OPTIMUM CONTINUOUS CONTROL
BY A
METHOD OF PARAMETERIZATION

A Dissertation
Presented to
the Faculty of The Cullen College of Engineering
University of Houston

In Partial Fulfillment
of the Requirements for the Degree
Doctor of Philosophy

By

J. Denton Tarbet
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ABSTRACT

A method is developed in this thesis which utilizes a parameter search iteration scheme to determine the continuous control which will yield an optimal cost for a multistage process. The method of parameterization presented is defined to be a technique which considers the state at any stage of the process to be a function of the state at the preceding stage, the process equations derived for the control variables, and the parameters which are chosen to yield extremal values of some cost model. The free parameters are the initial conditions for the process equations of control and the duration of the process. The process equations for the control are obtained by applying the necessary conditions for optimality and converting the problem to a two point boundary value problem. The parameters (the missing initial conditions and the duration of the process) are determined by numerically selecting a set of values such that the sum of the squares of the dissatisfactions in meeting the necessary conditions for optimality and the boundary conditions is minimized.

The problems of multiburn spacecraft transfers can be solved by using the method of parameterization. An obstacle to obtaining optimal control for such problems is the discontinuity in the control which occurs when the engine is shut down or restarted. For such cases the method of parameterization converges to an optimal cost with little sensitivity to errors in the assumed initial conditions for the free parameters.

To demonstrate the widely different orbit transfer problems which can be solved using the method of parameterization developed in this report, the following cases were used as examples:

1. A transfer between coplanar, near circular orbits. The results are compared with those obtained by using a quasilinearization method.

2. Two transfers from an initial staging altitude to circular orbits of three and ten earth radii, respectively. The ten earth radii transfer was extremely sensitive to perturbations in the free parameters.

3. A three-dimensional transfer. The boundary conditions in this case were near circular orbits inclined 45 degrees to each other.

4. The continuous burn transfer with no points of discontinuity. The transfer was from the end of a launch phase to a circular orbit.

The assumed initial conditions were derived in all but the last case from the solution of the same transfer by an available optimal impulse routine. The method of parameterization converged in every example attempted using the initial conditions so derived. The optimal impulsive solution is the theoretical minimum velocity change transfer. Comparisons were made between the velocity change required for the finite burn case and that of the impulsive solution. In all cases of orbit to orbit transfer the optimal finite burn velocity change was extremely close to that for the optimal impulsive solution.

Initial conditions for the impulsive starter are an initial and terminal orbit definition and an estimate of the total transfer time required for the mission. Use of the optimal impulsive solutions as starting estimates eliminates the need to specify initial conditions for a set of Lagrange multipliers. Hence, it is possible to obtain solutions for orbit transfer problems without a depth of knowledge in the area of optimization of continuous systems.

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CHAPTER 1

INTRODUCTION

1.1 General Discussion

During the past 30 years, a number of interesting and significant activities have been defined in terms of a set of variables which is processed by a definite law from some epoch through a series of sequential stages, or time periods, to a specified set of terminal conditions. This class of problems is generally referred to as multistage problems. The variables which progress according to a set law from a given set of initial conditions to a set of terminal conditions are called the state variables for the system. If it is possible to exert influence, or control, over the state of the system as it progresses to obtain a desired result, a decision problem exists. For example, if it is desired to row a boat from shore to an island in a river with a constant current, it will be necessary to control the boat properly. In such cases the future state of the system is a function of the present state, the control exerted on the process, and the duration of the process. The dynamic system can be described by a process equation, such as:

$$\frac{dx}{dt} = f(x,u,t) \qquad \text{1-1}$$

$$x\left(t_o\right) = C_1$$

where x is an n-vector of state variables, u is an m-vector of control variables, and t is the independent variable.

The general decision problem requires the definition of a cost model or performance index for the system. The performance index is usually a scalar function of some combination of the initial and/or terminal state and the duration of the process. Extremizing the performance index will be the criterion upon which the selection of the control variables will be made. For example, the performance index to be extremized for an industrial complex is often the cost of operating the process by an "optimal" allocation of men and/or materials. In space flight applications it is often advantageous to perform a given vehicle maneuver in such a manner as to use the least amount of fuel and still satisfy the specified boundary conditions.

The decision problem involves selecting the control variables in a manner such that an extremal value is obtained from the performance index. Simultaneously, the state must be forced to satisfy the boundary conditions on the system. In parameter optimization the control parameters are: the missing initial conditions if any, perturbations which may be made to the control at intermediate stages of the process, and the duration of the process. In variational optimization the control must be determined and applied continuously over the interval of interest for the continuous problem. Applying the necessary conditions for optimality, the control can be expressed as a function of the initial conditions and the duration of the process. This will allow parameter and variational optimization to be considered in a similar manner.

The most general form of a cost model is the Bolza[1]* form, that is, minimize J, where

$$J = \phi(t,x,x',u)\Big]_a^b + \int_a^b L(t,x,x',u)dt \qquad 1\text{-}2$$

where a and b are the lower and upper limits on the independent variable, $(a \leq t \leq b)$, $x' = dx/dt$, ϕ is a scalar function of the boundary conditions, and the symbol $]_a^b$ implies a summation of the evaluation of the function at $t = a$ and $t = b$. If ϕ is identically zero, the cost model is referred to as being in Lagrange form. A Mayer form results if the instantaneous cost L is identically zero over the interval $(a \leq t \leq b)$ while $\phi \neq 0$.

Until the advent of the computer, only the simplest of dynamic problems requiring application of the calculus of variations in order to obtain solutions to Eq. 1-2 were considered. Since the early fifties, a noticeable advance in numerical techniques has given rise to solutions, through the use of approximations, to very complex systems.

1.2 Methods of Solution

Extremal solutions to the multistage process are obtained through the application of dynamic programming,[2] linear programming,[3] direct optimization methods such as a gradient technique,[4] and indirect methods such as the method of perturbation functions or quasilinearization.[5] As defined by Edelbaum[6] "an indirect method is considered to be a method where the minimum is sought by means of a necessary condition for the minimum. A direct method, on the other hand, is a method

*The superscript numeral refers to an entry in the bibliography.

that depends upon direct comparison of the values of the function at two or more points." Generally, dynamic programming is considered to be of limited usefulness due to excessive core storage requirements for multi-dimensional problems. The linear programming algorithm is restricted to linear cost models and constraining equations. The requirement for linear approximations for dynamic models would be too severe to allow anything but limited use of linear programming.

The definition of indirect optimization methods infers the need to establish a set of necessary conditions on the solution which will be satisfied when a local extremal value of the performance index is obtained. The extremal solution must also satisfy the boundary conditions on the state at both the initial and terminal times. Simultaneously, the solution must obey the process equations throughout the duration of the process. A classical technique to accomplish all three sets of conditions is to adjoin the process equations to the performance index with a set of unknown, time varying multipliers, or co-state variables. These co-state variables are the classical Lagrange multipliers. The decision problem becomes a two-point boundary value problem since the state is usually known at the initial time while the co-state is usually known at the terminal time.

Determining the missing initial conditions on the co-state variables presents a difficult problem.(5) The usual techniques for determining the missing initial conditions and other control variables in order to extremize the cost model are iterative schemes often based on Newton-Raphson type methods. Experience has shown that the problem of selecting a set of missing initial conditions which will yield solutions within the relatively small convergence space of most iteration schemes becomes extremely difficult for a wide class of problems.[5] The direct methods, of which the gradient method is probably the best known, will tend toward convergence within the limits of accuracy on the computer used. Convergence of such methods is often very slow as a local extremal is approached. For example, gradient methods often oscillate excessively as the state approaches the true extremal state.

1.3 Developments in Optimization

The classical problem is to extremize a cost model $J = f(x,x',u,t)$ subject to the process equations for the system and the given boundary conditions. The optimization of dynamic systems, especially in aerospace applications, has taken two paths. The classical calculus of variations and the use of dynamic models for the process equations will be referred to as the first approach. The methods of quasilinearization and perturbation fall into this class. The second approach utilizes approximations which, due to the simplifications, often allow analytical solutions to be calculated between the boundary conditions rather than being forced to rely upon numerical solutions. In trajectory analysis the impulsive approximations to the burns of a spacecraft are of this class.

The two approaches have advantages and disadvantages. The advantage of the first approach is a more realistic real-time control with the capability of increasing the complexity of the modeling equations rather easily. However, obtaining initial conditions which will converge to a feasible solution by using the iteration techniques available is an extremely difficult problem. An advantage of the second approach is that usually the search for an extremal solution can be reduced to a parameter search. The searching techniques which have been developed for parameter searching are powerful. The approximations, however, must be carefully examined to determine if the reduced solution truly represents a limiting solution for the desired dynamic model.

The two approaches have developed simultaneously. Both approaches are widely used, especially in the study of aerospace problems. A particular use has been the design of missions for the National Aeronautics and Space Administration space programs. As optimal use of fuel becomes more critical due to the increasing complexity of future missions and the requirement for bigger payloads, more efficient optimization techniques will be required. An approach is needed which yields the accuracy and flexibility of the first approach. The approach must also have the sure convergence from easily derived initial conditions which is characteristic of the second approach. An aerospace technician would not have to be an expert in optimization theory to use such an approach in mission design analysis. The development of such an approach, referred to as a method of parameterization, is the purpose of this report.

Understanding the concepts used in parameterization as applied to the multiburn problem requires a background in both approaches. A short literature survey of both methods is included as the ground work for the technique of parameterization.

1.3.1 Variational Optimization of Thrust Limited Trajectories

A paper published in 1965 by Paiewonsky[7] presents a comprehensive review and bibliography of the literature then available in the field of variational optimal control of aerospace systems. In June 1966 Lewallen[5] gave a historical review of the numerical optimization techniques for continuous systems by using both direct and indirect methods. The paper also gives an excellent comparison of some of the more common convergence techniques applied to aerospace systems. A thorough discussion of the relative convergence characteristics of each method is also included.

The multiple-burn problem for a spacecraft in a transfer orbit presents the added problem of determining the switch times (times at which the engine is turned on or off) as well as the duration of the total transfer. The difficulty is the discontinuity in the control, that is, the start up and shut down of the engine. McCue,[8] using approximations suggested by Robbins,[9] obtained starting guesses for the missing initial conditions and switch times to use in a quasilinearization solution to a two-burn coplanar elliptical transfer. McCue made note of the extreme sensitivity of the solution to slight perturbations in the switch times. The convergence envelope was very small, requiring extremely accurate first guesses at the value of the missing initial conditions.

Using a function representing the switch times (called the switch function), Kern[10] developed a two-loop optimization technique for the coplanar elliptical transfers. A two-loop optimization is a process where a part of the control variables is extremized as an inside loop and then held constant while the remaining control variables are extremized on the outer loop. Kern's approach was to assume a set of switch times as outer loop variables and then solve the continuous, thrust limited, minimum fuel transfer using those switch times. The outer loop, that is, the determination of the optimal switch times, was then extremized by perturbing the initial value of the switch function in a manner to reduce the performance index and to satisfy simultaneously the conditions on the switch function which must be met during the process. Kelley, et al.(11) applied a second-order technique to the multiburn problem with the time of the coast period as a parameter in an outer loop and the number of burns assumed known. O'Neill(12) applied an indirect technique to the two-burn trajectory for an interplanetary problem using the switch times as parameters. However, the actual corrections to reduce the value of the cost model were made as a function of the midpoint and the duration of the coast. The criterion used to determine the best switch times was the difference in the switch function evaluated at the ends of the coast arc.

1.3.2 Optimization of Finite Thrust Trajectories Using Impulsive Approximations

In the early sixties, Lawden(13) derived a set of necessary conditions which would hold on an optimal trajectory if the finite burns were replaced by impulsive approximations. A discussion of the "Primer Vector," the vector of multipliers associated with the velocity vector for optimal and nonoptimal trajectories, was made in 1967 by Lion and Handelsman.(14) Jezewski and Rozendaal(15) incorporated a Davidon-type conjugate gradient algorithm into the impulsive analysis to obtain a very efficient method for converging to locally optimal multi-impulse transfers.

The conjugate gradient algorithms are efficient methods for obtaining extremal values of a cost model, which is a function of a set of variables. The multi-impulse analysis uses the Davidon method as derived by Davidon.[16] The proof of theoretical convergence of the Davidon algorithm was given by Fletcher and Powell.[17] The algorithm was programmed and used by Johnson.[18] The Davidon algorithm begins in a gradient approach to reduce the cost model and builds internally a matrix which in the limit approaches the inverse of the matrix of second partials similar to the matrix which would be required by a second-order iteration process. The algorithm, through approximations, yields the best qualities of both direct and indirect optimization schemes, that is, certain initial convergence in a gradient sense and approximately second-order convergence in the terminal stages. The conjugate gradient iterator will yield a local extremal value for the cost model for every set of initial conditions. It should be noted that the number of parameters to be searched affects the rate of convergence.

Early investigation of numerical techniques for obtaining optimal solutions to a scalar cost model pointed to the fact that initial conditions which would allow a solution are very difficult to obtain. A method is definitely needed which will remove the need for "guessing" the missing initial conditions and allow convergence in general to a feasible solution.

Handelsman[19] formulated the starting multipliers for a continuous solution using the results of an optimal impulsive approximation. Robbins[9] determined an initial guess for the burn time and hence the switch times from an assumption that the impulsive maneuver occurs at the centroid of the burn time.

1.4 Purpose of the Investigation

The problem of obtaining accurate initial approximations to the missing initial conditions is inherent in most trajectory optimization problems. If the accuracy of the final solution obtained by using an indirect optimization approach is desired, a more powerful starting technique is necessary to give the method usefulness in a mission design sense. The impulsive approximations to the multiburn transfers used by Jezewski and Rozendaal[15] are very efficient and yield a local extremal. A requirement of the Davidon algorithm used to obtain a local extremal is a cost model that is a function of a set of parameters instead of some time varying control. The indirect techniques have the more accurate time varying control but poor initial convergence characteristics. The impulsive approximations have powerful convergence capabilities but become accurate in the limit as the ratio of burn time to trip time goes to zero.

