, in the last part of Chapter I, outlines his plan for rebuilding stochastic theory from a consideration of physical data and for casting stochastic theory into a form which can be compared with quantum theory.

Chapter II contains a paper submitted for publication to the Journal of Mathematical Physics, by the author and Professor Collins, which represents the major portion of the research of this thesis.

In this paper, classical stochastic theory is formulated from a consideration of the nature of data. It is shown that the conventional predictive stochastic formalism as well as a *new* retrodictive stochastic formalism follows directly from these elementary considerations, and that the random walk equations, previously thought to be valid only for Markoffian processes, are in general true for the non-Markoffian case.

Since quantum theory is formulated from axioms totally dissimilar to the axioms of the stochastic process, the compatibility of these two theories can be tested only if we can succeed in casting the equations of motion for stochastic theory into a form which can be compared with the dynamic structure of quantum theory. We then do this by demonstrating the existance of a structure in a separable Hilbert space $\mathcal{H}$ which will, *in all cases*, reproduce the dynamical structure of stochastic processes, both *predictive* and *retrodictive*. This has never before been accomplished, and it is hoped that this accomplishment will do for stochastic theory what the $\mathcal{H}$ formulation by Dirac did for quantum theory. In any case, with this stochastic $\mathcal{H}$ formulation we are able to compare directly the dynamical structure of classical stochastic theory with the dynamical structure of quantum theory. It is shown by this comparison that classical stochastic theory is compatible with quantum theory in only a trivial circumstance.

Chapter III, entitled "Discussion", is intended as a physical exposition of the mathematical results of Chapter II. The Journal of Mathematical Physics, to which the paper of Chapter II was submitted, demands that papers submitted to it, be written in a rigorous mathematical setting. Thus physical examples, minus the mathematical rigor, will be presented in Chapter III to enhance the understanding gained by reading Chapter II.

In this chapter we will investigate the nature of the transition probability and point out by a simple thought experiment, the conditions for this transition probability to be Markoffian. We will then investigate time-reversal in stochastic processes, by deriving the predictive and retrodictive diffusion equation from the predictive and retrodictive random walk equation. We will show the retrodictive diffusion equation to be the time-reversed form of the predictive diffusion equation and will thus be able to investigate the time-reversal characteristics of diffusion.

We will then derive the Pauli master equation from the stochastic formalism and compare it to the form of the master equation derived from quantum theory. We will show from this that stochastic theory seems to afford a more general description of ensemble systems than does quantum theory.

Finally we will examine a simple experimental situation which can be described by stochastic theory. We will show that the probabilistic interpretation of quantum theory fails to describe the most general type of data available from this experiment.

Chapter IV, the summary, will review the results of this dissertation, discuss the major differences between quantum evolution and stochastic evolution, and will propose new research to formulate a more general evolution structure for physical systems.

In the bibliography there is included, with many of the references, a brief statement of the content of the work undertaken which would relate specifically to this research.

CHAPTER I

BACKGROUND

Quantum theory and the theory of classical stochastic processes have shared a strange history of conflict in marriage since the introduction (1905-1927) of quantum theory be deBroglie, Planck, Einstein, Bohr, Heisenberg, Born, Schrodinger, Jordan and Dirac. Many physicists, including the originators of quantum theory, have tried with only partial success to resolve the conflict between these two highly successful theories. Part of the difficulty as we shall see in this section is the unintuitive nature of the quantum formulation, the impreciseness of previous formulations of stochastic theory and the failure to compare directly the dynamical laws of both theories.

At the outset, the foundations of quantum theory seemed unintuitive to many in its prescription of a dual nature to both electromagnetic and particle phenomena. The early formulations of Heisenberg and Schrodinger seemed to set the stage for a series of unintuitive although precise formulations by Dirac(1), von Neumann(2), Birkhoff (3), Jauch(4), Piron(5), Mackey(6) and others.

However the brilliant success that quantum theory enjoyed in describing the results of a large number of experiments, motivated many attempts to rationalize the foundations of the theory with experience. Born(7) in 1926 gave birth to the so-called "Copenhagen" or "orthodox" interpretation of quantum theory which provided the first interpretative connections between classical probability theory and quantum theory. Born's interpretations lead to a lengthy rational-emotive exchange between Bohr and Einstein[8],[9],[10],[11] arguing the applicability of quantum theory to anything other than ensemble systems. This controversy remains to be settled as can be seen in more recent publications[12],[13],[14],[15].

Other attempts to deny or confirm the Born hypothesis came in the form of hidden variable arguments. The Einstein, Podolsky, Rosen (EPR) paper[10] was the first to present an argument suggesting that quantum mechanics was not a complete theory. This paper led others[16],[17],[18],[19] to suggest that the "dispersion free" states of quantum theory could be specified not only by the quantum mechanical state vector but also by additional "hidden variables".

Many attempts were made to defeat the hidden variable idea mathematically (i.e. show it inconsistent with quantum theory). Von Neumann[2] was the first to present such a proof and more recently Jauch and Piron[5] have presented another version of this proof.

However, Bohm[20], Bell[21] and Bub[22] claim that von Neumann's assumptions are restrictive and of limited relevance and in addition have shown the axioms of Jauch and Piron to be unreasonable.

Several attempts at the explicit construction of a hidden variable theory have been undertaken. Bohm[20] has constructed a hidden variable theory which is non-local in nature and Bell [23] has purported to prove that a local hidden variable theory is not possible. Based on Bell's argument, Clauser et.al.[24] have proposed an experiment to test Bell's theorem. However Collins [25] has shown that one of Bell's assumptions is unnecessary and that without this assumption, an explicit local hidden variable theory can be constructed.

At the roots of this lengthy and continuing controversy seems to be a general lack of intuitive feeling for the quantum axioms; thus other approaches [13],[19],[26],[27],[28] rationalize the quantum axioms in terms of the measurement process but final and decisive interpretations have not been forthcoming.

Many physicists have sought to overcome the intuitive difficulties of quantum theory by deriving certain results of quantum theory from a classical basis. Heisenberg's uncertainty principle and the correspondence principle motivated many early attempts [29]-[47] to generalize Newtonian mechanics with a statistical approach to obtain the structure of quantum theory.

The similarities of the classical Hamiltonian formalism to the quantum Heisenberg operator "picture" <u>and</u> of the classical diffusion and Langevin equations to the quantum Schrodinger "picture" have motivated more recent attempts to relate quantum theory to classical physics.

Leibowitz [48],[49] derived a differential equation for the time dependence of the density of solutions for the Hamilton-Jacobi equation and showed that it could be put into the form of the time-independent Schrodinger equation. He was not able to obtain the time-dependent Schrodinger equation however.

Nelson[50],[51] examined a collection of electrons interacting with an external field and assumed that the collection is described by Brownian motion with a diffusion coefficient given by $\hbar/2m$. With this model he derived the time-dependent Schrodinger equation. He showed that the classical paths for the electron are so discontinuous that time derivatives do not exist; thus the particle momenta are not well defined. He also showed that the Brownian hypothesis led to the correct energy levels for bound states of the Hydrogen atom and interpreted these levels as states of dynamical equilibrium.

De la Pena-Auerbach and Garcia-Colin[52] generalized Newtonian mechanics in a stochastic Brownian model and showed that the time-dependent Schrödinger and the Fokker-Planck equations followed if the non-Markoffian terms of the stochastic equations are ignored. The same authors in another article[53] suggested a generalization of Schrödinger's equation based on a stochastic model and purport to give a counter example to von Neumann's hidden variable proof.

Kursunoglu[54] derives the classical Markoff formulation of the random walk problem in phase space in a manner analogous to the Feynmann path-integral formulation[55] and compares it to the result obtained by Chandrasekhar[56].

There have been a few attempts[57],[58],[59],[60] to study the differences between the algebras of quantum theory and classical probability theory. Some of the difficulties of generalizing the algebra of probability theory to the algebra of quantum theory lie in the definition of conditional probability. Several publications by Watanabe[61],[62],[63] summarize these difficulties and comment on the physical implications of these difficulties.

Thus we see that while connections between quantum theory and stochastic theory have been noted in certain special cases, no overall analysis of the relationship of these two theories has been attempted. Before we propose such an analysis let us examine briefly the development and present "state" of stochastic theory.

Classical stochastic theory, while perhaps not as broadly applicable as quantum theory, is extremely successful in some areas, and has as its foundations one of the most lucid and intuitive of all physical theories: classical probability theory.

That classical probability theory is so lucidly formulated is probably due largely to its experiential genesis in the works of Pascal, Fermat, Huygens, Bernoulli, de Moivre and Laplace*, who generated probability theory to describe data which were more subject to direct experience than were the subtle microscopic data which motivated quantum theory.

As probability theory began to intrigue mathematicians the subject took on important developments in the works of Tshebysheff, Markoff and Lia Pounoff. The fundamental work which set the stage for the modern measure-theoretic approach was done by Kolmogoroff[64]. Since that time many books have appeared; three of the most outstanding are those by Doob[65], Feller[66], and Loeve[67].

Before we discuss the modern developments of stochastic theory we should perhaps distinguish stochastic theory from probability theory. Although repeated trials are fundamental in validating the results of probability theory, probability theory does not concern itself explicitly with the dynamical behavior of the data. Stochastic theory does concern itself with this dynamical behavior and thus can be viewed as the temporal generalization of probability theory.

---

* A short history of the early development of probability theory can be found in Introduction to Probability Theory, by Uspensky, McGraw Hill (1937).

Stochastic theory unfortunately has not had the same intuitively lucid and mathematically rigorous treatment as has classical probability theory. An example which characterizes one of the more rigorous treatments of stochastic theory is the text by Doob [65]. In this text and most others, the stochastic generalization of probability theory comes about by considering probabilities for sequences of random variables. Such sequences and their probabilities are assumed to exist and be well defined independent of any observational considerations. Certainly no mention is made of the observational processes necessary to validate the axioms for the treatment of random variable sequences. It is not surprising therefore that the application of stochastic theory to the description of physical phenomena is a difficult and ill-defined procedure.

For example, once physical considerations are brushed aside and the mathematical structure for general stochastic processes has been established, most texts define the Markoff chain (sequence) and devote the majority of their considerations to the study of Markoff processes. Although the definition of the Markoff chain is usually made quite rigorous mathematically (eg. Chung, ref. 68) the class of measurement sequences which contain such chains is not defined. Consequently the range of applicability of Markoffian stochastic theory to the dynamical behavior of physical systems is unknown. Such unintuitive treatments of stochastic theory have, in the opinion of this author, made impossible an exhaustive comparison of the dynamical descriptions of stochastic and quantum theory.

From this review of the research in quantum theory and stochastic theory it is easily seen that no one has yet reconciled or even successfully compared these two theories in any satisfactory way; only bits and pieces of connections have been observed for rather restricted cases. It is this author's contention that these attempts have failed for three major reasons:

(1) The author agrees with those who feel that the quantum axioms, historically and in more recent formulations, are not derived from a consideration of data, but are postulated and then are shown, after the fact, to agree with certain experiments. The axiom which the founders of the modern axiomatic approach are most concerned about is the complex Hilbert space axiom; more precisely, <u>the partially ordered set of all questions in quantum mechanics is isomorphic to the partially ordered set of all closed subspaces of a separable, infinite dimensional Hilbert space</u>.

The mathematician George Mackey in his treatise "Mathematical Foundations of Quantum Mechanics"[(6)] says of this axiom, "Why do we make it?....We make it because it "works", that is it leads to a theory which explains physical phenomena and successfully predicts the results of experiments. It is conceivable that a quite different assumption would do likewise but this is a possibility that no one seems to have explored." Then a physicist Josef Jauch in his elegant formulation of quantum theory, "Foundations of Quantum Mechanics"[(4)] says of the Hilbert space axiom, "...We begin the building of the bridge which connects the general quantum theory, as an abstract proposition system, with conventional quantum mechanics in a complex Hilbert space. This bridge is not yet complete. There are no convincing empirical grounds why our Hilbert space should be constructed over the field of complex numbers."

(2) Mathematicians have made little or no appeal to the nature of physical data in their axiomatic formulation of stochastic theory. As a consequence, the class of physical systems to which stochastic theory is applicable is unknown. Nelson in his monograph, "Dynamical Theories of Brownian Motion"[(5)], discusses the "doldrums" in which stochastic processes is in today; "Physicists lost interest in the phenomena of Brownian motion about thirty or forty years ago. If a modern physicist is interested in Brownian motion, it is because the mathematical theory of Brownian motion has proved useful as a tool in the study of some models of quantum field theory and in quantum statistical mechanics".

(3) Stochastic theory has never been formulated in a way which would allow its dynamical structure to be compared with the dynamical structure of quantum theory.

It is this state of affairs in quantum theory and stochastic theory which invites the line of investigation undertaken in this thesis.

The author's first direction for investigation consisted of a detailed study of the work of Jauch[(4)] to determine whether or not the Hilbert space axiom could be rationalized in terms of direct experience. Failing this, the author decided that if one was to compare directly stochastic theory with quantum theory, then the dynamical equations of stochastic theory would have to be cast into a form similar to the dynamical equations of quantum theory.

The first task then, was to formulate stochastic theory from an examination of the nature of data so that at least the class of experiments for which stochastic theory applies could be known. As mentioned earlier, several new results for stochastic theory were a result of this investigation.

The next step then was to cast the dynamical structure of stochastic theory into a format comparable with the dynamical structure of quantum theory. This was accomplished by deducing the existance of a structure in a separable complex Hilbert space which is always capable of describing the stochastic process. This result is totally mathematical in nature and the purpose for choosing the complex Hilbert space representation, as stated above, was to allow a direct comparison between quantum theory and stochastic theory.

To the best of the author's knowledge, this approach to the $\mathcal{H}$ formulation for stochastic processes is unique. It is interesting however to compare the approach of this thesis with the approach by Jauch[4]. Such a comparison will lend some insight into the basic differences between the stochastic formalism in $\mathcal{H}$ and the quantum formalism in $\mathcal{H}$.

Jauch begins by assuming that all questions that one could ask about a physical system form a basis for the set of all propositions about a physical system. The analogous collection in this thesis is the event space which is assumed to contain only those questions which can be answered by a measurement. Jauch then assumes that the propositions for a system form a mathematical collection known as a lattice. In the present work however, the event space forms a mathematical collection called a σ algebra. This structure is chosen because it contains only those propositions which are answerable by a direct observation, whereas the lattice contains what Jauch calls "incompatible propositions", i.e., propositions not answerable by a direct observation.

Jauch chooses as his time evolution model, a temporal mapping which leaves the lattice of propositions invariant. This assumption seems reasonable since, if one can imagine a set of propositions (answerable or not) for a system at t_o, then he can certainly imagine the same set of propositions for a system at $t_1 > t_o$. In contrast, the evolution model introduced here is built to reproduce <u>only the data which is gathered</u> in a time ordered sequence of measurements. Obviously the event space then need not remain invariant with time since the type of process used for data gathering may change in the time sequence. Finally, Jauch, with no empirical grounds (as he himself claims) shows mathematically that his axioms have a representation in a separable, complex, Hilbert space which is identical with conventional quantum theory. Here, in an entirely different way from Jauch, it is shown that mathematically, the stochastic structure obtained from the stochastic axiomatic set also has a representation in a separable, complex, Hilbert space but that the stochastic evolution picture in $\mathcal{H}$ differs considerably from the quantum evolution picture in $\mathcal{H}$.

This comparison however, suggests a more general evolution picture in $\mathcal{H}$ which includes quantum evolution and stochastic evolution as special cases.

CHAPTER II

STOCHASTIC PROCESSES AND THEIR REPRESENTATIONS IN $\mathcal{H}$

This chapter consists of a paper submitted to the Journal of Mathematical Physics. The paper is intact with the exception of its original bibliography, which has been incorporated into the main bibliography.

Stochastic Processes and Their Representations in Hilbert Space*

F. G. Hall and R. E. Collins

The Department of Physics

University of Houston, Houston, Texas

ABSTRACT

Beginning with an intuitive consideration of sequences of measurements, we define a time-ordered event space representing the collection of all imaginable outcomes for measurement sequences.

We then postulate the generalized distributive relation on the event space and examine the class of measurements for which this relation can be experimentally validated. The generalized distributive relation is shown to lead to a σ-additive conditional probability on the event space and to a predictive and retrodictive formalism for stochastic processes.

We then show that this formalism has a predictive and a retrodictive representation in a separable Hilbert space $\mathcal{H}$, which has no counterpart in unitary quantum dynamics.

*Supported in part by a Frederick-Gardner Cottrell Grant-in-Aid from Research Corporation.

Stochastic Processes and Their Representations in Hilbert Space

F. G. Hall and R. E. Collins

The Department of Physics

University of Houston, Houston, Texas

Introduction

A recent series of papers[(69) - (74)] has developed the idea that much of the formal mathematical structure of physical theory can be deduced directly from the statistical nature of experimental data. The present paper presents that portion of these studies which bears directly on the evolution of irreversible physical processes.

We begin the study of the evolution of a system by insisting that, if we are to say we have observed the dynamic behavior of the system then we must monitor the system by a sequence of time-documented measurements $\{M_0 \rightarrow M_1 \rightarrow M_2 \rightarrow \cdots \rightarrow M_L\}$

With each of the measurements in the sequence, we associate in our mind a collection of possible outcomes. The collection being determined, of course, by the properties of the measuring apparatus. We may also associate a collection of possible outcomes with the entire measurement sequence.

We assume that all experimental data is statistical in nature, i.e., each outcome in the collection of possible outcomes is a random event. This assumption leads us to consider probability theory as a mathematical model for the kinematics of a system.

Since our imagination, at least for physical-experimental situations, seems to be conditioned by conventional logic we will assume that a σ-algebra describes the collection of imaginable outcomes (event space) of a measurement and that the frequency of outcomes can be described by a σ-additive measure of unit norm whose domain is the σ-algebra.

This approach does not differ from conventional approaches except, as we will show , in the definition of the σ-algebra of possible outcomes for measurement sequence and the conditional probability defined on this σ-algebra.

We will show that an equivalence relation must be defined on the σ-algebra for the measurement sequence in order to obtain the predictive and retrodictive random walk formulation for stochastic processes. This equivalence relation the <u>generalized distributive relation</u>, is empirical in nature, and is not deducible from the logical structure of the mathematics describing the measurement sequence.

