ANALYSIS OF FLUID FLOW IN THE ENTRANCE REGION OF A DUCT WITH AN ECCENTRIC ANNULAR CROSS-SECTION

A Dissertation

Presented to

the Faculty of the Department of Mathematics

University of Houston

Houston, Texas

In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

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by

John Terry Wilson

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ABSTRACT

The purpose of this dissertation is to analyze the velocity development of laminar incompressible flow in the entrance region of a straight duct with an unchanging eccentric annular cross-section. A linearized version of the governing equations is solved under the assumptions that the velocity is zero on the duct wall and that the initial velocity profile is uniform across the cross-section. The analysis leads to a two-dimensional eigenvalue value problem which is then posed in the appropriate Hilbert space. Galerkin's method is shown to converge for, and then applied to, this eigenvalue problem.

December 1974

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CHAPTER 1

INTRODUCTION

In a straight duct with an arbitrary but unchanging cross-section, the laminar flow of an incompressible fluid undergoes a velocity development from an initial profile at the entrance of the duct to a fully developed profile far downstream. The region where this development occurs is called the entrance region of the duct. Analytical expressions for the fully developed velocity have been obtained for ducts of various geometries. However, due to the nonlinearity of the governing equations, an analytical expression for the velocity in the entrance region has never been obtained even for the simplest geometries.

In 1964, Sparrow et. al. [1]* proposed a linearized version of the momentum equation and then proceeded, after making several assumptions, to solve the entrance region problem for two ducts: the parallel-plate channel and the circular tube. Since their solutions compare favorably with experimental results, their linearization procedure has been applied to ducts of other shapes. In particular,

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*Numbers in brackets refer to references in the bibliography.

Wiginton and Wendt [2] have generalized this procedure and have obtained results for the rectangular duct. A report by Shah and London [3] contains a summary of research in this and other areas.

Using the Sparrow linearization procedure, we propose to analyze the velocity development in the entrance region of a duct with an eccentric annular cross-section. This analysis leads to a two dimensional boundary value problem which we solve by the method of Galerkin. Since the Sparrow procedure is applicable to ducts of various shapes, we now state the problem and present part of the analysis for a duct with an arbitrary cross-section.

We consider steady state laminar incompressible flow in a straight duct with an arbitrary but unchanging cross-section. The duct axis is taken to be in the direction of the positive z-axis and the duct entrance is assumed to lie in the xy-plane (see Fig. 1)*. The equations which govern the fluid motion in the duct are the z-direction momentum equation and the continuity equation. For this type of flow, they simplify to

$$\overline{V} \cdot \text{grad } w = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\nabla^2 w \tag{1-1}$$

and

---

*The figures for Chapters 1 and 2 are contained in Appendix A.

$$\text{div } \bar{V} = 0 \ , \tag{1-2}$$

respectively (see Appendix B). In these equations, $\bar{V} = (u,v,w)$ is the velocity vector, $p\,(x,y,z)$ is the pressure, ρ is the density, and ν is the kinematic viscosity. The symbol ∇^2 represents the three dimensional Laplacian operator in x, y, and z. We now make the following assumptions:

(i) the term $\dfrac{\partial^2 w}{\partial z^2}$ is negligible compared to

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \ .$$

(ii) the static pressure $p(x,y,z) = p(z)$ has z dependence only.

Equation (1-1) becomes

$$\bar{V} \cdot \text{grad } w = - \frac{1}{\rho} \frac{dp}{dz} + \nu \nabla^2 w \tag{1-3}$$

where the symbol ∇^2 now represents the two dimensional Laplacian operator in x and y. At this point of the analysis, we replace equation (1-3) with the Sparrow linearized version:

$$\varepsilon(z) W \frac{\partial w}{\partial z} = \Omega(z) + \nu \nabla^2 w \tag{1-4}$$

where W is the average velocity over the cross-section A, $\varepsilon(z)$ is a to-be-determined function which weights the average velocity W, and $\Omega(z)$ is another to-be-determined function which includes the pressure term and the residual of the inertia terms. Integration of equation (1-4) over the cross-section A yields

$$\Omega(z) = -\frac{\nu}{A}\int_A \nabla^2 w \, dA \quad .$$

Thus by Green's theorem,

$$\Omega(z) = -\frac{\nu}{A}\oint_C \frac{\partial w}{\partial N}\, dS \tag{1-5}$$

where C represents the boundary of the cross-section and N represents the outward normal. Since $\varepsilon(z)$ is unknown, we temporarily suppress it by introducing a stretched axial coordinate z^* defined by

$$dz = \varepsilon(z)\, dz^* \quad . \tag{1-6}$$

Using equations (1-5) and (1-6), equation (1-4) becomes

$$W \frac{\partial w}{\partial z^*} + \frac{\nu}{A}\oint_C \frac{\partial w}{\partial N}\, dS = \nu\nabla^2 w \quad . \tag{1-7}$$

The problem undertaken in this dissertation is to solve equation (1-7) for w(x,y,z), the z component of the velocity vector, under the following conditions:

(i) the function $w(x,y,z) = 0$ on the duct wall.

(ii) the initial profile at the duct entrance is uniform across the cross-section, i.e., $w(x,y,0) = W$.

We now introduce the following dimensionless variables:

(i) $\phi = \frac{w}{W}$

(ii) $\xi = \frac{x}{\sqrt{A}}$

(iii) $\eta = \frac{y}{\sqrt{A}}$

(iv) $\beta = \frac{z^*}{\sqrt{A}\ R}$

(v) $n = \frac{N}{\sqrt{A}}$

(vi) $s = \frac{S}{\sqrt{A}}$

(vii) $R = \frac{W\sqrt{A}}{\nu}$.

Substitution of these variables into equation (1-7) yields

$$\frac{\partial \phi}{\partial \beta} + \oint_{\overline{C}} \frac{\partial \phi}{\partial n}\, ds = \nabla^2 \phi \qquad (1\text{-}8)$$

where the symbol ∇^2 represents the two dimensional

Laplacian operator in ξ and η. We seek a solution to equation (1-8) of the form

$$\phi(\xi,\eta,\beta) = \phi_e(\xi,\eta,\beta) + \phi_f(\xi,\eta) \tag{1-9}$$

where $\phi_f(\xi,\eta)$ is the fully developed velocity and $\phi_e(\xi,\eta,\beta)$ is a difference velocity. Substitution of equation (1-9) into equation (1-8) yields

$$\begin{aligned}&[\nabla^2\phi_e - \frac{\partial\phi_e}{\partial\beta} - \oint_{\overline{C}} \frac{\partial\phi_e}{\partial n}\, ds] \\ &\qquad + [\nabla^2\phi_f - \oint_{\overline{C}} \frac{\partial\phi_f}{\partial n}\, ds] = 0 \,.\end{aligned} \tag{1-10}$$

The fully developed velocity $\phi_f(\xi,\eta)$ is a solution of

$$\nabla^2\phi_f(\xi,\eta) = \frac{dp}{dz} \tag{1-11}$$

where

$$\phi_f(\xi,\eta) = 0 \qquad \text{on} \qquad \overline{C} \,. \tag{1-12}$$

The pressure drop $\frac{dp}{dz}$ is assumed z-independent. Using Green's theorem and equation (1-11) we obtain

$$\oint_{\bar{C}} \frac{\partial \phi_f}{\partial n}\, ds = \int_{\bar{A}} \nabla^2 \phi_f \, d\bar{A}$$

$$= \int_{\bar{A}} \frac{dp}{dz}\, d\bar{A}$$

$$= \frac{dp}{dz}$$

since the pressure $p(x,y,z) = p(z)$ is assumed to be a function of z only. This shows that the second term in equation (1-10) is zero, i.e.,

$$\nabla^2 \phi_f - \oint_{\bar{C}} \frac{\partial \phi_f}{\partial n}\, ds = 0 \ . \tag{1-13}$$

Equation (1-10) becomes

$$\nabla^2 \phi_e - \frac{\partial \phi_e}{\partial \beta} - \oint_{\bar{C}} \frac{\partial \phi_e}{\partial n}\, ds = 0 \ . \tag{1-14}$$

Clearly, the difference velocity $\phi_e(\xi,\eta,\beta)$ is of significance only in the entrance region, i.e.,

$$\lim_{\beta\to\infty} \phi_e(\xi,\eta,\beta) = 0 \ . \tag{1-15}$$

With condition (1-15) in mind, we seek a solution to equation (1-14) of the form

$$\phi_e(\xi,\eta,\beta) = \sum_{j=1}^{\infty} c_j g_j(\xi,\eta) \exp(-\alpha_j^2 \beta) \tag{1-16}$$

where c_j, g_j, and α_j are to be determined. Substitution of equation (1-16) into equation (1-14) yields equations of the form

$$\nabla^2 g_j + \alpha_j^2 g_j = \oint_{\overline{C}} \frac{\partial g_j}{\partial n}\, ds \tag{1-17}$$

for the determination of the g_j's. The g_j's are to satisfy the boundary condition

$$g_j(\xi,\eta) = 0 \quad \text{on} \quad \overline{C}. \tag{1-18}$$

The above analysis does not depend on a particular cross-sectional geometry. We now consider an eccentric annular cross-section (see Fig. 2). We have two boundary value problems to solve. One of them is the eigenvalue problem

$$-\nabla^2 g(\xi,\eta) + \oint_{\overline{C}} \frac{\partial g}{\partial n}\, ds = \lambda g(\xi,\eta) \tag{1-19}$$

where

$$g(\xi,\eta) = 0 \quad \text{on} \quad \overline{C} \tag{1-20}$$

(see Fig. 3). The other equation to solve is the fully developed velocity equation:

$$\nabla^2 \phi_f(\xi,\eta) - \oint_{\overline{C}} \frac{\partial g}{\partial n}\, ds = 0 \qquad (1\text{-}21)$$

where

$$\phi_f(\xi,\eta) = 0 \quad \text{on} \quad \overline{C} \,. \qquad (1\text{-}22)$$

We do not choose to solve equations (1-19) and (1-21) on the annulus. Instead, in Chapter 2 we transform the annulus conformally onto a rectangle and transform the eigenvalue and fully developed velocity equations accordingly.

In Chapter 3, we formulate our eigenvalue problem in a Hilbert space setting. First, we define the appropriate Hilbert spaces and an operator L. The operator L is then shown to be symmetric and positive definite. Finally, we define the energy space which is the space where Galerkin's method converges.

In Chapter 4, we find a linearly independent complete sequence for the energy space. We use linear combinations of elements of this sequence to approximate solutions of equation (2-26).

Chapter 5 contains an explanation of the method of Galerkin for eigenvalue problems and the proof of convergence for our particular operator L. The approximate eigenvalues of equation (2-26)are found by setting the Galerkin determinant equal to zero. In Chapter 6, we obtain expressions for the inner products required in the Galerkin determinant.

In Chapter 7, we solve the transformed fully developed velocity equation. The method of solution is to expand ϕ_f in a Fourier series.

In Chapter 8, we give a procedure for finding approximations to the dimensionless velocity $\phi(\xi,\eta,\beta)$.

CHAPTER 2

THE TRANSFORMATION

In Chapter 1, we stated that we are seeking a solution ϕ_e of equation (1-14) of the form

$$\phi_e(\xi,\eta,\beta) = \sum_{j=1}^{\infty} c_j g_j(\xi,\eta) \exp(-\alpha_j^2 \beta)$$

(see Fig. 2). Since our flow is laminar, our geometry is symmetric with respect to the $\xi\beta$-plane, and the effect of gravity is neglected, ϕ_e is symmetric with respect to the $\xi\beta$-plane. These assumptions also imply that $\frac{\partial \phi_e}{\partial \eta} = 0$ on $\overline{B}$ (see Fig. 3). Thus we may assume that the $g_j(\xi,\eta)$ are symmetric with respect to the ξ-axis so that we need consider only the lower half of the cross-section of the duct when solving

$$\nabla^2 g(\xi,\eta) - \oint_{\overline{C}} \frac{\partial g}{\partial n} ds = \lambda g(\xi,\eta) \tag{2-1}$$

where

$$g(\xi,\eta) = 0 \quad \text{on} \quad \overline{C} . \tag{2-2}$$

By Green's theorem and the symmetry of g, we have

$$\oint_{\bar{C}} \frac{\partial g}{\partial n}\, ds = \int_{\bar{A}} \nabla^2 g\, d\bar{A}$$

$$= 2\int_{\bar{A}^L} \nabla^2 g\, d\bar{A}^L$$

$$= 2\oint_{\bar{B},\bar{C}^L} \frac{\partial g}{\partial n}\, ds$$

where $\bar{A}^L$, $\bar{B}$, and $\bar{C}^L$ are as indicated in Figure 4. Thus we must solve the equation

$$-\nabla^2 g(\xi,\eta) + 2\int_{\bar{A}^L} \nabla^2 g\, d\bar{A}^L = \lambda g(\xi,\eta) \qquad (2\text{-}3)$$

where

$$g(\xi,\eta) = 0 \quad \text{on} \quad \bar{C}^L \qquad (2\text{-}4)$$

and

$$\frac{\partial g(\xi,\eta)}{\partial \eta} = 0 \quad \text{on} \quad \bar{B}\,. \qquad (2\text{-}5)$$

We choose not to solve equation (2-3) on the lower half of the eccentric annulus but to transform this region onto a rectangle and solve the transformed equation. We use a composition of transformations to achieve our purpose. The first transformation maps the lower half of the eccentric annulus onto the upper half of a concentric annulus with center at the origin.

The second transformation maps the upper half of the concentric annulus onto a rectangle. We now present the details of these transformations.

Considering Figure 3, we recall that nothing has been said about c_1 and c_2, i.e., about the location of the eccentric annulus on the ξ-axis. We locate the eccentric annulus in such a way that the two boundary circles form a coaxial system of circles with the ξ-axis as the line of centers and the η-axis as the radial axis. It is shown in [5] that such a location is possible and that the circles have the following equations:

$$C_1: \quad \xi^2 + \eta^2 + 2\frac{c_1}{\sqrt{A}}\xi + c = 0 \tag{2-6}$$

$$C_2: \quad \xi^2 + \eta^2 + 2\frac{c_2}{\sqrt{A}}\xi + c = 0 \tag{2-7}$$

where c_1, c_2, and c are to be determined. We have that

$$(\xi + \frac{c_1}{\sqrt{A}})^2 + \eta^2 = \frac{c_1^2}{A} - c = \frac{r_1^2}{A}$$

and

$$(\xi + \frac{c_2}{\sqrt{A}})^2 + \eta^2 = \frac{c_2^2}{A} - c = \frac{r_2^2}{A} \quad .$$

Solving the equations

$$c_1^2 - cA = r_1^2$$

$$c_2^2 - cA = r_2^2$$

$$c_2 - c_1 = \rho$$

for c_1, c_2, and c yields

$$c_1 = \frac{r_2^2 - r_1^2 - \rho^2}{2\rho} \tag{2-8}$$

$$c_2 = \frac{r_2^2 - r_1^2 + \rho^2}{2\rho} \tag{2-9}$$

$$c = \frac{c_2^2 - r_2^2}{A} = \frac{c_1^2 - r_1^2}{A} \quad . \tag{2-10}$$

For convenience in this and later chapters, we set

(i) $K_1 = \sqrt{c_1 + r_1}$

(ii) $K_2 = \sqrt{c_1 - r_1}$

(iii) $K_3 = \sqrt{c_2 + r_2}$

(iv) $K_4 = \sqrt{c_2 - r_2}$

(v) $$K_5 = \ln \left[\frac{K_3 + K_4}{K_3 - K_4} \right]$$

(vi) $$K_6 = \ln \left[\frac{K_1 + K_2}{K_1 - K_2} \right]$$

(vii) $$K_7 = K_6 - K_5$$

(viii) $$K_8 = \frac{K_1 - K_2}{K_1 + K_2}$$

(ix) $$K_9 = \frac{(K_1 - K_2)^2}{2K_1K_2}$$

(x) $$K_{10} = \frac{(K_1^2 - K_2^2)^2}{4K_1^2K_2^2}$$

(xi) $$K_{11} = \frac{K_1^2 + K_2^2}{2K_1K_2}$$

(xii) $$K_{12} = \frac{K_3 - K_4}{K_3 + K_4}$$

(xiii) $$K_{13} = \frac{(K_3 - K_4)^2}{2K_3K_4}$$

(xiv) $$K_{14} = \frac{(K_3^2 - K_4^2)^2}{4K_3^2K_4^2}$$

(xv) $$K_{15} = \frac{K_3^2 + K_4^2}{2K_3K_4} .$$

We now show that the bilinear transformation

$$W_1(z) = \xi_1 + i\eta_1 = \frac{z + p}{z - p} \qquad (2\text{-}11)$$

where $z = \xi + i\eta$ and $p = \sqrt{c}$ maps C_1 and C_2 ($\xi\eta$ - plane) onto circles ($\xi_1\eta_1$-plane) whose centers are at the origin. Solving equation (2-11) for ξ and η yields

$$\xi = \frac{p[(\xi_1^2 - 1) + \eta_1^2]}{(\xi_1 - 1)^2 + \eta_1^2} \qquad (2\text{-}12)$$

and

$$\eta = \frac{-2p\eta_1}{(\xi_1 - 1)^2 + \eta_1^2} \quad . \qquad (2\text{-}13)$$

Substitution of equations (2-12) and (2-13) into equations (2-6) and (2-7) yields

$$C_1' \ (\xi_1\eta_1\text{-plane}): \quad \xi_1^2 + \eta_1^2 = \frac{c_1 - \sqrt{c_1^2 - r_1^2}}{c_1 + \sqrt{c_1^2 + r_1^2}} \qquad (2\text{-}14)$$

$$C_2' \ (\xi_1\eta_1\text{-plane}): \quad \xi_1^2 + \eta_1^2 = \frac{c_2 - \sqrt{c_2^2 - r_2^2}}{c_2 + \sqrt{c_2^2 + r_2^2}} \quad . \qquad (2\text{-}15)$$

Figure 5 shows the location of C_1 and C_2 in the $\xi_1\eta_1$-plane (C_1' and C_2' , respectively). Thus $W_1(z)$ maps the circles C_1 and C_2 onto circles C_1' and C_2' with centers at the origin. That $W_1(z)$ maps the lower half of the eccentric annulus onto the upper half of the concentric annulus is seen from the equations

$$\xi_1 = \frac{\xi^2 + \eta^2 - p^2}{(\xi-p)^2 + \eta^2} \tag{2-16}$$

$$\eta_1 = \frac{-2p\eta}{(\xi-p)^2 + \eta^2} \tag{2-17}$$

which are found by solving equation (2-11) for ξ_1 and η_1.