The goal of this study was to find a technique to apply the parameter searching method, such as used in ref. 18, to multiburn transfer problems with finite thrust. The result was required to be comparable to using an indirect optimization method, but it did not require the accurate initial approximations to the missing initial conditions.

Careful analysis of the convergence scheme presented by Johnson[18] indicated a way to achieve the goal. If the control of the multiburn problem in question could be expressed as a set of parameters rather than time varying functions, the Davidon method could be used. Treating the set of unknown initial conditions, the duration of the process, and the points of occurrence of any discontinuities as free parameters, a pseudo cost model was formed which expressed the dissatisfaction in meeting the boundary conditions, the necessary conditions for optimality throughout the process, and the criteria which must hold across the points of discontinuity. Driving this function to zero insured satisfaction of all of the trajectory requirements and the necessary conditions for optimality. In the limit of convergence, the solution is the same as one obtained by using a variational analysis.

Problems chosen to study the technique were multiburn transfer trajectories. By using results from an optimal impulse solution for the transfers, starting conditions were obtained for the finite thrust bang-bang control problem. The method of parameterization is based on work done by Brown, Harrold, and Johnson[20] who used a Newton-Raphson type iteration scheme. A set of gradients was derived, relating reductions in the pseudo cost model to perturbations in the free parameters. The Davidon algorithm was then applied to the parameter search with the result that an extremal value could be obtained with no requirement for guessing the missing initial values of the Lagrange multipliers.

1.5 Scope of the Investigation

Solutions are given for several example problems to illustrate the technique and at the same time show the ease with which solutions are obtained. As the first example, the problem presented by McCue[8] was used for comparison purposes. Parameterization yielded results which were the same as the solutions given by McCue. However, parameterization used as initial conditions the multipliers and switch times derived directly from one application of the optimal multi-impulse program of ref. 21. The next two examples studied were transfers to a three-earth-radii circular and a ten-earth-radii circular orbit from a staging altitude. The three-earth-radii transfer is typical of transfers which will be required for shuttle-type spacecraft and required approximately 2.5 minutes of Univac 1108 time. The ten-earth-radii transfer required approximately 30 minutes. However, a ten-earth-radii terminal orbit would probably be an upper limit for realistic problems. The time is approximate because computer runs were made in short stages to allow monitoring of the convergence, and a large amount of printing was done. An out-of-plane problem was solved as the fourth example to show the three-dimensional capability of the method. The final example studied was a problem of a continuous burn launch to orbit. This result was obtained to assure the capability of solving continuous burn trajectories.

CHAPTER 2

FORMULATING A CONTINUOUS OPTIMIZATION PROBLEM FOR SOLUTION BY PARAMETERIZATION

2.1 The Method of Parameterization

The problem of parameterizing the control of a continuous system in order to obtain an optimal value of the cost model has two major parts. First, the time varying control variables must be expressed as a set of parameters at some epoch in time. Second, it is necessary to associate a cost constraint with each parameter that is to be free. The method of changing the initial estimate of the free parameters is derived in a manner to force the cost model to within an acceptable tolerance of zero. In the limit the extremal value of the performance index obtained by the parameter search method must also satisfy the first necessary conditions for an optimal process and force the terminal state to meet the specified boundary conditions.

2.2 Necessary Conditions for Optimization of a Parameterized Control Problem

Necessary conditions for the set of parameters which extremize a cost model are derived by ordinary calculus. For the parameter situation the process equations are of the form

$$x_{i+1} = f\left(x_i, u\right) \qquad 2\text{-}1$$

where x is the n-vector of the state variables, u is the m-vector of control parameters, and the subscript i denotes the $i^{\underline{th}}$ stage of an N-stage process. The scalar performance index has the form

$$J = g\left(x_i, u\right) \qquad 2\text{-}2$$

The extremal value of J is obtained by varying the set of parameters u. For u* to be a set of parameters which yield a minimum for J, it is necessary that the vector of the first derivative of J with respect to the control

$$\frac{dJ}{du} = \frac{d\ g\left(x_i, u\right)}{du} \qquad 2\text{-}3$$

evaluated at u = u* be identically zero, that is,

$$\left.\frac{d\ g\left(x_i, u\right)}{du}\right|_{u=u^*} = 0 \qquad 2\text{-}4$$

A typical example where Eq. 2-4 might be applied to obtain an optimal result is the case of an allocation problem. For example, an industrial firm may have a given product to produce with a certain resource of raw material and capital. The cost model (J) would be to maximize the long-term return. The state (x) would be the current status of the amount of raw material and capital available. The control vector (u) would be the decision vector of percentages of each corresponding resource to allocate to the production of the given product. The process equations (Eq. 2-1) give the state at the $(i+1)^{\underline{st}}$ stage in terms of the state at the $i^{\underline{th}}$ stage and the decisions made at that time. Eq. 2-4 is a first necessary condition for an optimal process which is a function of a set of parameters.

2.3 Necessary Conditions for Optimization of a Continuous Problem

The necessary conditions for optimal control of a continuous system can be derived in a manner similar to Lewallen.[5] Consider the problem of determining the history of the control variables of a nonlinear system in order to yield an optimal value for some scalar cost model of the system. The solution must at the same time force the system to satisfy a given set of boundary conditions. The cost model in the general Bolza form is a function of the terminal state, the duration of the process, and some integral function of the state during the process.

The nonlinear process equations of the system may be of several types, such as:

1.	$dx/dt = g(x,t)$	differential equations
2.	$x(t) = g(x(t - \Delta t))$	difference equations
3.	$dx/dt = h(x(t),x(t - \Delta t))$	differential difference equations
4.	$x(t) = u(t) + \int g(x(s),t)ds$	integral equations

Each situation requires a particular type of process equation. The problems solved in this investigation are of type 1. The first necessary conditions are derived for such a process. The extension to the other types of process equations would follow a similar approach.

The differential equations of motion for trajectory analysis are generally of second order. Reducing the equations to first order allows the problem to be formulated as a first-order, nonlinear, ordinary, vector differential equation.

$$\dot{x} = f(x,u,t) \qquad 2\text{-}5$$

where x is an n-vector of time varying state variables, f is an n-vector of known process equations, u is an m-vector of control variables, and t is the independent variable time. The performance index in the general Bolza form is

$$J(x,u,t) = \phi\left(x_f,t_f\right) + \int_{t_o}^{t_f} L(x,t)dt \qquad 2\text{-}6$$

where ϕ is a scalar function of the terminal state and L is the instantaneous rate of cost at any value of the independent variable $t_o \le t \le t_f$.

The process is subject to a set of p initial conditions

$$\eta = \eta\left(x_o,t_o\right) \qquad 2\text{-}7$$

and a set of q terminal conditions

$$\psi = \psi\left(x_f,t_f\right) \qquad 2\text{-}8$$

The boundary conditions and the intermediate constraining equations of motion, Eq. 2-5, may be adjoined to the cost model through the use of a set of Lagrange multipliers. The cost model can now be written

$$J(x,u,t) = R\left(x_o,t_o,x_f,t_f,\mu,\nu\right) + \int_{t_o}^{t_f} \left[H(x,u,\lambda,t) - \lambda^T\dot{x}\right]dt \qquad 2\text{-}9$$

where

$$R = \phi\left(x_f,t_f\right) + \mu^T\eta\left(x_o,t_o\right) + \nu^T\psi\left(x_f,t_f\right) \qquad 2\text{-}10$$

The p-vector μ is the constant vector of Lagrange multipliers associated with the initial condition vector, ν is the q-vector of constant

Lagrange multipliers adjoining the terminal constraints, and λ is an n-vector of time dependent Lagrange multipliers. H is the generalized Hamiltonian for the system

$$H = L(x,u,t) + \lambda^T f(x,u,t) \tag{2-11}$$

The functional equation (Eq. 2-9) is now expanded in a Taylor series about some nominal trajectory

$$dJ = dJ' + dJ'' + \ldots \tag{2-12}$$

where dJ' is the first variation of J about the nominal path. A first necessary condition for a weak relative extremal is that the first variation of the functional vanish, that is,

$$dJ' = dR + d\left\{\int_{t_o}^{t_f} \left[H - \lambda^T \dot{x}\right] dt\right\} = 0 \tag{2-13}$$

Taking the indicated derivatives, Eq. 2-13 may be written

$$dJ' = \left(R_\mu\, d\mu + R_\nu\, d\nu + R_x\, dx + R_t\, dt\right) + \int_{t_o}^{t_f} \left[dH - d\left(\lambda^T \dot{x}\right)\right] dt \tag{2-14}$$

where

$$dH = \frac{\partial H}{\partial u}\, du + \frac{\partial H}{\partial \lambda}\, d\lambda + \frac{\partial H}{\partial x}\, dx + \frac{\partial H}{\partial t}\, dt$$

Examining the integral term of Eq. 2-14

$$\int_{t_o}^{t_f} \left[dH - d\left(\lambda^T \dot{x}\right)\right] dt = \int_{t_o}^{t_f} \left[dH - d\lambda^T \dot{x} - \lambda^T\, d\dot{x}\right] dt \tag{2-15}$$

Separating the integral and integrating the final term by parts

$$-\int_{t_o}^{t_f} \lambda^T d\dot{x}\ dt = -\left[\lambda^T dx\right]\Big|_{t_o}^{t_f} + \int_{t_o}^{t_f} \left[\dot{\lambda}^T dx\right] dt \qquad \text{2-16}$$

Eq. 2-14 can now be written

$$dJ'(x,u,t) = \left(R_\mu d\mu\right)_{t_o} + \left(R_\nu d\nu\right)_{t_f} + \left[\left(R_t + H\right)dt + \left(R_x - \lambda^T\right)dx\right]_{t_o}^{t_f}$$

$$+ \int_{t_o}^{t_f} \left[\left(H_x + \dot{\lambda}^T\right)dx + H_\mu d\mu + \left(H_\lambda - \dot{x}\right)d\lambda^T\right]dt \qquad \text{2-17}$$

Recall that for an extremal value of J, it is necessary that dJ' = 0. Each term of Eq. 2-17 is independent; hence, each term must vanish identically. It follows that a set of necessary conditions for an extremal of a continuous cost model under variational control are:

At $t = t_o$

1. $\left(\phi_t + \mu^T \eta_t + H\right) = 0$ 2-18

2. $\left(\phi_x + \mu^T \eta_x - \lambda^T\right) = 0$ 2-19

3. $\eta\left(x_o, t_o\right) = 0$ 2-20

At $t = t_f$

4. $\left(\phi_t + \nu^T \psi_t + H\right) = 0$ 2-21

5. $\left(\phi_x + \nu^T \psi_x - \lambda^T\right) = 0$ 2-22

6. $\psi\left(x_f, t_f\right) = 0$ 2-23

For $t_o \leq t \leq t_f$

$$7. \quad \dot{x} = H_\lambda^T(x,u,\lambda,t) \qquad 2\text{-}24$$

$$8. \quad \dot{\lambda}^T = -H_x^T(x,u,\lambda,t) \qquad 2\text{-}25$$

$$9. \quad H_u^T(x,u,\lambda,t) = 0 \qquad 2\text{-}26$$

The 2n+2+p+q conditions given by Eqs. 2-18 through 2-26 are in theory sufficient to solve for the 2n histories x(t) and λ(t) from the 2n first-order process equations, the values of t_o and t_f, and the p+q Lagrange multipliers. The m relations, Eq. 2-26, yield the process equations for the histories of the m control variables u.

2.4 Expressing the Control as a Set of Parameters

The control of a continuous or multistage system is an m-vector of time dependent variables. The application of parameterization requires that the time varying control be expressed in terms of a parameter at some time epoch. This can be done by expressing the time varying control as a function of the initial state and/or co-state. The state and co-state obey a set of process equations from the initial state to the terminal state. Hence, if a set of initial values can be determined which yields an extremal value for the cost model, the time varying control can be calculated for all t in $t_o \leq t \leq t_f$. Eqs. 2-24, 2-25, and 2-26 must be satisfied at every point on the extremal trajectory. The classical optimality condition

$$H_u(x,u,\lambda,t) = \frac{\partial H}{\partial u} = 0 \qquad 2\text{-}27$$

may be solved for the m control variables in terms of the state and co-state variables and time. The state and co-state variables obey a set of process equations from t_o to any time t, and if the initial conditions for both are known, the control is determined. However, since the problem is reduced to a two-point boundary value problem, some of the initial conditions are missing. These missing initial conditions are considered as free parameters.

The reduction of the problem to a classical two-point boundary value problem can be accomplished in a manner similar to that of ref. 5. The problem is reformulated in terms of an ordinary, first-order, nonlinear, vector differential equation

$$\dot{z} = F(z,t) \qquad 2\text{-}28$$

where z is a 2n-vector made up of the n state variables and their associated n co-state variables of Lagrange multipliers, that is,

$$\dot{z} = \begin{pmatrix} \dot{x} \\ \hdashline \dot{\lambda} \end{pmatrix} = \begin{pmatrix} H_\lambda^T(x,\lambda,t) \\ \hdashline -H_x(x,\lambda,t) \end{pmatrix} = F(z,t) \qquad 2\text{-}29$$

Note that the control shown in the process Eqs. 2-24 and 2-25 has been eliminated by using the optimality condition $H_u = 0$. The set of p initially specified constraints

$$\eta\left(z_o,t_o\right) = 0 \qquad 2\text{-}30$$

is assumed to be satisfied. Hence, only n-p of the initial relations must be obtained from the transversality conditions of Eqs. 2-18, 2-19, and 2-20. The resulting n conditions at the initial time are represented as

$$g\left(z_o,t_o\right) = 0 \qquad 2\text{-}31$$

The remaining conditions necessary to solve the two-point boundary value problem are evaluated at the terminal time. The result is a set of parameters, that is, the missing initial conditions, which must be determined in a manner which satisfies all the necessary conditions. The time varying control, $u(t)$, can then be calculated from the trajectory generated.

Additional parameters are the duration of the process (t_f) and the points of occurrence of any discontinuities in various elements of the state of the system. The trajectory problems assume the initial state is known at time $t = t_o = 0$. Points of discontinuities in trajectory problems are the switch times (t_j) when the engine is started or shut down.

