ANALYSIS OF AN OPTIMAL REGULATOR WITH A JUMP PARAMETER

A Thesis

Presented to the Faculty of the Graduate School of
The University of Houston in Partial Fulfillment
of the Requirements
for the Degree of

Master of Science in Electrical Engineering

By
Harry Andrew Herwig
Houston, Texas
May, 1970

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ABSTRACT

This thesis examines a first order linear sensitivity and state regulator to illustrate system performance when changes in the plant parameter take place suddenly. It is shown that using a closed loop sensitivity and state feedback design approach to reduce trajectory deviations can result in significant savings in total control effort when compared to the state feedback design approach which neglects sensitivity.

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I. INTRODUCTION

The objective of this thesis is to investigate the performance of a first-order linear sensitivity and state regulator when changes in plant parameters take place suddenly. A comprehensive introduction to the class of systems where nominal values for the parameters are known, but their exact values are unknown and unmeasurable is given by Dougherty, Lee, and DeRusso [1].

When the optimal controller is designed using standard time-domain optimization techniques [2] for a set of nominal plant parameters, and these parameters are subject to uncertainties, the control in most cases will not be optimal. Present approaches to introduce parameter sensitivity constraints consist of reformulating the problem by including sensitivity functions in the cost function to be minimized along with functions of state and control variables. Several papers [1] [3] [4] [5] have been published demonstrating results of this approach.

Higginbotham in his paper [6] has pointed out that it is not possible to obtain a closed form solution without neglecting or omitting certain sensitivity terms. As an alternate approach to synthesizing an optimal controller, Higginbotham has proposed the introduction of a sensitivity forcing function term in the cost function to replace those terms which are not computationally realizable [7]. The resulting system is closed loop and attempts to compensate for uncertainties in both state and control coefficient matrices. The state sensitivity problem is formulated so that independent sensitivity variable control vectors are determined along with finding the conventional state control vector u(t) that minimizes a specified performance functional. This technique results in closed-loop state and sensitivity control laws. Some work [7] [8] has already been done to illustrate the performance of the system using this approach. This work has only considered the case when the uncertain parameter, although different from the assumed value, was constant over the time of system operation.

This thesis presents an analysis of the closed loop sensitivity and state feedback approach for the case when a plant parameter is subject to an abrupt change in value during system operation. Three first-order examples are analyzed to show trends resulting from the use of the combined sensitivity and state feedback approach. The examples are as follows:

Example 1 - Several abrupt changes occur in the plant parameter during the time of highest system sensitivity.

Example 2 - One abrupt change occurs in the plant parameter at peak system sensitivity.

Example 3 - The plant parameter is exactly known and is constant during system operation.

The combined sensitivity and state feedback approach is compared with the state feedback approach which neglects sensitivity to illustrate system performance which results when system sensitivity is controlled. In addition, the analysis determines tradeoff's between control effort required and trajectory error reduction for each of the examples.

II. MATHEMATICAL DEVELOPMENT OF PROBLEM

The problem is to find optimal state and sensitivity control laws which will keep the state of a given plant near the origin during real-time operation, while at the same time reduce trajectory deviations resulting from abrupt changes in a plant parameter. This objective is to be accomplished with minimum control effort.

The plant to be controlled is a deterministic first order linear regulator, represented by,

$$\dot{x}(t) = ax(t) + bu(t); \; x(t_o) = x_o \tag{1}$$

where the parameters "a" and "b" are subject to uncertainty.

The sensitivity dynamics are determined from the sensitivity function [9]. There will be a sensitivity term for each uncertain parameter. Therefore the two sensitivity terms will be

$$\text{for an uncertain "a"} \quad v_1 = \frac{\partial x}{\partial a} \tag{2}$$

$$\text{for an uncertain "b"} \quad v_2 = \frac{\partial x}{\partial b} \tag{3}$$

Differentiating Equations (2) and (3) with respect to time and using Equation (1) the sensitivity equations become

$$\dot{v}_1 = a_o v_1 + x + b_o \frac{\partial u}{\partial a} \; ; \; v_1(t_o) = 0 \qquad (4)$$

$$\dot{v}_2 = a_o v_1 + u + b_o \frac{\partial u}{\partial b} \; ; \; v_2(t_o) = 0 \qquad (5)$$

where a_o and b_o are the nominal values of a and b respectively. Anticipating a control which will be functions of both state and sensitivity, expansions of the terms $b_o \frac{\partial u}{\partial a_o}$ and $b_o \frac{\partial u}{\partial b_o}$ yield

$$b_o \frac{\partial u}{\partial a} = b_o \left[\frac{\partial u}{\partial x} v_1 + \frac{\partial u}{\partial v_1} \cdot \frac{\partial v_1}{\partial a} + \frac{\partial u}{\partial v_2} \cdot \frac{\partial v_2}{\partial a}\right] \quad (6)$$

$$b_o \frac{\partial u}{\partial b} = b_o \left[\frac{\partial u}{\partial x} v_2 + \frac{\partial u}{\partial v_2} \cdot \frac{\partial v_2}{\partial b} + \frac{\partial u}{\partial v_1} \cdot \frac{\partial v_1}{\partial b}\right] \quad (7)$$

The terms involving the partial derivatives of sensitivity with respect to the uncertain parameters which appear in Equations (6) and (7) have been neglected by some authors [3] [10]. The approach suggested by Higginbotham [7] is to treat these computationally unrealizable terms as sensitivity forcing functions and include them in the cost function to be minimized along with state and control functions.

By defining

$$m_1(t) \triangleq \frac{\partial u}{\partial v_1} \cdot \frac{\partial v_1}{\partial a} + \frac{\partial u}{\partial v_2} \cdot \frac{\partial v_2}{\partial a} \qquad (8)$$

$$m_2(t) \stackrel{\Delta}{=} \frac{\partial u}{\partial v_2} \cdot \frac{\partial v_2}{\partial b} + \frac{\partial u}{\partial v_1} \cdot \frac{\partial v_1}{\partial b} \tag{9}$$

and combining Equations (6), (7), (8), and (9) with Equations (4) and (5) the sensitivity differential equations become

$$\dot{\sigma}_1(t) = [a_o + b_o \frac{\partial u}{\partial x}]\, \sigma_1(t) + x(t) + b_o m_1(t) \ ; \ \sigma_1(t_o) = 0 \tag{10}$$

$$\dot{\sigma}_2(t) = [a_o + b_o \frac{\partial u}{\partial x}]\, \sigma_2(t) + u(t) + b_o m_2(t) \ ; \ \sigma_2(t_o) = 0 \tag{11}$$

An augmented system z(t) can be written by letting

$$z(t) = [x^T(t), \sigma_1{}^T(t), \sigma_2{}^T(t)]^T$$

where T denotes transpose. Equations (1), (10) and (11) can then be written as

$$\begin{bmatrix} \dot{x} \\ \dot{\sigma}_1 \\ \dot{\sigma}_2 \end{bmatrix} = \begin{bmatrix} a & 0 & 0 \\ 1 & [a_o + b_o \frac{\partial u}{\partial x}] & 0 \\ 0 & 0 & [a_o + b_o \frac{\partial u}{\partial x}] \end{bmatrix} \begin{bmatrix} x \\ \sigma_1 \\ \sigma_2 \end{bmatrix} + \begin{bmatrix} b & 0 & 0 \\ 0 & b_o & 0 \\ 1 & 0 & b_o \end{bmatrix} \begin{bmatrix} u \\ m_1 \\ m_2 \end{bmatrix} \tag{12}$$

or

$$\dot{z}\,(t) = A_1 z(t) + B_1 u_1(t) \ ; \ \ z(t_o) = z_o \tag{13}$$

where

$$u_1(t) = [u^T(t), m_1{}^T(t), m_2{}^T(t)]^T \tag{14}$$

A quadratic functional is chosen as the performance index. Thus

$$J = \frac{1}{2}\int_0^{t_f} [qx^2(t) + s_1\sigma_1^{\,2}(t) + s_2\sigma_2^{\,2}(t) + r_1u^2(t) + r_2m_1^{\,2}(t) + r_3m_2^{\,2}(t)]\ dt \quad (15)$$

or in terms of the augmented system z(t)

$$J = \frac{1}{2}\int_0^{t_f} [z^TQz + u_1^{\,T}R_1u_1]\ dt \qquad (16)$$

where the terminal time t_f is fixed and specified. The problem is to find the control $u_1(t)$ that minimizes the performance index (16) and thus determine the optimal state control u(t) and the sensitivity controls $m_1(t)$ and $m_2(t)$.

III. PROBLEM SOLUTION

Using the technique outlined in Athens and Falb [2], Chapter 9, the solution proceeds as follows.

The Hamiltonian H for the system (13) and the cost J of Equation (16) is

$$H = \frac{1}{2}\left[z^T(t)Qz(t) + u_1^T(t)R_1u_1(t)\right] + \lambda^T(t)\left[A_1z(t) + B_1u_1(t)\right] \quad (17)$$

The costate vector $\lambda(t)$ is the solution of the vector differential equation

$$\dot{\lambda}(t) = -\frac{\partial H}{\partial z(t)} \quad (18)$$

which reduces to

$$\dot{\lambda}(t) = -Qz(t) - A_1^T\lambda(t) \quad (19)$$

Along the optimal trajectory, we must have

$$\frac{\partial H}{\partial u_1(t)} = 0 \quad (20)$$

which implies that

$$\frac{\partial H}{\partial u_1(t)} = R_1u_1(t) + B_1^T\lambda(t) = 0 \quad (21)$$

or

$$u_1(t) = -R_1^{-1} B_1^T \lambda(t) \tag{22}$$

Assuming $\lambda(t)$ in the form

$$\lambda(t) = K(t)\ z(t) \tag{23}$$

Then

$$u_1(t) = -R_1^{-1} B_1^T K(t) z(t) \tag{24}$$

Athens and Falb [2] show that K(t) must satisfy the matrix differential equation

$$\dot{K}(t) = -K(t)A_1 - A_1^T K(t) + K(t)B_1R_1^{-1} B_1^T K(t) - Q \tag{25}$$

where A_1 Contains $k_{11}(t)$. Equation 25 has the boundary conditions of $K(t_f) = 0$.

The state of the optimal system is then the solution of the linear differential equation

$$\dot{z}(t) = [A_1 - B_1R_1^{-1} B_1^T K(t)]\ z(t)\ ;\ z(t_0) = z_0 \tag{26}$$

To simulate the system dynamics, it will be necessary to expand the vector matrix equations into their components.

Expansion of the optimal control vector $u_1(t)$ yields

$$u_1(t) = -R_1^{-1} B_1^T K(t) z(t)$$

$$\begin{bmatrix} u(t) \\ m_1(t) \\ m_2(t) \end{bmatrix} = \begin{bmatrix} -\frac{1}{r_1} & 0 & 0 \\ 0 & -\frac{1}{r_2} & 0 \\ 0 & 0 & -\frac{1}{r_3} \end{bmatrix} \begin{bmatrix} b_o & 0 & 1 \\ 0 & b_o & 0 \\ 0 & 0 & b_o \end{bmatrix} \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix} \begin{bmatrix} X \\ \sigma_1 \\ \sigma_2 \end{bmatrix} \quad (27)$$

$$u(t) = -\frac{1}{r_1} [b_o(k_{11}x + k_{12}\sigma_1 + k_{13}\sigma_2) + k_{31}x + k_{32}\sigma_1 + k_{33}\sigma_2] \quad (28)$$

$$m_1(t) = -\frac{b_o}{r_2} [k_{12}x + k_{22}\sigma_1 + k_{23}\sigma_2] \quad (29)$$

$$m_2(t) = -\frac{b_o}{r_3} [k_{13}x + k_{23}\sigma_1 + k_{33}\sigma_2] \quad (30)$$

Next the $\frac{\partial u}{\partial x}$ term may be evaluated

$$\frac{\partial u}{\partial x} = -\frac{1}{r_1} [b_o k_{11} + k_{31}] \quad (31)$$

