A NOVEL APPROACH TO ROBUST DESIGN USING RECENT ADVANCES IN ROBUST AND MULTIOBJECTIVE OPTIMIZATION METHODS

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In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

in Mechanical Engineering

by

Gregory Joseph

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Abstract

Current advances in the field of Robust Optimization (RO) from such authors as Azarm, Ben-Tal, Elishakoff, Zhang, Renaud and others have led to new and interesting approaches to the treatment of uncertainty in traditional engineering problems. This paper presents the Budget of Uncertainty (BoU) design method; a new method by which such approaches can be applied in a manner which balances the need for optimization with the desire for robust solutions.

Where previous work has focused on immunizing an optimization problem against pre-set uncertainty ranges, the BoU method adds additional design variables in an effort to solve for an appropriate uncertainty range. The BoU method simultaneously determines an optimum solution and an allowed uncertainty budget within a restricted feasibility space. The result is a solution that guarantees first order satisfaction of uncertain constraints and provides a measure of problem sensitivity to its uncertain parameters. This provides additional insight to early problem development, and can potentially create alternatives to traditional approaches such as Monte Carlo analysis.

Within this work we will present a summary of current RO research and introduce the BoU method. We will then apply the BoU method to a simple 2D geometric problem to illustrate its application. Finally, we tackle two well-studied engineering design problems, the Golinksi Speed Reducer and the simple Helical Spring design problem to show a more realistic application of the new method.

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1 Overview

When considering the solution to a given engineering design problem, it is customary to assume design parameters are fixed, representing measured or known quantities that have been evaluated empirically or selected through engineering judgement. These fixed parameters are then used to formulate the nominal design problem. Upon determining a solution to the nominal problem, it is then customary to consider uncertainty in the design parameters as a secondary activity, usually accounting for variation with factor of safety and/or Monte Carlo simulation approaches.

This thesis will investigate alternative methods to these traditional approaches using methods from the latest research in the area of robust optimization. In particular, the Budget of Uncertainty (BoU) method will focus on engineering design problems where the uncertainty is considered as part of the initial problem formulation. Since many engineering design parameters are not known to sufficient degree to determine an appropriate gaussian distribution, design parameters will be considered in the form of closed convex set uncertainty. The end result will be a first order guarantee that design constraints are satisfied over the full range of potential uncertainty.

The Zhang and Ben-Tal methods discussed in this thesis were not derived here, but rather collected from the most current research in this area. However, this thesis does something which the prevailing literature is missing; it connects the latest applied mathematics of robust optimization to current down-to-earth engineering examples and bridges the gap from research to application. This is done using the

BoU method. The overall intent is to provide a reference for engineers and scientists that both surveys the most current methods of robust optimization and provides a clear method applied to specific case studies.

For each example problem, once a solution has been determined using the BoU robust optimization techniques, this thesis will consider the tradeoff that occurred as a result of the robust consideration. In some cases, it will be evident that the robust formulation does a good job of trading optimality for robust feasibility, in other cases, it will be apparent that the robust formulation is overly conservative. In both cases, a clear understanding of the method will be presented and the results made clear to the reader.

By the conclusion of this thesis, we will present a new method for robust design. We will show that designers can both consider a budget of uncertainty as well as optimal design points, and that by balancing optimality conditions and robust needs, designers can arrive at solutions which provide the necessary flexibility, but retain critical characteristics of optimal design as well.

2 Review of Literature

2.1 Introduction

Literature on Robust Optimization is remarkably recent, with most relevant rsearch only published in the last decade and the most current publications as recent as the last year. This alone can be claimed as evidence that the field of robust optimization is a vibrant field of study. In fact, it could be said that this field is just now receiving the mainstream recognition it deserves. Advances in computational technology as well as improved algorithms for finding global and local minima for well-posed optimization problems have aided this research. Two recent contributors to literature in the area of robust optimization are Ben-Tal [1, 2] and Zhang [3, 4]. Ben-Tal's work focuses on solving the inner maximization problems when considering uncertain constraints and on immunizing problems from pre-selected uncertainty in the design parameters. The application of this method is limited to concave inequality constraints. Zhang's focus has been on a more general approach to robust optimization that does not necessarily require particular forms of constraint. Zhang has posed a method by which all uncertainty is reduced to a single perturbation parameter. This parameter is then used to pose a new robust counterpart problem. Finally, the most recent work in this area is by Ben-Tal et al., [5] in which a general method for obtaining the robust counterpart to a general concave nonlinear inequality constraint is described.

In recent years, many advancements have been made in the field of Robust Optimization by authors such as Chen [6]; Allen [7]; Papalambros [8, 9]; Chan [10];

Han [11]; Kokkolaras [12]; Pomrehn [13]; Athan [14]; Azarm [15, 16]; Li [17]; Hu [18]; Gunawan [19]; Besharati [20]; Gu [21, 22]; Agarwal [23–25]; and Renaud [26]. In this field, the goal is to retain an acceptable design even when faced with uncertain parameters; that is to say, the constraints must be met under any realization of the system. Robust Optimization methods generally refer to solutions as 'Here and Now' decisions, meaning that a design must be chosen before the value of the uncertain parameters is known exactly. Such is the case with Ben-Tal [5] and Zhang [3, 4].

These current RO methods require at least an estimation of the uncertain parameters. They then seek to immunize the solution under these conditions. The method proposed in this document attempts to include uncertain parameter estimation as part of the original design problem. The method will provide, as part of its solution set, an acceptable variation for each specified uncertain parameter for which the problem constraints will still be met.

The method discussed here requires modifications to the objective function to include the maximization of uncertainty for selected variables in addition to the minimization of the objective function. This idea of competing objective function elements is similar to Goal Programming and also the powerful Compromise Decision Support models studied by Mistree [27]; Bras [28, 29]; Vadde [30]; Bascaran [31]; and Shupe [32] as well as their collaborators. The technique used to balance the competing objectives of optimality and robustness in this thesis is most similar to Schniederjans [33].

Uncertainty affects more than just optimization design problems. Other areas, such as robust control based on input error estimation [34], robust compensators and controllers [35–38], robust feasibility design [39–41], and reliability analsis have been proposed to compensate for uncertainty in various applications.

A solution which provides information regarding control variables as well as uncertain parameter variations allows a system designer to better visualize and understand the problem feasibility space in terms of the uncertain parameters and to make design decisions accordingly. For example, if the problem is found to be insensitive to the material property of a particular component, a less expensive material might be selected to reduce cost. Likewise, a parameter for which the sensitivity is high should be prioritized early in case the required tolerance is not achievable through existing manufacturing methods or procurement channels.

Although the most recent work is dominated by Ben-Tal and Zhang, older research on this subject is still relevant. Perhaps the best example of this is the work by Ben-Haim et al. [42]. In this work, several engineering design problems are studied when perturbed by specific sets of uncertain parameters. The methods used are not as sophisticated as the robust optimization methods we see in the more recent literature, but the approach is straightforward and obviously presented more for the engineering applications than for presenting a new mathematical method.

This thesis attempts to provide a similiar resource for designers; a straightforward application of the most recent optimization tools available for solving general nonlinear robust optimization problems.

2.2 Design Under Uncertainty

The concept of design under uncertainty has been well understood by field engineers for many years, but has not been treated by engineering mathematics until fairly recently. In nearly every design, engineers must understand and account for manufacturing tolerances, material variability, and other factors to ensure that a completed product or design functions as intended. In traditional engineering disciplines, such as bridge design, engineers use methods such as factor of safety to ensure a design can function even when some variables are not well known. Other disciplines use an iterative approach, or accept the consequence that a post-manufacturing screening process may be needed to identify a low number of failed components. With the advent of Robust Optimization techniques, it is now possible to consider uncertainty as part of the design process, eliminating the need for screening processes and potentially costly over conservatism. While not yet widely adopted, robust optimization techniques provide a window to what future engineering designs may hold. When uncertainty is included in the design process, it is possible to achieve higher efficiencies, better performance, and lower cost than with traditional conservative engineering approaches.

The last several decades have produced many contributions in the area of Robust Optimization and Robust Design. Perhaps the first mentions of uncertainty stem from Taguchi [43] who sought a means to increase manufacturing quality through a better understanding of design uncertainties. Much of today's robust optimization theory ultimately stem from his initial case studies and approach.

As computational capability has grown, increasingly complex problems have become solvable. This increase in computational capability brings with it new challenges, such as how to pose increasingly complex problems in forms that are solvable through computer programs. In particular, with the advent of parallel computing, some researchers have developed methods such as genetic algorithms which take advantage of several CPUs at once to more readily generate optimal solutions.

In addition to parallel computing methods, algorithmic efficiency and definitions of uncertainty have proved to be fruitful areas of research. Several papers have developed Collaborative Optimization methods which combine inputs from several subsystems to provide an optimal solution to systems of problems or particularly complex programs.

Stemming from the discussion above, this chapter will outline robust optimization fundamentals, including common definitions, a short survey of key research within the past several decades, and a discussion of some of the more common applications of this research. Finally, the motivation for this research will be presented, including the key aspects of this work that represent new additions to the robust optimization community.

2.3 Robust Optimization vs Stochastic Optimization

A key component of robust optimization is the determination of the uncertain parameter set. This determination is key to the conclusions from a given analysis. Two types of uncertainty prevail in current optimization research. The first is stochastic uncertainty, otherwise known as reliability theory or probabilistic uncertainty and the focus of Stochastic Optimization (SO), similar to the definition adopted by Bertsimas et al. [44, 45]. Uncertainties are assumed to be known in terms of a probability distribution function (PDF), generally approximated as a "normal" distribution [46]. Some current research uses Belief and Plausibility (Dempster-Schafer theory) [23, 24] to provide a second order approximation of these uncertainties. In general, probabilistic uncertainties are treated as statistically independent; however, some research has tested such assumptions by providing for correlation matrices as part of the stochastic optimization theory [47], [48]. Regardless of the stochastic definition, these methods seek to balance reliability and optimality, and avoid the penalties of a true "worst-case" analysis by providing a high level of design confidence at reduced objective cost [12, 49].

In contrast, some researchers have opted for a more conservative approach. Such methods, deemed robust optimization (RO), treat the uncertainty as a closed set with no defined distribution, assuming that the goal of the robust optimization is to ensure constraint satisfaction for ***any*** value of the uncertain parameters [50, 51]. When solutions of this type are determined, they are worst case solutions (in terms of the objective function value), and as such, they protect for each parameter simultaneously taking their worst case value in terms of the objective cost. While these solutions are generally more costly, they provide solutions for problems that cannot tolerate constraint violations. A bridge design, for example, cannot be 99% safe, it must be 100% safe, and such problems must use this type of interval uncertainty. For other problems, reliable stochastic information may not be available and as a result, probabilistic uncertainties may provide overly optimistic (or just plain incorrect) solutions. A key element of RO is the definition of a convex uncertainty set. Such sets can be box-type, ellipsoidal-type or other convex forms, and can be generated from any dataset [52].

In general, the SO and RO methods are treated separately and a particular example problem will contain only a single type of uncertainty. However, some research has begun to investigate problems in which both methods are used [24, 53]. In such cases, SO theory allows for inclusion of generally accepted random variations that are uncontrollable such as temperature variation. Other conditions, such as machining tolerance, are considered as interval uncertainties, with no guarantee that mean-valued parameters are more likely. Ultimately, this balance produces solutions which account for each type of uncertainty with a defined reliability factor. In this way, the combined SO and RO uncertainties suffer from some of the same drawbacks as the SO methods alone in that they produce solutions with the potential for constraint violation. Only problems with exclusively interval uncertain variables can be guaranteed to be feasible for any realization of those variables.

2.4 A Definition of Robust Optimization

Robust Optimization, as defined in this text and used herein, seeks to find a solution that is feasible for ANY realization of the uncertain parameters within their uncertainty range. The optimum in this case is defined as the minimum objective function value that still guarantees satisfaction of all uncertain constraints. The uncertainties considered in this thesis, and more generally in the field of Robust Optimization as a whole are deterministic and set-based rather than probabilistic.

Within this thesis, when discussing robust design solutions, we claim that a given design is immunized against a given uncertainty condition if the chosen solution results in no constraint violations for the entire range of uncertainty.

Solutions to Robust Optimization problems are treated as "here and now" decisions, meaning that a solution must be chosen before the particular realization of the uncertainty set is known.

2.5 Definition of Terms Commonly Used in Robust Optimization Literature

Throughout this document, several terms will be used specific to robust optimization literature. Each term will be described and defined below.

Type I Robust Design – This design method focuses on uncontrollable uncertainties and attempts to immunize an objective function against these uncontrollable variations

Type II Robust Design – This design method focuses on controllable design parameters and studies how their definition and correlation can affect objective function values

Feasibility robustness – When variations within the uncertain parameters do not produce constraint violations, the solution is said to be "feasibly robust". The result can be thought of as the ***lowest objective value of the worst case***

realization. This concept is a key component of this thesis.

Sensitivity Analysis – Separate from SO or RO methods, sensitivity analysis refers to post optimization tools used to quantify a change in objective function cost due to uncertain parameter variations. This can be similar to feasibility robustness described above, but occurs as a secondary activity after optimization solutions have already been determined.

2.6 Sources of Uncertainty

In the following section, we describe types and sources of uncertainty. These definitions will be used later in the text to provide clear descriptions of the uncertain parameters.

Aleatory Uncertainty – When an uncertain parameter is an inherent variation of a physical system, it is said to be aleatory, Examples of aleatory uncertainty might include temperature/pressure variations for a given system.

Epistemic Uncertainty – When an uncertain parameter is uncertain due to ignorance or lack of knowledge, it is said to be epistemic. Examples of epistemic uncertainty might include roundoff error, or model error.

Propagated Uncertainty – Propagated uncertainty arises when errors from a previous iteration increase errors for successive iterations. These errors occur in addition to aleatory and epistemic uncertainties within a given problem. Simulations which involve the integration of time varying functions suffer from propagated uncertainty.

2.7 Survey of Robust Optimization Work Over the Last Several Decades

Nearly all of the current SO and RO work can be traced back in some form to Taguchi [43]. Taguchi was the first to formally consider uncertainty and its effect on final product reliability. He used a deterministic approach to develop the concept of a robust design. However, due to computational limitations and the very new ideas developed by Taguchi, very little robust design work was generated for some time.

Another early contribution to robust design considerations was Elishakoff and Ben-Haim [42]. They were able to show that for a subset of problems, monotonicity allows for efficient solutions to specific uncertain problems. They applied this method to common engineering problems and showed that design under uncertainty could be achieved for well posed problems.

More recently, Azarm, Li, and others [15, 54] have exploited monotonicity methods to find solutions to many uncertain systems. This work has been applied to many common engineering example problems and provides a means by which many problems can be simplified for robust considerations.

Another common method in current literature is the Compromise Decision Support Problem (DSP) which introduces additional deviation parameters d^+ and d^- to modify system goals to allow for uncertainties and even collaboration between multiple subsystems [6].

Multiobjective Robust Optimization has been a topic of much research, providing methods such as Collaborative Optimization (CO), MultiObjective Robust Optimization (MORO) [16], MultiObjective Collaborative Robust Optimization (McRO)

[17], and even the newer AA-McRO method [18]. All of this research has generated methods by which multi-discipline engineering problems can be considered, even in the presence of uncertain parameters [23, 24, 55]. Many of these current methods utilize genetic algorithms to balance multiple objectives. Within the field of multi-objective robust optimization, some methods provide solutions that are "all-at-once" (i.e. consider all objectives simultaneously) and others use multilevel approaches to iterate to a collaborative robust optimal design.

Some research has focused specifically on chemical plant design, using a 2-stage approach to first immunize against uncontrollable uncertainties and then, once more information becomes available, fine tuning the design using control variable modification [56,57]. Not unlike the BoU method presented here, this 2-stage approach allows uncertainty to be refined as the design becomes more complete.

Some researchers have gone so far as to apply game theory methods to robust optimization [58]. These methods use Cooperative, NonCooperative, and Sequential game player models to achieve robust optimal designs. Still others have focused on evaluating the worst case propagation of uncertain parameters through multiple systems [21, 22, 59].

Other researchers have focused on new and innovative ways to determine objective value, both in the presence of constraints and without them. These methods, largely grounded in simulation-based design, have provided new computational techniques for identifying robustly feasible solutions that provide the best case result in the presence of known uncertainty [60, 61]. The Bertsimas, Nohadani, and Teo method, in contrast to other methods, deals nearly exclusively with the objective value, minimizing that quantity in the presence of uncertain parameters [60]. This iterative method is in contrast to the first order methods of Zhang [3] and the convex constraint modification of Ben-Tal [5]. Both Zhang and Bertsimas et al., allow for nonconvex problems, however while Zhang takes a first order approximation method, Bertsimas et al., takes an iterative approach to zero in on the optimum solution.

Perhaps some of the current research most directly related to the work covered within this volume, has been performed by Xu [62], Tan [63], and Shukla [64]. Shukla provides a cellular network approach to trading robustness and optimality, attempting to balance the cost of flexibility in the network against the true optimum. Tan and Xu, similar to the method presented here, use a modified objective function based in goal programming techniques. Although these references explore the relationship between optimality and robustness, the work to date stops short of determining a true budget of uncertainty. This extension of the current research underlies the concept within this work.

2.8 The Budget of Uncertainty and Robust Design

In all of the literature described above, it is assumed that the designer begins with at least an estimate of the uncertain parameters. This estimation may be based on expert opinion, detailed statistical data, or engineering judgement, but in all cases, the RO methods rely on this information as input data. Uncertainties can be described as box-type, ellipsoidal-type, or some other bounded set. Each method examines the design reliability, constraint satisfaction, and robust design objective to determine a final design solution, which can then be compared against Monte Carlo simulations utilizing these same uncertain parameter inputs. However, there is a need for a more comprehensive approach. What can be done in early phases of design when uncertain parameter values are still unknown?

In this thesis, we explore one possible method by which the uncertain parameters can be included as ***additional design variables***. This produces a solution which not only provides a feasibly robust design, but also quantifies the allowable uncertainty within each uncertain parameter. These uncertainty intervals can then be used to specify machine tolerances or component specifications. This provides a common sense way by which a designer can begin to account for design uncertainty even when little is known about the design parameters and then refine the design as more information becomes available.

The contributions of this work are as follows:

1. The "Budget of Uncertainty" (BoU) method provides a straightforward approach for understanding the design feasibility space.

2. The BoU method can be implemented in the earliest design stages when uncertainty intervals are not yet known, and indeed can be used to begin to quantify acceptable uncertainty intervals.

3. The BoU method allows for a regimented and iterative approach to refine the feasibility space as additional design information becomes available.

4. The BoU method can be applied alongside current research methods such as

multidisciplinary problems, multilevel approaches, and various robust counterpart problem realizations.

5. The BoU method systematically provides for the implementation and evolution of a budget of uncertainty for RO problems.

2.9 Parallels to Goal Programming

The foundation of our new BoU method for robust design lies in goal programming techniques. These techniques have been developed by many authors, most notably Schniederjans and Charnes [33,65]. Goal programming allows the balance of several objective function components in such as way as to allow simultaneous consideration to each. The BoU method uses an amended objective function to provide a way to optimize based on uncertainty tolerance, and the amended objective function is generated using principles common in goal programming.

2.10 Zhang Robust Optimization Method - First Order Robust Nonlinear Programming Formulation

The first of the current RO methods discussed in this text is a general formulation for robust optimization first presented by YH. Zhang [3]. This method uses the L_1 norm of the uncertainty vector measured from the nominal and uses this single scalar value as a measure of the uncertainty variation of the problem. This factor is then used with a linearized version of the original problem. The method is not without drawbacks, since due to linearization, it can generate solutions with small violations of the constraints within the proposed uncertainty. However, Zhang presents methods by which these excursions can be bounded and reformulation of the problem can remove these violations.

Ultimately, the Zhang method is intended for problems which by their nature, do not lend themselves well to robust formulations. In particular, the method is useful for situations where a given constraint is non-concave. Zhang's method can be applied to nonlinear and nonconvex problems and its solution is valid in the vicinity of the nominal point. This limited applicability of the solution is due in part to the linearization of the problem, and in part to the assumpion that the uncertainties are only moderate and do not alter the problem significantly from the nominal.

The Zhang method can be summarized as follows:

For a general nonlinear optimization problem:

$$\min_{y \in R^{N_y}, u \in R^{N_u}} \quad \phi(y, u, s), \tag{1a}$$

$$s.t \quad F(y, u, s) = 0, \text{ and} \tag{1b}$$

$$G(y, u, s) \leq 0, \tag{1c}$$

where $s \in \Re^{N_s}$ is the vector of uncertain system parameters, $y \in \Re^{N_y}$ is the state variable vector, and $u \in \Re^{N_u}$ is the design variable vector constrained in a set $U \subset \Re^{N}$. $F(y, u, s) = 0$ is the state equation for $F \in \Re^{N_y}$, and $G(y, u, s) \leq 0$ is the vector of safety constraints for $G \in \Re^{m}$. The funtions defined by F and G must be continuously differentiable.

The robust counterpart formulation is:

$$\min_{y \in R^{N_y}, u \in R^{N_u}} \quad \phi(y, u, \hat{s}), \tag{2a}$$

$$s.t \quad F(y, u, \hat{s}) = 0, \tag{2b}$$

$$\tau(F_y y_s + F_s) = 0, \text{ and} \tag{2c}$$

$$\operatorname{diag}(G)E \pm \tau(G_y y_s + G_s) \leq 0, \tag{2d}$$

where $y_s \in \Re^{N_y \times N_s}$ is a new variable representing the unknown Jacobian of y with respect to s, $E \in \Re^{m \times N_s}$ is the matrix of all ones, and the matrix inequalities are elementwise. The functions F_y, F_s, G_y, G_s are all evaluated at $(y, u, \hat{s})$.

In recent published work [4], Zhang has used this method to solve several example problems of various types. In particular he has presented results for a simple 3-bar truss, a heat exchanger network, and a reactor-separator system [4]. In these results he shows the method in action and the level of conservatism expected. The Zhang method has been only preliminarily compared to other robust formulations, specifically results from Kwak and Huang [66].

2.11 Ben-Tal Robust Optimization Method

Ben-Tal et al have recently proposed a general method by which nonlinear concave inequalities can be reformulated to their robust counterparts [5]. The method presented involves the use of Fenchel duality, the support function of the uncertainty region, and the conjugate function of the original constraint function. For simple problems this reduces to a fairly straightforward derivation of the robust constraint, but in the presence of more complicated constraint functions that are not linear in the uncertain variables, the complexity can rise rapidly. Regardless, this method is one of the few methods available which treat the generation of a robust counterpart with a general approach and without need for simplifying assumptions. Additionally, without the need for linearization, the applicability of the solution is general, and all constraints are guaranteed to be satisfied for the entire space of uncertain problem realizations.

Theorem 2 from Ben-Tal [5]: *Let a^0 be regular. Then the vector $x \in \mathfrak{R}^n$ satisfies (RC) if and only if $x \in \mathfrak{R}^n$, $v \in \mathfrak{R}^m$ satisfy the single inequality*

$$(RC) \qquad \min_{a \in U} f(a,x) \leq 0 \text{ and} \tag{3a}$$

$$(FRC) \qquad (a^0)^T v + \delta^*(A^T v|Z) - f_*(v,x) \leq 0 \tag{3b}$$

in which the support function δ^* and the partial concave conjugate function f_* are defined later within this document.

Because this Theorem refers to specific conclusions drawn within the paper, namely the support function and the partial conjugate function, the following paragraphs provide a restatement of that theorem with the necessary components collected here. For a detailed explanation on the theory, see the references [5].

For a general inequality constraint:

$$f(a,x) \leq 0, \tag{4}$$

where $x \in \mathfrak{R}^n$ is the optimization variable, $f(.,x)$ is concave in a for $x \in \mathfrak{R}^n$, and $a \in \mathfrak{R}^m$ is an uncertain vector which is known only to reside in a set U.

Its robust counterpart is determined by:

$$(a^0)^T v + \delta^*(A^T v|Z) - f_*(v,x) \leq 0, \tag{5a}$$

$$a = a^0 + A\zeta|\zeta \in Z \subset \mathfrak{R}^L, \tag{5b}$$

$$\delta^*(y|S) = \sup_{x \in S} y^T x, \text{ and} \tag{5c}$$

$$g_*(v,x) = \inf_{a \in \mathfrak{R}^m} \{a^T v - g(v,x)\}, \tag{5d}$$

where $v \in \mathfrak{R}^m$, δ^* defined in Equation 5c denotes the support function, and f_* defined in Equation 5d denotes the partial conjugate function. a^0 is the nominal value and Z is a given nonempty, convex and compact set with $0 \in Z$.

2.12 Tradeoffs Between Optimal And Robust Formulations

As with any treatment of uncertainty, there is a cost associated with accounting for model variation in the sense that the feasibility space decreases with added variations in the design parameters. As the feasibility set reduces, the resulting optimal solution objective increases with respect to the nominal solution. The final robust result indicates the best solution that still guarantees satisfaction of the constraints. In fact, this result is essentially the worst case objective function value permitted by the given uncertainty space. Depending on the model sensitivity to the uncertainty, this cost can range from moderate to substantial. An obvious immediate result is that nearly every model has a finite uncertainty budget that can be accomodated before the problem becomes infeasible.

The reduction in feasibility space is is to be expected, and as a result it is important to consider realistic values for each of the uncertain parameters. Later chapters will discuss how these competing objectives can be balanced so as to preserve some meaasure of optimality while also protecting for required uncertainties.

For the examples in this thesis, uncertainty in the problem parameters will be treated as part of the problem statement. In order to apply the most general application of uncertainty many parameters will be considered uncertain, but the author assumes that in practice, only a few design variables will need substantial uncertainty. Other variables can be considered with a much smaller uncertainty region.

2.13 Sensitivity to Uncertainty

The sensitivity of a problem to parameter uncertainty is twofold. The uncertainty can affect the objective function value as compared with the nominal solution, or the uncertainty can reduce slack within specific constraints (i.e. reduce the feasibility space). With enough uncertainty, the feasibility space can eventually be reduced to zero, creating a nontractable problem. This will be clearly evident from the later examples in this thesis. Even simple geometric examples at first glance may appear to permit vast amounts of uncertainty, but nonetheless, seemingly small variations can make the problem completely unsolvable. This is not an uncommon characteristic of robust design problems; often there is a well defined limit to how much uncertainty will permit a viable solution.

In other examples, this sensitivity to uncertainty can be even more pronounced. The reader is advised to consider any robust optimization method as a balancing act, balancing the objective function value and the overall uncertainty budget and feasibility space.

2.14 Drawbacks of Probabilistic and Factor of Safety Approaches

Nearly all current engineering design work involves probabilistic and/or factor of safety approaches to engineering design. For extremely complicated systems, often Monte Carlo analysis is coupled with factor of fafety approaches to produce a final design. While these methods do work, and have been used for decades, they may not always be the best path to an optimum design. In fact, the current process may drive overly conservative designs due to lack of specific insights that could be determined through robust optimization methods. Both Monte Carlo/probabilistic methods and factor of safety methods have significant drawbacks that more modern methods can partially mitigate. These drawbacks hinge on the fact that both factor of safety and Monte Carlo approaches do not take advantage of model characteristics. In addition, Monte Carlo methods generally require substantial processing time and substantial post-processing of generated data to obtain the desired result.

When a probabilistic approach to uncertainty is taken, it requires knowledge of the probabilistic uncertainty. Without an extremely large sample size, it can be almost impossible to know the true statistical variance of the design parameter, and this lack of knowledge can result in an overly optimistic projection for the ultimate probability of failure in the final design. Indeed the distribution of uncertainty may not even be Gaussian and probabilistic methods may break down entirely. Even with perfect knowledge of the statistical probability for the design variables, the final solution is not assured to be feasible for all possible uncertain realizations, merely a significant percentage of them based on the assumptions mentioned above. Although the statistical probability of failure can be shown to be low, it is still possible and this is not always an acceptable result. Consider bridge design for a moment, is it good engineering practice to allow any realizations of loading where the bridge structure fails?

Factor of safety approaches, while traditional, have the drawback of nearly always being overly conservative. In structural design, a factor of safety approach can quickly result in additional material costs, additional weight in the final design, and potentially additional labor costs during assembly due to the increased size and mass. This added cost and complexity may not be necessary, and modern computational tools can now begin to illustrate this with methods such as those discussed here.

3 The BoU Method

3.1 Origins of Robust Design

Even with sophisticated computational methods and immense numerical computational capability, solutions to design problems are always laced with the uncertainty of the application. This uncertainty takes many forms, from simple measurement uncertainty, to estimated or calculated parameter uncertainty, imperfect processes, physical variations such as temperature and pressure, and other contributing causes for which there are no adequate control. As an example, when designing spacecraft control systems, any single spacecraft thruster firing varies from other firings due to imperfect combustion, subtle temperature/pressure variations within the nozzle, density/pressure fluctuations within the propellant system, and other factors. Most work in the field of optimization neglects these generally small uncertainties in favor of considering the larger problem and determining an optimum solution for the nominal or expected problem.

Robust design—and robust optimization methods in particular—considers these uncertainties and seeks to determine solutions that account (and correct) for the resulting problem variation. Robust optimization seeks to determine a best worst-case objective so as to guarantee a minimum performance even in the presence of uncertainty. This approach requires considering the competing objectives of optimality and robustness in addition to uncertainty variation imposed by real-world considerations.