We now apply the transformation

$$W_2(z_1) = \xi_2 + i\eta_2 = \text{Log } W_1 = \log_e|W_1| + i \arg(W_1). \tag{2-18}$$

Since the Log transformation maps circles onto straight lines, the upper half of Figure 5 is transformed onto the rectangle shown in Figure 6. Hence the transformation

$$W_2(z) = \text{Log } \frac{z + p}{z - p} \tag{2-19}$$

maps the region shown in Figure 4 onto the region shown in Figure 6.

The Jacobian J of the transformation $W_2(z)$ is given by

$$J(\xi_2,\eta_2) = (\xi_{\xi_2}\eta_{\eta_2} - \xi_{\eta_2}\eta_{\xi_2})^{-1}$$

where,e.g., ξ_{ξ_2} represents the derivative of ξ with respect to ξ_2 [6]. A straight forward, but tedious calculation shows that

$$J(\xi_2,\eta_2) = \frac{A}{K_1^2K_2^2}(\cosh\xi_2 - \cos\eta_2)^2 . \tag{2-20}$$

Some properties of the Jacobian are presented in Appendix C. In terms of the coordinates ξ_2 and η_2, equation (2-3) is given by

$$-J(\xi_2,\eta_2)\,\nabla^2 g\,(\xi_2,\eta_2) + 2\int_{\bar{A}_2}\nabla^2 g\, d\bar{A}_2 = \lambda g(\xi_2,\eta_2) . \tag{2-21}$$

Conditions (2-4) and (2-5) are given by

$$g(\xi_2,\eta_2) = 0 \quad \text{on} \quad \bar{C}_2 \tag{2-22}$$

and

$$\frac{\partial g(\xi_2,\eta_2)}{\partial\eta_2} = 0 \quad \text{on} \quad \bar{B}_2 , \tag{2-23}$$

respectively (see Fig. 6).

We now make the transformation

$$\xi_3 = \xi_2 - K_5 \tag{2-24}$$

$$\eta_3 = \eta_2 \ . \tag{2-25}$$

The transformed region is shown in Figure 7. The transformed equation is

$$-J(\xi_3,\eta_3)\ \nabla^2 g\ (\xi_3,\eta_3) + 2\int_{\bar{A}_3} \nabla^2 g\ d\bar{A}_3 = \lambda g(\xi_3,\eta_3) \tag{2-26}$$

where

$$g(\xi_3,\eta_3) = 0 \qquad \text{on} \qquad \bar{C}_3 \tag{2-27}$$

and

$$\frac{\partial g(\xi_3,\eta_3)}{\partial \eta_3} = 0 \qquad \text{on} \qquad \bar{B}_3 \ . \tag{2-28}$$

We note that

$$\int_{\bar{A}_3} \frac{1}{J(\xi_3,\eta_3)}\ d\bar{A}_3 = \bar{A}^L = \tfrac{1}{2} \ . \tag{2-29}$$

In terms of the transformed variables ξ_3 and η_3, the equation for the fully developed velocity is given by

$$-J(\xi_3,\eta_3)\ \nabla^2\phi_f(\xi_3,\eta_3) + 2\int_{\bar{A}_3} \nabla^2\phi_f\ d\bar{A}_3 = 0 \qquad (2\text{-}30)$$

where

$$\phi_f\ (\xi_3,\eta_3) = 0 \qquad \text{on} \qquad \bar{C}_3 \qquad (2\text{-}31)$$

and

$$\frac{\partial\phi_f(\xi_3,\eta_3)}{\partial\eta_3} = 0 \qquad \text{on} \qquad \bar{B}_3\ . \qquad (2\text{-}32)$$

For convenience in later chapters, we make the following notation changes:

(i) $\xi_3 \rightarrow x$

(ii) $\eta_3 \rightarrow y$

(iii) $\bar{A}_3 \rightarrow \bar{A}$

(iv) $\bar{B}_3 \rightarrow B$

(v) $\bar{C}_3 \rightarrow C$.

CHAPTER 3

A HILBERT SPACE SETTING FOR $L(g) = \lambda g$

Let us formulate our eigenvalue problem in a Hilbert space setting. Consider the Hilbert space $H_1 = \{L_2(\bar{A});(\ ,\ )\}$ of all square integrable real-valued functions defined on $\bar{A}$. The usual inner product for $L_2(\bar{A})$* is defined by

$$(g,h) = \int_{\bar{A}} gh\ d\bar{A}$$

but for our purposes it is advantageous to consider the inner product

$$(g,h)_J = \int_{\bar{A}} \frac{gh}{J}\ d\bar{A}$$

where $J = J(x,y) = \dfrac{A}{K_1^2K_2^2}\ (\cosh(x+K_5) - \cos y)^2$. We set

$||g|| = \sqrt{(g,g)}$ and $||g||_J = \sqrt{(g,g)_J}$.

LEMMA 3-1: Let $q(x,y)$ be an element of $L_2(\bar{A})$ and $p(x,y)$ be a continuous function defined on $\bar{A}$. Then pq is an element of $L_2(\bar{A})$.

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*We, of course, consider functions equivalent if they differ only on a set of measure zero. In the sequel we will usually require continuous representations from the resulting equivalence classes.

PROOF:

Since $p(x,y)$ is continuous on $\bar{A}$, there exists a real number M such that $|p(x,y)|^2 < M$. Thus $\int_{\bar{A}} (pq)^2\, d\bar{A}$, which is less than $M \int_{\bar{A}} q^2\, d\bar{A}$, is finite since $\int_{\bar{A}} q^2\, d\bar{A}$ is finite.

THEOREM 3-1: $H_2 = \{L_2(\bar{A}); (\ ,\ )_J\}$ is a Hilbert space.

PROOF:

Let g and h be elements of H_2. By Lemma 3-1, $\frac{g}{\sqrt{J}}$ and $\frac{h}{\sqrt{J}}$ are also in H_2. This implies that

$$(g,h)_J = \int_{\bar{A}} \frac{gh}{J}\, d\bar{A} = \int_{\bar{A}} \frac{g}{\sqrt{J}} \frac{h}{\sqrt{J}}\, d\bar{A}$$

is finite, i.e., $(g,h)_J$ is defined. It is easy to see that

(1) $(g,h)_J = (h,g)_J$

(2) $(g + h,k)_J = (g,k)_J + (h,k)_J$

(3) $(\alpha g,h)_J = \alpha(g,h)_J$

where g, h, and k are in H_2 and α is a real number. Also,

$$(g,g)_J = \int_{\bar{A}} \frac{g^2}{J}\, d\bar{A} > 0$$

if $g \neq 0$ since $J(x,y)$ is positive on $\bar{A}$. Thus $(\ ,\ )_J$ defines an inner product. We now prove completeness. Let $\{g_k\}_{k=1}^{\infty}$ be a Cauchy sequence in H_2. This implies that

$$||\frac{g_m}{\sqrt{J}} - \frac{g_n}{\sqrt{J}}||^2 = \int_{\bar{A}} \frac{(g_m - g_n)^2}{J} \, d\bar{A} = ||g_m - g_n||_J$$

converges to zero as m and n approach infinity. Since $\{\frac{g_k}{\sqrt{J}}\}_{k=1}^{\infty}$ is Cauchy is H_1, there exists a g' in H_1 so that

$$||g_n - g||_J = ||\frac{g_n}{\sqrt{J}} - \frac{g}{\sqrt{J}}|| = ||\frac{g_n}{\sqrt{J}} - g'||$$

converges to zero where $g = \sqrt{J}\, g'$. Since g is an element of H_2, H_2 is complete and hence H_2 is a Hilbert space.

THEOREM 3-2: The mapping $T\colon H_1 \to H_2$ defined by

$$T(g) = \sqrt{J}\, g$$

is a Hilbert space isomorphism.

PROOF:

Clearly T defines an isomorphism between H_1 and H_2. Also,

$$(T(g),T(h))_J = \int_{\bar{A}} \frac{(\sqrt{J}g)(\sqrt{J}h)}{J} \, d\bar{A} = (g,h) \; .$$

Thus T defines a Hilbert space isomorphism.

Define a subspace M of H_2 as follows: Let g be an element of M if and only if

(i) g, the first partial derivatives of g, and the second partial derivatives of g are continuous in the closure of $\bar{A}$.

(ii) $g = 0$ on C and $\frac{\partial g}{\partial y} = 0$ on B .

Now define an operator $L: M \to H_2$ by

$$L(g) = -J\nabla^2 g + 2 \int_{\bar{A}} \nabla^2 g \, d\bar{A} .$$

Note that L(g) is continuous in the closed region. Our problem then is to solve the eigenvalue problem

$$L(g) = \lambda g \tag{3-1}$$

where g is an element of M and M is contained in H_2.

THEOREM 3-3: If g is an element of M, then $(1,L(g))_J = 0$.

PROOF:

For g in M, we have that

$$(1,L(g))_J = \int_{\bar{A}} \frac{1}{J} \left(-J\nabla^2 g + 2 \int_{\bar{A}} \nabla^2 g \, d\bar{A}\right) d\bar{A}$$

$$= -\int_{\bar{A}} \nabla^2 g \, d\bar{A} + 2 \int_{\bar{A}} \nabla^2 g \, d\bar{A} \int_{\bar{A}} \frac{1}{J} \, d\bar{A}$$

$$= 0$$

since $\int_{\bar{A}} \frac{1}{J} \, d\bar{A} = \frac{1}{2}$.

COROLLARY 1: If λ, g is a solution of (3-1) for which $\lambda \neq 0$, then $(1,g)_J = 0$.

PROOF:

We have that $\lambda(1,g)_J = (1,\lambda g)_J = (1,L(g))_J = 0$.

THEOREM 3-4: Let λ,g and μ,h be solutions of (3-1) for which $\lambda \neq \mu$. Then

$$(g,h)_J = 0.$$

PROOF:

We have that

$$\begin{aligned}
\lambda(g,h)_J &= (\lambda g,h)_J \\
&= (L(g),h)_J \\
&= \int_{\bar{A}} \frac{(-J\nabla^2 g + 2\int_{\bar{A}} \nabla^2 g \, d\bar{A})\, h}{J} \, d\bar{A} \\
&= -\int_{\bar{A}} h\nabla^2 g \, d\bar{A} + 2\int_{\bar{A}} \nabla^2 g \, d\bar{A} \int_{\bar{A}} \frac{h}{J} \, d\bar{A} \\
&= -\int_{\bar{A}} h\nabla^2 g \, d\bar{A}
\end{aligned}$$

since $(1,h)_J = 0$. Similarly,

$$\mu(h,g)_J = -\int_{\bar{A}} g\nabla^2 h \, d\bar{A} \quad .$$

By Green's Theorem,

$$(\lambda - \mu)\ (g,h)_J = \int_{\bar{A}} (g\nabla^2 h - h\nabla^2 g)d\bar{A}$$

$$= \oint_S (g\frac{\partial h}{\partial n} - h\frac{\partial g}{\partial n})\ ds$$

$$= 0$$

since

$$g = 0 = h \quad \text{on} \quad C$$

and

$$\frac{\partial g}{\partial n} = \frac{\partial g}{\partial y} = 0 = \frac{\partial h}{\partial y} = \frac{\partial h}{\partial n} \quad \text{on} \quad B\ .$$

This completes the proof.

Set $H_3 = \{g \text{ in } H_2 |\ (1,g)_J = 0\}$ and $N = M \bigcap H_3$. Then Theorem 3-3 and Corollary 1 allow us to solve

$$L(g) = \lambda g \tag{3-2}$$

where g is an element of N and N is contained in H_3. We assume from now on that the domain of L is N, i.e., $L:N \to H_3$. Note that H_3 is a Hilbert space since it is a closed subspace of a Hilbert space.

We now show that our operator $L:N \to H_3$ is symmetric, positive, and positive definite. The definition of these properties assumes that the domain of the operator is dense in the space. In the next chapter we will construct a set which is complete in H_3 and whose span is in H; this remark implies that N is dense in H_3.

THEOREM 3-5: The operator $L:N \to H_3$ is symmetric, i.e.,

$$(L(g),h)_J = (g,L(h))_J$$

for all g and h in N.

PROOF:

Let g and h be elements of M. The proof of the Theorem 3-4 shows that

$$(L(g),h)_J = -\int_{\bar{A}} h\nabla^2 g \, d\bar{A}$$

and

$$(g,L(h))_J = -\int_{\bar{A}} g\nabla^2 h \, d\bar{A} \quad .$$

By Green's Theorem

$$\begin{aligned}(L(g),h)_J - (g,L(h))_J &= \int_{\bar{A}} (g\nabla^2 h - h\nabla^2 g) \, d\bar{A} \\ &= \oint_S \left(g\frac{\partial h}{\partial n} - h\frac{\partial g}{\partial n}\right) ds \\ &= 0 \quad .\end{aligned}$$

Hence L is symmetric.

THEOREM 3-6: The operator $L:N \to H_3$ is positive, i.e.,

$$(L(g),g)_J > 0$$

for all g in N, $g \neq 0$.

PROOF:

Let g be an element of N. By Green's Theorem

$$\begin{aligned}(L(g),g)_J &= -\int_{\bar{A}} g\nabla^2 g \, d\bar{A} \\ &= \int_{\bar{A}} \{(\frac{\partial g}{\partial x})^2 + (\frac{\partial g}{\partial y})^2\} \, d\bar{A} - \oint_S g\frac{\partial g}{\partial n} \, ds \\ &= \int_{\bar{A}} \{(\frac{\partial g}{\partial x})^2 + (\frac{\partial g}{\partial y})^2\} d\bar{A} \\ &\geq 0\end{aligned}$$

since $g = 0$ on C and $\frac{\partial g}{\partial n} = 0$ on B. If $(L(g),g)_J = 0$, then $\frac{\partial g}{\partial x} = 0 = \frac{\partial g}{\partial y}$ on $\bar{A}$. This implies that g is a constant function. Since $g = 0$ on C, g must be the zero function. Therefore $(L(g),g)_J > 0$ if $g \neq 0$. The proof is now complete.

THEOREM 3-7: The operator $L:N \to H_3$ is positive definite, i.e., there exists a positive number γ such that

$$(L(g),g)_J \geq \gamma^2 ||g||_J^2$$

for all g in N.

PROOF:

Let g be an element of N. For every (x,y) in $\bar{A}$, we have

$$g(x,y) = g(x,y) - g(0,y)$$
$$= \int_o^x \frac{\partial g(\xi,y)}{\partial \xi} \, d\xi \; .$$

Using the Cauchy-Schwartz inequality, we obtain

$$|g(x,y)|^2 = \left|\int_o^x \frac{\partial g}{\partial \xi} \, d\xi\right|^2$$
$$\leq \int_o^x 1^2 \, d\xi \int_o^x \left(\frac{\partial g}{\partial \xi}\right)^2 d\xi$$
$$\leq K_7 \int_o^{K_7} \left(\frac{\partial g}{\partial \xi}\right)^2 d\xi \; .$$

Hence

$$||g||_J^2 = \int_o^\pi \int_o^{K_7} \frac{|g(x,y)|^2}{J(x,y)} \, dx \, dy$$
$$\leq \int_o^\pi \int_o^{K_7} \left\{K_7 \int_o^{K_7} \left(\frac{\partial g}{\partial \xi}\right)^2 d\xi\right\} \frac{1}{\text{Min } J} \, dx \, dy$$
$$= \frac{K_7^2}{\text{Min } J} \int_o^\pi \int_o^{K_7} \left(\frac{\partial g}{\partial \xi}\right)^2 d\xi \, dy$$
$$= \frac{K_7^2}{\text{Min } J} \int_o^\pi \int_o^{K_7} \left(\frac{\partial g}{\partial x}\right)^2 dx \, dy$$
$$\leq \frac{K_7^2}{\text{Min } J} \int_{\bar{A}} \left\{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2\right\} d\bar{A}$$
$$= \frac{-K_7^2}{\text{Min } J} \int_{\bar{A}} g\nabla^2 g \, d\bar{A}$$
$$= \frac{K_7^2}{\text{Min } J} (L(g),g)_J \; .$$

Setting

$$\gamma = \frac{\sqrt{\text{Min } J}}{K_7} \tag{3-3}$$

we obtain

$$(L(g),g)_J \geq \gamma^2 ||g||_J^2 .$$

Therefore L is positive definite.