The expanded state equations then become

$$\dot{x}(t) = [a - \frac{bb_o}{r_1} k_{11} - \frac{b}{r_1} k_{31}]\, x(t) + [-\frac{bb_o}{r_1} k_{12} - \frac{b}{r_1} k_{23}]\sigma_1(t)$$

$$+ [-\frac{bb_o}{r_1} k_{13} - \frac{b}{r_1} k_{33}]\sigma_2(t) \; ; \; x(t_o) = x_o \quad (32)$$

$$\dot{\sigma}_1(t) = [a_o - \frac{b_o^2}{r_1} k_{11} - \frac{b_o}{r_1} k_{13} - \frac{b_o^2}{r_2} k_{22}]\sigma_1(t) + [1 - \frac{b_o^2}{r_2} k_{12}]x(t)$$

$$- \frac{b_o^2}{r_2} k_{23}\sigma_2 \ ; \quad \sigma_1(t_o) = 0 \tag{33}$$

$$\dot{\sigma}_2(t) = [a_o - \frac{b_o^2}{r_1} k_{11} - 2\frac{b_o}{r_1} k_{13} - \frac{b_o^2}{r_3} k_{33} - \frac{1}{r_1} k_{33}]\sigma_2(t)$$

$$+[-\frac{b_o^2}{r_3} k_{23} - \frac{b_o}{r_1} k_{12} - \frac{1}{r_1} k_{23}]\sigma_1(t)$$

$$+[-\frac{b_o^2}{r_3} K_{13} - \frac{b_o}{r_1} k_{11} - \frac{1}{r_1} k_{13}]x(t) \ ; \quad \sigma_2(t_o) = 0 \tag{34}$$

Expansion of the gain matrix $\dot{K}(t)$ yields

$$\begin{bmatrix} \dot{k}_{11} & \dot{k}_{12} & \dot{k}_{13} \\ \dot{k}_{21} & \dot{k}_{22} & \dot{k}_{23} \\ \dot{k}_{31} & \dot{k}_{32} & \dot{k}_{33} \end{bmatrix} = -\begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix}\begin{bmatrix} a_o & 0 & 0 \\ 1 & [a_o - \frac{b_o^2}{r_1}k_{11} - \frac{b_o}{r_1}k_{13}] & 0 \\ 0 & 0 & [a_o - \frac{b_o^2}{r_1}k_{11} - \frac{b_o}{r_1}k_{13}] \end{bmatrix}$$

$$- \begin{bmatrix} a_o & 1 & 0 \\ 0 & [a_o - \frac{b_o^2}{r_1}k_{11} - \frac{b_o}{r_1}k_{13}] & 0 \\ 0 & 0 & [a_o - \frac{b_o^2}{r_1}k_{11} - \frac{b_o}{r_1}k_{13}] \end{bmatrix}\begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix}$$

$$
+\begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix} \begin{bmatrix} b_o & 0 & 0 \\ 0 & b_o & 0 \\ 1 & 0 & b_o \end{bmatrix} \begin{bmatrix} \frac{1}{r_1} & 0 & 0 \\ 0 & \frac{1}{r_2} & 0 \\ 0 & 0 & \frac{1}{r_3} \end{bmatrix} \begin{bmatrix} b_o & 0 & 1 \\ 0 & b_o & 0 \\ 0 & 0 & b_o \end{bmatrix} \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix}
$$

$$
- \begin{bmatrix} q & 0 & 0 \\ 0 & s_1 & 0 \\ 0 & 0 & s_2 \end{bmatrix}
$$

$$
\dot{k}_{11}(t) = -2a_o k_{11} - 2k_{12} + \frac{b_o^2}{r_1} k_{11}^2 + 2\,\frac{b_o}{r_1} k_{11}k_{13} + \frac{1}{r_1} k_{13}^2 + \frac{b_o^2}{r_2} k_{12}^2
$$

$$
+ \frac{b_o^2}{r_3} k_{13}^2 - q \tag{35}
$$

$$
\dot{k}_{12}(t) = -\,2a_o k_{12} + 2\,\frac{b_o^2}{r_1} k_{11}k_{12} + 2\,\frac{b_o}{r_1} k_{12}k_{13} - k_{22} + \frac{b_o}{r_1} k_{11}k_{23}
$$

$$
+ \frac{1}{r_1} k_{13}k_{33} + \frac{b_o^2}{r_2} k_{12}k_{22} + \frac{b_o^2}{r_3} k_{13}k_{23} \tag{36}
$$

$$
\dot{k}_{13}(t) = -2a_o k_{13} + 2\,\frac{b_o^2}{r_1} k_{11}k_{13} + 2\,\frac{b_o}{r_1} k_{13}^2 - k_{23} + \frac{b_o}{r_1} k_{11}\,k_{23}
$$

$$
+ \frac{1}{r_1} k_{13}k_{33} + \frac{b_o^2}{r_2} k_{12}\,k_{23} + \frac{b_o^2}{r_3} k_{13}k_{33} \tag{37}
$$

$$\dot{k}_{22}(t) = -2a_o k_{22} + 2\frac{b_o^2}{r_1}k_{11}k_{22} + 2\frac{b_o}{r_1}k_{13}k_{22} + \frac{b_o^2}{r_1}k_{12} + 2\frac{b_o}{r_1}k_{12}k_{23}$$

$$+ \frac{1}{r_1}k_{23}^2 + \frac{b_o^2}{r_2}k_{22}^2 + \frac{b_o^2}{r_3}k_{23}^2 - s_1 \tag{38}$$

$$\dot{k}_{23}(t) = -2a_o k_{23} + 2\frac{b_o^2}{r_1}k_{11}k_{23} + 3\frac{b_o}{r_1}k_{13}k_{23} + \frac{b_o^2}{r_1}k_{12}k_{13} + \frac{b_o}{r_1}k_{12}k_{33}$$

$$+ \frac{1}{r_1}k_{23}k_{33} + \frac{b_o^2}{r_2}k_{22}k_{23} + \frac{b_o^2}{r_3}k_{23}k_{33} \tag{39}$$

$$\dot{k}_{33}(t) = -2a_o k_{33} + 2\frac{b_o^2}{r_1}k_{11}k_{33} + 4\frac{b_o}{r_1}k_{13}k_{33} + \frac{b_o^2}{r_1}k_{13}^2 + \frac{1}{r_1}k_{33}^2$$

$$+ \frac{b_o^2}{r_2}k_{23}^2 + \frac{b_o^2}{r_3}k_{33} - s_2 \tag{40}$$

$$\dot{k}_{12}(t) = \dot{k}_{21}(t) \tag{41}$$

$$\dot{k}_{13}(t) = \dot{k}_{31}(t) \tag{42}$$

$$\dot{k}_{23}(t) = \dot{k}_{32}(t) \tag{43}$$

The computation of the gain matrix K(t) is reviewed in Appendix A. Figure 1 shows the structure of the closed loop sensitivity and state regulating system. The "thickened" part outlines the solution which is obtained from the conventional approach.

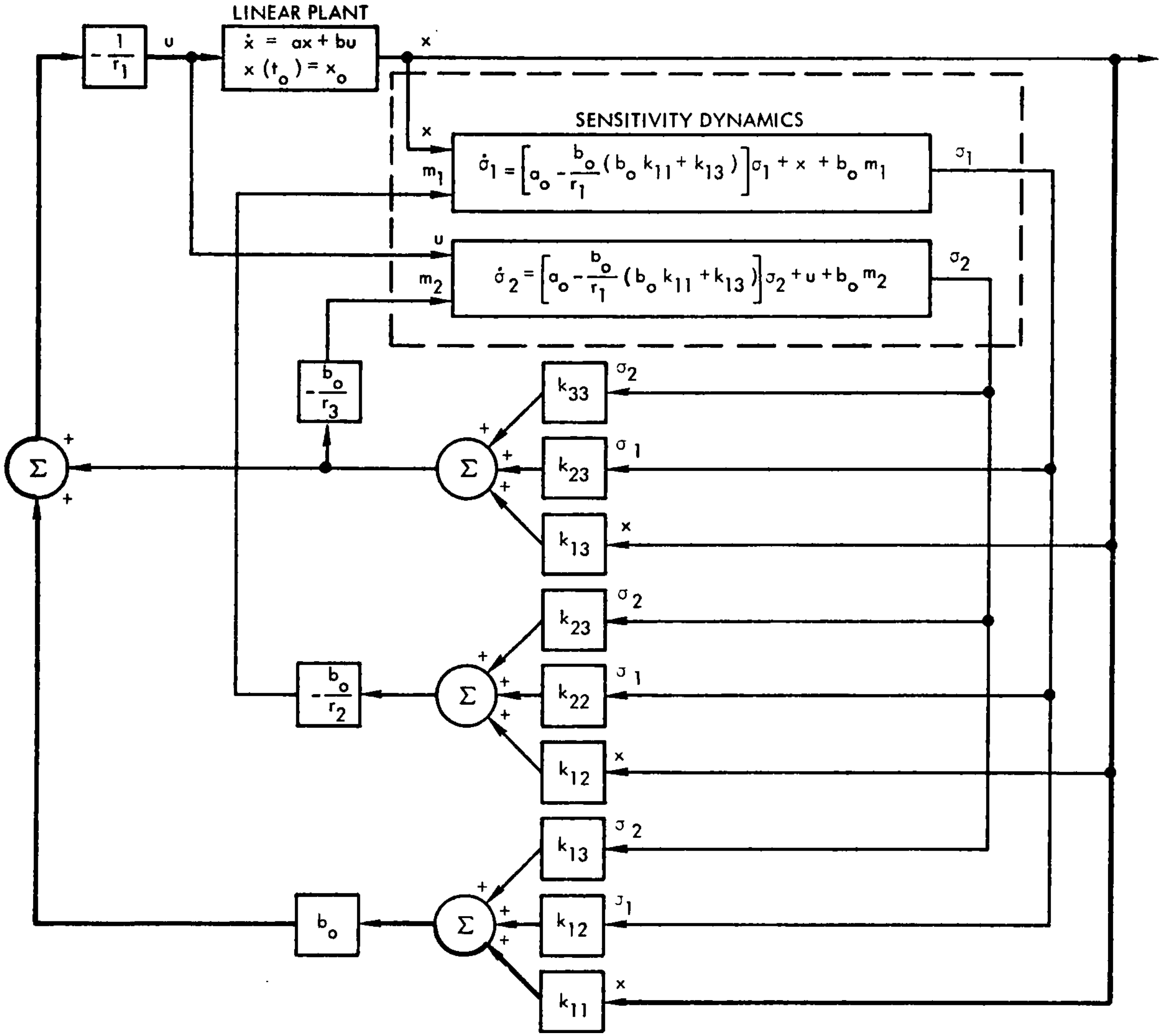

Figure 1. Structure of the Optimal Linear Sensitivity and State Controlled Regulator

IV. SYSTEM PERFORMANCE ANALYSIS

This section presents the system performance of the closed loop sensitivity and state feedback design for the reduction of trajectory deviations resulting from a suddenly changing plant parameter. The system studied in this thesis has the state coefficient 'a' subject to abrupt changes which occur when the plant sensitivity is the greatest. The combined sensitivity and state feedback approach is compared with the approach of using only state feedback to determine tradeoffs between control energy requirements and trajectory deviation reduction. For the state feedback approach, trajectory error was investigated with weighting on state in the cost functional as a parameter. In the combined approach, the weightings on sensitivity and the sensitivity forcing function in the cost functional are the parameters which are varied.

The analysis was performed with the following numerical values. The plant equation is

$$\dot{x}(t) = ax(t) + u(t) \quad ; \quad x(t_0) = 1.0$$

with the assumed value $a_0 = -2.0$ and $b = b_0 = 1.0$

The cost functional for the state feedback approach is

$$J = \frac{1}{2} \int_0^4 [qx^2(t) + u^2(t)] \, dt$$

with the weighting on state, q varying from 0.2 to 8.0.

The cost functional for the combined sensitivity and state feedback approach is

$$J = \frac{1}{2} \int_0^4 [0.2qx^2(t) + u^2(t) + s_1\sigma_1{}^2(t) + r_2m_1{}^2(t)]dt$$

with the weighting on sensitivity s_1 varying from 0.0 to 100.0 and the weighting on the sensitivity forcing function, r_2 varying from 0.2 to 500.0. The value of s_2 and r_3 are zero because the control coefficient 'b' for this analysis is assumed to be known and constant. Three examples have been analyzed with different parameter time histories. For Example 1 the state coefficient 'a' has several abrupt changes during the sytem operation.