Robust optimization methods have been explored by several researchers including Azarm [15–18, 54], Ben-Tal [1, 2, 5, 50, 67–76], Papalambros [8–12, 53], and others and a comprehensive discussion of this work is given in Chapter 2. Beyond Robust Optimization, we also consider the aspect of Robust Design. Robust Design as defined within this work considers the continuous application of robust optimization methods throughout an evolving design process. Robust Optimization results serve to inform design decisions which ultimately refine the uncertainty set. Robust design then becomes a specific application of robust optimization concepts in a way that can be applied readily to many engineering disciplines. We formally define ***Robust Design*** as the application of robust optimization methods iteratively or continuously so as to inform and evolve a real world design or to make a determination of acceptable uncertainty characteristics for the final design.

Within this chapter, we explore first a novel take on current robust optimization methods, considering the uncertainty set itself to be part of the design variable set, and then apply this approach for general robust design. We will present the method as well as results for three examples, a simple two-dimensional geometric problem, the Golinski Speed Reducer, and a helical spring.

3.2 An Introduction to Robust Optimization

The standard form for a general nonlinear optimization problem is well known, and serves as the necessary starting point for the introduction of the "Budget of Uncertainty" (BoU) method. This form is represented as

$$\min_{x \in R^{N_x}} \qquad \phi(x), \tag{6a}$$

$$s.t \qquad F(x) = 0, \text{ and} \tag{6b}$$

$$G(x) \leq 0, \tag{6c}$$

and shows the minimization of the objective function subject to the equality and inequality contraints. In the presence of uncertainty some modification of this form is required. In the presence of uncertainty, the nominal solution does not prevent violations of inequality constraints, nor does it necessarily retain (or even limit) the value of the objective function. In fact, as a general rule, a nominally optimized problem ***will*** violate one or more inequality constraints when uncertainty is added. In addition, in the presence of uncertainty, equality constraints become meaningless in the sense that equality can not be guaranteed for multiple realizations. For this reason, the incorporation of uncertainty requires that equality constraints be either (a) removed through substitution or (b) rewritten in an inequality form. As a means of illustration, take a simple parabolic example:

$$\min_{[x,y] \in R^2} \qquad y \text{ and} \tag{7a}$$

$$s.t \qquad x^2 - y \leq 0. \tag{7b}$$

This problem has a nominal solution of $[x, y] = [0, 0]$ with a nominal objective value of 0. If we introduce uncertainty in x in the form $[x - dx, x + dx]$, then our only constraint is immediately violated at either of the boundary conditions for any value of $dx \neq 0$! Even for this simple example, the traditional form is insufficient for our needs. Stated more clearly, we desire the solution to the uncertain problem to be "immunized" against uncertainty such that inequality constraints are not violated AND the objective value is never greater than the robust solution. To our benefit, this is well studied and authors such as Zhang and Ben-Tal provide means by which we can modify the traditional form and create what we will call the robust counterpart of the optimization problem. For either method, the key elements of this robust counterpart are (1) ensuring that each inequality constraint becomes a "worst case" constraint and (2) ensuring that the objective value is a "worst case" value. These two elements ensure that for any realization of the problem within the defined uncertainty set, the objective is equal to or less than the calculated value, and constraints remain satisfied for all realizations. Of course, equality constraints with uncertain parameters are not possible in this context. In its simplest form, this alters our simple parabolic example to

$$\min_{[x,y]\in R^2} \quad y \text{ and} \tag{8a}$$

$$s.t \quad (x+dx)^2 - y \leq 0. \tag{8b}$$

So, if our situation dictated $dx = 0.1$ (that is to say x is contained in the interval $[x-0.1, x+0.1]$), then our objective value becomes $y = 0.01$. This is the worst case objective value which also guarantees the inequality constraint is satisfied over the entire range of uncertainty. In addition, since this is a worst case objective value, the true objective value for any realization of the problem within the specified uncertainty interval is better or equal to our objective value.

Zhang and Ben-Tal each provide much more in depth studies of uncertainty set types, problem types, and constraint types [3, 4, 72], however within this work, we will focus on interval "box-type" uncertainty and so will restrict ourselves to that discussion. Certainly, the methods presented here can be applied to ellipsoidal or other uncertainty distributions. Also, although we use a first order approximation here for the BoU method, we could also use the more general approaches presented by Ben-Tal. The BoU method is independent of these choices and so is best illustrated through examples contained within this document.

3.3 The BoU Method of Robust Design

While Zhang, Ben-Tal and others have shown comprehensive and enlightening insights to the solution of robust optimization problems, to date there has been little investigation of true robust design as defined in Section 3.1. Robust design does not generally allow knowledge of the uncertainty set in advance, but rather is the attempt by a designer to determine an appropriate uncertainty set for a particular design. This can be visualized as a "Budget of Uncertainty" for a particular design space. Take our initial parabolic example as a starting point, what if we did not know in advance that the uncertainty in our "x" variable was ± 0.1? How could we reformulate our problem to determine what uncertainty would be appropriate? Here in lies the motivation for this work.

We begin by considering the type of problems for which a budget of uncertainty is a useful consideration. Immediately, in our parabolic example, we see already that we may have some problems. First of all, the feasibility space is unbounded! Without some bounds, if we merely added the variable "dx" as part of our design variable set in Equation 8, we would quickly see that "dx" can take any value and in fact will be driven to zero as a result of our objective function. Also, we must consider penalties for our uncertainty. If we solve for the uncertainty set, how can we be sure that we are not "throwing away" our optimality? We will address these two concerns separately.

3.3.1 A Bounded Feasibility Space

For the BoU method, we can bound the feasibility space in two distinct ways, direct and indirect. The direct method is to add a constraint to the original problem in terms of the original design variables. For example, we could construct our parabolic example as follows:

$$\min_{[x,y]\in R^2, dx\in R^1} \quad y, \tag{9a}$$

$$s.t \quad (x+dx)^2 - y \leq 0, \tag{9b}$$

$$y - 2 \leq 0, \text{ and} \tag{9c}$$

$$-dx \leq 0. \tag{9d}$$

In this way, we have created a bounded feasibility space, and when we attempt to solve for the allowed uncertainty in x, we will be bounded by our additional constraint and the feasibility space itself. It is important to note that the feasibility space must only be bounded in terms of the uncertain variables. Without uncertainty, we have no need for this potentially artificial bound. Figure 1 provides a geometric visualization of the unbounded and bounded feasibility space.

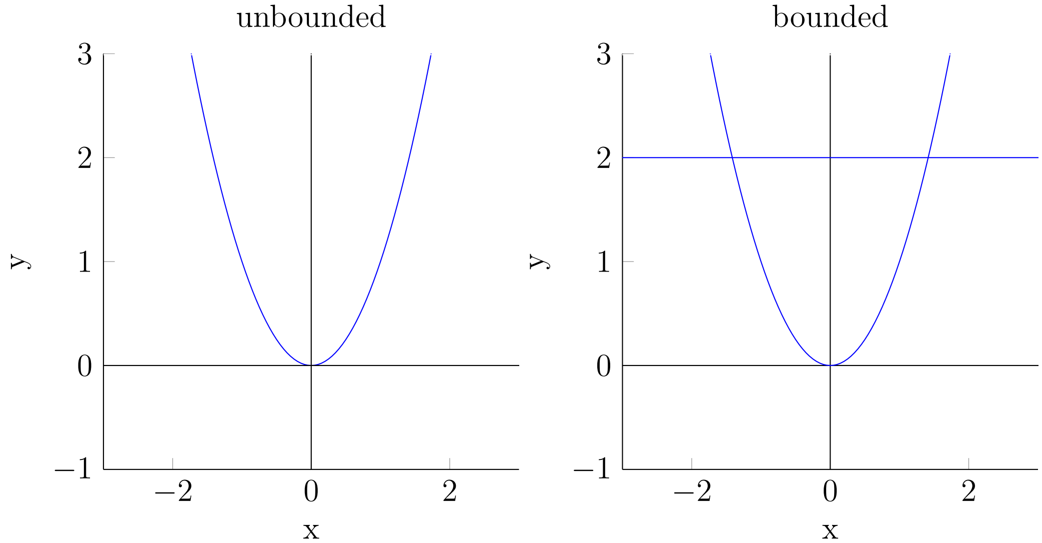

Figure 1: Feasibility Space for Simple Parabola Problem

The indirect method of bounding the feasibility space is to constrain uncertainty variables through specific constraints. Rather than constrain the design variables directly to close the feasibility space, we could have added constraints such as $dx \leq 0.2$ and $-dx \leq 0$.

Note that at this point, we still have not truly generated our desired outcome. The problem above will merely default to an "allowed" uncertainty budget of 0, and provide our nominal solution. We must add uncertainty as an additional optimization objective!

3.3.2 Balancing Optimality and Robustness - the BoU Objective Function

For our BoU method, we must balance the original optimal objective and our desire for uncertainty. This is a delicate balance. Too much importance in the optimal solution will return us to the nominal solution with no allowance for uncertainty.

Likewise, too much importance on uncertainty will drive us far from the nominal solution and potentially create an unnecessarily poor design.

For the solution to this dilemma, we turn to recent developments in goal programming and multi-objective methods. A survey of multi-objective methods is provided by Marler and Arora [77] and this provides us several possibilities for our approach. We will discuss several possibilities in detail in later chapters, but for now, we will select an appropriate combined objective for our simple problem:

$$\min_{[x,y]\in R^2, s\in R^1} \quad \alpha(y/\overline{y}-1)+(\alpha-1)sw, \tag{10a}$$

$$s.t \quad (x+dx)^2-y\le 0, \tag{10b}$$

$$y\le 0, \text{ and} \tag{10c}$$

$$\text{where} \quad dx = xs. \tag{10d}$$

Our new objective function now has two distinct components and we will refer to these two components as the "optimal objective component" $(y/\overline{y}-1)$ and the "robust objective component" $(\sum_{i=1}^{1} s_i w_i = sw)$. The first term, the optimal objective component, is an expression of the ratio of the nominal objective value ($\overline{y}$ is the nominal objective without uncertainty consideration). Since we expect $y \ge \overline{y}$ in the presence of uncertainty, this term approaches a value of 0 as the solution approaches the original nominal solution and increases in value as the solution moves away from its nominal value. In the special case where $\overline{y}$ is identically 0, we use $y+\epsilon$ to avoid the singularity. We expect this component to take reasonably small values since for most design problems substantial increase of the objective function is undesired. In fact, we have found that for the design problems investigated to date, the value of $(y/\overline{y} - 1)$ is often between 0 and 0.5.

Now we turn our attention to the second term, the robust objective component. This component is expressed in terms of multiplicative uncertainty using the uncertain parameter vector, s. To this point, we have (for convenience) defined our uncertainty as additive in the form of dx and dy, however the use of a vector, s provides a more reasonably scaled objective function component. When multiple uncertain parameters are present, we need a way to normalize the uncertainty so that large variables do not dominate the uncertainty budget when considered alongside small valued variables. We will see examples of this in later chapters. A simple variable transformation allows us to convert between multiplicative and additive uncertainty, and the result provides a compact objective function that simultaneously attempts to minimize the variation from the nominal optimal point and maximize the potential uncertainty.

In our current form the relative importance of optimality and robustness can be adjusted by altering the value of α between 0 and 1. However, we have additional discretion in the secondary weighting vector, w. The secondary weighting vector allows the designer to assign lexicographic weighting for each uncertain parameter, biasing the optimizer to allocate more variation to more senstive or more important parameters. It is worth noting that even when only the secondary weights are changed it is possible to arrive at a new solution. That is to say, adjusting only the secondary weights may not only shift the uncertainty budget between uncertain parameters, but also provide a higher or lower worst case objective value. This is because even when the primary weight does not change, adjusting the secondary weighting parameters may change the relative magnitude of the robust objective component which in turn may result in a new optimal point. This is especially true if uncertain parameters are particularly 'costly' and likely to affect the original objective function value.

The objective function form above is the default form, and is used throughout this document unless otherwise described. However, other objective function forms are possible, and may for some models be preferred. Potential varations are discussed in Chapter 4.

The term α, otherwise known as our primary weight, provides a tuning parameter through which a designer can trade between optimality and robustness. In fact, considering varying values of alpha produces a pareto front through which the solution space can be analyzed. For our purposes, we restrict $0 < \alpha < 1$.

3.4 Inequality Constraint Inner Maximum

With the robust and optimal components of the new objective function settled, along with the allowance for primary and secondary weighting elements, we must now revisit our inequality constraints. To avoid constraint violations in the presence of uncertainty, we must determine the inner maximum for each constraint. This can be done in general for all concave constraints. The BoU method, in an attempt to retain generality, applies a first order approximation method. This approximation for the j^{th} constraint takes the form of $\sum_{i=1}^{M} \frac{\partial f_j(x,s)}{\partial s_i} x_i s_i$, where M denotes the number of uncertain parameters.

The new term represents an uncertainty 'penalty' for each inequality. This penalty ensures that even when uncertain parameters take values such that the inequality constraint is at its maximum, this maximum value does not violate the inequality. The development of a general robust counterpart has been studied extensively by Ben-Tal and others. The method presented here most similar to that found in Zhang and is selected to provide a straightforward method for general application.

3.5 General BoU Form

Summarizing our conclusions above, we can synthesize a general form for our approach. In general form, the above method can be stated as follows:

$$\min_{x \in R^n, s \in R^m} \quad \alpha(\phi(x)/\overline{\phi(x)} - 1) + (\alpha - 1)\sum_{i=1}^{m}(w_i s_i), \tag{11a}$$

$$s.t \quad f(x,s) + \delta f(x,s) \leq 0, \tag{11b}$$

$$g(x) = 0, \text{ and} \tag{11c}$$

$$\text{where,} \quad \delta f(x,s) = \sum_{i=1}^{M} \frac{\partial f(x,s)}{\partial s_i} x_i s_i. \tag{11d}$$

This form allows for many inequality constraints, varying forms of objective function, and many uncertainty parameters. An additional set of tuning parameters are added, w_i, to provide lexicographic adjustment based on the relative importance of uncertainty values. The term $\delta f(x,s)$ represents a term to capture the first order approximation of the worst case deviation for each constraint. The feasibility space must be bounded and there must be no uncertain parameters in the equality constraints.

The general form shown above, and in fact the systematic approach to design of the uncertainty space is presented here as the Budget of Uncertainty (BoU) method. In the next chapters, we show how the primary and secondary tuning values can be used to the benefit of the designer, how the objective function can be modified to suit varying problem forms and sensitivities, and the application of this method to several design problems.

4 BoU Illustrative Example 1 - 2D Geometric Optimization

In the previous chapter, we introduced the BoU method of Robust Design. This method combines elements of multi-objective optimization and goal programming with recent advances in robust optimization to produce a means by which engineering designers can balance a need for robustness against the need for an optimal design and solve directly for a budget of uncertainty in selected varables. This chapter will explore variations of the BoU method which allow for alternative formulations of objective function, weighting considerations and uncertainty sets. As our method of illustration, throughout this chapter we will refer to a straightforward optimization problem with a well-defined, regular feasibility space. The nominal optimization problem,

$$\min_{[x,y]\in R^2} \quad x^2 + y^2, \tag{12a}$$

$$s.t \quad 2 - x \leq 0, \tag{12b}$$

$$y - 5 \leq 0, \tag{12c}$$

$$px + (2 - 2p) - y \leq 0, \text{ and} \tag{12d}$$

$$\text{where} \quad p > 0, \tag{12e}$$

will form the basis for our discussion.

The problem reduces to the minimization of distance to the origin for a right triangular feasibility space with a varable hypotenuse angle. In this problem, the parameter p represents the slope of the triangle's hyponenuse. The straightforward geometric visualization is shown below in Figure 2 for several values of p. The optimal point is shown as a filled red dot and the feasibility space is bounded by the lines representing the three constraints.

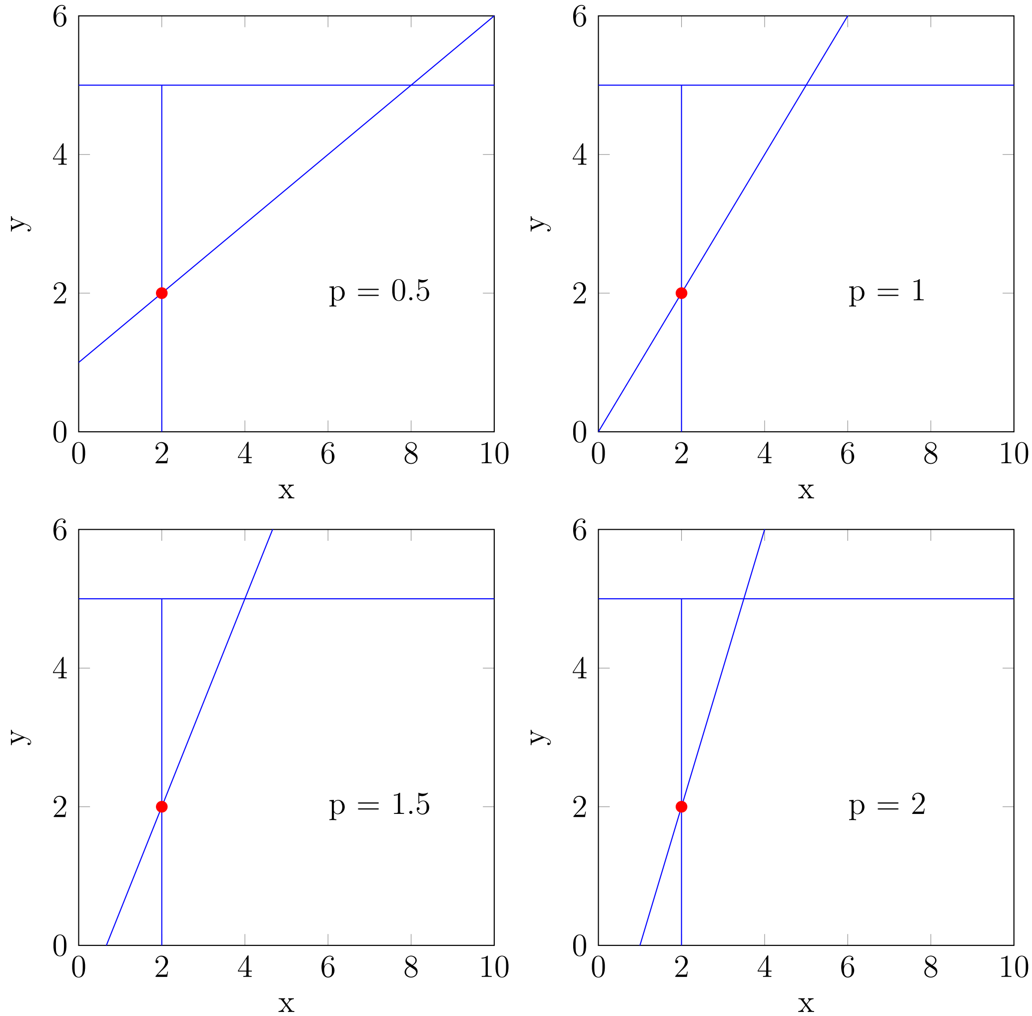

Figure 2: Geometric Representation of Simple Triangle Problem

The problem is straightforward to solve using any readily available numerical solver, analytically through the application of KKT conditions for optimality, or through geometric inspection. The optimum point lies at the point $(x, y) = (2, 2)$ and the problem constraints have been written such that even if the positive parameter p is varied, the optimal point remains fixed at this location. However, we wish to investigate the robust counterpart of this problem, which can be expressed using the BoU method described in Chapter 3. In this form of our example,

$$\min_{[x,y,v]\in R^3, s\in R^2} \quad \alpha(\frac{v}{8}-1)+(\alpha-1)\sum_{i=1}^{2} w_i s_i, \tag{13a}$$

$$s.t \quad 2-x+\delta x \leq 0, \tag{13b}$$

$$y-5+\delta y \leq 0, \tag{13c}$$

$$px+(2-2p)-y+p\delta x+\delta y \leq 0, \tag{13d}$$

$$(x^2+y^2)-v+2x\delta x+\delta x^2+2y\delta y+\delta y^2 \leq 0, \text{ and} \tag{13e}$$

$$s_i \leq 0.5, \tag{13f}$$

$$\text{where} \quad p > 0, \tag{13g}$$

$$\delta x = x s_1, \text{ and} \tag{13h}$$

$$\delta y = y s_2, \tag{13i}$$

we consider both x and y as uncertain parameters.

In this robust formulation, we are able to solve not only for a best worst-case solution in terms of x and y, but we can also simultaneously determine a range of acceptable values for x and y for each solution. Note that we limit the maximum uncertainty in each parameter through the additional constraint $s_i \leq 0.5$. In general, a 50% uncertainty range for a particular variable is sufficient, and this allows the optimizer to avoid assigning the entire uncertainty budget to a single variable. We can, through our choice of α, weight the relative importance of optimality versus robustness. Finally, we can, using the lexicographic weights w_i, assign relative importance to x or y as needed. For example, with values of $\alpha = 0.05$, $w_1 = 0.5$, $w_2 = 0.5$, and $p = 1$, we produce the solution represented graphically in the upper left of Figure 3. The point (x, y) is shown as a star surrounded by the uncertainty region shown as a blue box. Note that even at the extreme value of uncertainty, all constraints remain satisfied. Other possible solutions with varied values for α and w_i are also shown in Figure 3.

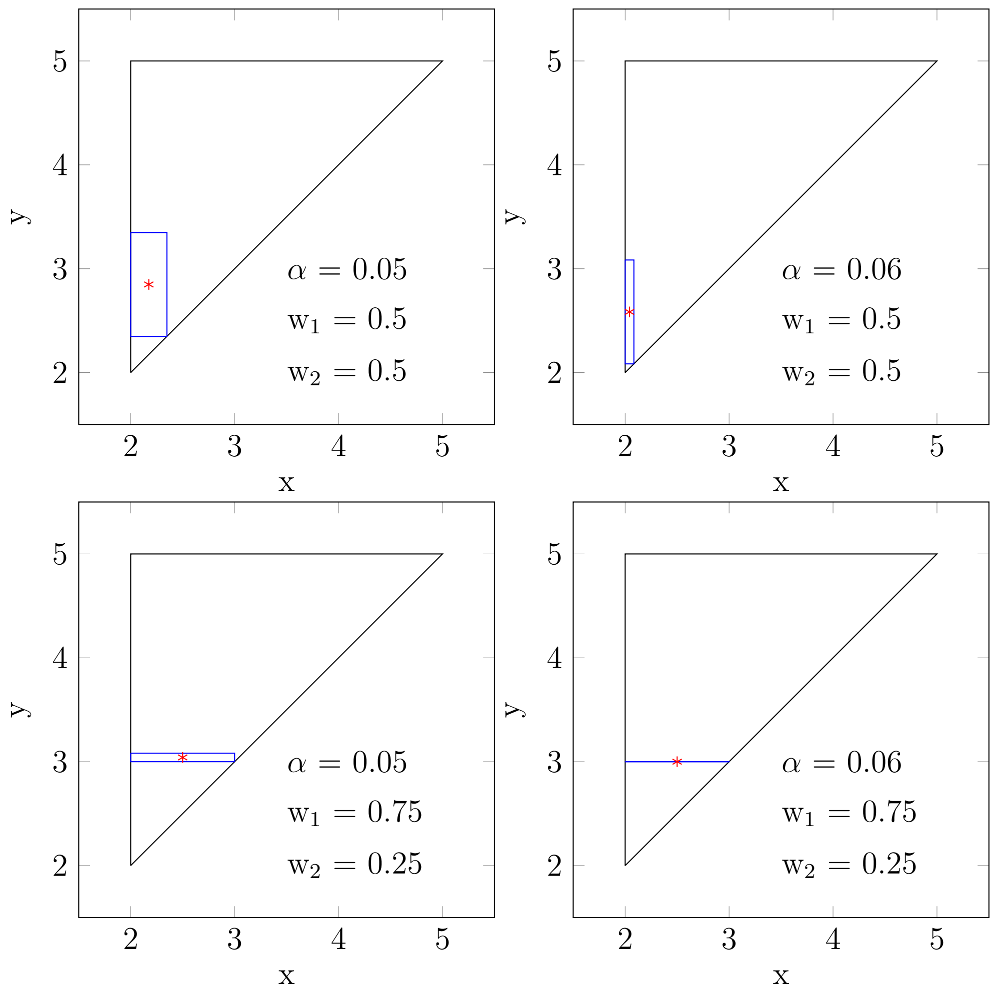

Figure 3: Various BoU Solutions for the Simple Triangle Problem

Figure 3 highlights how changing both the primary and secondary weights generate varying solutions, each with a defined uncertainty budget for the parameters specified in the s vector. In the two upper subplots, we see that when both w_1 and w_2 are equal, the uncertainty level for y is substantially greater than x. This is because the cost of uncertainty in y is lower. The bottom two subplots show solutions where w_1 is increased to generate a larger uncertainty in x at the expense of uncertainty in y. Comparing the left two subplots to the right two subplots, we see the total uncertainty reduces as α increases. We will explore this characteristic in the more complex example problems to follow. For now, it is sufficient to highlight both the lexicographic potential in the w vector and the overall uncertainty control available in α.

4.1 BoU Variations - Tuning Parameter Considerations

Armed with our robust counterpart to the nominal Triangle problem (shown in Equation 13), we now wish to consider the solution variations that are possible as the tuning parameters are adjusted. The tuning variables considered include α as the primary weighting factor and w_i as the lexicographic weighting factors (or secondary weighting factors).

4.1.1 The BoU Pareto Front (Primary Weighting Factor, α)

As we have shown previously, the parameter α determines the balance between optimality and robustness. When α is set to 0, the relative importance of uncertainty is maximized, producing the largest possible uncertainty set at the expense of optimality. When α is set to 1, the relative importance of optimality is maximized, and the nominal solution is obtained with uncertainty limited to slack conditions that do not affect objective function value. Using a fixed value of p and varying only α in our simple example, we produce the pareto front shown in Figure 4. This pareto front shows, for various choices of line slope, p, the objective value as a function of α. This shows the smooth transition from the optimal solution (high values of α) to the fully robust solution (lower values of α).

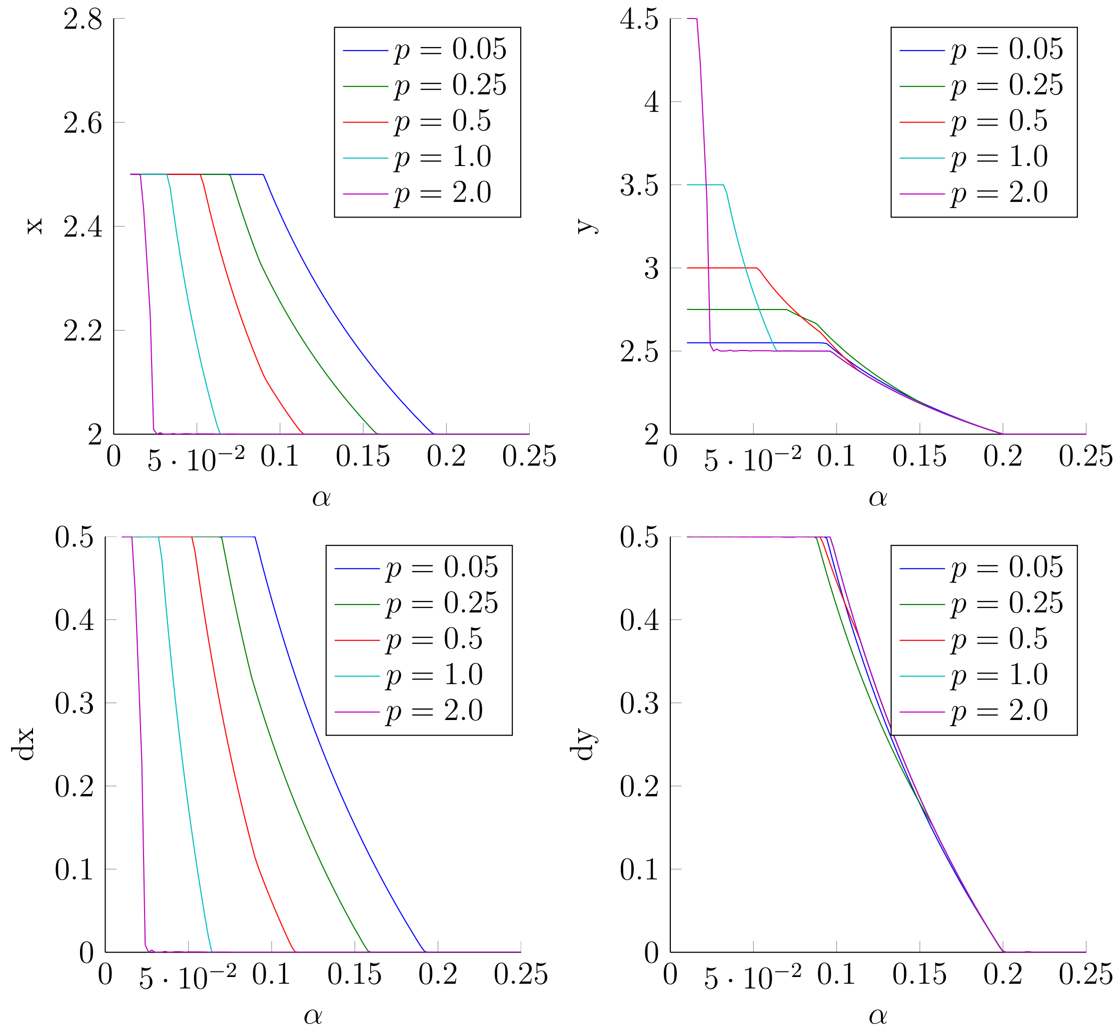

Figure 4: Pareto Fronts for Varying Slope (p) as a Function of α

In the left two subplots of Figure 4 we see that as the value of p increases, the cost of uncertainty in x also increases. We see in the right two subplots that the uncertainty in y is essentially unaffected by the value of α and p. This is expected given the geometric orientation of our triangle. Since the value of p affects the hypotenuse angle of the triangle, higher values of p mean that ensuring uncertainty in x is more costly than a similar level of uncertainty in y. Given the values of x, y, dx, and dy do not change significantly as α increases beyond 0.25, the four subplots in Figure 4 provide a clear indication of potential uncertainty space as a function of α.