We have previously defined an operator L on a dense subspace N (the denseness will be shown in Chapter 4) of a Hilbert space H_3 and have shown that it is symmetric and positive definite. A new inner product $(\, , \,)_E$ for N is defined by setting

$$(g,h)_E = (L(g),h)_J .$$

The completion of N with respect to $(\, , \,)_E$ will be denoted by H_4 and is called the energy space associated with the operator L. It can be shown that H_4 is contained in H_3 [7]. It is in the energy space where convergence of Galerkin's method will be established.

CHAPTER 4

A COMPLETE SEQUENCE FOR THE ENERGY SPACE H_4

The purpose of this chapter is to construct a complete linearly independent sequence for H_4. With this in mind, we first construct a complete sequence for H_2. It is well known that

$$\left\{\sqrt{\frac{2}{K_7}}\,\sin\frac{n\pi x}{K_7}\right\}_{n=1}^{\infty}$$

and

$$\left\{\sqrt{\frac{1}{\pi}}\,,\ \sqrt{\frac{2}{\pi}}\,\cos my\right\}_{m=1}^{\infty}$$

are complete orthonormal sequences for $L_2(0,K_7)$ and $L_2(0,\pi)$, respectively. Hence

$$S_1 = \left\{\sqrt{\frac{2}{\pi K_7}}\,\sin\frac{n\pi x}{K_7}\right\}_{n=1}^{\infty} \cup$$

$$\left\{\frac{2}{\sqrt{\pi K_7}}\,\sin\frac{n\pi x}{K_7}\cos my\right\}_{m,n=1}^{\infty}$$

is a complete orthonormal sequence for H_1 [8]. Since

H_1 and H_2 are isometrically isomorphic via multiplication by $\sqrt{J(x,y)}$,

$$S_2 = \{\sqrt{\frac{2}{\pi K_7}}\sqrt{J(x,y)} \sin \frac{n\pi x}{K_7}\}_{n=1}^{\infty} \bigcup$$

$$\{\frac{2}{\sqrt{\pi K_7}}\sqrt{J(x,y)} \sin \frac{n\pi x}{K_7} \cos my\}_{m,n=1}^{\infty}$$

is a complete orthonormal sequence for H_2.

THEOREM 4-1: The sequence

$$S_3 = \{\sqrt{\frac{2}{\pi K_7}} J(x,y) \sin \frac{n\pi x}{K_7}\}_{n=1}^{\infty} \bigcup$$

$$\{\frac{2}{\sqrt{\pi K_7}} J(x,y) \sin \frac{n\pi x}{K_7} \cos my\}_{m,n=1}^{\infty}$$

is complete in H_2.

PROOF:

Let g be an element of H_2. Then $g' = \frac{g}{\sqrt{J}}$ is an element of H_2. Let ε be a positive number and set $\varepsilon' = \varepsilon^2 \text{ Min } (\frac{1}{J})$. Denote the elements of S_1 by s_k. Since S_2 is complete in H_2, there exists an element $\sum_{k=1}^{q} c_k \sqrt{J}\, s_k$ in the Span (S_2) such that

$$||g' - \sum_{k=1}^{q} c_k \sqrt{J}\, s_k||_J^2 < \varepsilon' \ .$$

We have that

$$\varepsilon' > ||g' - \sum_{k=1}^{q} c_k \sqrt{J}\, s_k||_J^2$$

$$= \int_{\bar{A}} \frac{\frac{1}{J}\,(g - \sum_{k=1}^{q} c_k\, Js_k)^2}{J}\, d\bar{A}$$

$$\geq [\mathrm{Min}\ (\tfrac{1}{J})] \int_{\bar{A}} \frac{(g - \sum_{k=1}^{q} c_k\, Js_k)^2}{J}\, d\bar{A}$$

so that

$$[\mathrm{Min}\ (\tfrac{1}{J})]\ ||g - \sum_{k=1}^{q} c_k\, Js_k||_J^2 < \varepsilon' \ .$$

This implies that

$$||g - \sum_{k=1}^{q} c_k\, Js_k||_J < \varepsilon$$

where $\sum_{k=1}^{q} c_k\, Js_k$ is an element of the Span (S_3). Therefore S_3 is complete in H_2.

The next theorem shows one way of using S_3 to form a complete sequence for H_3.

THEOREM 4-2: Let h be an element of a Hilbert space $\{H;(\ ,\ )\}$ and C the orthogonal complement of h:

$$C = \{g \text{ in } H \mid (h,g) = 0\} .$$

Assume that $S \cup T = \{s_k\}_{k=1}^{\infty} \cup \{t_j\}_{j=1}^{\infty}$ is linearly independent and complete in H where S is contained in C and T is contained in $H \backslash C$. If

$$g = \sum_{k=1}^{\infty} a_k\, s_k + \sum_{j=1}^{\infty} b_j\, t_j$$

is an element of C, then g is an element of the $\overline{\text{Span}\ (S \cup T')}$ where

$$T' = \left\{\frac{t_j}{(h,t_j)} - \frac{t_{j+1}}{(h,t_{j+1})}\right\}_{j=1}^{\infty} .$$

PROOF:

Let

$$g = \sum_{k=1}^{\infty} a_k\, s_k + \sum_{j=1}^{\infty} b_j\, t_j$$

be an element of C. We first consider the special case where only a finite number of the b_j are different from zero. Then g has the form

$$g = \sum_{k=1}^{\infty} a_k \, s_k + \sum_{j=1}^{q} b_j \, t_j$$

and so we must find c_j's such that

$$\sum_{j=1}^{q} b_j \, t_j = \sum_{j=1}^{q-1} c_j \left\{ \frac{t_j}{(h,t_j)} - \frac{t_{j+1}}{(h,t_{j+1})} \right\}$$

$$= c_1 \frac{t_1}{(h,t_1)} + (c_2 - c_1) \frac{t_2}{(h,t_2)}$$

$$+ \cdot \cdot \cdot + (c_{q-1} - c_{q-2}) \frac{t_{q-1}}{(h,t_{q-1})}$$

$$- c_{q-1} \frac{t_q}{(h,t_q)} \quad .$$

By equating coefficients, we obtain

$$b_1 = \frac{c_1}{(h,t_1)}$$

$$b_2 = \frac{c_2 - c_1}{(h,t_2)}$$

$$\vdots$$

$$b_{q-1} = \frac{c_{q-1} - c_{q-2}}{(h,t_{q-1})}$$

$$b_q = \frac{-c_{q-1}}{(h,t_q)} \quad .$$

Solving for the c_j's we obtain

$$c_1 = b_1(h,t_1)$$

$$c_2 = c_1 + b_2(h,t_2) = \sum_{j=1}^{2} b_j(h,t_j)$$

$$\vdots$$

$$c_{q-1} = c_{q-2} + b_{q-1}(h,t_{q-1}) = \sum_{j=1}^{q-1} b_j(h,t_j) .$$

Also $c_{q-1} = -b_q(h,t_q)$. It is clear that we can find $c_1,\cdots,c_{q-2}$ but we have two equations for finding c_{q-1}. It is easy to see that both equations yield the same value for c_{q-1}. In fact, we have that

$$0 = (h,g)$$

$$= \sum_{k=1}^{\infty} a_k (h,s_k) + \sum_{j=1}^{q} b_j (h,t_j)$$

$$= \sum_{j=1}^{q} b_j (h,t_j)$$

which implies that

$$\sum_{j=1}^{q-1} b_j(h,t_j) = -b_q(h,t_q).$$

Thus $\sum_{j=1}^{q} b_j t_j$ is in the set $\overline{\text{Span }(T')}$.

We now prove the general case by showing that

$$\sum_{j=1}^{\infty} b_j t_j = \sum_{j=1}^{\infty} c_j \left\{\frac{t_j}{(h,t_j)} - \frac{t_{j+1}}{(h,t_{j+1})}\right\}$$

where $c_j = \sum_{k=1}^{j} b_k (h,t_k)$. We have that

$$\left|\left|\sum_{j=1}^{\infty} b_j t_j - \sum_{j=1}^{\infty} c_j \left\{\frac{t_j}{(h,t_j)} - \frac{t_{j+1}}{(h,t_{j+1})}\right\}\right|\right|$$

$$= \left|\left|\sum_{j=1}^{\infty} b_j t_j - \sum_{j=1}^{\infty} \left(\sum_{k=1}^{j} b_k (h,t_k)\right) \left\{\frac{t_j}{(h,t_j)} - \frac{t_{j+1}}{(h,t_{j+1})}\right\}\right|\right|$$

$$= \left|\left|\sum_{j=1}^{\infty} b_j t_j - \sum_{j=1}^{\infty} \left(\sum_{k=1}^{j} b_k (h,t_k)\right) \frac{t_j}{(h,t_j)}\right.\right.$$

$$\left.\left. + \sum_{j=2}^{\infty} \left(\sum_{k=1}^{j-1} b_k (h,t_k)\right) \frac{t_j}{(h,t_j)}\right|\right|$$

$$= \left|\left|\sum_{j=2}^{\infty} b_j t_j - \sum_{j=2}^{\infty} (b_j (h,t_j)) \frac{t_j}{(h,t_j)}\right|\right|$$

$$= 0 \quad .$$

This completes the proof.

COROLLARY 1: If $S \cup T$ is a complete orthonormal sequence in H, then $S \cup T'$ is complete in C.

COROLLARY 2: Let

$$S_4 = \left\{ \sqrt{\frac{2}{\pi K_7}}\, J(x,y) \sin \frac{n\pi x}{K_7} \right\}_{\text{even } n} \cup$$

$$\left\{ \frac{2}{\sqrt{\pi K_7}}\, J(x,y) \sin \frac{n\pi x}{K_7} \cos my \right\}_{m,n=1}^{\infty}$$

and

$$S_5 = \left\{ \frac{1}{2K_7}\, J(x,y) \left\{ n \sin \frac{n\pi x}{K_7} - (n+2) \sin \frac{(n+2)\pi x}{K_7} \right\} \right\}_{\text{odd } n} .$$

Then $S_4 \cup S_5$ is complete in H_3 .

PROOF:

Let g be an element of H_3. Since S_2 is a complete orthonormal sequence for H_2 ,

$$\frac{g}{\sqrt{J}} = \sqrt{\frac{2}{\pi K_7}} \sum_{n=1}^{\infty} a_n \sqrt{J(x,y)} \sin \frac{n\pi x}{K_7} +$$

$$\frac{2}{\sqrt{\pi K_7}} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_{mn} \sqrt{J(x,y)} \sin \frac{n\pi x}{K_7} \cos my .$$

Therefore $g = \sqrt{\frac{2}{\pi K_7}} \sum_{n=1}^{\infty} a_n\, J(x,y) \sin \frac{n\pi x}{K_7} +$

$$\frac{2}{\sqrt{\pi K_7}} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_{mn}\, J(x,y) \sin \frac{n\pi x}{K_7} \cos my .$$

Thus each element of H_3 can be represented as an infinite linear combination of elements of S_3. The set S_3 is linearly independent and complete in H_3. It is easy to see that each element of S_4 is orthogonal to the constant function one and that

$$\left(1, \sqrt{\frac{2}{\pi K_7}}\, J(x,y) \sin \frac{n\pi x}{K_7}\right)_J = \frac{2}{n}\sqrt{\frac{2K_7}{\pi}}$$

if n is odd. Also if n is odd,

$$\frac{\sqrt{\frac{2}{\pi K_7}}\, J(x,y) \sin \frac{n\pi x}{K_7}}{\frac{2}{n}\sqrt{\frac{2K_7}{\pi}}} - \frac{\sqrt{\frac{2}{\pi K_7}}\, J(x,y) \sin \frac{(n+2)\pi x}{K_7}}{\frac{2}{(n+2)}\sqrt{\frac{2K_7}{\pi}}}$$

$$= \frac{1}{2K_7} J(x,y) \left\{ n \sin \frac{n\pi x}{K_7} - (n+2) \sin \frac{(n+2)\pi x}{K_7} \right\} .$$

The conclusion now follows from the theorem by setting $H = H_2$, $(\,,\,) = (\,,\,)_J$, $h=1$, $C = H_3$, $S = S_4$, $T = S_3 \setminus S_4$, and $T' = S_5$.

COROLLARY 3: The domain of the operator

$$L(g) = - J\nabla^2 g + 2 \int_{\bar{A}} \nabla^2 g \; d\bar{A}$$

is dense in H_3, ie, $\bar{N} = H_3$.

PROOF:

It is sufficient to show that $S_4 \cup S_5$ is contained in N since N is a subspace. Clearly each element of $S_4 \cup S_5$ satisfies condition (i) and the first part of (ii) in the definition of M(see Chapter 3). Let

$$f(x,y) = \sqrt{\frac{2}{\pi K_7}}\ J(x,y)\ \sin\frac{n\pi x}{K_7}\ .$$

Then

$$\frac{\partial f}{\partial y} = \sqrt{\frac{2}{\pi K_7}}\,\frac{\partial J}{\partial y}\,\sin\frac{n\pi x}{K_7}\ .$$

Since $\frac{\partial J}{\partial y} = 0$ on B, $\frac{\partial f}{\partial y} = 0$ on B.

Let

$$g(x,y) = \frac{2}{\sqrt{\pi K_7}}\ J(x,y)\ \sin\frac{n\pi x}{K_7}\cos my\ .$$

Then

$$\frac{\partial g}{\partial y} = \frac{2}{\sqrt{\pi K_7}}\sin\frac{n\pi x}{K_7}\left\{ -m\ J(x,y)\ \sin my + \frac{\partial J}{\partial y}\cos my\right\}\ .$$

Thus $\frac{\partial g}{\partial y} = 0$ on B. Hence each element in $S_4 \cup S_5$ satisfies the last part of (ii). Also, each element of $S_4 \cup S_5$ is orthogonal to the constant function equal to 1. Therefore N is dense in H_3.

We are now in a position to construct a complete set for H_4. The construction is based on the following:

THEOREM 4-3: Let $L: N \to H_3$ be a symmetric positive definite operator and let $\{g_n\}_{n=1}^{\infty}$ be a subset of N. If $\{Lg_n\}_{n=1}^{\infty}$ is complete in H_3, then $\{g_n\}_{n=1}^{\infty}$ is complete in H_4, the energy space associated with H_3 [9].

For our operator

$$L(g) = -J\nabla^2 g + 2\int_{\bar{A}} \nabla^2 g \, d\bar{A}$$

it is easy to see that

$$L(S_6) = S_4$$

and

$$L(S_7) = S_5$$

where

$$S_6 = \left\{ \sqrt{\frac{2}{\pi K_7}} \left(\frac{K_7}{n\pi}\right)^2 \sin \frac{n\pi x}{K_7} \right\}_{\text{even } n} \bigcup$$

$$\left\{ \frac{2}{\sqrt{\pi K_7}} \frac{1}{\left(\frac{n\pi}{K_7}\right)^2 + m^2} \sin \frac{n\pi x}{K_7} \cos my \right\}_{m,n=1}^{\infty}$$

and

$$S_7 = \{\frac{1}{2K_7}\left[n\left(\frac{K_7}{n\pi}\right)^2 \sin\frac{n\pi x}{K_7} - (n+2)\left(\frac{K_7}{(n+2)\pi}\right)^2 \sin\frac{(n+2)\pi x}{K_7}\right]\}_{\text{odd } n} .$$

Since $S_4 \cup S_5$ is complete in H_3, Theorem 4-3 implies that $S_6 \cup S_7$ is complete in H_4. We state this result as

THEOREM 4-4: $S_6 \cup S_7$ is complete in H_4.

The next theorem follows from the definition of $S_6 \cup S_7$ and the linear independence of S_1.

THEOREM 4-5: $S_6 \cup S_7$ is linearly independent.

CHAPTER 5

THE GALERKIN METHOD AND ITS CONVERGENCE

Galerkin's method is a technique for finding approximate solutions of certain types of operator equations defined on Hilbert spaces. We now present a particular formulation of this method and state conditions sufficient for convergence. Then we will show that these conditions hold for our operator equation.