The state response of the system for Example 1 using only state feedback is shown in Figure 2. Also given in Figure 2 is the time history of the plant parameter 'a' for example 1. As the weighting on state q is increased the trajectory deviation from a perfectly model system with constant parameters is reduced. The state response of the combined sensitivity and state feedback approach for Example 1 is given in Figure 3. When the sensitivity weighting s_1 is zero, sensitivity is not considered and the state response for both approaches with q = 0.2 are identical. Using a weighting on the sensitivity forcing function m(t) of 10, and increasing the sensitivity weighting s_1 the trajectory error is decreased. The responses shown in Figure 3 were chosen to demonstrate that it is possible to have comparable trajectories between the two

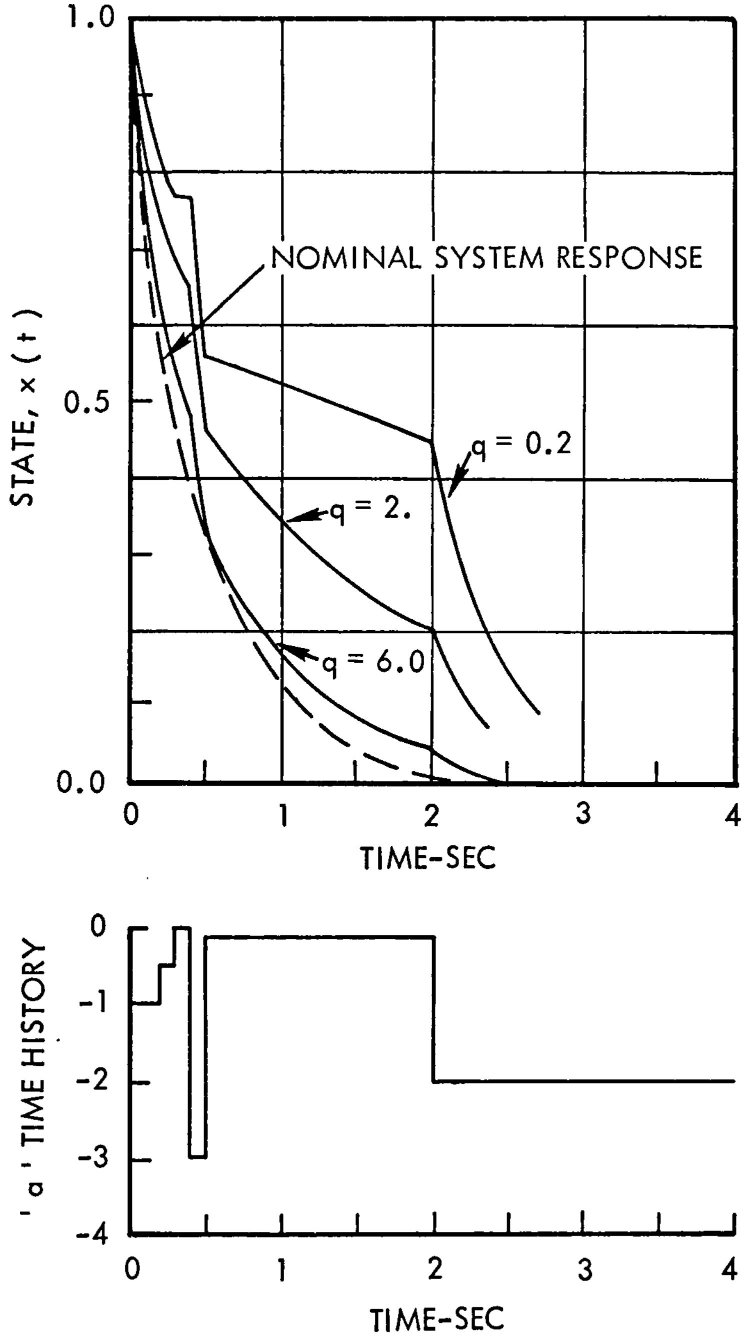

Figure 2. Example 1 - State Response Using State Feedback Only

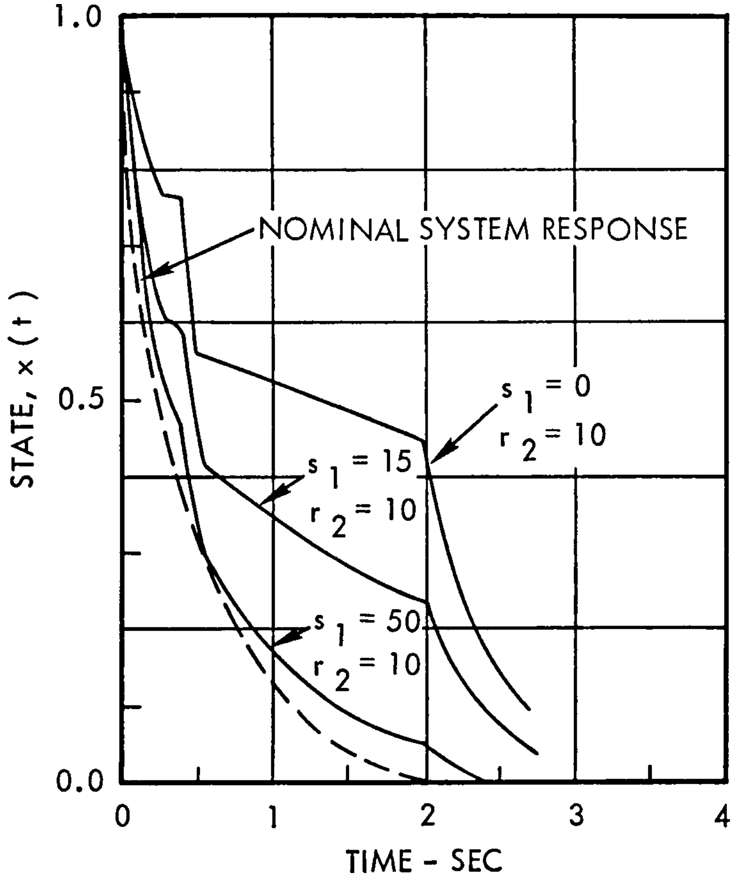

Figure 3. Example 1 - State Response Using Sensitivity and State Feedback

approaches. Figure 4 shows the corresponding total control effort required to reduce trajectory error for both the state feedback approach and the combined sensitivity and state feedback approach.

The results given in Figure 4 show for Example 1 that significant saving in total control effort can be realized if a combined sensitivity and state controller is used to reduce trajectory deviations. In the event it was desirable to reduce the trajectory error to that of a perfectly modeled system with constant plant parameters, the sensitivity and state feedback design would require 67 percent less control energy than if only state feedback was used.

In Example 2, the state coefficient 'a' has only one abrupt change during system operation. This example was chosen to see if the combined sensitivity and state approach would still require less control energy if the plant parameter had a less active time listing. The state response using only state feedback is shown in Figure 5. Also given in Figure 5 is the time history of the plant parameter 'a' for Example 2. Notice that a lower value of state weighting, q, is required in this example to achieve a reduction in trajectory error than was required in Example 1. The state response for the combined sensitivity and state feedback approach is given in Figure 6. Figure 7 shows the corresponding total control effort required to reduce trajectory for both the state feedback approach and the combined sensitivity and state feedback approach.

Figure 7 shows that the combined sensitivity and state feedback approach is only slightly better than the state feedback only approach.

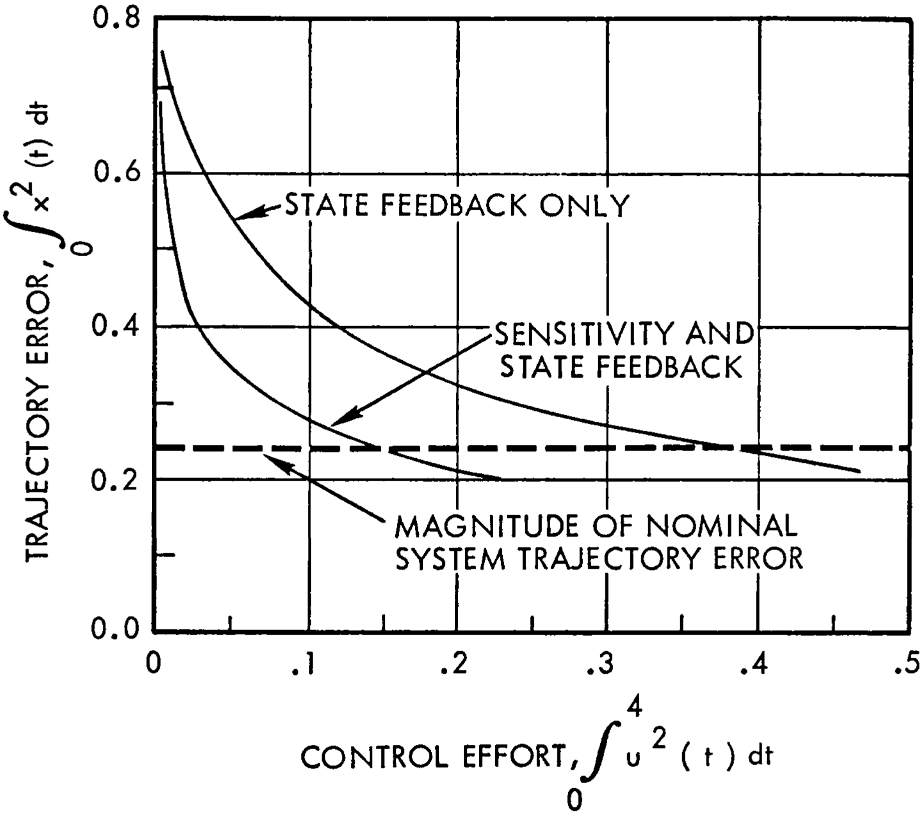

Figure 4. Example 1 - Tradeoff Between Trajectory Error Reduction and Control Effort