4.1.2 Secondary Tuning Parameters

In addition to the use of α to vary the relative weighting of optimality and robustness considerations, it is also possible to tune the BoU method to allow for lexicographic considerations. The secondary weights, w_i, allow for an increase in the relative weight of specific uncertain design variables and can guide the optimizer to re-allocate uncertainty to more desired areas. Often in this thesis, we choose weights where $\sum w_i = 1$, however this normalization of the secondary weights is not required. For problems with high sensitivity to particular uncertain parameters is may be necessary to assign secondary weights with values larger (and perhaps much larger) than 1.

It is important to note that the potential uncertainty for a particular parameter is dependent on the feasibility space. In our simple example, we limit the maximum value of dy to 0.5; the structure of the model is such that this value cannot be exceeded no matter how much weight we assign to dy using w_2. A similar limit exists for dx. The value of α at which x and y reach their maximum values is dependent upon the value of p. Even if we had not limited the maximum value of dx and dy, this would not be unexpected. Since we require that the feasibility space be bounded as a pre-requisite for the BoU method, a limit to potential uncertainty should (and for the BoU method to succeed, must) always exist for each parameter. Beyond that point, assigning additional weight in the form of the secondary weight factors will provide no further insight or model flexibility.

In Figure 5, we show the values of dx and dy as a function of the two tuning parameters w_1 and w_2. In this example, $w_1 = 1 - w_2$, $\alpha = 0.05$, and $p = 1$. Values of $w_1 < 0.35$ and $w_1 > 0.85$ are not shown because there is no appreciable change beyond the values shown.

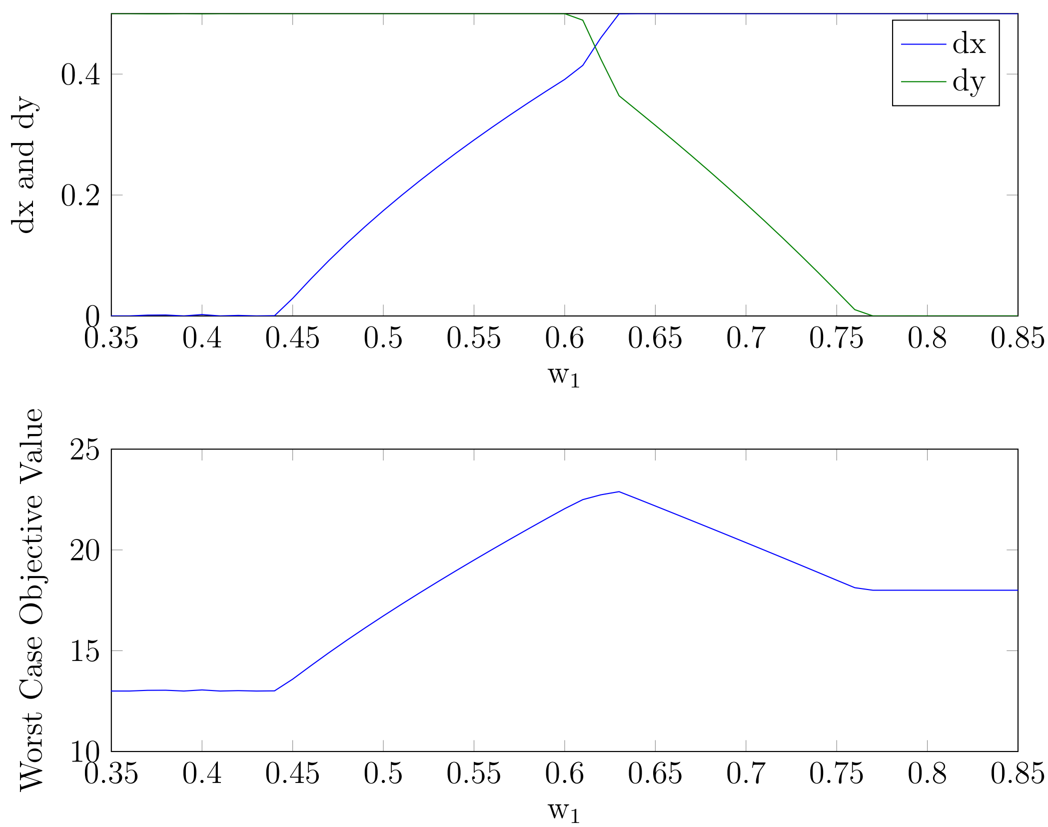

Figure 5: Evolution of dx, dy, and Worst Case Objective Value as a Function of Secondary Weighting, w_i

From this image, we can see that as the relative importance of uncertainty in x increases, the uncertainty budget is allocated more and more toward x (increasing

dx) at the expense of uncertainty in y (decreasing dy). This method of secondary weight tuning affords a designer a means by which to prioritize particular variables at the expense of other, potentially less important or less costly design considerations.

Also within Figure 5, it is clear that the worst case objective can increase or decrease even when only the values of w_i change. This is a direct result of the cost of each uncertain parameter. It is possible to specify an additional constraint to limit the increase in the robust objective function component, however the observation regarding objective function value is critical to a full understanding of the BoU method application.

Finally, in Figure 5 we see a critical point at $w_1 = 0.62$. At this value, we see the worst case objective value reaches its maximum value. While this point may be different for other values of α, the trend shown in Figure 5 remains fixed. At this value, we reach the worst case maximum for $x^2 + y^2$. This critical point provides additional insight to the problem in terms of the feasibility space, and more specifically the potential for a Budget of Uncertainty.

4.2 BoU Variations – Objective Variations

To this point, we have presented the BoU method with a particular objective function form as shown in Equation 14. This form is

$$\min_{x \in R^n, s \in R^m} \quad \alpha\phi_1(x) + (\alpha - 1)\phi_2(x), \tag{14a}$$

$$\text{where,} \quad \phi_1(x) = y/\overline{y} - 1, \text{ and} \tag{14b}$$

$$\phi_2(x) = \sum_{i=1}^{m} w_i s_i, \tag{14c}$$

where α represents the primary weighting, $\phi_1(x)$ represents the optimal component and $\phi_2(x)$ represents the robust component.

This form is commonly known as the Weighted Sum Method. While a good starting point, much research has addressed potential shortcomings to this approach. The shortcomings are best summarized by Marler and Arora [77], who themselves cite numerous authors such as Koski [78], Stadler [79, 80], Athan and Papalambros [14], Das and Dennis [81,82], and Messac [83–85]. Specifically in reference to the Weighted Sum method, Marlor and Arora [77] have this to say:

> *"First, despite the many methods for determining weights, a satisfactory, a priori selection of weights does not necessarily guarantee that the final solution will be acceptable; one may have to resolve the problem with new weights...*
>
> *The second problem with the weighted sum approach is that it is impossible to obtain points on non-convex portions of the Pareto optimal set in the criterion space...*
>
> *The final difficulty with the weighted sum method is that varying the weights consistently and continuously may not necessarily result in an even distribution of Pareto optimal points and an accurate, complete representation of the Pareto optimal set..."*

While we have found that the weighted sum is a good starting point for most problems and provides a good initial guess (and often an entirely satisfactory solution), some alternate objective function forms are also possible to address these shortcomings. Each of these alternative objective function forms shares the following characteristics:

- The objective function is made of a robust component and an optimal component similar to $\phi_1(x)$ and $\phi_2(x)$
- A primary weighting method exists similar (or identical to) α as presented in previous sections
- The objective function value is minimal when the optimal component is minimized and the robust component is maximized

In an effort to present the BoU method as a method somewhat independent of the specific objective format, we show, in the next two subsections two alternate formulations of objective function which also serve as solution methods in similar spirit to the BoU. The two methods shown are the Weighted Product Method and the Exponential Weighting Method. As a baseline for comparison, Figure 6 shows the Pareto front for the weighted sum method, as well as the respective values of dx and dy as a function of α. A figure of this form will be provided for each objective function discussed in this section. For each figure, we use the Triangle example described earlier in this chapter as the data source with p set to a value of 1.

4.2.1 The BoU Weighted Product Objective Form

In Equation 15, we show the weighted product objective form. In this form, as in our weighted sum form, we have a primary weighting factor however in this case, the primary weighting is in the form of a two element vector, α. In addition, similar to our original weighted sum form, $\phi_1(x)$ is the optimal component, and $\phi_2(x)$ is the robust component. This form is shown as

$$\min_{x \in R^n, s \in R^m} \quad \frac{\phi_1(x)^{\alpha_1}}{\phi_2(x)^{\alpha_2}}, \tag{15a}$$

$$\phi_1(x) = y/\overline{y} - 1, \text{ and} \tag{15b}$$

$$\phi_2(x) = \sum_{i=1}^{m} w_i s_i. \tag{15c}$$

The same approach applies when using this objective form as for the previous weighted sum form. α_1 and α_2 can be used to generate the Pareto optimal front, $\phi_1(x)$ is minimized, and $\phi_2(x)$ is maximized. Using our Triangle example, we once again illustrate the Pareto optimal solutions in Figure 7. In Figure 7, α_1 is set to 1 and α_2 is varied from 1 to 3.

4.2.2 The BoU Exponential Weighted Objective Form

In addition to the Weighted Sum and Weighted Product forms, the last form presented here is the Exponential Weighted form. This form, in addition to the α_1 and α_2 weighting parameters, also includes an additional parameter, p. While p can take any value, in practice higher values of p have been shown to be more advantageous [77]. The BoU objective, modified using the Exponential Weighting form, is shown as follows:

$$\min_{x \in R^n, s \in R^m} \quad (e^{p\alpha_1} - 1)e^{p\phi_1(x)} + (e^{p\alpha_2} - 1)e^{p\phi_2(x)}, \tag{16a}$$

$$\phi_1(x) = y/\overline{y} - 1, \text{ and} \tag{16b}$$

$$\phi_2(x) = \sum_{i=1}^{m} w_i s_i. \tag{16c}$$

In the Exponential Weighted form, once again we see α_1 and α_2 as the primary weighting elements. Increasing values of α_1 increase the relative weighting of the optimal objective component; likewise, increasing values of α_2 increase the relative weighting of the robust objective component. $\phi_1(x)$ and $\phi_2(x)$ exist similarly to the previous two objective variations. In Figure 8, we set $\alpha_2 = 1$ and $p = 1$. We vary the value of α_1 from 1 to 15 and show the results in Figure 8.

Reviewing Figures 6, 7, and 8, we clearly see that the choice of the objective function is not unique for the BoU method. The method accommodates nearly any multi-objective or goal programming approach to combining the two competing objectives of optimality and robustness. The BoU method merely applies these ideas as a means to generate design flexibility and insight regarding potential uncertainties.

For all the methods shown here, the Weighted Sum, Weighted Product, and Exponential Weighting methods, we have shown how dx, dy, and the Worst Case Objective value vary as a function of primary weighting. In every variation, we see the Worst Case Objective value reach a maximum value of 25 and a minimum value at the original nominal value of 8. The maximum value is a direct result of our choice to limit both dx and dy to a maximum value of 0.5. In each of the three variations, we can see that the trend from maximum to original nominal value changes slightly. In the Weighted Sum example, we see first a sharp drop, then a slower decrease to the nominal value as α varies from 0 to 0.2. In the Weighted Product example, we see a stairstep behavior as α_2 varies from 1.2 to 2.6. In contrast, the Exponential Weighted method shows a smooth transition from the maximum to original nominal objective value. Similarly, dx and dy show a similar trending behavior. The results shown here support the conclusions drawn by Marler [77] which suggests the Exponential Weighting method is a superior method for objective function combination of this type. We will, for the remainder of this document continue to use the Weighted Sum method, however any of the three methods mentioned here can be used interchangeably as the model requires.

4.3 BoU Variations – Robust Counterpart Variations

Throughout this work, we choose to present only box uncertainty sets for uncertain parameters. This selection is made because it is the most conservative and requires the least amount of knowledge from a designer. This is, in fact, the intended BoU application scenario. However, current research in the field of robust optimization, in particular work by Ben-Tal, Hertog, and Vial [5] has considered more novel forms of uncertainty, including ellipsoidal uncertainties, polyhedral uncertainty, and other more novel choices of uncertainty regions. While not directly discussed in this work, the methods described by Ben-Tal et al can be directly applied to the BoU process. Once a problem has been expressed as its robust counterpart (using nearly any robust optimization method for immunizing each inequality constraint), and an objective function form has been selected, the BoU method can be applied. In this way, the BoU method can be applied to a wide range of engineering and other design problems, can account for various models in the choice of objective function, and can permit simple box uncertainty sets as well as more novel and complex uncertainty sets. The limit of the BoU method is therefore set only by the designer's imagination and the limitations presented earlier.

Within the following Chapters, we apply the BoU method to several example problems, solving for an optimal solution as well as a potential uncertainty budget. For each example, we provide the full problem, a sample solution set, and detailed results to permit replications of this work. The examples include the Golinski Speed Reducer and a helical spring design. The Speed Reducer shows the BoU method applied in an example problem with many uncertain parameters with a large number of constraints. The helical spring design problem, while including only a small number of uncertain parameters, includes highly nonlinear constraints and highlights the first-order nature of the BoU method.

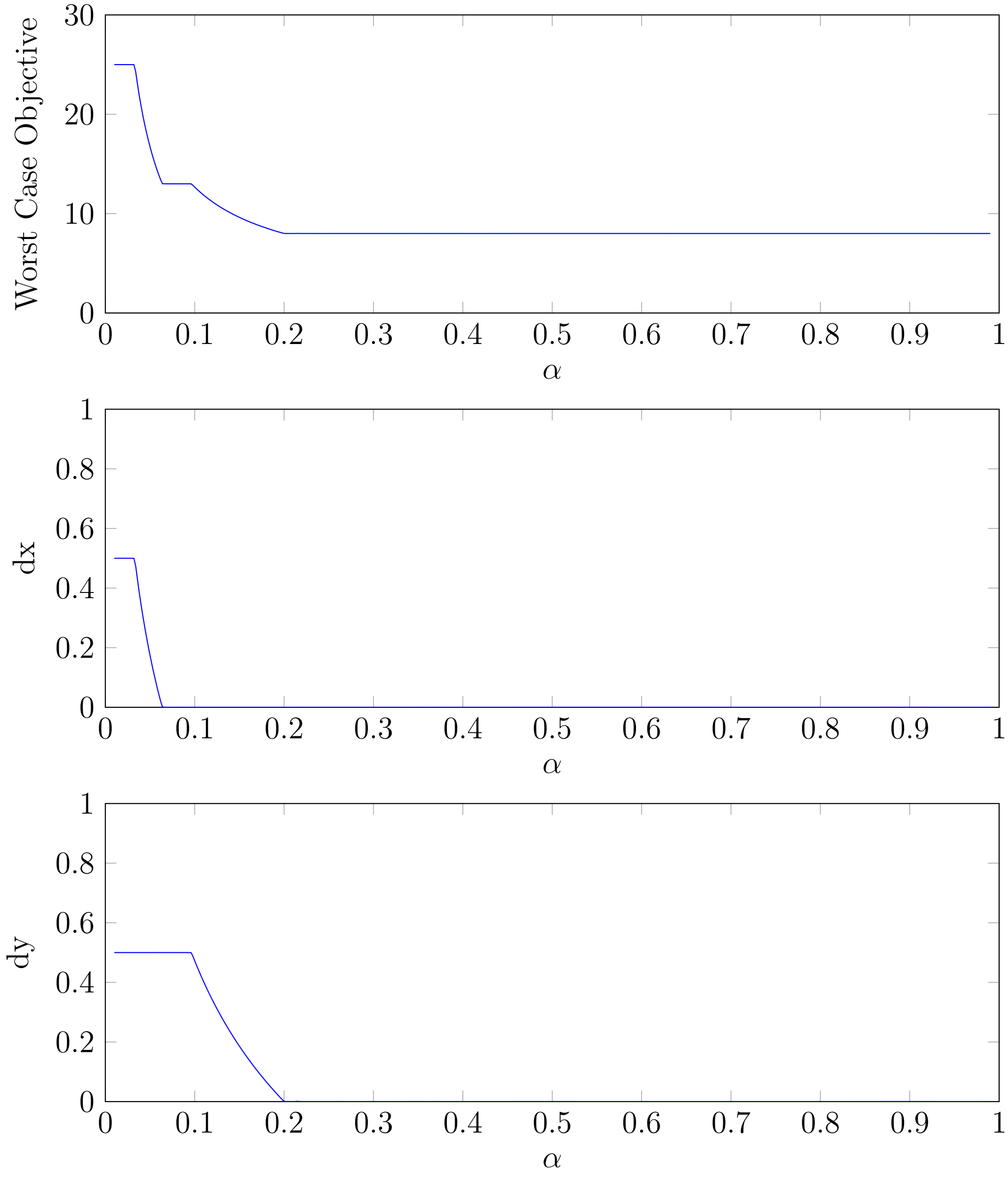

Figure 6: Weighted Sum Method Variation as Function of α

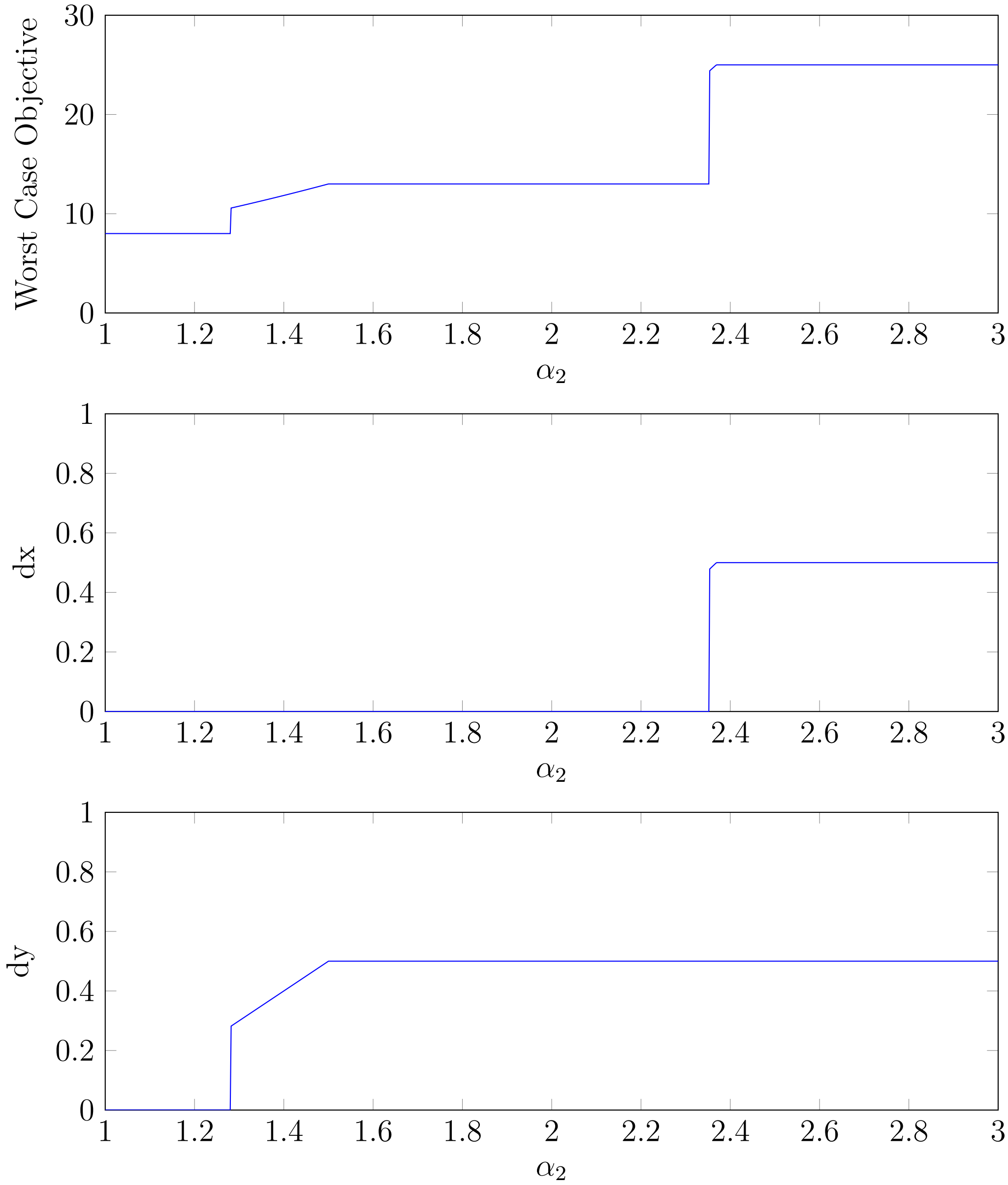

Figure 7: Weighted Product Method Variation as Function of α_2

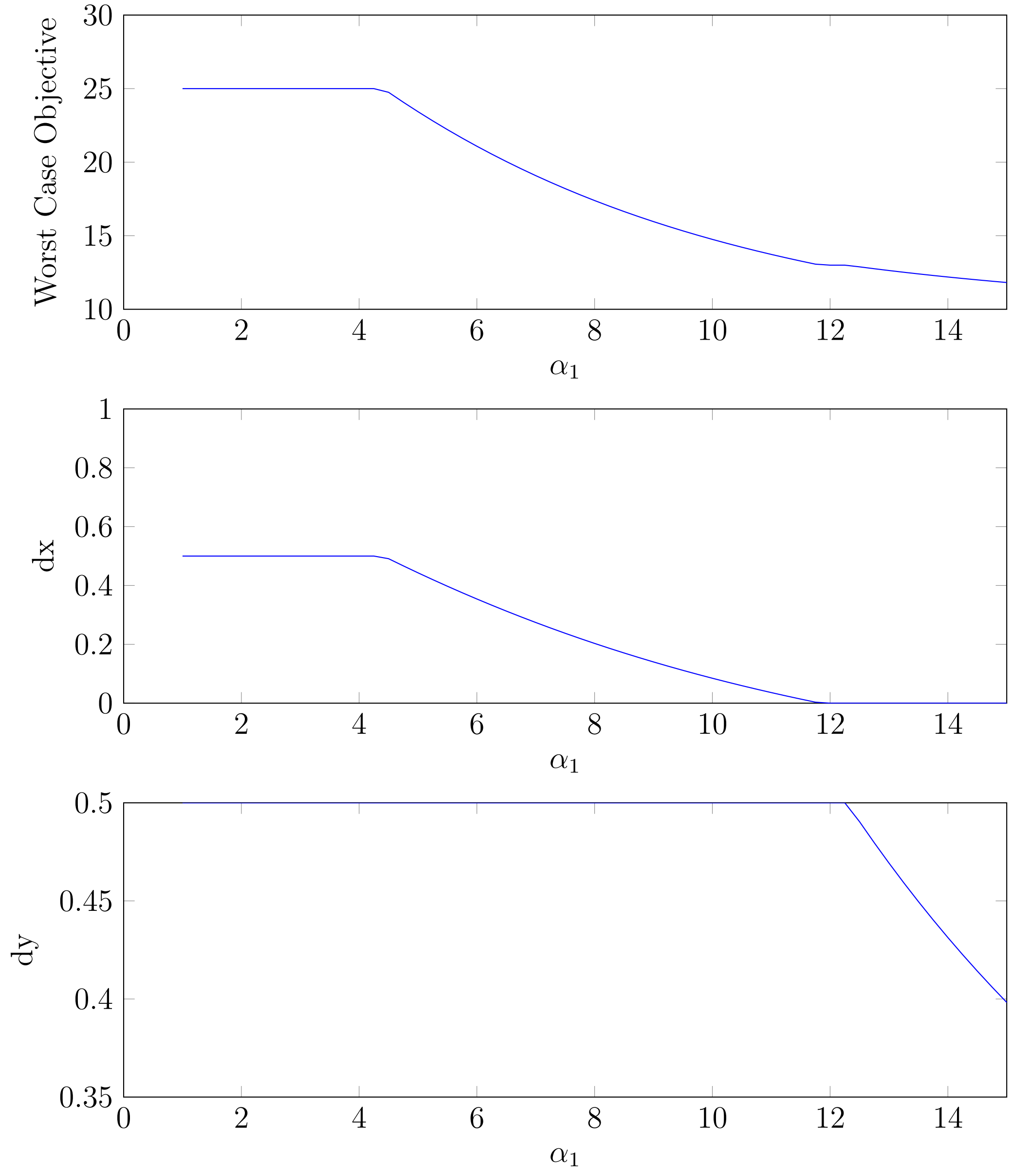

Figure 8: Exponential Weighted Sum Method Variation as Function of α

5 BoU Illustrative Example 2 - Gearbox Example

Having shown the BoU method and its application to a simple 2-dimensional problem, we now begin the presentation of the method to more realistic problems including multi-dimensional feasibility spaces, nonlinear constraints, and non-intuitive uncertain parameters. Our first example of this type is the well-studied Golinksi Speed Reducer problem [15, 17, 54, 86], hereafter referred to as the "Gearbox" example.

This problem has been studied by many researchers in the fields of multi-objective optimization, however, it is rarely studied from a robust optimization perspective as we will show here. This problem is well-suited to the BoU method due to the large number of constraints and its use of parameters which take on engineering values. These parameters are estimated based on historical experience but are likely not exact and could deviate by small amounts from their predicted value. Most literature in which the gearbox problem is mentioned treats these parameters as known, referencing their value based on historical prediction, then focusing on the problem solution with these known values. Here, we will consider the inherent uncertainty within those values, and will (1) study the sensitivity (relative to the objective function value) to parameter knowledge and (2) show that it is possible to generate a solution for which constraints are not violated when the uncertain parameters vary from their predicted values.

Figure 9 shows a sketch of the gearbox geometry. Within this figure, the control variables are defined as follows: x_1 represents the width of the gear face; x_2

represents the teeth module; x_3 represents the number of pinion teeth; x_4 and x_5 represent the shaft 1 and shaft 2 length between bearings; and x_6 and x_7 represent the diameter of shaft 1 and shaft 2. All dimensions are in centimeters.

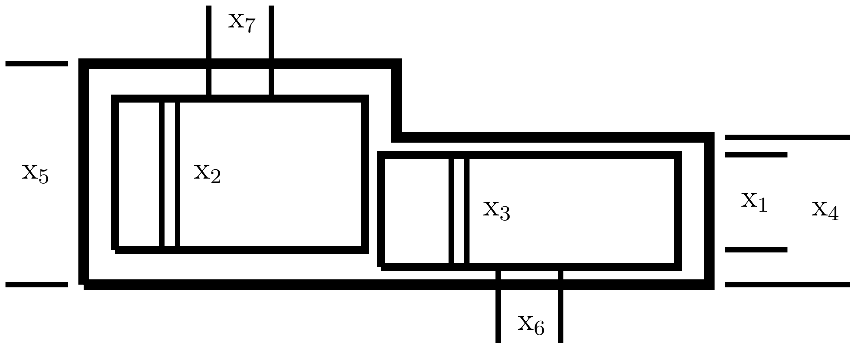

Figure 9: Gearbox Illustration

The problem definition is to minimize Equation 17 subject to gear teeth bending and contact stresses, transverse deflection of shafts, stress in each shaft, and dimensional requirements/restrictions. The upper and lower bounds for each of the seven control variables are listed in Equation 19. Finally, nominal values for each of the necessary constants are listed in Table 1. The form shown in Equations 17 - 19 is the general form in existing research. For the purposes of our example, we slightly modify the form for our purposes and present this new form in Equation 21. This new form is a mathematically equivalent presentation, it has merely been transformed for our convenience. For future reference, we label the 11 significant constraints as C1-C11. The remaining constraints will be reformulated as bounds to the optimizer.