2.5 Solution Techniques

Parameterization yields a problem of a multivariable search for a set of parameters which will yield a local extremal value for the cost model and consequently also will satisfy the first necessary conditions for optimality. The search is made by determining a set of constraints (Y^*) which must satisfy the first necessary conditions and the boundary conditions on the problem. A constraint must exist for each free parameter. An initial estimate of the parameters yields a value (Y) which in general will not satisfy the constraints Y^*. Defining a pseudo cost model as

$$\tilde{J} = \sum_{j=1}^{M} \left(Y_j - Y_j^*\right)^2 \qquad \text{2-32}$$

the problem is that of determining that set of M-parameters which will minimize $\tilde{J}$. If a set of parameters is found such that $\tilde{J}$ is within a small tolerance of zero, the solution will be a locally optimal solution.

Solving for the zeroes of Eq. 2-32 may be considered a problem of minimizing a function of a set of M-parameters. The solution may be obtained by the use of any of the classical multivariable searching techniques. Because of the favorable convergence characteristics, a form of the conjugate gradient method was used. The Davidon algorithm used by Jezewski and Rozendaal[15] has proved to be a powerful iteration scheme for solving such multivariable search problems. The initial search is based on a gradient direction in the function space. The algorithm builds, by approximation, an inverse matrix of partial derivatives similar to that used by a Newton-Raphson convergence scheme. Hence, in the limit, the convergence of the Davidon algorithm is second order.

The gradient-like initial searching of the Davidon algorithm allows large errors in guessing the missing initial conditions. Experience has shown that an order of magnitude error in the initial conditions of the vector of Lagrange multipliers can be tolerated and can still allow convergence to an extermal. The actual limits on allowable errors in the missing initial conditions are problem dependent. For example, a multiburn spacecraft trajectory with an extremely long coast arc separating short burn periods is very sensitive to errors in guessing the time of the first engine shut down. The convergence of the Davidon method, like all iterative searching methods, is adversely affected by too many free parameters. However, problems have been solved with up to 20 free parameters.

2.6 An Example of Parameterization

It is not necessary to solve an optimization problem by using only parameterization. The method can be used on any portion of a solution which gives difficulty when approached by classical indirect optimization methods. The classical Brachistochrone problem is well known and solutions for the case where the terminal conditions are to be specified points are readily available.(22, 23) Consider the case where it is desired to terminate the Brachistochrone on a specified constraint.

$$y_f = ax_f + b \qquad \text{2-33}$$

The solution for the example will be made as a two-loop optimization. The inner loop will be the solution of the Brachistochrone from an initial point to a specified terminal point, (x_f, y_f). The terminal point, (x_f, y_f), will be determined by the outer loop of the optimization process by using a method of parameterization. An off-the-shelf quasilinearization method is used as the technique to obtain the inner-loop solutions. Parameterization is applied to the outer loop to determine the point (x_f^*, y_f^*) which will satisfy the necessary conditions for optimality and at the same time will meet the terminal boundary constraint, Eq. 2-33.

The terminal value for y can be determined from Eq. 2-33 if x_f is known. Hence, x_f will be the parameter. One parameter free requires one constraint and therefore one term in Eq. 2-32. The constraint used is derived from the transversality condition which must hold at the final point. For the Brachistochrone to give the minimum time path from the point (x_o,y_o) to the terminal constraint of Eq. 2-33, it is necessary that the Brachistochrone intersect the terminal constraint perpendicularly (see Wilde).[24] Defining the slope on the Brachistochrone curve as

$$y' = dy/dx \tag{2-34}$$

Eq. 2-32 becomes

$$\tilde{J} = \left(y_f' - 1/a\right) \tag{2-35}$$

The parameterization is now complete with x_f the free parameter. The numerical solution is as follows:

1. Guess a value of the parameter x_f, say x_f^*.
2. Determine a corresponding y_f^* from Eq. 2-33.
3. Obtain the quasilinearization solution for a Brachistochrone from (x_o,y_o) to (x_f^*,y_f^*).
4. Evaluate y'^* and then $\tilde{J}$.
5. If $\tilde{J} \neq 0$ perturb x_f^* in some manner to decrease the value of $\tilde{J}$. (The Derivative Interpolating Method[24] was used to control the perturbations to x_f^*.)
6. Repeat steps 2 through 4 until the value of $\tilde{J}$ is less than some preselected minimum value.

The solution for a zero of Eq. 2-35 was obtained using the Derivative Interpolating Method derived by Tarbet.[25] The method is similar to cubic fit methods but retains second-order convergence with no matrix inversion required. The search method requires three starting guesses for x_f. The solution must be bracketed by the starting guesses.

The example presented in Fig. 2.1 is for an initial point of

$$x_o = 0 \text{ and } y_o = 1$$

with the terminal constraint being

$$2x_f + y_f = 4.25$$

The three initial guesses for x_f^* required by the search method were 0.5, 1.0, and 1.5. Table 2-1 shows the numerical iterations. The magnitude of the error in the terminal state is the dissatisfaction in meeting the transversality condition, that is, the value of $\tilde{J}$. The solution was made in single precision in a Univac 1108.

TABLE 2-1

THE ITERATIONS TO A SOLUTION OF THE BRACHISTOCHRONE TERMINATING ON A GIVEN LINE

	Final State			Magnitude of the Error in the Terminal State
	$x(T)$	$y(T)$	$\dot{y}$	
1.	0.5	3.25	3.6037	3.1037
2.	1.5	1.25	-.1185	.6185
3.	1.0	2.25	.95169	.45169
4.	1.1374	1.975	.60123	.10123
5.	1.1805	1.8889	.5039	.0039
6.	1.1832	1.8853	.499973	.000027
7.	1.18231	1.88537	.49999996	$.7 \times 10^{-6}$

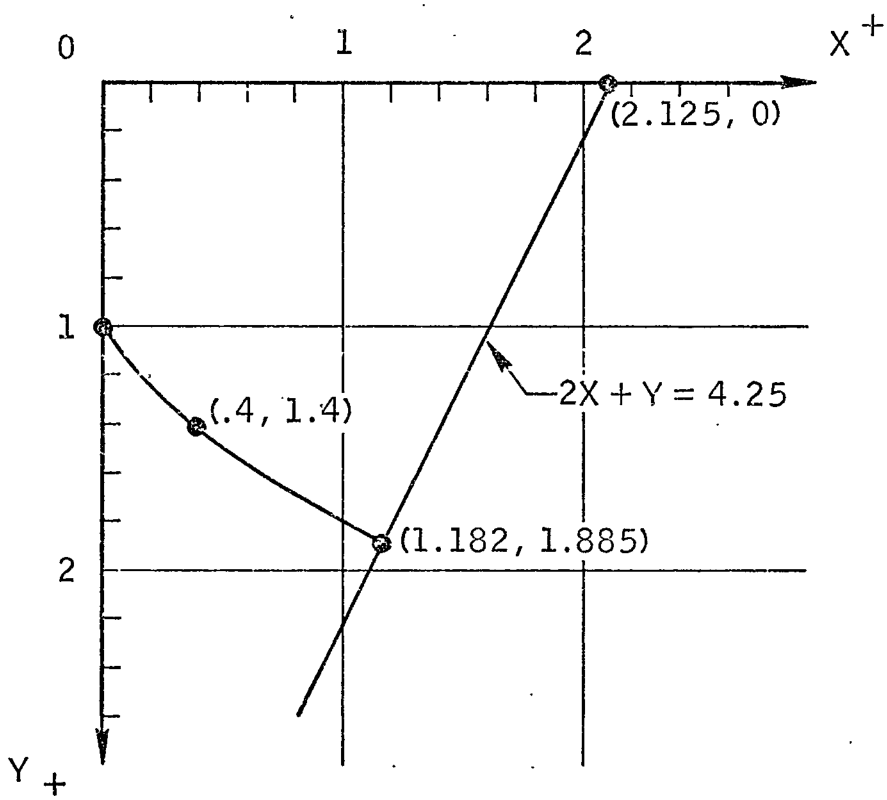


Figure 2.1.- The Brachistochrone solution terminating on a given line.

CHAPTER 3

OPTIMAL FINITE BURN SPACECRAFT TRANSFERS

3.1 Numerical Techniques for Optimal Trajectories

The multitude of numerical optimization schemes used at the present time for solving optimal spacecraft transfer problems have evolved from the simple "shooting methods." Shooting methods are optimization schemes in which the missing initial conditions for the two-point boundary value problem (described in Chapter 2) are used as parameters. A set of missing initial conditions is guessed, and a corresponding trajectory is generated. In general the terminal boundary conditions will not be satisfied by that trajectory. The process is repeated, changing the initial conditions in an attempt to satisfy the terminal boundary conditions. The various schemes such as perturbation methods or quasilinearization are approaches to improve on the shooting method. The improvement is usually made in the manner in which the guessed initial conditions are iteratively moved toward a set of values which will yield a trajectory, satisfying both the boundary conditions and the first necessary conditions for optimality.

Recent advances in the area of indirect optimization schemes have been very successful when applied to continuous burn transfers. Convergence envelopes have been expanded, and convergence speed has been improved. The solutions for a local extremal are still dependent upon the assumed initial conditions. Experience with the indirect techniques improves the user's capability to guess a set of values for the missing conditions which will converge. The convergence envelope within which the assumed initial conditions must lie is problem and formulation dependent. The convergence to a solution is difficult if corrections during the iteration process significantly violate the linearity assumptions made when the cost model is linearized to obtain the first necessary conditions for optimality.

The additional capability of engine shut down and restart complicates the picture. At a point where the thrust level is switched from max-thrust to zero thrust and back, a discontinuity in the acceleration components of the state occurs. The occurrence of a discontinuity is called a corner point in variational analysis. The methods widely used for multiburn transfers usually rely on the outside loop of a two-loop process to determine the points of occurrence of the corners. Generally, the optimal location of the corner is determined as a function of the switch function ($\mathcal{S}$); see McCue,[8] Kern,[10] or O'Neill.[12]

The problem of obtaining starting guesses which will converge to an optimal solution still exists. The problem is greatly increased in difficulty due to the sensitivity of most solutions to the duration of the burns. A method to overcome the problem is suggested by the earliest of optimization schemes, the shooting method. Formulating the multiburn problem so that the variational control is a function of the initial conditions for the Lagrange multipliers and the switch times, a parameter search can be made for a set of the initial conditions and switch times which extremizes the cost model. Powerful methods, based on the conjugate gradient algorithms, for numerically locating minimums of a function of several parameters have been developed. It is possible to solve for the near optimum starting conditions for the multiburn transfer trajectory by parameterization based on the shooting method. The Davidon algorithm is the conjugate gradient method used to control the numerical search process.

3.2 Defining the Problem

The differential equations of motion for a space vehicle in a central gravitational field may be expressed as:

$$\dot{r} = v \tag{3-1}$$

$$\dot{v} = -\mu r/R^3 + (cL/m)u \tag{3-2}$$

$$\dot{m} = -L \tag{3-3}$$

where r is the position vector, R is the magnitude of the vector r, v is the velocity vector, u is the unit vector in the direction of the thrust, μ is the gravitational constant, and L is the time rate of change of vehicle mass due to propellant expenditure such that

$$0 \leq L \leq L_{max} \tag{3-4}$$

The problem in general is to determine that set of controls, u, and the duration of the process, which will allow a spacecraft moving according to the process equations (Eqs. 3-1 to 3-3) to satisfy the boundary conditions with a minimum fuel consumption. In this instance, since mass loss is due to fuel expenditure, the problem is to maximize the final mass. The inequality constraint on the mass flow rate is rewritten in a form more easily adjoined to the cost model, that is,

$$L\left(L_{max} - L\right) - \alpha^2 = 0 \tag{3-5}$$

where α is a real variable.

The differential system, composed of the eight Eqs. 3-1, 3-2, 3-3, and 3-5, is the constraints placed on the independent variable time (t), the twelve dependent variables (r, v, m, L, u, α), and the four degrees of freedom (the duration and direction of the thrust) by the physical system. For the problems considered, the free variables will be determined which, at least locally, maximize final mass. (Note: This is equivalent to minimizing the mass loss.) The solution, which can be found by using a Lagrange formulation for the calculus of variations problem, is that set of burn duration times and thrust angle histories for those burns which will minimize

$$J = +\int_{t_o}^{t_f} L\,dt \tag{3-6}$$

and simultaneously satisfy a given set of boundary conditions.