We will then show that the predictive and retrodictive random walk formulations for the dynamics of a physical system, have representations in a separable Hilbert space $\mathcal{H}$, which differs considerably from the conventional quantum representation. It appears that the dynamical laws of conventional quantum theory are not the most general representation of the random walk formulation in $\mathcal{H}$.

The Measurement

For the sake of clarity and brevity in the following discussions we will begin by defining the measurement process.

We assume that an experimental situation may be completely described by a countable, functionally independent set of real valued functions $(\mathfrak{f}_1, \mathfrak{f}_2, \mathfrak{f}_3, \cdots)$, which may be arbitrarily partitioned into two functionally independent sets; one set, a K-tuple $(\mathfrak{f}_1', \mathfrak{f}_2', \cdots, \mathfrak{f}_K')$ describing the results of K simultaneous measurements and one set $(\mathfrak{f}_1'', \mathfrak{f}_2'', \mathfrak{f}_3'', \cdots)$ describing the environment conditioning the measurement. (This simply states that we must be satisfied to determine a finite number of system properties.)

We suppose that a measurement is always limited to some finite resolution and thus each of $\mathfrak{f}_1', \mathfrak{f}_2', \cdots, \mathfrak{f}_K'$ has a countable range $R_{\mathfrak{f}_1}, R_{\mathfrak{f}_2}, \cdots R_{\mathfrak{f}_K}$ respectively. Since each of $\mathfrak{f}_1', \mathfrak{f}_2', \cdots \mathfrak{f}_K'$ has a countable range, there exists a countable collection $\{(p_1, p_2, \cdots, p_K)\}_{p_1 \in R_{\mathfrak{f}_1}, p_2 \in R_{\mathfrak{f}_2}, \cdots p_K \in R_{\mathfrak{f}_K}}$ of K-tuples of real numbers (denoted $\{\hat{p}_k\}_{k=1,2,3,\cdots}$)

which contains all possible K-tuples of real numbers in the range of $(f_1', f_2', \cdots f_K')$.

Such assumptions lead us to make the following definitions:

A measurement of a system is an operation performed on a system which assigns a configuration $\hat{p}_k \in \{\hat{p}_k\}_{k=1,2,3,\ldots}$ to the system. For example, if we are interested in the pressure and volume of a system, then a configuration assigned to the system is a 2-tuple of real numbers (P_i, V_i) in the range of the functions P and V respectively.

The spectrum of a measurement is the collection of all possible configurations $\{\hat{p}_k\}_{k=1,2,3,\ldots}$.

We may now define the event space as the collection of all imaginable outcomes for a measurement. Let C_i denote the spectrum of a measurement process M_i. The event space $\{E_i(C_i)\}$ is the σ-algebra (75) of subsets of C_i. The motivations for such a choice for the event space are discussed in several texts (66), (67) ; arguments against such a choice have been discussed by Jauch (4). We will assume the σ-algebra to be a valid representation since as we will see there seem to be many physical situations for which the σ-algebra is appropriate and yields results not obtainable by conventional quantum theory.

Here we will refer to the members of $\{E_i(C_i)\}$ as <u>events</u> and define the probability for an event as a σ-additive measure P of unit norm on $\{E_i(C_i)\}$. Such a function has the following properties:

(i) if $E \in \{E_i(C_i)\}$; $0 \leq P(E) \leq 1$

(ii) $P(\phi)=0$; ϕ is the null event corresponding to the empty set in $\{E_i(C_i)\}$

(iii) $P(C_i)=1$; $C_i = \bigcup_{k_i=1}^{\infty}(\hat{P}_{k_i})$ (set union is interpreted as logical or)

(iiii) if $\{E_j\}_{j=1,2,3,\cdots}$ is a disjoint sequence of sets in $\{E_i(C_i)\}$ then $P(\bigcup_{j=1}^{\infty} E_j) = \sum_{j=1}^{\infty} P(E_j)$

There is a much wider agreement on the properties of P than the event space because of obvious physical interpretations. Axioms (i) and (ii) follow from the operational definition of probability. Axiom (iii) simply states that some value in the spectrum must be obtained as a result of M_i and axiom (iiii) is the mathematical statement of the familiar mutually exclusive rule in probability theory.

With this brief introduction we may now consider sequences of measurements operations.

<u>Sequences of Measurements</u>

We wish now to consider the time-documented sequence of measurements $\{M_0 \rightarrow M_1 \rightarrow M_2 \rightarrow \cdots \rightarrow M_L\}$. By time documented we mean M_i occurs at t_i and in case $i < j$, then $t_i < t_j$.

Since each M_i has an associated event space $\{E_i(C_i)\}$, the collection of all imaginable outcomes for the ordered sequence $\{M_i\}_{i=1,L}$ is a physically meaningful notion; thus we proceed to define the event space $\{E(C)\}$ for $\{M_i\}_{i=1,L}$. Let C denote the cartesian product space for the sequence of σ-algebras $\{\{E_0(C_0)\} \to \{E_1(C_1)\} \to \{E_2(C_2)\} \to \cdots \to \{E_L(C_L)\}\}$ i. e.,

(1) $$C = \{E_0(C_0)\} \otimes \{E_1(C_1)\} \otimes \cdots \otimes \{E_L(C_L)\}$$

$\{E(C)\}$ is the event space for $\{M_i\}_{i=1,L}$ means, E is an event in $\{E(C)\}$ only in case E is a subset of C.

That $\{E(C)\}$ contains the imaginable paths of outcomes for the measurement sequence, can be seen from recognizing that $\{E(C)\}$ contains the collection $\{S_m\}$ of simple paths $\{(\hat{P}_{k_0} \to \hat{P}_{k_1} \to \cdots \to \hat{P}_{k_L})\}$, (which are read as " $\hat{P}_{k_0}$ occurred then $\hat{P}_{k_1}$ occurred then, ...,then $\hat{P}_{k_L}$ occurred"), the compound paths such as $\{(\bigcup_{k_0=1}^{\ell_0} \hat{P}_{k_0} \to \bigcup_{k_1=1}^{\ell_1} \hat{P}_{k_1} \to \cdots \to \bigcup_{k_L=1}^{\ell_L} \hat{P}_{k_L})\}$ and the unions and intersections of the compound paths, for example, $(\hat{P}_{k_0} \to \bigcup_{k_1=1}^{\ell_1} \hat{P}_{k_1} \to \cdots \to \hat{P}_{k_L}) \cup (\hat{P}'_{k_0} \to \bigcap_{k_1=1}^{\ell_1} \hat{P}_{k_1} \to \cdots \to \bigcup_{k_L=1}^{\ell_L} \hat{P}_{k_L})$.

Notice that in contrast to the usual route in probability theory[66] we have not defined $\{E(C)\}$ to be the cartesian product space of the σ-algebras $\{E_0(C_0)\} \otimes \{E_1(C_1)\} \otimes \cdots \otimes \{E_L(C_L)\}$. Such a choice is not the most general one since it requires that set operations in $\{E(C)\}$ be defined in terms of set operations in $\{E_i(C_i)\}$. For our definition of $\{E(C)\}$ we see that C does not form a σ-algebra since it contains no unions of members of C. One can however, by choosing an equivalence relation between members of C and the compliment of C in $\{E(C)\}$, "induce" a σ-algebra on C. As we shall see in the next section, such a choice is empirical and seems necessary in order to produce the stochastic process.

PROBABILITY ON $\{E(C)\}$

We now turn our attention to probability functions on $\{E(C)\}$ and in particular conditional probabilities. We will assume in the following discussions that the environment for the sequence $\{M_i\}_{i=1,L}$ is fixed and described by $\hat{\mathfrak{g}}$. We will tacitly require that all probability functions on $\{E(C)\}$ be conditioned by $\hat{\mathfrak{g}}$.

The unit norm condition for P on $\{E_i(C_i)\}$ is given by

(2) $P(C_i) = 1$

which was interpreted as the probability for some event to occur during M_i. In view of this it would seem reasonable that for the sequence of measurements

(3) $P(C_0 \to C_1 \to \cdots \to C_L) = 1$

and for the simple paths $\{S_m\}_{m=1,2,\ldots}$ in $\{E(C)\}$

(4) $P(\bigcup_{m=1}^{\infty} S_m) = 1$

which is interpreted as <u>some simple path</u> must occur. In order for (3) and (4) to be true, we must postulate the following relation:

If $E_0 \in \{E_0(C_0)\}, E_1 \in \{E_1(C_1)\}, \ldots$, both $E_i' \,\&\, E_i'' \in \{E_i(C_i)\}, \cdots, E_L \in \{E_L(C_L)\}$

then

$$(5) \quad (E_0 \to E_1 \to \cdots \to E_i' \cup E_i'' \to \cdots \to E_L) = (E_0 \to E_1 \to \cdots \to E_i' \to \cdots \to E_L) \cup (E_0 \to E_1 \to \cdots \to E_i'' \to \cdots \to E_L)$$

We will also require the class of measurements that we are investigating to obey

$$(6) \quad P((\hat{p}_{k_0} \to \hat{p}_{k_1} \to \cdots \to \hat{p}_{k_i} \to \cdots \to \hat{p}_{k_L}) \cap (\hat{p}_{k_0} \to \hat{p}_{k_1} \to \cdots \to \hat{p}_{k_i'} \to \cdots \to \hat{p}_{k_L})) = \begin{cases} 0, & k_i' \neq k_i \\ P(\hat{p}_{k_0} \to \hat{p}_{k_1} \to \cdots \to \hat{p}_{k_i} \to \cdots \to \hat{p}_{k_L}), & k_i' = k_i \end{cases}$$

which simply states that only one configuration may be obtained as the result of a measurement. Statements (4), (5), and (6) must be <u>a postiori</u> in nature, not derivable from any <u>a priori</u> consideration. To clarify this point, consider the following measurement situation.

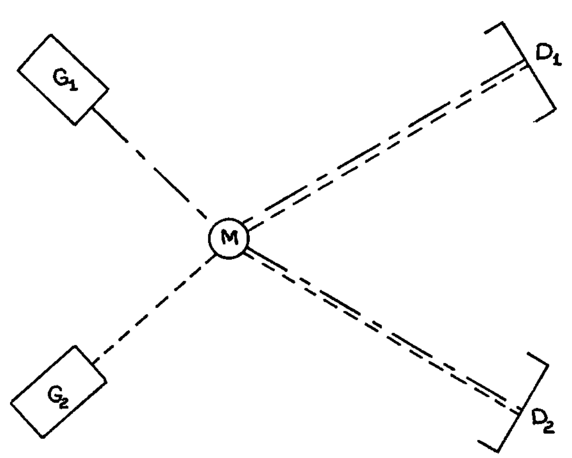

Fig. 1. Electron Gun Aparatus

The schematic in figure 1 describes two electron guns

G_1 and G_2 firing at a fixed target M. These electrons are scattered from M and detected at D_1 or D_2. The entire apparatus is placed in a cloud chamber so that the track of each electron can be monitored if desired. Such a device will serve to examine the generalized σ-algebra $\{E(C)\}$ and equations (5) and (6).

Let M_0 denote detection of the firing of the guns, M_1 denote detection of scattering from the target, M_2 denote detection at D_1 or D_2. We may now build $\{E(C)\}$ for the sequence $\{M_0 \to M_1 \to M_2\}$. The σ-algebras $\{E_0(C_0)\}$, $\{E_1(C_1)\}$, $\{E_2(C_2)\}$ are given by

$$\begin{aligned} &\{E_0(C_0)\} = \{(G_1), (G_2), (G_1 \cup G_2), (G_1 \cap G_2), \phi\} \\ (7) \quad &\{E_1(C_1)\} = \{M, \phi\} \\ &\{E_2(C_2)\} = \{(D_1), (D_2), (D_1 \cup D_2), (D_1 \cap D_2), \phi\} \end{aligned}$$

C as defined earlier is given by the cartesian product space $\{E_0(C_0)\} \otimes \{E_1(C_1)\} \otimes \{E_2(C_2)\}$ and $\{E(C)\}$, the event space for $\{M_0 \to M_1 \to M_2\}$ is the σ-algebra of subsets of C.

If we form $\{E(C)\}$ by the prescription given above we will see that $\{E(C)\}$ contains events such as $(G_1 \to M \to D_1)$, $(G_1 \to M \to D_2)$, $(G_2 \to M \to D_1)$, $(G_2 \to M \to D_2)$, the union of these $(G_1 \to M \to D_1) \cup (G_1 \to M \to D_2) \cup \cup (G_2 \to M \to D_1) \cup (G_2 \to M \to D_2)$, and $(G_1 \cup G_2 \to M \to D_1 \cup D_2)$. It is quite natural to interpret each of the events in the collection $\{(G_i \to M \to D_j)\}$ as the event for a certain <u>simple path</u> to be observed in the cloud chamber. The union of these simple paths would of course be interpreted as the event for one or another of the simple paths to occur. However, the event $(G_1 \cup G_2 \to M \to D_1 \cup D_2)$ would appear to have no simple interpretation as an event independent of the events for simple paths.*

We do see however that equations (3) and (4) can be satisfied for $\{E(C)\}$ only in case equation (5) is valid on $\{E(C)\}$. Equation (5) defines the event $(G_1 \cup G_2 \to M \to D_1 \cup D_2)$ in terms of the simple paths in $\{E(C)\}$ i.e., by equation (5)

$$
\begin{aligned}
(8)\quad (G_1 \cup G_2 \to M \to D_1 \cup D_2) &= (G_1 \to M \to D_1 \cup D_2) \cup (G_2 \to M \to D_1 \cup D_2) \\
&= (G_1 \to M \to D_1) \cup (G_1 \to M \to D_2) \cup (G_2 \to M \to D_1 \cup D_2) \\
&= (G_1 \to M \to D_1) \cup (G_1 \to M \to D_2) \cup (G_2 \to M \to D_1) \cup (G_2 \to M \to D_2)
\end{aligned}
$$

and therefore the requirement for $\{E(C)\}$ that P(c) is unity is consistent with equations (3) and (4).

We will call equation (5) the generalized destributive relation of the set operation, $\bullet$, with respect to the ordering operation $\to$. We see that this relation is a postiori in nature i.e., it is not required by the structure of $\{E(C)\}$. Only when we require equation (3) or equation (4) to be valid must we require the generalized distributive relation. The validity of equation (4) can be tested only if each of the simple paths are observable, thus the generalized distributive relation is ultimately a postiori in nature.

It should be evident that equation (5) "induces" a

---

* The event $(G_1 \cup G_2 \to M \to D_1 \cup D_2)$ seems a likely candidate for a "superposition" event defined by Jauch(4) if the σ-algebraic structure of $\{E(C)\}$ is modified. This investigation will constitute another publication.

σ-algebra on C and thus reduces $\{E(C)\}$ to the conventional σ-algebra of simple paths.* We will see however that the generalized notation obtained from generalizing $\{E(C)\}$ leads to some new notions in stochastic processes.

Let us return to the experiment of figure 1, assuming that the generalized distributive relation is valid for this experiment. We see in general that the probability for $G_1 \cap G_2$ and $D_1 \cap D_2$ is non-zero. However if we suppose that G_1 and G_2 never fire simultaneously and that D_1 and D_2 never detect simultaneously, then equation (6) is satisfied; thus we see that equation (6) is a requirement motivated by a postiori knowledge.

From equation (6), equation (8) and the additive property of P we see that

(9) $$P(G_1 \cup G_2 \to M \to D_1 \cup D_2) = P(G_1 \to M \to D_1) + P(G_2 \to M \to D_1) + P(G_1 \to M \to D_2) + P(G_2 \to M \to D_2)$$

from equation (9) we conclude that

(10) $$P(G_1 \cup G_2 \to M \to D_j) = P(G_1 \to M \to D_j) + P(G_2 \to M \to D_j)$$

thus we are provided with the definition

(11) $$P(D_j) = P(C_0 \to C_1 \to D_j) = \sum_{i=1,2} P(G_i \to C_1 \to D_j)$$

---

* If equations (3) and (4) are to be consistent with the requirement $P(c)=1$, then the generalized distributive relation must be valid for both union and intersection with respect to ordering.

for the <u>unconditional</u> <u>probability</u> to detect a particle at D_j this definition may be generalized to an L-term measurement sequence, i.e. for the L-term measurement sequence $\{M_0 \to M_1 \to M_2 \to \cdots \to M_L\}$, the <u>unconditional</u> <u>probability</u> for a result $\hat{r}_{k_i}$ during M_i, $0 \le i \le L$ is given by

$$(12)\ P(\hat{r}_{k_i}) \triangleq P(c_0 \to c_1 \to \cdots \to \hat{r}_{k_i} \to c_{i+1} \to \cdots \to c_L) = P\Big(\bigcup_{k_0=1}^{\infty} \hat{r}_{k_0} \to \bigcup_{k_1=1}^{\infty} \hat{r}_{k_1} \to \cdots \to \hat{r}_{k_i} \to \bigcup_{k_{i+1}=1}^{\infty} \hat{r}_{k_{i+1}} \to \cdots \bigcup_{k_L=1}^{\infty} \hat{r}_{k_L}\Big)$$

and using the generalized distributive relation, the disjointness of the simple paths in $\{E(C)\}$ and the σ-additivity of P, we see that equation (12) may be written in the more familiar form

$$(13)\ P(\hat{r}_{k_i}) = \sum_{k_0=1}^{\infty} \sum_{k_1=1}^{\infty} \cdots \sum_{k_{i-1}=1}^{\infty} \sum_{k_{i+1}=1}^{\infty} \cdots \sum_{k_L=1}^{\infty} P(\hat{r}_{k_0} \to \hat{r}_{k_1} \to \cdots \to \hat{r}_{k_i} \to \cdots \to \hat{r}_{k_L})$$

that is the unconditional probability for $\hat{r}_{k_i}$ is the sum of the probabilities of all simple paths containing $\hat{r}_{k_i}$.