We plan to minimize our objective,

$$\begin{aligned}\min_{x_i \in \Re} \quad & f = C_{f1} x_1 x_2^2 \left(C_{f2} x_3^2 + C_{f3} x_3 - C_{f4} \right) \\ & - C_{f5} \left(x_6^2 + x_7^2 \right) x_1 + C_{f6} \left(x_6^3 + x_7^3 \right) \\ & + C_{f1} \left(x_4 x_6^2 + x_5 x_7^2 \right), \end{aligned} \tag{17}$$

subject to the constraints,

$$\frac{C_{g1}}{x_1 x_2^2 x_3} \leq 1.0: \quad \texttt{C1 - Gear 1 Tooth Bending Stress}, \tag{18a}$$

$$\frac{C_{g2}}{x_1 x_2^2 x_3^2} \leq 1.0: \quad \texttt{C2 - Gear 2 Tooth Bending Stress}, \tag{18b}$$

$$\frac{C_{g3} x_4^3}{x_2 x_3 x_6^4} \leq 1.0: \quad \texttt{C3 - Transverse Shaft 1 Deflection}, \tag{18c}$$

$$\frac{C_{g4} x_5^3}{x_2 x_3 x_7^4} \leq 1.0: \quad \texttt{C4 - Transverse Shaft 2 Deflection}, \tag{18d}$$

$$\frac{\sqrt{\frac{C_{A12}^2 x_4^2}{x_2^2 x_3^2} + C_{A1}}}{C_{g5} C_B x_6^3} \leq 1.0: \quad \texttt{C5 - Shaft 1 Stress}, \tag{18e}$$

$$\frac{\sqrt{\frac{C_{A12}^2 x_5^2}{x_2^2 x_3^2} + C_{A2}}}{C_{g6} C_B x_7^3} \leq 1.0: \quad \texttt{C6 - Shaft 2 Stress}, \tag{18f}$$

$$\frac{x_2 x_3}{C_{g7}} \leq 1.0: \quad \texttt{C7 - Dimensional Restriction}, \tag{18g}$$

$$\frac{C_{g8} x_2}{x_1} \leq 1.0: \quad \texttt{C8 - Dimensional Restriction}, \tag{18h}$$

$$\frac{x_1}{C_{g9} x_2} \leq 1.0: \quad \texttt{C9 - Dimensional Restriction}, \tag{18i}$$

$$\frac{C_{g24} x_6 + C_{g245}}{x_4} \leq 1.0: \quad \texttt{C10 - Shaft Design Condition}, \text{ and} \tag{18j}$$

$$\frac{C_{g25} x_7 + C_{g245}}{x_5} \leq 1.0: \quad \texttt{C11 - Shaft Design Condition}, \tag{18k}$$

and the design variable limits,

$$C_{g10} \leq x_1 \leq C_{g11}, \tag{19a}$$

$$C_{g12} \leq x_2 \leq C_{g13}, \tag{19b}$$

$$C_{g14} \leq x_3 \leq C_{g15}, \tag{19c}$$

$$C_{g16} \leq x_4 \leq C_{g17}, \tag{19d}$$

$$C_{g18} \leq x_5 \leq C_{g19}, \tag{19e}$$

$$C_{g20} \leq x_6 \leq C_{g21}, \text{ and} \tag{19f}$$

$$C_{g22} \leq x_7 \leq C_{g23}. \tag{19g}$$

Table 1: Nominal Values for Gearbox Constants

Parameter	**Value**	**Parameter**	**Value**	**Parameter**	**Value**
C_{f1}	0.7854	C_{f3}	14.9334	C_{f5}	1.5079
C_{f2}	3.3333	C_{f4}	43.0934	C_{f6}	7.477
C_{g1}	27.0	C_{g7}	40.0	C_{g17}	8.3
C_{g2}	397.5	C_{g8}	5.0	C_{g18}	7.3
C_{g3}	1.93	C_{g9}	12.0	C_{g19}	8.3
C_{g4}	1.93	C_{g10}	2.6	C_{g20}	2.9
C_{g5}	1100.0	C_{g11}	3.6	C_{g21}	3.9
C_{A12}	745.0	C_{g12}	0.7	C_{g22}	5.0
C_{A1}	0.169×10^8	C_{g13}	0.8	C_{g23}	5.5

Table 1: (continued)

C_B	0.1	C_{g14}	17	C_{g24}	1.5
C_{A2}	0.1575×10^9	C_{g15}	28	C_{g25}	1.1
C_{g6}	850.0	C_{g16}	7.3	C_{g245}	1.9

For the purposes of this thesis, we will consolidate several of the uncertain parameters to equivalent representations shown by Equation 20. The new nominal value table is then shown by Table 2. This representation is mathematically equivalent to the original problem shown above. We perform this variable transformation to generate a set of constraints affine in the uncertain variables. When the constraints are affine in the uncertain parameters, the BoU method produces an exact solution and can guarantee constraint satisfaction. This property of the BoU method is discussed in detail in Chapter 3. The new uncertain parameters are represented by

$$C'_{A12} = C_{A12}^2, \tag{20a}$$

$$C'_{g5} = C_{g5}^2 C_B^2, \text{ and} \tag{20b}$$

$$C'_{g6} = C_{g6}^2 C_B^2. \tag{20c}$$

Table 2: Nominal Values for Gearbox Constants - Adjusted

Parameter	**Value**	**Parameter**	**Value**	**Parameter**	**Value**
C_{f1}	0.7854	C_{f3}	14.9334	C_{f5}	1.5079
C_{f2}	3.3333	C_{f4}	43.0934	C_{f6}	7.477

Table 2: (continued)

C_{g1}	27.0	C_{g7}	40.0	C_{g17}	8.3
C_{g2}	397.5	C_{g8}	5.0	C_{g18}	7.3
C_{g3}	1.93	C_{g9}	12.0	C_{g19}	8.3
C_{g4}	1.93	C_{g10}	2.6	C_{g20}	2.9
C'_{g5}	**12100.0**	C_{g11}	3.6	C_{g21}	3.9
C'_{A12}	**555025.0**	C_{g12}	0.7	C_{g22}	5.0
C_{A1}	0.169×10^8	C_{g13}	0.8	C_{g23}	5.5
		C_{g14}	17	C_{g24}	1.5
C_{A2}	0.1575×10^9	C_{g15}	28	C_{g25}	1.1
C'_{g6}	**7225.0**	C_{g16}	7.3	C_{g245}	1.9

The final problem realization accounting for the adjusted variables as well as the replacement of constraints with upper and lower bounds is shown in Equations 21 and 22 using nominal parameter values as shown in Table 2. The final problem realization is expressed as

$$\begin{aligned} \min_{x \in R^7} \quad & f = C_{f1} x_1 x_2^2 \left(C_{f2} x_3^2 + C_{f3} x_3 - C_{f4} \right) \\ & -C_{f5} \left(x_6^2 + x_7^2 \right) x_1 + C_{f6} \left(x_6^3 + x_7^3 \right) \\ & +C_{f1} \left(x_4 x_6^2 + x_5 x_7^2 \right), \end{aligned} \tag{21a}$$

$$s.t. \quad C1: \quad \frac{C_{g1}}{x_1 x_2^2 x_3} \leq 1.0, \tag{21b}$$

$$C2: \quad \frac{C_{g2}}{x_1 x_2^2 x_3^2} \leq 1.0, \tag{21c}$$

$$C3: \quad \frac{C_{g3}x_4^3}{x_2x_3x_6^4} \leq 1.0, \tag{21d}$$

$$C4: \quad \frac{C_{g4}x_5^3}{x_2x_3x_7^4} \leq 1.0, \tag{21e}$$

$$C5: \quad \frac{C'_{A12}x_4^2}{x_2^2x_3^2} + C_{A1} - C'_{g5}x_6^6 \leq 0.0, \tag{21f}$$

$$C6: \quad \frac{C'_{A12}x_5^2}{x_2^2x_3^2} + C_{A2} - C'_{g6}x_7^6 \leq 0.0, \tag{21g}$$

$$C7: \quad \frac{x_2x_3}{C_{g7}} \leq 1.0, \tag{21h}$$

$$C8: \quad \frac{C_{g8}x_2}{x_1} \leq 1.0, \tag{21i}$$

$$C9: \quad \frac{x_1}{C_{g9}x_2} \leq 1.0, \tag{21j}$$

$$C10: \quad \frac{C_{g24}x_6 + C_{g245}}{x_4} \leq 1.0, \text{ and} \tag{21k}$$

$$C11: \quad \frac{C_{g25}x_7 + C_{g245}}{x_5} \leq 1.0, \tag{21l}$$

with design variable constraints

$$C_{g10} \leq x_1 \leq C_{g11}, \tag{22a}$$

$$C_{g12} \leq x_2 \leq C_{g13}, \tag{22b}$$

$$C_{g14} \leq x_3 \leq C_{g15}, \tag{22c}$$

$$C_{g16} \leq x_4 \leq C_{g17}, \tag{22d}$$

$$C_{g18} \leq x_5 \leq C_{g19}, \tag{22e}$$

$$C_{g20} \leq x_6 \leq C_{g21}, \text{ and} \tag{22f}$$

$$C_{g22} \leq x_7 \leq C_{g23}. \tag{22g}$$

5.1 Nominal Solution

The nominal solution to the gearbox problem is well studied and documented in the current literature [15, 17, 54, 86]. As a result, we will spend very little time discussing this particular solution. To generate a solution, the solver is presented with the objective, constraints, and bounds as shown in Equations 21 and 22, and the upper and lower bounds are used to directly inform the optimizer (rather than considering them as additional constraints as in some literature references). The solution obtained is shown in Table 3 and the objective function value shown is the value obtained from nominal parameter values.

This solution agrees with previous documented solutions. Of note, before we move to robust formulations for this problem, is the fact that x_2, x_3, and x_4 all take values at their respective bounds.

Table 3: Gearbox Nominal Solution

x_1	x_2	x_3	x_4	x_5	x_6	x_7	**Objective**
3.5	0.7	17.0	17.3	7.7153	3.3502	5.2867	2994

In the remaining sections of this Chapter, we will consider variations in the uncertain parameters and we will use, for the purposes of discussion, a goal of 5% uncertainty allowance in each of the uncertain parameters. However, before we can do this, we must first select the parameters which we consider to be uncertain. Table 4 shows the selected uncertain parameters. We have deliberately chosen nearly all the engineering values for potential uncertainty due to (1) lack of historical experience insight and (2) as a means of illustrating the utility of the BoU method in the presence of large numbers of uncertain parameters. We have specifically chosen to exclude parameters used essentially as bounds for $x_1 - x_7$ since those values could very easily be modified for any solution method. Similarly C_{g8} and C_{g9} are excluded since these represent merely a desired range of $\frac{x_1}{x_2}$ and themselves represent a bound for that relationship.

Table 4: Gearbox Uncertain Parameters

C_{g1}	C_{g2}	C_{g3}
C_{g4}	C'_{g5}	C'_{g6}
C_{g7}	C_{g24}	C_{g25}
C_{g245}	C_{A1}	C_{A2}
C'_{A12}		

As a point of comparison, we apply a random selection of 10,000 realizations of these parameters to the nominal solution. To generate the random realizations we use a uniform distributed sampling method with a 5% variation for all the parameters listed in Table 4. In Figure 10 we show first the total number of constraint violations for all cases. This shows the expected number of total constraint violations for any specific case is expected to be either 1 or 2. Only a small percentage of realizations result in full constraint satisfaction implying that our design is infeasible most of the time when subjected to only a 5% uncertainty range in the selected engineering values. The bottom of Figure 10 shows the total number of violations for each constraint summed over all 10,000 realizations. Here, we see that we are most likely to violate constraints 5, 6, and 11. So, when subjected to 5% uncertainty in the engineering parameters, our solution is likely to violate shaft stress constraints (Constraints C5 and C6) or design considerations (Constraint C11). As we investigate the traditional approach to such uncertainty and our BoU methodology, we certainly would expect to improve upon the 10% success rate shown here. Finally, in Table 5, we provide a summary of the Monte Carlo results showing the number of successful and unsuccessful cases for this solution.

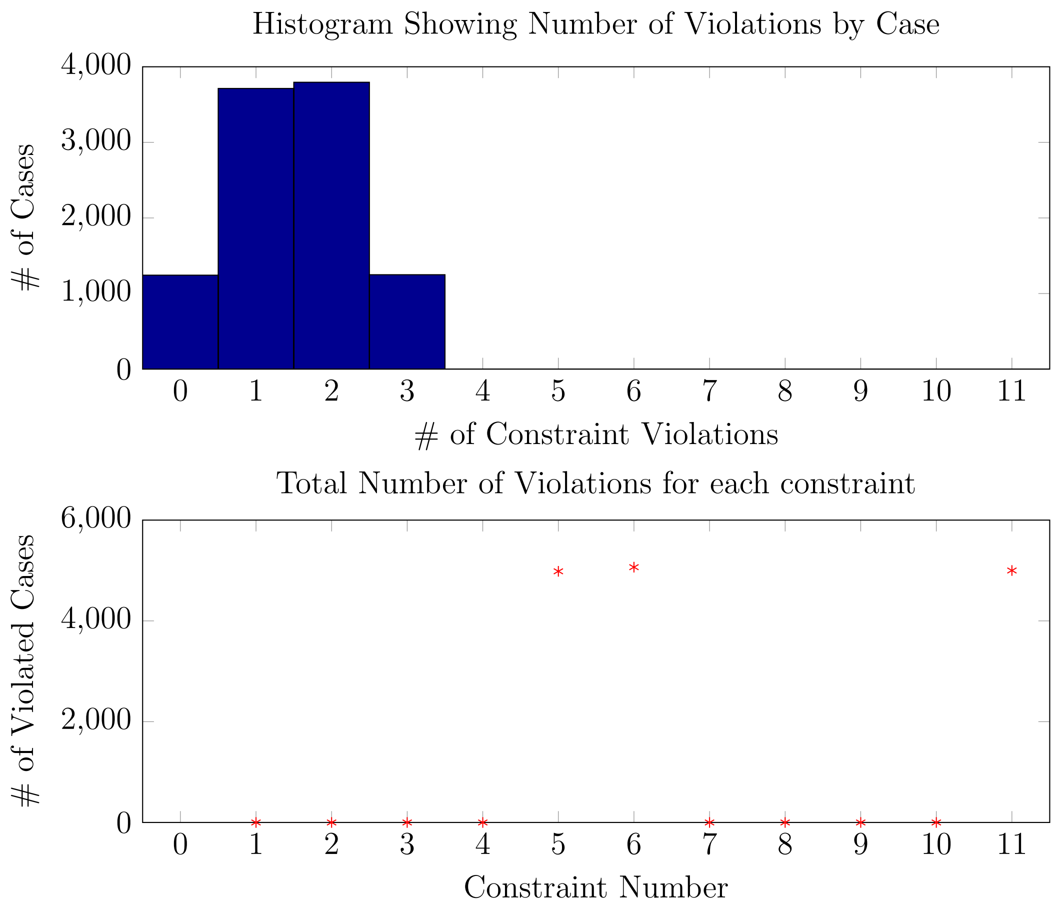

Figure 10: Summary of Results when Nominal Solution is Confronted with 5% Uncertainty

Table 5: Nominal Performance Under 5% Uncertainty

	Number of Cases
Acceptable Cases	1243
Cases with Constraint Violations	8757

5.2 Traditional Monte Carlo Approach to Robust Optimization

Lacking model insight, the traditional approach to engineering problems with several varying parameters is to vary them and evaluate a Monte Carlo dataset. We investigate this method here to see how it compares to our nominal and eventually our BoU solution.

We start by generating 10,000 random problem realizations and we solve as if each is a nominal problem with a slightly altered parameter set. We then tabulate our results for each parameter. The results of this monte carlo analysis are shown in Figures 11 and 12. Within these figures, we illustrate the upper and lower bounds for each variable with red horizontal dashed lines. We see clearly that x_5 and to a lesser extent x_7 show large effects as the uncertain parameters vary. Additionally, we can see that for any particular problem realization, the objective function remains bounded by 2700 and 3300.

Reviewing Figures 11 and 12, we are faced with the problem of selecting the most appropriate values for each design parameter. For this, we must rely on problem insight. In deference to our stress and deflection constraints, we choose maximum values for x_1, x_2, x_6, and x_7. To minimize stress, we also choose a minimum value for x_3. x_4 and x_5 are a bit tougher to choose since there is engineering justification for both maximal and minimal values. Through trial and error, we checked results for x_4 and x_5 set to their maximum, minimum, and average values. We found the fewest constraint violations when these parameters were both set to their maximums, so this will be the chosen set. The solutions as a result of these selections can be seen

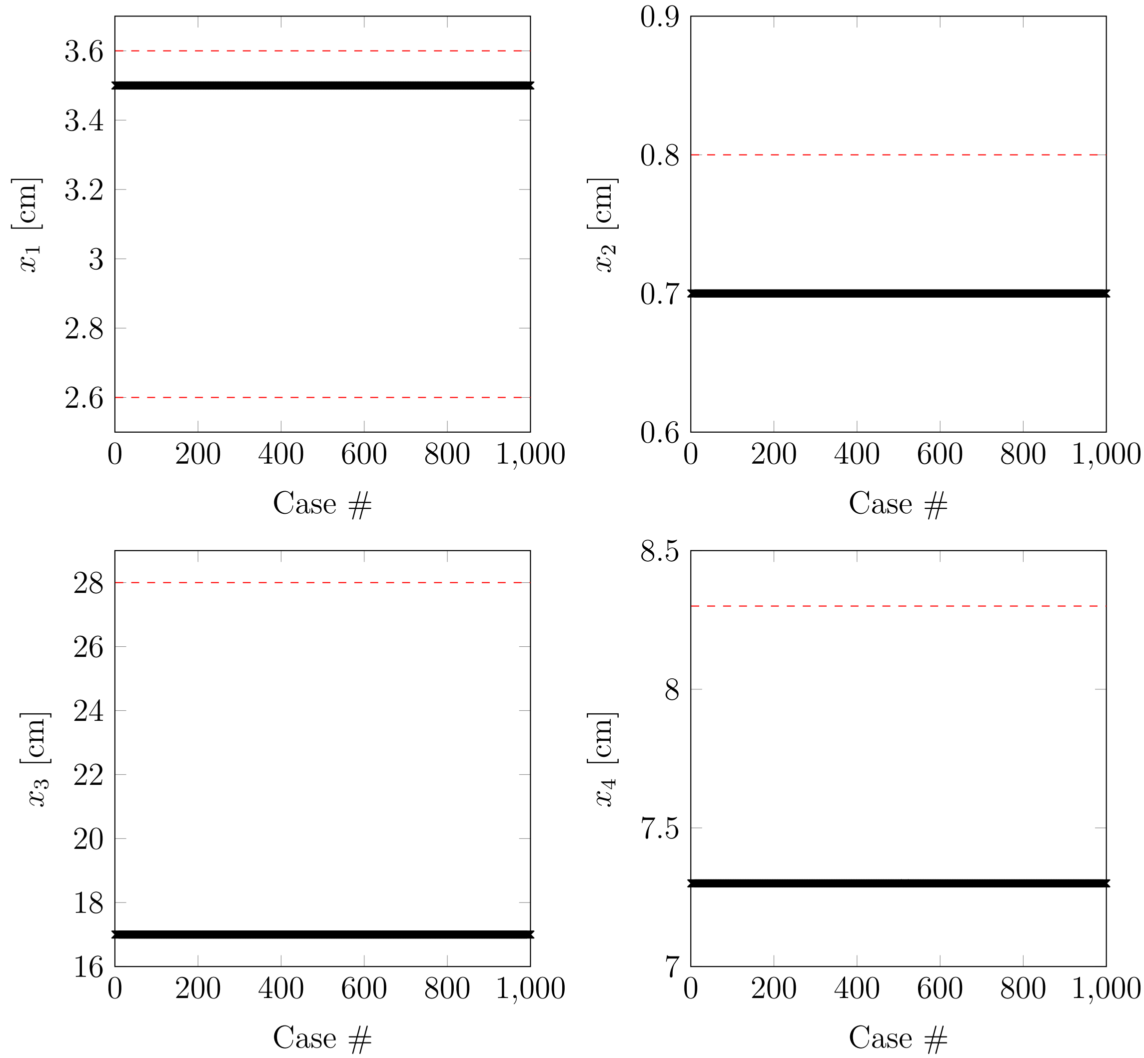

Figure 11: Summary of Results from Traditional Monte Carlo Investigation 1 of 2

in Table 6. The objective value shown is obtained from nominal parameter values.

We then apply the solution in Table 6 to 10,000 random problem realizations (the same 10,000 realizations shown in Figure 10). The results are shown in Figure 13. We see few violations; more than 99% of the realizations show no constraint violations. The computational cost for this solution was high however, requiring first the nominal solution of 10,000 cases to determine a potential range of values for x_1 – x_7, and then the application of knowledge and insight (along with a small amount

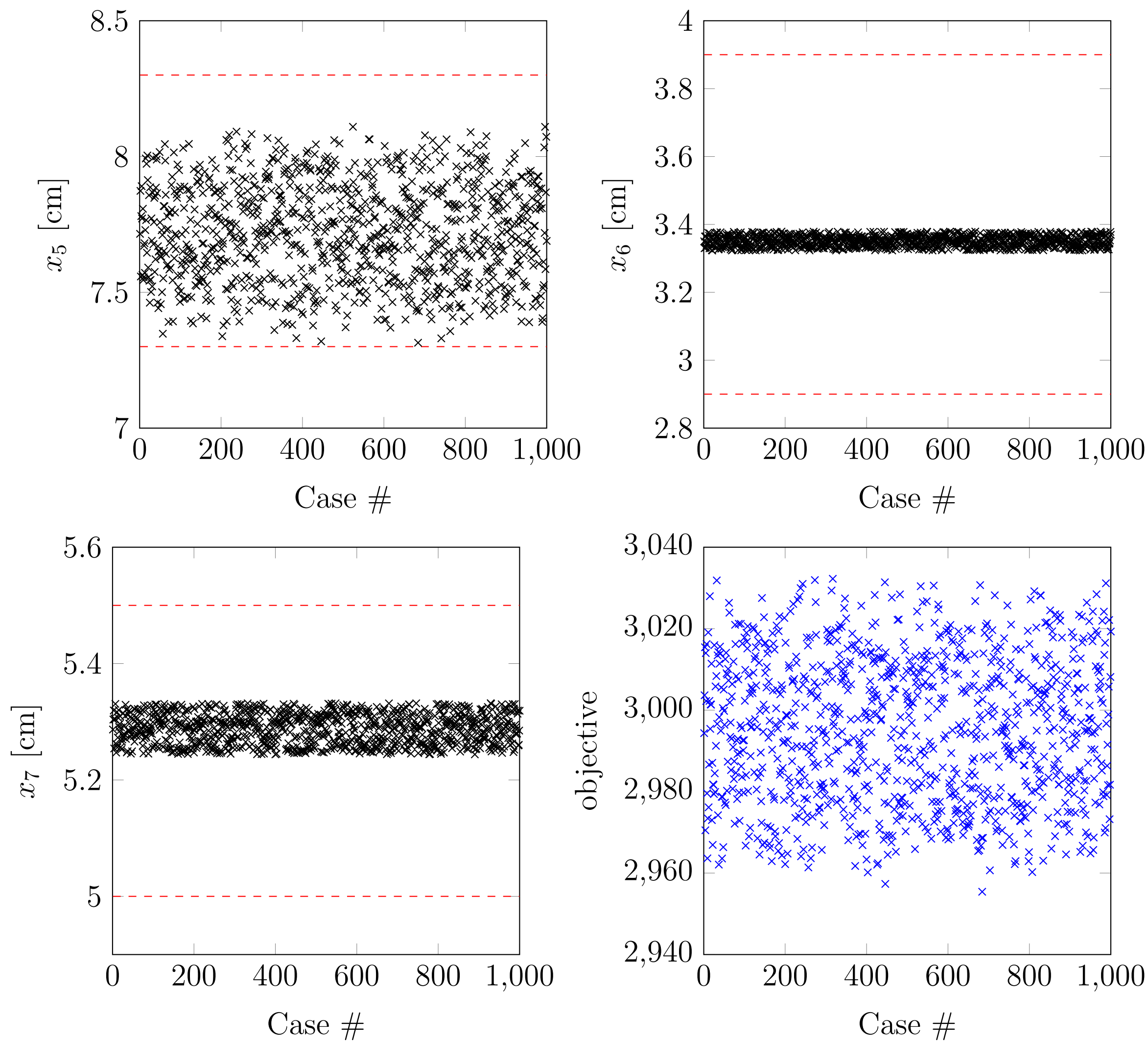

Figure 12: Summary of Results from Traditional Monte Carlo Investigation 2 of 2

of trial and error) to determine the final solution. With MATLAB on a quad-core intel processor and 2 GB of RAM, a single nominal solution requires approximately 0.03s of processor time. Multipled by 10,000 cases, this yields approximately 300s of computation. We will compare this statistic to the BoU method later in this Chapter. Similar to the summary data shown for our nominal solution, Table 7 provides summary information for the traditional method.

Table 6: Gearbox Monte Carlo Solution

Variable	x_1	x_2	x_3	x_4	x_5	x_6	x_7	**Objective**
Selection	Max	Max	Min	Max	Max	Max	Max	–
Value	3.5003	0.7	17.0	7.3014	8.1324	3.3790	5.3321	3040

Table 7: Traditional Monte Carlo Performance Under 5% Uncertainty

	Number of Cases
Acceptable Cases	9972
Cases with Constraint Violations	28

5.3 BoU Solution

Given our revised variable set (shown in Table 2) and the full problem definition in Equations 21 and 22, the application of the BoU method is straightforward. The robust version of our problem is shown in Equations 23 and 24. Within these Equations additional optimizer variables are denoted by s_i. These variables, similar to our earlier Triangle Illustration problem, provide a consistent means to optimize parameter uncertainty. The initial lexicographic weighting for the uncertain parameters is shown in Table 8. We provide the following initial BoU model:

$$\min_{x \in R^7, s \in \Re^{13}} \quad \alpha\left(\frac{f}{2994} - 1\right) + (\alpha - 1)\sum w_i s_i, \tag{23a}$$

$$s.t. \quad C1: \quad \frac{C_{g1}}{x_1 x_2^2 x_3} + \frac{\delta C_{g1}}{x_1 x_2^2 x_3} \leq 1.0, \tag{23b}$$

$$C2: \quad \frac{C_{g2}}{x_1 x_2^2 x_3^2} + \frac{\delta C_{g2}}{x_1 x_2^2 x_3^2} \leq 1.0, \tag{23c}$$

$$C3: \quad \frac{C_{g3} x_4^3}{x_2 x_3 x_6^4} + \frac{\delta C_{g3} x_4^3}{x_2 x_3 x_6^4} \leq 1.0, \tag{23d}$$

$$C4: \quad \frac{C_{g4} x_5^3}{x_2 x_3 x_7^4} + \frac{\delta C_{g4} x_5^3}{x_2 x_3 x_7^4} \leq 1.0, \tag{23e}$$

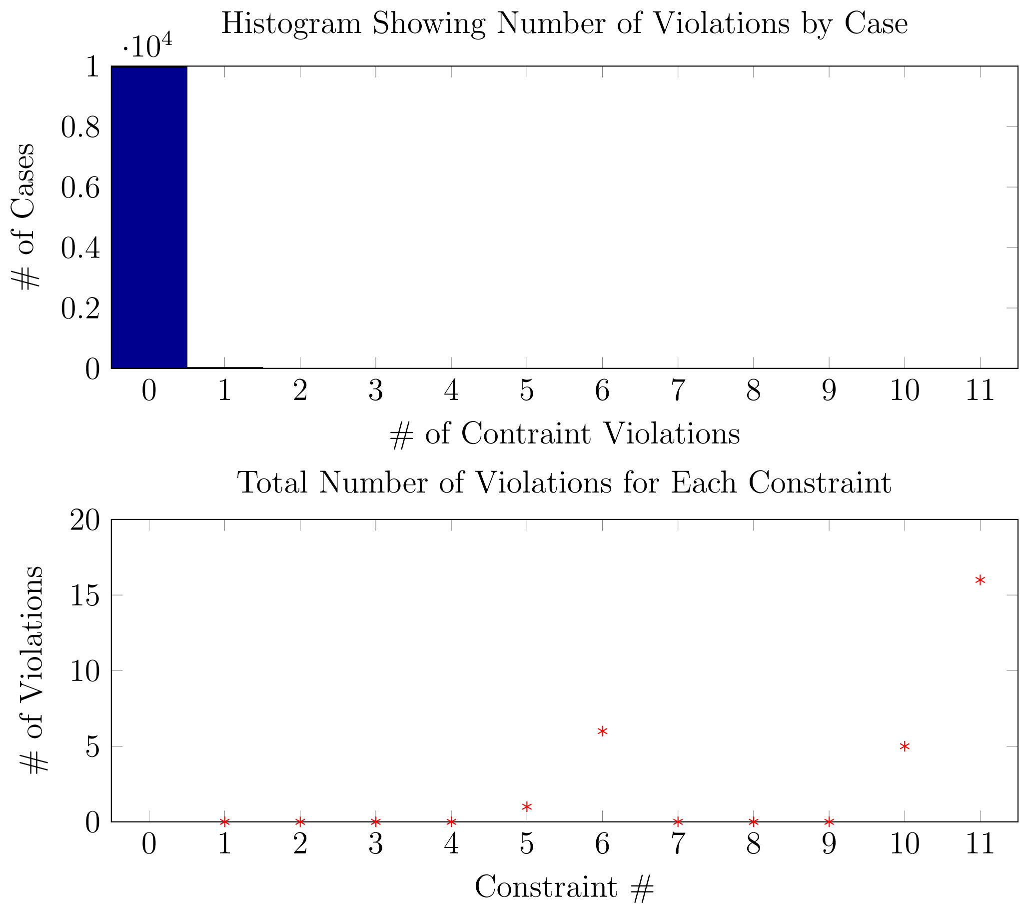

Figure 13: Summary of Results when Traditional Monte Carlo Solution is Confronted with 5% Uncertainty

$$C5: \quad \frac{C'_{A12}x_4^2}{x_2^2x_3^2} + C_{A1} - C'_{g5}x_6^6 + \frac{\delta C'_{A12}x_4^2}{x_2^2x_3^2} + \delta C_{A1}\delta C'_{g5}x_6^6 \leq 0.0, \tag{23f}$$

$$C6: \quad \frac{C'_{A12}x_5^2}{x_2^2x_3^2} + C_{A2} - C'_{g6}x_7^6 + \frac{\delta C'_{A12}x_5^2}{x_2^2x_3^2} + \delta C_{A2} + \delta C'_{g6}x_7^6 \leq 0.0, \tag{23g}$$

$$C7: \quad x_2x_3 - C_{g7} + \delta C_{g7} \leq 0.0, \tag{23h}$$

$$C8: \quad \frac{C_{g8}x_2}{x_1} \leq 1.0, \tag{23i}$$

$$C9: \quad \frac{x_1}{x_2} - C_{g9} \leq 1.0, \tag{23j}$$

$$C10: \quad \frac{C_{g24}x_6 + C_{g245}}{x_4} + \frac{\delta C_{g24}x_6}{x_4} + \frac{\delta C_{g245}}{x_4} \leq 1.0, \tag{23k}$$

$$C11: \quad \frac{C_{g25}x_7 + C_{g245}}{x_5} + \frac{\delta C_{g25}x_7}{x_5} + \frac{\delta C_{g245}}{x_5} \leq 1.0, \tag{23l}$$

$$\delta C_{g1} = s_1 C_{g1}, \delta C_{g2} = s_2 C_{g2}, \delta C_{g3} = s_3 C_{g3}, \delta C_{g4} = s_4 C_{g4}, \delta C'_{g5} = s_5 C'_{g5}, \tag{23m}$$

$$\delta C'_{g6} = s_6 C'_{g6}, \delta C_{g7} = s_7 C_{g7}, \delta C_{g24} = s_8 C_{g24}, \delta C_{g25} = s_9 C_{g25}, \tag{23n}$$

$$\delta C_{g245} = s_{10} C_{g245}, \delta C_{A1} = s_{11} C_{A1}, \delta C_{A2} = s_{12} C_{A2}, \text{ and } \delta C'_{A12} = s_{13} C'_{A12}, \tag{23o}$$

with design variable limits of

$$C_{g10} \leq x_1 \leq C_{g11}, \tag{24a}$$

$$C_{g12} \leq x_2 \leq C_{g13}, \tag{24b}$$

$$C_{g14} \leq x_3 \leq C_{g15}, \tag{24c}$$

$$C_{g16} \leq x_4 \leq C_{g17}, \tag{24d}$$

$$C_{g18} \leq x_5 \leq C_{g19}, \tag{24e}$$

$$C_{g20} \leq x_6 \leq C_{g21}, \text{ and} \tag{24f}$$

$$C_{g22} \leq x_7 \leq C_{g23}. \tag{24g}$$

Table 8: Gearbox Initial Lexicographic Weighting of Uncertain Parameters

Uncertain Parameter	**Weight**	**Uncertain Parameter**	**Weight**
C_{g1}	1	C_{g2}	1
C_{g3}	1	C_{g4}	1
C'_{g5}	1	C'_{g6}	1
C_{g7}	1	C_{g24}	1
C_{g25}	1	C_{g245}	1

As a first step, we solve the BoU method for varying values of alpha. This determines an initial pareto front for our solution space. For the purposes of this plot, we solve the BoU problem for increasing values of alpha and seed each iteration with the previous iteration result. The first iteration uses the nominal solution for the initial guess. The initial guess is critical since there is no mathematical guarantee of a unique BoU solution. These results are presented in Figures 14, 15, 16, and 17.