Let L be a symmetric positive definite linear operator defined on a dense subspace N of the Hilbert space $\{H_3; (\ , \ )_J\}$ of real-valued square integrable functions defined on a closed domain $\bar{A}$ contained in E^2 which are orthogonal to the constant function one. Consider the eigenvalue operator equation

$$L(g) = \lambda g \tag{5-1}$$

where g must satisfy certain homogeneous boundary conditions. Select a linearly independent sequence $F = \{f_i\}_{i=1}^{\infty}$ of functions with the following properties:

(i) each element of F is in the domain of L .

(ii) each element of F satisfies the same homogeneous boundary conditions as does g .

(iii) F is complete with respect to the energy space $\{H_4; (\, , \,)_E\}$ associated with H_3.

The set F is called a set of coordinate functions for the eigenvalue problem. We call a function h_{nj} of the form

$$h_{nj} = \sum_{i=1}^{n} a_{ij}^{n} f_i$$

the n'th Galerkin approximation of a solution g_j of equation (5-1) with respect to the n'th Galerkin approximation λ_{nj} of the j'th eigenvalue λ_j if the a_{ij}^{n} and λ_{nj} are defined as follows: first, we substitute h_{nj} for g in equation (5-1) and require that

$$(L(h_{nj}) - \lambda h_{nj}, f_m)_J = 0 \tag{5-2}$$

for $m = 1, 2, \cdots, n$. We have that

$$\begin{aligned}
&(L(h_{nj}) - \lambda h_{nj}, f_m)_J \\
&= (L(\sum_{i=1}^{n} a_{ij}^{n} f_i) - \lambda \sum_{i=1}^{n} a_{ij}^{n} f_i, f_m)_J \\
&= \sum_{i=1}^{n} a_{ij}^{n} \{(L(f_i), f_m)_J - \lambda (f_i, f_m)_J\} \\
&= \sum_{i=1}^{n} a_{ij}^{n} \{(f_i, f_m)_E - \lambda (f_i, f_m)_J\} \quad .
\end{aligned}$$

Thus equation (5-2) yields the homogeneous system

$$\sum_{i=1}^{n} a_{ij}^{n} \{(f_i, f_m)_E - \lambda (f_i, f_m)_J\} = 0 \qquad (5\text{-}3)$$

$$(m = 1, 2, \cdots, n)$$

of n equations in n unknowns. For it to have a unique solution it is necessary and sufficient that the determinant D_n of the system be equal to zero, where D_n is as shown on the next page. Setting $D_n = 0$ yields a polynomial in λ, the roots of which we denote by λ_{nj} $(j = 1, \cdots, n)$. Without loss of generality we assume $\lambda_{n1} \leq \lambda_{n2} \leq \cdots \leq \lambda_{nn}$. Now to find the a_{ij}^{n} corresponding to λ_{nj}, we simply solve system (5-3) with λ replaced by λ_{nj}.

We now state a sequence of theorems which lead to the convergence of Galerkin's method. S.G. Mikhlin has proved the following theorems:

THEOREM 5-1: Assume that every bounded set in H_4 is compact in H_3. Then

(i) there exists a countable number of eigenvalues of $L(g) = \lambda g$: $0 < \lambda_1 \leq \lambda_2 \leq \cdots \lambda_j \leq \cdots$; $\lim_{j \to \infty} \lambda_j = \infty$.

(ii) the sequence $\{g_j\}_{j=1}^{\infty}$ of eigenvectors is complete in both H_3 and H_4 [7].

$$\begin{vmatrix} (f_1,f_1)_E - \lambda(f_1,f_1)_J & (f_1,f_2)_E - \lambda(f_1,f_2)_J & \cdots & (f_1,f_n)_E - \lambda(f_1,f_n)_J \\ (f_2,f_1)_E - \lambda(f_2,f_1)_J & (f_2,f_2)_E - \lambda(f_2,f_2)_J & & (f_2,f_n)_E - \lambda(f_2,f_n)_J \\ \vdots & & & \vdots \\ (f_n,f_1)_E - \lambda(f_n,f_1)_J & \cdots & & (f_n,f_n)_E - \lambda(f_n,f_n)_J \end{vmatrix}$$

COROLLARY 1: L^{-1} is completely continuous in H_4.

THEOREM 5-2: Assume that L^{-1} is completely continuous in H_4. Then for each j, λ_{nj} converges to an eigenvalue λ_j of equation (5-1) as $n \to \infty$. Conversely, every eigenvalue of equation (5-1) is the limit point of one of the Galerkin approximate sequences, i.e., of $\{\lambda_{nj}\}_{n=1}^{\infty}$ for some j [10].

N.I. Pol'skii has proved the following:

THEOREM 5-3: Assume that L^{-1} is completely continuous in H_4. Let $\{\lambda_{nj}\}_{n=1}^{\infty}$ be a Galerkin approximate sequence of eigenvalues and $\{h_{nj}\}_{n=1}^{\infty}$ the corresponding Galerkin approximate sequence of normalized eigenvectors. Then $\{h_{nj}\}_{n=1}^{\infty}$ is compact in H_4 and every limit point of this set is an eigenvector of equation (5-1). Conversely, every eigenvector of equation (5-1) is a limit point of one of the Galerkin approximate sequences of normalized eigenvectors [11].

COROLLARY 1: Assume that g_j is a limit point of $\{h_{nj}\}_{n=1}^{\infty}$ so that $h_{n_k j}$ converges to g_j in H_4. Then $h_{n_k j}$ converges to g_j in H_3, i.e., in the mean.

We now consider the case where L is the operator

$$L(g) = -J\nabla^2 g + 2 \int_{\bar{A}} \nabla^2 g \, d\bar{A}$$

and where N, $(\, , \,)_J$, H_3, etc. are defined as in Chapter 3. Since we have already shown that this operator L is symmetric and positive definite on H_3 and that its domain N is dense in H_3, the previous theorems show that Galerkin's method will converge if any set bounded in H_4 is compact in H_3. As an aid in proving this we introduce the Hilbert space $W_2^{(1)}(\bar{A}) = W$ of functions in $H_1 = L_2(\bar{A})$ having generalized first derivatives (see Appendix G for details). The usual norm for W, $||g||_W$, is not convenient for our purposes. We use an equivalent norm for W, $||g||_{W^*}$, defined by

$$||g||^2_{W^*} = |B(g)|^2 + \int_{\bar{A}} \{D_x^{*2} g + D_y^{*2} g\} \, d\bar{A}$$

where B(g) can be any bounded linear functional with respect to $||g||_W$ which does not vanish for the constant function $h(x,y) = 1$ [2]. We define $B(g) = \int_C g \, ds$; note that we are using our particular $\bar{A}$ to define B(g). We now show that this B(g) satisfies the required conditions. We have that

$$|B(g)|^2 \leq (\int_C |g| \, ds)^2 .$$

Using the Cauchy-Schwartz inequality, we obtain

$$|B(g)|^2 \leq 2\pi \int_C g^2 \, ds$$

$$\leq 2\pi \int_S g^2 \, ds$$

$$\leq 2\pi k\{ \int_{\bar{A}} \{g^2 + D_x^{*2} g + D_y^{*2} g\} \, d\bar{A} \}$$

$$= 2\pi k \, ||g||_W^2$$

where k is some positive real number [7]. Also,

$$B(h) = \int_C h \, ds = 2\pi \neq 0 .$$

Thus our B(g) satisfies the required conditions.

THEOREM 5-4: For $L(g) = -J\nabla^2 g + 2 \int_{\bar{A}} \nabla^2 g \, d\bar{A}$, H_4 is isometrically isomorphic to a subspace of $W_2^{(1)}$ $(\bar{A})$.

PROOF:

For g in N, we have that

$$||g||_{W^*}^2 = |\int_C g \, ds|^2 + \int_{\bar{A}} \{(\frac{\partial g}{\partial x})^2 + (\frac{\partial g}{\partial y})^2\} \, d\bar{A}$$

$$= \int_{\bar{A}} \{(\frac{\partial g}{\partial x})^2 + (\frac{\partial g}{\partial y})^2\} \, d\bar{A}$$

$$= (L(g), g)_J$$

$$= ||g||_E^2 \quad . \qquad (5\text{-}4)$$

Let g be an element of $H_4 \setminus N$. Then there exists a sequence $\{g_k\}_{k=1}^{\infty}$ contained in N such that $||g_k - g||_E \to 0$ as $k \to \infty$. Since $\{g_k\}_{k=1}^{\infty}$ is Cauchy in H_4, inequality (5-4) shows that it is Cauchy in W. Thus there exists a $\hat{g}$ in W such that $||g_k - \hat{g}||_{W^*} \to 0$ as $k \to \infty$. We define

$P\colon H_4 \to W$ by

$$P(g) = \begin{cases} g & \text{if } g \text{ is in } N \\ \hat{g} & \text{if } g \text{ is in } H_4 \setminus N \end{cases} .$$

It is not difficult to prove that P is well defined and linear. To see that P is one-to-one, assume that $P(g) = \hat{g}$ and $P(h) = \hat{g}$. Then there exist $\{g_k\}_{k=1}^{\infty}$ and $\{h_k\}_{k=1}^{\infty}$ contained in N such that $||g_k - g||_E \to 0$ and $||h_k - h||_E \to 0$. Since $P(g - h) = 0$, $||g_k - h_k||_{W^*} \to 0$. Thus $g_k - h_k$ converges to 0 weakly in W and hence in H_4. Thus for p in N, $(g_k - h_k, p)_E \to 0$, ie, $(g - h, p)_E = 0$. Since N is dense in H_4 we have that $g = h$; therefore P is one-to-one. Thus P is an isomorphism into W. The proof that P is an isometry is as follows. Let $P(g) = \hat{g}$ and $\{g_k\}_{k=1}^{\infty}$ be a sequence in N which converges to g in H_4. Then

$$
\begin{aligned}
||g||_E &= \lim_{k\to\infty} ||g_k||_E \\
&= \lim_{k\to\infty} ||g_k||_{W^*} \\
&= ||\hat{g}||_{W^*} \quad .
\end{aligned}
$$

The proof is now complete.

COROLLARY 1: For all g in H_4

$$||g||_E = ||g||_{W^*} \quad .$$

The importance of being able to consider H_4 as a subspace of W is seen in the following theorem:

THEOREM 5-5 (Imbedding Theorem): The injection mapping from W into $H_1 = L_2(\bar{A})$ is completely continuous [12].

From a previous remark, the next theorem establishes convergence of Galerkin's method for our eigenvalue problem.

THEOREM 5-6: For $L(g) = -J\nabla^2 g + 2 \int_{\bar{A}} \nabla^2 g \, d\bar{A}$, any set bounded in H_4 is compact in H_3.

PROOF:

Let G be a bounded set in H_4. By Corollary 1 of Theorem 5-4, G is bounded in W. Hence by the Imbedding Theorem, G is compact in H_1. Since H_1 and H_2 are isometrically isomorphic, G is compact in H_2 and hence in H_3. This completes the proof.

CHAPTER 6

THE GALERKIN DETERMINANT

We now construct the Galerkin determinant for our eigenvalue problem. The set $S_6 \cup S_7$ (see Chapter 4) is linearly independent and complete in H_4. Each of its elements is in the domain of our operator

$$L(g) = -J(x,y)\nabla^2 g(x,y) + 2 \int_{\bar{A}} \nabla^2 g \, d\bar{A}$$

and satisfies the required homogeneous boundary conditions. We choose this set to be our set of coordinate functions. For convenience, we set

$$F = \{f_i\}_{i=1}^{\infty} = S_6 \cup S_7 .$$

Since the elements of the Galerkin determinant D_n (see Chapter 5) are of the form $(f_i,f_j)_E - \lambda(f_i,f_j)_J$, it is necessary to evaluate the inner products $(f_i,f_j)_J$ and $(f_i,f_j)_E$ for various combinations of elements of $S_6 \cup S_7$.

The possibilities for the inner product $(f_i,f_j)_J$ are given by

$$\text{(i)}\quad \frac{2}{\pi K_7}\left(\frac{K_7}{n\pi}\right)^2\left(\frac{K_7}{p\pi}\right)^2\frac{K_1^2K_2^2}{A}\int_0^{K_7}\int_0^{\pi}\frac{\sin\frac{n\pi x}{K_7}\sin\frac{p\pi x}{K_7}}{(\cosh(x+K_5)-\cos y)^2}\,dy\,dx$$

(n,p even)

$$\text{(ii)}\quad \frac{2\sqrt{2}}{\pi K_7}\left(\frac{K_7}{n\pi}\right)^2\frac{1}{\left(\frac{p\pi}{K_7}\right)^2+q^2}\frac{K_1^2K_2^2}{A}\int_0^{K_7}\int_0^{\pi}\frac{\sin\frac{n\pi x}{K_7}\sin\frac{p\pi x}{K_7}\cos qy}{(\cosh(x+K_5)-\cos y)^2}\,dy\,dx$$

(n even)

$$\text{(iii)}\quad \sqrt{\frac{2}{\pi K_7}}\left(\frac{K_7}{n\pi}\right)^2\frac{K_1^2K_2^2}{A}\left\{\frac{p}{2K_7}\left(\frac{K_7}{p\pi}\right)^2\right.$$

$$\int_0^{K_7}\int_0^{\pi}\frac{\sin\frac{n\pi x}{K_7}\sin\frac{p\pi x}{K_7}}{(\cosh(x+K_5)-\cos y)^2}\,dy\,dx$$

$$-\frac{p+2}{2K_7}\left(\frac{K_7}{(p+2)\pi}\right)^2$$

$$\left.\int_0^{K_7}\int_0^{\pi}\frac{\sin\frac{n\pi x}{K_7}\sin\frac{(p+2)\pi x}{K_7}}{(\cosh(x+K_5)-\cos y)^2}\,dy\,dx\right\}$$

(n even, p odd)

$$\text{(iv)}\quad \frac{4}{\pi K_7}\frac{1}{\left(\frac{n\pi}{K_7}\right)^2+m^2}\frac{1}{\left(\frac{p\pi}{K_7}\right)^2+q^2}\frac{K_1^2K_2^2}{A}$$

$$\int_o^{K_7}\int_o^{\pi} \frac{\sin\frac{n\pi x}{K_7}\cos my \sin\frac{p\pi x}{K_7}\cos qy}{(\cosh(x+K_5) - \cos y)^2}\, dy\, dx$$

(v) $$\frac{2}{\sqrt{\pi K_7}} \frac{1}{(\frac{n\pi}{K_7})^2 + m^2} \frac{K_1^2 K_2^2}{A}$$

$$\{\frac{p}{2K_7}\left(\frac{K_7}{p\pi}\right)^2 \int_o^{K_7}\int_o^{\pi} \frac{\sin\frac{n\pi x}{K_7}\cos my \sin\frac{p\pi x}{K_7}}{(\cosh(x+K_5) - \cos y)^2}\, dy\, dx$$

$$- \frac{(p+2)}{2K_7}\left(\frac{K_7}{(p+2)\pi}\right)^2 \int_o^{K_7}\int_o^{\pi} \frac{\sin\frac{n\pi x}{K_7}\cos my \sin\frac{(p+2)\pi x}{K_7}}{(\cosh(x+K_5) - \cos y)^2}\, dy\, dx\ \}$$

(p odd)

(iv) $$\frac{1}{(2K_7)^2} \frac{K_1^2 K_2^2}{A} \{n\left(\frac{K_7}{n\pi}\right)^2 p\left(\frac{K_7}{p\pi}\right)^2$$

$$\int_o^{K_7}\int_o^{\pi} \frac{\sin\frac{n\pi x}{K_7}\sin\frac{p\pi x}{K_7}}{(\cosh(x+K_5) - \cos y)^2}\, dy\, dx$$

$$-n\left(\frac{K_7}{n\pi}\right)^2 (p+2)\left(\frac{K_7}{(p+2)}\right)^2$$

$$\int_o^{K_7}\int_o^{\pi} \frac{\sin\frac{n\pi x}{K_7}\sin\frac{(p+2)\pi x}{K_7}}{(\cosh(x+K_5) - \cos y)^2}\, dy\, dx$$

$$-(n+2)\ \left(\frac{K_7}{(n+2)\pi}\right)^2\ p\left(\frac{K_7}{p\pi}\right)^2$$

$$\int_0^{K_7}\int_0^{\pi} \frac{\sin\frac{(n+2)\pi x}{K_7}\ \sin\frac{p\pi x}{K_7}}{(\cosh(x+K_5)\ -\ \cos\ y)^2}\ dy\ dx$$

$$+\ (n+2)\ \left(\frac{K_7}{(n+2)\pi}\right)^2\ (p+2)\ \left(\frac{K_7}{(p+2)\pi}\right)^2$$

$$\int_0^{K_7}\int_0^{\pi} \frac{\sin\frac{(n+2)\pi x}{K_7}\ \sin\frac{(p+2)\pi x}{K_7}}{(\cosh(x+K_5)\ -\ \cos\ y)^2}\ dy\ dx\ \}$$

(n, p odd).