With a suitable definition of <u>conditional</u> <u>probability,</u> equation (12) provides the general mathematical structure for a stochastic process. <u>Conditional</u> <u>probability</u> on $\{E(C)\}$ may be defined by analogy with the traditional definition. Conventionally the probability for "E_i^l is observed if E_i^k is observed" is given by

$$(14)\ P_c(E_i^l \mid E_i^k) \triangleq \frac{P(E_i^l \cap E_i^k)}{P(E_i^k)}$$

for the conventional event space, such a definition suffers from causal ambiguities; however for the time-ordered event space such ambiguities disappear.

In addition to the simultaneous events of equation (14) we wish to consider the conditional probability for the time-seperated events E_i^{ℓ}, E_j^{k}; $i \neq j$ By analogy with the conventional definition, equation (14), we define

$$(15) \quad P(E_j^{\ell} \mid E_i^{k}) = P(C_0 \to C_1 \to \cdots \to E_j^{\ell} \to C_{j+1} \to \cdots \to C_L \mid C_0 \to C_1 \to \cdots \to E_i^{k} \to C_{i+1} \to \cdots \to C_L)$$
$$= \frac{P((C_0 \to C_1 \to \cdots \to E_j^{\ell} \to C_{j+1} \to \cdots \to C_L) \cap (C_0 \to C_1 \to \cdots \to E_i^{k} \to C_{i+1} \to \cdots \to C_L))}{P(C_0 \to C_1 \to \cdots \to E_i^{k} \to C_{i+1} \to \cdots \to C_L)}$$

we see that this is well defined, independent of the magnitude of i with respect to j. Let us examine this definition for the case where i<j and the case where i = j.

When i<j, equation (15) becomes

$$(16) \quad P(E_j^{\ell} \mid E_i^{k}) = \frac{P((C_0 \to C_1 \to \cdots \to C_i \to \cdots \to E_j^{\ell} \to C_{j+1} \to \cdots \to C_L) \cap (C_0 \to C_1 \to \cdots \to E_i^{k} \to C_{i+1} \to \cdots \to C_j \to \cdots \to C_L))}{P(C_0 \to C_1 \to \cdots \to E_i^{k} \to C_{i+1} \to \cdots \to C_j \to \cdots \to C_L)}$$

thus $P(E_j^{\ell} \mid E_i^{k})$; $i<j$ has the obvious interpretation "the conditional probability for the event E_j^{ℓ} to occur at time t_j if E_i^{k} is known to have occurred at an earlier time t_i".

Now $P(E_i^{k} \mid E_j^{\ell})$ is also well defined by equation (15). Let us examine the nature of this conditional probability. Equation (15) yields

$$(17) \quad P(E_i^{k} \mid E_j^{\ell}) = \frac{P((C_0 \to C_1 \to \cdots \to E_i^{k} \to C_{i+1} \to \cdots \to C_j \to \cdots \to C_L) \cap (C_0 \to C_1 \to \cdots \to E_j^{\ell} \to C_{j+1} \to \cdots \to C_L))}{P(C_0 \to C_1 \to \cdots \to E_j^{\ell} \to C_{j+1} \to \cdots \to C_L)}$$

Which in view of the nature of the sequenced event space can only be interpreted as "the conditional probability for the event E_i^{k} to occur at t_i if E_j^{ℓ} is known to have occurred at a later time t_i".

In case i = j, we see from equations (16) and (17) that our definition of equation (15) is the analog of the conventional definition given by equation (14).

It is our claim, and we discuss this more fully in the sections to follow, that the sequenced formalism clearly distinguishes and defines both "types" of conditional probabilities as given in equations (16) and (17). We will demonstrate that the conditional probability of equation (17) can be the "inverse: or "time-reversed" form of the conditional probability of equation (16), only in case the system follows a deterministic path through the measurement sequence. We also will see that $P(E_i^k \mid E_j^l); i<j$ is definable only because of the <u>a postiori</u> nature of the data from a measurement sequence.

We will postpone this discussion until we have more fully developed the stochastic equations describing the measurement sequence.

<u>THE RANDOM WALK</u>

Now that we have developed the definitions for conditional probability and unconditional probability, we are able to consider the measurement sequence as a generalized random walk problem. We will, in this section, develop the random walk equation which determines the probability for the statement, "the simple event $\hat{p}_{k_i}$ is the outcome of M_i, regardless of the outcomes of the rest of the measurements in the sequence", in terms of the conditional probabilities of $\hat{p}_{k_i}$ with respect to the outcomes of other measurements in the sequence.

We accomplish this by beginning with the definition in equation (12) of the unconditional probability. From this we may write

$$(18)\quad P(\hat{p}_{k_j}) = \sum_{k_i} P(c_0 \to c_1 \to \cdots \to \hat{p}_{k_i} \to c_{i+1} \to \cdots \hat{p}_{k_j} \to c_{j+1} \to \cdots \to c_L) \quad ; \quad j > i$$

Since the conditional probability is defined for each member of $\{E(c)\}$ we may write, from equation (15)

$$(19)\quad P(\hat{p}_{k_j} | \hat{p}_{k_i}) = \frac{P((c_0 \to c_1 \to \cdots \to \hat{p}_{k_j} \to c_{j+1} \to \cdots \to c_L) \cap (c_0 \to c_1 \to \cdots \to \hat{p}_{k_i} \to c_{i+1} \to \cdots \to c_L))}{P(c_0 \to c_1 \to \cdots \to \hat{p}_{k_i} \to c_{i+1} \to \cdots \to c_L)}$$

Using the generalized distributive relation, the numerator of equation (19) may be reduced so that equation (19) becomes

$$(20)\quad P(\hat{p}_{k_j} | \hat{p}_{k_i}) = \frac{P(c_0 \to c_1 \to \cdots \to \hat{p}_{k_i} \to c_{i+1} \to \cdots \to p_{k_j} \to c_{j+1} \to \cdots \to c_L)}{P(c_0 \to c_1 \to \cdots \to \hat{p}_{k_i} \to c_{i+1} \to \cdots \to c_L)}$$

Since the numerator of equation (20) is exactly the term inside the sum of equation (18), we may employ equation (20) to write equation (18) as,

$$(21)\quad P(\hat{p}_{k_j}) = \sum_{k_i} P(\hat{p}_{k_j} | \hat{p}_{k_i}) \, P(c_0 \to c_1 \to \cdots \to \hat{p}_{k_i} \to c_{i+1} \to \cdots \to c_L)$$

Before we "expose" this as the random walk equation, let us consider the unconditional probability for $\hat{p}_{k_i}$. From equation (12) we may write

$$(22)\quad P(\hat{p}_{k_i}) = \sum_{k_j} P(c_0 \to c_1 \to \cdots \hat{p}_{k_i} \to c_{i+1} \to \cdots \to \hat{p}_{k_j} \to c_{j+1} \to \cdots \to c_L) \quad ; \quad j > i$$

and as we saw in the development of equation (20), we may write from equation (15)

$$(23)\quad P(\hat{p}_{k_i} | \hat{p}_{k_j}) = \frac{P(c_0 \to c_1 \to \cdots \to \hat{p}_{k_i} \to \cdots \to \hat{p}_{k_j} \to c_{j+1} \to \cdots \to c_L)}{P(c_0 \to c_1 \to \cdots \to \hat{p}_{k_i} \to c_{i+1} \to \cdots \to c_L)} \quad ; \quad j > i$$

which allows us to write equation (22) as

$$(24)\quad P(\hat{p}_{k_i}) = \sum_{k_j} P(\hat{p}_{k_i} | \hat{p}_{k_j}) P(c_0 \to c_1 \to \cdots \to \hat{p}_{k_i} \to c_{i+1} \to \cdots \to c_L) \quad ; \; j > i$$

and in the simplified notation provided by the definition of unconditional probability, equation (21) may be written

$$(25)\quad P(\hat{p}_{k_j}) = \sum_{k_i} P(\hat{p}_{k_j} | \hat{p}_{k_i}) P(\hat{p}_{k_i}) \quad ; \; j > i$$

and equation (24) may be written

$$(26)\quad P(\hat{p}_{k_i}) = \sum_{k_j} P(\hat{p}_{k_i} | \hat{p}_{k_j}) P(\hat{p}_{k_j}) \quad ; \; j > i$$

which we will name the <u>predictive random walk equation</u> and the <u>retrodictive random walk equation</u> respectively. This is an obvious choice of terminology since equation (25) calculates probability distributions for events occurring at t_j in terms of the probability distributions for events occurring at an <u>earlier</u> time t_i and equation (26) calculates probability distributions for events occurring at t_i in terms of the probability distributions for events occurring at a <u>later</u> time t_j.

We may go a step futher in adapting our notation to the standard notation by defining the <u>predictive transition probability</u>

$$(27)\quad T_{k_j k_i} \triangleq P(\hat{p}_{k_j} | \hat{p}_{k_i}) = P(\hat{p}_{k_j} | c_0 \to c_1 \to \cdots \to \hat{p}_{k_i} \to c_{i+1} \to \cdots \to c_j \to \cdots \to c_L)$$

and the <u>retrodictive transition probability</u>

$$(28)\quad T'_{k_i k_j} \triangleq P(\hat{p}_{k_i} | \hat{p}_{k_j}) = P(\hat{p}_{k_i} | c_0 \to c_1 \to \cdots \to c_i \to \cdots \to \hat{p}_{k_j} \to c_{j+1} \to \cdots \to c_L)$$

so that the predictive random walk equation becomes

$$(29)\quad P(\hat{p}_{k_j}) = \sum_{k_i} T_{k_j k_i} P(\hat{p}_{k_i})$$

and the retrodictive random walk equation becomes

$$(30)\quad P(\hat{p}_{k_i}) = \sum_{k_j} T'_{k_i k_j} P(\hat{p}_{k_j})$$

We see from the preceding analysis that equation (29) is a generalized form of the conventional Markoff random walk equation. It is generalized in the sense that $T_{k_j k_i}$ is not Markoffian.

We also see that equation (30) is not at all conventional since it implies that if we know the probability set $\{P(\hat{p}_{k_j})\}$ at t_j and the set of retrodictive transition probabilities $\{T'_{k_j k_i}\}$ then we may calculate the probability set $\{P(\hat{p}_{k_i})\}$ even when $t_i < t_j$. Such a result is completely consistent with the <u>a postiori</u> nature of data. We will discuss this property of data in the conclusion section of this paper.

<u>PROPERTIES OF THE STOCHASTIC PROCESS</u>

In this section we will examine the temporal behavior of the stochastic process in terms of prediction and retrodiction. This examination will clarify the relationship between the predictive dynamics and the retrodictive dynamics and will provide a foundation for our examination of the $\mathcal{H}$ representation of stochastic processes.

Each measurement pair $M_i, M_j; i<j$ in the measurement sequence $\{M_0 \to M_1 \to \cdots \to M_L\}$ defines a collection of predictive transition probabilities, $\{T_{k_j k_i}\}$ a collection of retrodictive transition probabilities $\{T'_{k_i k_j}\}$ and a collection of _simultaneous conditional probabilities_ $\{T_{k_i k_i'}\}$.

$T(j,i)$ is the _predictive transition matrix_ for the measurement pair $M_i, M_j; i<j$ means, $T(j,i)$ is a matrix such that $T_{k_j k_i}$ is the k_jth row and the k_ith column element of $T(j,i)$.

$T'(i,j)$ is the _retrodictive transition matrix_ for the measurement pair $M_i, M_j; i<j$ means, $T'(i,j)$ is a matrix such that $T'_{k_i k_j}$ is the k_ith row and the k_jth column element of $T'(i,j)$.

$T(i,i)$ is the _simultaneous conditional probability matrix_ for the measurement M_i means, $T(i,i)$ is a matrix such that $T_{k_i k_i'}$ is the k_ith row and the k_i'th column element of $T(i,i)$.

We see then that an L-term measurement sequence defines $\frac{L(L+1)}{2}$ measurement pairs $M_i, M_j; i<j$ and thus defines $\frac{L(L+1)}{2}$ retrodictive transition matrices, $\frac{L(L+1)}{2}$ predictive transition matrices and L simultaneous conditional probability matrices.

Let $\{T(j,i)\}$ denote the collection of predictive transition matrices, $\{T'(i,j)\}$ denote the collection of retrodictive transition matrices and $\{T(i,i)\}$ denote the collection of simultaneous conditional probability matrices. Let

$\{\tau(j,i)\}$ denote the collection of members of $\{T(j,i)\}$, $\{T'(i,j)\}$ and $\{T(i,i)\}$.

We will now investigate the conditions, if any, for the collections $\{T(j,i)\}$, $\{T'(i,j)\}$ and $\{\tau(j,i)\}$ to form either groups or semi-groups with respect to matrix multiplication.

First, we note that equation (6) requires that the collection $\{T(i,i)\}$ be the collection of unit matrices $\{I_i\}$. In general, each member of $\{I_i\}$ is of a different dimension, depending on the spectrum of M_i . In this investigation, we will assume that each spectrum is countably infinite, and thus each member of $\{I_i\}$ will be of the same dimension.

It is not difficult to see that matrix multiplication between certain members of $\{T(j,i)\}$ produces a transition matrix in $\{T(j,i)\}$. To show this we simply use equation (29) to write the following equation set;

$$P(\hat{p}_{k_1}) = \sum_{k_0} T_{k_1 k_0} P(\hat{p}_{k_0})$$

$$P(\hat{p}_{k_2}) = \sum_{k_1} T_{k_2 k_1} P(\hat{p}_{k_1}) = \sum_{k_0} T_{k_2 k_0} P(\hat{p}_{k_0})$$

(31) $$\vdots$$

$$P(\hat{p}_{k_L}) = \sum_{k_{L-1}} T_{k_L k_{L-1}} P(\hat{p}_{k_{L-1}}) = \sum_{k_{L-2}} T_{k_L k_{L-2}} P(\hat{p}_{k_{L-2}}) = \cdots = \sum_{k_1} T_{k_L k_1} P(\hat{p}_{k_1}) = \sum_{k_0} T_{k_L k_0} P(\hat{p}_{k_0})$$

Substituting the first equation of the set (31) into the second equation in the set we obtain

(32) $$P(\hat{p}_{k_2}) = \sum_{k_0} P(\hat{p}_{k_0}) \sum_{k_1} T_{k_2 k_1} T_{k_1 k_0} = \sum_{k_0} P(\hat{p}_{k_0}) T_{k_2 k_0}$$

which implies by comparison the Chapman-Kolmogorov relation

(33) $$T_{k_2 k_0} = \sum_{k_1} T_{k_2 k_1} T_{k_1 k_0}$$

This procedure may be repeated for the entire set (31) to obtain

(34) $$T_{k_L k_0} = \sum_{k_{L-1}} \sum_{k_{L-2}} \cdots \sum_{k_1} T_{k_L k_{L-1}} T_{k_{L-1} k_{L-2}} \cdots T_{k_1 k_0}$$

Since $T_{k_L k_0}$ is the $k_L th$ row and $k_0 th$ column of $T(L,0)$ we see that equation (34) provides a multiplication theorem for transition matrices

(35) $$T(L,0) = T(L,L-1)\,T(L-1,L-2)\cdots T(1,0)$$

From the retrodictive equation (30) we may write an equation set similar to the equation set (31) and derive the multiplication theorem for the retrodictive transition matrices

(36) $$T'(0,L) = T'(0,1)\,T'(1,2)\cdots T'(L-1,L)$$

In addition, equations (29) and (30) can be combined for various integers i and j so that multiplication is defined between members of $\{T(j,i)\}$ and $\{T'(i,j)\}$. For example consider the integers q, s, t such that $0 \le q < s < t \le L$. Equations (29) and (30) then define the products

(37)
$$\begin{aligned}
T'(q,s)\,T'(s,t) &= T'(q,t)\\
T'(q,t)\,T(t,s) &= T'(q,s)\\
T(s,q)\,T'(q,t) &= T'(s,t)\\
T(s,t)\,T(t,q) &= T(s,q)\\
T(t,q)\,T'(q,s) &= T(t,s)
\end{aligned}$$

However we also obtain from this process

$$(38)\quad \begin{aligned} P(\hat{p}_{k_\Delta}) &= \sum_{k'_\Delta} P(\hat{p}_{k'_\Delta}) \sum_{k'_t} T_{k_\Delta k'_t} T'_{k'_t k'_\Delta} = \sum_{k'_\Delta} M_{k_\Delta k'_\Delta} P(\hat{p}_{k'_\Delta}) \\ P(\hat{p}_{k_t}) &= \sum_{k'_t} P(\hat{p}_{k'_t}) \sum_{k'_\Delta} T_{k_t k'_\Delta} T'_{k'_\Delta k'_t} = \sum_{k'_t} M_{k_t k'_t} P(\hat{p}_{k'_t}) \end{aligned}$$

Equations (38) define the matrix products

$$(39)\quad \begin{aligned} M(\Delta,\Delta) &= T'(\Delta,t)\,T(t,\Delta) \\ M(t,t) &= T(t,\Delta)\,T'(\Delta,t) \end{aligned}$$

The immediate inclination is to identify the collection $\{M(i,i)\}_{i=0,L}$ as the collection $\{T(i,i)\}$ of simultaneous conditional probability matrices. However such an identification would require that

$$(40)\quad \begin{aligned} M(\Delta,\Delta) &= I_\Delta = T'(\Delta,t)\,T(t,\Delta) \\ M(t,t) &= I_t = T(t,\Delta)\,T'(\Delta,t) \end{aligned}$$

and if the dimension of I_Δ is the dimension of I_t then equations (40) imply that

$$(41)\quad T'(\Delta,t) = [T(t,\Delta)]^{-1}$$

Wu[76] has shown however that since each member of $T(t,\Delta)$ is positive, then its inverse transition matrix $[T(t,\Delta)]^{-1}$ must have at least one <u>negative</u> member, unless of course $T(t,\Delta)$ has only one non-zero member. Since $T'(\Delta,t)$ is itself a transition matrix, equation (41) and thus equations (40) can be satisfied only in case $T(t,\Delta)$ has only one non-zero member.* Thus we see that in general, $M(\Delta,\Delta)$ cannot be

---

* In this case $T(t,\Delta)$ would describe a <u>deterministic process</u>.

identified as the matrix $T(s,s)$ of simultaneous conditional probabilities.

With multiplication defined in $\{T(j,i)\}$ and $\{T'(i,j)\}$ we may proceed to examine these collections as groups or semi-groups.