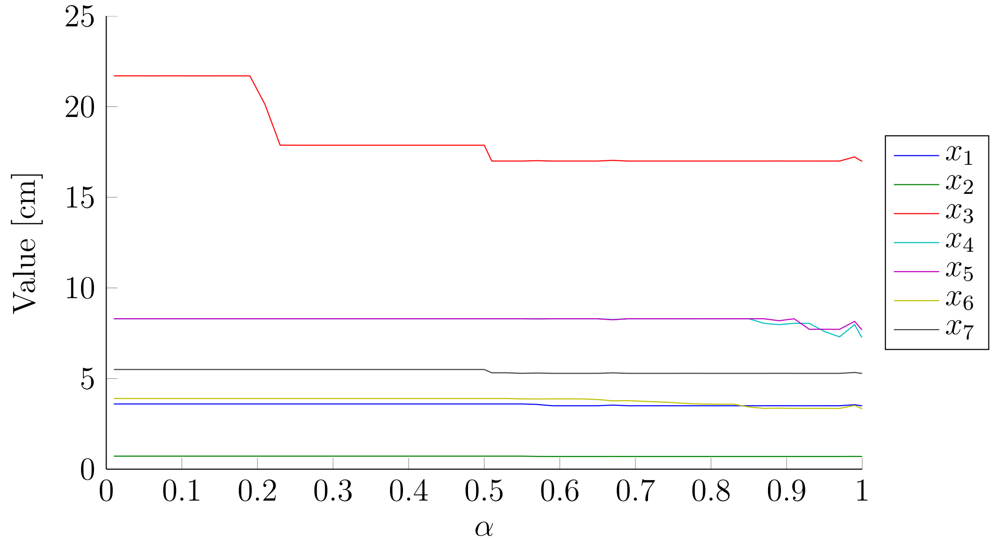

Figure 14: Initial BoU Pareto Front - Control Variables

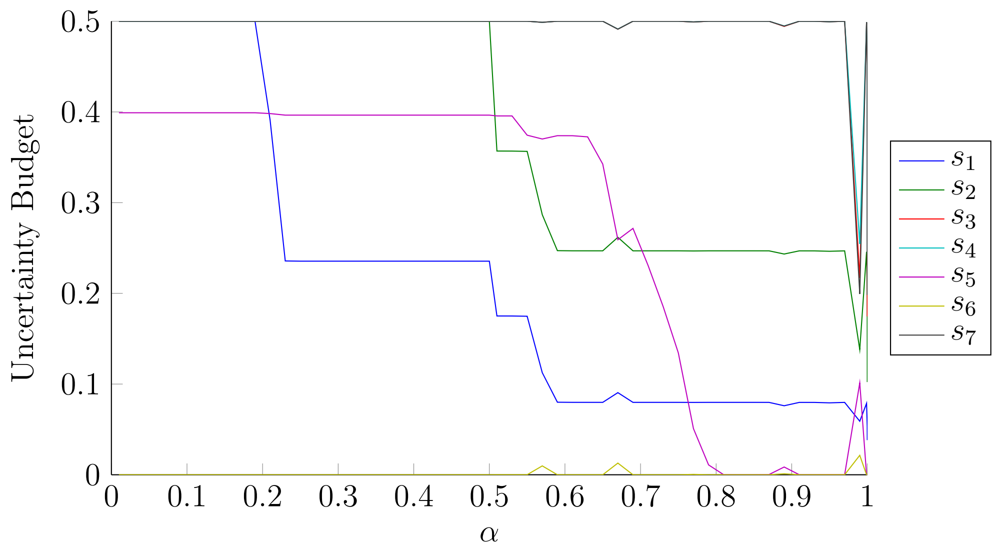

Figure 15: Initial BoU Pareto Front - Uncertain Parameters 1 of 2

As stated previously, the BoU method does not guarantee a unique solution,

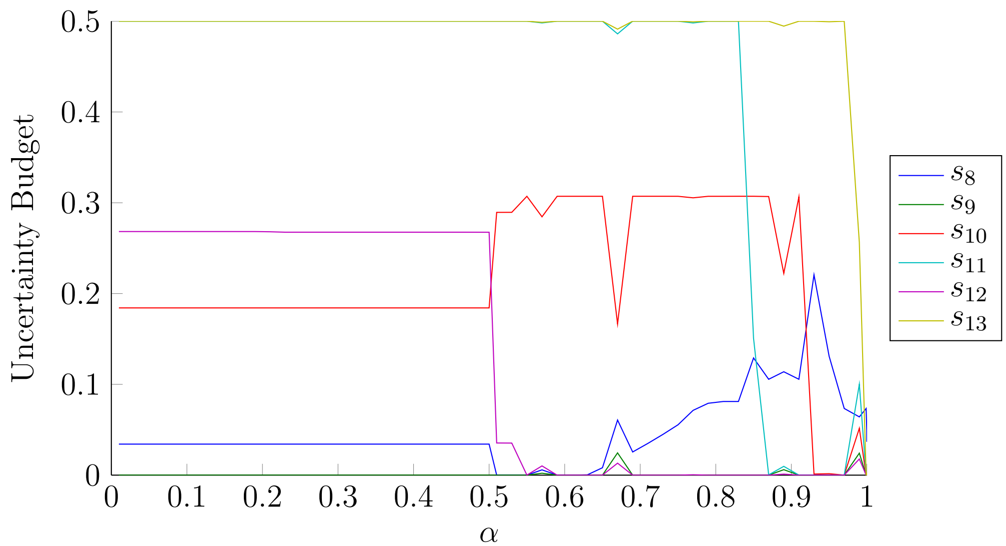

Figure 16: Initial BoU Pareto Front - Uncertain Parameters 2 of 2

and it may be possible to achieve similarly minimized objective values with slightly different choices of uncertainty. This is evident in Figure 16 near $\alpha = 0.7$. Here, the uncertainty allowance for the variable C_{g245} (represented by s_{10}) is reduced while C_{g24} (s_8), C_{A1} (s_{11}) and C_{A2} (S_{12}) are increased. However, as α increases, we see uncertain parameters return to near previous values. Near this value, C_{g24} (s_8) begins a steadily increasing trend of uncertainty. This implies the existence of at least two solutions near $\alpha = 0.7$ for which the BoU problem can be solved. In general, this is not a problem since the goal of the BoU method is to explore the feasibility space in the presence of uncertainty. Here we gain additional insight on the tradeoff between these parameters.

From Figure 14, we observe the design variables, aside from x_3, change very little as a function of α. This is in stark contrast to our Monte Carlo analysis, however, when we review the permitted uncertainty for each uncertain parameter in Figures 15 and 16, it is quickly apparent that our initial BoU solution does not allow 5%

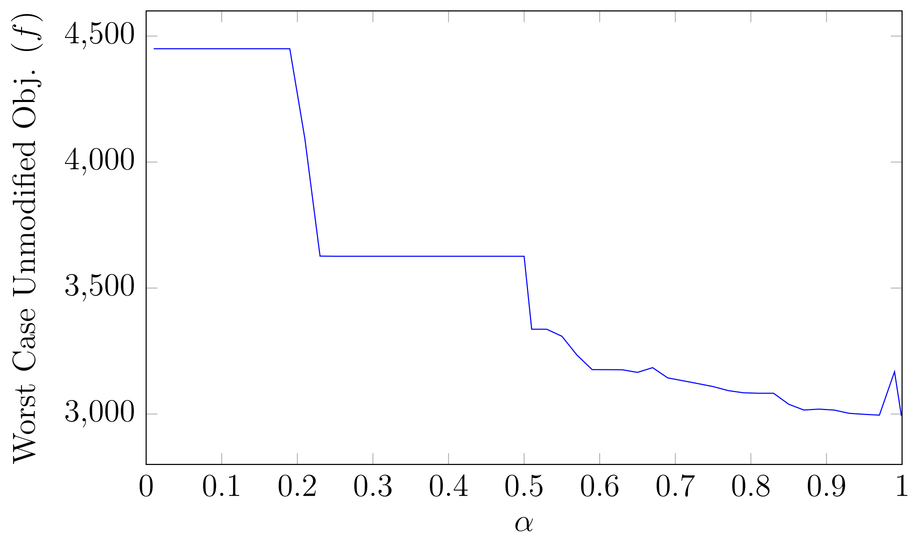

Figure 17: Initial BoU Pareto Front - Worst Case Objective Function

uncertainty in all variables. In fact, even for very low values of α (implying high weight on the robust component of the objective function) we do not achieve 5% uncertainty. Clearly, we will need to revise our BoU solution if we are to account for the full desired range of uncertainty.

Figure 17 shows our objective function value varies as expected over the range of α, approaching (and in fact matching) the nominal solution when $\alpha = 1$. An interesting result of our BoU solution can be seen when α is very near a value of 1. Here, we find the worst case objective function equal to that of the nominal solution, but we also see permitted levels of uncertainty in several uncertain parameters. These values represent the "slack" in the constraints. Any constraint with slack has the potential for increased uncertainty without violation of the constraint. Evaluating the BoU solution near $\alpha = 1$ provides us with insight on how slack in constraints can translate to permitted levels of uncertainty. The BoU solution for the chosen value of $\alpha = 0.5$ is shown in Table 9. Within this table, we also show the nominal solution for comparison.

Table 9: Gearbox Budget of Uncertainty Solution

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	**Objective**
BoU	3.6	0.72	17.8744	8.3	8.3	3.9	5.5	3626
Nom	3.5	0.70	17.0	17.3	7.7153	3.3502	5.2867	2994

Applying the solution in Table 9, Figure 18 shows how our initial BoU result compares to the desired uncertainty of 5%. For this figure, we have chosen an α value of 0.5 (consistent with the solution shown in Table 9, equally weighting optimality and robustness. As we expect given our initial results, our method does not as yet provide the desired level of uncertainty in key variables, fairing similarly to the traditional method based on Monte Carlo results. We see constraint violations in constraint C11. A summary of our test results is provided in Table 10.

Table 10: Initial BoU Performance Under 5% Uncertainty

	Number of Cases
Acceptable Cases	9901
Cases with Constraint Violations	99

5.4 Lexicographic BoU Solution

In an attempt to gain additional uncertainty in critical variables, we now modify our BoU problem to lexicographically weight specific variables. Given our initial result in Figure 18, we increase the relative secondary weights on C_{g24} and C_{g25}. The weighting for all uncertain parameters is shown in Table 11. Note the weights shown in Table 11 are normalized to 1 when used within the method and are shown as whole numbers for illustration purposes only. All other elements of the BoU

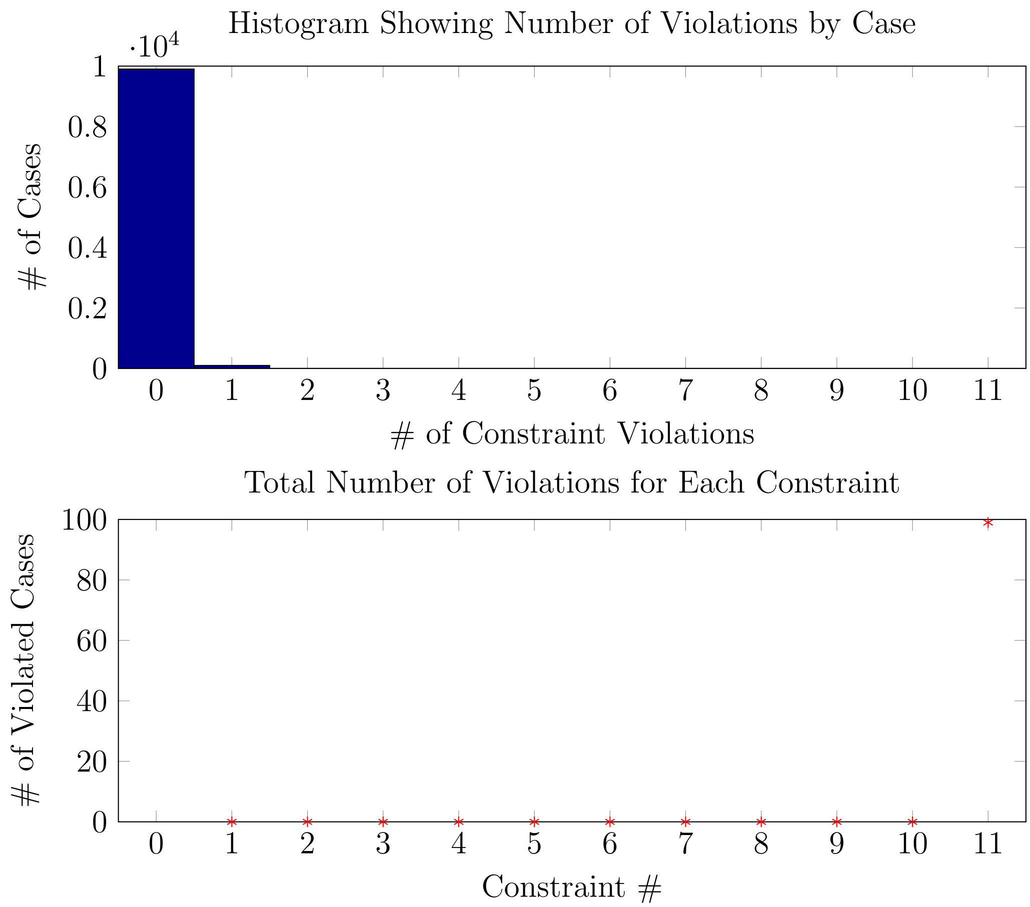

Figure 18: Summary of Results when Initial BoU Solution is Confronted with 5% Uncertainty, $\alpha = 0.5$

setup remain the same as shown in Equations 23 and 24. The resulting pareto front solution is provided in Figures 19, 20, 21, and 22.

Table 11: Gearbox Modified Lexicographic Weighting of Uncertain Parameters

Uncertain Variables	**Weight**	**Uncertain Variables**	**Weight**
C_{g1}	1	C_{g2}	1
C_{g3}	1	C_{g4}	1
C'_{g5}	1	C'_{g6}	1
C_{g7}	1	C_{g24}	2

Table 11: (continued)

C_{g25}	2	C_{g245}	1
C_{A1}	1	C_{A2}	1
C'_{A12}	1		

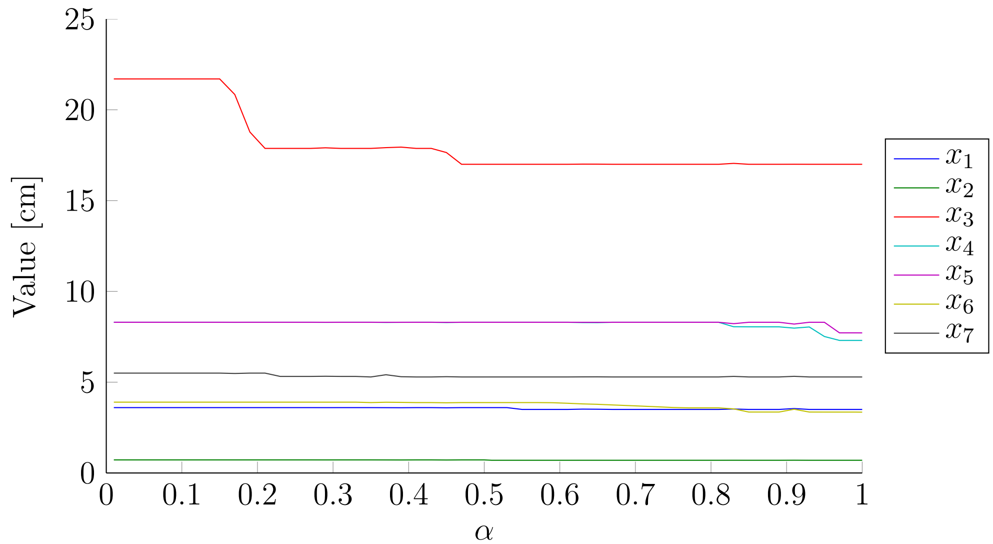

Figure 19: Lexicographic BoU Pareto Front - Control Variables

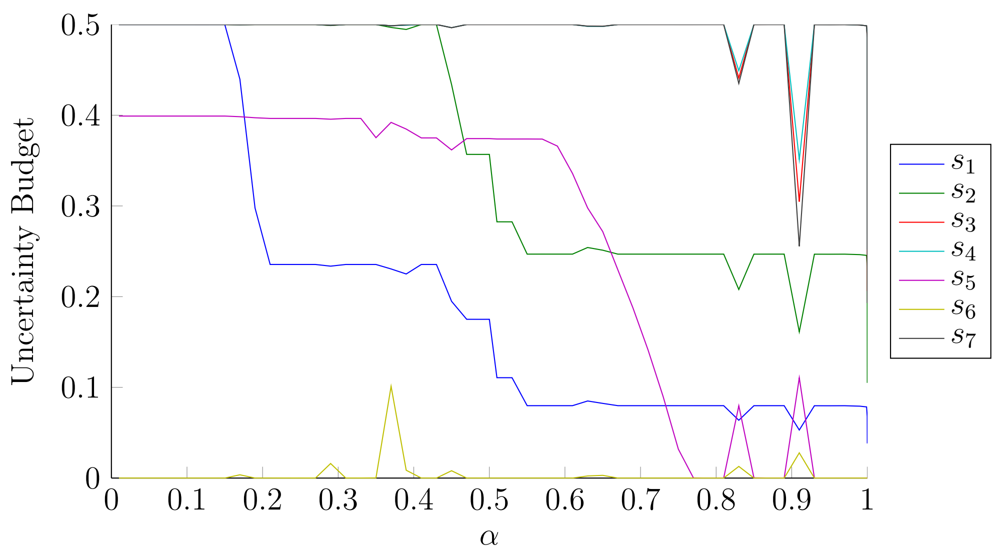

Figure 20: Lexicographic BoU Pareto Front - Uncertain Parameters 1 of 2

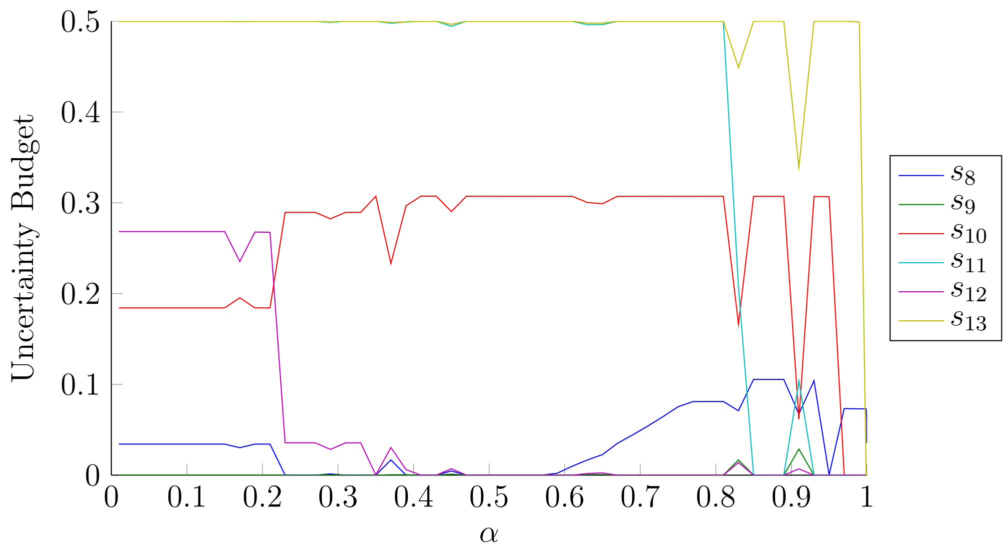

Figure 21: Lexicographic BoU Pareto Front - Uncertain Parameters 2 of 2

As we expect, the permitted uncertainty for C_{g24} and C_{g25} is increased and for many values of α even shows a level above 5% as we desire. However, we clearly see that permitted uncertainty in other variables is sacrificed, and for our desired solution, this will still not be acceptable. Table 12 shows the solution for $\alpha = 0.5$, Figure 23 shows the results when this solution is submitted to 5% uncertainty, and Table 13 provides summary information. By attempting to prioritize specific parameters, we have improved the situation for C11, but created a much more serious violation for C6.

Certainly a near infinite amount of trial and error is possible at this point, and for some problems it may even be a desired feature to trade within the uncertainty design space, however, we need something more. In the next section we will provide yet a third variant for the BoU method.

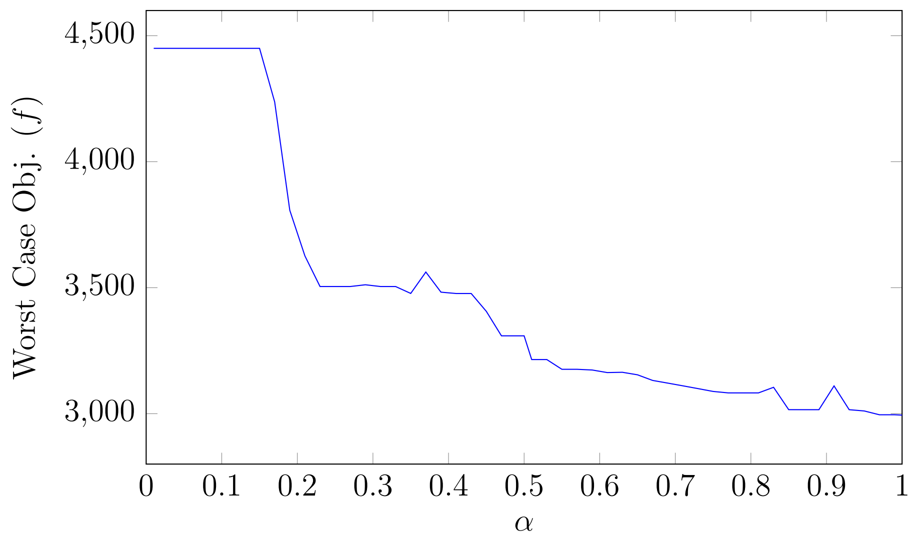

Figure 22: Lexicographic BoU Pareto Front - Worst Case Objective Function

Table 12: Gearbox Lexicographic Budget of Uncertainty Solution

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	**Objective**
BoU	3.6	0.72	17.8744	8.3	8.3	3.9	5.5	3626
Lexi	3.6	0.72	17.0000	8.3	8.3	3.8775	5.2875	3309
Nom	3.5	0.70	17.0	17.3	7.7153	3.3502	5.2867	2994

5.5 BoU with Minimum Uncertainty - MinP Variation

Our next application of the BoU method inserts additional constraints in our uncertain variables in the form of lower bounds. Using a lower bound of $s_i > 0.05$ for our uncertain variables, we can ensure our final results will at least meet this criteria. We will, of course, suffer in our objective function value, but we can ensure all constraints are met and remove the tedious process of lexicographic tuning. This BoU variation will hereby be refered to as the 'MinP' variation. Results for this variation are shown in Figures 24, 25, 26, and 27.

Unlike our previous two BoU iterations, the addition of lower bound constraints on the uncertain parameters means that we do not return to the optimal

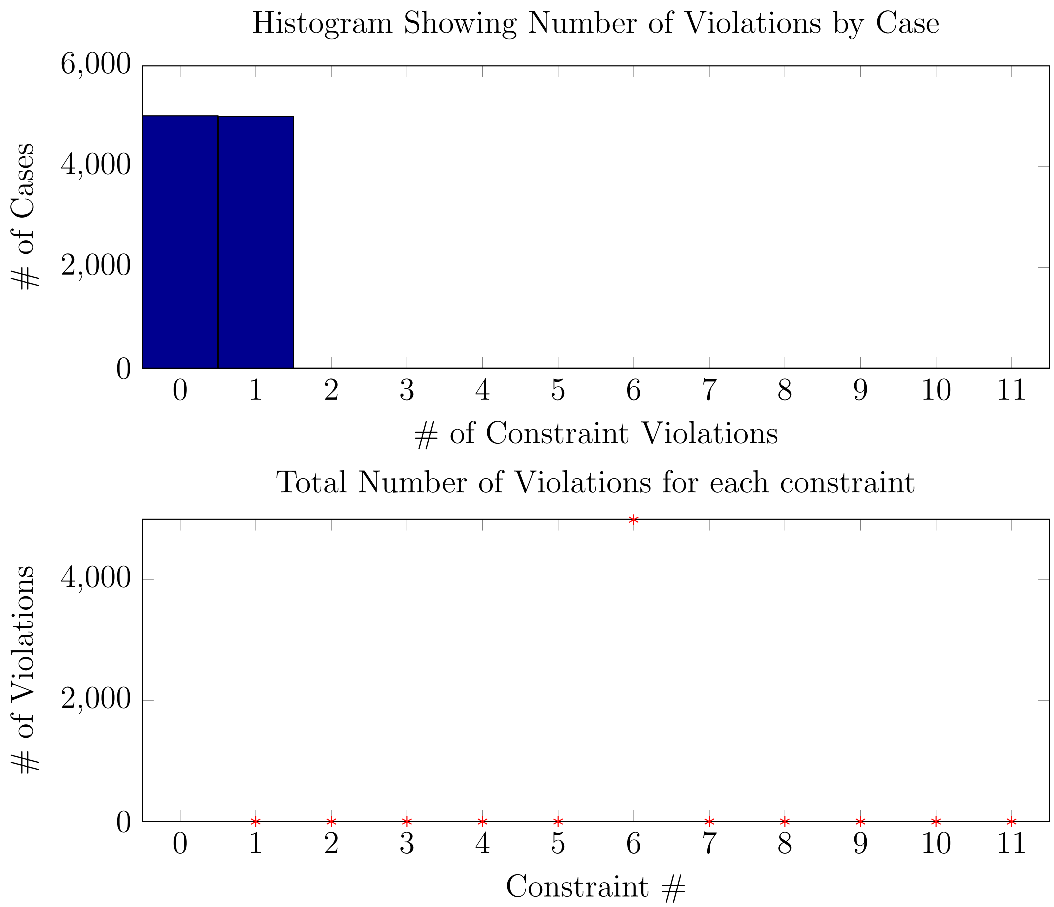

Figure 23: Summary of Results when Lexicographic BoU Solution is Confronted with 5% Uncertainty, $\alpha = 0.5$

objective function value at $\alpha = 1$. In this case, rather than choose $\alpha = 0.5$ as we have done previously, the application of the additional bounds make $\alpha = 1$ a better choice. We provide this solution in Table 14. Applying this solution, we achieve the Monte Carlo results in Figure 28. Results are summarized in Table 15. This special case of the BoU method, when a fixed lower bound is used for s_i, is very similar to the results shown recently by Ben-Tal.