Thus each inner product $(f_i,f_j)_J$ requires the evaluation of an integral of the form

$$I = \int_0^{K_7}\int_0^{\pi} \frac{\sin\frac{t\pi x}{K_7}\cos uy\ \sin\frac{v\pi x}{K_7}\cos wy}{(\cosh(x+K_5)\ -\ \cos\ y)^2}\ dy\ dx$$

where t and v are positive integers and u and w are nonnegative integers. A closed form expression for I is derived in Appendix E.

The possibilities for the inner products $(f_i,f_j)_E$

are given by

(vii) $$\frac{2}{\pi K_7} \left(\frac{K_7}{p\pi}\right)^2 \int_o^{K_7}\int_o^{\pi} \sin \frac{n\pi x}{K_7} \sin \frac{p\pi x}{K_7}\, dy\, dx$$

(n, p even)

(viii) $$\frac{2\sqrt{2}}{K_7} \frac{1}{\left(\frac{p\pi}{K_7}\right)^2 + q^2} \int_o^{K_7}\int_o^{\pi} \sin \frac{n\pi x}{K_7} \sin \frac{p\pi x}{K_7} \cos qy\, dy\, dx$$

(n even)

(ix) $$\sqrt{\frac{2}{\pi K_7}} \left\{\frac{p}{2K_7} \left(\frac{K_7}{p\pi}\right)^2 \int_o^{K_7}\int_o^{\pi} \sin \frac{n\pi x}{K_7} \sin \frac{p\pi x}{K_7}\, dy\, dx\right.$$

$$\left. - \frac{p+2}{2K_7} \left(\frac{K_7}{(p+2)\pi}\right)^2 \int_o^{K_7}\int_o^{\pi} \sin \frac{n\pi x}{K_7} \sin \frac{(p+2)\pi x}{K_7}\, dy\, dx\right\}$$

(n even, p odd)

(x) $$\frac{4}{\pi K_7} \frac{1}{\left(\frac{p\pi}{K_7}\right)^2 + q^2} \int_o^{K_7}\int_o^{\pi} \sin \frac{n\pi x}{K_7} \cos my \sin \frac{p\pi x}{K_7} \cos qy\, dy\, dx$$

(xi) $$\frac{2}{\sqrt{\pi K_7}} \left\{\frac{p}{2K_7} \left(\frac{K_7}{p\pi}\right)^2 \int_o^{K_7}\int_o^{\pi} \sin \frac{n\pi x}{K_7} \cos my \sin \frac{p\pi x}{K_7}\, dy\, dx\right.$$

$$- \frac{(p+2)}{2K_7} \left(\frac{K_7}{(p+2)\pi}\right)^2$$

$$\left.\int_o^{K_7}\int_o^{\pi} \sin \frac{n\pi x}{K_7} \cos my \sin \frac{(p+2)\pi x}{K_7}\, dy\, dx\right\}$$

(p odd)

(xii) $\frac{1}{4K_7^2}$ { $n\ p\ (\frac{K_7}{p\pi})^2$

$$\int_0^{K_7}\int_0^{\pi} \sin \frac{n\pi x}{K_7} \sin \frac{p\pi x}{K_7}\, dy\, dx$$

$$- n(p+2)\ (\frac{K_7}{(p+2)\pi})^2$$

$$\int_0^{K_7}\int_0^{\pi} \sin \frac{n\pi x}{K_7} \sin \frac{(p+2)\pi x}{K_7}\, dy\, dx$$

$$- (n+2)\ p(\frac{K_7^2}{p\pi})^2$$

$$\int_0^{K_7}\int_0^{\pi} \sin \frac{(n+2)\pi x}{K_7} \sin \frac{p\pi x}{K_7}\, dy\, dx$$

$$+ (n+2)\ (p+2)\ (\frac{K_7}{(p+2)\pi})^2$$

$$\int_0^{K_7}\int_0^{\pi} \sin \frac{(n+2)\pi x}{K_7} \sin \frac{(p+2)\pi x}{K_7}\, dy\, dx\ \}$$

(n, p odd) .

The integrals in (viii), (ix), and (xi) are always zero. The expression in (vii) is zero unless $n = p$ in which case it is equal to $\frac{K_7^2}{\pi n^2}$. The expression in

(x) is zero unless n = p and m = q in which case it is equal to $\frac{1}{(\frac{n\pi}{K_7})^2 + m^2}$. The possible values for the expression in (xii) are given by the following

THEOREM 6-1: If f_i and f_j are elements of S_7, then

$$(f_i, f_j)_E = \begin{cases} -\frac{K_7}{8} & \text{if } n = p - 2 \\ \frac{K_7}{4} & \text{if } n = p \\ -\frac{K_7}{8} & \text{if } n = p + 2 \\ 0 & \text{otherwise} . \end{cases}$$

We have now determined closed form expressions for all of the inner products required in the Galerkin determinant.

Once we have chosen n elements from F and have evaluated the appropriate inner products, we must determine the λ's which satisfy $D_n = 0$. Solving $D_n = 0$ for λ is equivalent to solving the matrix eigenvalue problem

$$A_n x = \lambda B_n x \tag{6-1}$$

where

$$A_n = \begin{bmatrix} (f_1,f_1)_E & (f_1,f_2)_E & \cdots & (f_1,f_n)_E \\ (f_2,f_1)_E & (f_2,f_2)_E & & \vdots \\ \vdots & & & \vdots \\ (f_n,f_1)_E & \cdots & & (f_n,f_n)_E \end{bmatrix}$$

and

$$B_n = \begin{bmatrix} (f_1,f_1)_J & (f_1,f_2)_J & \cdots & (f_1,f_n)_J \\ (f_2,f_1)_J & (f_2,f_2)_J & & \vdots \\ \vdots & & & \vdots \\ (f_n,f_1)_J & & \cdots & (f_n,f_n)_J \end{bmatrix}.$$

If λ_{nj} is an eigenvalue of equation (6-1) and x_{nj} is the corresponding eigenvector, then

$$x_{nj} = (a^n_{ij},\ a^n_{2j},\ \cdots,\ a^n_{nj})$$

gives the coefficients for the approximation

$$h_{nj} = \sum_{i=1}^{n} a_{ij}^{n} f_i \; .$$

Since F is linearly independent and the determinants of A_n and B_n are Gram determinants, A_n and B_n are nonsingular [9]. In fact, since the operator

$$L(g) = -J\nabla^2 g + 2 \int_{\bar{A}} \nabla^2 g \, d\bar{A}$$

is positive definite, the matrix A_n is positive definite. This implies that A_n can be written as

$$A_n = C_n C_n^T$$

for some nonsingular matrix C_n. Defining $\mu = \frac{1}{\lambda}$, we now write equation (6-1) as

$$B_n x = \mu C_n C_n^T x$$

which we in turn write as

$$G_n y = \mu y \tag{6-2}$$

where

$$G_n = C_n^{-1} B_n C_n^{-T}$$

and

$$y = C_n^T x \quad .$$

If μ_{nj} is an eigenvalue of equation (6-2) and y_{nj} is the corresponding eigenvector, then

$$\lambda_{nj} = \frac{1}{\mu_{nj}}$$

is an eigenvalue of equation (6-1) and

$$x_{nj} = C_n^{-T} y_{nj}$$

is the corresponding eigenvector. Thus we have reduced the problem of finding the λ_{nj}'s which satisfy $D_n = 0$ and the corresponding a_{ij}^n's for h_{nj} to the problem of finding the eigenvalues and eigenvectors of the real symmetric matrix G_n; techniques for solving this latter problem are widely known.

The number n of coordinate functions to use in a particular approximation as well as which f_i to choose from S_6 and which f_i to choose from S_7 is arbitrary. The next theorem shows, however, that approximations with all the f_i in S_6 do not contribute to the solution for ϕ_e, the difference velocity.

LEMMA 6-1: The coefficients c_j for the difference velocity ϕ_e are given by

$$c_j = \frac{-(\phi_f, g_j)_J}{||g_j||_J^2} \quad . \tag{6-3}$$

PROOF:

From Chapter 1 and Chapter 2, we have that

$$\text{(i)} \quad \phi_e(x,y,0) = \sum_{j=1}^{\infty} c_j g_j(x,y)$$

$$\text{(ii)} \quad \phi_o = \phi(x,y,0) = 1$$

$$\text{(iii)} \quad \phi_e(x,y,0) = \phi_o - \phi_f(x,y) \quad .$$

This implies that

$$\int_{\bar{A}} \frac{(\phi_o - \phi_f)g_j}{J} = \int_{\bar{A}} \frac{\phi_e g_j}{J} \, d\bar{A}$$

$$= \sum_{k=1}^{\infty} c_k \int_{\bar{A}} \frac{g_k g_j}{J} \, d\bar{A}$$

$$= \sum_{k=1}^{\infty} c_k (g_k, g_j)_J$$

$$= c_j \, ||g_j||_J^2$$

by Theorem 3-4. We also have that

$$\int_{\bar{A}} \frac{(\phi_o - \phi_f)g_j}{J} = \int_{\bar{A}} \frac{\phi_o g_j}{J}\, d\bar{A} - \int_{\bar{A}} \frac{\phi_f g_j}{J}\, d\bar{A}$$

$$= (1, g_j) - (\phi_f, g_j)_J$$

$$= -(\phi_f, g_j)_J$$

by Corollary 1 of Theorem 3-3. Therefore

$$c_j = \frac{-(\phi_f, g_j)_J}{||g_j||_J^2} \qquad .$$

THEOREM 6-2: Let

$$g_j = \sum_{k=1}^{\infty} a_k s_k$$

be a solution of $L(g) = \lambda g$ where s_k is in S_6 for every k. Then the coefficient c_j corresponding to g_j in the series expansion of ϕ_e is zero.

PROOF:

We set $\nabla^2 s_k = b_k s_k$. We have that

$$\lambda(\phi_f, g_j)_J = (\phi_f, -J\nabla^2 g_j + 2 \int_{\bar{A}} \nabla^2 g_j \, d\bar{A})_J$$

$$= (\phi_f, -J\nabla^2 g_j + 2 \sum_{k=1}^{\infty} a_k b_k \int_{\bar{A}} s_k \, d\bar{A})_J$$

$$= (\phi_f, -J\nabla^2 g_j)_J$$

since $\int_{\bar{A}} s_k \, d\bar{A} = 0$ for every k. By Green's theorem,

$$(\phi_f, -J\nabla^2 g_j) = - \int_{\bar{A}} \phi_f \nabla^2 g_j \, d\bar{A}$$

$$= - \int_{\bar{A}} g_j \nabla^2 \phi_f \, d\bar{A} + \oint_S (g_j \frac{\partial \phi_f}{\partial n} - \phi_f \frac{\partial g_j}{\partial n}) \, ds$$

$$= - \int_{\bar{A}} g_j \nabla^2 \phi_f \, d\bar{A}$$

since $g_j = 0 = \phi_f$ on C and $\frac{\partial g_j}{\partial n} = 0 = \frac{\partial \phi_f}{\partial n}$ on B.

We have that

$$- \int_{\bar{A}} g_j \nabla^2 \phi_f \, d\bar{A} = - \int_{\bar{A}} J\nabla^2 \phi_f \frac{g_j}{J} \, d\bar{A}$$

$$= -\int_{\bar{A}} \frac{dp}{dz} \frac{g_j}{J} \, d\bar{A}$$

$$= 0$$

since $\frac{dp}{dz}$ is a constant and $\int_{\bar{A}} \frac{g_j}{J}\, d\bar{A} = (1,g_j)_J = 0.$

Therefore $(\phi_f,g_j)_J = 0$. This completes the proof.

CHAPTER 7

THE FULLY DEVELOPED VELOCITY

The fully developed velocity ϕ_f (in dimensionless variables) is a solution of

$$\nabla^2 \phi_f(\xi,\eta) = \frac{dp}{dz} \tag{7-1}$$

where

$$\phi_f(\xi,\eta) = 0 \quad \text{on} \quad \overline{C} \tag{7-2}$$

(see Fig. 2 of Appendix A). Heyda [14] has solved the dimensional form of equation (7-1). His method of solution was to write ϕ_f as the integral of the Green's function for the Laplacian operator defined on an eccentric annulus. The evaluation of this integral leads to a series solution for ϕ_f. Heyda's solution is defined on the original annulus. In order to have a solution defined on the rectangle, we now solve the transformed fully developed velocity equation. We use the method of Fourier to obtain an expression for ϕ_f in terms of the coordinate functions which were used in Galerkin's method.

In Chapter 2, we saw that the transformed fully developed velocity $\phi_f(x,y)$ is a solution of the equation

$$-J(x,y)\nabla^2 g(x,y) + 2\int_{\bar{A}} \nabla^2 g(x,y)\, d\bar{A} = 0 \tag{7-3}$$

where

$$g(x,y) = 0 \quad \text{on} \quad C \tag{7-4}$$

and

$$\frac{\partial g(x,y)}{\partial y} = 0 \quad \text{on} \quad B\ . \tag{7-5}$$

Since $2\int_{\bar{A}} \nabla^2 g(x,y)\, d\bar{A}$ represents the pressure drop,

we can assume that $\int_{\bar{A}} \nabla^2 g\, d\bar{A} \neq 0$. Under this assumption equation (7-3) can be written as

$$\nabla^2 \bar{g} = \frac{2}{J(x,y)} \tag{7-6}$$

where

$$\bar{g} = \frac{g}{\int_{\bar{A}} \nabla^2 g\, d\bar{A}}\ .$$

The boundary conditions become

$$\bar{g}(x,y) = 0 \quad \text{on} \quad C \tag{7-7}$$

and

$$\frac{\partial \bar{g}(x,y)}{\partial y} = 0 \qquad \text{on} \qquad B \quad . \tag{7-8}$$

We consider the Hilbert space $H_1 = \{L_2(\bar{A});(\ ,\ )\}$ which was defined in Chapter 3. Since a complete orthonormal sequence for H_1 is given by

$$S_1 = \{\sqrt{\frac{2}{\pi K_7}} \sin \frac{n\pi x}{K_7}\}_{n=1}^{\infty} \cup \{\frac{2}{\sqrt{\pi K_7}} \sin \frac{n\pi x}{K_7} \cos my\}_{m,n=1}^{\infty} \quad ,$$

the function $\bar{g}(x,y)$ can be represented in the form

$$\bar{g}(x,y) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{K_7}$$

$$+ \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} b_{mn} \sin \frac{n\pi x}{K_7} \cos my \quad . \tag{7-9}$$

Since each element of S_1 satisfies conditions (7-7) and (7-8), so does $\bar{g}(x,y)$. Substituting equation (7-9) into equation (7-6) yields

$$- \sum_{n=1}^{\infty} \left(\frac{n\pi}{K_7}\right)^2 a_n \sin \frac{n\pi x}{K_7}$$

$$- \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left\{ \left(\frac{n\pi}{K_7}\right)^2 + m^2 \right\} b_{mn} \sin \frac{n\pi x}{K_7} \cos my = \frac{2}{J(x,y)} .$$

Since the elements of S_1 are orthonormal, it follows that

$$\text{(i)} \quad a_n = \frac{-4K_7 K_1^2 K_2^2}{n^2 \pi^3 A} \int_0^{K_7} \int_0^{\pi} \frac{\sin \frac{n\pi x}{K_7}}{(\cosh(x+K_5) - \cos y)^2} \, dy \, dx$$

$$\text{(ii)} \quad b_{mn} = \frac{-8K_7 K_1^2 K_2^2}{\pi\{(n\pi)^2 + (mK_7)^2\} A}$$

$$\int_0^{K_7} \int_0^{\pi} \frac{\sin \frac{n\pi x}{K_7} \cos my}{(\cosh(x+K_5) - \cos y)^2} \, dy \, dx .$$

Closed form expressions for the integrals contained in (i) and (ii) are derived in Appendix F. Using these results, the Fourier series solution of equation (7-6) can be written in closed form.

CHAPTER 8

APPROXIMATIONS OF THE VELOCITY $\phi(\xi,\eta,\beta)$

The purpose of this chapter is to outline a procedure for finding approximations to $\phi(\xi,\eta,\beta)$, the z-component of the velocity in dimensionless form. Recall that in Chapter 2, we made the following notation changes:

(i) $\xi_3 \to x$

(ii) $\eta_3 \to y$

(iii) $\bar{A}_3 \to \bar{A}$

(iv) $\bar{B}_3 \to B$

(v) $\bar{C}_3 \to C$.

We now revert to the original notation so that (ξ_3,η_3) again refers to a point on the rectangle (see Fig. 7).*

The first step in our procedure is the selection of the eccentric annulus parameters: r_1, r_2, and ρ. The symbol r_1 represents the radius of the inner circle, r_2

*The figures for Chapter 8 are contained in Appendix A.

represents the radius of the outer circle and ρ represents the eccentricity, i.e., the distance between the centers. Once r_1, r_2, and ρ are chosen, we calculate the following additional parameters;

(i) The area A of the original annulus:

$$A = \pi(r_2^2 - r_1^2).$$

(ii) The location of the annulus on the ξ-axis: c_1, c_2 (see Equations (2-8) and (2-9) and Fig. 2).