Since $\{T(j,i)\}$ can form a group only in case each member $T(r,q) \in \{T(j,i)\}$ has an inverse $[T(r,q)]^{-1} \in \{T(j,i)\}$, we see from the preceding arguments that neither $\{T(j,i)\}$ nor $\{T'(i,j)\}$ can form a group.

We also see that the collection $\{\Upsilon(j,i)\}$ cannot form a semi group since the product M given by equation (40) is not a member of $\{\Upsilon(j,i)\}$ unless for each positive integer i such that $i \leq L$, ,

(42) $M(i,i) = T(i,i) = \mathbf{I}$

which as we argued is possible only for a deterministic system.

Let us now examine the conditions for $\{T(j,i)\}$ and $\{T'(i,j)\}$ to form semi-groups. Suppose $\{T(j,i)\}$ forms a semi-group. In this case, closed associative multiplication must be defined between each pair in $\{T(j,i)\}$. We see from equation (35) that left multiplication of $T(r,s)$ by $T(q,r)$ yields $T(q,s)$ thus the product $T(q,r)T(r,s)$ is a member of $\{T(j,i)\}$ and the multiplication is closed. Since this multiplication is matrix multiplication it is also associative.

We see however that multiplication of the two matrices

$T(p,q)T(r,s)$ produces a transition matrix in $\{T(j,i)\}$ only in case $q=r$ or $p=s$. This fact motivates us to define the following notion: Two transition matrices $T(p,q)$ and $T(r,s)$ are <u>adjacent</u> means, either $p=s$ or $q=r$. It is clear then, that if each pair of matrices in $\{T(j,i)\}$ can be made adjacent, then $\{T(j,i)\}$ will form a semi-group.

If each member of $\{T(j,i)\}$ has the property that

(43) $T(l,k) = T(x,y)$ in case $|l-k| = |x-y|$

then any two matrices $T(p,q) \in \{T(j,i)\}$ and $T(r,s) \in \{T(j,i)\}$ can be made adjacent simply by relabeling $T(r,s)$ as $T(q,s')$ where $T(q,s') \in \{T(j,i)\}$ and $|q-s'|$ equals $|r-s|$ so that

(44) $T(p,q)T(r,s) = T(p,q)T(q,s') = T(p,s')$

Thus the collection $\{T(j,i)\}$ can form a semi-group in case the matrices in the collection are all conformable and equation (43) is satisfied for each matrix in the collection. The same argument applies for the collection $\{T'(i,j)\}$. If in addition we include the collection $\{T(i,i)\}$ in $\{T(j,i)\}$ we see that $\{T(j,i)\}$ can form a <u>monoid</u> semi-group. The same argument applies for $\{T'(i,j)\}$.

We see then that the predictive collection $\{T(j,i)\}$ and the retrodictive collection $\{T'(i,j)\}$ can each form a group only in case each member in $\{T(j,i)\}$ and each member in $\{T'(i,j)\}$ describes a deterministic system. However each of $\{T(j,i)\}$ and $\{T'(i,j)\}$ can form a semi-group in case each member of

$\{T(j,i)\}$ and each member of $\{T'(i,j)\}$ satisfies equation (43). Physically, equation (43) restricts the transition probabilities to be a function only of the number of measurements between M_i and M_j; this requires that each $T(j,i)$ be a function only of the relative time difference between M_i and M_j. Thus equation (43) is analogous to the quantum requirement that $\mathbf{U}(t_2,t_1)$ be a function only of $|t_2-t_1|$ if $\mathbf{U}$ is to be a member of the unitary group.

We also demonstrated that a retrodictive transition matrix is not the inverse of the corresponding predictive transition matrix. However the equations resulting from the sequenced event space clearly define and distinguish between retrodiction and prediction and show that one may always predict or retrodict the stochastic process.

PROBABILITY FUNCTIONS IN ℓ^2

In this section we will demonstrate that probabilities for simple paths in $\{E(C)\}$ may be represented as products of complex functions in l^2, the space of square summable sequences. From the isomorphism of l^2 to a separable Hilbert space $\mathcal{H}$ we deduce the existance of a continuous linear operator in $\mathcal{H}$ which corresponds to the transition probability of equation (27). Hilbert space representations for probabilities of simple paths in $\{E(C)\}$ are shown to be possible because of the positive definite, unit norm and σ-additive properties of P.

Since $P(\hat{p}_{k_j})$ is positive definite, there exists a complex function α_{k_j} such that for each $\hat{p}_{k_j}$

(45) $P(\hat{p}_{k_j}) = \alpha^*_{k_j}\,\alpha_{k_j}$

and the phase of α_{k_j} is <u>arbitrary</u>.

Using the unit norm property and the generalized distributive relation we see that

(46) $\sum\limits_{k_j} P(\hat{p}_{k_j}) = 1 = \sum\limits_{k_j=1}^{\infty} \alpha^*_{k_j}\alpha_{k_j}$

Thus the sequence $\{\alpha_{k_j}\}_{k_j=1,2,\cdots}$ is square summable and is a member of 1^2. If we now consider the vector $|\alpha(j)\rangle$ defined by

(47) $|\alpha(j)\rangle = \sum\limits_{k_j=1}^{\infty} C_{k_j}|k_j\rangle$

where $\{|k_j\rangle\}_{k_j=1,2,\ldots}$ is an orthonormal basis for a separable Hilbert space $\mathcal{H}$, then $|\alpha(j)\rangle \in \mathcal{H}$ only in case $\{C_{k_j}\}$ is a square summable sequence.(77) Thus if we define C_{k_j} as

(48) $C_{k_j} \triangleq \alpha_{k_j}\,(\langle\alpha(j)|\alpha(j)\rangle)^{1/2}$

we see that $\{C_{k_j}\}$ is square summable therefore $|\alpha(j)\rangle$ defined by

(49) $|\alpha(j)\rangle = (\langle\alpha(j)|\alpha(j)\rangle)^{1/2} \sum\limits_{k_j} \alpha_{k_j}|k_j\rangle$

is a member of $\mathcal{H}$. Thus we see that for each square summable sequence $\{\alpha_{k_j}\}$ there exists a vector $|\alpha(j)\rangle \in \mathcal{H}$ such that each member of $\{\alpha_{k_j}\}$ has a representation in H given by

(50) $\alpha_{k_j} = \dfrac{\langle k_j|\alpha(j)\rangle}{(\langle\alpha(j)|\alpha(j)\rangle)^{1/2}}$

Thus we have established an $\mathcal{H}$ representation for each member in the collection $\{\alpha_{k_j}\}$ and therefore for $\{P(\hat{P}_{k_j})\}$

Now let us examine the transition probability $T_{k_j k_i}$. Since $T_{k_j k_i}$ is positive there exists a complex function for each k_j and k_i such that

(51) $T_{k_j k_i} = K^*_{k_j k_i} K_{k_j k_i}$

and since $\{T_{k_j k_i}\}$ is singly stochastic the sequence $\{K_{k_j k_i}\}_{k_j = 1,2,\ldots}$ is square summable for each k_i. Therefore there exists a countable orthonormal basis $\{|k_j\rangle\}$ and a member $|Q_{k_i}\rangle \in \mathcal{H}$ such that for each k_i

(52) $K_{k_j k_i} = \dfrac{\langle k_j | Q_{k_i}\rangle}{(\langle Q_{k_i} | Q_{k_i}\rangle)^{1/2}}$

We see from (51) and (52) that for a given basis $\{|k_j\rangle\}$, each member of the countable collection $\{|Q_{k_i}\rangle\}$ is determined only to within a phase.

$K_{k_j k_i}$ may be written in a different form since we may associate with the collection $\{|Q_{k_i}\rangle\}$ an orthonormal basis $\{|k_i\rangle\}$ in $\mathcal{H}$ by an operator $\mathbf{K}(j,i)$ mapping $\mathcal{H}_i$ onto $\mathcal{H}_i$, i.e. for each k_i

(53) $|Q_{k_i}\rangle = \mathbf{K}(j,i)\,|k_i\rangle$

thus we may write (52) as

(54) $K_{k_j k_i} = \dfrac{\langle k_j | \mathbf{K}(j,i) | k_i\rangle}{(\langle k_i | \mathbf{K}^+ \mathbf{K} | k_i\rangle)^{1/2}}$

with these representations for $T_{k_j k_i}$ and P we may write the

$\mathcal{H}$ representation for the predictive random walk equation as

$$(55)\quad \frac{\langle k_j|\alpha(j)\rangle\langle\alpha(j)|k_j\rangle}{\langle\alpha(j)|\alpha(j)\rangle} = \sum_{k_i} \frac{\langle k_j|\mathbf{K}(j,i)|k_i\rangle\langle k_i|\mathbf{K}^{+}(j,i)|k_j\rangle}{\langle k_i|\mathbf{K}^{+}(j,i)\mathbf{K}(j,i)|k_i\rangle} \frac{\langle k_i|\alpha(i)\rangle\langle\alpha(i)|k_i\rangle}{\langle\alpha(i)|\alpha(i)\rangle}$$

clearly, from this development, an $\mathcal{H}$ representation can be generated for the retrodictive equation (30). This equation would be given by

$$(56)\quad \frac{\langle k_i|\alpha(i)\rangle\langle\alpha(i)|k_i\rangle}{\langle\alpha(i)|\alpha(i)\rangle} = \sum_{k_j} \frac{\langle k_i|\mathbf{K}'(i,j)|k_j\rangle\langle k_j|\mathbf{K}'^{+}(i,j)|k_i\rangle}{\langle k_j|\mathbf{K}'^{+}(i,j)\mathbf{K}(i,j)|k_j\rangle} \frac{\langle k_j|\alpha(j)\rangle\langle\alpha(j)|k_j\rangle}{\langle\alpha(j)|\alpha(j)\rangle}$$

where the operator $\mathbf{K}'(i,j)$ is constructed so that

$$(57)\quad T'_{k_i k_j} = \frac{\langle k_i|Q_{k_j}\rangle\langle Q_{k_j}|k_i\rangle}{\langle Q_{k_j}|Q_{k_j}\rangle} = \frac{\langle k_i|\mathbf{K}'(i,j)|k_j\rangle\langle k_j|\mathbf{K}'^{+}(i,j)|k_i\rangle}{\langle k_j|\mathbf{K}'(i,j)\,\mathbf{K}'^{+}(i,j)|k_j\rangle}$$

the retrodictive transition probability is reproduced. Thus we have established $\mathcal{H}$ representations for both the retrodictive and predictive random walk equations.

RANDOM WALK AND TIME EVOLUTION IN $\mathcal{H}$

Now that we have established an $\mathcal{H}$ representation for the random walk equation, we may employ a phase choice theorem established in a previous paper[1] to establish another $\mathcal{H}$ representation for the random walk equation which will allow us to compare the dynamics of stochastic and quantum theory.

This theorem demonstrates the existance of choices for the phases of the sequence of products $\{K_{k_j k_i}\alpha_{k_i}\}_{k_i=1,2,\ldots}$ such that equation (55) factors to yield (see appendix A for this theorem and its connection here).

$$(58)\quad \frac{\langle k_j|\alpha(j)\rangle}{(\langle\alpha(j)|\alpha(j)\rangle)^{1/2}} = \sum_{k_i} \frac{\langle k_j|\mathbf{K}(j,i)|k_i\rangle\langle k_i|\alpha(i)\rangle}{\langle k_i|\mathbf{K}^{+}\mathbf{K}|k_i\rangle(\langle\alpha(i)|\alpha(i)\rangle)^{1/2}}$$

Equation (58) provides a very simple representation in $\mathcal{H}$ for the dynamics of classical probability theory, i.e. equation (58) may be written

$$(59) \quad |\alpha'(j)\rangle = \sum_{k_i} \mathbf{K}(j,i) \frac{|k_i\rangle\langle k_i|}{a^j_{k_i}} |\alpha'(i)\rangle$$

where

$$(60) \quad a^j_{k_i} \triangleq (\langle k_i | \mathbf{K}^\dagger(j,i)\mathbf{K}(j,i)|k_i\rangle)^{1/2} \quad , \quad |\alpha'(j)\rangle = \frac{|\alpha(j)\rangle}{(\langle\alpha(j)|\alpha(j)\rangle)^{1/2}}$$

and (59) can be futher simplified by defining the operator $\mathbf{S}(j,i)$ as

$$(61) \quad \mathbf{S}(j,i) \triangleq \sum_{k_i} \mathbf{K}(j,i) \frac{|k_i\rangle\langle k_i|}{a^j_{k_i}}$$

so that equation (59) becomes

$$(62) \quad |\alpha'(j)\rangle = \mathbf{S}(j,i)\,|\alpha(i)\rangle$$

and we see that in a similar manner we may construct this representation for the retrodictive case which is

$$(63) \quad |\alpha'(i)\rangle = \mathbf{S}'(i,j)\,|\alpha'(j)\rangle$$

Equation (62) is similar in form to the evolution equation of quantum theory, although as we will see in the discussion to follow, the stochastic operator $\mathbf{S}(j,i)$ differs strikingly from the quantum evolution operator $\mathbf{U}(t_j,t_i)$. In addition to equation (62) we have equation (63), the retrodictive evolution equation. No such formalism appears in conventional quantum theory.

Thus we see that for the measurement sequence $\{M_0 \rightarrow M_1 \rightarrow \cdots \rightarrow M_L\}$, there exists a collection $\{\mathbf{S}(j,i)\}$ of $\frac{L(L+1)}{2}$ predictive stochastic operators and a collection $\{\mathbf{S}'(i,j)\}$ of $\frac{L(L+1)}{2}$ retrodictive stochastic operators. Let us now examine the properties of $\{\mathbf{S}'(i,j)\}$ and $\{\mathbf{S}(j,i)\}$.

First we see from equation (61) that

$$(64) \quad \langle k_j | \mathbf{S}(j,i) | k_i \rangle = \frac{\langle k_j | \mathbf{K}(j,i) | k_i \rangle}{(\langle k_i | \mathbf{K}^{+}(j,i) \mathbf{K}(j,i) | k_i \rangle)^{1/2}}$$

If we multiply equation (64) by its complex conjugate and sum over all $|k_j\rangle$ then we obtain the isometric property for $\mathbf{S}$

$$(65) \quad \mathbf{S}^{+}(j,i)\,\mathbf{S}(j,i) = \mathbf{I}$$

However multiplying equation (64) from the right by its complex conjugate, we see that $\mathbf{S}$ is unitary ($\mathbf{S}^{+}\mathbf{S} = \mathbf{S}\mathbf{S}^{+} = \mathbf{I}$) only in case $\mathbf{K}$ is unitary. Thus we see that $\mathbf{S}$ is automatically isometric by construction, but can be unitary only if $\mathbf{K}$ is unitary. This relationship of $\mathbf{S}$ to $\mathbf{K}$, as we will see, has important physical implications. In order to see these implications we must explore the properties of the collections $\{\mathbf{S}(j,i)\}$ and $\{\mathbf{S}'(i,j)\}$.

The approach to the examination of $\{\mathbf{S}(j,i)\}$ and $\{\mathbf{S}'(i,j)\}$ will be almost identical to our earlier approach when we examined the collections $\{\mathbf{T}(j,i)\}$ and $\{\mathbf{T}'(i,j)\}$ and not suprisingly, the results will be almost identical. The complex analogs to equations (31) are by the phase choice theorem

(66) $$\alpha_{k_1} = \sum_{k_0} \langle k_1 | S(1,0) | k_0 \rangle \alpha_{k_0}$$

$$\alpha_{k_2} = \sum_{k_1} \langle k_2 | S(2,1) | k_1 \rangle \alpha_{k_1} = \sum_{k_0} \langle k_2 | S(2,0) | k_0 \rangle \alpha_{k_0}$$

$$\vdots$$

$$\alpha_{k_L} = \sum_{k_{L-1}} \langle k_L | S(L,L-1) | k_{L-1} \rangle \alpha_{k_{L-1}} = \sum_{k_{L-2}} \langle k_L | S(L,L-2) | k_{L-2} \rangle \alpha_{k_{L-2}} = \cdots = \sum_{k_0} \langle k_L | S(L,0) | k_0 \rangle \alpha_{k_0}$$

Substituting the first of equations (66) into the second equation in the set and comparing we obtain

(67) $$\sum_{k_0} \alpha_{k_0} \sum_{k_1} \langle k_2 | S(2,1) | k_1 \rangle \langle k_1 | S(1,0) | k_0 \rangle = \sum_{k_0} \langle k_2 | S(2,0) | k_0 \rangle \alpha_{k_0}$$

so that we obtain the H-representation of equation (33)

(68) $$\langle k_2 | S(2,0) | k_0 \rangle = \sum_{k_1} \langle k_2 | S(2,1) | k_1 \rangle \langle k_1 | S(1,0) | k_0 \rangle$$

which implies the multiplication theorem

(69) $$S(2,0) = S(2,1)\, S(1,0)$$

This procedure may be repeated for the entire set (66) to obtain the general multiplication theorem for the stochastic operator set $\{S(j,i)\}$ i.e.,

(70) $$S(L,0) = S(L,L-1)\, S(L-1,L-2) \cdots S(2,1)\, S(1,0)$$

and similarly for the retrodictive set

(71) $$S'(0,L) = S'(0,1)\, S'(1,2) \cdots S'(L-2,L-1)\, S'(L-1,L)$$

In addition we have the set $\{S(i,i)\}$ which by equation (64) and the definition of $\{T(i,i)\}$ is given by

(72) $\{\mathbf{S}(i,i)\} = \{\mathbf{I}_i\}$

Suppose $\{\mathbf{S}(j,i)\}$ forms a subset of a group. It must be true then that each member of $\{\mathbf{S}(j,i)\}$ has an inverse. We show in appendix B, that in case $\mathbf{S}^{-1}(j,i)$ exists, then

(73) $P(\hat{p}_{k_i}) = \delta_{k_i k_i'}$

that is, the state of the system at M_i must be precisely determined. Consider the predictive random walk equation in case $\mathbf{S}^{-1}(j,i)$ exists for each measurement pair in the sequence.