Formulating the problem in a manner similar to the two-point boundary value problem of Chapter 2, the Hamiltonian for the system is

$$H = L + \lambda^T\dot{x} + \nu\left(1 - u^Tu\right) + \eta\left[L\left(L_{max} - L\right) - \alpha^2\right] \tag{3-7}$$

where L is the instantaneous rate of cost ($-\dot{m}$), x is the state vector (v, r, m), λ is the set of adjoint or co-state variables (p, q, w) for the system, and ν and η are constants used to adjoin the remaining constraints. Expanding Eq. 3-7

$$H = L + q^Tv + p^T\left(-\mu r/R^3 + (cL/m)u\right) - wL + \nu\left(1 - u^Tu\right)$$
$$+ \eta\left[L\left(L_{max} - L\right) - \alpha^2\right] \tag{3-8}$$

From Eq. 2-22 necessary conditions for optimality with respect to the control u are given by the following equation.

$$\frac{\partial H}{\partial u} = H_u = p^T \frac{cL}{m} - 2\nu u = 0 \qquad 3\text{-}9$$

By solving the above system of linear equations, the necessary conditions are satisfied if

$$u = \pm p/P \qquad 3\text{-}10$$

where $P = |p|$. The choice of which sign to use in Eq. 3-10 is determined by applying the Weierstrass E-condition which requires that the Hamiltonian on the extremal curve be greater than the Hamiltonian on any nearby admissable curve for a maximum to occur. This would imply that

$$H_{uu} = \partial^2 H/\partial u^2 \leq 0 \qquad 3\text{-}11$$

Hence it follows that

$$u = +p/P \qquad 3\text{-}12$$

Substituting Eq. 3-12 into Eq. 3-8, the Hamiltonian becomes

$$H = -\dot{m}\left(+1 - w + P\frac{c}{m}\right) + q^T v - \frac{\mu}{R^3}\left(p^T r\right) \qquad 3\text{-}13$$

If a switch function is defined by

$$\mathcal{S} = +1 - w + \frac{c}{m} P \qquad 3\text{-}14$$

then H becomes

$$H = -\dot{m}\mathcal{S} + q^T v - \frac{\mu}{R^3}\left(p^T r\right) \qquad 3\text{-}15$$

The Hamiltonian is to be maximized over the remaining free parameter $\dot{m}$. By examining Eqs. 3-3, 3-5, and 3-15, it is noted that for values of $\mathcal{S} < 0$ any value of $\dot{m}$ other than 0 will decrease H. Hence for maximum H, the logical choice would be $\dot{m} = 0$, the lower limit. For

$\mathcal{S} > 0$, $\dot{m}$ should be chosen with as large a magnitude as possible, that is, $\dot{m} = -L_{max}$. Note that the sign of the quantity $\mathcal{S}$ or its vanishing along the trajectory, determine whether the appropriate thrust program is one of maximum thrust, null thrust, or intermediate thrust. $\mathcal{S}$ is normally termed the thrust magnitude switching function. The assumption that $\mathcal{S}$ is never zero for a finite time is made for this study. Dreyfus[26] has shown an important class of problems for which this assumption holds. Generally control having only maximum thrust or null thrust is referred to as bang-bang control. As discussed in Chapter 5 of ref. 6, the assumption that $\mathcal{S} \neq 0$ for a finite time cannot be shown to hold in general when aerodynamic forces must be considered. For example, consider the case of an atmospheric flight such as the minimum climb problem for an attack aircraft. In such a case the ability to control the vehicle with the throttle is needed as well as the ability to control the direction of thrust. Mathematically, if arcs for which $\mathcal{S} = 0$ for a finite time are included, the Weierstrass E-function, from which the necessary conditions at the points of discontinuity in the control are derived, will not yield useful information.[6]

The process equations for the co-state variables are derived by using the Euler Lagrange equation for the system (Eq. 2-25).

$$\dot{q} = r\left(\frac{-3\mu}{R^5} r^T p\right) + p\,\frac{\mu}{R^3} \qquad 3\text{-}16$$

$$\dot{p} = -q \qquad 3\text{-}17$$

$$\dot{w} = -P\left(c\dot{m}/m^2\right) \qquad 3\text{-}18$$

The problem is now posed in the form suggested in Chapter 2. The process equations are defined for both the state and co-state. The control, u, has been expressed in terms of the time varying vector, λ. The vector λ has a set of known process equations (3-16, 3-17, and 3-18); hence, only the initial values need to be determined. The terms of the psuedo cost model, for the method of parameterization, needed to help determine the optimal time of the switch from an arc of maximum thrust to a coast arc or vice versa, are derived from the switch function (Eq. 3-14).

3.3 Parameterizing the Optimization Problem

The problem of determining locally extremized multiburn transfers of a space vehicle can now be formulated as a parameterization problem. The approach of Chapter 2 has been used to express the control, u, in terms of the co-state vector. The initial conditions of the process equations for the co-state vectors can be used as free parameters. The philosophy of the method of parameterization for a multiburn trajectory optimization problem is:

1. Determine the control variables in terms of the co-state variables which obey a set of process equations from a given initial state.

2. Consider the initial values of the co-state variables, the time of occurrence of engine shut down or start up, and the total trip time as free parameters.

3. Determine a set of conditions which, if satisfied, guarantee that the boundary conditions as well as the first necessary conditions for optimality are satisfied. The same number of conditions as parameters must be determined.

4. Determine a pseudo cost model to be used by the conjugate gradient type algorithm which will be defined as the sum of squares of the dissatisfactions incurred because the trajectory does not meet the conditions from step 3.

5. Determine a gradient vector which relates changes in the pseudo cost model of step 4 to changes in the initial parameter values.

6. Apply the Davidon searching algorithm to minimize the cost function defined in step 4. The Davidon algorithm requires as inputs the cost model of step 4 and the gradient vector of step 5.

The multiburn cases considered are composed of burn and coast arcs separated by corners, that is, switching times. In Chapter 5 of ref. 6, it is shown that the thrust over the burn arcs can in general be considered constant; hence, the thrust is under bang-bang control. The time varying control variables are the thrust direction vector and its time rate of change, the duration of the coast arcs (that is, the time of engine shut down and restart), and the duration of the total transfer for the cases with a nonspecified terminal time.

The solutions by the method of parameterization are local extremal solutions. A mathematical argument to verify that the results are not global optimals is given by Brown, Harrold, and Johnson (ref. 20). To examine the problem of determining global optimality from a realistic viewpoint, consider the following: The trajectory problems are solved assuming that coasting motion is Keplerian, thrust is proportional to mass flow rate, the terminal constraints are not time dependent, and no penalty is incurred for adding coast and burn arcs; hence, the possibility of globally optimum transfers is precluded for most trajectory transfers.

For example, at any point in a trajectory, a coast of one orbit may be inserted into a burn arc with no additional cost. The result would be a different transfer with a longer total trip time but no increase in cost. In reality there are constraints on trip time, the number of engine restarts, and even the duration of the coast arcs. Existing chemical rocket engines actually lose a small amount of mass during coasting due to venting of vaporized fuel. The astronauts and onboard equipment can only function for a limited time in space. Even the first assumption would not hold over extended periods. Planet oblateness, third body effects, and other space flight perturbations would slightly influence periodicity.

Several methods may be used to remove the ambiguities created.

1. A terminal time constraint could be added.

2. A nonzero rate of cost could be forced on the coast arc.

3. A limit to the number of separate burn and coast arcs could be imposed.

Depending on the type of mission, any of the three constraints could be used to change the problem definition so that a unique optimum exists. The third constraint was used in this study. The solutions that were obtained satisfied the process equations, the imposed boundary conditions, and the necessary conditions for optimality. In addition, since terminal time was considered to be free, the added condition that the Hamiltonian (H) be identically zero over the extremal arc is satisfied.

3.4 The Switch Function

The assumption has been made that the multiburn solutions will have bang-bang control on the thrust level. The requirement of a function which will locate the switch times in an optimal manner is satisfied by the switch function as defined by Eq. 3-14, that is,

$$\mathcal{S} = +1 - w + P\frac{c}{m} \qquad \text{3-19}$$

where $P = |p|$

The necessary conditions for optimality which are placed on $\mathcal{S}$ when the thrust program is a series of either maximum or null thrusts is

1. When $\mathcal{S} > 0$, the engine is at maximum thrust.
2. When $\mathcal{S} < 0$, the engine is at null thrust.
3. $\mathcal{S} = 0$ at the instant the engine is switched on or off.

From Eq. 3-19, it follows that

$$\dot{\mathcal{S}} = -\dot{w} - P\frac{c\dot{m}}{m^2} + \frac{c}{m}\frac{dP}{dt} \qquad \text{3-20}$$

From Eq. 3-17 and 3-18

$$\dot{w} = -P\frac{cm}{m^2} = -P\frac{cm}{m^2}$$

and

$$\frac{dP}{dt} = +\frac{p\dot{p}}{P} = +\frac{p^Tq}{P}$$

Hence

$$\dot{\mathcal{S}} = +\frac{c}{m}\left(\frac{p^Tq}{P}\right) = +\frac{c}{m}\left(\frac{p^Tq}{P}\right) \qquad \text{3-21}$$

Since p, q, and m are continuous and c is a constant, $\mathcal{S}$ and $\dot{\mathcal{S}}$ are necessarily continuous functions.

Fig. 3.1 is a plot of a typical thrust program for bang-bang control of a burn-coast-burn transfer. The initial time period (t_0 to t_1) is a burn arc of maximum thrust. The null thrust arc (t_1 to t_2) is a coast arc. The final period (t_2 to t_f) is the terminal burn period at maximum thrust.

Assuming that the thrust program represented by Fig. 3.1 yields a locally optimal transfer trajectory, then $\mathscr{S}$ is positive for $t_0 < t < t_1$, negative for $t_1 < t < t_2$, and positive for $t_2 < t < t_f$. Recalling that both $\mathscr{S}$ and $\dot{\mathscr{S}}$ are continuous, the plot of the switch function will appear similar to that of Fig. 3.2. The optimal location of the t_j, the switch times, can be determined from the necessary conditions on the switch function ($\mathscr{S}$). For example, guess a switch time for the end of a burn arc (t_j^*); generate the trajectory using t_j^* as the switch time; evaluate $\mathscr{S}$ over the trajectory; check to see if $\mathscr{S}(t_j^*)$ is identically zero; if not, a new guess at t_j should be made so that $\mathscr{S}(t_j^*)_{new} < \mathscr{S}(t_j^*)_{old}$.

Another utilization of the necessary conditions on $\mathscr{S}$ for an optimal transfer would be in making a decision to add a burn or coast period. A solution for an N-burn trajectory which satisfies the boundary conditions, the optimality conditions, and forces $\mathscr{S}(t_j) = 0$ for all switch times, is locally optimal for the N-burns. However, the N-burn solution could be improved by changing to an (N+1)-burn solution if the value of $\mathscr{S}(t)$ becomes positive or zero over a coasting arc, or negative or zero over a burn arc.

The conditions for an extremal multiple burn arc follow directly from the assumption that the problems considered are time open, the necessary conditions for an extremal developed in Section 2.2, and the preceding discussion of the switch function and its derivative.

The necessary conditions are:

1. The Hamiltonian must be constant on the optimal trajectory. The time open solution requires that $H_t = 0$; hence, the constant value for the Hamiltonian is zero.

2. The switch function ($\mathcal{S}$) must:

   a. Be continuous

   b. Have continuous first derivatives

   c. Be positive on burn arcs

   d. Be negative on coast arcs

   e. Be zero at junction points of the arcs.

The most recent efforts at optimizing multiburn transfers have relied on these conditions. Usually the solution is obtained by guessing the values of the switch times. The optimal solution to the multiburn transfer with fixed switch times is then obtained. The value of $\mathcal{S}$ is calculated and, as an outside loop of the optimization process, corrections are made to the switch times in an attempt to reduce the switch function at the switch times to zero. Each time a correction to the switch times is made, a new optimal solution to the transfer with fixed switch times must be made.

The method of parameterization is an attempt to reduce the multiburn problem to a multiparameter search problem; hence, the switch times will be used as parameters along with the other missing conditions. A constraint to be satisfied must be included in the pseudo cost model for each switch time added as a free parameter to the parametric control of the optimization process.

3.5 Numerical Procedures of the Parameterization Process

The formulation of the problem into a parameterized version is accomplished in a manner suggested in Chapter 2. The necessary constraints are obtained from the boundary conditions on the problem and the necessary conditions for optimality.

The trajectory problems considered are of the class with fixed initial state, terminal constraints defined by a specified orbit, and the true anomaly at entrance into the terminal orbit free. The (2N+5) parameters, for an N-burn problem with the first arc a burn arc, are the six initial values of the co-state, the (2N-2) switch times, and the terminal time. The case with an initial coast arc yields (2N+6) parameters, since there are (2N-1) switch times.

Each parameter requires a constraint. The first five of the constraints come from the terminal boundary conditions. The first three constraints are the components of the angular momentum vector on the desired terminal orbit.

$$h = r \times v \qquad 3\text{-}22$$

Two more constraints come from the first two elements of the vector whose magnitude is the eccentricity of the orbit and whose direction is aligned with the argument of periapsis.

$$e = \left(-\frac{r}{R} + \frac{h \times v}{\mu}\right) \qquad 3\text{-}23$$

A sixth condition is taken from the transversality condition (T_v) which holds at t_f on the optimal arc.

$$T_v\left(t_f\right) = q^T v - \mu p^T r / R^3 \qquad 3\text{-}24$$

Recall from the necessary conditions on the extremal arc that the Hamiltonian must be constant over the extremal arc.

$$H = -\dot{m}\mathcal{S} + q^T v - \frac{\mu p^T r}{R^3} \qquad 3\text{-}25$$

or

$$H = -\dot{m}\mathcal{S} + T_v(t) \qquad 3\text{-}26$$

Since the intermediate parameters are the switch times, the value of H is of interest at the switch times. On the optimal trajectory, $\mathcal{S} = 0$ at each switch point; hence, in the limit as the trajectory converges to an extremal trajectory, $\mathcal{S}(t_j)$ tends to zero. $\mathcal{S}(t_j) \rightarrow 0$ implies that

$$H\left(t_j\right) = T_v\left(t_j\right) \qquad 3\text{-}27$$

where t_j is the time of the j^{th} switch. Eq. 3-26 yields the remaining N-1 constraints. Since terminal time open implies that $H_t = 0$, then $H = 0$ everywhere on the arc. Eq. 3-26 implies that $T_v(t_j)$ tends to zero at the switch times as convergence to an optimal solution occurs.

The motivation for accepting the possible slower convergence arising from neglecting the term $\dot{m}\mathcal{S}$ in evaluating H at the end of the burns is obvious if the equation for $\mathcal{S}$ is examined.

$$\mathcal{S} = +1 - w + P\frac{c}{m}$$

Evaluation of $\mathcal{S}(t)$ would require evaluating $w(t)$. Since the equation for $\mathcal{S}$ is the only requirement for $w(t)$, numerical integration of the co-state variable w is not required during the convergence process, provided the value of $\mathcal{S}$ is not required except at switch points. The execution time is decreased by the amount required to carry along the extra integration and data. This is beneficial since the state equations are integrated, as shown below, in their second-order form, and a special integration package would be needed for $\dot{w}$.