(iii) The parameters related to the Jacobian:

$K_1, \cdots, K_4$ (see Chapter 2).

(iv) The parameters related to the rectangle:

$K_5, \cdots, K_7$ (see Chapter 2) .

(v) The parameters related to the $(f_i, f_j)_J$ inner products and the coefficients of the fully developed velocity: $K_8, \cdots, K_{15}$.

(see Chapter 2, Appendix E, and Appendix F).

(vi) The parameter p defined in the transformation $W_1(z)$ (see Equation (2-11)).

(vii) The minimum and maximum values of the Jacobian defined on the rectangle: Min(J), Max(J) (see Chapter 2 and Fig. 7).

(viii) A positive definite bound γ for our operator L(g) (see Equation (3-3)).

In terms of the coordinates ξ_3, η_3, and β, the velocity ϕ is given by

$$\phi(\xi_3,\eta_3,\beta) = \phi_e(\xi_3,\eta_3,\beta) + \phi_f(\xi_3,\eta_3)$$

(see Equation (1-19)). The fully developed velocity $\phi_f(\xi_3,\eta_3)$ is given by

$$\phi_f(\xi_3,\eta_3) = \sum_{p=1}^{\infty} a_p \sin \frac{p\pi x}{K_7} + \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} b_{pq} \sin \frac{p\pi x}{K_7} \cos qy$$

(see Equation (7-9)). We approximate it by truncating this series when the coefficients a_p, b_{pq} become smaller than some preassigned value. We now approximate the difference velocity ϕ_e:

$$\phi_e(\xi_3,\eta_3,\beta) = \sum_{j=1}^{\infty} c_j g_j(\xi_3,\eta_3)\exp(-\lambda_j\beta)$$

(see Equation (1-16)). First, we select the number, say n, of g_j's to use in our approximation of ϕ_e. Then we select n coordinate functions from $F = S_6 \bigcup S_7$ to use to approximate the g_j's (see Chapter 4). The approximation h_{nj} of g_j is given by

$$h_{nj} = \sum_{i=1}^{n} a_{ij}^{n} f_i \qquad (j = 1,\cdots, n)$$

(see Chapter 5). We must determine the approximation λ_{nj} to λ_j and the a_{ij}^{n}'s for $j = 1,\cdots, n$. To do this we first calculate the inner products $(f_i,f_j)_J$ and $(f_i,f_j)_E$ for $i, j = 1,\cdots, n$. (see Chapter 6 and Appendix E). We then form the matrices

$$A_n = \begin{bmatrix} (f_1,f_1)_E & (f_1,f_2)_E & \cdots & (f_1,f_n)_E \\ (f_2,f_1)_E & (f_2,f_2)_E & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \\ (f_n,f_1)_E & \cdots & & (f_n,f_n)_E \end{bmatrix}$$

and

$$B_n = \begin{bmatrix} (f_1,f_1)_J & (f_1,f_2)_J & \cdot & \cdot & \cdot & (f_1,f_n)_J \\ (f_2,f_1)_J & (f_2,f_2)_J & & & & \cdot \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ \cdot & & & & & \\ (f_n,f_1)_J & \cdot & \cdot & \cdot & & (f_n,f_n)_J \end{bmatrix} .$$

Next, we find a Cholesky decomposition of A_n

$$A_n = C_n C_n^T$$

and form the matrix

$$G_n = C_n^{-1} B_n C_n^{-T}$$

(see Chapter 6). We then find the eigenvalues μ_{nj} and the eigenvectors y_{nj} of G_n $(j = 1,\cdots, n)$.

It follows that

$$\lambda_{nj} = \frac{1}{\mu_{nj}}$$

and

$$C_n^{-T} y_{nj} = (a_{ij}^n, \cdots, a_{nj}^n) \qquad (j = 1, \cdots, n).$$

To complete our approximation of ϕ_e, we need approximations for the c_j's. We find these by using the formula

$$c_j = \frac{-(\phi_f, g_j)_J}{||g_j||_J^2}$$

wherein we replace ϕ_f with its truncated series approximation and g_j with h_{nj} (see Equation (6-3)).

At this point, we have our approximation of ϕ in terms of the coordinates ξ_3, η_3, and β, i.e., we have $\phi(\xi_3, \eta_3, \beta)$. Since we want $\phi(\xi, \eta, \beta)$, we make the following substitutions:

$$\text{(i)} \quad \xi_3 = \tfrac{1}{2} \ln \left[\frac{(\xi+p)^2 + \eta^2}{(\xi-p)^2 + \eta^2}\right] - K_5$$

$$\text{(ii)} \quad \eta_3 = \arctan \left[\frac{-2p\eta}{\xi^2 + \eta^2 - p^2}\right]$$

(see Chapter 2). Recall from Chapter 1 that

$$\beta(z^*) = \frac{z^*}{\sqrt{A}\ R} \ .$$

We want the dimensionless variable β to be a function of z, i.e., we want $\beta(z) = \dfrac{z}{\sqrt{A}\ R}$. It is shown in [2] that

$$\beta(z) = \frac{z}{\sqrt{A}\ R} = \frac{\int_o^\beta \varepsilon(\beta)\ d\beta}{\sqrt{A}\ R}$$

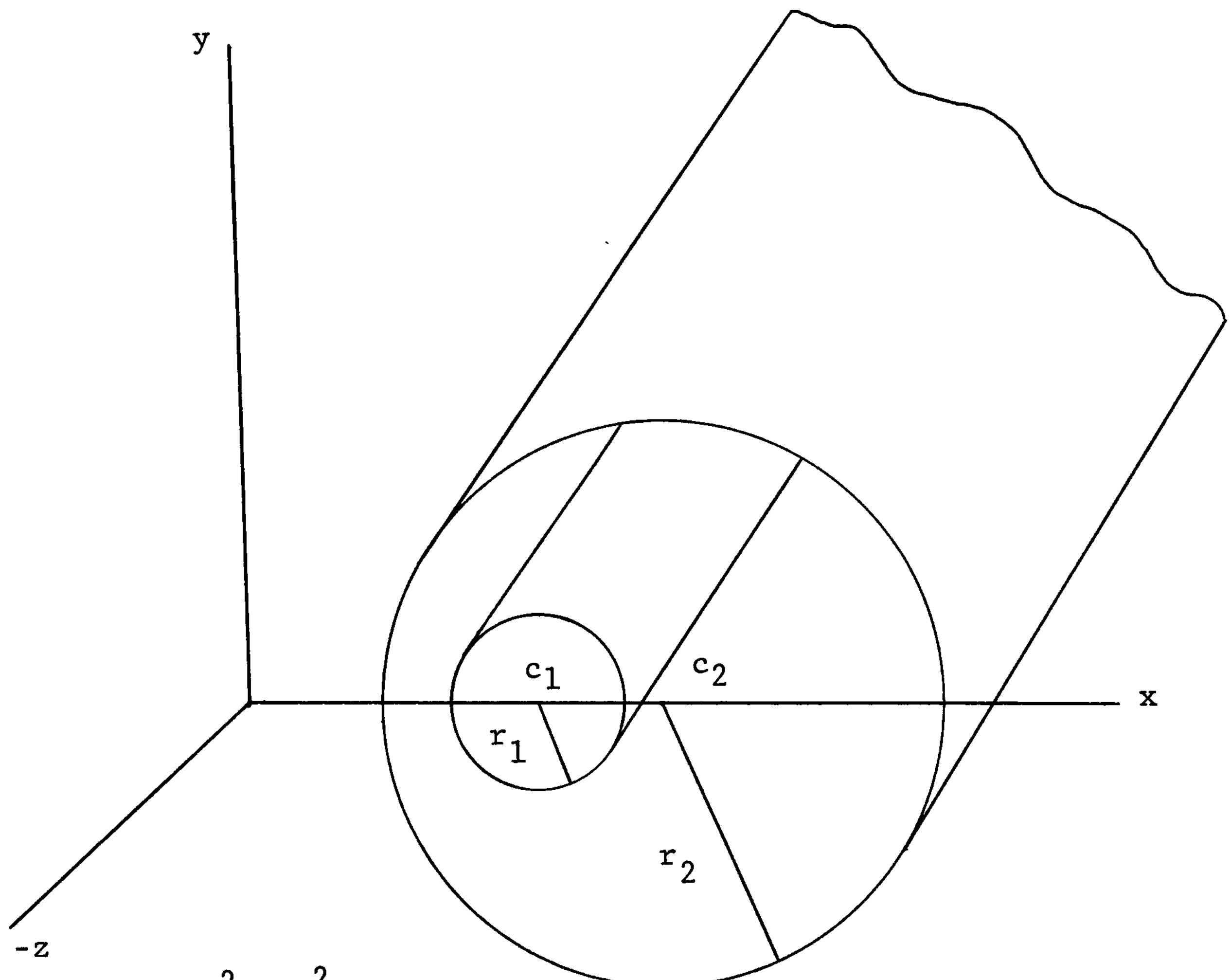
where

$$\varepsilon(\beta) = \frac{\frac{3}{2}\int_A \phi^2(\xi,\eta,\beta)\frac{\partial\phi_e}{\partial\beta}\ dA}{\int_A \phi(\xi,\eta,\beta)\frac{\partial\phi_e}{\partial\beta}\ dA} - 2\ .$$

Using $\beta = \beta(z)$ in $\phi(\xi,\eta,\beta)$, we have the desired approximation to the velocity.