We finally achieve what we have desired all along. We see all constraints met, for the various realizations under 5% uncertainty. We also see a modest cost in objective function value for this assurance (3078 from a nominal value of 2994 -

Table 13: Lexicographic BoU Performance Under 5% Uncertainty

	Number of Cases
Acceptable Cases	5008
Cases with Constraint Violations	4992

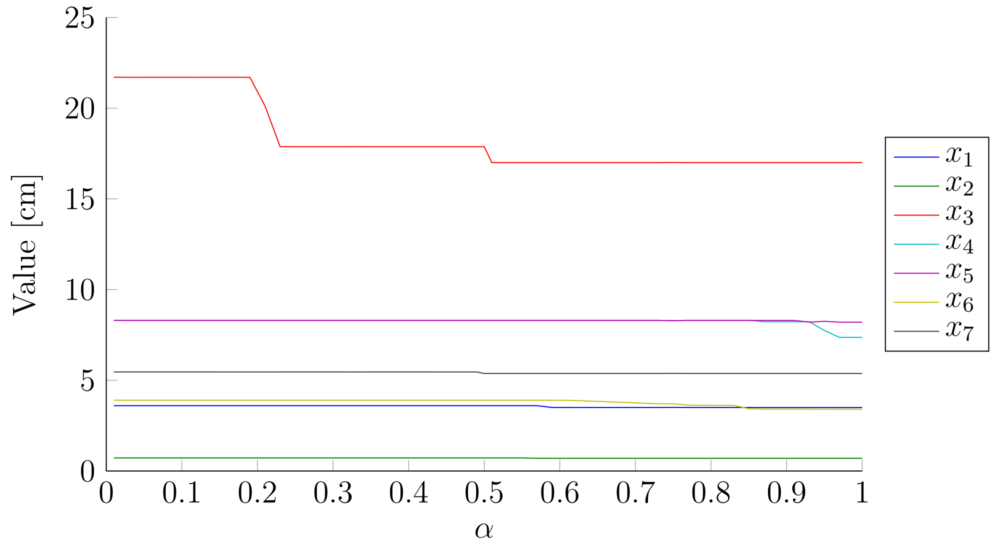

Figure 24: Minimum Uncertainty BoU Pareto Front - Control Variables

roughly a 3% increase). Clearly, we have generated a solution which both tolerates the required uncertainty and provides an assurance that the objective function will remain bounded below its worst case value.

5.6 Summary of Results

We now compare the results of all methods. Table 16 shows the solution for all methods discussed in this chapter, including all three iterations of the BoU method. For convenience we refer to the BoU variants as “initial”, “lexicographic”, and “MinP”.

From these results, we see the following performance when we subject each solution to a 5% uncertainty in each of several selected uncertain variables (see Table

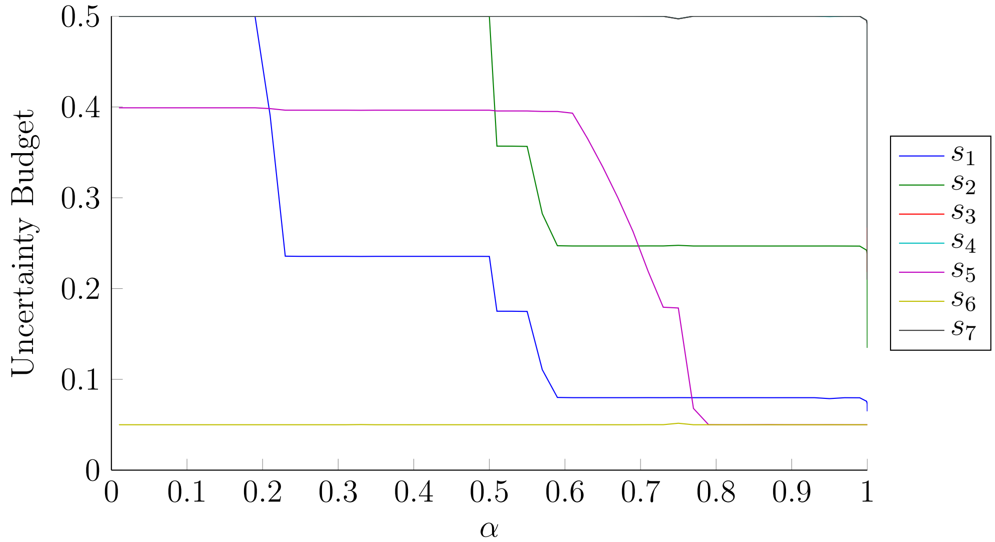

Figure 25: Minimum Uncertainty BoU Pareto Front - Uncertain Parameters 1 of 2

Table 14: Gearbox Minimum Uncertainty Budget of Uncertainty Solution

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	**Objective**
BoU	3.6	0.72	17.8744	8.3	8.3	3.9	5.5	3626
Lexi	3.6	0.72	17.0000	8.3	8.3	3.8775	5.2875	3309
MinP	3.5	0.7	17.0	7.3605	8.2040	3.4067	5.3758	3078
Nom	3.5	0.70	17.0	17.3	7.7153	3.3502	5.2867	2994

4 for a list of these variables).

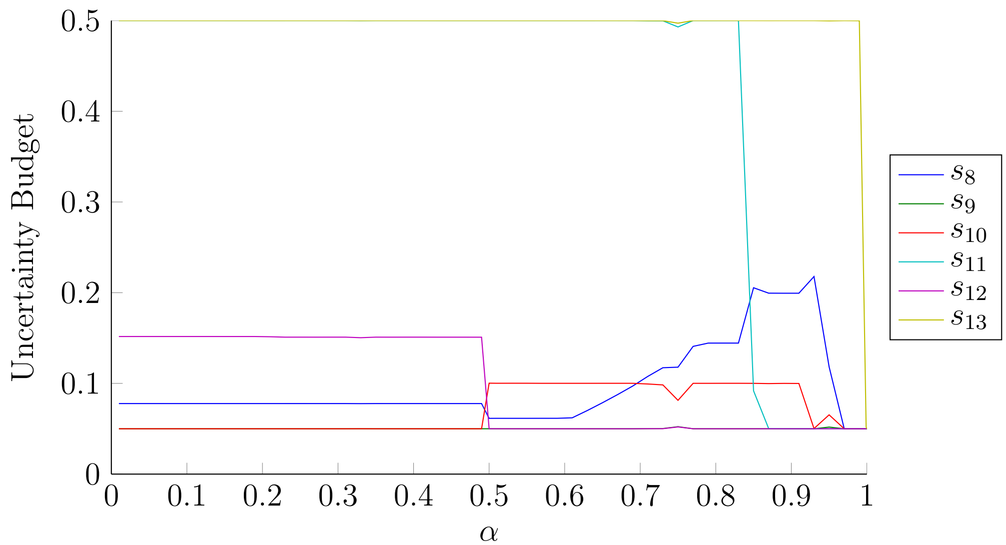

Figure 26: Minimum Uncertainty BoU Pareto Front - Uncertain Parameters 2 of 2

Table 15: BoU MinP Performance Under 5% Uncertainty

	Number of Cases
Acceptable Cases	10000
Cases with Constraint Violations	0

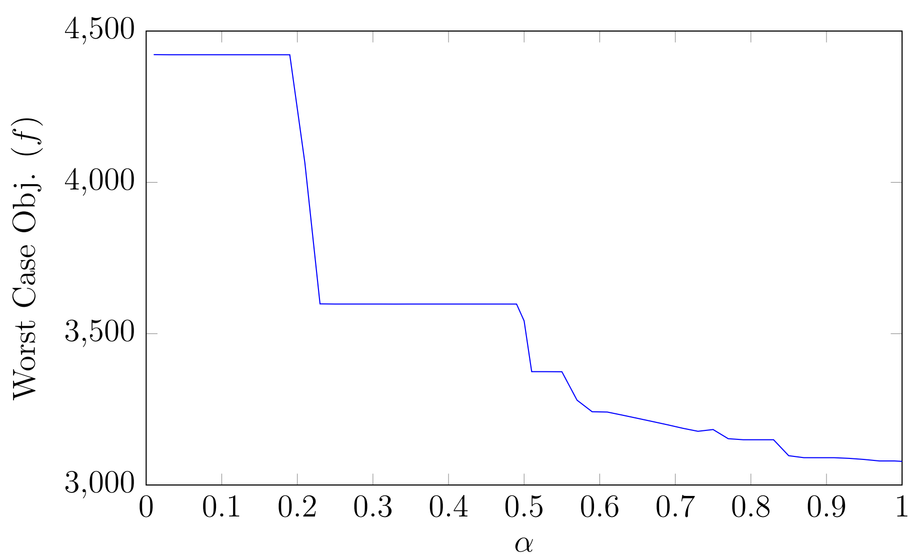

Figure 27: Minimum Uncertainty BoU Pareto Front - Worst Case Objective Function

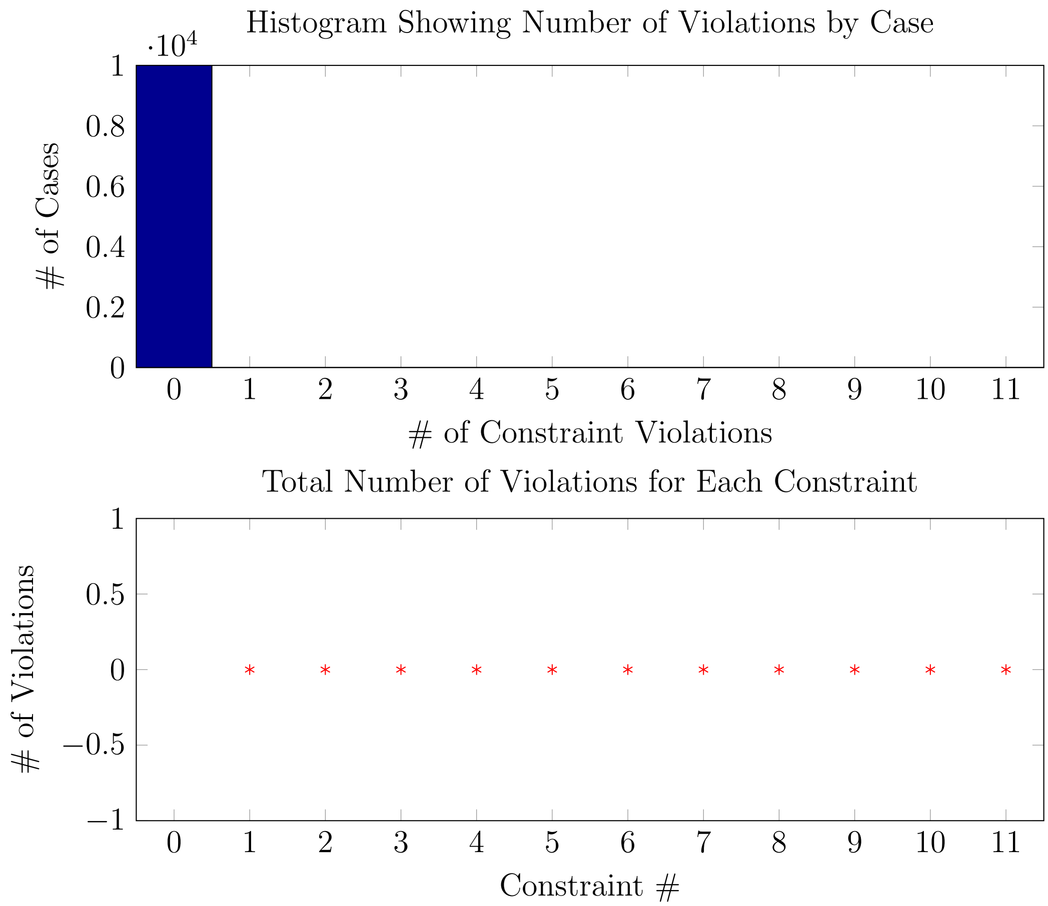

Figure 28: Summary of Results when Minimum Uncertainty BoU Solution is Confronted with 5% Uncertainty, $\alpha = 1.0$

Table 16: Gearbox Solution Comparison

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	**Obj**	% Pass
Nom	3.5	0.7	17.0	17.3	7.7153	3.3502	5.2867	2994	12.43
Trad	3.5	0.7	17.0	7.3546	8.1981	3.4080	5.3779	3040	99.72
BoU	3.6	0.72	17.8744	8.3	8.3	3.9	5.5	3626	99.01
Lexi	3.6	0.72	17.0001	8.3	8.3	3.8208	5.2875	3309	50.08
MinP	3.6	0.72	17.8732	8.3	8.3	3.9	5.3762	3078	100

6 BoU Illustrative Example 3 - Helical Spring Example

Our next example contains a smaller number of constraints, but they are highly nonlinear functions. The example used here is based on the form presented in Azarm's doctoral thesis [54] and will be referred to hereafter as the "helical spring" problem.

Our objective is to select the wire diameter and coil diameter for a helical spring such that specific design conditions are met, such as the minimum number of coils, spring constant value, deflection distance, and material constraints (e.g. buckling). The objective function is a representation of the total material volume resulting in the minimum material usage for a given specification. The problem is best visualized by Figure 29.

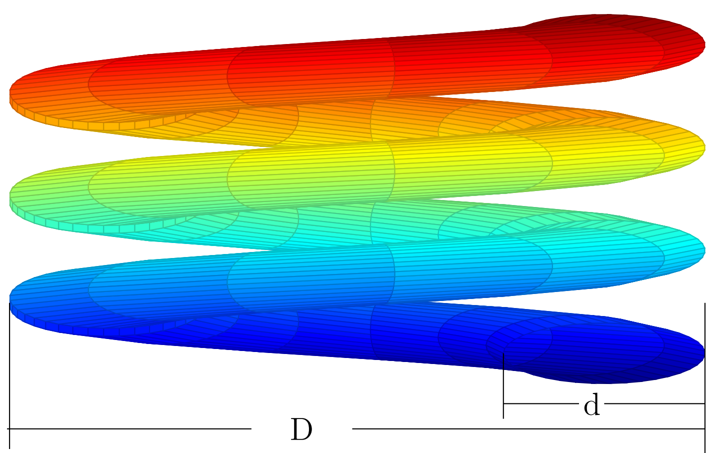

Figure 29: Illustration of a Helical Spring

The nominal problem is shown in Equation 25. This formulation uses the wire diameter, d, and the coil diameter, D, as the control variables rather than the Spring index ($C = \frac{D}{d}$) and the wire diameter, d as presented by Azarm. This modification is mathematically equivalent and was chosen to better illustrate the uncertainty investigation for this example. For the helical spring problem, we minimize the volume of wire (Equation 25a) subject to the constraints in Equation 25b-25l. Table 17 shows the constants used for the nominal solution. Equation 26 shows intermediate calculations for the problem constants. We solve the NLP,

$$\min_{[D,d]\in\Re^2} \quad 2.04\left(\frac{F_U - F_L}{C_1 N C^{B_1}} + \frac{F_U + F_L}{C_2}\right)\frac{C^{0.86}}{d^{(2+A_1)}}, \tag{25a}$$

$$s.t \quad G1: \quad K_1 G d^3 - D \leq 0: \text{ Surging}, \tag{25b}$$

$$G2: \quad K_2 G d^5 - D^5 \leq 0: \text{ Buckling}, \tag{25c}$$

$$G3: \quad \frac{K_3 D^3}{G} - d^4 \leq 0: \text{ Min. No. of Coils}, \tag{25d}$$

$$G4: \quad G K_6 d^5 + L_6 d D^3 - D^3 \leq 0: \text{ Pocket Length}, \tag{25e}$$

$$G5: \quad K_7 D + K_7 d - 1 \leq 0: \text{ Outside Diameter}, \tag{25f}$$

$$G6: \quad d + K_8 - D \leq 0: \text{ Inside Diameter}, \tag{25g}$$

$$G7: \quad \frac{K_{11} D^3}{G} - d^5 \leq 0: \text{ Clash Allowance}, \tag{25h}$$

$$G8: \quad I_L d - D \leq 0: \text{ Spring Index Bound}, \tag{25i}$$

$$G9: \quad D - d I_U \leq 0: \text{ Spring Index Bound}, \tag{25j}$$

$$G10: \quad d_{min} - d \leq 0: \text{ Wire Diameter Bound, and} \tag{25k}$$

$$G11: \quad d - d_{max} \leq 0: \text{ Wire Diameter Bound}, \tag{25l}$$

with the intermediate variables

$$K_1 = \frac{f\Delta}{112800(F_U - F_L)}, \tag{26a}$$

$$K_2 = \frac{F_U(1+A)}{22.3k^2}, \tag{26b}$$

$$K_3 = 8kN_{min}, \tag{26c}$$

$$K_6 = \frac{(1+A)}{8kL_m}, \tag{26d}$$

$$L_6 = \frac{Q}{L_m}, \tag{26e}$$

$$K_7 = \frac{1}{OD}, \tag{26f}$$

$$K_8 = ID, \text{ and} \tag{26g}$$

$$K_{11} = \frac{0.8(F_U - F_L)}{A}. \tag{26h}$$

Table 17: Nominal values for Spring Constants

Parameter	**Value**	**Unit**	**Description**
Q	2	–	Number of Inactive Coils
L_m	1.25	in	Max Spring Length at Max Force
A	0.4	–	Percentage of Wire Diameter Between Adjacent Coils Under Max Force
OD	3	in	Max Outside Diameter
ID	0.75	in	Min Inside Diameter
N_{min}	3	–	Min Number of Coils
I_L	4	–	Min Spring Index
I_U	20	–	Max Spring Index
Δ	0.25	in	Spring Deflection
G	11.5e6	psi	Shear Modulus

Table 17: (continued)

F_U	30	lb	Max Force
F_L	18	lb	Min Force
NC	1e6	–	Number of Cycles to Failure
f	500	Hz	Min Allowable Natural Frequency
d_{min}	0.004	in	Min Wire Diameter
d_{max}	0.25	in	Max Wire Diameter
A_1	-0.14	–	Material Constant
C_1	630500	–	Material Constant
C_2	160000	–	Material Constant
B_1	-0.2137	–	Material Constant

Although relatively straightforward, this problem contains highly nonlinear constraints. For the purposes of our discussion in this chapter, we will consider the following scenario:

Let us assume that we are a spring manufacturer and have just received an order from a potential customer to produce a large volume of specially designed springs. The new customer plans to order a large quantity of springs and has provided several very specific constraints. As the manufacturer, we must determine what quality of wire must be purchased. More expensive wire sources come with assurances of minimal material variation whereas less expensive wire sources come with higher variations. We must determine if a wire source with an expected 5% variation in material properties can be used and still satisfy the customer constraints.

Given this scenario, Table 18 shows the parameters for which we will consider uncertainty.

Table 18: Helical Spring Uncertain Parameters

A_1	B_1	C_1
C_2	G	

6.1 Nominal Solution

The nominal solution to the helical spring problem is shown in Table 19. Clearly, this solution provides a satisfaction of all constraints under nominal conditions, however when we investigate the solution under the desired 5% material uncertainty, we immediately see that this solution falls well short of the desired robust solution. To test the robustness of the solution we evaluate constraint satisfaction for a random selection of 10,000 realizations of the material parameters. Figure 30 shows almost 100% of the Monte Carlo realizations fail either constraint 1 (G1) or G3. Clearly we need a better solution if we are to purchase from a less expensive wire manufacturer. Table 20 summarizes the results of this Monte Carlo investigation. As in the previous chapter we will use these values as a point of comparison.

Table 19: Helical Spring Nominal Solution

D	**d**
0.95202	0.096422

Table 20: Nominal Performance Under 5% Uncertainty

	Number of Cases
Acceptable Cases	26
Cases with Constraint Violations	9974

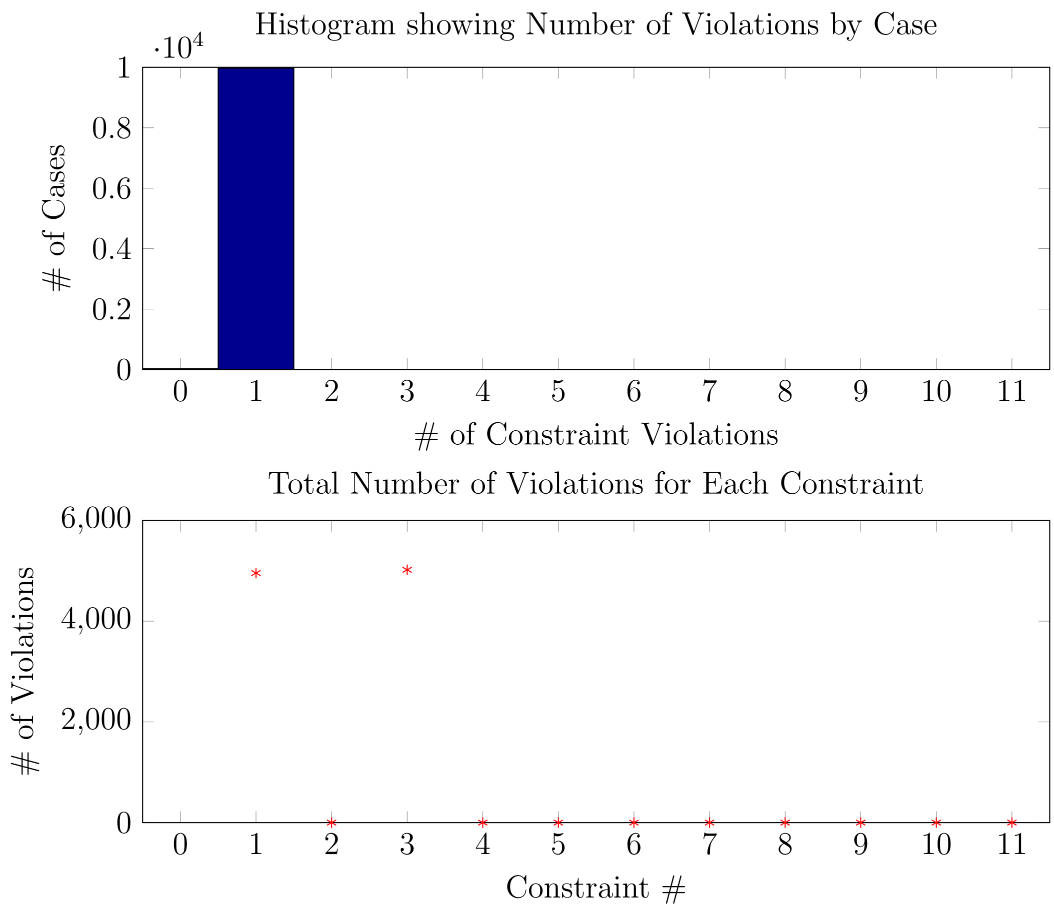

Figure 30: Summary of Results when Nominal Solution is Confronted with 5% Uncertainty

6.2 Traditional Monte Carlo Approach to Robust Optimization

Without a general method, traditional engineering practice is to use a monte carlo approach. For this approach, the designer will generate many random realizations with the desired uncertainty, solve each problem independently, and then select a final solution from statistical analysis or extreme values. The results of such an investigation are shown in Figures 31-32. Figure 31 shows the values for D and d respectively and Figure 32 shows the objective function value for each realization. Only a representative set of 1000 cases are shown in these figures, however a total sample of 10,000 was used.

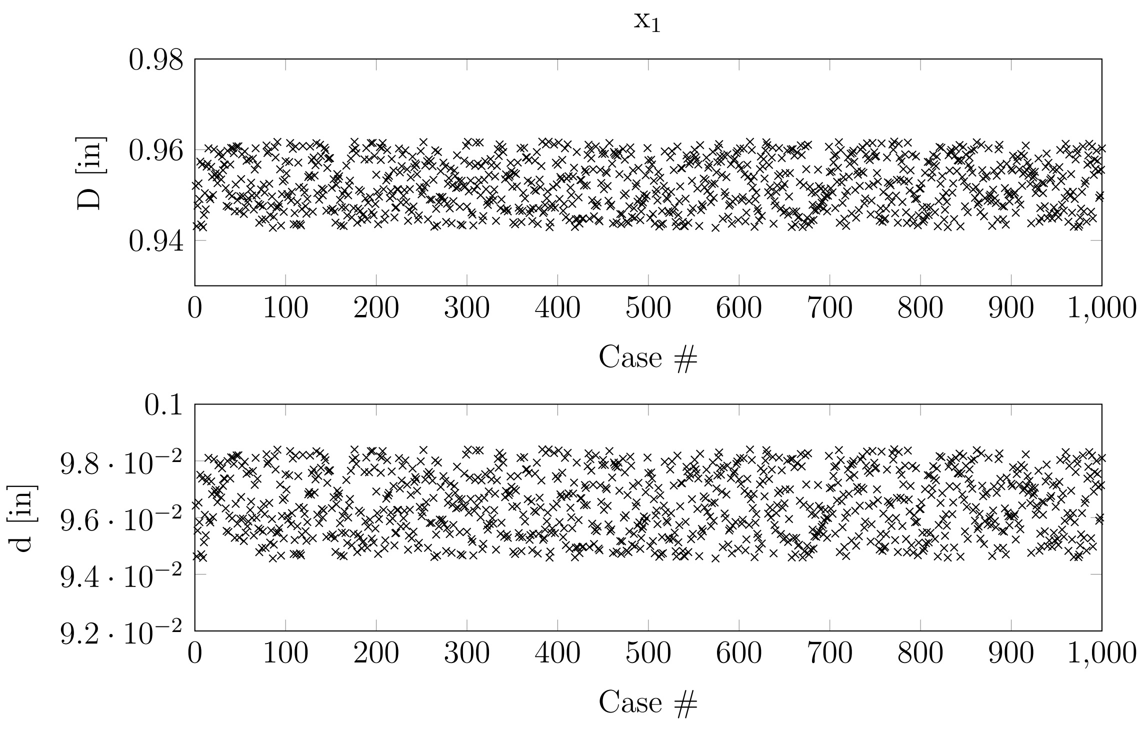

Figure 31: Summary of Results from Traditional Monte Carlo Investigation 1 of 2

After completing the Monte Carlo analysis, we must choose values for D and d to see if we have achieved our goal of allowing the desired material uncertainty.

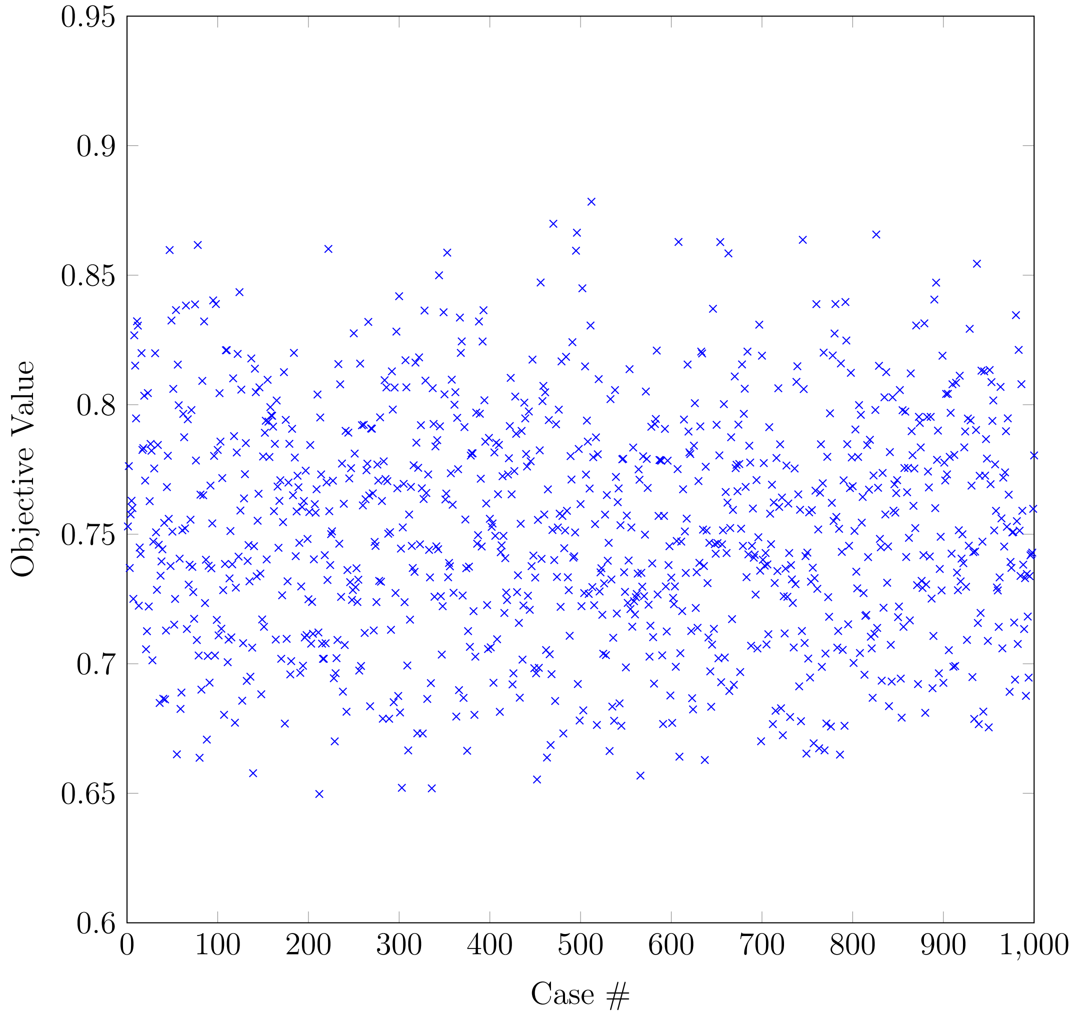

Figure 32: Summary of Results from Traditional Monte Carlo Investigation 2 of 2

Without specific insight, we reviewed maximum, minimum, and average values for both D and d and reviewed the performance of each permutation against another 10,000 sample set of realizations. Unlike the gearbox example, this problem is not readily solved with this method. Ultimately we elect to use the minimum values of D and d from our Monte Carlo analysis. Table 21 shows the best solution based on this method and Figure 33 shows this solution measured against our desired 5% uncertainty.