Additional N-1 constraints arise from the conditions on $\mathcal{S}$ at the start and end of the coast arcs. At each switch time (t_j), $\mathcal{S} = 0$. Since over a coast arc $\dot{m} = 0$, the values of w and m are constant. The only variable over the coast arcs is P. As P varies, $\mathcal{S}$ will vary over the coast period. Since $\mathcal{S}$ at the ends of the coast arc on the optimal trajectory has the same value, that is, $\mathcal{S}(t_j) = \mathcal{S}(t_{j+1}) = 0$, the value of P at the start and end of a coast must be identical, that is,

$$P\left(t_j\right) - P\left(t_{j+1}\right) = 0 \qquad 3\text{-}28$$

The next N-1 or N constraints, depending upon whether the first arc is a burn or a coast arc, are derived from Eq. 3-28. The final constraint is a pseudoconstraint used to force the homogeneous six vector of co-state differential equations to a unity vector.

The boundary conditions can be summarized by the following (2N+5) or (2N+6) vector Y^*. Y^* is a (2N+5) vector if the first arc is a burn arc. It is a (2N+6) vector if the first arc is a coast arc. Y^* is a vector of desired values and is based upon the boundary conditions and necessary conditions of the particular problem under investigation.

$$
\begin{pmatrix} Y_1^* \\ Y_2^* \\ Y_3^* \end{pmatrix} = \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} = \begin{pmatrix} r \times v \end{pmatrix} \qquad 3\text{-}29
$$

$$
\begin{pmatrix} Y_4^* \\ Y_5^* \end{pmatrix} = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} -r/R + h \times v/\mu \end{pmatrix} \qquad 3\text{-}30
$$

$$
Y_6^* = T_v\left(t_f\right) = 0 \qquad 3\text{-}31
$$

$$
\begin{pmatrix} Y_7^* \\ \cdot \\ \cdot \\ Y_{6+(N-1)}^* \end{pmatrix} = \begin{pmatrix} T_v\left(t_1\right) \\ \cdot \\ \cdot \\ T_v\left(t_{f-1}\right) \end{pmatrix} = 0 \qquad 3\text{-}32
$$

$$
\begin{pmatrix} Y_{6+N}^* \\ \cdot \\ \cdot \\ Y_{2N+4}^* \end{pmatrix} = \begin{pmatrix} P\left(t_1\right) - P\left(t_2\right) \\ \cdot \\ \cdot \\ P\left(t_{N-1}\right) - P\left(t_n\right) \end{pmatrix} = 0 \qquad 3\text{-}33
$$

$$
Y_{2N+5}^* = \left(\sum_{i=1}^{6} \lambda_i^2 - 1\right) = 0 \qquad 3\text{-}34
$$

If the initial arc is a coast arc, Eq. 3-33 has one more element Y_{2N+5}^*, and Eq. 3-34 is the constraint Y_{2N+6}^*. In general, the last 2N or 2N+1 terms of Y* are zero for the time open problem. If time were fixed or if the problem included a rendezvous, the value of the Hamiltonian would not be zero but some other constant across the optimal arc.

From an initial guess at the (2N+5) or (2N+6) parameters, generate a trajectory and the associated (2N+5) or (2N+6) vector Y. The vector Y has as its components the same elements as Y* except that they are evaluated on the assumed trajectory. Letting M be either 2N+5 or 2N+6 depending on the type of trajectory specified, the psuedo cost model is

$$\tilde{J} = \sum_{i=1}^{M} \left(Y_i - Y_i^*\right)^2 \qquad 3\text{-}35$$

The problem has been cast into a form to which the Davidon algorithm may be applied to search for the unknown parameters which yield $\tilde{J} = 0$.

3.6 Applying the Davidon Algorithm

The free parameters and the corresponding cost model have been defined. The remaining input needed for the Davidon algorithm is a gradient vector for the cost model as a function of the free parameters. To compute the gradient vector, it is necessary to compute numerically a trajectory and its first variations with respect to the free parameters. The trajectory is computed as a sequence of burn and coast arcs. The trajectory is numerically integrated over burn arcs and propagated analytically on coast arcs.

The second-order form of both the state and co-state equations as well as the first variations are integrated by a fourth-order Runge-Kutta scheme particularly suited to such equations. (See ref. 27, page 237.) In addition, a Richardson extrapolation technique is used to control the step size.[28] In order to control truncation error by controlling step size, a special combination of Runge-Kutta steps of size h/3 of the state equations with an overlapping step of h of both state and co-state is used. The Richardson extrapolation gives a method to determine h based upon the error which appears between the forward integration by a step of h versus a series of steps of size h/3 which should propagate the state to the same position.

The state, co-state, and their partial derivatives with respect to time are propagated along the coast arcs by a general conic formulation. For each coast arc, values of r, v, q, and p are needed at the end of the coast arc. In addition, partial derivatives of the final values with respect to the initial values are needed. The partials are a link in the chain which will eventually yield the needed gradient vector for the Davidon method.

A general closed-form solution for propagating the state in a second-order form over a coasting arc is used. In a similar manner, the computation of the state transition matrix $\partial x(t)/\partial x(t_o)$ is made. A good formulation for the closed-form solution is Goodyear's.[29] The complete formulation of the equation giving the transition matrices over coasting arcs for the co-state and the equations of the first variations are given in ref. 20.

To complete the derivation of the gradient vector, consider the following: Define the 2N+5 or 2N+6 vector of free parameters as $z^T = (\lambda_1,\ldots,\lambda_6,t_1,\ldots,t_f)$. To obtain $\partial\tilde{J}$ with respect to z, it is necessary to derive the vector

$$\frac{\partial\tilde{J}}{\partial z} = \frac{\partial\sum_{i=1}^{M}\left(Y_i - Y_i^*\right)^2}{\partial z} \qquad 3\text{-}36$$

Recall that Y* is a vector of constants for each problem situation so that

$$\frac{\partial \tilde{J}}{\partial z} = 2\sum_{i=1}^{M} \left(Y_i - Y_i^*\right) \frac{\partial Y_i}{\partial z} \qquad 3\text{-}37$$

From Eq. 3-37 the only unknown term is the partial vector relating Y_i to the set of parameters in the vector z. The problem is to determine the matrix of partials, $\partial Y_i/\partial z$. $Y(z)_{t_f}$ is the set of final values obtained from the arc flown, using as initial conditions the parameter vector z. The partial matrix will relate desired changes in terminal conditions to necessary changes in the initial conditions. Using the chain rule of differentiation

$$\frac{\partial Y(z)_{t_f}}{\partial z} = \frac{\partial Y(z)_{t_f}}{\partial x_{t_f}} \cdot \frac{\partial x(t)}{\partial z} + \frac{\partial Y(z)_{t_f}}{\partial \lambda_{t_f}} \cdot \frac{\partial \lambda(t)}{\partial z} \qquad 3\text{-}38$$

The partials obtained from Eq. 3-38 are the keys which allow the solution of the very complex multiburn transfer to be optimized in a method applicable to the Davidon algorithm. Eq. 3-38 relates the desired changes in initial conditions to needed changes in terminal constraints and the necessary conditions for optimality. The results are obtained in the following manner. By using the numerical methods described above, the second term of the differential equation on the right $(\partial x(t)/\partial z)$ is integrated forward simultaneously with the state and co-state equations over the burn arcs and propagated forward explicitly over the coast arcs. The first term, that is, the partial of the terminal conditions with respect to the terminal state, evaluated from the calculated transfer orbit, is evaluated at the terminal time (t_f). The second term related changes in the state at any time (t) to changes in the initial conditions. Multiplying the two terms together at t_f by the chain rule of differentiation yields the partials of the terminal conditions evaluated on the calculated transfer orbit with respect to the free parameters.

The vector of partials $\partial x(t)/\partial z$ is obtained as follows. If a vector system of differential equations

$$\dot{x} = f(x,t) \tag{3-39}$$

has an initial value dependent on some independent vector of parameters, z, then the partial derivative of the solution x(t) with respect to that vector satisfies the equation

$$\frac{d\{\partial x(t)/\partial z\}}{dt} = \frac{\partial f}{\partial x} \cdot \frac{\partial x(t)}{\partial z} \tag{3-40}$$

If the initial time (t_o) is not itself a function of the vector of free parameters (z), then the initial conditions for Eq. 3-40 are

$$\left.\frac{\partial x(t)}{\partial z}\right|_{t=t_o} = \frac{\partial x\left(t_0\right)}{\partial z} \tag{3-41}$$

The solution to the set of differential equations which relate the switch times to the terminal condition vector is obtained in the same manner as Eq. 3-40 except for the determination of the initial conditions. The initial conditions for those differential equations

$$\frac{\partial x(t)}{\partial t_i} = \dot{x}\left(t_i^-\right) - \dot{x}\left(t_i^+\right) \tag{3-42}$$

where the $\dot{x}(t_i^+)$ refers to the value of $\dot{x}$ a delta time step before t_i, while $\dot{x}(t_i^-)$ refers to the value of $\dot{x}$ a delta time step after t_i. To verify Eq. 3-42 consider the following.

The initial conditions for the parameters of Eq. 3-40 which are switch times are derived by observing that it is desired to obtain initial values for a continuous solution to an equation of the form of Eq. 3-40, that is,

$$\left.\frac{\partial x(t)}{\partial t_i}\right|_{t=t_i} = \underset{t\to t_i}{\text{Limit}} \left(\frac{\partial x(t)}{\partial t_i}\right) \qquad 3\text{-}43$$

Assuming that $\dot{x} = f_1(x(t))$ directly prior to t_i and $\dot{x} = f_2(x(t))$ immediately after t_i and relying on equation 3-43, the relation between x at t immediately after t_i and x at t' immediately before t_i is

$$x(t) = x(t') + f_1\Big(x\big(t_1\big)\Big)\Big(t_i - t'\Big) + f_2\Big(x\big(t_2\big)\Big)\Big(t - t_i\Big) \qquad 3\text{-}44$$

with $t_1 \varepsilon (t', t_i)$ and $t_2 \varepsilon (t_i, t)$

Differentiating x(t) with respect to t_i as needed yields

$$\frac{\partial x(t)}{\partial t_i} = f_1\Big(x\big(t_1\big)\Big) - f_2\Big(x\big(t_2\big)\Big) + 0\Big(t_i - t'\Big) + 0\Big(t - t_i\Big) \qquad 3\text{-}45$$

which in the limit as $t \to t_i$ gives

$$\frac{\partial x\big(t_i\big)}{\partial t_i} = f_1\Big(x\big(t_i^-\big)\Big) - f_2\Big(x\big(t_i^+\big)\Big) \qquad 3\text{-}46$$

With Eqs. 3-41 and 3-43 as initial conditions, the right-hand term of Eq. 3-38 can be evaluated.

The actual iteration process of the Davidon algorithm is a sequence of cycles which reduce the cost by proper selection of the free parameters. The iterator has two parts; part one is the sequence of cycles, while part two is the set of iterations which occur during each cycle. A cycle is the iteration step at which the Davidon algorithm requires, from the trajectory package, a value of the cost model, and a new set of gradients for the cost model with respect to the initial conditions. The iterations during the cycle require only values of the cost model. During the cycle, the Davidon iterator is searching for the least value of the cost model possible by using the gradient direction defined at the start of the cycle to control the search direction. When that minimum is found, the iterator will start a new cycle requiring a new set of gradients and corresponding cost model. The gradient vector will be perturbed by the Davidon iterator to allow the search process to approach that of a second-order iteration method. The trajectory package must work in two parts; part one will be executed when each cycle is started, and part two will be executed during the iterations internal to the cycle. Part one will be required to calculate the cost model and gradient vector while part two will only be required to calculate a value of the cost model. The process of solving the multiburn transfer problem by an application of the method of Davidon iterative search for a minimum is as follows:

1. A boolean switch from the iterator will signal when the gradients are required. If the gradients are required, the trajectory is generated by integrating the state and co-state equations forward numerically over the burn arcs and explicitly over the coast arcs using as initial conditions the initial state and the current values of the parameter vector z. At the same time Eq. 3-40, that is, the differential equation for the partial of the state at any time with respect to the free parameter vector, is integrated forward over the burn arc and propagated analytically over the coast arcs. The initial conditions for Eq. 3-40 are given by Eq. 3-41 for the parameters from z related to the

co-state. Eq. 3-43 gives the initial conditions for those terms related to the parameters which are switch times.

2. On the iterations internal to the cycling, only the new value of the cost model is required; hence, Eq. 3-40 will not be updated.

3. At the end of each cycle, the Davidon iterator will recycle and update the initial conditions to start from that set of initial conditions which yielded the minimum in the one-dimensional search. A new gradient will be required and the search direction will be updated.

4. The process will continue to cycle until an appropriate cut-off criterion is reached. In the case of the multiburn orbit problem using a normalized coordinate system, three checks for convergence were made.

a. If the magnitude of the gradient vector became less than 1×10^{-5}

b. If the cost model did not decrease more than 1×10^{-9} over five cycles

c. If the routine made more than 2000 iterations

The first cut-off criterion is the nominal stopping conditions. At this value of the gradient magnitude, the cost model usually has a value less than 10^{-16}. During preliminary analysis if convergence was slow or a problem developed and the cost model did not decrease on five successive cycles, the routine was halted to check for data problems or bad problem definition. Finally, an overall iteration limit was established to keep the routine from using an indefinite amount of computer time if an error in convergence should occur.

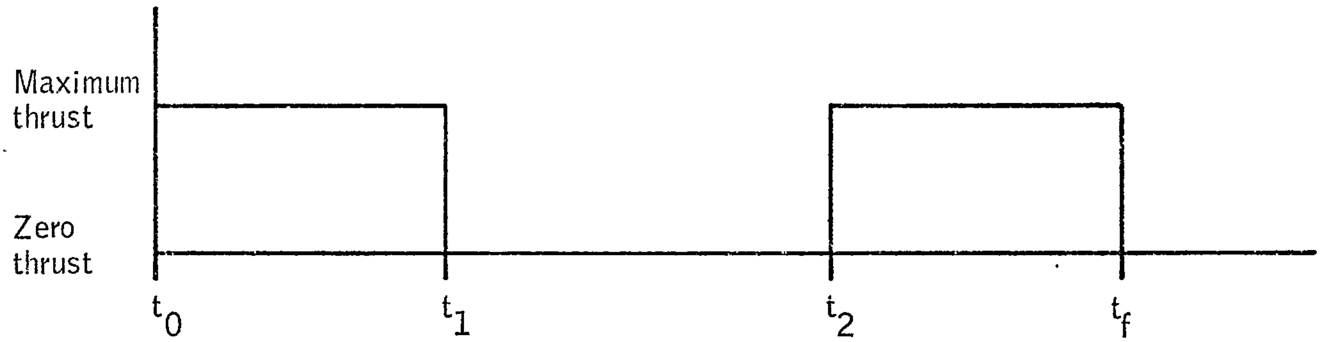


Figure 3.1.- Typical thrust program for bang-bang control.