APPENDIX A

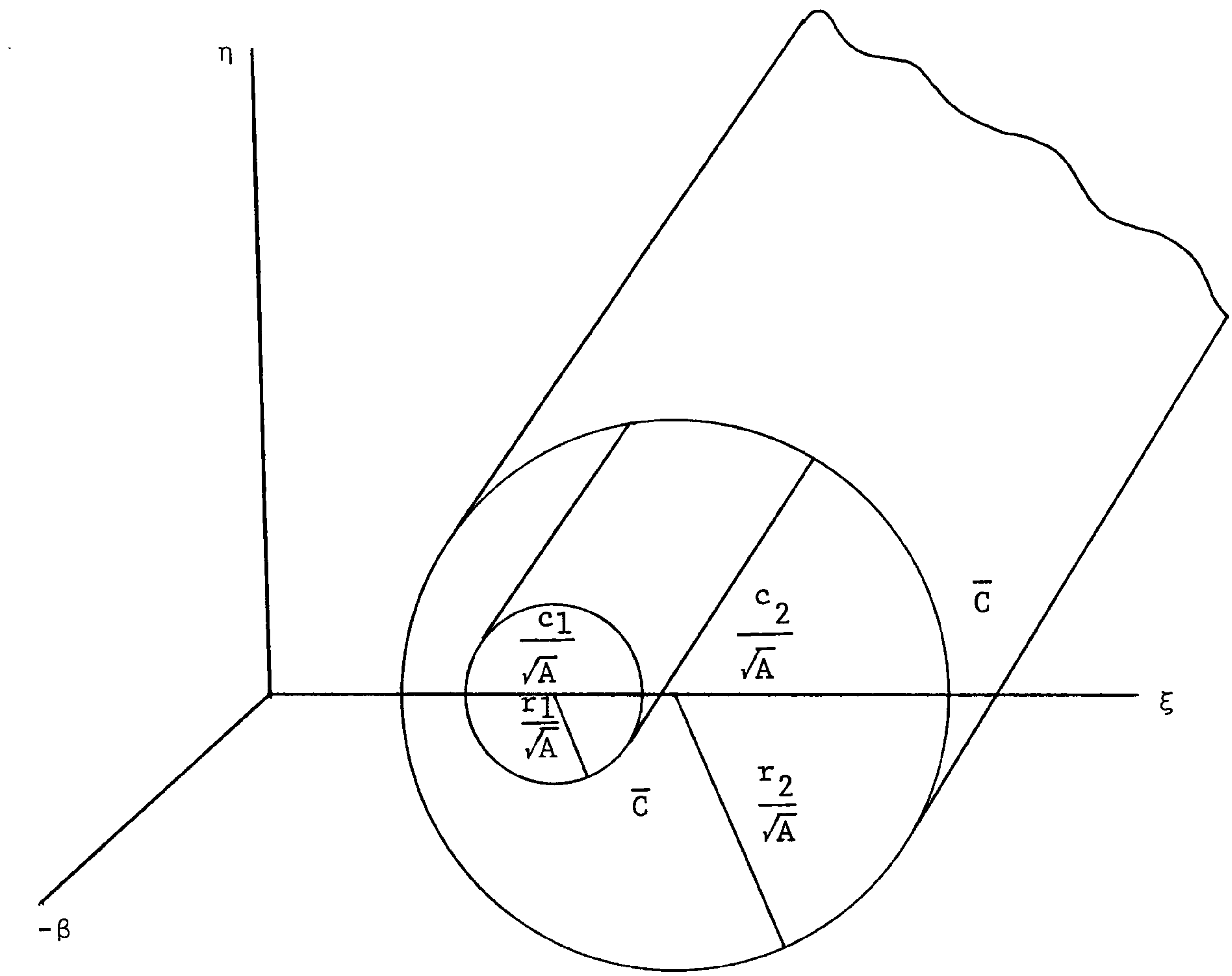
FIGURES



$$A = \pi(r_2^2 - r_1^2)$$

$$\rho = c_2 - c_1$$

FIGURE 1



$$\xi = \frac{x}{\sqrt{A}}$$

$$\eta = \frac{y}{\sqrt{A}}$$

$$\bar{A} = \frac{\pi}{A}\,(r_2^2 - r_1^2) = 1$$

FIGURE 2

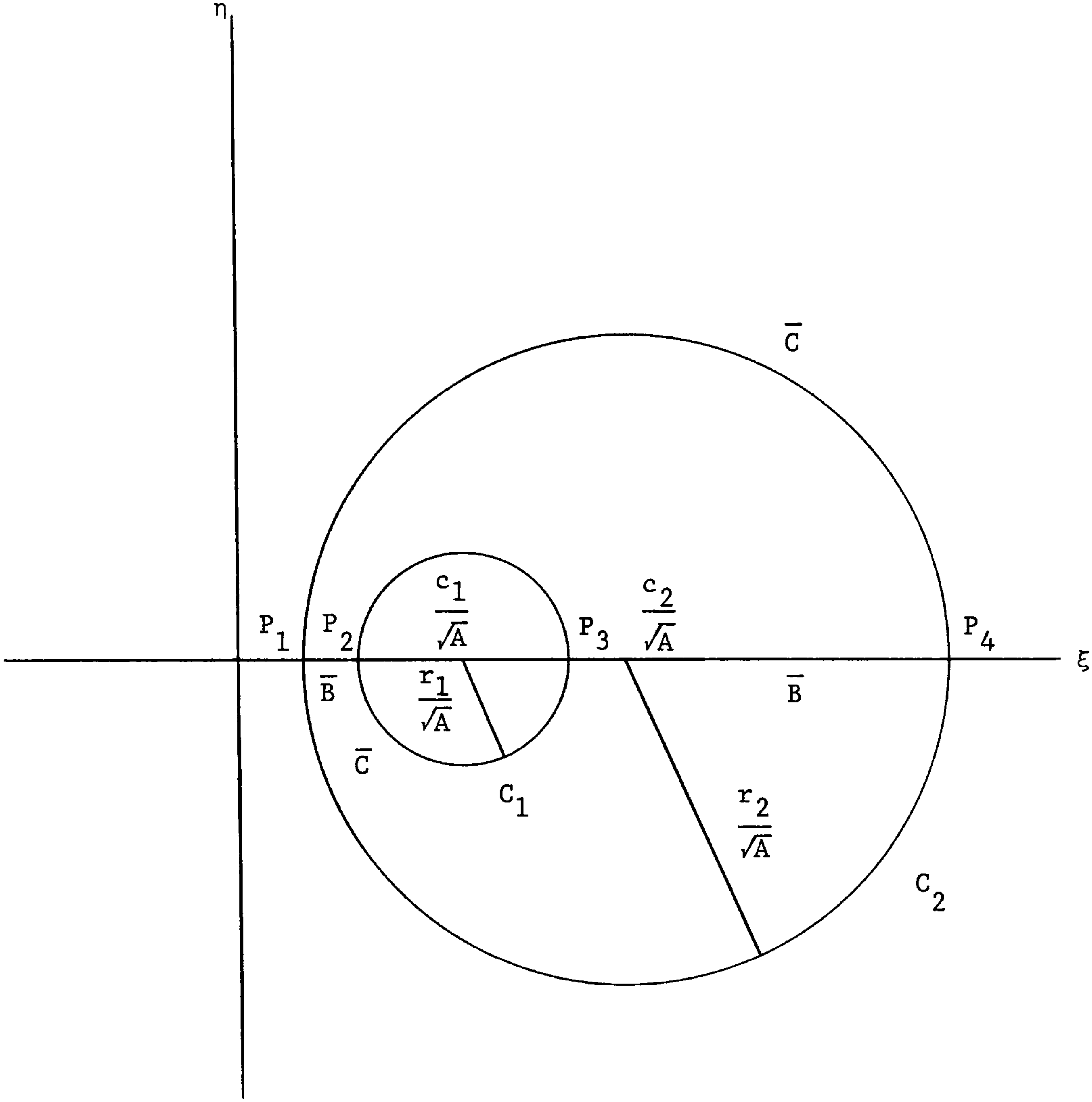
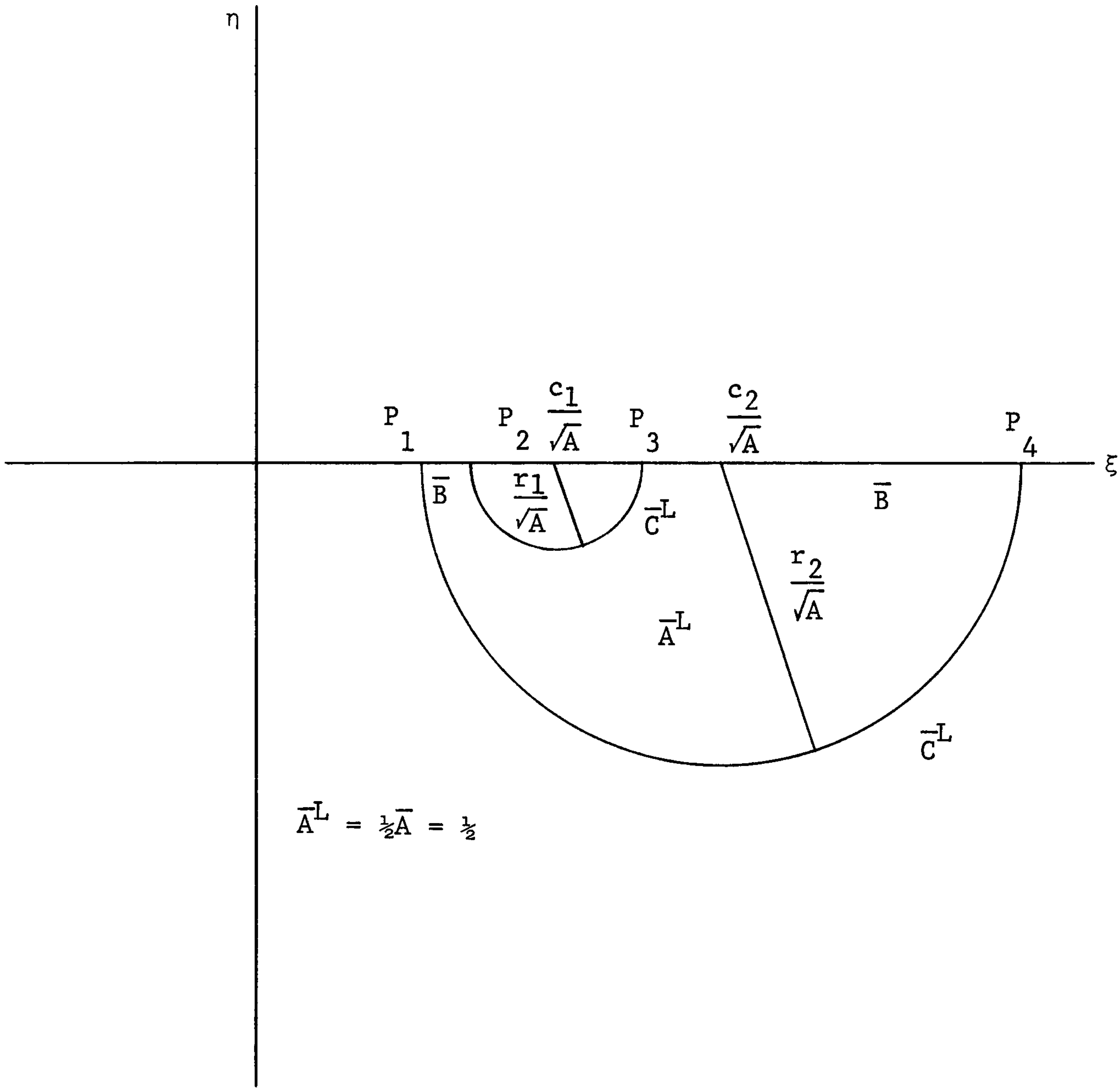

η
ξ
P_1
P_2
P_3
P_4
$\frac{c_1}{\sqrt{A}}$
$\frac{c_2}{\sqrt{A}}$
$\frac{r_1}{\sqrt{A}}$
$\frac{r_2}{\sqrt{A}}$
$\bar{B}$
$\bar{B}$
$\bar{C}$
$\bar{C}$
C_1
C_2

FIGURE 3



η
ξ
P_1
P_2
$\frac{c_1}{\sqrt{A}}$
P_3
$\frac{c_2}{\sqrt{A}}$
P_4
$\bar{B}$
$\frac{r_1}{\sqrt{A}}$
$\bar{C}^L$
$\bar{B}$
$\frac{r_2}{\sqrt{A}}$
$\bar{A}^L$
$\bar{C}^L$
$\bar{A}^L = \frac{1}{2}\bar{A} = \frac{1}{2}$

FIGURE 4

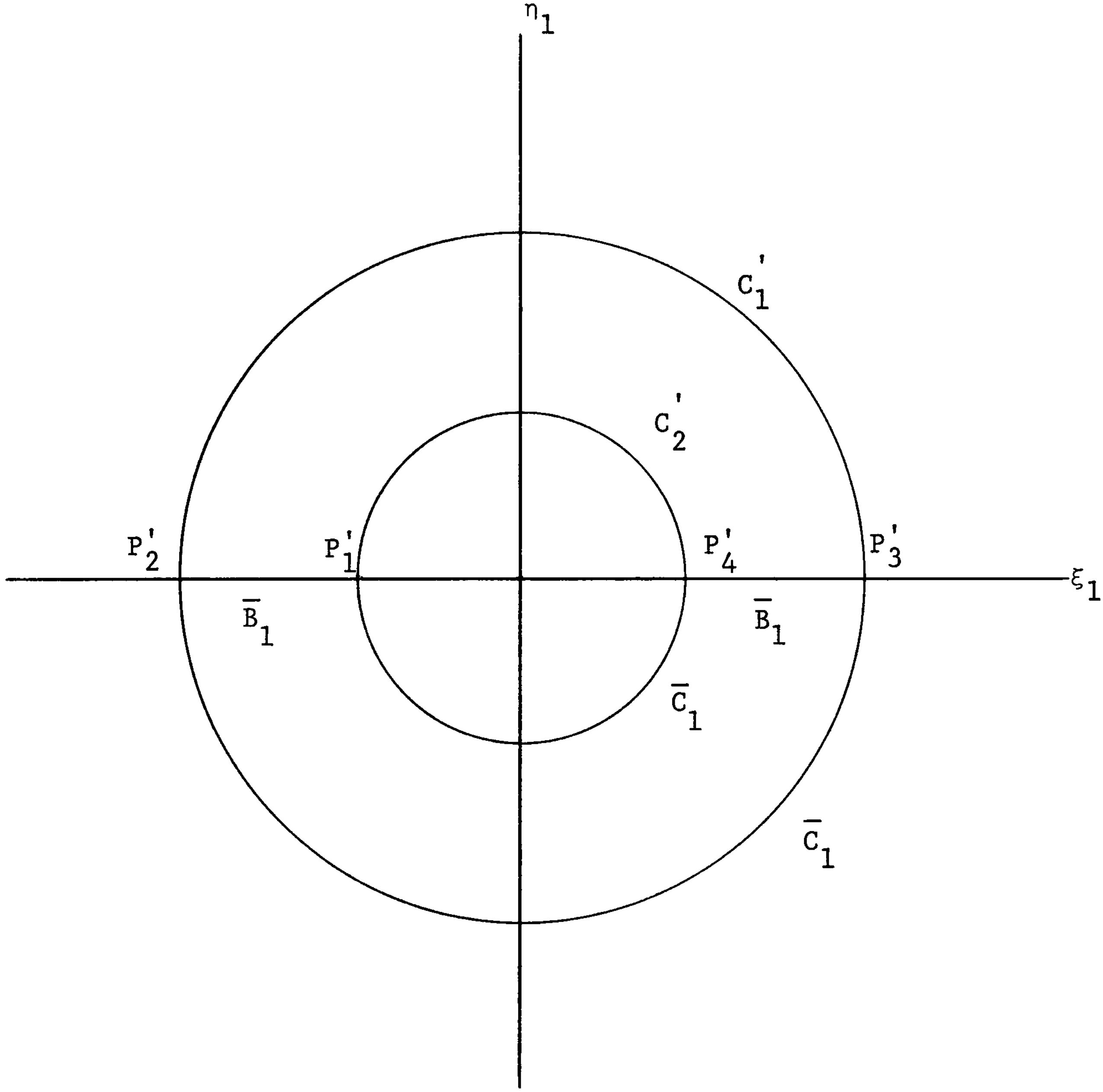


FIGURE 5

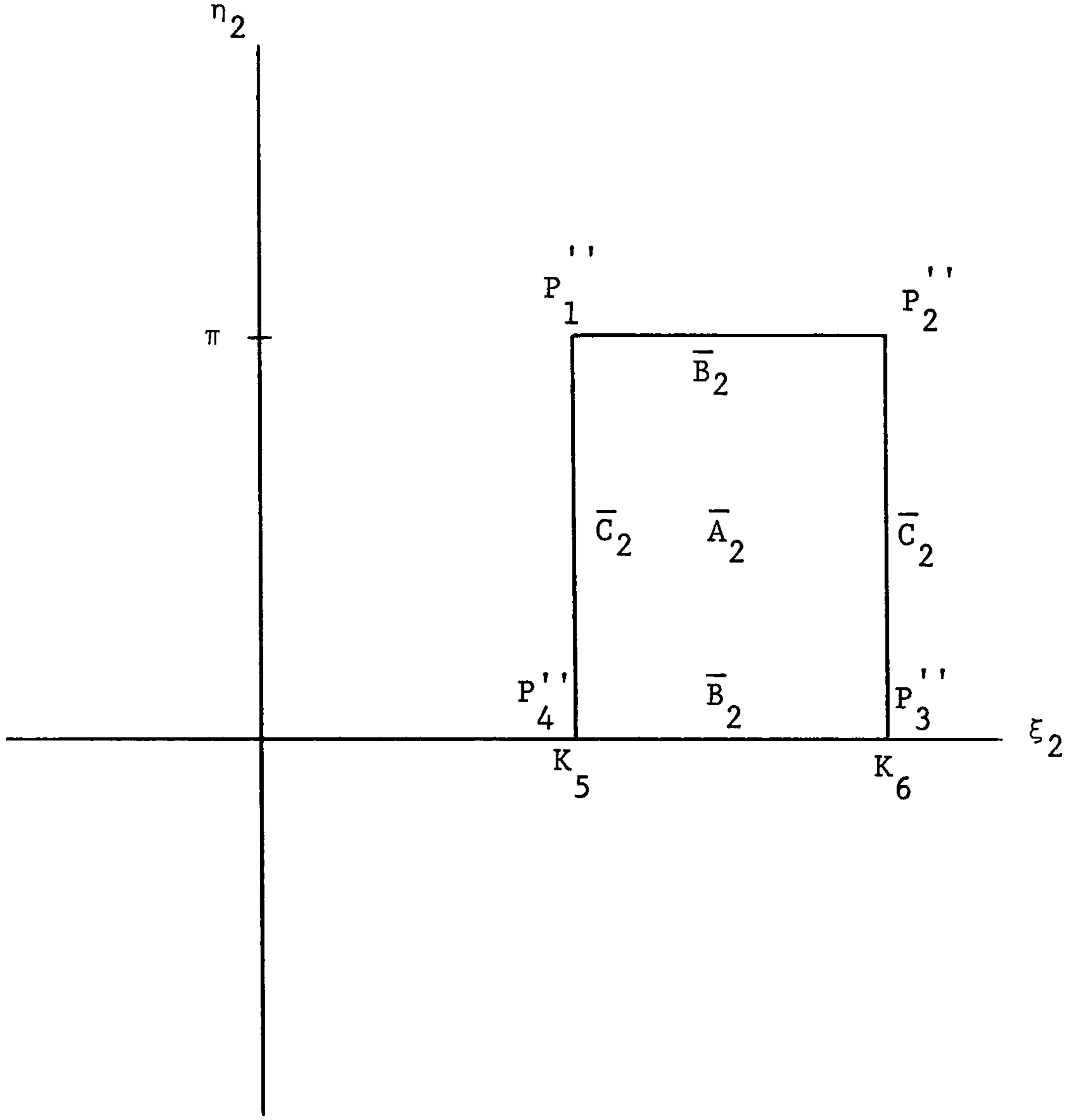


FIGURE 6

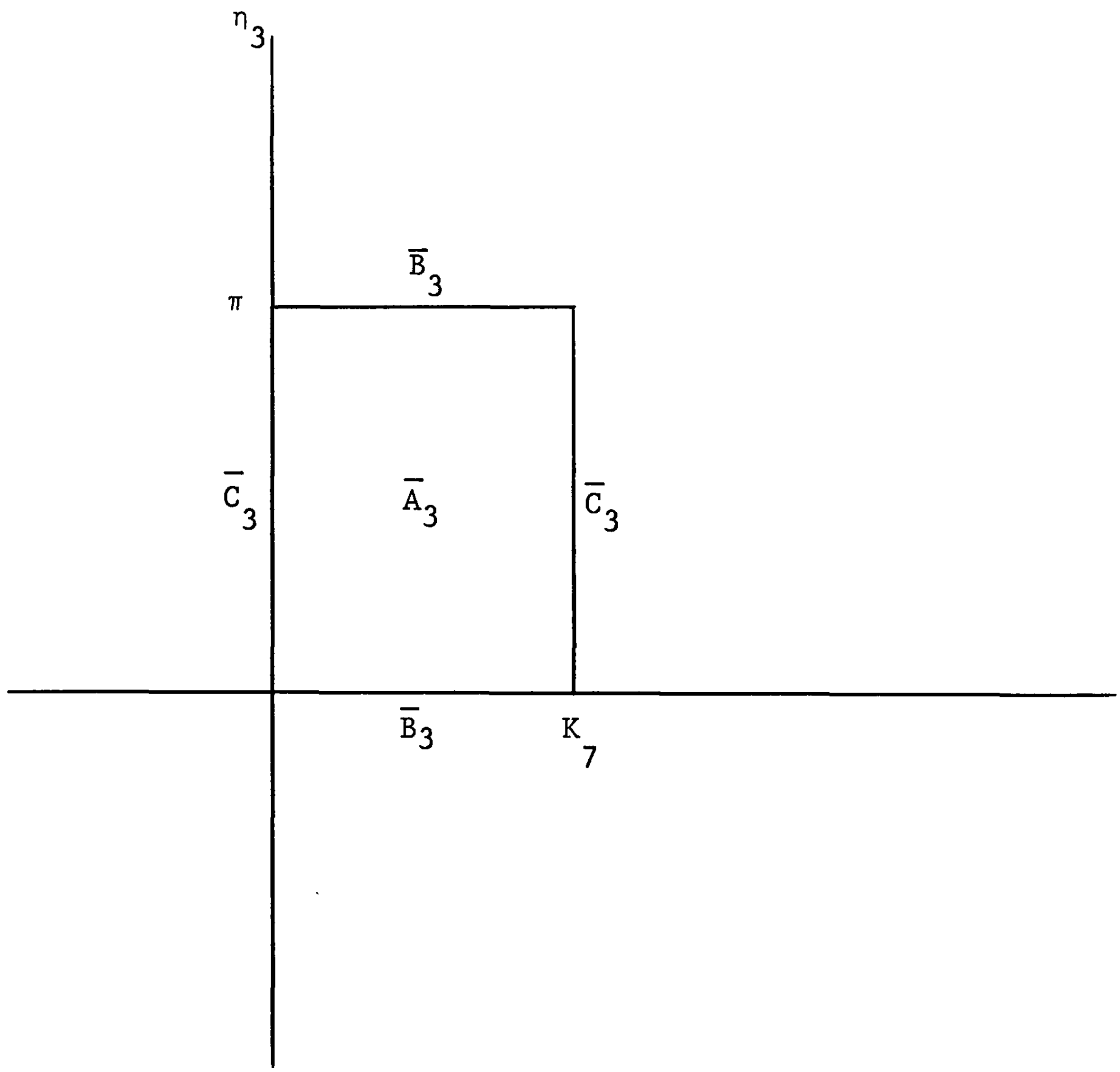


FIGURE 7

APPENDIX B

SIMPLIFICATION OF THE GOVERNING EQUATIONS

The equations governing laminar flow along the axis of a duct (see Fig. 1) are the continuity equation

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\overline{V}) = 0 \qquad \text{(B-1)}$$

and the z-direction momentum equation

$$\rho\frac{Dw}{Dt} = f_z - \frac{\partial p}{\partial z} + \frac{\partial}{\partial z}\left[\mu\left(2\frac{\partial w}{\partial t} - \frac{2}{3}\operatorname{div}\overline{V}\right)\right]$$

$$+ \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right]$$

$$+ \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right] \qquad \text{(B-2)}$$

where

(i) $\overline{V} = (u,v,w)$ represents the velocity

(ii) t represents time

(iii) $\rho(x,y,z,t)$ represents the density

(iv) $\mu(x,y,z,t)$ represents the viscosity

(v) $p(x,y,z,t)$ represents the pressure

(vi) $f = (f_x, f_y, f_z)$ represents the external body force

(vii) $\frac{D}{Dt}$ represents the substantial derivative [4] .