(74) $P(\hat{p}_{k_L}) = \sum_{k_{L-1}} \sum_{k_{L-2}} \cdots \sum_{k_0} T_{k_L k_{L-1}} T_{k_{L-1} k_{L-2}} \cdots T_{k_1 k_0} P(\hat{p}_{k_0})$

which by (73) must reduce to

(75) $P(\hat{p}_{k_L}) = T_{k_L k_{L-1}} \delta_{k_{L-1} k_{L-1}'} \delta_{k_{L-2} k_{L-2}'} \cdots \delta_{k_0 k_0'}$

Equation (75) is the random walk equation for a system which is <u>deterministic</u> from M_0 through M_{L-1}. We see from this that in case $\{\mathbf{S}(j,i)\}$ is a subset of a group, then the members of $\{\mathbf{S}(j,i)\}$ cannot describe the most general class of stochastic processes. The same argument applies for $\{\mathbf{S}'(i,j)\}$.

Let $\{\mathcal{S}(j,i)\}$ denote the collection of members of $\{\mathbf{S}(j,i)\}$, $\{\mathbf{S}'(i,j)\}$ and $\{\mathbf{S}(i,i)\}$. As we did for the transition matrices we may define multiplication between members of $\{\mathbf{S}(j,i)\}$ and $\{\mathbf{S}'(i,j)\}$ and show that for t and s each a positive integer such that t>s

$$(76)\quad \alpha_{k_t} = \sum_{k'_t} \langle k_t | \mathbf{S}(t,s)\,\mathbf{S}'(s,t) | k'_t \rangle \alpha_{k'_t}$$

$$\alpha_{k_s} = \sum_{k'_s} \langle k_s | \mathbf{S}'(s,t)\,\mathbf{S}(t,s) | k'_s \rangle \alpha_{k'_s}$$

Equations (76) are satisfied in case

$$(77)\quad \begin{aligned} \mathbf{S}(t,s)\,\mathbf{S}'(s,t) &= \mathbf{S}'(t,t) = \mathbf{I} \\ \mathbf{S}'(s,t)\,\mathbf{S}(t,s) &= \mathbf{S}(s,s) = \mathbf{I} \end{aligned}$$

but can be satisfied, as could equations (38), without the conditions imposed by equations (77). In fact if equations (77) are required of each **S**(j,i) and each **S**'(j,i) then the system described by the collection $\{\mathcal{S}(j,i)\}$ would, by equation (75), be completely deterministic. In addition we see that if $\{\mathcal{S}(j,i)\}$ is to form a semi-group, then equations (77) must be satisfied if multiplication between **S**(j,i) and **S**'(j,i) is to be closed in $\{\mathcal{S}(j,i)\}$. Therefore if $\{\mathcal{S}(j,i)\}$ forms a semi-group then it must form a group, and this group must be a <u>unitary group</u> since each $\mathcal{S} \in \{\mathcal{S}(j,i)\}$ is isometric and has an inverse.

Now suppose that $\{\mathbf{S}(j,i)\}$ forms a semi-group. As with $\{\mathbf{T}(j,i)\}$, we must require that

$$(78)\quad \mathbf{S}(\ell,k) = \mathbf{S}(x,y) \qquad |\ell-k| = |x-y|$$

that is $\{\mathbf{S}(j,i)\}$ can form a semi-group only if each $\mathbf{S} \in \{\mathbf{S}(j,i)\}$ is a function of the relative time. The same argument applies for $\{\mathbf{S}'(i,j)\}$.

We are now in a position to fully appreciate the difference between stochastic dynamics and quantum dynamics. First we note that the stochastic evolution operator $\mathbf{S}$ is in general isometric while the quantum evolution operator $\mathbf{U}$ is always unitary.

We see that in case the collection of stochastic operators $\{\mathcal{S}(j,i)\}$ for the measurement sequence $\{M_0 \to M_1 \to \cdots \to M_L\}$ forms a unitary group, then a system must follow a deterministic path through the measurement sequence. We also see from appendix B that in case each member of the collection $\{\mathbf{S}(j,i)\}$ has an inverse then $\{\mathbf{S}(j,i)\}$ is a unitary collection and equation (75) implies that each measurement in the sequence, except the last, yields a unique result.

Since the quantum evolution operator $\mathbf{U}$ always has an inverse, we see that the quantum evolution equation, when subjected to the phase choice of appendix A, can only describe evolution corresponding to equation (75). In case the quantum evolution operators form a unitary group then unitary evolution in $\mathcal{H}$ can only describe a deterministic stochastic process when the phase choice is imposed. Thus we see that quantum dynamics, i.e. unitary evolution in $\mathcal{H}$, can never reproduce the random walk structure of stochastic processes.

QUANTUM AND STOCHASTIC DYNAMICS IN A SINGLE $\mathcal{H}$ REPRESENTATION

From the preceding section we see that quantum dynamics and stochastic dynamics in $\mathcal{H}$ are identical only in case the quantum evolution equation is subject to the phase choice of Appendix A and the stochastic operator $\mathbf{S}$ is unitary. However, if the quantum evolution equation is subject to the phase choice, then the peculiar probability structure produced by the "square" of this equation disappears; on the other hand, if the stochastic operator $\mathbf{S}$ is unitary, then the more general singly stochastic structure of the transition matrices of stochastic processes is restricted to the doubly stochastic structure of quantum theory. Furthermore, if the phase choice is imposed on unitary evolution in $\mathcal{H}$, then the ensuing dynamical model in $\mathcal{H}$ can reproduce only a special case, given by equation (75), of the random walk equation (29).

In view of this, it is interesting to note that Nelson[(50),(51)] has derived the time-dependent Schrödinger equation from the diffusion equation. However, one may readily see from Chandrasekhar's[(56)] derivation of the diffusion equation that the diffusion format follows from the random walk equation (29) only in case $T(j,i)$ is <u>doubly stochastic</u>.

Such a result emphasizes the pecularity of the doubly stochastic "transition" matrix of quantum theory. The quantum "transition" matrix is clearly doubly stochastic since its elements are given by

(79) $$T_{k_j k_i} \triangleq |\langle k_j | \mathbf{U}(t_j, t_i) | k_i \rangle|^2$$

and we see from this equation that since $\mathbf{U}$ is unitary

(80) $$\sum_{k_j} T_{k_j k_i} = \sum_{k_i} T_{k_j k_i} = 1$$

However the stochastic representation with elements

(81) $$T_{k_j k_i} = |\langle k_j | \mathbf{S}(j,i) | k_i \rangle|^2$$

is in general not doubly stochastic since in general $\mathbf{S}$ is only isometric and not unitary.

The above properties of the evolution equations and the transition matrices of quantum and stochastic dynamics provide the motivation for a more general mathematical structure in $\mathcal{H}$ which will include both stochastic and quantum dynamics as a special case. To do this we simply hypothesize that each "state" of a physical system has a representation by a member of a seprable Hilbert space $\mathcal{H}$, and that the dynamical evolution of the system is described by

(82) $$|\alpha(t)\rangle = \mathbf{S}(t,t_0)\,|\alpha(t_0)\rangle$$

where $\mathbf{S}$ is in general isometric. The quantum dynamical description is given by a unitary $\mathbf{S}$ and the stochastic dynamical description is given by applying the phase choice theorem to equation (82). In this way we encompass both the peculiar probability structure provided by quantum theory and the singly stochastic transition matrix of classical stochastic theory.

CONCLUSION

We have discussed in this paper a novel formulation for the σ-algebra of stochastic chains and have seen how the sequenced event space leads to the notions of both prediction and retrodiction in stochastic theory. We have shown also that the equations for stochastic dynamics have a representation in a seprable Hilbert space $\mathcal{H}$ which in general is distinct from the conventional quantum representation in $\mathcal{H}$. The stochastic picture in $\mathcal{H}$ suggests a more general evolution picture in $\mathcal{H}$ which includes quantum evolution and stochastic evolution as special cases.

That retrodiction in stochastic theory is possible, is not surprising and in fact is necessary when one considers the definitions upon which stochastic theory is built. For example, consider the measuring sequence $\{M_0 \to M_1 \to \cdots \to M_L\}$. Suppose we let N systems pass this sequence one at a time, so that a moving picture camera may record the configurations assigned to a system as it passes through the sequence. Let the ith frame on the film record the result of M_i. Then the passage of a single system through the L-term measurement sequence will be recorded on an L-frame strip of film, each frame containing the result of one measurement. Suppose we record each system's passage through the sequence until we obtain N, L-frame strips of motion picture film. Suppose we mark the first frame of each strip to identify the direction of time passage for each strip. We may now place the N strips into a box and shuffle them. If the configuration of the environment is fixed for the N systems, then we may operationally define the unconditional probability for some $\hat{p}_{k_i}$ during M_i, as the number of strips $m(\hat{p}_{k_i})$ which have the configuration $\hat{p}_{k_i}$ on the ith frame divided by the total number N of strips, i.e.,

$$(82) \qquad P(\hat{p}_{k_i}) = \frac{m(\hat{p}_{k_i})}{N}$$

The unconditional probability for the sequence $(C_0 \to C_1 \to \cdots \hat{p}_{k_i} \to C_{i+1} \to \cdots \to \hat{p}_{k_j} \to C_{j+1} \to \cdots \to C_L)$ then is simply

$$(83) \quad P(\hat{p}_{k_i} \to \hat{p}_{k_j}) = \frac{m(\hat{p}_{k_i} \to \hat{p}_{k_j})}{N}$$

and the predictive conditional probability is given by

$$(84) \quad P(\hat{p}_{k_j} \mid \hat{p}_{k_i}) = \frac{P(\hat{p}_{k_i} \to \hat{p}_{k_j})}{P(\hat{p}_{k_i})} = \frac{m(\hat{p}_{k_i} \to \hat{p}_{k_j})}{m(\hat{p}_{k_i})} \; ; \; i < j$$

With these operational definitions it is then absolutely reasonable to define the retrodictive conditional probability

$$(85) \quad P(\hat{p}_{k_i} \mid \hat{p}_{k_j}) = \frac{P(\hat{p}_{k_i} \to \hat{p}_{k_j})}{P(\hat{p}_{k_j})} = \frac{m(\hat{p}_{k_i} \to \hat{p}_{k_j})}{m(\hat{p}_{k_j})}$$

which as we see from our example is not anti-causal in nature but is a simple result of the <u>a postiori</u> nature of the film data.

From the above example, we see that we may interpret the predictive and the retrodictive random walk equations in the following way; the predictive random walk equation will describe the diffusion of a drop of cream placed in a cup of coffee. If we film this process, then the retrodictive random walk equation will describe the "reverse diffusion process" as it appears on a projection screen when the film is run in reverse. We saw however, from the analysis of the transition matrices, that the retrodictive transition matrix is the inverse of the predictive transition matrix, only for deterministic systems.

When the stochastic equations were cast into their respective $\mathcal{H}$ representations, we saw that the predictive evolution operator $\mathbf{S}$ and the retrodictive evolution operator $\mathbf{S}'$ defined predictive and retrodictive evolution in $\mathcal{H}$. We saw that $\mathbf{S}$ and $\mathbf{S}'$ are isometric, but that $\mathbf{S}'$ is $\mathbf{S}^{-1}$ only for deterministic systems. Futhermore we saw that in contrast to conventional quantum theory, $\mathbf{S}$ is unitary only for systems described by equation (75). Thus we saw that the stochastic $\mathcal{H}$ representation is distinct from the quantum representation so that stochastic processes can not be considered as a special case of quantum evolution.

We then postulated a mathematical structure, equation (82), in $\mathcal{H}$ which would include both quantum evolution and stochastic evolution as special cases. No basis was given for such a structure, but it is envisioned that a more general definition of the event space $\{E(C)\}$ might well produce the more general postulated structure. Recall that we required $\{E(C)\}$ to be a σ-algebra and further imposed the generalized distributive relation on $\{E(C)\}$. It is hoped that a removal of the generalized distributive requirement, or a mathematical generalization of the σ-algebraic structure of $\{E(C)\}$, or both, will produce the more general evolution picture in $\mathcal{H}$.

APPENDIX A

Suppose each of $\{T_{k_j k_i}\}_{k_i=1,2,\ldots}$ and $\{P(\hat{p}_{k_i})\}_{k_i=1,2,\ldots}$ is a sequence of positive real numbers and there exists a real number $P(\hat{p}_{k_j})$ such that

$$P(\hat{p}_{k_j}) = \sum_{k_i=1}^{\infty} T_{k_j k_i} P(\hat{p}_{k_i}) \tag{1A}$$

Then there exists a sequence of complex numbers $\{K_{k_j k_i}\}_{k_i=1,2,\ldots}$ and a sequence of complex numbers $\{\alpha_{k_i}\}_{k_i=1,2,\ldots}$ and a complex number α'_{k_j} such that the following equations are consistent:

$$\alpha_{k_j} = \sum_{k_i=1}^{\infty} K_{k_j k_i} \alpha_{k_i} \tag{2A}$$

$$P(\hat{p}_{k_j}) = \alpha^*_{k_j} \alpha_{k_j} \tag{3A}$$

and for each positive integer k_i

$$T_{k_j k_i} = K^*_{k_j k_i} K_{k_j k_i} \tag{4A}$$

$$P(\hat{p}_{k_i}) = \alpha^*_{k_i} \alpha_{k_i} \tag{5A}$$

This theorem thus states that phases for the sequence $\{K_{k_j k_i} \alpha_{k_i}\}_{k_i=1,2,\ldots}$ can be found such that the double sum formed from the square of equation (2A) reduces to a single sum of real numbers. Equations (55) and (58) are nothing more than the $\mathcal{H}$ representations of equations (1A) and (2A) respectively.

APPENDIX B

THEOREM: Suppose that S is a linear continuous operator such that S^{-1} exists and I is a collection of positive integers such that k_i belongs to I only in case $\alpha_{k_i} \neq 0$. Then the equations

$$\alpha^*_{k_j}\alpha_{k_j} = \sum_{k_i} \langle k_j|S(j,i)|k_i\rangle\langle k_i|S^+(j,i)|k_j\rangle \alpha^*_{k_i}\alpha_{k_i} \tag{1B}$$

and

$$\alpha_{k_j} = \sum_{k_i} \langle k_j|S(j,i)|k_i\rangle \alpha_{k_i} \tag{2B}$$

are consistent only in case I has only one member, i.e.

$$\alpha_{k_i} = \delta_{k_i k_i'} \tag{3B}$$

PROOF: Substitution of (2B) into (1B) for α_{k_j} produces

$$\alpha_{k_j} \sum_{k_i \in I} \langle k_j|S(j,i)|k_i\rangle \alpha_{k_i} = \sum_{k_i \in I} \langle k_j|S(j,i)|k_i\rangle\langle k_i|S^+(j,i)|k_j\rangle \alpha^*_{k_i}\alpha_{k_i} \tag{4B}$$

Rearranging we obtain

$$\sum_{k_i \in I} (\alpha^*_{k_j} - \langle k_i|S^+(j,i)|k_j\rangle \alpha^*_{k_i})\alpha_{k_i}\langle k_j|S(j,i)|k_i\rangle = 0 \tag{5B}$$

which may be written

$$\sum_{k_i \in I} \alpha_{k_i}(\alpha^*_{k_j} - \langle k_i|S^+(j,i)|k_j\rangle \alpha^*_{k_i})S(j,i)|k_i\rangle = 0 \tag{6B}$$

If S^{-1} exists, the collection $\{S|k_i\rangle\}_{k_i \in I}$ is a linearly independent set so that (6B) is satisfied only in case

$$\alpha_{k_i}(\alpha^*_{k_j} - \langle k_i|S^+(j,i)|k_j\rangle \alpha^*_{k_i}) = 0 \quad ; \quad k_i \in I \tag{7B}$$

Since α_{k_i} is non-zero for each k_i in I , (7B) is satisfied only in case

$$\alpha^*_{k_j} = \langle k_i | S^+(j,i) | k_j \rangle \alpha^*_{k_i} \quad ; k_i \in I \tag{8B}$$

or

$$\alpha_{k_j} = \langle k_j | S(j,i) | k_i \rangle \alpha_{k_i} \quad ; k_i \in I \tag{9B}$$

Thus we see that (9B) is consistent with (2B) only in case the set $\{\alpha_{k_i}\}$ has only one non-zero member i.e., I has only one member.

Let us examine the implications of this in terms of probabilities. Since

$$P(\hat{p}_{k_i}) = |\langle k_i | \alpha(i) \rangle|^2 = \alpha^*_{k_i} \alpha_{k_i} = \delta_{k_i k_i'} \tag{10B}$$

We see that the system must be in some initial state of M_i . We also see then that the random walk equation yields

$$P(\hat{p}_{k_j}) = P(\hat{p}_{k_j} | \hat{p}_{k_i'}) \delta_{k_i k_i'} \tag{11B}$$

CHAPTER III

DISCUSSION

In this chapter, we will cast some of the more mathematical points of chapter II into physical terms. This will greatly enhance the comprehension of the mathematical apparatus and results presented in Chapter II.

First we will demonstrate by means of a physical example, that the random walk equations, retrodictive and predictive, are in general not Markoffian as supposed by Chung[68] but describe the evolution of a system for the most general type of transition probability; this example illustrates that the transition matrix reduces to a Markoffian transition matrix in certain instances.

We will examine the time reversible behavior of the stochastic structure by considering a special case of the predictive and retrodictive random walk equation. This special case will be the diffusion equation which we will show to be reversible only for equilibrium conditions.

We will show that the Pauli master equation follows directly from the stochastic formalism. This will allow us to compare time reversal in quantum theory to time reversal in stochastic theory, by comparing the quantum time-reversed master equation with the stochastic time-reversed master equation. In this analysis we also point out that the stochastic transition rates are calculated from singly stochastic transition matrices, while the quantum transition rates are calculated from doubly stochastic transition matrices.

Finally we examine a simple experiment which is described by a singly stochastic transition matrix and examine the description of this experiment in terms of quantum theory and then in terms of stochastic theory.

THE MARKOFF PROCESS

In Chapter II we claimed that equation (29) was not in general Markoffian. We would like to explore this claim here.