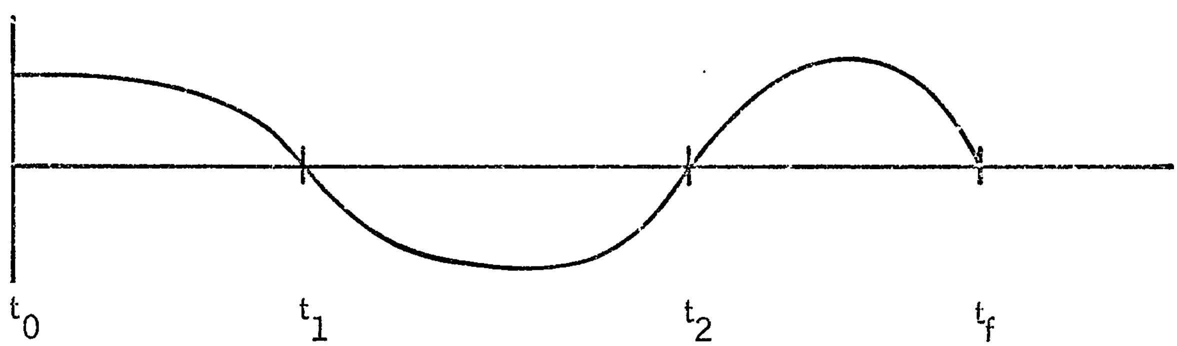


Figure 3.2.- Switch function corresponding to the thrust program of figure 3.1.

CHAPTER 4

APPLICATION OF PARAMETERIZATION TO TRAJECTORY TRANSFER PROBLEMS

4.1 General Discussion

A method has been formulated to obtain solutions to complex multi-stage control problems using parameterization. The usefulness of the technique can be determined by the ease of application and the speed of the numerical convergence to a solution. The optimal multiple burn transfer trajectory of a space vehicle is of interest in application to studies of missions such as those for a shuttle-type spacecraft. The problems chosen to verify the capabilities of the method of parameterization, therefore, were earth orbit transfers similar to those which will be analyzed in the near future for shuttle missions.

The method of parameterization, as is the case with other numerical techniques for determining optimal solutions to control problems, yields only a local extremal when convergence is obtained. In the case of coplanar orbital transfers, there are generally at least four locally optimal trajectories which satisfy the first necessary conditions of optimality.[30, 31] The type of solution obtained depends upon the orientation of the initial and terminal orbits. The cases for which the orbits are nearly coaxial but with perigee, the point of closest approach, rotated approximately 180 degrees, yield an approximate perigee to perigee transfer (Fig. 4.1 (a)) and an approximate apogee to apogee transfer (Fig. 4.1 (b)). For the case of a near coaxial set of orbits with perigee points nearly the same, the two solutions are an apogee to perigee transfer (Fig. 4.1 (c)) and a perigee to apogee transfer (Fig. 4.1 (d)).

The method of parameterization utilizing an optimal impulse starter allows an easy study of the various locally optimum solutions. Specifying initial and terminal positions near perigee of the initial and terminal orbits of coaxial orbits 180 degrees out of phase will yield a finite-burn solution near to a perigee to perigee transfer. Similarly initiating positions near apogee would yield an apogee to apogee solution. The other types of solutions may be obtained in a similar manner.

Using the Optimal Multi-Impulse for Rendezvous (OMIR - Ref. 21) Program available at the NASA Manned Spacecraft Center as a starter, the required data for each case studied are the initial and terminal state vectors, or the orbital elements of the respective orbits, a guessed value of the total transfer time, and data concerning the particular rocket engine used. The guessed transfer time used was the half period of the larger orbit.

The ease with which the solutions to the four relative minimums can be obtained would be a definite advantage for mission analysis of coplanar elliptical transfers. Knowing that there are theoretically four minimums to the coplanar elliptical transfer, it is possible to specify these points as initial conditions for the impulsive starter. The finite-burn solution will converge to the local extremal trajectory corresponding to the initial and terminal conditions used as starters for the impulsive solution.

4.2 Numerical Results

The actual numerical data and results for the cases to be discussed in the following sections are presented in Table 4-1. The data are broken into three parts, that is, the characteristics of the rocket used, the definition of the initial orbit, and the definition of the terminal orbit. The rocket data needed for the finite-burn study are the specific impulse (I_{sp}) in seconds which is a characteristic of the engine and fuel used, and the initial thrust to weight ratio of the vehicle. The orbit descriptions are given in terms of the usual six orbital elements.

1. The eccentricity of the orbit (e)
2. The semi-latus rectum (p) in earth radii
3. The longitude of the ascending node (Ω) in degrees
4. The true anomaly (θ) in degrees
5. The inclination of the orbit (i) in degrees
6. The argument of perigee (ω) in degrees

All the cases considered except case 4 were coplanar, that is, the inclination was zero degree, while in case 4 the inclination of the terminal orbit was 45 degrees.

In each case the argument of perigee was considered as zero degree, and the epoch time of perigee passage was $t_o = 0$. Case 1 was broken into two parts, A and B, with the only difference being an order of magnitude increase in the I_{sp} of the engine.

4.3 A Coplanar Elliptical Transfer

The first case studied was a two-burn problem previously studied by McCue.[8] McCue used an impulsive solution to obtain the initial conditions for a quasilinearization solution approach. The transfer was between coplanar ellipses with the same eccentricities of 0.2. The semi-latus rectum of the initial orbit was 5000 miles and of the terminal orbit was 6000 miles. The arguments of perigee were rotated by 120 degrees. Results of the solution are presented as case 1 in Table 4-1. The results are exactly the same as those obtained by McCue. Kern[10] also solved the problem using his two-loop optimization and obtained similar results.

For the parameterization method the OMIR Program was used to obtain the optimal impulsive solution as a starting guess for the location, duration, and initial direction of thrust of each needed burn. Convergence to the optimal finite burn transfer was obtained easily and used less than 2.25 minutes of Univac 1108 execution time, including the time to solve the impulsive problem and convert the data to input required by the finite thrust portion of the program. McCue commented on the difficulties of obtaining accurate enough starting conditions to allow convergence using quasilinearization. His two-loop optimization technique had slow convergence and required considerable manipulation from the user. He concluded the method was satisfactory but rather inefficient. A great deal of knowledge about the method and the theory of optimization would be required to apply such a technique.

In the case of parameterization, solutions were obtained with relative ease. Each solution was obtained by using as initial conditions a set of initial and terminal state vectors near perigee or apogee of the initial and terminal orbits, depending upon whether a perigee to apogee or an apogee to perigee transfer was desired. The OMIR impulse results were converted by an internal routine to the necessary initial Lagrange multipliers and switch times for the parameterization method. In all cases convergence was obtained with no additional user influence. The method could easily be used by anyone familiar enough with orbit transfer problems to recognize the types of solutions which are locally optimal and to investigate each separately. Since no assumed initial multipliers or switch times are needed as inputs, the optimization by parameterization is less of an "art" than most of the second-order techniques.

4.4 Transfer From Staging Altitude to Circular Orbit

Convergence of iterative solutions for multiburn transfers is hampered by long coast arcs. Two transfers from a staging altitude to circular orbits were studied to show the limits of the technique of parameterization. The first terminated at an altitude of three earth radii and the second at ten earth radii. Case 2 of Table 4-1 is the transfer to three earth radii while case 3 is the transfer to ten earth radii.

The three-earth-radii problem is very typical of the altitudes which would be of practical use in studies for shuttle craft. The ten-earth-radii problem is probably of higher terminal altitude than would ordinarily be studied. The problems were interesting in their respective convergence. Approximately 10 minutes of Univac 1108 time were required for the three-earth-radii case. In the ten-earth-radii case, the sensitivity of the terminal stage due to variations in the initial conditions induced by the exceedingly long coast arcs yielded a convergence time of approximately 22 minutes on the Univac 1108.

Parameterization yielded solutions for the transfer to three earth radii, which is the same as that obtained by Kern.(10) The power of the convergence method becomes apparent in such cases.

The reason for the long execution times is in part the fault of the starting conditions. Normally if the initial state is on an orbit, the state is backed up to allow the centroid of the burn to occur at approximately the time of the first impulse derived by the optimal impulsive solution. For the problems considered in cases 2 and 3, the initial conditions are suborbital, and the initial state could not be backed up; hence, the initial values for the co-state and switch times derived from the optimal impulsive transfer generate a finite-burn trajectory which yields a much larger magnitude for the psuedo cost model than transfers with the initial state free as in case 1. An improvement in the convergence time could probably be made by iterating the starting conditions one time in the following manner. By using the optimal impulsive results from one application, the approximate burn time of the first burn could be calculated. Determine the centroid time, that is, the time to burn from the initiation of the burn to the time of the centroid of the burn. By using conic arcs the state could be propagated forward to a time equal to $t_o + t_c$ where t_c is the centroid time. Consider the state at $t_o + t_c$ to be the initial state, and obtain the optimal impulsive solution from that state to the desired terminal state. The result from the second application of the optimal impulsive solution could then be used as a more accurate starting condition for the finite-burn solution, allowing the initial state to be backed up an appropriate amount to allow the centroid of the burn to occur at approximately the same time as the first impulse occurred.

The velocity change required for the finite-burn case in this problem turned out slightly less than the velocity change required for the impulsive solution. (See the corresponding columns of Table 4-1.) Part of the reason for the apparent inconsistency is due to the difference of the transfers. The initial state was suborbital; hence, the first impulse of the impulsive solution occurred at the start of the first burn rather than at the centroid of the burn. The two converged trajectories are different. The impulsive transfer is approximately 120 degrees while the finite-burn transfer is approximately 220 degrees. The results imply that the transfer of the finite burn is slightly more efficient, probably due to the gravity model assumed.

4.5 Three-Dimensional Orbit Transfers

The method will solve a three-dimensional transfer. This was validated by using a transfer between inclined orbits. The orbits had eccentricities of 0.2. The semi-latus rectum of orbit one was 1.2 earth radii, while on the terminal orbit it was 1.78 earth radii. The orbits were inclined 45 degrees to each other. The method converged with no difficulty in this case, with a relatively short execution time.

4.6 Continuous Burn Orbit Transfers

A simple one-burn launch phase was included as case 5 in Table 4-1. The continuous burn transfer shows the capability of the method of parameterization to solve those problems which have traditionally been solved by some of the indirect optimization methods which have been developed over the past few years. Case 5 was a set of data with the initial point at a suborbital altitude and the terminal point on a highly elliptic orbit at 1.03 earth radii. Since the solution is for one continuous burn, the optimal impulsive starter was not applicable. Initial conditions for the co-state were assumed in such a manner as to align the thrust vector with the velocity vector, while the derivative of the primer $\dot{\lambda}$ was chosen to be in an opposite direction of the primer vector, λ. The elements of $\dot{\lambda}$ were in proportion to the elements of λ. The magnitude of λ was 0.993, and the magnitude of $\dot{\lambda}$ was 0.1212. The method was able to converge in this example for any set of initial conditions on the co-state, as long as the thrust vector did not point to the earth or backward in orbit.

Results of the study are given in the last three columns of Table 4-1. The desired result of the problem was to minimize the mass loss due to expenditure of fuel. The required delta velocity change for each mission is given, both for the finite burn and the optimal impulsive approximation solutions. The last column gives the time in seconds which the engine was required to burn to accomplish the mission. The respective columns of Table 4-1 for the characteristic velocity change required by the finite and impulsive solutions yield an important result. In each case tested, the delta velocity change required is nearly the same by both methods.

The transfer for case 1A is plotted in Fig. 4.2 to illustrate the type of transfers involved. The solid curve is the initial orbit. The terminal orbit is represented by the broken curve. Arrows are used to show the direction and location of the impulses but not the magnitude since the plot is not to scale. The shaded area represents the burn arcs for the finite-burn solution.

Parameterization has been used to solve the finite-burn transfer problems. Normal mission design would require knowledge of the actual real-time control, that is, the thrust level and some angle showing the direction of thrust. The cases studied assumed bang-bang control; hence, the thrust level is maximum thrust. The control angle can be calculated from the co-state vector along the solution orbit.

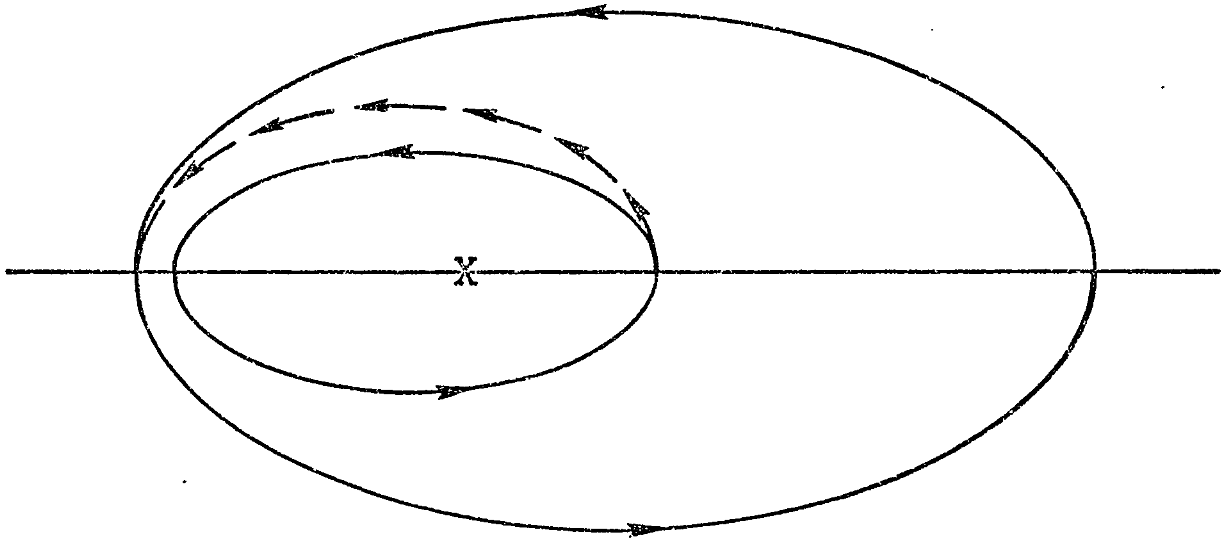
Figs. 4.3 to 4.6 are plots of the time varying control angle or direction of thrust to be used on the burn arcs. The angle plotted is the pitch angle, that is, the angle between the thrust vector and the local horizontal plane on the optimal burn arc. Each burn arc is plotted as a separate figure. Control of the direction of the thrust vector is not exerted during coast arcs; hence, only the burn arcs were considered for the plots. The figures presented refer to representative data cases described in Table 4-1. The horizontal axis is the time axis in seconds, and the vertical axis is the pitch axis in degrees.