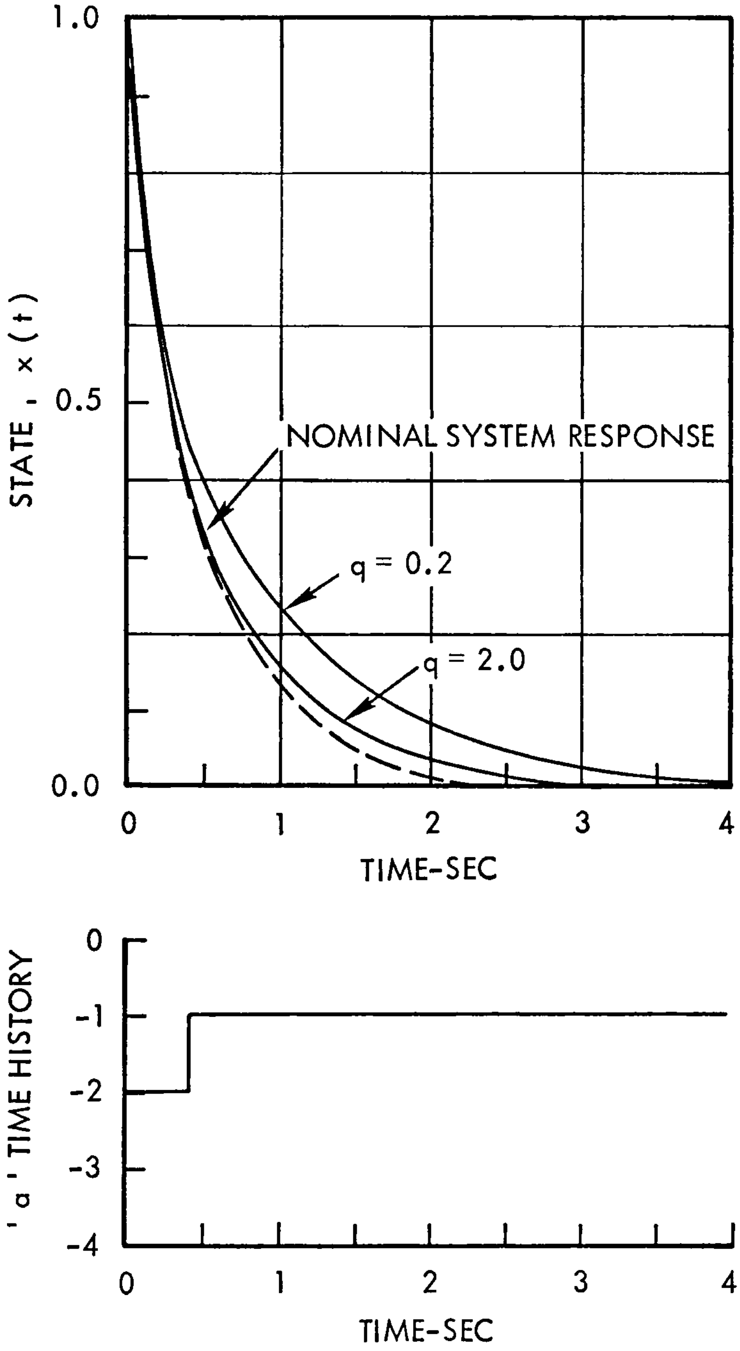

Figure 5. Example 2 - State Response Using State Feedback Only

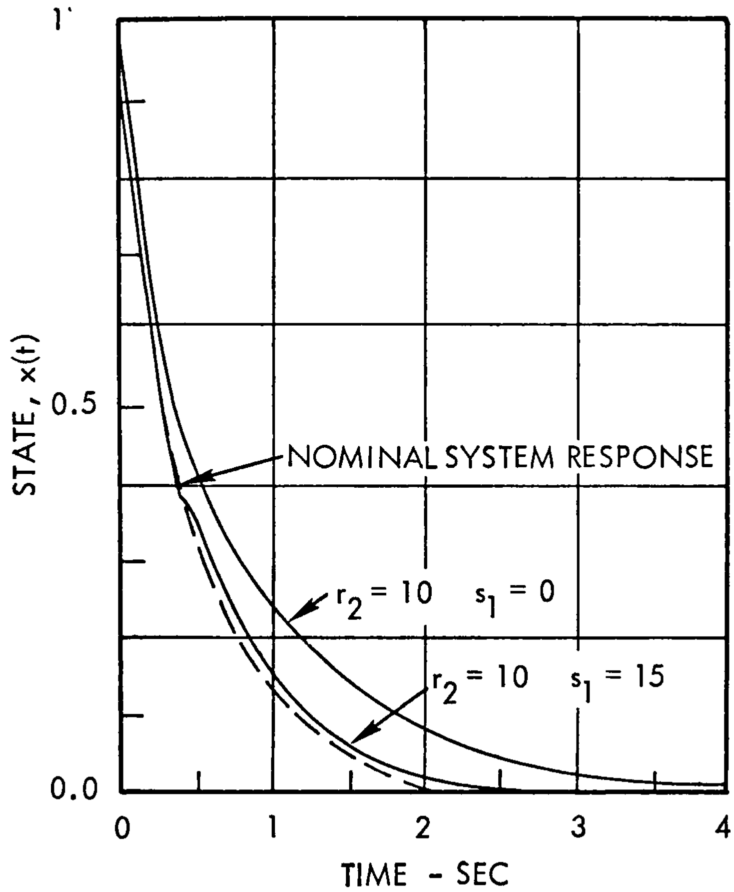

Figure 6. Example 2 - State Response Using Sensitivity and State Feedback

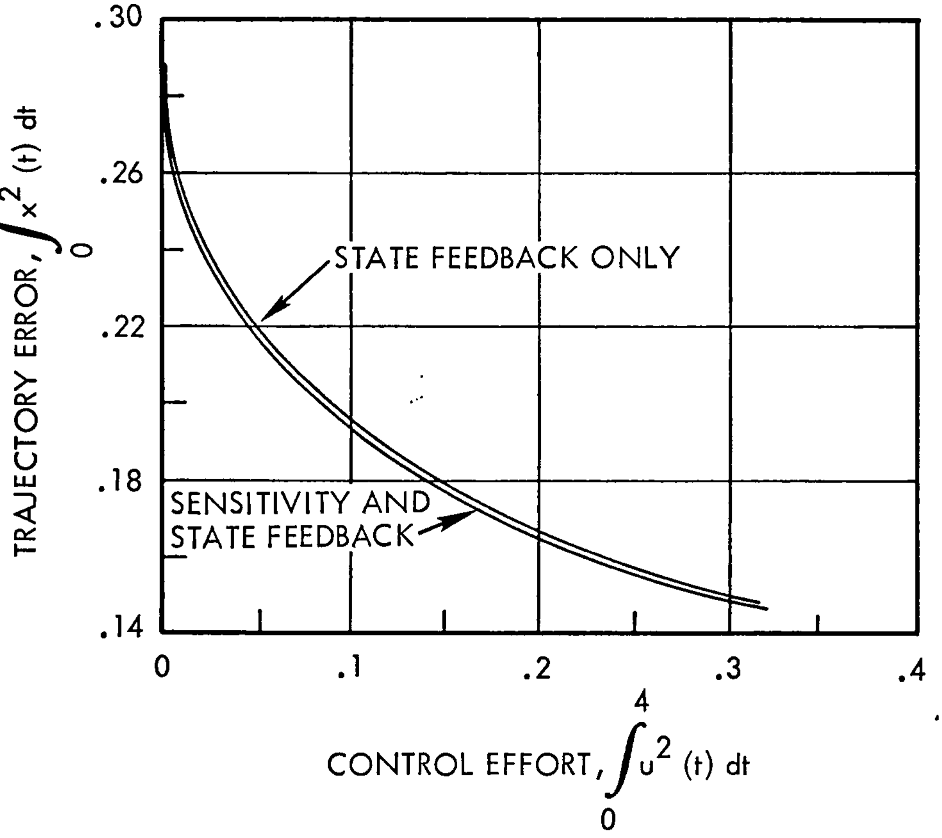

Figure 7. Example 2 - Tradeoff Between Trajectory Error Reduction and Control Effort

This might be expected, because as the plant behaves more closely to the model, less trajectory error results. Consequently, the amount of sensitivity feedback required is less and the two systems approach each other.

For Example 3, the state coefficient 'a' has only one abrupt change during system operation. Figure 8 presents the tradeoff between trajectory error reduction and total control effort for both approaches when the actual and assumed plant parameters are the same. For this perfectly modeled system, both approaches require approximately the same control effort for a given state error reduction. This would indicate that if a plant parameter was expected to change abruptly as in Example 1 or 2 but, in fact, remained constant, the combined sensitivity and state feedback approach would not require more control effort than the state feedback approach. Yet, if the plant parameter did change suddenly, the combined sensitivity and state feedback approach would be able to more effectively reduce the trajectory error with less control effort than if the state feedback approach was used.

It is interesting to note the difference between the state control, u(t) for the two approaches. Figure 9 shows the form of the state control for both the state feedback design and the combined sensitivity and state feedback design. For the system with only state feedback, the control starts with a greater value than the combined approach and decays exponentially. The system sensitivity of the unforced system begins at zero and rises to a peak early in the system operation and then goes

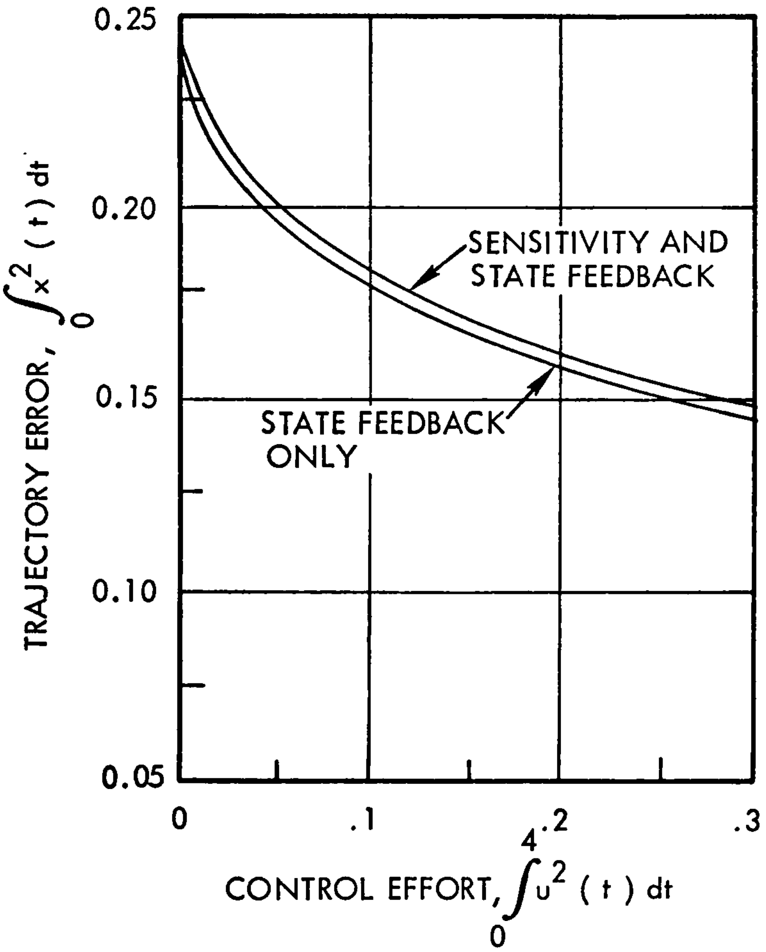

Figure 8. Example 3 - Tradeoff Between Trajectory Error Reduction and Control Effort

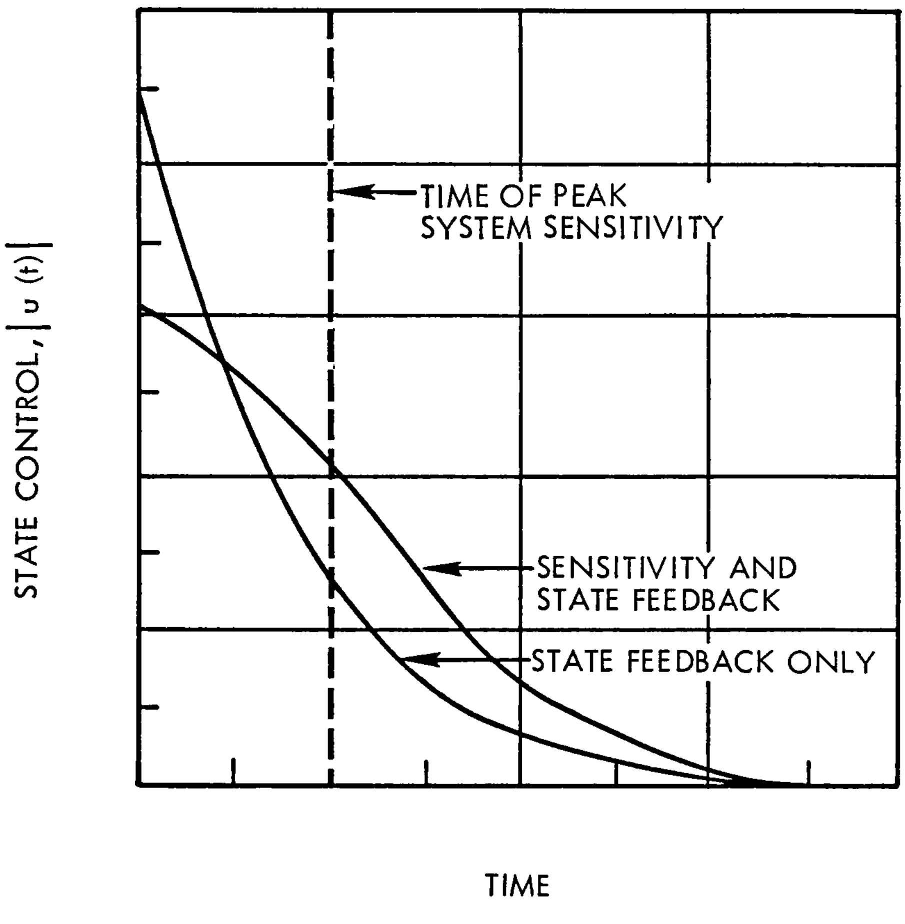

Figure 9. Form of the Optimal State Control

to zero as time increases. In figure 9, note how the combined sensitivity and state feedback control starts lower (sensitivity is small) and remains higher until sensitivity peaks and then quickly goes to zero. The fact that the combined approach state control is greater during high system sensitivity enables it to better compensate for the changing parameter. The following section discusses some of the practical limitations of implementing the optimal control for the combined sensitivity and state feedback design.

V. COMPUTER MEMORY REQUIREMENTS

The combined sensitivity and state feedback design will have greater computer storage requirements than the state feedback alone approach. This is due to the fact that there are inherent difficulties in the real-time calculation of the optimal control because the forward time numerical solution of the differential gain equations are subject to computational instability. Presently, it is required to first solve off-line the gains backward in time and then store them in memory to be used at the appropriate time. For higher order systems, with the potential of a large number of uncertain parameters, the computer memory requirements may become excessive to store the required time varying gains.

However, a computational technique has been proposed which will allow the gains to be computed in an on-line manner [11]. This technique uses a recursive algorithm which can integrate the differential gain equations forward in time without the associated computational instability, thereby eliminating costly memory storage. Further investigation needs to be done in this area to develop the full potentiality of implementing this computational technique with the state and sensitivity feedback gain requirements.