The definition of a Markoff chain is given by Chung[68] as a chain $(\hat{p}_{k_0} \to \hat{p}_{k_1} \to \cdots \to \hat{p}_{k_L})$ for which

$$(86) \quad P(\hat{p}_{k_L} \mid \hat{p}_{k_0} \to \hat{p}_{k_1} \to \cdots \to \hat{p}_{k_{L-1}}) = P(\hat{p}_{k_L} \mid \hat{p}_{k_{L-1}})$$

That is, the conditional probability for $\hat{p}_{k_L}$ at M_L is dependent on only the previous outcome, the outcome of M_{L-1}. The conventional treatments of Markoff processes begin by defining a σ-algebra of simple paths (as opposed to our more general $\{E(C)\}$ and derive the Markoff form of the random walk equation, equation (29), by defining the unconditional probability for $\hat{p}_{k_L}$ as the sum over all simple paths containing $\hat{p}_{k_L}$

$$(87) \quad P(\hat{p}_{k_L}) = \sum_{k_0=1} \sum_{k_1=1} \cdots \sum_{k_{L-1}=1} P(\hat{p}_{k_0} \to \hat{p}_{k_1} \to \cdots \to \hat{p}_{k_{L-1}} \to \hat{p}_{k_L})$$

Then the conventional definition of conditional probability is invoked to write equation (87) as

$$(88) \quad P(\hat{p}_{k_L}) = \sum_{k_0=1} \sum_{k_1=1} \cdots \sum_{k_{L-1}=1} P(\hat{p}_{k_L} \mid \hat{p}_{k_0} \to \hat{p}_{k_1} \to \cdots \to \hat{p}_{k_{L-1}}) P(\hat{p}_{k_0} \to \hat{p}_{k_1} \to \cdots \to \hat{p}_{k_{L-1}})$$

The conventional treatments then consider the chains to each be Markoffian so that equation (88) becomes

$$(89)\quad P(\hat{p}_{k_L}) = \sum_{k_{L-1}} P(\hat{p}_{k_L} | \hat{p}_{k_{L-1}}) \sum_{k_0} \sum_{k_1} \cdots \sum_{k_{L-2}} P(\hat{p}_{k_0} \to \hat{p}_{k_1} \to \cdots \to \hat{p}_{k_{L-2}} \to \hat{p}_{k_{L-1}})$$

and the multiple sum of equation (89) is by definition the unconditional probability for $\hat{p}_{k_{L-1}}$ so that equation (89) may be written

$$(90)\quad P(\hat{p}_{k_L}) = \sum_{k_{L-1}=1} P(\hat{p}_{k_L} | \hat{p}_{k_{L-1}}) \, P(\hat{p}_{k_{L-1}})$$

which is the Markoffian form of equation (29).

We see however that equation (29), as derived in Chapter II, made no assumptions concerning the nature of the paths in $\{E(C)\}$. Consequently we see that the conditional probability of equation (29), as given in equation (27) is a function of the entire sequence of measurements and is in general a non-Markoffian transition probability. Thus equation (29) is the non-Markoffian generalization of equation (90).

We may now examine our definition for conditional probability and ascertain what conditions are necessary for it to be Markoffian. First we see from our definition, equation (20) that the Markoffian definition, equation (86), is satisfied only in case

$$(91)\quad \frac{P(\hat{p}_{k_0} \to \hat{p}_{k_1} \to \cdots \hat{p}_{k_{L-1}} \to \hat{p}_{k_L})}{P(\hat{p}_{k_0} \to \hat{p}_{k_1} \to \cdots \to \hat{p}_{k_{L-1}} \to C_L)} = \frac{P(C_0 \to C_1 \to \cdots \to \hat{p}_{k_{L-1}} \to \hat{p}_{k_L})}{P(C_0 \to C_1 \to \cdots \to \hat{p}_{k_{L-1}} \to C_L)}$$

independent of the values for the integers $(k_0, k_1, \cdots, k_{L-2})$, thus it must be true that the ratio on the left hand side of equation (91) be independent of the path taken from M_0 to M_{L-1}, therefore we may write

$$(92)\quad \frac{P(\hat{p}_{k_0'} \to \hat{p}_{k_1'} \to \cdots \to \hat{p}_{k_{L-1}} \to \hat{p}_{k_L})}{P(\hat{p}_{k_0'} \to \hat{p}_{k_1'} \to \cdots \to \hat{p}_{k_{L-1}'} \to C_L)} = \frac{P(\hat{p}_{k_0} \to \hat{p}_{k_1} \to \cdots \to \hat{p}_{k_{L-1}} \to \hat{p}_{k_L})}{P(\hat{p}_{k_0} \to \hat{p}_{k_1} \to \cdots \to \hat{p}_{k_{L-1}} \to C_L)}$$

This equation may be manipulated into simpler forms, but the Markoff process is better illustrated by proceeding with a simple physical example.

Consider the example of figure 1 in Chapter II. Equation (92) for this example becomes

$$\frac{P(G_1 \to M \to D_1)}{P(G_1 \to M \to D_1 \cup D_2)} = \frac{P(G_2 \to M \to D_1)}{P(G_2 \to M \to D_1 \cup D_2)} \tag{93}$$

Using the generalized distributive relation, equation (93) may be written as

$$\frac{P(G_1 \to M \to D_1)}{P(G_1 \to M \to D_1) + P(G_1 \to M \to D_2)} = \frac{P(G_2 \to M \to D_1)}{P(G_2 \to M \to D_1) + P(G_2 \to M \to D_2)} \tag{94}$$

this equation may be rearranged to obtain

$$\frac{1}{1 + \frac{P(G_1 \to M \to D_2)}{P(G_1 \to M \to D_1)}} = \frac{1}{1 + \frac{P(G_2 \to M \to D_2)}{P(G_2 \to M \to D_1)}} \tag{95}$$

which may be written

$$\frac{P(G_1 \to M \to D_2)}{P(G_1 \to M \to D_1)} = \frac{P(G_2 \to M \to D_2)}{P(G_2 \to M \to D_1)} \tag{96}$$

which must be satisfied if the scattering process of figure 1 is to be Markoffian. Equation (96) becomes clearer if we rewrite the probabilities in terms of their numerical definitions.

Let $n(G_i \to M \to D_i)$ denote the number of electrons that are observed to follow the track $(G_i \to M \to D_i)$. Suppose that G_1 fired N_1 electrons and G_2 fired N_2 electrons during the experiment. In terms of numbers of electrons, equation (96) becomes

$$(97)\quad \frac{m(G_1 \to M \to D_2)}{m(G_1 \to M \to D_1)} = \frac{m(G_2 \to M \to D_2)}{m(G_2 \to M \to D_1)}$$

However we see that

$$(98)\quad \begin{aligned} N_1 &= m(G_1 \to M \to D_1) + m(G_1 \to M \to D_2) \\ N_2 &= m(G_2 \to M \to D_1) + m(G_2 \to M \to D_2) \end{aligned}$$

thus equation (97) may be written

$$(99)\quad \frac{N_1 - m(G_1 \to M \to D_1)}{m(G_1 \to M \to D_1)} = \frac{N_2 - m(G_2 \to M \to D_1)}{m(G_1 \to M \to D_1)}$$

which reduces to

$$(100)\quad \frac{m(G_1 \to M \to D_1)}{N_1} = \frac{m(G_2 \to M \to D_1)}{N_2}$$

now suppose that $N_1 = N_2$, that is an equal number of electrons are fired from each gun. In this case we may multiply both sides of equation (100) by the fraction 1/2 to obtain

$$(101)\quad \frac{m(G_1 \to M \to D_1)}{2N} = \frac{m(G_2 \to M \to D_1)}{2N}$$

and by definition of the unconditional probability, equation (101) becomes

$$(102)\quad P(G_1 \to M \to D_1) = P(G_2 \to M \to D_1)$$

For our simple example this result is exactly what we would expect the Markoff process to be; a process for which the probability of a particle to be scattered from M to D_1 is independent of the gun from which it was fired. We see from equation (96) that in case $N_1 \neq N_2$, then the ratio of equation (96) must be satisfied if the process is Markoffian.

Thus we see that our definition of conditional probability, in terms of the sequenced event space, provides a clear and intuitive method for examining and classifying physical processes as Markoffian or non-Markoffian.

THE RETRODICTIVE DIFFUSION EQUATION

In the conclusion section of Chapter II we suggested that the retrodictive equation (30) would describe the time reversed stochastic process. For example, had we filmed the diffusion process of a drop of cream in a cup of coffee, then the retrodictive equation would describe the film as it was viewed while being run in reverse. Let us examine this idea more fully.

To do this we will investigate the diffusion process in a manner similar to the derivation given by Chandrasekhar[56]. Whereas Chandrasekhar begins with a form of the predictive random walk equation (29), we will begin our investigation with the retrodictive equation (30).

Let M_j determine $\{\vec{R}_{k_j}\}$, the collection of possible position vectors for a particle on its jth step. The position vector $\vec{R}_{k_j}$ represents the k_jth configuration of the particle at step j. Let us suppose that the collection of position vectors which a particle may assume at each step is the same for each step so that the spectrum for each measurement in the sequence $\{M_0 \to M_1 \to \cdots \to M_L\}$ is given by $\{\vec{R}_k\}$. Let $P(\vec{R}_k, j)$ denote the probability for a particle to have the configuration $\vec{R}_k$ at step j and let $T'(\vec{R}_k, i \mid \vec{R}_{k'}, j)$ denote the retrodictive transition probability for a particle to have the configuration $\vec{R}_k$ at step i if it is going to be at $\vec{R}_{k'}$ at step j, i<j.

The retrodictive random walk equation then becomes

$$(103) \quad P(\vec{R}_k, i) = \sum_{k'} T'(\vec{R}_k, i \mid \vec{R}_{k'}, j)\, P(\vec{R}_{k'}, j) \quad ; \quad i<j$$

Now let us assume that our formalism is valid in case the spectrum $\{\vec{R}_k\}$ is continuous instead of discreet. Let $\vec{R}_{k'} - \vec{R}_k = \Delta\vec{R}_{k'}$ so that equation (103) may be written for the continuous case as

$$(104) \quad P(\vec{R}, i) = \int T'(\vec{R}, i \mid \vec{R}+\Delta\vec{R}, j)\, P(\vec{R}+\Delta\vec{R}, j)\, d(\Delta\vec{R})$$

We may expand $P(\vec{R}+\Delta\vec{R}, j)$ in a Taylor series to obtain

$$(105) \quad P(\vec{R}+\Delta\vec{R}, j) = P(\vec{R}, j) + \sum_{\ell} \left.\frac{\partial P(\vec{R}, j)}{\partial x_\ell}\right|_{\vec{R}} \Delta x_\ell$$

$$+ \frac{1}{2} \sum_{\ell} \sum_{k} \left.\frac{\partial^2 P(\vec{R}, j)}{\partial x_\ell \partial x_k}\right|_{\vec{R}} \Delta x_\ell \Delta x_k + \cdots$$

substituting equation (105) into equation (104) and ignoring higher order terms, we obtain

(106) $$P(\vec{R},i) = P(\vec{R},i)\int T'(\vec{R},i\,|\,\vec{R}+\Delta\vec{R},j)\,d(\vec{\Delta R})$$

$$+\sum_{\ell}\frac{\partial P(\vec{R},j)}{\partial X_\ell}\int T'(\vec{R},i\,|\,\vec{R}+\vec{\Delta R},j)\,\Delta X_\ell\,d(\vec{\Delta R})$$

$$+\sum_{\ell}\sum_{k}\frac{\partial^2 P(\vec{R},j)}{\partial X_\ell\,\partial X_k}\cdot\frac{1}{2}\int T'(\vec{R},i\,|\,\vec{R}+\vec{\Delta R},j)\,\Delta X_\ell\,\Delta X_k\,d(\vec{\Delta R})$$

we may now define the symbols

$$\vec{U}^{\Delta'}_{\ell}(\vec{R},j) \triangleq \int T'(\vec{R},i\,|\,\vec{R}+\vec{\Delta R},j)\,\Delta X_\ell\,d(\vec{\Delta R})$$

(107)

$$D^{\Delta'}_{\ell k}(\vec{R},j) \triangleq \frac{1}{2}\int T'(\vec{R},i\,|\,\vec{R}+\Delta\vec{R},j)\,\Delta X_\ell\,\Delta X_k\,d(\vec{\Delta R})$$

and if $T'(\vec{R},j\,|\,\vec{R}+\vec{\Delta R},i)$ is <u>doubly</u> <u>stochastic</u>, then

(108) $$\int T'(\vec{R},i\,|\,\vec{R}+\Delta\vec{R},j)\,d(\vec{\Delta R}) = 1$$

so that equation (106) may be written

(109) $$P(\vec{R},i) = P(\vec{R},j) + \vec{U}^{\Delta'}\cdot\nabla P(\vec{R},j) + \sum_{\ell,k} D^{\Delta'}_{\ell k}(\vec{R},j)\frac{\partial^2 P(\vec{R},j)}{\partial X_\ell\,\partial X_k}$$

Now let us assume that a film was taken of the diffusion process and that we are viewing it in reverse. As we view the film, we will assign t_{-i} to step i and t_{-j} to step j such that $t_{-j} < t_{-i}$; i<j.

Let us divide equation (109) by $(t_{-i}-t_{-j})$ to obtain

$$\frac{P(\vec{R},t_{-i})-P(\vec{R},t_{-j})}{(t_{-i}-t_{-j})} = \frac{\vec{U}^{\Delta'}(\vec{R},t_{-j})}{(t_{-i}-t_{-j})}\cdot\bar{\nabla}P(\vec{R},t_{-j}) + \sum_{l,k}\frac{D^{\Delta}_{lk}(\vec{R},t_{-j})}{(t_{-i}-t_{-j})}\frac{\partial^2 P(\vec{R},t_{-j})}{\partial X_l \partial X_k} \tag{110}$$

To the viewers, equation (110) will describe the "diffusion" process as the film is run in reverse. Taking the limit of equation (110) as $t_{-j}\to t_{-i}$, and denoting the time reversed paramater t as t_-, we obtain

$$\frac{\partial P(\vec{R},t_-)}{\partial t_-} = \sum_{l,k} D'_{lk}(\vec{R},t_-)\frac{\partial^2 P(\vec{R},t_-)}{\partial X_l \partial X_k} + \vec{U}'(\vec{R},t_-)\cdot\bar{\nabla}P(\vec{R},t_-) \tag{111}$$

where

$$\lim_{t_{-j}\to t_{-i}}\frac{\vec{U}^{\Delta'}(\vec{R},t_-)}{(t_{-i}-t_{-j})} \triangleq \vec{U}'(\vec{R},t_-)\ ;\ \lim_{t_{-j}\to t_{-i}}\frac{D^{\Delta'}_{lk}(\vec{R},t_-)}{(t_{-i}-t_{-j})} \triangleq D'_{lk}(\vec{R},t_-) \tag{112}$$

a similar derivation from the predictive equation yields

$$\frac{\partial P(\vec{R},t_+)}{\partial t_+} = \sum_{l,k} D_{lk}(\vec{R},t_+)\frac{\partial^2 P(\vec{R},t_+)}{\partial X_l \partial X_k} - \vec{U}(\vec{R},t_+)\cdot\nabla P(\vec{R},t_+) \tag{113}$$

where

$$\left[\vec{U}(\vec{R},t_+)\right]_l \triangleq \lim_{t_{+i}\to t_{+j}}\frac{1}{(t_{+j}-t_{+i})}\int T(\vec{R},j\,|\,\vec{R}-\Delta\vec{R},i)\,\Delta X_l\, d(\Delta\vec{R})$$

(114)

$$D_{lk}(\vec{R},t_+) \triangleq \lim_{t_{+i}\to t_{+j}}\frac{1}{(t_{+j}-t_{+i})}\int T(\vec{R},j\,|\,\vec{R}-\Delta\vec{R},i)\,\Delta X_l\,\Delta X_k\, d(\Delta\vec{R})$$

From equations (107) and (114) we see that $\vec{U}(\vec{R}, t+)$ and $\vec{U}'(\vec{R}, t-)$ are drift terms generated by probability bias to move in a preferred direction, and that $D_{\ell k}(\vec{R}, t+)$ and $D'_{\ell k}(\vec{R}, t-)$ are diffusion coefficients representing the tendency for the initial distribution of particles to "smear" out. We see that $\vec{U}'(\vec{R}, t+)$ and $D'_{\ell k}(\vec{R}, t-)$ are calculated from the retrodictive transition probability whereas $\vec{U}(\vec{R}, t+)$ and $D_{\ell k}(\vec{R}, t-)$ are calculated from the predictive transition probability.

From this we see that some major differences exist between the <u>predictive diffusion equation</u> (113) and the <u>retrodictive diffusion</u> equation (111). These equations differ not only in their coefficients, but also we see that the drift term in equation (111) differs in sign from the drift term in equation (113). This seems reasonable however, if we suppose that the drift term $\vec{U}(\vec{R}, t+)$ is generated by a fluid motion impressed on the random particle motion. In such a case, since the coordinate system is invariant to the direction of time motion and since the fluid motion would appear to reverse as the direction of the motion picture film is reversed, the sign of the drift vector would necessarily change when the direction of time flow is changed. From this discussion we see that time reversal invariance is not a characteristic of the stochastic diffusion equations.

Let us now intuitively examine the differences between the coefficients $\vec{U}$, $\vec{U}'$ and $D_{\ell k}$, $D'_{\ell k}$ for a special case: one dimensional diffusion. Suppose the initial distribution for the predictive diffusion equation is a delta function $\delta(x-x_0)$. Let us suppose for the time forward case, that a particle located at the position x at time t will have an equally likely probability of being at x-Δx or at x+Δx at time t+Δt. This would require the predictive transition probability to be Gaussian[56], i.e.,

$$(115)\quad T(x,\, t+\Delta t \mid x-\Delta x,\, t) \propto e^{-\frac{(\Delta x)^2}{a}}$$

and we see from equation (114) that the drift term $\vec{v}$ is zero for T Gaussian. For the time reversed case however, we know that the particles are more likely to move toward x_0 as t_- increases, thus particles are more likely to move toward x_0 than away from x_0 so that the retrodictive transition probability cannot be Gaussian.