TABLE 4-1

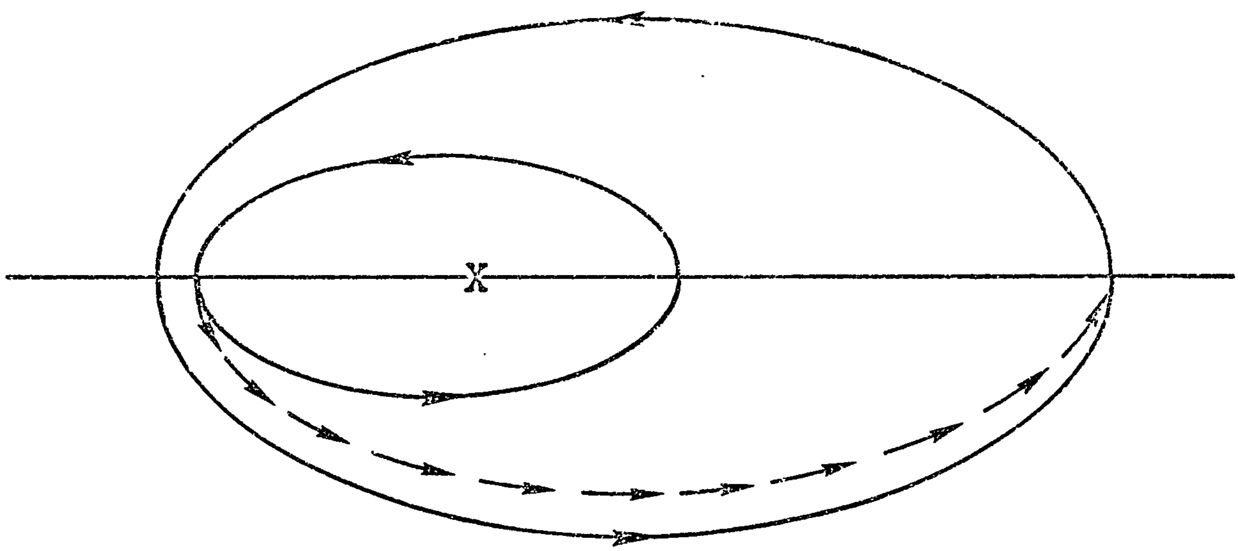
NUMERICAL RESULTS FOR DATA CASES 1 THROUGH 5

Case	Rocket		Initial Orbit				Terminal Orbit				Finite ΔV, ft/sec	Impulsive ΔV, ft/sec	Burn Time, sec
	I_{sp}, sec	T/W_o	e	p(e.r.)	Ω, deg	θ, deg	e	p(e.r.)	Ω, deg	θ, deg			
1A	400	0.4	.2	1.25	90	150.4	.2	1.5	210.0	206.8	3679.1	3677.7	248.5
1B	4000	.4	.2	1.25	90	150.4	.2	1.5	210.0	206.8	3679.3	3677.7	281.6
2	427	.8	.9	1.0	0	0	0	3	235.3	148	28679.4	28719.7	470.8
3	427	.8	.9	1.0	0	0	0	10	129.7	132.4	31159.6	31401.9	481.8
4	427	.8	.2	1.2	180	318.5	.2	1.78	180	274.6	15156.9	15144.9	356.5
5	460	1.5	.84	1.0	0	--	.956	1.03	180	90	24857.4	N/A	256.4

Note: Case 4 has a 45-degree inclination of the terminal orbit to the initial orbit. The argument of the ascending node is 180 degrees and 90 degrees on initial and terminal orbits, respectively.

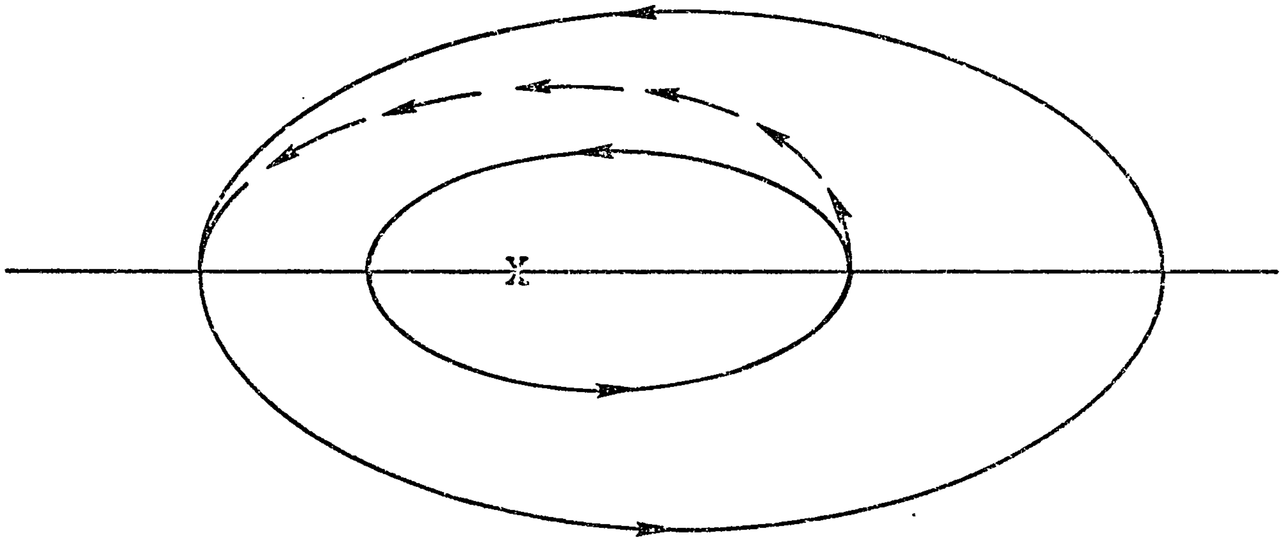


(a) Perigee to perigee transfer.

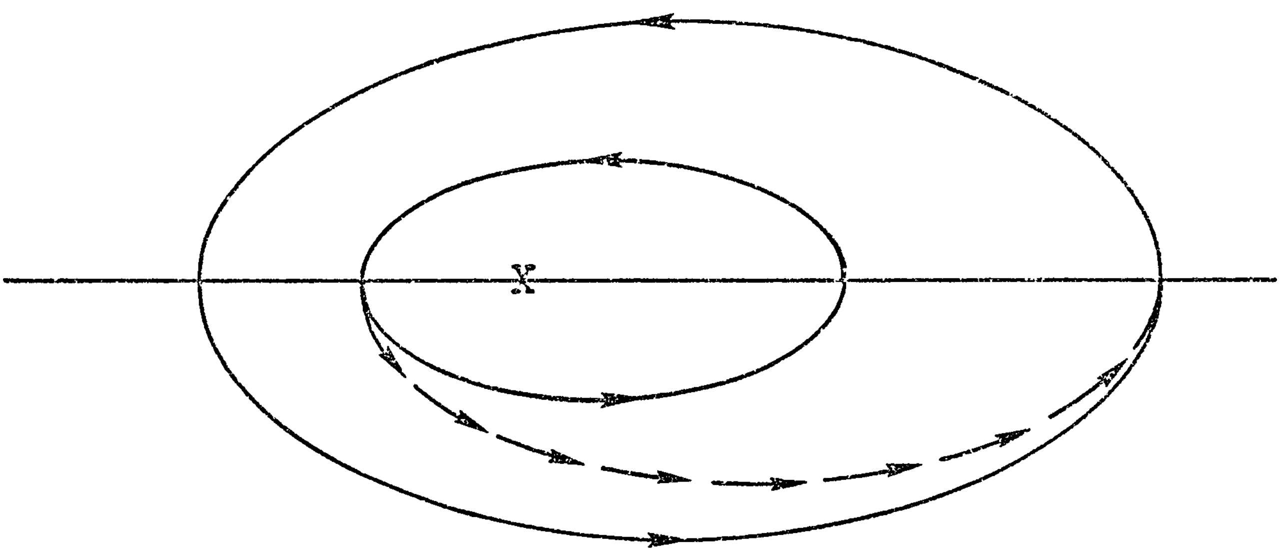


(b) Apogee to apogee transfer.

Figure 4.1.- Typical types of orbital transfers.



(c) Apogee to perigee transfer.



(d) Perigee to apogee transfer.

Figure 4.1.- Concluded.

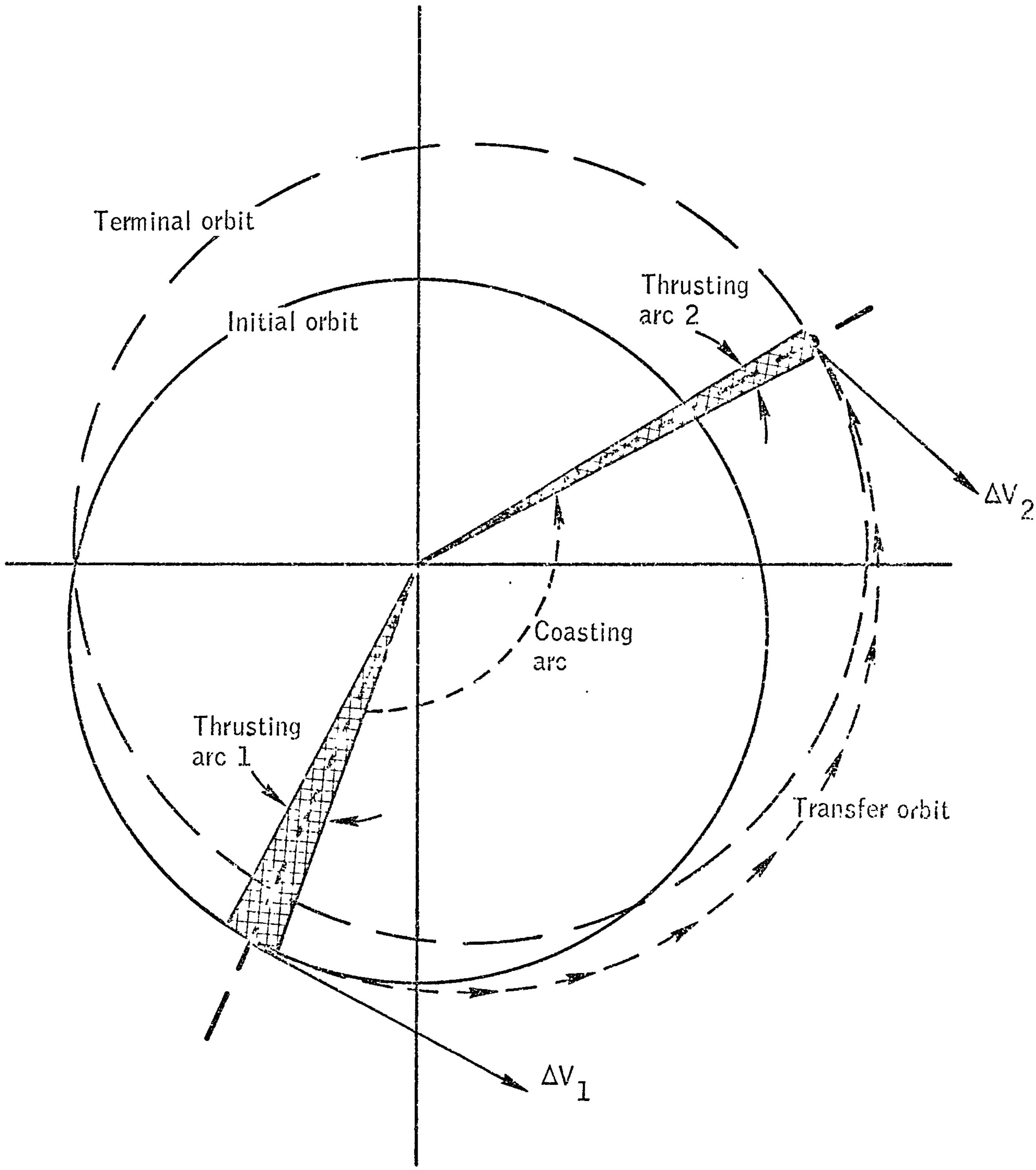
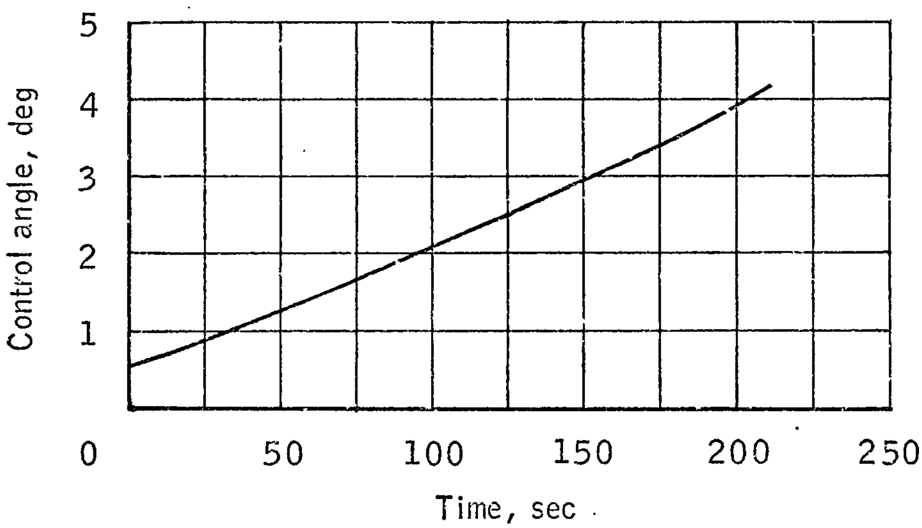
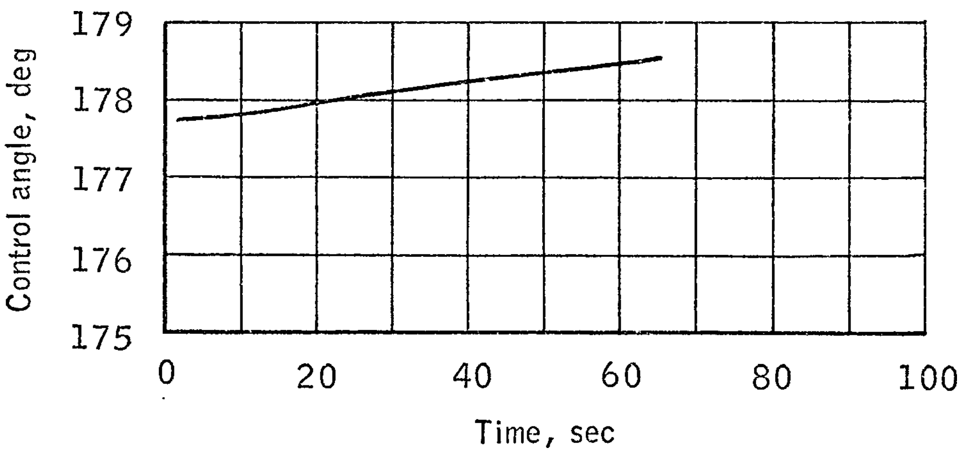