VI. SUMMARY AND CONCLUSIONS

In this thesis, an approach for the reduction of trajectory deviation resulting from a parameter with abrupt changes has been studied and compared to the conventional state feedback approach. The new approach was a system which had feedback of a sensitivity measure as well as state feedback. Three first order examples were analyzed. It was shown that the combined sensitivity and state feedback approach has the potential for using substantially less control energy to reduce trajectory error than an approach using only state feedback where the plant parameter has several abrupt changes. In addition, it was demonstrated that for the case where the parameters are known exactly, both approaches require essentially the same control energy to effect a reduction in trajectory error.

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APPENDIX A

COMPUTATION OF THE GAIN MATRIX, K(t) AND THE STATE AND SENSITIVITY EQUATIONS

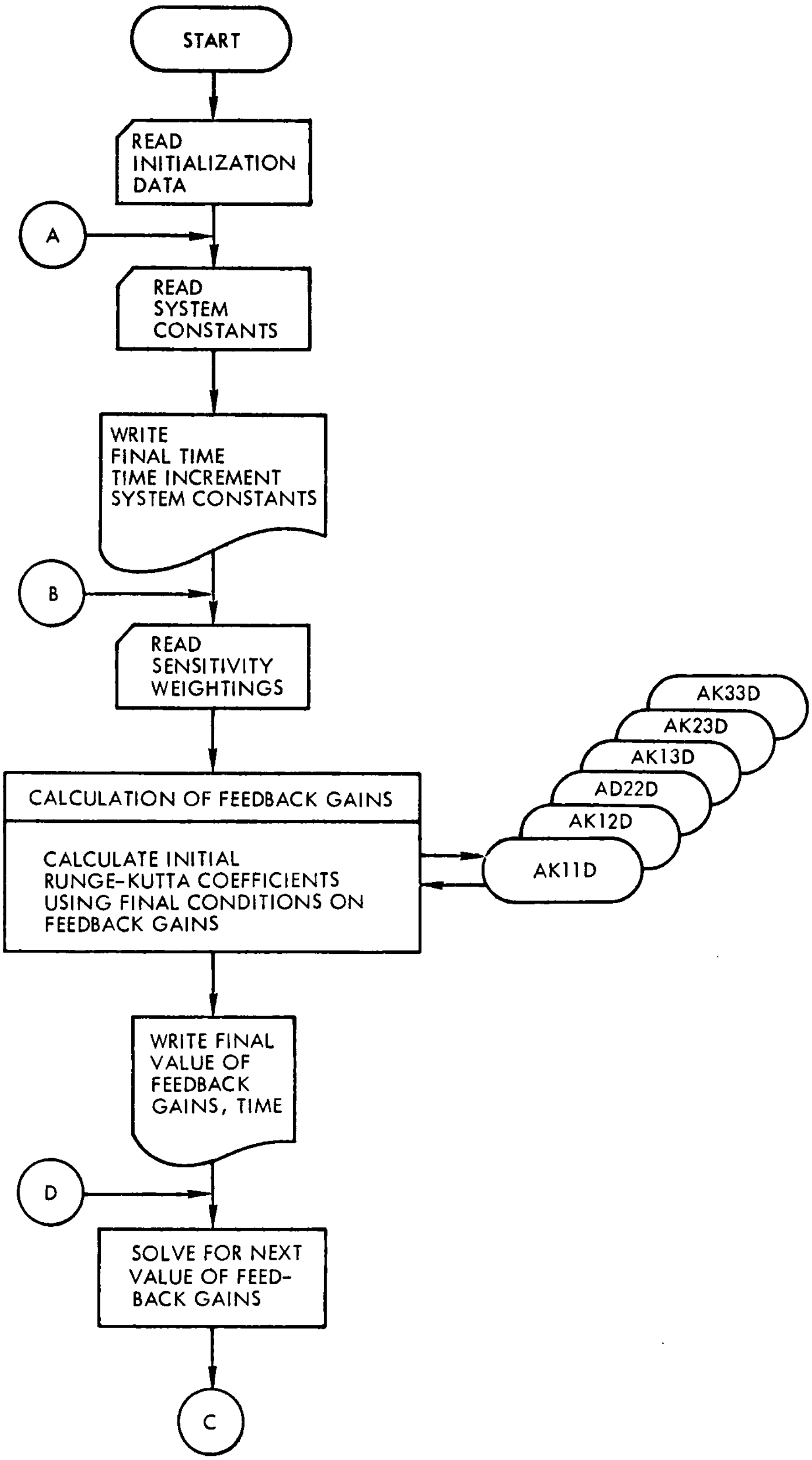
The analysis of the state and sensitivity system requires the solution of several non-linear differential equations. The technique for solving these equations will be to use the method of Runge-Kutta.

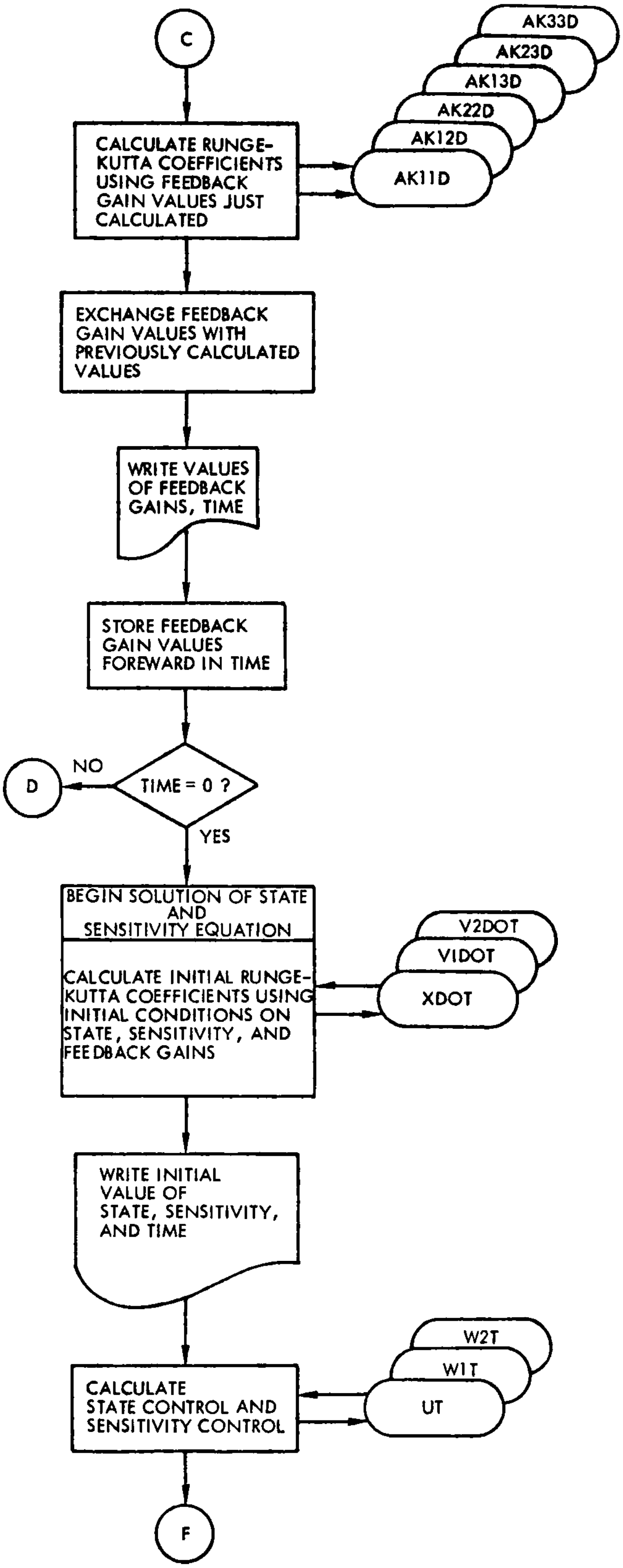
First the feedback gains (Equations 35 through 39) will have to be solved backward in time because only the final conditions on K(t) are known. The feedback gains solutions will be stored forward in time and used in the calculation of the state and sensitivity Equations 32, 33, and 34.

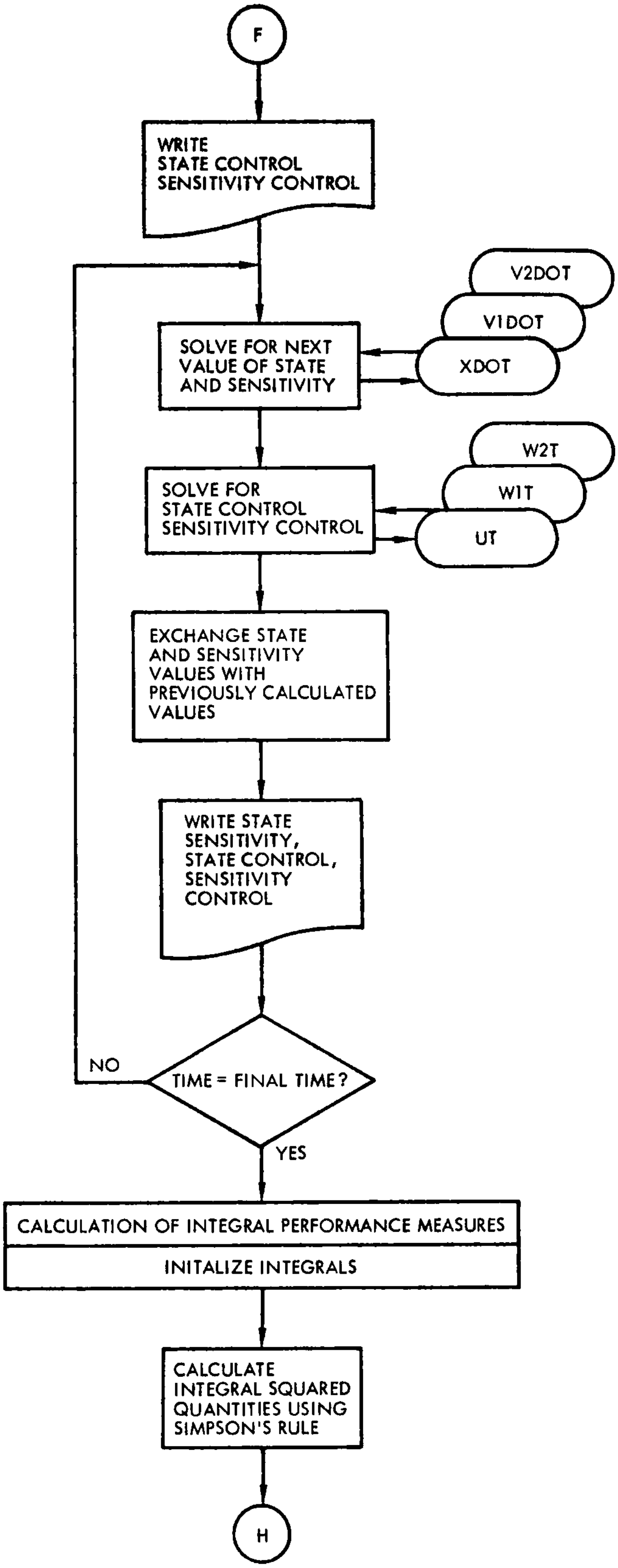
After the state and sensitivity are known, the integral squared performance indices are computed by using Simpson's integration rule.

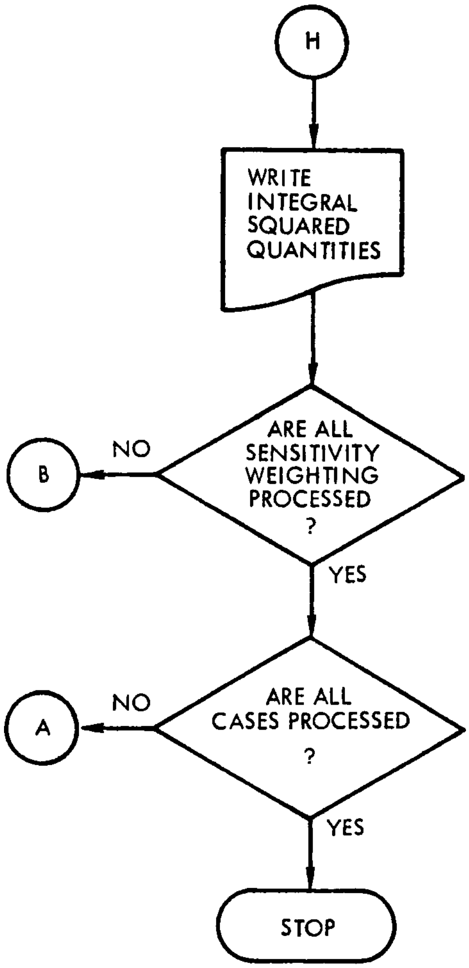
The flow chart for the computer simulation follows.