Clearly, Monte Carlo will not be enough for this problem. We still violate

Table 21: Helical Spring Traditional Solution

Variable	**D**	**d**
Selection	Min	Min
Value	0.9428	0.0946

constraints for an overwhelming majority of the test cases. Unfortunately, choosing the average or maximum values for D and d result in similarly poor robust performance. Table 22 summarizes our findings. Certainly more involved traditional methods could be employed for this problem - perhaps a systematic exploration of the design space with a concurrent investigation of robust performance, but for our purposes we have shown already the opportunity for an improved method. We will now turn our attention to the BoU method and see if we can improve this situation.

Table 22: Traditional MC Performance Under 5% Uncertainty

	Number of Cases
Acceptable Cases	11
Cases with Constraint Violations	9989

6.3 BoU Solution

As with the gearbox example, we first apply the BoU method with equal secondary weights, w_i, and by exploring the pareto front as the primary weight, α, progresses from 0 (most robust) to 1 (most optimal). We use a modified version of the original problem as shown in Equations 27 - 29. The initial lexicographic weighting is shown in Table 23. We solve the NLP,

$$\min_{[x,y,v]\in R^3, s\in R^5} \quad \alpha\left(\frac{v}{0.7501}-1\right)+(\alpha-1)\sum_{i=1}^{5} w_i s_i, \tag{27a}$$

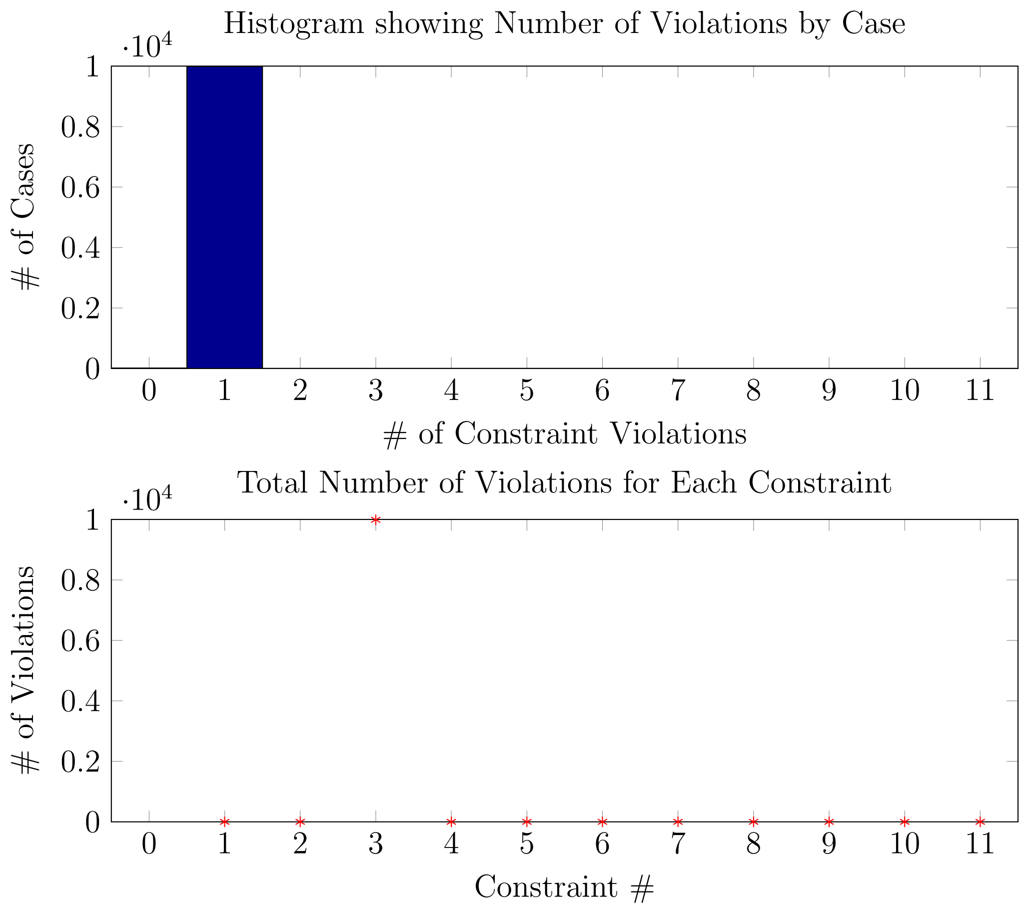

Figure 33: Summary of Results when Traditional Monte Carlo Solution is Confronted with 5% Uncertainty

$$s.t.\quad G1: \quad K_1Gd^3 - D + K_1d^3\delta G \leq 0, \tag{27b}$$

$$G2: \quad K_2Gd^5 - D^5 + K_2d^5\delta G \leq 0, \tag{27c}$$

$$G3: \quad \frac{K_3D^3}{G} - d^4 + \frac{K_3D^3\delta G}{G^2} \leq 0, \tag{27d}$$

$$G4: \quad GK_6d^5 + L_6dD^3 - D^3 + K_6d^5\delta G \leq 0, \tag{27e}$$

$$G5: \quad K_7D + K_7d - 1 \leq 0, \tag{27f}$$

$$G6: \quad d + K_8 - D \leq 0, \tag{27g}$$

$$G7: \quad \frac{K_{11}D^3}{G} - d^5 + \frac{K_{11}D^3\delta G}{G^2} \leq 0, \tag{27h}$$

$$G8: \quad I_Ld - D \leq 0, \tag{27i}$$

$$G9: \quad D - dI_U \leq 0, \tag{27j}$$

$$G10: \quad d_{min} - d \leq 0, \tag{27k}$$

$$G11: \quad d - d_{max} \leq 0, \text{ and} \tag{27l}$$

$$\begin{aligned} &\frac{K_0 D^{0.86}}{d^{2.86+A_1}} + \frac{K_0 D^{0.86} ln(d) \delta A_1}{d^{2.86+A_1}} + \frac{2.04 D^{0.86}(F_U - F_L) ln(NC) \delta B_1}{d^{2.86+A_1} C_1 N C^{B_1}} \\ &\quad + \frac{2.04 D^{0.86}(F_U - F_L) \delta C_1}{d^{2.86+A_1} C_1^2 N C^{B_1}} + \frac{(F_U + F_L) D^{0.86} \delta C_2}{d^{2.86+A_1} C_2^2} - v \leq 0, \end{aligned} \tag{27m}$$

where uncertain parameters are defined by,

$$\delta G = G s_1, \tag{28a}$$

$$\delta A_1 = A_1 s_2, \tag{28b}$$

$$\delta B_1 = B_1 s_3, \tag{28c}$$

$$\delta C_1 = C_1 s_4, \text{ and} \tag{28d}$$

$$\delta C_2 = C_2 s_5, \tag{28e}$$

and intermediate constants are

$$K_1 = \frac{f \Delta}{112800(F_U - F_L)}, \tag{29a}$$

$$K_2 = \frac{F_U(1 + A)}{22.3k^2}, \tag{29b}$$

$$K_3 = 8kN_{min}, \tag{29c}$$

$$K_6 = \frac{(1 + A)}{8kL_m}, \tag{29d}$$

$$L_6 = \frac{Q}{L_m}, \tag{29e}$$

$$K_7 = \frac{1}{OD}, \tag{29f}$$

$$K_8 = ID, \tag{29g}$$

$$K_{11} = \frac{0.8(F_U - F_L)}{A}, \text{ and} \tag{29h}$$

$$K_0 = 2.04\frac{(F_U - F_L)}{C_1 N C^{B_1}} + \frac{(F_U + F_L)}{C_2}. \tag{29i}$$

Table 23: Helical Spring Initial Lexicographic Weighting

Uncertain Variables	**Lexicographic Weight**
G	1
A_1	1
B_1	1
C_1	1
C_2	1

Solving this problem, we produce the pareto front shown in Figures 34-36. Clearly, the uncertainty required for the parameter G will require the most work since the allowed uncertainty for this parameter, δG, remains quite low throughout the pareto front. This shows G has a high cost, or stated differently, the problem is sensitive to changes in G.

Selecting $\alpha = 0.5$, we investigate this solution against 10,000 random problem realizations. This solution is shown in Table 24. Again, in Figure 37, we see violations in contraints G1 and G3. This result is not surprising however, because when $\alpha = 0.5$, the allowed uncertainty for the parameter G is much lower than 5% and consequently, we expect to see constraint violations in constraints which contain this parameter. To correct this, we will attempt to weight this parameter and evolve our BoU solution to a lexicographic representation. A summary of our results is shown in Table 25.

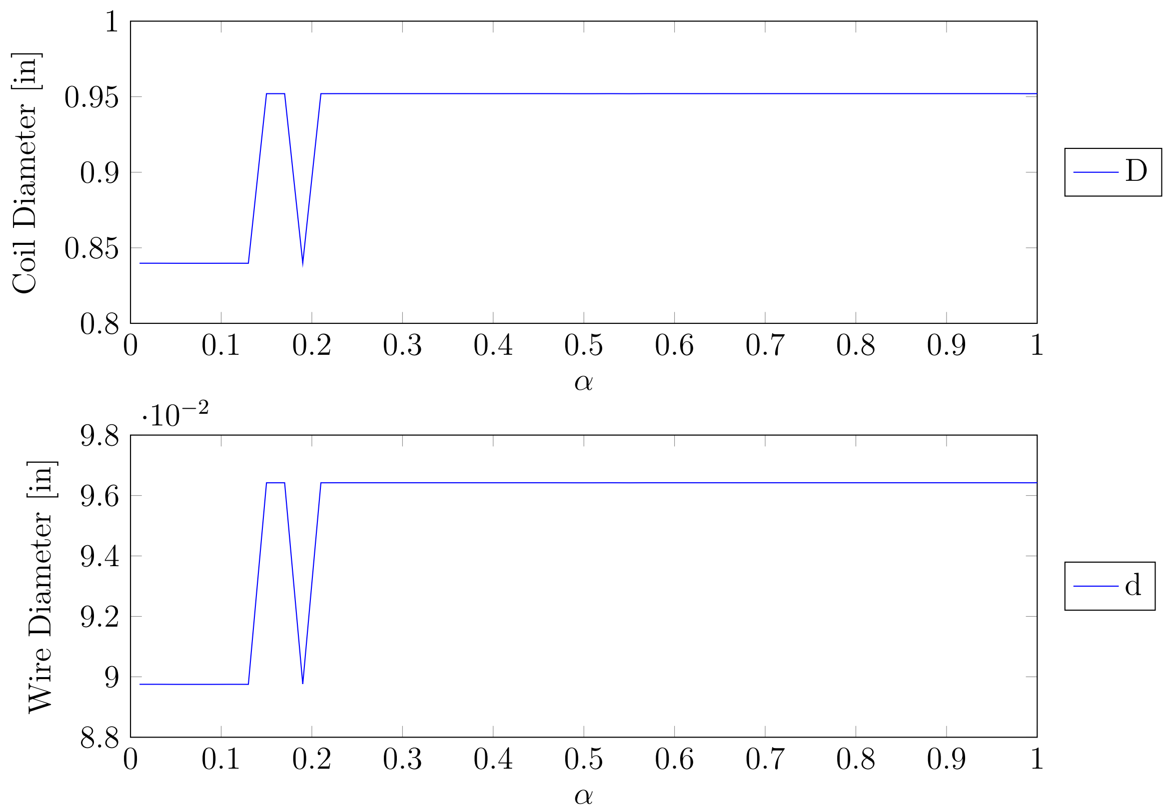

Figure 34: Initial BoU Pareto Front - Control Variables

Table 24: Helical Spring BoU Solution

D	d	s_1	s_2	s_3	s_4	s_5
0.9520	0.0964	0.0	0.4999	0.00	0.00	0.5

6.4 Lexicographic BoU Solution

In an attempt to improve our results, we choose a new set of lexicographic weights, shown in Table 26. These weights attempt to increase the available uncertainty for G, while maintaining the uncertainty we enjoy for the other uncertain parameters. We specifically reduce the weighting for wire and coil diameter since these can be easily screened in the case of wire diameter or tuned through manufacturing means. The results are shown in Figures 38 - 40. Again, the numbers shown in Table 26 are normalized when used directly in the BoU solution.

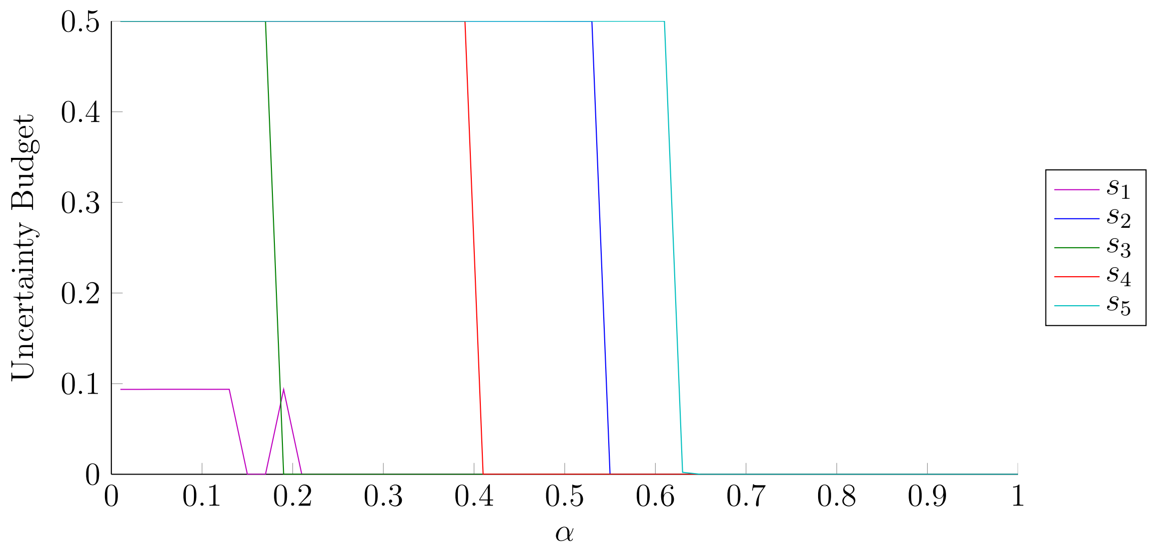

Figure 35: Initial BoU Pareto Front - Uncertain Parameters 2 of 2

Table 25: BoU Performance Under 5% Uncertainty

	Number of Cases
Acceptable Cases	32
Cases with Constraint Violations	9968

The solution obtained when $\alpha = 0.5$ is shown in Table 27. In Figure 39, we clearly see increased levels of acceptable uncertainty in the variable G at the expense of other parameters. Since this parameter exists in multiple constraints, we expect to see an improvement in the tolerance to uncertainty. Figure 41 shows how this solution holds up with the desired 5% uncertainty and Table 28 shows a summary of these results. Although we see an acceptable solution already (no constraint violations), we will continue to investigate to see if we can reduce the worst case objective value while still maintaining the desired immunization to uncertainty.

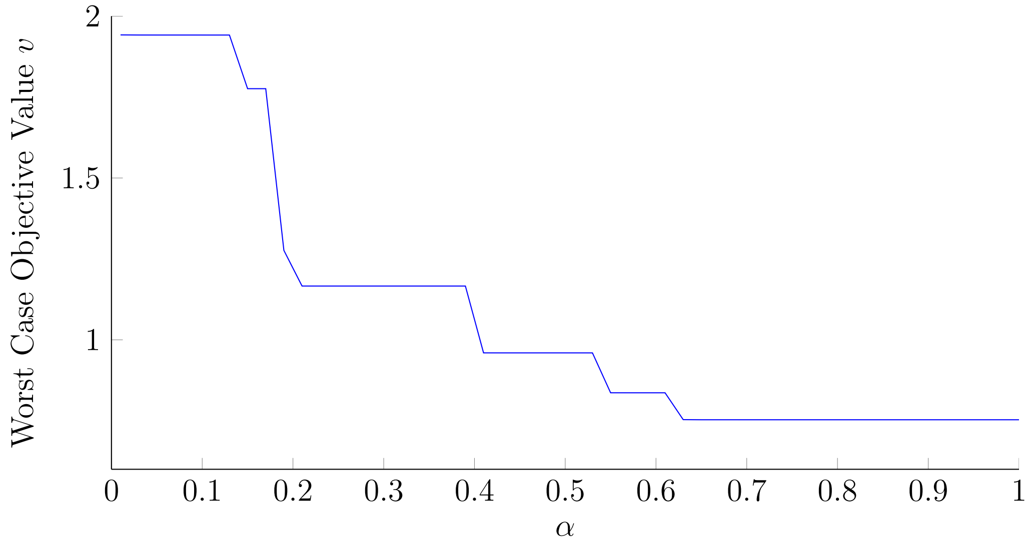

Figure 36: Initial BoU Pareto Front - Worst Case Objective Function

Table 26: Helical Spring Lexicographic Weighting

Uncertain Variables	**Lexicographic Weight**
G	10
A_1	1
B_1	1
C_1	1
C_2	1

6.5 BoU with Minimum Uncertainty - MinP Variation

As our final BoU iteration, we insert lower bounds in our uncertain parameters and again solve the BoU problem. Results are shown in Figures 42 - 44. We choose a lower bound of 0.05 for each of the uncertain parameters, excluding the wire and coil diameters

In this variation, we see all the desired uncertain parameters now show an uncertainty level of 0.05 even when $\alpha = 1$. This solution should also allow the full 5% uncertainty. Choosing $\alpha = 1$ we arrive at the solution shown in Table 29.

Unlike previous examples, we do not see a "guarantee" of constraint satisfac-

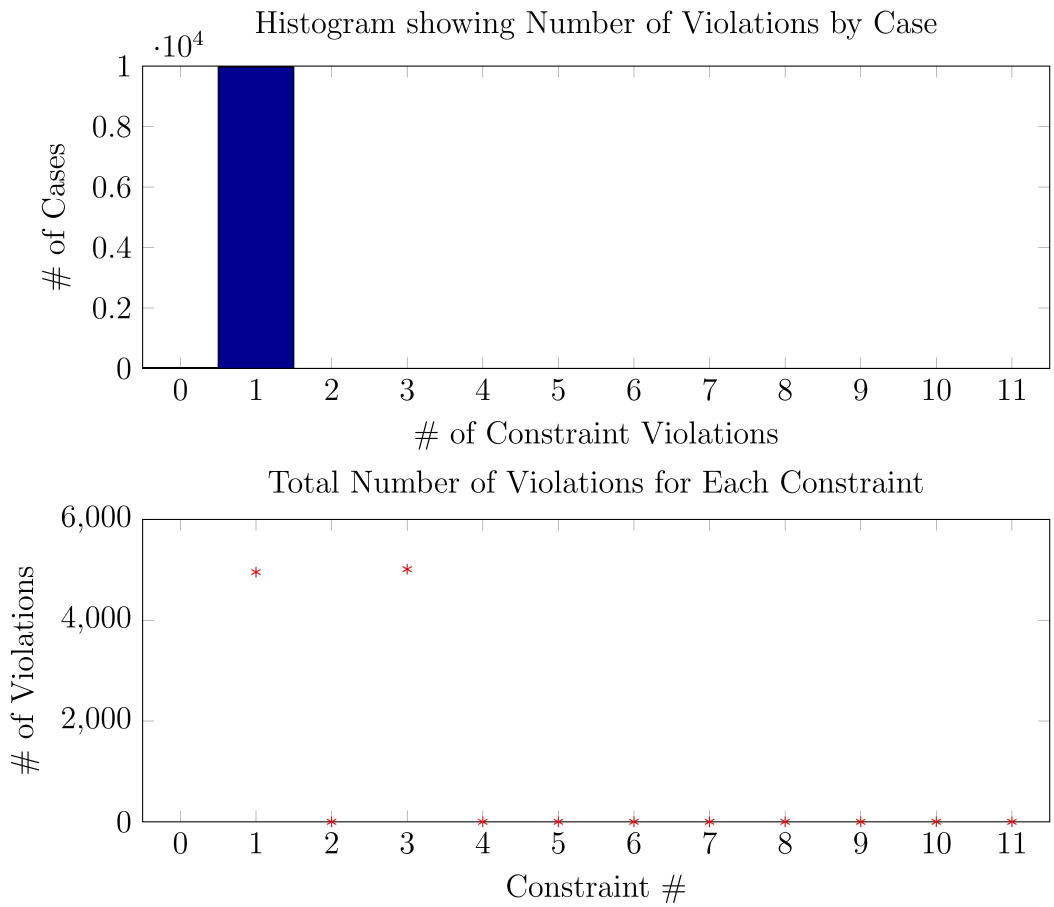

Figure 37: Summary of Results when Initial BoU Solution is Confronted with 5% Uncertainty, $\alpha = 0.5$

tion even though $s_i \geq 0.05$ for necessary parameters. The violations shown in G3 are a direct result of our first order approximation of the BoU method. Similar to results shown in Zhang [4], highly nonlinear constraints are merely changed to an approximation of their worst case value for a given level of uncertainty and consequently, when the solution is tested, we see some small number of violations. We summarize our results in Table 30.

Even with a specified minimum uncertainty in the previous solution, we still find our solution to show violations for the desired level of uncertainty. In our simple example, we could perform a trade study to see if the number of rejected springs

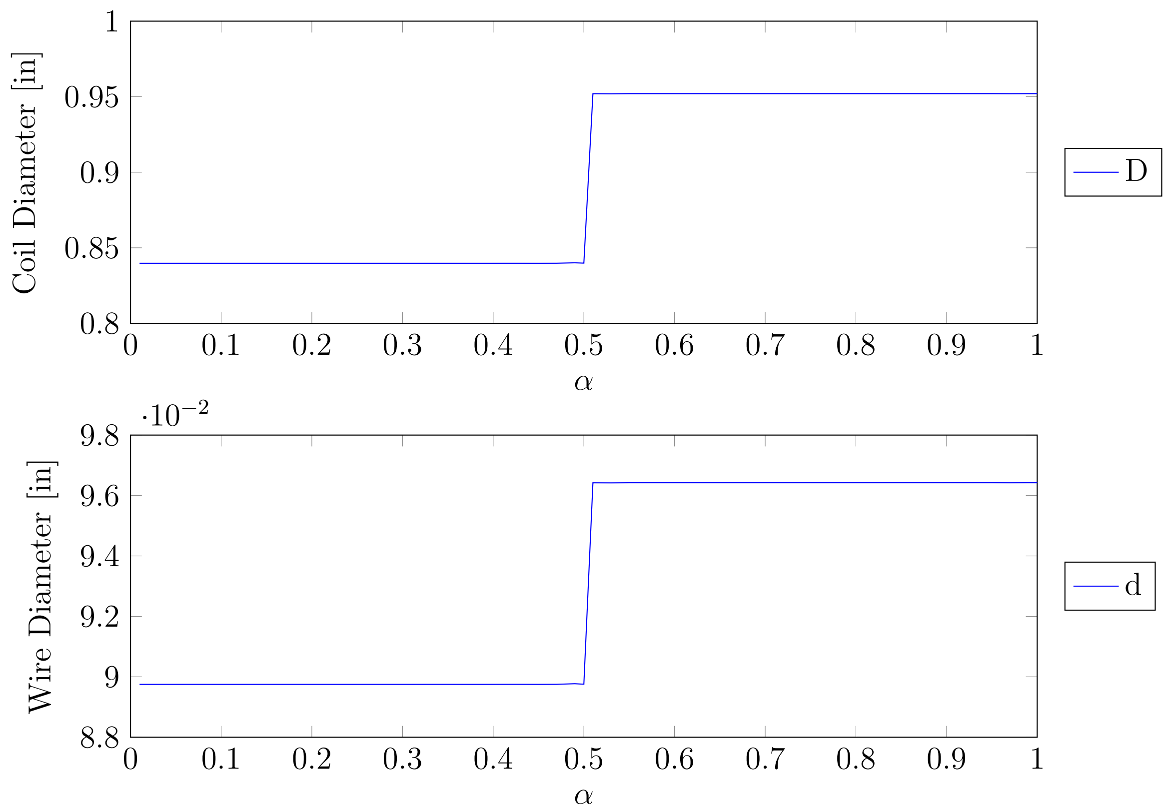

Figure 38: Lexicographic BoU Pareto Front - Control Variables

Table 27: Helical Spring BoU Lexicographic Solution

D	d	s_1	s_2	s_3	s_4	s_5
0.8398	0.0898	0.0937	$1.88e^{-6}$	3.06^{-7}	$1.02e^{-6}$	$3.55e^{-6}$

(in this case 252 of them) and the cost of testing them would outweigh the benefits of more expensive material with a lower level of material uncertainty. Similarly, we could revise our problem to consider a desired uncertainty level of 0.06 for the problem parameter, G, to protect against the approximation error. These results are shown in Table 31 and Figures 46 - 48.

Results are of course similar to our previous solution, but with slightly more uncertainty provided for G. Reviewing this new solution against 10,000 realizations, produces the desired results shown in Figure 49. A summary of these results is shown

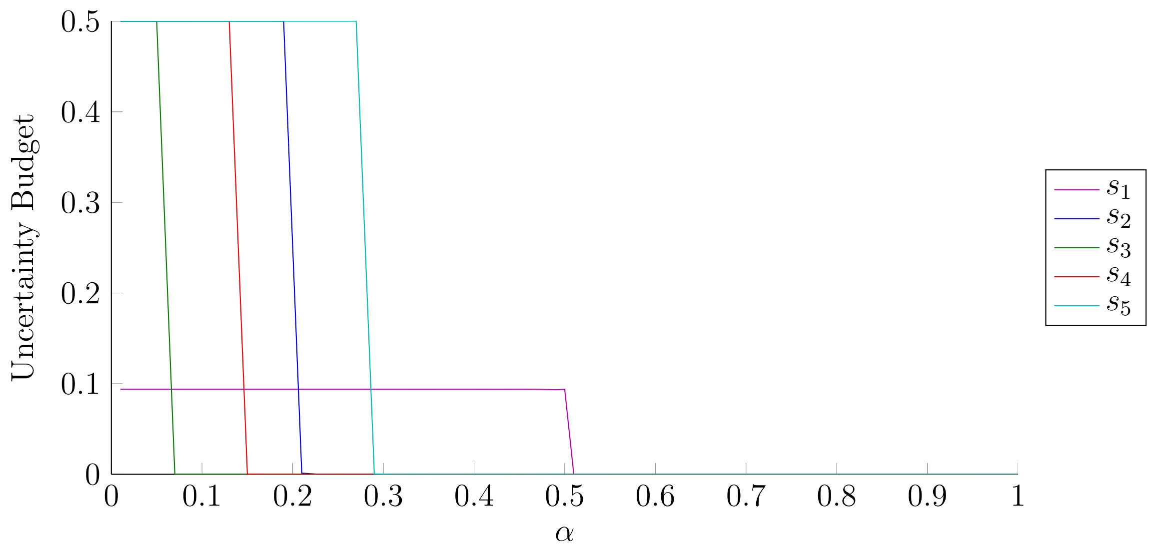

Figure 39: Lexicographic BoU Pareto Front - Uncertain Parameters 2 of 2

Table 28: BoU Performance Under 5% Uncertainty

	Number of Cases
Acceptable Cases	0
Cases with Constraint Violations	10000

in Table 32.