We are considering Newtonian fluid under incompressible steady state laminar conditions. We assume that there are no external body forces present. The effect of these assumptions on equations (B-1) and (B-2) will now be determined. The assumptions of incompressible flow and Newtonian fluid mean that ρ and μ are constants, respectively. The continuity equation now simplifies to

$$\operatorname{div} \bar{V} = 0 \tag{B-3}$$

and the z-direction momentum equation to

$$\rho \frac{Dw}{Dt} = f_z - \frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) . \tag{B-4}$$

The substantial derivative of w, $\frac{Dw}{Dt}$, is given by

$$\frac{Dw}{Dt} = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} .$$

Since body forces are neglected , $\bar{f} = \bar{0}$. The assumption of steady state flow means that the flow in independent of time. The z-direction momentum equation now simplifies to

$$\bar{V} \cdot \text{grad } w = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\nabla^2 w \qquad \text{(B-5)}$$

where

$$\nu = \frac{\mu}{\rho}$$

is the kinematic viscosity.

APPENDIX C

PROPERTIES OF THE JACOBIAN

We now establish several properties of the Jacobian

$$J(x,y) = \frac{A}{K_1^2 K_2^2} (\cosh(x+K_5) - \cos y)^2 .$$

THEOREM C-1: For a fixed x in $[0,K_7]$, $J(x,y)$ is increasing on $[0,\pi]$ and for a fixed y in $[0,\pi]$, $J(x,y)$ is increasing on $[0,K_7]$.

PROOF:

Let x be an element of $[0,K_7]$. Then

$$\frac{\partial J}{\partial y} = \frac{2A}{K_1^2 K_2^2} (\cosh(x+K_5) - \cos y) \sin y .$$

Since x is nonnegative and K_5 is positive, $\cosh(x+K_5)$ is positive so that $(\cosh(x+K_5) - \cos y)$ is positive. On $(0,\pi)$, $\sin y$ is positive and for $y = 0,\pi$, $\sin y = 0$.

Therefore $\frac{\partial J}{\partial y}$ is positive on $(0,\pi)$ which implies that $J(x,y)$ is increasing on $[0,\pi]$ for a fixed x. Let y be an element of $[0,\pi]$. Then

$$\frac{\partial J}{\partial x} = \frac{2A}{K_1^2 K_2^2} (\cosh(x+K_5) - \cos y) \sinh(x+K_5) .$$

Since $\sinh(x+K_5)$ is positive on $[0,K_7]$, we have that $\frac{\partial J}{\partial x}$ is positive on $[0,K_7]$. Hence $J(x,y)$ is increasing on $[0,K_7]$ for a fixed y. The proof is now complete.

COROLLARY 1: We have the following:

(i) $\dfrac{\partial J(x,0)}{\partial y} = 0$ for x in $[0,K_7]$.

(ii) $\dfrac{\partial J(x,\pi)}{\partial y} = 0$ for x in $[0,K_7]$.

COROLLARY 2: We have the following:

(i) The minimum of $J(x,y)$ on the rectangle is

$$J(0,0) = \frac{\pi(r_2^2 - r_1^2)([r_2 - \rho]^2 - r_1^2)}{r_2^2([r_2 + \rho]^2 - r_1^2)} .$$

(ii) The maximum of $J(x,y)$ on the rectangle is

$$J(K_7,\pi) = \frac{\pi(r_2^2 - r_1^2)(r_2^2 - [r_1 - \rho]^2)}{r_1^2(r_2^2 - [r_1 + \rho]^2)} .$$

APPENDIX D

CALCULATIONS OF IMPORTANT INTEGRALS

The purpose of this appendix is to evaluate the following integrals:

$$\text{(i)}\quad I_1 = \int_{-\pi}^{\pi} \frac{\cos(ry)}{(\cosh(x+K_5) - \cos y)^2}\, dy$$

$$\text{(ii)}\quad I_2 = \int_{o}^{K_7} \frac{\exp(-rx)\sin\frac{s\pi x}{K_7}[\cosh(x+K_5) + r\sinh(x+K_5)]}{\sinh^3(x+K_5)}\, dx$$

$$\text{(iii)}\quad I_3 = \int_{o}^{K_7} \frac{\exp(-rx)\cos\frac{s\pi x}{K_7}[\cosh(x+K_5) + r\sinh(x+K_5)]}{\sinh^3(x+K_5)}\, dx$$

where r and s are integers. We have that

$$I_1 = \mathrm{Re}\int_{-\pi}^{\pi} \frac{\exp(iry)}{(\cosh(x+K_5) - \cos y)^2}\, dy \quad .$$

For convenience we set

$$z_o = \exp(x+K_5)$$

and

$$z_1 = \exp(-(x+K_5)) \ .$$

If we make the substitution $z = \exp(iy)$, then I_1 becomes

$$I_1 = 4 \text{ Re } \{-i \oint_C f(z)\, dz\} \qquad \text{(D-1)}$$

where

$$f(z) = \frac{z^{r+1}}{(z-z_o)^2 (z-z_1)^2}$$

and

$$C: \ |z| = 1$$

is the unit circle. Since x is nonnegative and K_5 is positive, the Residue Theorem implies that

$$\oint_C f(z)\, dz = 2\pi i \text{ Res}(z_1) \qquad \text{(D-2)}$$

where

$$\text{Res}(z_1) = \lim_{z \to z_1} \frac{d}{dz} \{(z-z_1)^2 f(z)\} \ .$$

It is not difficult to show that

$$\mathrm{Res}(z_1) = \frac{(r-1)z_o^{(r+1)} - (r+1)z_o^{(r-1)}}{(z_o - z_1)^3} \quad . \qquad \text{(D-3)}$$

Substitution of equation (D-3) into equation (D-2) and (D-2) into equation (D-1) yields, after some manipulation,

$$I_1 = \frac{2\pi \exp(-r(x+K_5)) \{\cosh(x+K_5) + r \sinh(x+K_5)\}}{\sinh^3 (x+K_5)} . \qquad \text{(D-4)}$$

We now evaluate I_2. Consider the following functions:

(i) $f(x) = -\frac{1}{2} \dfrac{\exp(-rx) \sin \frac{s\pi x}{K_7}}{\sinh^2(x+K_5)}$

(ii) $g(x) = -\frac{r}{2} \exp(-rx) \sin \frac{s\pi x}{K_7} \coth (x+K_5)$

(iii) $h(x) = -\frac{s\pi}{2K_7} \exp(-rx) \cos \frac{s\pi x}{K_7} \coth (x+K_5)$.

A straight forward calculation shows that

$$I_2 = [f(x) + g(x) + h(x)]_o^{K_7}$$

$$- \tfrac{1}{2} \{r^2 + (\tfrac{s\pi}{K_7})^2\} I \qquad \text{(D-5)}$$

where

$$I = \int_o^{K_7} \exp(-rx) \sin \frac{s\pi x}{K_7} \coth(x+K_5)\, dx \ .$$

Using the well-known fact that for $x + K_5 > 0$,

$$\coth(x+K_5) = 1 + 2 \sum_{j=1}^{\infty} \exp(-2j(x+K_5)$$

we obtain

$$I = \int_o^{K_7} \exp(-rx) \sin \frac{s\pi x}{K_7}\, dx$$

$$+ 2 \sum_{j=1}^{\infty} \exp(-2jK_5) \int_o^{K_7} \exp(-(r+2j)x) \sin \frac{s\pi x}{K_7}\, dx \ . \tag{D-6}$$

Evaluation of $[f(x) + g(x) + h(x)]_o^{K_7}$ and equation (D-6) leads to, after considerable simplification, the following expression for I_2:

$$I_2 = - \frac{s\pi}{2K_7} \{(-1)^s \left(\frac{K_8}{K_{10}}\right)^r K_{11} - K_{15}\}$$

$$- \frac{s\pi}{2K_7} \{1 + (-1)^{s+1} \left(\frac{K_8}{K_{12}}\right)^r\}$$

$$- \frac{s\pi}{K_7} \{r^2 + \left(\frac{s\pi}{K_7}\right)^2\} \sum_{j=1}^{\infty} K_{12}^{2j} \left\{ \frac{1+(-1)^{s+1} \left(\frac{K_8}{K_{10}}\right)^{r+2j}}{(r+2j)^2 + \left(\frac{s\pi}{K_7}\right)^2} \right\} \ . \tag{D-7}$$

We now evaluate I_3. Consider the following functions:

$$\text{(i)} \quad f(x) = \tfrac{1}{2} \frac{\exp(-rx)\cos\frac{s\pi x}{K_7}}{\sinh^2 (x+K_5)}$$

$$\text{(ii)} \quad g(x) = -\frac{r}{2}\exp(-rx)\cos\frac{s\pi x}{K_7}\coth(x+K_5)$$

$$\text{(iii)} \quad h(x) = \frac{s\pi}{K_7}\exp(-rx)\sin\frac{s\pi x}{K_7}\coth(x+K_5) \ .$$

It is not difficult to show that

$$I_3 = [f(x) + g(x) + h(x)]_o^{K_7}$$

$$- \tfrac{1}{2}\{r^2 + (\frac{s\pi}{K_7})^2\}\int_o^{K_7} \exp(-rx)\cos\frac{s\pi x}{K_7}\coth(x+K_5)\,dx \ .$$

Calculations similar to those made in the last paragraph then show that

$$I_3 = -\tfrac{1}{2}\{(-1)^s K_{10} (\frac{K_8}{K_{12}})^r - K_{14}\}$$

$$- \frac{r}{2}\{(-1)^s (\frac{K_8}{K_{12}})^r K_{11} - K_{15}\}$$

$$-\{r^2 + (\frac{s\pi}{K_7})^2\} \sum_{j=1}^{\infty} K_{12}^{2j} \left\{ \frac{(r+2j)\{(-1)^{s+1}(\frac{K_8}{K_{10}})^{r+2j} + 1\}}{(r+2j)^2 + (\frac{s\pi}{K_7})^2} \right\} \ . \qquad \text{(D-8)}$$

APPENDIX E

THE $(f_i, f_j)_J$ INNER PRODUCTS FOR THE GALERKIN DETERMINANT

The $(f_i, f_j)_J$ inner products needed for the Galerkin determinant require evaluation of an integral of the form

$$I = \int_o^{K_7}\int_o^{\pi} \frac{\sin \frac{t\pi x}{K_7} \cos uy \sin \frac{v\pi x}{K_7} \cos wy}{(\cosh(x+K_5) - \cos y)^2} \, dy \, dx$$

where t and v are positive integers and u and w are nonnegative integers. Using equation (D-4) and the trigonometric identities

(i) $\cos uy \cos wy = \frac{1}{2}(\cos(u-w)y + \cos(u+w)y)$

(ii) $\sin \frac{t\pi x}{K_7} \sin \frac{v\pi x}{K_7} = \frac{1}{2}(\cos \frac{(t-v)\pi x}{K_7} - \cos \frac{(t+v)\pi x}{K_7})$,

we can show that

$$I = \frac{\pi}{4} (I_1 - I_2 + I_3 - I_4)$$

where I_1, I_2, I_3, and I_4 are of the form

$$I' = \exp(-rK_5)\int_o^{K_7} \frac{\exp(-rx)\ \cos\frac{s\pi x}{K_7}\{\cosh(x+K_5) + r\ \sinh(x+K_5)\}}{\sinh^3(x+K_5)}dx .$$

Using the equation (D-8) , I' simplifies to

$$I' = -\tfrac{1}{2}\{(-1)^s\ K_{10}K_8^r - K_{14}K_{12}^r\}$$

$$- \frac{r}{2}\ \{(-1)^s\ K_8^r K_9 - K_{12}^r K_{13}\}$$

$$- \{r^2 + (\frac{s\pi}{K_7})^2\}\sum_{j=1}^{\infty}(r+2j)\left\{\frac{K_{12}^{r+2j} - (-1)^s K_8^{r+2j}}{(r+2j)^2 + (\frac{s\pi}{K_7})^2}\right\} .$$

The values of r and s for I_1, I_2, I_3, and I_4 are given by

(i) I_1: $r = u - w;\ s = t - v$

(ii) I_2: $r = u - w;\ s = t + v$

(iii) I_3: $r = u + w;\ s = t - v$

(iv) I_4: $r = u + w;\ s = t + v$.

APPENDIX F

THE COEFFICIENTS OF THE FULLY DEVELOPED VELOCITY

The coefficients a_n and b_{mn} in the Fourier series expansion of the fully developed velocity require evaluation of integrals of the form

$$I = \int_o^{K_7} \int_o^{\pi} \frac{\sin \frac{n\pi x}{K_7} \cos my}{(\cosh(x+K_5) - \cos y)^2} \, dy \, dx$$

where n is a positive integer and m is a nonnegative integer. We established in Appendix D that

$$\int_o^{\pi} \frac{\cos my}{(\cosh(x+K_5) - \cos y)^2} \, dy$$

$$= \tfrac{1}{2} \int_{-\pi}^{\pi} \frac{\cos my}{(\cosh(x+K_5) - \cos y)^2} \, dy$$

$$= \frac{\pi \exp(-m(x+K_5))\{\cosh(x+K_5) + m \sinh(x+K_5)\}}{\sinh^3(x+K_5)} .$$

Using another result from Appendix D, we obtain

$$I = \pi \exp(-mK_5)\int_o^{K_7} \frac{\exp(-mx)\ \sin\frac{n\pi x}{K_7}\{\cosh(x+K_5)+m\ \sinh(x+K_5)\}}{(\cosh(x+K_5)\ -\ \cos\ y)^2}\,dx$$

$$= -\ \frac{n\pi^2}{2K_7}\ \{(-1)^n K_8^m K_9\ -\ K_{12}^m K_{13}\}$$

$$-\ \frac{n\pi^2}{K_7}\ \{m^2 + (\frac{n\pi}{K_7})^2\}\ \sum_{j=1}^{\infty}\left\{\frac{K_{12}^{m+2j}\ -(-1)^n\ K_8^{m+2j}}{(m+2j)^2 + (\frac{n\pi}{K_7})^2}\right\}\ .$$

APPENDIX G

THE HILBERT SPACE $W_2^{(1)}(\bar{A})$

Consider $L_2(\bar{A})$ where $\bar{A}$ is a compact domain in E^2. We say that an element $g(x,y)$ of $L_2(\bar{A})$ vanishes in a boundary layer if there exists a positive number δ such that for all (x,y) in the set

$$\bar{A}_\delta = \{(x,y) \text{ in } \bar{A} \mid \text{dist}\,[(x,y), \text{Bdy}(\bar{A})] < \delta\} ,$$

$g(x,y) = 0$. Define $M_1(\bar{A})$ to be the set of all continuously differentiable functions in $L_2(\bar{A})$ which vanish in a boundary layer. For g in $M_1(\bar{A})$,

$$D_x g = \frac{\partial g}{\partial x}$$

exists. Since $M_1(\bar{A})$ is dense in $L_2(\bar{A})$, the adjoint D_x^* of D_x exists [7,13]. We call the operator $-D_x^*$ the operation of generalized differentiation with respect to x. If h is an element of the domain of D_x^*, we call $-D_x^* h$ the generalized first derivative of h with respect to x. If h has continuous derivatives in $\bar{A}$, then

$$-D_x^* h = \frac{\partial h}{\partial x}$$

i.e., the generalized derivative coincides with the ordinary derivative. Similar remarks can be made with x replaced with y.

Consider the set $W_2^{(1)}(\bar{A}) = W$ of all elements of $L_2(\bar{A})$ which have generalized first derivatives. An inner product on W is defined by

$$(g,h)_W = \int_{\bar{A}} gh \, d\bar{A} + \int_{\bar{A}} D_x^* g \, D_x^* h \, d\bar{A}$$

$$+ \int_{\bar{A}} D_y^* g \, D_y^* h \, d\bar{A}$$

and the corresponding norm by

$$||g||_W^2 = \int_{\bar{A}} g^2 \, d\bar{A} + \int_{\bar{A}} D_x^{*2} g \, d\bar{A}$$

$$+ \int_{\bar{A}} D_y^{*2} g \, d\bar{A} .$$

With these definitions, W becomes a Hilbert space [7].

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