STATE AND SENSITIVITY FEEDBACK SIMULATION PROGRAM




C
CALCULATE RUNGE-KUTTA COEFFICIENTS USING FEEDBACK GAIN VALUES JUST CALCULATED
AK33D
AK23D
AK13D
AK22D
AK12D
AK11D
EXCHANGE FEEDBACK GAIN VALUES WITH PREVIOUSLY CALCULATED VALUES
WRITE VALUES OF FEEDBACK GAINS, TIME
STORE FEEDBACK GAIN VALUES FOREWARD IN TIME
TIME = 0 ?
NO
D
YES
BEGIN SOLUTION OF STATE AND SENSITIVITY EQUATION
CALCULATE INITIAL RUNGE-KUTTA COEFFICIENTS USING INITIAL CONDITIONS ON STATE, SENSITIVITY, AND FEEDBACK GAINS
V2DOT
V1DOT
XDOT
WRITE INITIAL VALUE OF STATE, SENSITIVITY, AND TIME
CALCULATE STATE CONTROL AND SENSITIVITY CONTROL
W2T
W1T
UT
F


F
WRITE
STATE CONTROL
SENSITIVITY CONTROL
SOLVE FOR NEXT
VALUE OF STATE
AND SENSITIVITY
V2DOT
V1DOT
XDOT
SOLVE FOR
STATE CONTROL
SENSITIVITY CONTROL
W2T
W1T
UT
EXCHANGE STATE
AND SENSITIVITY
VALUES WITH
PREVIOUSLY CALCULATED
VALUES
WRITE STATE
SENSITIVITY,
STATE CONTROL,
SENSITIVITY
CONTROL
NO
TIME = FINAL TIME?
YES
CALCULATION OF INTEGRAL PERFORMANCE MEASURES
INITALIZE INTEGRALS
CALCULATE
INTEGRAL SQUARED
QUANTITIES USING
SIMPSON'S RULE
H


H
WRITE
INTEGRAL
SQUARED
QUANTITIES
ARE ALL
SENSITIVITY
WEIGHTING
PROCESSED
?
NO
B
YES
ARE ALL
CASES PROCESSED
?
NO
A
YES
STOP

APPENDIX B

PROGRAM VERIFICATION

Athans and Falb [1] page 777, equation 9-155, gives a closed form solution for the feedback gain for the first-order system without sensitivity considerations.

Program verification will consist of comparing the Runge-Kutta solution of $k_{11}(t)$ with the closed form solution.

The optimal state gain is given by

$$k_{11}(t) = \frac{r\left[\beta + a + (\beta - a)\,\frac{f/r - a - \beta}{f/r - a + \beta}\,e^{2\beta(t - T)}\right]}{1 - \frac{f/r - a - \beta}{f/r - a + \beta}\,e^{2\beta(t - T)}}$$

The system constants are for the fixed time case, $t_f = 1$ sec

$$a = -1$$

$$\beta = \sqrt{\frac{q}{r} + a^2} = \sqrt{2}$$

$$q = 1$$

$$r = 1$$

$$f = 0$$

$$t = 0$$

$$T = 1$$

Equation (1) becomes

$$k_{11}(0) = \frac{\sqrt{2} - 1 + (\sqrt{2} + 1)\ \frac{1 - \sqrt{2}}{1 + 2}\ e^{-2\sqrt{2}}}{1 - \frac{1 - \sqrt{2}}{1 + \sqrt{2}}\ e^{-2\sqrt{2}}}$$

$$k_{11}(0) = \frac{1 - e^{-2\sqrt{2}}}{1 + \sqrt{2} - (1 - \sqrt{2})e^{-2\sqrt{2}}}$$

$$e^{-2\sqrt{2}} = .059487$$

$$1 + \sqrt{2} = 2.414214$$

$$\sqrt{2} - 1 = .4142.4$$

$$k_{11}(0) = \frac{1 - .059487}{2.4.42.4 + .414214\ (.059487)} = .385837$$

The program calculated value for $k_{11}(0)$ = .385818

For the infinite time case Equation (1) becomes

$$k_{11}(0) = \sqrt{2} - 1 = .414214$$

The program calculated value of $k_{11}(0)$ for the time infinite case is .414214. The results of the Runge-Kutta solution of the feedback gain are considered acceptable.