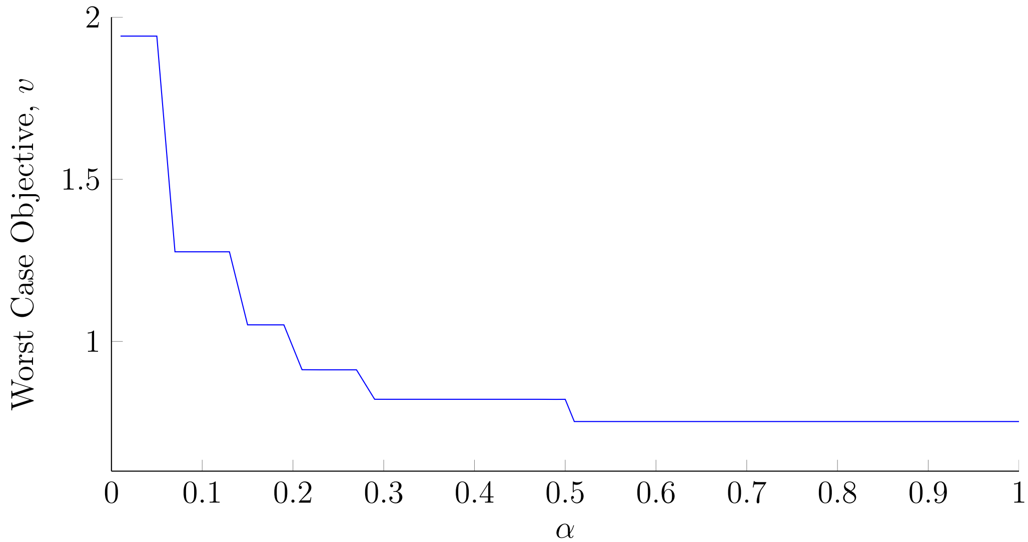

Figure 40: Lexicographic BoU Pareto Front - Worst Case Objective Function

Table 29: Helical Spring BoU Minimum Uncertainty Solution

D	d	s_1	s_2	s_3	s_4	s_5
0.8892	0.0927	0.05	0.05	0.05	0.05	0.05

Table 30: BoU Minimum Uncertainty Performance Under 5% Uncertainty

	Number of Cases
Acceptable Cases	9748
Cases with Constraint Violations	252

Table 31: Helical Spring BoU Minimum Uncertainty Solution 2

D	d	s_1	s_2	s_3	s_4	s_5
0.8775	0.0920	0.06	0.05	0.05	0.05	0.05

Table 32: BoU Minimum Uncertainty Performance 2 Under 5% Uncertainty

	Number of Cases
Acceptable Cases	10000
Cases with Constraint Violations	0

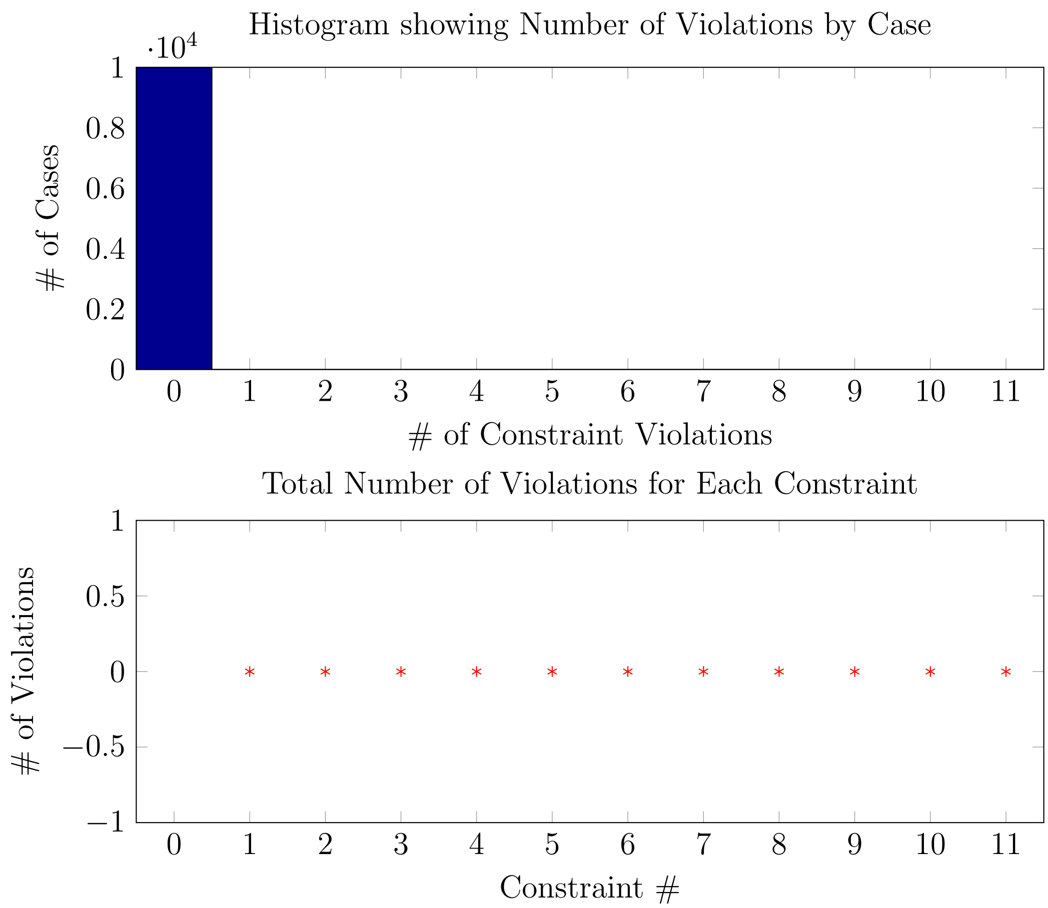

Figure 41: Summary of Results when Lexicographic BoU Solution is Confronted with 5% Uncertainty, $\alpha = 0.5$

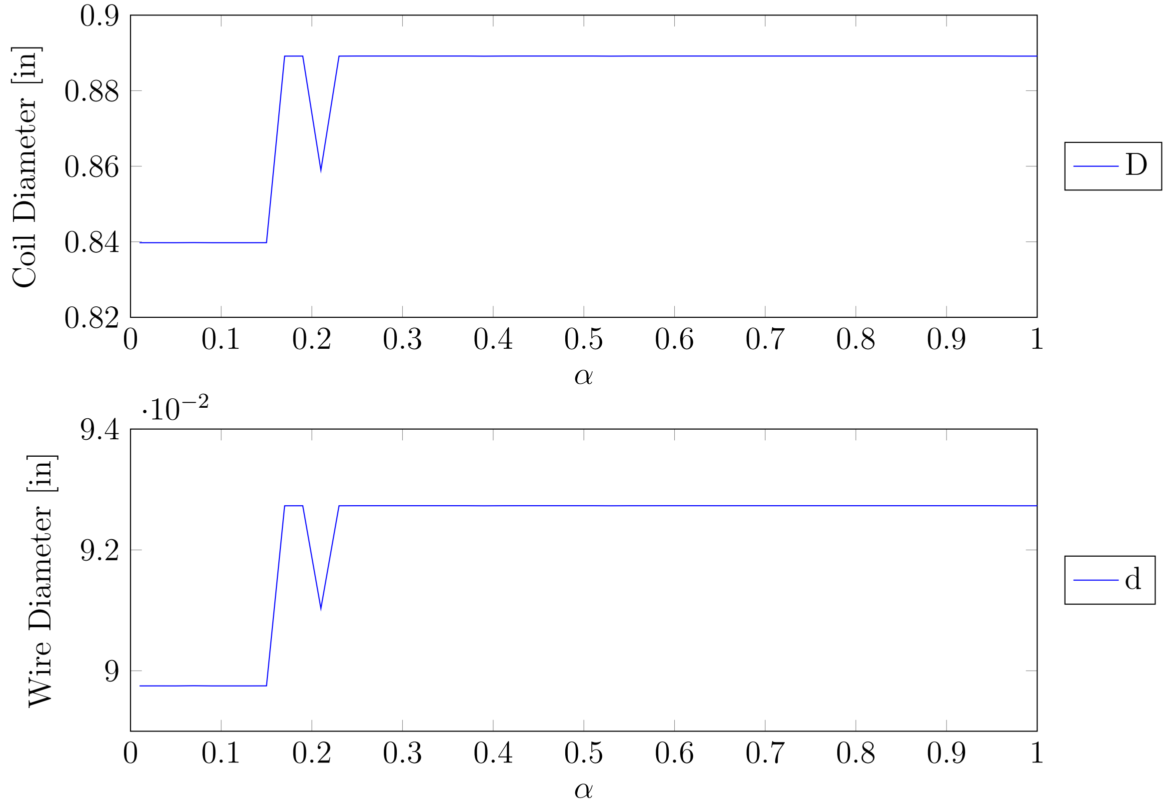

Figure 42: Minimum Uncertainty BoU Pareto Front - Control Variables

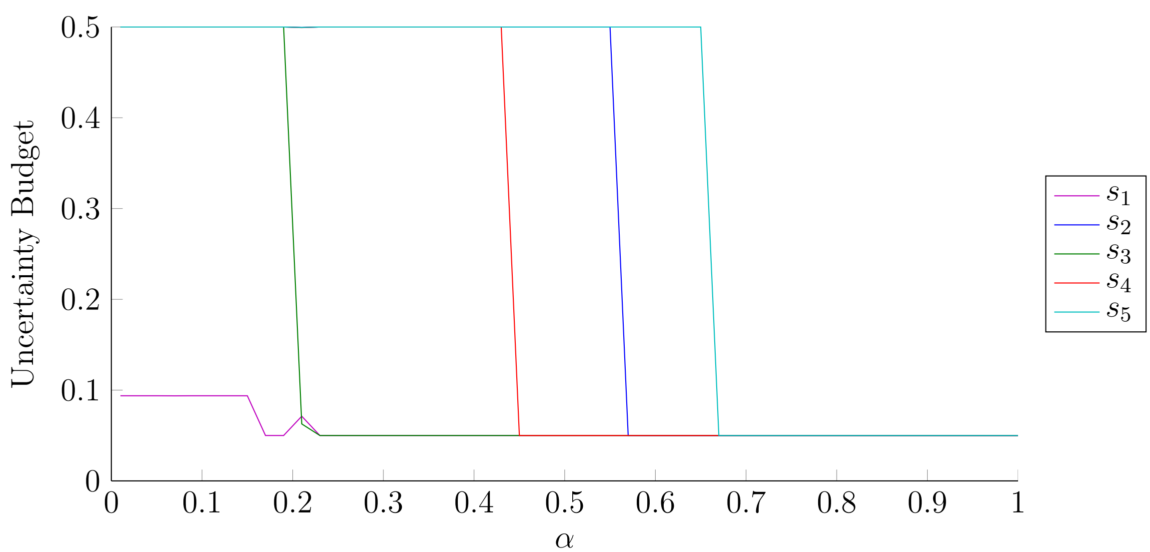

Figure 43: Minimum Uncertainty BoU Pareto Front - Uncertain Parameters 2 of 2

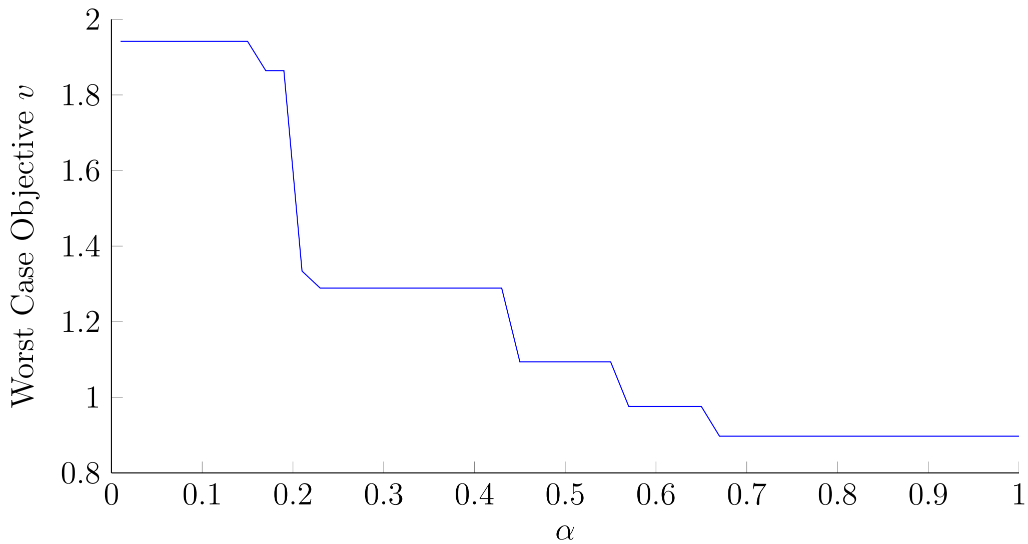

Figure 44: Minimum Uncertainty BoU Pareto Front - Worst Case Objective Function

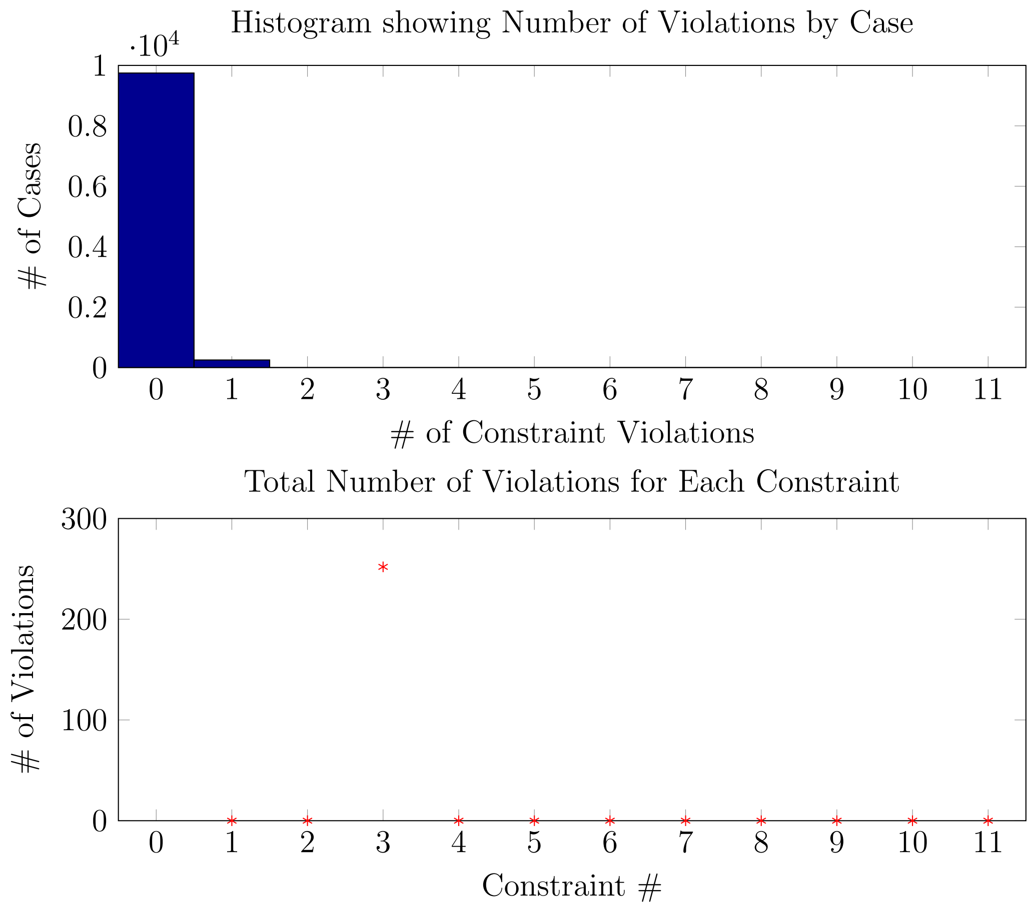

Figure 45: Summary of Results when Minimum Uncertainty BoU Solution is Confronted with 5% Uncertainty, $\alpha = 1.0$

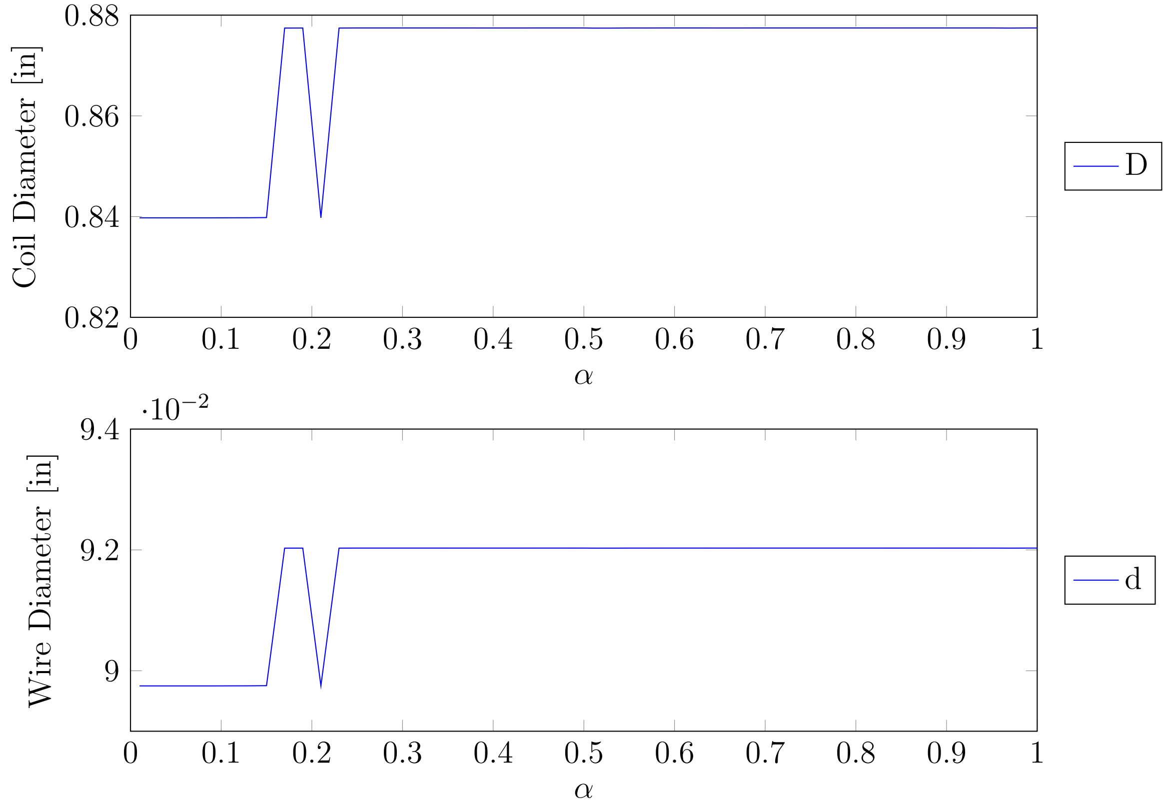

Figure 46: Minimum Uncertainty BoU Pareto Front - Control Variables

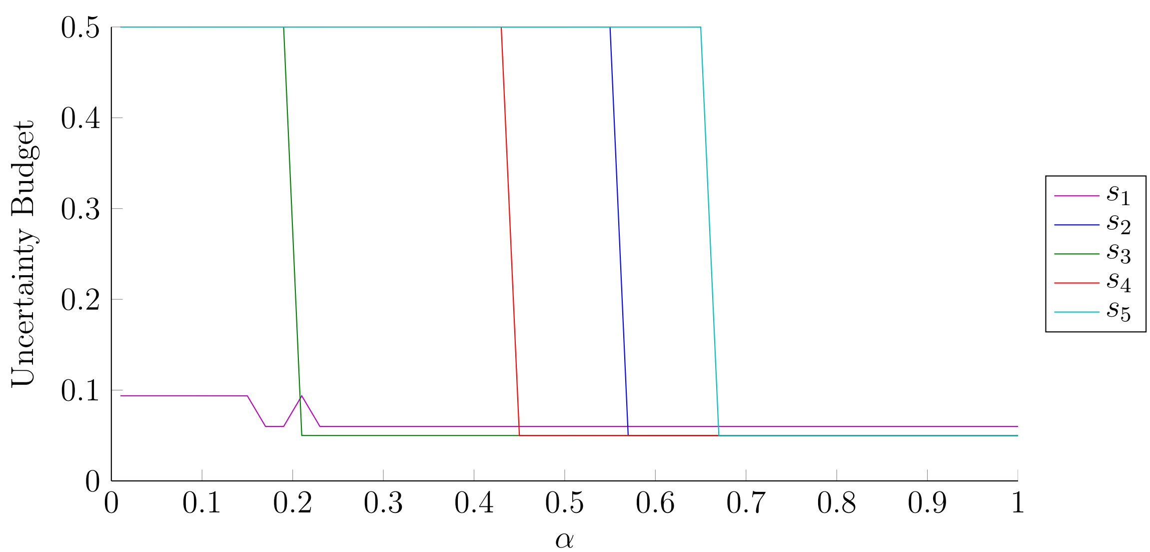

Figure 47: Minimum Uncertainty BoU Pareto Front - Uncertain Parameters 2 of 2

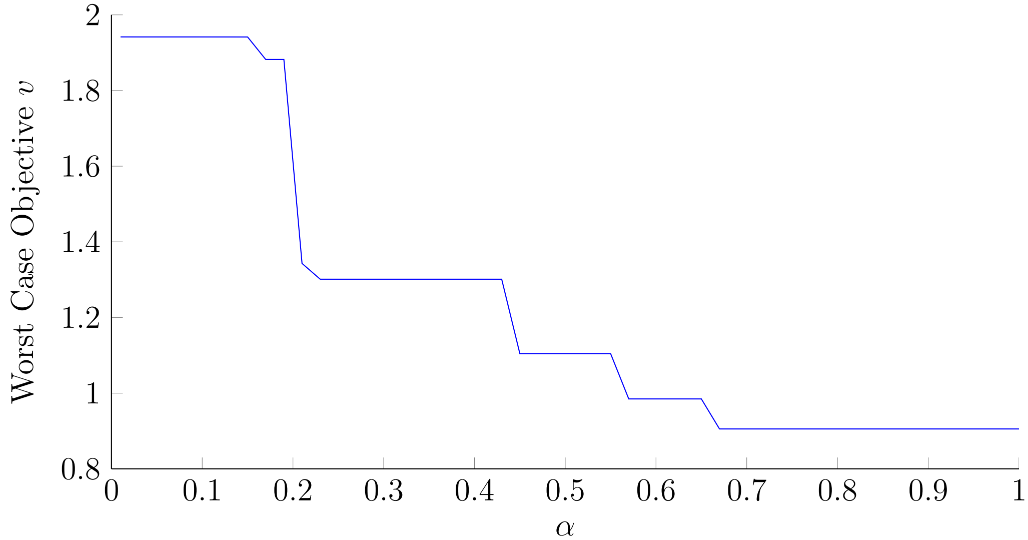

Figure 48: Minimum Uncertainty BoU Pareto Front - Worst Case Objective Function

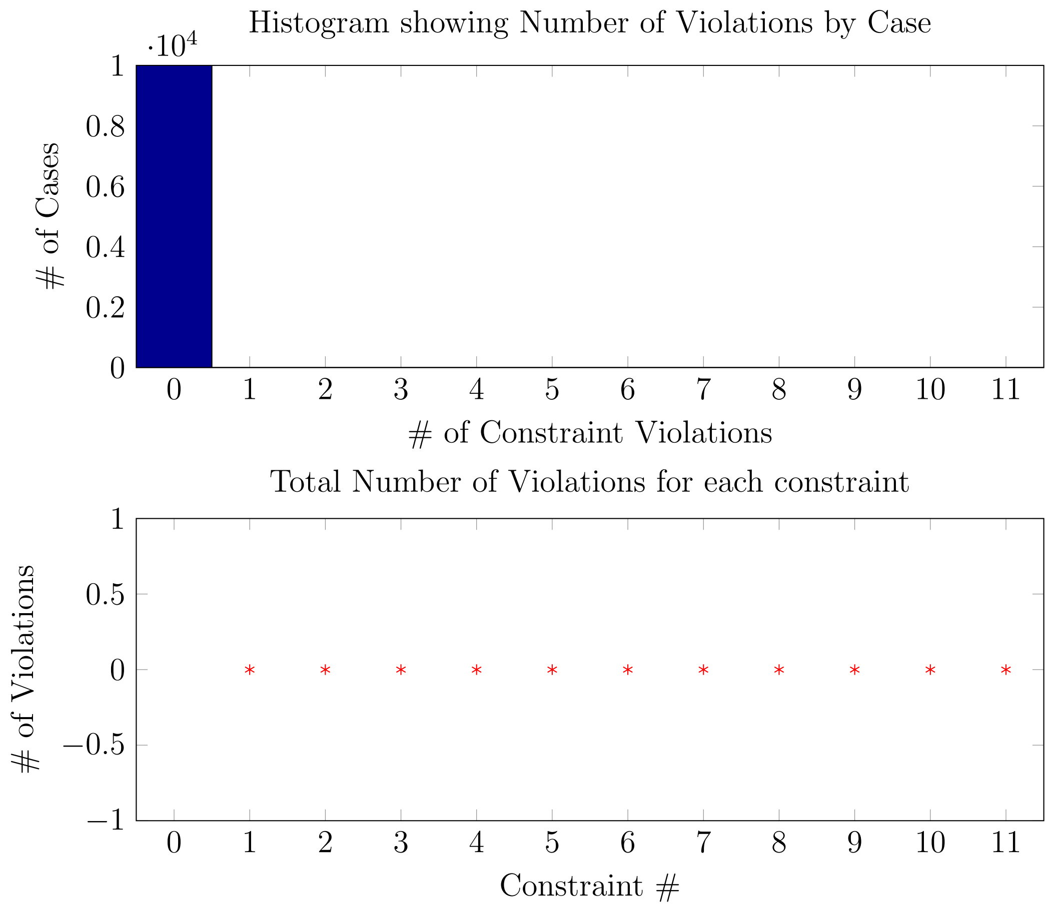

Figure 49: Summary of Results when Minimum Uncertainty BoU Solution is Confronted with 5% Uncertainty, $\alpha = 0.5$

7 Summary and Conclusions

The BoU method presented in this work not only provides a means to determine robust solutions to models with uncertain parameters, but is also capable of providing a budget of uncertainty for those same uncertain parameters. We have shown this method build on, and in some ways generalizes current RO research from authors such as Zhang, Ben-Tal, Azarm, and others in the sense that we determine robust constraint counterparts in a similar fashion and can impose constraints to generate similar solutions to previous work. However, the BoU method is a completely new concept in that it does not require knowledge of the uncertain parameters a-priori.

The BoU method, as presented here allows for several variations. The first type of variation is in the revised objective function. Within this work, we have focuses on the Weighted Sum objective formulation, but have presented the Weighted Product and Exponentially Weighted forms. The secondary variations allow for lexicographic tuning of uncertain parameters and additional constraints to further refine the solution set as needed.

Within our discussion, we have shown robust solutions for three sample problems which show moderately more expensive solutions (in terms of the original objective function value) with the ability to absorb specific uncertainty. We have shown the BoU method produces a worst case objective function value so as to quantify the true cost of the proposed uncertainty budget.

For our example problems, we have explored the range of solutions along a Pareto front where we investigate solutions ranging from the original optimum solution, to a highly robust formulation. In each case, we clearly show the impact on uncertainty budget, the tradeoff between uncertain parameters, and the ultimate solution.

In the first example problem we showed the BoU applied to a simple 2 dimensional geometric problem. Through this example, we showed the method's ability to determine an initial solution, refine that solution lexicographically to overcome problem sensitivity, and finally to fix the desired uncertainty budget and determine an effective design.

Next, in our Gearbox example, we showed the BoU method applied to a problem with many constraints and a large number of uncertain variable. Through this example we showed the BoU method is capable of managing a large number of uncertain variables and can generate fully immunized solutions.

Finally, in the Spring example, we apply the BoU method to a problem with highly nonlinear constraints and show the limitations of our first order approximation. Even in the presence of this limitation, we show the ability to compensate and ultimately generate a robust solution, but in so doing, we highlight the current limitations of our approach.

What is perhaps not shown clearly to this point is the intended application of the BoU method. This method is intended to be used very early in the design process, when uncertain parameters are poorly characterized. When used at this point, the model provides insight to the potential uncertainty in each parameter and can help guide engineering development work, procurement decisions, and machine tolerance requirements. As more information becomes available, the designer continues to refine the BoU solution, imposing additional uncertain parameter constraints and adjusting the weighting factors to refine the uncertainty budget estimate. Ultimately, near the completion of the design, the resulting solution provides not only a worst case objective function value, but also an allowed uncertainty for each selected parameter. This uncertainty budget can then be used as part of a comprehensive quality and inspection activity throughout manufacture and certification of the ultimate design.

Similarly, if this technique were used for non-physical systems, it would be possible to determine the accuracy needed for specific parameters. Again, this method provides an inherent approach to uncertainty requiring little insight from the designer.

7.1 Forward Work

Currently, the BoU method seems to show promise as a comprehensive approach to Robust Optimization, however it is currently limited in its first order approximation. Future work will focus on improving the development of robust constraint counterparts for concave constraints. In addition, the method has been so far applied to physical and geometric problems, but has not been applied to fields such as electrical engineering or portfolio management. These are potential areas of application which may generate additional capability or insight.

Finally, future work may include the development of a software tool to allow automatic BoU application to problems formulated as standard optimization problems. This tool could facilitate the inclusion of additional example problems or perhaps facilitate the use of the BoU method by other researchers.

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