Thus in case T is Gaussian, equations (111) and (113) become for one dimensional diffusion

$$(117)\quad \frac{\partial P(x,t_-)}{\partial t_-} = D_x'(x,t_-)\frac{\partial^2 P(x,t_-)}{\partial x^2} + v_x'(x,t_-)\frac{\partial P(x,t_-)}{\partial x}$$

$$(118)\quad \frac{\partial P(x,t_+)}{\partial t_+} = D_x(x,t_+)\frac{\partial^2 P(x,t_+)}{\partial x^2}$$

The retrodictive equation (117) will obviously have different solutions than does equation (118). The initial conditions for the retrodictive equation (117) will be the solution of equation (118) evaluated at the final time, and the solution to equation (117) will represent the solution to equation (118) as it would be viewed with the time parameter reversed.

Thus we see that the equations of stochastic theory do not have a time-reversal invariance but instead express the more realistic view that the equations of physical theory are nothing more than a short-hand way of writing time-parameterized data; that is the solutions to the time reversed equations must appear as the time-forward data viewed in reverse.

THE PAULI MASTER EQUATION

We will now show that the Pauli Master Equation can be obtained in a general way from the predictive random walk equation and in addition we will derive the time reversed form of the Master Equation from the retrodictive random walk equation. Once this time reversed form has been obtained, we can compare the time reversal characteristics of stochastic theory to the time reversal characteristics of quantum theory.

The Master Equation has been derived in many ways(78)- (87). Swenson(81) has shown the Master Equation, given below, to be valid for an arbitrary quantum system.

$$\frac{\partial \rho(\hat{p}_k,t)}{\partial t} = \sum_{k'} \{ \omega_{kk'}\, \rho(\hat{p}_{k'},t) - \omega_{k'k}\, \rho(\hat{p}_k,t) \} \tag{117}$$

where $\rho(\hat{p}_k,t)$ is the diagonal element of the quantum mechanical density matrix* and $\omega_{kk'}$, $\omega_{k'k}$ are quantum transition rates defined by

---

* The bibliography lists the historical papers as well as more modern mathematical treatments of the density matrix. See references (88)-(95).

$$\omega_{kk'} \triangleq \lim_{\Delta t \to 0} \frac{|\langle k | U(t_j, t_i) | k' \rangle|^2}{\Delta t} \tag{118}$$
$$\omega_{k'k} \triangleq \lim_{\Delta t \to 0} \frac{|\langle k' | U(t_j, t_i) | k \rangle|^2}{\Delta t}$$

Notice that since U is unitary, the transition rates are built from doubly stochastic matrices.

We may also derive the Master Equation from the stochastic formalism. Equation (117) is built for a system which has a time-invariant spectrum. Thus for our stochastic formalism we may denote each spectrum $\{\hat{p}_{k_i}\}$ by $\{\hat{p}_k\}$. With this notation the predictive random walk equation becomes

$$P(\hat{p}_k, t_j) = \sum_{k'} T(\hat{p}_k, t_j | \hat{p}_{k'}, t_i)\, P(\hat{p}_{k'}, t_i) \quad ; \; t_i < t_j \tag{119}$$

subtracting $P(\hat{p}_k, t_i)$ from both sides of equation (119) we obtain

$$P(\hat{p}_k, t_j) - P(\hat{p}_k, t_i) = \sum T(\hat{p}_k, t_j | \hat{p}_{k'}, t_i)\, P(\hat{p}_{k'}, t_i) - P(\hat{p}_k, t_i) \tag{120}$$

since $T(\hat{p}_{k'}, t_j | \hat{p}_k, t_i)$ is singly stochastic

$$\sum_{k'} T(\hat{p}_{k'}, t_j | \hat{p}_k, t_i) = 1 \tag{121}$$

so that equation (120) may be written

$$P(\hat{p}_k, t_j) - P(\hat{p}_k, t_i) = \sum_{k'} \left\{ T(\hat{p}_k, t_j | \hat{p}_{k'}, t_i) P(\hat{p}_{k'}, t_i) - T(\hat{p}_{k'}, t_j | \hat{p}_k, t_i) P(\hat{p}_k, t_i) \right\} \tag{122}$$

dividing both sides of equation (122) by $(t_j - t_i)$, taking the limit as $t_i \to t_j$ and defining

$$\text{(123)} \quad \begin{aligned} \omega_{kk'} &\triangleq \lim_{t_i \to t_j} \frac{T(\hat{p}_k, t_j \mid \hat{p}_{k'}, t_i)}{t_j - t_i} \\ \omega_{k'k} &\triangleq \lim_{t_i \to t_j} \frac{T(\hat{p}_{k'}, t_j \mid \hat{p}_k, t_i)}{(t_j - t_i)} \end{aligned}$$

we obtain the Master Equation derived from quantum theory

$$\text{(124)} \quad \frac{\partial P(\hat{p}_k, t_+)}{\partial t_+} = \sum_{k'} \left\{ \omega_{kk'} P(\hat{p}_{k'}, t_+) - \omega_{k'k} P(\hat{p}_k, t_+) \right\}$$

There is at least one important difference between equation (124) and equation (117) and that is the fact that the transition rates for equation (124) are built on transition probabilities which may form either singly stochastic or doubly stochastic transition matrices. Thus we see that the quantum Master Equation is not the most general form derivable and since this equation was built to describe ensemble systems, it seems questionable that it is capable of describing the most general ensemble system.

We now will note another very important difference between quantum and stochastic theory; the time-reversible descriptions.

Suppose once again that we have recorded on film the process described by the Master Equation. Suppose we then view the film as its motion is reversed so that t_+ becomes t_-. As with the diffusion example the derivative on the left hand side of equation (117) becomes $\frac{\partial P(\hat{p}_k, t_-)}{\partial t_-}$ and we see from equations (118) that

(125) $$\frac{\partial \rho(\hat{p}_k, t_-)}{\partial t_-} = \lim_{\Delta t \to 0} \sum_{k'} \left\{ \frac{|\langle k|U(-\Delta t)|k'\rangle|^2}{-\Delta t} \rho(\hat{p}_{k'}, t_-) - \frac{|\langle k'|U(-\Delta t)|k\rangle|^2}{-\Delta t} \rho(\hat{p}_k, t_-) \right\}$$

but the quantum evolution operator $\mathbf{U}$ has the property[4] that

(126) $$\mathbf{U}(-\Delta t) = \mathbf{U}^{+}(\Delta t)$$

thus we see that

(127) $$\begin{aligned} \frac{|\langle k|\mathbf{U}(-\Delta t)|k'\rangle|^2}{-\Delta t} &= -\frac{|\langle k|\mathbf{U}^{+}(\Delta t_-)|k'\rangle|^2}{\Delta t_-} = -\omega_{k'k} \\ \frac{|\langle k'|\mathbf{U}(-\Delta t)|k\rangle|^2}{-\Delta t} &= -\frac{|\langle k'|\mathbf{U}^{+}(\Delta t_-)|k\rangle|^2}{\Delta t_-} = -\omega_{kk'} \end{aligned}$$

so that equation (125) becomes

(128) $$\frac{\partial \rho(\hat{p}_k, t_-)}{\partial t_-} = \sum_{k'} \left\{ \omega_{kk'} \rho(\hat{p}_{k'}, t_-) - \omega_{k'k} \rho(\hat{p}_k, t_-) \right\}$$

which is identical to equation (117) thus demonstrating the time-reversal invariance of quantum theory. Notice that equation (128) does not describe a process as if it were being viewed in reverse. Instead it claims that the phenomena is totally independent of the time direction.

Now let us consider the Master Equation as it is derived from the retrodictive random walk equation. For the situation we are describing, the retrodictive random walk equation yields

$$(129)\quad P(\hat{p}_k, t_i) = \sum_{k'} T'(\hat{p}_k, t_i \mid \hat{p}_{k'}, t_j)\, P(\hat{p}_{k'}, t_j) \quad ; \; t_i < t_j$$

Subtracting $P(\hat{p}_k, t_j)$ from both sides of equation (129) we obtain

$$(130)\quad P(\hat{p}_k, t_i) - P(\hat{p}_k, t_j) = \sum_{k'} \{ T'(\hat{p}_k, t_i \mid \hat{p}_{k'}, t_j) P(\hat{p}_{k'}, t_j) - T'(\hat{p}_{k'}, t_i \mid \hat{p}_k, t_j) P(\hat{p}_k, t_j) \}$$

dividing equation (130) by $t_i - t_j$ we see that for the time-reversed case, $t_i > t_j$, this equation becomes in the limit as $t_j \to t_i$

$$(131)\quad \frac{\partial P(\hat{p}_k, t_-)}{\partial t_-} = \sum_{k'} \{ \omega'_{kk'} P(\hat{p}_{k'}, t_-) - \omega'_{k'k} P(\hat{p}_k, t_-) \}$$

where

$$(132)\quad \begin{aligned} \omega_{kk'} &\triangleq \lim_{t_j \to t_i} \frac{T'(\hat{p}_k, t_i \mid \hat{p}_{k'}, t_j)}{t_i - t_j} \\ \omega'_{k'k} &\triangleq \lim_{t_j \to t_i} \frac{T'(\hat{p}_{k'}, t_i \mid \hat{p}_k, t_j)}{t_i - t_j} \end{aligned}$$

Thus we see, that we have a peculiar form of stochastic time-reversal invariance for the Master Equation. That is, the form of the equations remain invariant, but the <u>retrodictive transition rates are entirely different from the predictive transition rates</u>. We see also from equation (131) and equations (132), that in case

$$(133)\quad \begin{aligned} T'(\hat{p}_k, t_i \mid \hat{p}_{k'}, t_j) &= T(\hat{p}_k, t_j \mid \hat{p}_{k'}, t_i) \\ T'(\hat{p}_{k'}, t_i \mid \hat{p}_k, t_j) &= T(\hat{p}_{k'}, t_j \mid \hat{p}_k, t_i) \end{aligned} \qquad \text{for each } k$$

then the time-reversal invariance of stochastic theory is identical to the time reversal invariance of quantum theory.

THE TRANSITION MATRIX IN STOCHASTIC AND QUANTUM THEORY

We have mentioned in both Chapter II and Chapter III that the transition matrix of stochastic theory is in general singly stochastic while the matrix of "transition probabilities" in quantum theory is always doubly stochastic. We will now apply the stochastic formalism and the quantum formalism to a simple experiment and show that if one interprets the quantum matrix elements $\{|\langle k_j|\mathbf{U}(j,i)|k_i\rangle|^2\}$ as transition probabilities then the quantum formulation fails to describe the most general type of data available from this experiment.

Consider the apparatus shown below. Suppose the source S emits N identically prepared systems which contact the measurement pair M_i, M_j.

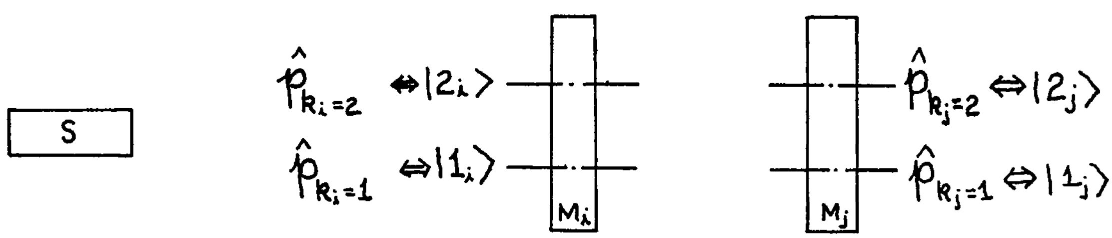

Figure 2. Measurement pair configuration

Throughout this discussion we will use the notation shown in figure 2 to distinguish between a configuration $\hat{p}_{k_i}$ of the system and the "state" $|k_i\rangle$ of a system. The <u>configuration</u> $\hat{p}_{k_i}$ as we have said earlier, is the <u>actual result</u> obtained by a measurement, while the <u>state</u> $|k_i\rangle$ is <u>constructed to reproduce the probability for the configuration</u> $\hat{p}_{k_i}$.

Let $m(\hat{p}_{k_i=1} \to \hat{p}_{k_j=1})$ denote the number of systems that are observed to have the configuration $\hat{p}_{k_i=1}$ at M_i and then the configuration $\hat{p}_{k_j=1}$ at M_j; we will use a similar notation for the other transitions. The number of systems $m(\hat{p}_{k_j=1})$ which are observed to have the configuration $\hat{p}_{k_j=1}$ at M_j is given by

$$(134) \quad m(\hat{p}_{k_j=1}) = m(\hat{p}_{k_i=1} \to \hat{p}_{k_j=1}) + m(\hat{p}_{k_i=2} \to \hat{p}_{k_j=1})$$

Dividing both sides of equation (134) by N we obtain

$$(135) \quad P(\hat{p}_{k_j=1}) = \frac{m(\hat{p}_{k_i=1} \to \hat{p}_{k_j=1})}{N} + \frac{m(\hat{p}_{k_i=2} \to \hat{p}_{k_j=1})}{N}$$

However the conditioned probability for a system to have the configuration $\hat{p}_{k_j=\ell}$ at M_j if it was observed to have the configuration $\hat{p}_{k_i=m}$ at M_i is given by

$$(136) \quad P(\hat{p}_{k_j=\ell} \mid \hat{p}_{k_i=m}) = \frac{m(\hat{p}_{k_i=m} \to \hat{p}_{k_j=\ell})}{m(\hat{p}_{k_i=m})}$$

where $m(\hat{p}_{k_i=m})$ is the number of systems observed to have the configuration $\hat{p}_{k_i=m}$ at M_i.

Now we may solve equation (136) for $m(\hat{p}_{k_i=m} \to \hat{p}_{k_j=\ell})$ and substitute this result into equation (135) to obtain

$$(137) \quad P(\hat{p}_{k_j=1}) = P(\hat{p}_{k_j=1} \mid \hat{p}_{k_i=1}) P_S(\hat{p}_{k_i=1}) + P(\hat{p}_{k_j=1} \mid \hat{p}_{k_i=2}) P(\hat{p}_{k_i=2})$$

where

$$(138) \quad P_S(\hat{p}_{k_i=m}) = \frac{m(\hat{p}_{k_i=m})}{N}$$

is the unconditional probability for observing a system (prepared by the source S) to have the configuration $\hat{p}_{k_i=m}$ at M_i.

Equation (136) can be easily recognized as the predictive random walk equation

$$(139) \qquad P(\hat{p}_{k_j=1}) = \sum_{m=1}^{2} T_{k_j=1\,k_i=m}\, P(\hat{p}_{k_i=m})$$

we may write the more general form of equation (139) with the compact notation of Chapter II

$$(140) \qquad P(\hat{p}_{k_j}) = \sum_{k_i=1}^{2} T_{k_j k_i}\, P(\hat{p}_{k_i}) \quad ; \; k_j = 1,2$$

Thus the predictive random walk formalism is simply a description of the data from the simple experiment of figure 2.

Now let us examine the quantum description of the data from this experiment. In this examination we will assume nothing more than the unitary evolution picture and the probabilistic interpretation of quantum theory. Let $|S\rangle$ denote the quantum state of each system as it leaves the source at t_0. Then the state $|\alpha(i)\rangle$ of the system, just prior to its measurement at M_i will be given by

$$(141) \qquad |\alpha(i)\rangle = \mathbf{U}(t_i, t_0)\,|S\rangle$$

Now according to the probability postulate of quantum theory, the unconditional probability for a system in the state $|\alpha(i)\rangle$ to be observed an instant later in the state $|k_i\rangle$, $(k = 1,2)$ is given by $|\langle k_i|\alpha(i)\rangle|^2$ or from equation (141)

$$(142) \qquad |\langle k_i|\alpha(i)\rangle|^2 = |\langle k_i|\mathbf{U}(t_i,t_0)|S\rangle|^2$$

If this interpretation is to agree with the data of the experiment, then

$$\text{(143)} \quad P_S(\hat{p}_{k_i}) = |\langle k_i|\mathbf{U}(t_i,t_0)|S\rangle|^2 = \frac{n(\hat{p}_{k_i})}{N}$$

where, as before, $n(\hat{p}_{k_i})$ is the number of systems observed to have the configuration $\hat{p}_{k_i}$ at M_i.

Now according to quantum theory, the system is either in the state $|1_i\rangle$ or the state $|2_i\rangle$ immediately after measurement by M_i; thus the state of the system just prior to measurement at M_j is given by either

$$\text{(144)} \quad |\alpha^{1_i}(j)\rangle = \mathbf{U}(t_j,t_i)|1_i\rangle$$

or

$$\text{(145)} \quad |\alpha^{2_i}(j)\rangle = \mathbf{U}(t_j,t_i)|2_i\rangle$$

depending on whether the system was to be observed in the state $|1_i\rangle$ at M_i or in the state $|2_i\rangle$ at M_i. Therefore according to quantum theory the probability for the system to be in the state $|k_j\rangle$ at M_j if it was observed to be in the state $|k_i\rangle$ at M_i is given by

$$\text{(146)} \quad P(\hat{p}_{k_j}|\hat{p}_{k_i}) \triangleq |\langle k_j|\alpha^{k_i}(i)\rangle|^2$$

or from equations (144) and (145) we may write

$$\text{(147)} \quad P(\hat{p}_{k_j}|\hat{p}_{k_i}) \triangleq |\langle k_j|\mathbf{U}(t_j,t_i)|k_i\rangle|^2$$

But if this is to agree with the data of the experiment then

$$\text{(148)} \quad |\langle k_j|\mathbf{U}(t_j,t_i)|k_i\rangle|^2 = \frac{n(\hat{p}_{k_i} \to \hat{p}_{k_j})}{n(\hat{p}_{k_i})}$$

where as before $m(\hat{p}_{k_i} \rightarrow \hat{p}_{k_j})$ is the number of systems observed to have the configuration $\hat{p}_{k_i}$ at M_i and then the configuration $\hat{p}_{k_j}$ at M_j.