APPENDIX C

PROGRAM LISTING

In this Appendix, the computer program used to perform the calculations described in this thesis is given.

```
C       * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *
C       *                                                               *
C       *  DESIGN OF SENSITIVITY AND STATE VARIABLE FEEDBACK OPTIMAL   *
C       *  CONTROL SYSTEMS.                                             *
C       *                                                               *
C       *    SYSTEM -
C       *            XDOT  = A*X + B*U                                  *
C       *           V1DOT  = X + (A0 -(B0/R1)*(B0*K11+K12))*V1 + B0*M1  *
C       *           V2DOT  = (A0 -(B0/R1)*(B0*K11)+K12))*V2 + U + B0*M2 *
C       *                                                               *
C       * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *
C....   NN = THE NUMBER OF S1 AND S2 WEIGHTINGS.
C       N = THE NUMBER OF ITERATIONS
      DIMENSION  AK11(300),BK11(300),AK12(300),BK12(300),AK13(300),
     1           BK13(300),AK22(300),BK22(300),AK23(300),BK23(300),
     2           AK33(300),BK33(300),XT(300),V1T(300),V2T(300),
     3           AXT(300),AV1T(300),AV2T(300),AUT(300),BUT(300),
     4           AM1T(300),BM1T(300),AM2T(300),BM2T(300)
      COMMON  A0, A , B0 ,B, R1, R2,  R3 ,  Q
       READ (5,90) NCASE
   90 FORMAT( I4)
       READ(5,100) DELTAT, TF, AK110, AK120, AK130, AK220, AK230, AK330,
     1            N, NN
  100 FORMAT( 8F8.0,2I5 )
       READ(5,102) XT0, V1T0, V2T0
       SAVETF = TF
       SAK110 = AK110
       SAK120 = AK120
       SAK130 = AK130
       SAK220 = AK220
       SAK230 = AK230
       SAK330 = AK330
       SAVXT0= XT0
       SAV1T0 = V1T0
       SAV2T0 = V2T0
       DO 30 LLLL = 1, NCASE
       READ(5,102) A0,A, B0, B, R1, R2, R3, Q
  102 FORMAT(10F8.0)
       TF = SAVETF
       WRITE(6,200) TF, DELTAT, N
  200 FORMAT( 1H1/ 10X,18HFINAL TIME........F7.3/
```

```
     1              10X,18HTIME INCREMENT....,F7.3/
     2              10X,18HNO. OF ITERATIONS.,I4   )
      WRITE(6,203) A0, A, B0, B, R1, R2, R3, Q
  203 FORMAT( ///// 10X, 9HCONSTANTS/ 10X,3HA0=,F6.2,2X,2HA=,F6.2,
     1          2X,3HB0=,F6.2,2X,2HB=,F6.2,2X /10X,3HR1=,F6.2,2X,
     2          3HR2=,F6.2,2X,3HR3=F6.2,2X,2HQ=,F6.2 )
      DO 20 LLLLL = 1, NN
      READ(5,101) S1, S2
  101 FORMAT( 2F10.0)
     TF = SAVETF
C....  SOLUTION OF FEEDBACK GAINS.
      AK110 = SAK110
      AK120 = SAK120
      AK130 = SAK130
      AK220 = SAK220
      AK230 = SAK230
      AK330 = SAK330
      AI0  = AK11D(AK110,AK120,AK130)
      AJ0  = AK12D(AK110,AK120,AK130,AK220,AK230)
      AK0  = AK13D(AK110,AK120,AK130,AK230,AK330)
      AL0  = AK22D(AK110,AK120,AK130,AK220,AK230,S1)
      AM0  = AK23D(AK110,AK120,AK130,AK220,AK230,AK330)
      AN0  = AK33D(AK110,AK130,AK230,AK330,S2)
      AI1  = AK11D(AK110+AI0*DELTAT/2., AK120+AJ0*DELTAT/2.,
     1              AK130+AK0*DELTAT/2.)
      AJ1  = AK12D(AK110+AI0*DELTAT/2., AK120+AJ0*DELTAT/2.,
     1              AK130+AK0*DELTAT/2., AK220+AL0*DELTAT/2.,
     2              AK230+AM0*DELTAT/2.)
      AK1  = AK13D(AK110+AI0*DELTAT/2., AK120+AJ0*DELTAT/2.,
     1              AK130+AK0*DELTAT/2., AK230+AM0*DELTAT/2.,
     2              AK330+AN0*DELTAT/2.)
      AL1  = AK22D(AK110+AI0*DELTAT/2., AK120+AJ0*DELTAT/2.,
     1              AK130+AK0*DELTAT/2., AK220+AL0*DELTAT/2.,
     2              AK230+AM0*DELTAT/2., S1)
      AM1  = AK23D(AK110+AI0*DELTAT/2., AK120+AJ0*DELTAT/2.,
     1              AK130+AK0*DELTAT/2., AK220+AL0*DELTAT/2.,
     2              AK230+AM0*DELTAT/2., AK330+AN0*DELTAT/2.)
      AN1  = AK33D(AK110+AI0*DELTAT/2., AK130+AK0*DELTAT/2.,
     1              AK230+AM0*DELTAT/2., AK330+AN0*DELTAT/2., S2)
      AI2  = AK11D(AK110+AI1*DELTAT/2., AK120+AJ1*DELTAT/2.,
     1              AK130+AK1*DELTAT/2.)
      AJ2  = AK12D(AK110+AI1*DELTAT/2., AK120+AJ1*DELTAT/2.,
     1              AK130+AK1*DELTAT/2., AK220+AL1*DELTAT/2.,
     2              AK230+AM1*DELTAT/2.)
      AK2  = AK13D(AK110+AI1*DELTAT/2., AK120+AJ1*DELTAT/2.,
     1              AK130+AK1*DELTAT/2., AK230+AM1*DELTAT/2.,
     2              AK330+AN1*DELTAT/2.)
      AL2  = AK22D(AK110+AI1*DELTAT/2., AK120+AJ1*DELTAT/2.,
     1              AK130+AK1*DELTAT/2., AK220+AL1*DELTAT/2.,
     2              AK230+AM1*DELTAT/2., S1)
      AM2  = AK23D(AK110+AI1*DELTAT/2., AK120+AJ1*DELTAT/2.,
     1              AK130+AK1*DELTAT/2., AK220+AL1*DELTAT/2.,
     2              AK230+AM1*DELTAT/2., AK330+AN1*DELTAT/2.)
```

```
      AN2 = AK33D(AK110+AI1*DELTAT/2., AK130+AK1*DELTAT/2.,
     1              AK230+AM1*DELTAT/2., AK330+AN1*DELTAT/2., S2)
      AI3 = AK11D(AK110+AI2*DELTAT,     AK120+AJ2*DELTAT,
     1              AK130+AK2*DELTAT)
      AJ3 = AK12D(AK110+AI2*DELTAT,     AK120+AJ2*DELTAT,
     1              AK130+AK2*DELTAT,     AK220+AL2*DELTAT,
     2              AK230+AM2*DELTAT)
      AK3 = AK13D(AK110+AI2*DELTAT,     AK120+AJ2*DELTAT,
     1              AK130+AK2*DELTAT,     AK230+AM2*DELTAT,
     2              AK330+AN2*DELTAT)
      AL3 = AK22D(AK110+AI2*DELTAT,     AK120+AJ2*DELTAT,
     1              AK130+AK2*DELTAT,     AK220+AL2*DELTAT,
     2              AK230+AM2*DELTAT, S1)
      AM3 = AK23D(AK110+AI2*DELTAT,     AK120+AJ2*DELTAT,
     1              AK130+AK2*DELTAT,     AK220+AL2*DELTAT,
     2              AK230+AM2*DELTAT,     AK330+AN2*DELTAT)
      AN3 = AK33D(AK110+AI2*DELTAT,     AK130+AK2*DELTAT,
     1              AK230+AM2*DELTAT,     AK330+AN2*DELTAT, S2)
      WRITE(6,201) S1, S2, TF, AK110, AK120, AK130, AK220, AK230,AK330
  201 FORMAT(  1H1,10X,42HWEIGHTING ON SENSITIVITY COEFFICIENT 1....,
     1 F7.3,/ 11X,42HWEIGHTING ON SENSITIVITY COEFFICIENT 2...., F7.3/
     2 3X,1HT,7X,5HK(11),8X,5HK(12),8X,5HK(13),8X,5HK(22),8X,5HK(23),
     3 8X,5HK(33)/1X, F6.3,2X,E11.6,2X,E11.6,2X,E11.6,2X,E11.6,2X,
     4 E11.6,2X,E11.6 )
      DO 10 I=1, N
      AK11(I) = AK110 + (DELTAT/6.)*(AI0+2.*(AI1+AI2)+AI3)
      AK12(I) = AK120 + (DELTAT/6.)*(AJ0+2.*(AJ1+AJ2)+AJ3)
      AK13(I) = AK130 + (DELTAT/6.)*(AK0+2.*(AK1+AK2)+AK3)
      AK22(I) = AK220 + (DELTAT/6.)*(AL0+2.*(AL1+AL2)+AL3)
      AK23(I) = AK230 + (DELTAT/6.)*(AM0+2.*(AM1+AM2)+AM3)
      AK33(I) = AK330 + (DELTAT/6.)*(AN0+2.*(AN1+AN2)+AN3)
      TF= TF + DELTAT
      IF(I-1) 7, 6, 7
    6 BK11(N) = AK110
      BK12(N) = AK120
      BK13(N) = AK130
      BK22(N) = AK220
      BK23(N) = AK230
      BK33(N) = AK330
    7 IF(N-I) 2, 1, 2
    1 BK110 = AK11(I)
      BK120 = AK12(I)
      BK130 = AK13(I)
      BK220 = AK22(I)
      BK230 = AK23(I)
      BK330 = AK33(I)
      GO TO 3
    2 BK11(N-I) = AK11(I)
      BK12(N-I) = AK12(I)
      BK13(N-I) = AK13(I)
      BK22(N-I) = AK22(I)
      BK23(N-I) = AK23(I)
      BK33(N-I) = AK33(I)
```

```
3   AI0 = AK11D(AK11(I), AK12(I), AK13(I))
    AJ0 = AK12D(AK11(I), AK12(I), AK13(I), AK22(I), AK23(I))
    AK0 = AK13D(AK11(I), AK12(I), AK13(I), AK23(I), AK33(I))
    AL0 = AK22D(AK11(I), AK12(I), AK13(I), AK22(I), AK23(I), S1)
    AM0 = AK23D(AK11(I), AK12(I), AK13(I), AK22(I), AK23(I), AK33(I))
    AN0 = AK33D(AK11(I), AK13(I), AK23(I), AK33(I), S2)
    AI1 = AK11D(AK11(I)+AI0*DELTAT/2., AK12(I)+AJ0*DELTAT/2.,
  1             AK13(I)+AK0*DELTAT/2.)
    AJ1 = AK12D(AK11(I)+AI0*DELTAT/2., AK12(I)+AJ0*DELTAT/2.,
  1             AK13(I)+AK0*DELTAT/2., AK22(I)+AL0*DELTAT/2.,
  2             AK23(I)+AM0*DELTAT/2.)
    AK1 = AK13D(AK11(I)+AI0*DELTAT/2., AK12(I)+AJ0*DELTAT/2.,
  1             AK13(I)+AK0*DELTAT/2., AK23(I)+AM0*DELTAT/2.,
  2             AK33(I)+AN0*DELTAT/2.)
    AL1 = AK22D(AK11(I)+AI0*DELTAT/2., AK12(I)+AJ0*DELTAT/2.,
  1             AK13(I)+AK0*DELTAT/2., AK22(I)+AL0*DELTAT/2.,
  2             AK23(I)+AM0*DELTAT/2., S1)
    AM1 = AK23D(AK11(I)+AI0*DELTAT/2., AK12(I)+AJ0*DELTAT/2.,
  1             AK13(I)+AK0*DELTAT/2., AK22(I)+AL0*DELTAT/2.,
  2             AK23(I)+AM0*DELTAT/2., AK33(I)+AN0*DELTAT/2.)
    AN1 = AK33D(AK11(I)+AI0*DELTAT/2., AK13(I)+AK0*DELTAT/2.,
  1             AK23(I)+AM0*DELTAT/2., AK33(I)+AN0*DELTAT/2., S2)
    AI2 = AK11D(AK11(I)+AI1*DELTAT/2., AK12(I)+AJ1*DELTAT/2.,
  1             AK13(I)+AK1*DELTAT/2.)
    AJ2 = AK12D(AK11(I)+AI1*DELTAT/2., AK12(I)+AJ1*DELTAT/2.,
  1             AK13(I)+AK1*DELTAT/2., AK22(I)+AL1*DELTAT/2.,
  2             AK23(I)+AM1*DELTAT/2.)
    AK2 = AK13D(AK11(I)+AI1*DELTAT/2., AK12(I)+AJ1*DELTAT/2.,
  1             AK13(I)+AK1*DELTAT/2., AK23(I)+AM1*DELTAT/2.,
  2             AK33(I)+AN1*DELTAT/2.)
    AL2 = AK22D(AK11(I)+AI1*DELTAT/2., AK12(I)+AJ1*DELTAT/2.,
  1             AK13(I)+AK1*DELTAT/2., AK22(I)+AL1*DELTAT/2.,
  2             AK23(I)+AM1*DELTAT/2., S1)
    AM2 = AK23D(AK11(I)+AI1*DELTAT/2., AK12(I)+AJ1*DELTAT/2.,
  1             AK13(I)+AK1*DELTAT/2., AK22(I)+AL1*DELTAT/2.,
  2             AK23(I)+AM1*DELTAT/2., AK33(I)+AN1*DELTAT/2.)
    AN2 = AK33D(AK11(I)+AI1*DELTAT/2., AK13(I)+AK1*DELTAT/2.,
  1             AK23(I)+AM1*DELTAT/2., AK33(I)+AN1*DELTAT/2., S2)
    AI3 = AK11D(AK11(I)+AI2*DELTAT,    AK12(I)+AJ2*DELTAT,
  1             AK13(I)+AK2*DELTAT )
    AJ3 = AK12D(AK11(I)+AI2*DELTAT,    AK12(I)+AJ2*DELTAT,
  1             AK13(I)+AK2*DELTAT,    AK22(I)+AL2*DELTAT,
  2             AK23(I)+AM2*DELTAT)
    AK3 = AK13D(AK11(I)+AI2*DELTAT,    AK12(I)+AJ2*DELTAT,
  1             AK13(I)+AK2*DELTAT,    AK23(I)+AM2*DELTAT,
  2             AK33(I)+AN2*DELTAT)
    AL3 = AK22D(AK11(I)+AI2*DELTAT,    AK12(I)+AJ2*DELTAT,
  1             AK13(I)+AK2*DELTAT,    AK22(I)+AL2*DELTAT,
  2             AK23(I)+AM2*DELTAT,    S1)
    AM3 = AK23D(AK11(I)+AI2*DELTAT,    AK12(I)+AJ2*DELTAT,
  1             AK13(I)+AK2*DELTAT,    AK22(I)+AL2*DELTAT,
  2             AK23(I)+AM2*DELTAT,  AK33(I)+AN2*DELTAT  )
    AN3 = AK33D(AK11(I)+AI2*DELTAT,    AK13(I)+AK2*DELTAT,
  1             AK23(I)+AM2*DELTAT,    AK33(I)+AN2*DELTAT,  S2)
```

```
      WRITE(6,204)  TF, AK11(I), AK12(I), AK13(I), AK22(I), AK23(I),
     1              AK33(I)
  204 FORMAT(1X,
     1        F6.3,2X,E11.6,2X,E11.6,2X,E11.6,2X,E11.6,2X,E11.6,2X
     1         E11.6  )
       AK110= AK11(I)
       AK120= AK12(I)