It is obvious from figure 2 that

$$(149) \qquad m(\hat{p}_{k_j}) = \sum_{k_i=1}^{2} m(\hat{p}_{k_i} \rightarrow \hat{p}_{k_j})$$

thus solving equation (148) for $m(\hat{p}_{k_i} \rightarrow \hat{p}_{k_j})$, equation (149) becomes

$$(150) \qquad m(\hat{p}_{k_j}) = \sum_{k_i=1}^{2} |\langle k_j | \mathbf{U}(t_j, t_i) | k_i \rangle|^2 \, m(\hat{p}_{k_i})$$

Dividing both sides of this equation by N we obtain

$$(151) \qquad \frac{m(\hat{p}_{k_j})}{N} = \sum_{k_i=1}^{2} |\langle k_j | \mathbf{U}(t_j, t_i) | k_i \rangle|^2 \, \frac{m(\hat{p}_{k_i})}{N}$$

and by the definition of unconditional probability, equation (151) becomes

$$(152) \qquad P(\hat{p}_{k_j}) = \sum_{k_i=1}^{2} |\langle k_j | \mathbf{U}(t_j, t_i) | k_i \rangle|^2 \, P(\hat{p}_{k_i})$$

It would appear that equation (152) is of the same form as the stochastic random walk equation (140). However we see that the quantum "transition probabilities" of equation (152) must be doubly stochastic since they are built from the unitary operator $\mathbf{U}(t_j, t_i)$. On the other hand the matrix of stochastic transition probabilities is in general not doubly stochastic since it is built from an isometric operator.

The only remaining question is, can the experiment of figure 2 be physically meaningful for a singly stochastic transition matrix? The answer is obviously yes, since the experiment is well defined for the singly stochastic transition matrix

$$(152)\quad T(j,i) = \begin{bmatrix} P(\hat{P}_{k_j=1}|\hat{P}_{k_i=1}) & P(\hat{P}_{k_j=1}|\hat{P}_{k_i=2}) \\ P(\hat{P}_{k_j=2}|\hat{P}_{k_i=1}) & P(\hat{P}_{k_j=2}|\hat{P}_{k_i=2}) \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{3}{4} & \frac{3}{4} \end{bmatrix}$$

So we must conclude that either there exists a physical experiment which quantum theory does not describe, or the probability postulate of quantum theory holds for only those experiments with doubly stochastic matrices. Certainly there are other avenues in quantum theory one could employ to calculate the numbers of particles making transitions from one state of M_i to a state of M_j. It is clear however that the quantum matrix elements $\{|\langle k_j|\mathbf{U}|k_i\rangle|^2\}$ can not in all cases have the probabilistic interpretation.

CHAPTER IV

SUMMARY

We see from the assumptions and mathematical development in the early part of Chapter II that the dynamical structure of conventional stochastic theory as well as a retrodictive stochastic formalism can be derived from nothing more than a consideration of how data is gathered from a sequence of measurements. Thus the dynamics of the stochastic process is simply a description of the temporal process of data collection. We see from this that physics enters the theory only when we "guess" values for the transition probabilities. By "guess" we mean an educated attempt to predict the values of the transition probabilities before the experiment is actually performed. The stochastic format is necessary to the guessing however since we must know the nature of the data we are going to collect before we can make any guesses.

By a careful consideration of the data process we were able to see that the random walk equation is in general valid for non-Markoffian transition probabilities. This knowledge will surely allow a larger class of physical problems to be handled by the random walk method.

Once the random walk structure was obtained, we were able to show that there existed a separable Hilbert space representation which is capable of reproducing the general random walk structure, both predictive and retrodictive. This result allowed us to compare directly the dynamical structure of stochastic theory with the dynamical structure of quantum theory. We have summarized these results in Table I, page 85. Item 1 displays the predictive and retrodictive formalism for both theories. We see for quantum theory that the retrodictive evolution operator is simply the adjoint of the predictive evolution operator. As we saw from the discussion of the master equation in Chapter III this form for $\mathbf{U}$ leads to time-reversal invariant behavior for quantum evolution. For the stochastic evolution operator we saw that $\mathbf{S}$ is built from the data in such a way as to allow the stochastic equations to be time-reversible invariant only at equilibrium. We saw from the motion picture discussion of the diffusion equation in Chapter III that this property of $\mathbf{S}$ is necessary if we are to say that stochastic dynamics describes the way in which data is collected. Because the structure of the quantum evolution equation cannot properly describe the time-reversal nature of data it is tempting for this author to suggest that quantum evolution can indeed not describe experiments which gather data in this way. We can continue our examination of Table I to further support this suggestion.

We see in item 3 of Table I that the quantum evolution operator is always unitary while the stochastic operator $\mathbf{S}$ is isometric. This fact implies item 4, that is, the transition matrices of quantum theory are doubly stochastic because they are built from a unitary evolution operator. The transition matrices of stochastic theory are in general <u>not</u> doubly stochastic since they are built from an isometric operator. We saw in the simple example of Chapter III that the doubly stochastic transition matrix cannot describe the most general type of data available from this experiment. Thus in one more instance, quantum evolution appears not to be built to describe certain types of experiments.

It is interesting to note in the same context, item 5 of Table I. Here we proved that in case $\mathbf{S}^{-1}$ exists and thus $\mathbf{S}$ is unitary, that the evolution equation could describe only data of a deterministic nature. By deterministic, we meant that we knew with certainty, before the experiment was performed, the outcome of the experiment. This is perfectly consistent in the context of item 4 since for the deterministic cast, T(j,i) has only one member and so we no longer must worry about the doubly stochastic property of T. This fact reinforces the plausibility of the strictly mathematical proof of Appendix B, which demonstrated that the quantum evolution equation can never reproduce the most general form of the random walk structure.

All these facts then lead us in Chapter II to postulate a more general evolution picture in $\mathcal{H}$. This more general evolution is isometric evolution, where the isometric evolution operator $\mathbf{S}$ is not required to produce the phase choice necessary to obtain the random walk structure from the evolution structure in $\mathcal{H}$. In this way we produce a covering structure which includes quantum evolution and stochastic evolution as special cases. When $\mathbf{S}^{-1}$ exists we obtain quantum evolution and when $\mathbf{S}$ is constructed to produce the proper phase choice, we obtain stochastic evolution. How this more general evolution picture is related to the nature of data is at this time not clear. It is a postulated evolution in $\mathcal{H}$ in the same sense that unitary evolution is postulated for quantum theory. It is conjectured that a more general formulation of the event space $\{E(C)\}$ for the measurement sequence will produce the more general evolution picture in $\mathcal{H}$. In any case, we are no worse off epistemologically, by postulating isometric evolution than we are by postulating unitary evolution. Furthermore, isometric evolution is clearly sound empirically since it reproduces the results of both quantum theory and stochastic theory and in addition such a structure provides a mathematical framework within which a search for a yet unknown class of phenomena may be pursued.

TABLE I

QUANTUM EVOLUTION-STOCHASTIC EVOLUTION

COMPARISON

QUANTUM EVOLUTION		STOCHASTIC EVOLUTION
$\vert\alpha(t_j)\rangle = U(t_j,t_i)\vert\alpha(t_i)\rangle$ $\vert\alpha(t_i)\rangle = U'(t_i,t_j)\vert\alpha(t_j)\rangle$	predictive 1 retrodictive	$\vert\alpha'(j)\rangle = S(j,i)\,\vert\alpha'(i)\rangle$ $\vert\alpha'(i)\rangle = S'(i,j)\,\vert\alpha'(j)\rangle$
$U'(t_i,t_j) = U^{+}(t_j,t_i)$	2	in general $S'(i,j) \neq S^{+}(j,i)$
$U(t_j,t_i)$ is unitary	3	$S(j,i)$ is isometric with phase choice restriction
Transition Matrices are doubly stochastic	4	Transition Matrices are singly stochastic
$\{U(t_j,t_i)\}$ forms a unitary group in case U is a function of relative time	5	$\{S(j,i)\},\{S'(i,j)\}$ form distinct semi groups; form unitary group only for determinism
Time reversal invariant	6	Time reversal invariant only at equilibrium

BIBLIOGRAPHY

BIBLIOGRAPHY

1. Dirac, P.A.M., The Principles of Quantum Mechanics 2nd. ed., Oxford (1936).

2. von Neumann, J., Mathematical Foundations of Quantum Mechanics, Princeton University Press, (1955).

3. Birkhoff, G. and von Neumann, J., Ann. Math., 37, 823 (1936).

4. Jauch, J. M., Foundations of Quantum Mechanics, Addison Wesley, Mass., (1968).

5. Piron, C., and Jauch, J. M., Helv. Phys. Acta, 36, 827 (1963).

6. Mackey, G. W., Mathematical Foundations of Quantum Mechanics W. A. Benjamin, N. Y., (1963).

7. Born, M. Z., Physik, 37 (1926).

8. Einstein, A., J. Franklin Inst., 221, 313 (1936); Einstein denys the Born hypothesis.

9. Bohr, N., J. Franklin Inst., 221, 235 (1936); Bohr's rebuttal to Einstein.

10. Einstein, A., Podolsky, B., and Rosen, N., Phys. Rev., 47, 777 (1935); The Einstein, Podolsky, Rosen Paradox.

11. Bohr, N., Phys Rev, 48, 696 (1935); Bohr's rebuttal to the EPR Paradox.

12. Everett, H., Rev Mod. Phys., 29, 454 (1957); presents an interpretation of quantum mechanics which includes the measurement device in the state vector.

13. Shimony, A., Am. J. Phys., 31, 755 (1963); a criticism of the orthodox theory and Everett's (12) work.

14. Pearl, Phillip, Am. J. Phys., 35, 742, (1967); generalizes Everett's (12) interpretation of quantum theory.

15. Park, J. L., Am. J. Phys. 36, 211, (1968); discusses the ensemble interpretation of quantum states.

16. de Broglie, L., Physicien et Panseur, Albin Micheal, Paris (1953).

17. Heisenberg, Neils Bohr and the Development of Physics, McGraw-Hill N. Y., (1955).

18. Hansen, N. R., The Concept of the Positron, Cambridge University Press, Cambridge (1963).

19. Bell, J. S., and Nauenberg, M., Preludes in Theoretical Physics, In Honor of V. Weisskopf, North Holland Pub. Co., Amsterdam (1966).

20. Bohm, D., Phys. Rev. 85, 166, 180, (1952).

21. Bell, J. S., Rev. Mod. Phys., 38, 3, 447, (1966).

22. Bohm, D., and Bub, J., Rev. Mod. Phys., 38, 3, 453, (1966).

23. Bell, J. S., Phys., 1, 3, 195, (1964).

24. Clauser, J. F.; Horne, A. M.; Shimony, A.; and Holt, R. A., Phys. Rev. lett., 23, 15, 880, (1969).

25. Collins, R. E., Critique of Proposed Experiment to Test Local Hidden Variable Theories, submitted to Phys. Rev. lett., (1969).

26. Wigner, E. P., Am. J. Phys., 31, 6, (1963).

27. Jauch, J. M., Helv. Phys. Acta, 37, 293, (1964).

28. Bohm, D., and Bub, J., Rev. Mod. Phys., 38, 3, 453, (1966).

29. Schrödinger, Ann. der Phys., 79, 361, (1926); early attempt to derive Schrödinger equation from classical concepts.

30. Wesley, J. P., Phys. Rev., 122, (1932); rejects the probabilistic view of classical mechanics.

31. Furth, R., Z. Phys., 81, 143, (1933); presents a formal analogy between Brownian motion and the Schrödinger equation.

32. Liebowitz, B., Phys. Rev., 66, 343, (1944); introduces the concept of orbital density.

33. Bohm, D., Phys. Rev., 85, 166, (1952); attempts a classical interpretation of quantum theory.

34. Fenyes, Z. Phys., 132, 81, (1952); examines quantum mechanics in terms of a Markoff process.

35. Weizel, W., Z. Phys., 134, 264, (1953).

36. Weizel, W., Z. Phys., 135, 270, (1953).

37. Weizel, W., Z. Phys,, 136, 582, (1954), proposes a model for interaction describing random fluxuations (interacting zerons).

38. Bohm, D., and Vigner, J. P., Phys. Rev., 96, 208, (1954); Introduced the notion of random fluxuations arising from interactions of a subquantum media.

39. Braffort, P., and Tzara, C., Compt. Rend., 239, 157, (1954); noted similiarities between stochastic processes and quantum theory.

40. Freistadt, H., Suppl Nuovo Cimento, 10, 1 (1957); bibliography of work relating classical mechanics and quantum theory.

41. de Broglie, L., Non-Linear Wave Mech., (Amsterdam 1960); noted the peculiar behavior of conditional probability in quantum theory.

42. Margenau and Hill, Progress Theor. Phys. 26, 722 (1961); noted the peculiar behavior of conditional probability in quantum theory.

43. Kershaw, D., Phys. Rev., 136, B1850 (1964); discusses Weizel's work (references 35, 36, and 37).

44. Comisar, G. G., Phys. Rev., 138, B1332 (1965); derives Schrödinger equation using imaginary diffusion coefficient.

45. Braffort, D., and Surdin, M., and Taroni, A., Compt. Rend., 261, 4339 (1965); note similarities between stochastic processes and quantum theory.

46. Marshall, T., Proc. Camb., Phil., Soc., 61, 537 (1965).

47. Bourret, R. C., Can J. Phys, 43, 619 (1965).

48. Liebowitz, B., Nuovo Cimento, 39, 1201 (1965).

49. Liebowitz, B., Nuovo Cimento, 1, 6, 1250 (1968).

50. Nelson, E., Phys. Rev., 150, 4 (1966).

51. Nelson, E., Dynamical Theories of Brownian Motion, Princeton Univ. Press, Princeton, 1967.

52. de la Pena-Auerbach, L., and Garcia-Colin, L. S., J. Math. Phys. 9, 6, 922 (1968).

53. de la Pena-Auerbach, L., and Garcia-Colin, L. S., J. Math. Phys. 9, 6, 922 (1968).

54. Kursunoglu, B., Am. J. Phys. (1966).

55. Feynmann, R. P., Rev. Mod. Phys., 20, 267 (1948).

56. Chandrasekhar, S., Rev. Mod. Phys., 15, 17 (1943).

57. Reichenbach, H., Philosophical Foundations of Quantum Mechanics, Univ. of Calif. Press (1948).

58. Birkhoff, G., and von Neumann, J., Ann. Math., 37, 823 (1936).

59. Montroll, E., Pure Appl. Math., 5, 415 (1952); a discussion of generalized probability and the connection with the Feynmann path - integral formulation.

60. Bellman, R., Proc. Natl. Sci. U. S., 39, 1075 (1953); a discussion of a generalization of probability theory and the use of Markoff chains.

61. Watanabe, S., Supp. of the Prog. of Theor. Phys. (1965).

62. Watanabe, S., Rev. Mod. Phys. 27, 26, 40 (1955).

63. Watanabe, S., Phys. Rev. 84, 1008 (1951).

64. Kolmogoroff, Grundbegriffe der Wahrseheinlichkeitsrechnung, Berlin (1933).

65. Doob, J. L., Stochastic Processes, N. Y. (1955).

66. Feller, W., An Introduction to Probability Theory, Vol. I & II N. Y. (1950).

67. Loeve, M., Probability Theory, N. Y. (1955).

68. Chung, K. L., Markov Chains with Stationary Transition Probabilities, p. 4, Springer Verlag, Berlin (1960).

69. Collins, R. E., Phys. Rev., 183, 5, 1081 (1969).

70. Collins, R. E., Proc. of Int. Conf. on Thermodynamics, Univ. of Pittsburg (1969).

71. Collins, R. E., "Generalized Quantum Theory", to be published in Phys. Rev.

72. Collins, R. E. and Hall, F. G., Proc of Int. Conf. on Thermodynamics, Univ. College, Cardiff, Wales (1970).

73. Collins, R. E., "Statistical Basis for the Many-Body Problem", Submitted to Phys. Rev., now pending 2nd. Review (1969).

74. Collins, R. E., "Statistical Basis for Quantum Theory; The Embedding Formalism", Submitted Phys. Rev. (1970).

75. Halmos, P. R., Measure Theory, Van Nostrand (1950).

76. Wu, Ta-You, Kinetic Equations of Gases and Plasmas, p. 20, Addison Wesley, Mass. (1966).

77. Berberian, Introduction to Hilbert Space, Oxford Press (1961).

78. Van Hove, Physica, 21, 517 (1955); derivation of the Master Equation with first order perturbation theory.

79. Van Hove, Physica, 23, 441 (1957); derivation of the generalized Master Equation valid for higher order perturbation.

80. Janner, A., Helv. Phys. Acta, 35, 47 (1962); derivation of Master Equation same result as Van Hove using different approach.

81. Swenson, R. J., J. Math. Phys., 3, 1017 (1962); shows Master Equation to be valid for an arbitrary quantum system.

82. Nakajima, S., Prog. Theor. Phys., 20, 948 (1958); derived non-Markoffian form of the Master Equation.

83. Zwanzig, R., Lectures in Theoretical Physics, Boulder (1960); derived non-Markoffian form of the Master Equation.

84. Zwanzig, R., Physica 30, 109 (1964); different treatment of the non-Markoffian Master Equation; demonstrated the equivalence of his results to those of Prigogine (85) and Resibois (86).

85. Prigogine, I., and Resibois, P., Physica 27, 629 (1961); standard reference on derivation of Master Equation.

86. Resibois, P., Physica, 29, 721 (1963).

87. Montroll, E. W., Fundamental Problems in Statistical Mechanics, North-Holland (1962).

88. Bloch, F., Z. Phys., 29, 58 (1927); introduced independently of Landau, & von Neumann concept of density operator.

89. von Neumann, J., Collected Works, Vol. I, p. 236, Oxford (1961); introduced independently of Bloch and Landau concept of density operator.

90. Landau, L., Z. Phys., 45, 430 (1927); introduced independently of Bloch and von Neumann concept of density operator.

91. Born, M., and Jordan, P., Elementare Quantenmechanik, Berlin (1930); first textbook to treat density matrix.

92. Landau, L., and Lifshitz, E., "Quantum Mechanics", Oxford (1959); further treatment of the density matrix.

93. Ter Haar, D., Rep. Prog. Phys., 24, 304 (1961); more general formulation of density matrix.

94. Segal, I. E., Ann. Math., 48, 930 (1947); original paper giving the density operator a rigorous formulation.

95. Mackey, G. W., Amer. Math. Mo., 64, 45 (1957); a rigorous formulation of the density operator.