       AK130= AK13(I)
       AK220= AK22(I)
       AK230= AK23(I)
       AK330= AK33(I)
   10 CONTINUE
   42 T=0.
       DELT=-DELTAT
       XTO= SAVXTO
       V1TO= SAV1TO
       V2TO= SAV2TO
       BIO = XDOT(BK110,BK120,BK130,BK230,BK330,  XTO, V1TO, V2TO)
       BJO =V1DOT(BK110,BK120,BK130,BK220,BK230,  XTO, V1TO, V2TO)
       BKO =V2DOT(BK110,BK120,BK130,BK230,BK330,  XTO, V1TO, V2TO)
       BI1 = XDOT(BK110,BK120,BK130,BK230,BK330,
     1            XTO+BIO*DELT/2., V1TO+BJO*DELT/2., V2TO+BKO*DELT/2. )
       BJ1 =V1DOT(BK110,BK120,BK130,BK220,BK230,
     1            XTO+BIO*DELT/2., V1TO+BJO*DELT/2., V2TO+BKO*DELT/2. )
       BK1 =V2DOT(BK110,BK120,BK130,BK230,BK330,
     1            XTO+BIO*DELT/2., V1TO+BJO*DELT/2., V2TO+BKO*DELT/2. )
       BI2 = XDOT(BK110,BK120,BK130,BK230,BK330,
     1            XTO+BI1*DELT/2., V1TO+BJ1*DELT/2., V2TO+BK1*DELT/2. )
       BJ2 =V1DOT(BK110,BK120,BK130,BK220,BK230,
     1            XTO+BI1*DELT/2., V1TO+BJ1*DELT/2., V2TO+BK1*DELT/2. )
       BK2 =V2DOT(BK110,BK120,BK130,BK230,BK330,
     1            XTO+BI1*DELT/2., V1TO+BJ1*DELT/2., V2TO+BK1*DELT/2. )
       BI3 = XDOT(BK110,BK120,BK130,BK230,BK330,
     1            XTO+BI2*DELT,     V1TO+BJ2*DELT,     V2TO+BK2*DELT     )
       BJ3 =V1DOT(BK110,BK120,BK130,BK220,BK230,
     1            XTO+BI2*DELT,     V1TO+BJ2*DELT,     V2TO+BK2*DELT     )
       BK3 =V2DOT(BK110,BK120,BK130,BK230,BK330,
     1            XTO+BI2*DELT,     V1TO+BJ2*DELT,     V2TO+BK2*DELT     )
       BUTO=UT( BK110,BK120,BK130,BK230,BK330, XTO,  V1TO,  V2TO )
       AUTO= BUTO * BUTO
       AXTO= XTO  *  XTO
       AV1TO= V1TO  * V1TO
       AV2TO= V2TO  * V2TO
       BM1TO=W1T( BK120, BK220, BK230,  XTO,  V1TO, V2TO )
       AM1TO=  BM1TO   *  BM1TO
       BM2TO=W2T( BK130, BK230, BK330,  XTO,  V1TO, V2TO )
       AM2TO = BM2TO   *  BM2TO
      WRITE(6,205)  T,  XTO, V1TO, V2TO, BUTO, BM1TO, BM2TO
  205 FORMAT( 1H1/11X,22HSYSTEM  TIME  RESPONSE,/  3X,1HT,7X,4HX(T),
     1        8X,5HV1(T),8X,5HV2(T),8X,4HU(T),9X,5HM1(T),8X,5HM2(T)/
     2  1X,   F6.3,2X,E11.6,2X,E11.6,2X,E11.6,2X,E11.6,2X,E11.6,
     3        2X,E11.6  )
      DO 15  I=1,N
       XT(I)= XTO  + DELT/6. * ( BIO +2.*(BI1 + BI2) + BI3 )
```

```
      V1T(I)=V1TO + DELT/6. * ( BJO +2.*(BJ1 + BJ2) + BJ3 )
      V2T(I)=V2TO + DELT/6. * ( BKO +2.*(BK1 + BK2) + BK3 )
      AXT(I)= XT(I) * XT(I)
      AV1T(I)=  V1T(I) * V1T(I)
      AV2T(I)=  V2T(I) * V2T(I)
      BUT(I)=UT(BK11(I),BK12(I),BK13(I),BK23(I),BK33(I),
     1            XT(I), V1T(I),  V2T(I) )
      AUT(I) = BUT(I) * BUT(I)
      BM1T(I)=W1T(BK12(I),BK22(I),BK23(I), XT(I), V1T(I),  V2T(I))
      AM1T(I) = BM1T(I)  *  BM1T(I)
      BM2T(I)=W2T(BK13(I),BK23(I),BK33(I),XT(I), V1T(I), V2T(I) )
      AM2T(I) = BM2T(I)  *  BM2T(I)
      T= T + DELT
      BIO = XDOT(BK11(I),BK12(I),BK13(I),BK23(I),BK33(I),
     1             XT(I), V1T(I), V2T(I)  )
      BJO =V1DOT(BK11(I),BK12(I),BK13(I),BK22(I),BK23(I),
     1             XT(I), V1T(I), V2T(I)  )
      BKO =V2DOT(BK11(I),BK12(I),BK13(I),BK23(I),BK33(I),
     1             XT(I), V1T(I), V2T(I)  )
      BI1 = XDOT(BK11(I),BK12(I),BK13(I),BK23(I),BK33(I),
     1             XT(I)+BIO*DELT/2.,V1T(I)+BJO*DELT/2.,
     2             V2T(I)+BKO*DELT/2. )
      BJ1 =V1DOT(BK11(I),BK12(I),BK13(I),BK22(I),BK23(I),
     1             XT(I)+BIO*DELT/2.,V1T(I)+BJO*DELT/2.,
     2             V2T(I)+BKO*DELT/2. )
      BK1 =V2DOT(BK11(I),BK12(I),BK13(I),BK23(I),BK33(I),
     1             XT(I)+BIO*DELT/2.,V1T(I)+BJO*DELT/2.,
     2             V2T(I)+BKO*DELT/2. )
      BI2 = XDOT(BK11(I),BK12(I),BK13(I),BK23(I),BK33(I),
     1             XT(I)+BI1*DELT/2.,V1T(I)+BJ1*DELT/2.,
     2             V2T(I)+BK1*DELT/2. )
      BJ2 =V1DOT(BK11(I),BK12(I),BK13(I),BK22(I),BK23(I),
     1             XT(I)+BI1*DELT/2.,V1T(I)+BJ1*DELT/2.,
     2             V2T(I)+BK1*DELT/2. )
      BK2 =V2DOT(BK11(I),BK12(I),BK13(I),BK23(I),BK33(I),
     1             XT(I)+BI1*DELT/2.,V1T(I)+BJ1*DELT/2.,
     2             V2T(I)+BK1*DELT/2. )
      BI3 = XDOT(BK11(I),BK12(I),BK13(I),BK23(I),BK33(I),
     1             XT(I)+BI2*DELT,   V1T(I)+BJ2*DELT,
     2             V2T(I)+BK2*DELT  )
      BJ3 =V1DOT(BK11(I),BK12(I),BK13(I),BK22(I),BK23(I),
     1             XT(I)+BI2*DELT,   V1T(I)+BJ2*DELT,
     2             V2T(I)+BK2*DELT  )
      BK3 =V2DOT(BK11(I),BK12(I),BK13(I),BK23(I),BK33(I),
     1             XT(I)+BI2*DELT,   V1T(I)+BJ2*DELT,
     2             V2T(I)+BK2*DELT  )
      WRITE(6,206)  T,  XT(I), V1T(I), V2T(I), BUT(I), BM1T(I), BM2T(I)
  206 FORMAT(1X,
     1        F6.3,2X,E11.6,2X,E11.6,2X,E11.6,2X,E11.6,2X,E11.6,2X
     1        E11.6  )
      XTO= XT(I)
      V1TO= V1T(I)
      V2TO= V2T(I)
   15 CONTINUE
```

```
      T=0.
      NM2= N-2
       AREAX=0.
       AREAV1=0.
       AREAV2=0.
      AREAVV=0.
       AREAU=0.
       AREAM1=0.
       AREAM2=0.
      COSTF=0.
      COSTF1=0.
      WRITE(6,207)  S1,S2, T, AREAX,AREAV1,AREAV2,AREAVV,AREAU,
     1              AREAM1,AREAM2, COSTF, COSTF1
  207 FORMAT( 1H1/ 10X,42HWEIGHTING ON SENSITIVITY COEFFICIENT 1....,
     1       F7.3/10X,42HWEIGHTING ON SENSITIVITY COEFFICIENT 2....,
     2       F7.3/ 3X,1HT,5X, 9HINT. X**2,3X,10HINT. V1**2,3X,
     3    10HINT. V2**2,4X,9H V1 + V2 ,4X,9HINT. U**2,3X,10HINT. M1**2,
     4    3X,10HINT. M2**2,3X,10HCOST FUNT.,3X,13HSEN.COST FUNT/
     5   1X, F6.3,2X,E11.6,2X,E11.6,2X,E11.6,2X,E11.6,2X,E11.6,
     6    2X,E11.6,2X,E11.6,2X,E11.6,2X,E11.6  )
      T=T+2.*DELT
      AREAX =  DELT/3. * ( AXTO  +4.* AXT(1)  +AXT(2) )
      AREAV1=  DELT/3. * ( AV1TO +4.* AV1T(1) +AV1T(2) )
      AREAV2=  DELT/3. * ( AV2TO +4.* AV2T(1) +AV2T(2) )
      AREAVV= AREAV1 + AREAV2
      AREAU =  DELT/3. * ( AUTO  +4.* AUT(1)  +AUT(2) )
      AREAM1=  DELT/3. * ( AM1TO +4.* AM1T(1) +AM1T(2) )
      AREAM2=  DELT/3. * ( AM2TO +4.* AM2T(1) +AM2T(2) )
      COSTF= .5 * ( Q*AREAX +R1*AREAU  )
      COSTF1=.5 * ( Q*AREAX +R1*AREAU +R2*AREAM1 +R3*AREAM2
     1                 +S1*AREAV1 +S2*AREAV2  )
      WRITE(6,208)  T, AREAX,AREAV1,AREAV2,AREAVV,AREAU,AREAM1,
     1              AREAM2, COSTF, COSTF1
  208 FORMAT(1X,
     1        F6.3,2X,E11.6,2X,E11.6,2X,E11.6,2X,E11.6,2X,E11.6,
     1        2X,E11.6,2X,E11.6,2X,E11.6,2X,E11.6  )
      DO 25  I=2,NM2,2
       T=T+2.*DELT
       AREAX = AREAX  +DELT/3.*( AXT(I) +4.*AXT(I+1) +AXT(I+2)  )
       AREAV1= AREAV1 +DELT/3.*( AV1T(I)+4.*AV1T(I+1)+AV1T(I+2) )
       AREAV2= AREAV2 +DELT/3.*( AV2T(I)+4.*AV2T(I+1)+AV2T(I+2) )
      AREAVV= AREAV1 + AREAV2
       AREAU = AREAU  +DELT/3.*( AUT(I) +4.*AUT(I+1) +AUT(I+2)  )
       AREAM1= AREAM1 +DELT/3.*( AM1T(I)+4.*AM1T(I+1)+AM1T(I+2) )
       AREAM2= AREAM2 +DELT/3.*( AM2T(I)+4.*AM2T(I+1)+AM2T(I+2) )
      COSTF= .5 * ( Q*AREAX +R1*AREAU  )
      COSTF1=.5 * ( Q*AREAX +R1*AREAU +R2*AREAM1 +R3*AREAM2
     1                 +S1*AREAV1 +S2*AREAV2  )
      WRITE(6,208)  T, AREAX,AREAV1,AREAV2,AREAVV,AREAU,AREAM1,
     1              AREAM2, COSTF, COSTF1
   25 CONTINUE
   20 CONTINUE
   30 CONTINUE
      STOP
      END
```

```
 FUNCTION  AK11D( AK11,AK12,AK13  )
 COMMON  A0, A, B0, B, R1, R2, R3,   Q
 AK11D=-2.*A0*AK11  -2.*AK12+B0*B0/R1*AK11*AK11  +2.*B0/R1*AK11*
1      AK13  +1./R1*AK13*AK13  +B0*B0/R2*AK12*AK12  +B0*B0/R3*
2      AK13*AK13   -Q
 RETURN
 END
 FUNCTION  AK12D( AK11,AK12,AK13,AK22,AK23 )
 COMMON  A0, A, B0, B, R1, R2, R3,   Q
 AK12D=-2.*A0*AK12  +2.*B0*B0/R1*AK11*AK12   +2.*B0/R1*AK12*AK13
1      -AK22   +B0/R1*AK11*AK23   +1./R1*AK13*AK23  +B0*B0/R2*AK12
2      *AK22   +B0*B0/R3*AK13*AK23
 RETURN
 END
 FUNCTION AK13D( AK11, AK12, AK13, AK23, AK33 )
 COMMON  A0,  A,  B0,  B,  R1,  R2,  R3,   Q
 AK13D=-2*A0*AK13   +2.*B0*B0/R1*AK11*AK13    +2.*B0/R1*AK13*AK13
1      -AK23   +B0/R1*AK11*AK33  +1./R1*AK13*AK33   +B0*B0/R2*
2       AK12*AK23   +B0*B0/R3*AK13*AK33
 RETURN
 END
 FUNCTION AK22D( AK11,AK12,AK13, AK22, AK23,  S1 )
 COMMON  A0, A, B0,B  ,R1, R2, R3,  Q
 AK22D=-2.*A0*AK22   +2.*B0*B0/R1*AK11*AK22   +2.*B0/R1*AK13*AK22
1      +B0*B0/R1*AK12*AK12   +2.   *B0/R1*AK12 *AK23  +1./R1*AK23*
2      AK23  +B0*B0/R2*AK22*AK22   +B0*B0/R3*AK23*AK23  -S1
 RETURN
 END
 FUNCTION  AK23D( AK11, AK12, AK13,  AK22,  AK23,  AK33 )
 COMMON  A0, A, B0, B,  R1, R2,  R3,   Q
 AK23D=-2.*A0*AK23   +2.*B0*B0/R1*AK11*AK23   +2.*B0/R1*AK13*AK23
1      +B0*B0/R1*AK12*AK13     +B0/R1*AK12*AK33   +B0/R1*AK23*AK13
2      +1./R1*AK23*AK33   +B0*B0/R2*AK22*AK23   +B0*B0/R3*AK23*AK33
 RETURN
 END
 FUNCTION  AK33D( AK11, AK13, AK23, AK33,  S2 )
 COMMON  A0, A, B0, B,  R1,  R2, R3,  Q
 AK33D=-2.*A0*AK33  +2.*B0*B0/R1*AK11*AK33  +3.*B0/R1*AK13*AK33
1      +B0*B0/R1*AK13*AK13  +B0/R1*AK13*AK33   +1./R1*AK33*AK33
2      +B0*B0/R2*AK23*AK23  +B0*B0/R3*AK33*AK33   -S2
 RETURN
 END
 FUNCTION  UT( AK11, AK12, AK13, AK23, AK33, X, V1 , V2 )
 COMMON   A0, A, B0, B, R1, R2, R3,   Q
 UT = -1./R1*( B0*(AK11*X +AK12*V1 +AK13*V2) +AK13*X  +AK23*V1
1                +AK33*V2 )
 RETURN
 END
 FUNCTION  W1T( AK12 , AK22 , AK23, X,  V1 , V2 )
 COMMON  A0,  A,  B0, B,  R1,  R2,  R3,  Q
 W1T = -B0/R2*( AK12*X  +AK22*V1  +AK23*V2 )
 RETURN
 END
```

```
      FUNCTION  W2T ( AK13,  AK23,  AK33,  X, V1 , V2)
      COMMON  AO, A,  BO,  B,   R1,  R2,  R3,   Q
      W2T =-BO/R3*( AK13*X  +AK23*V1  +AK33*V2 )
      RETURN
      END
      FUNCTION  XDOT( AK11,AK12, AK13, AK23, AK33,  X, V1, V2 )
      COMMON  AO, A,  BO, B, R1, R2, R3,  Q
      XDOT =  A*X     -B/R1*( BO*(AK11*X +AK12*V1 +AK13*V2 )  +AK13*X
     1                     +AK23*V1  +AK33*V2 )
      RETURN
      END
      FUNCTION  V1DOT( AK11, AK12, AK13, AK22, AK23 , X,  V1, V2 )
      COMMON  AO, A, BO,  B,  R1, R2,  R3,  Q
      V1DOT= ( AO -BO*BO/R1*AK11 -BO/R1*AK13 -BO*BO/R2*AK22)*V1
     1        +( 1.-BO*BO/R2*AK12)*X  -BO*BO/R2*AK23*V2
      RETURN
      END
      FUNCTION  V2DOT ( AK11, AK12,AK13, AK23, AK33, X, V1, V2)
      COMMON   AO, A, BO, B,  R1, R2, R3,  Q
      V2DOT= (AO -BO*BO/R1*AK11 -2.*BO/R1*AK13  -BO*BO/R3*AK33
     1       -1./R1*AK33)*V2  +(-BO*BO/R3*AK23 -BO/R1*AK12 -1./R1*AK23)
     2       *V1  +(-BO*BO/R3*AK13  -BO/R1*AK11 -1./R1*AK13)*X
      RETURN
      END
```