Figure 4.2.- Transfer orbit for Case 1.

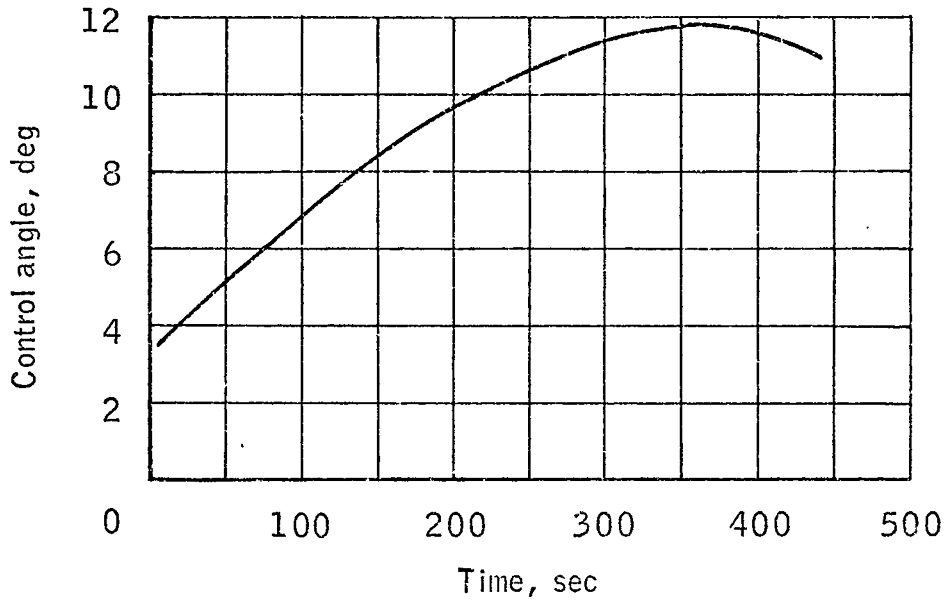


(a) Burn one.

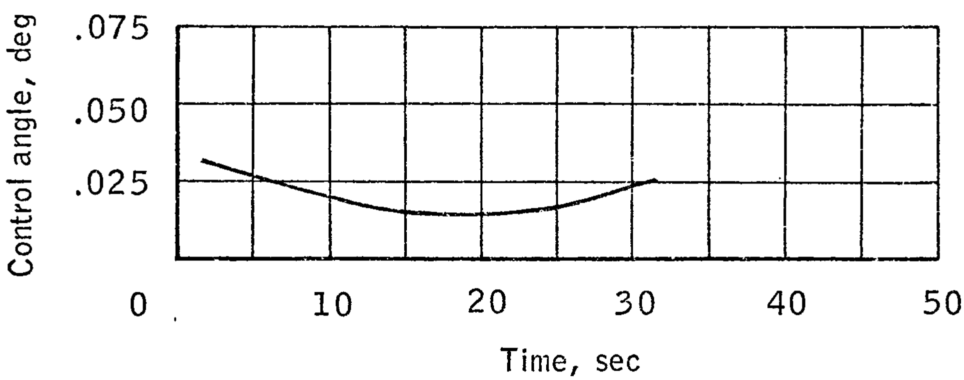


(b) Burn two.

Figure 4-3.- Control angle versus time for burns one and two of Case 1A.

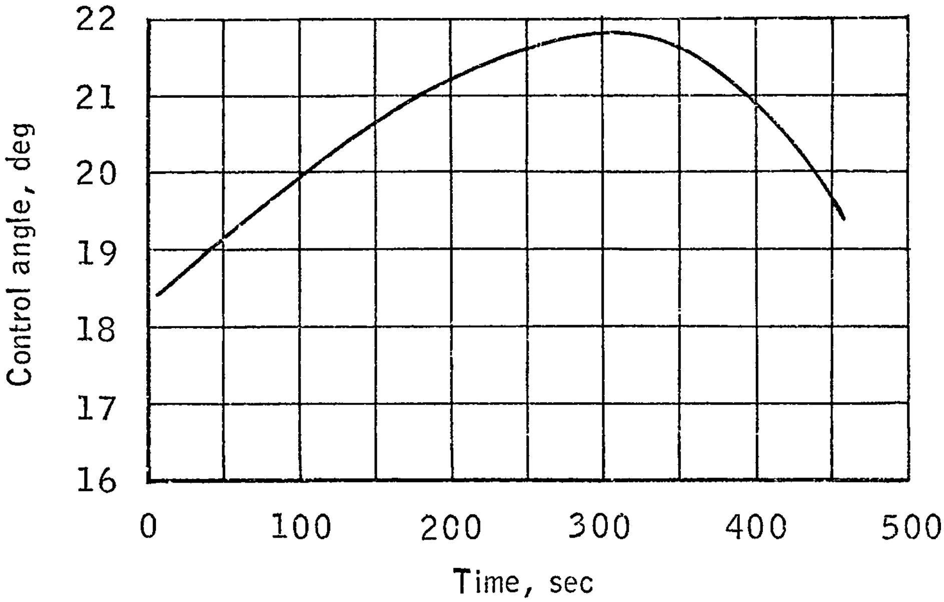


(a) Burn one.

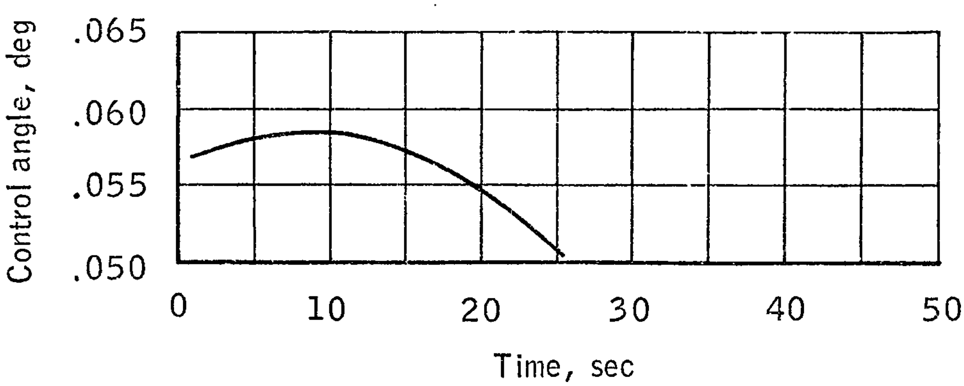


(b) Burn two.

Figure 4-4.- Control angle versus time for burns one and two of Case 2.



(a) Burn one.



(b) Burn two.

Figure 4-5.- Control angle versus time for burns one and two of Case 3.

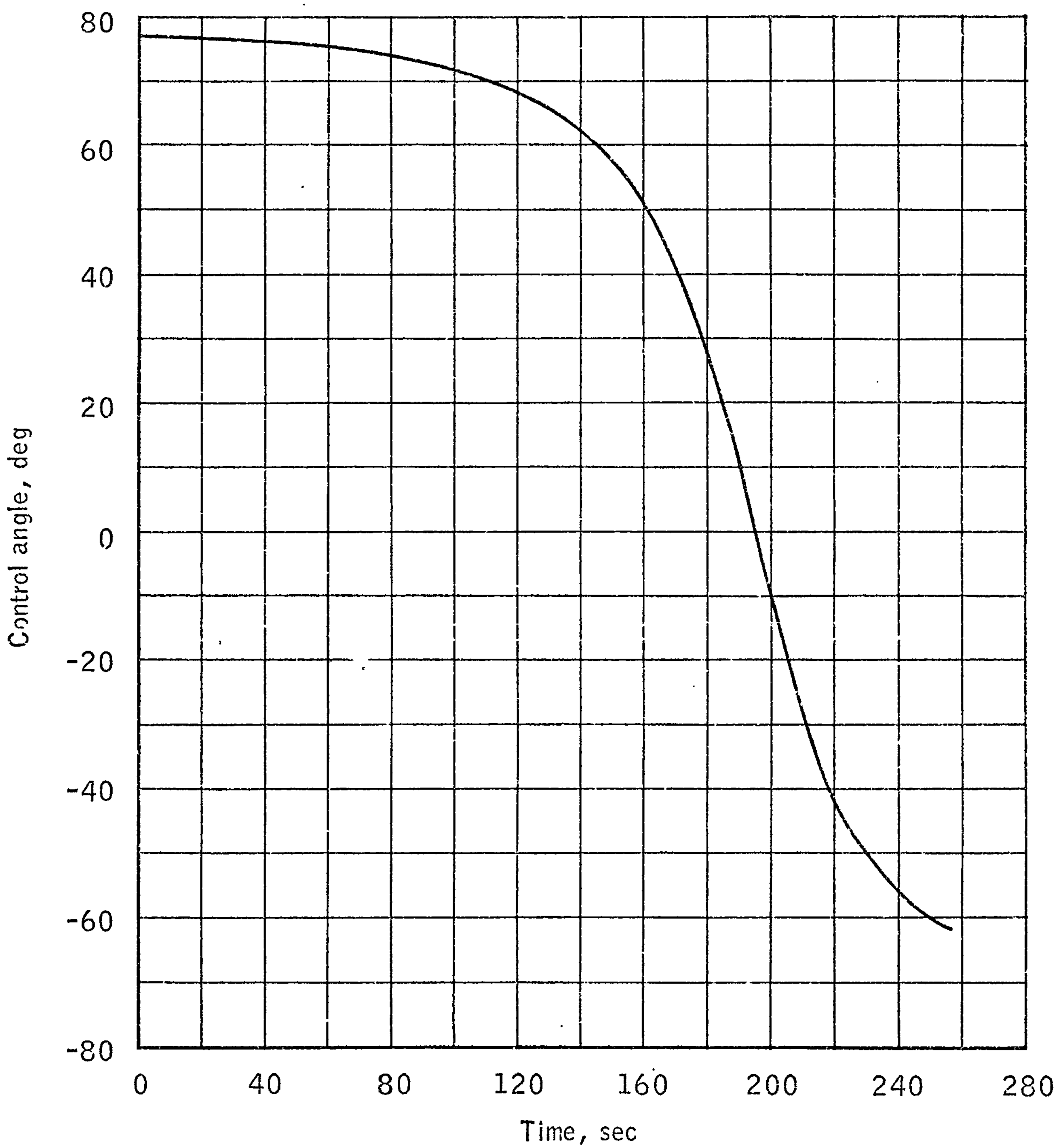


Figure 4-6.- Control angle versus time for one burn problem.

CHAPTER 5

CONCLUSIONS AND RECOMMENDATIONS

5.1 Conclusions

The method of parameterization used in the case of the multiburn trajectory optimization is an easily understood and straightforward technique. There is no requirement for the user to be an expert in the field of optimization or iteration algorithms. In every case tested, using only initial and terminal position vectors as inputs to the impulsive starter, a convergence to the optimal multiburn trajectory was obtained. The excellent convergence characteristics of the Davidon method used to extremize the cost function allowed a wide range of starting values for the parameters used. For example, in the case of the transfer to ten earth radii, the impulsive starter gave an initial coast time which was greater than a period on the coast orbit resulting from applying the first burn. When a converged solution was obtained, the initial burn period was nearly the same length as assumed, based on the delta velocity increase from the impulsive solution; however, the thrust was realigned to yield an intermediate orbit which was feasible. Every case attempted was successfully completed. The only problems encountered were convergence failures due to improper data, that is, improper vehicle or orbit description, and the sometimes slow convergence due to a very sensitive problem.

The method as applied to the multistage problems satisfies only necessary conditions for optimality; hence, as in other numerical optimization schemes, the method yields only local extremals. The failure to ensure global extremals is common to numerical methods and requires that a user be aware of the need to be certain the solution obtained is feasible for the mission as defined. A different local extremal might be obtained from a different set of starting conditions.

The parameterization method yielded solutions in the cases tested; however, the execution time was considerably greater than the execution time required to obtain an optimal impulsive approximation solution to the same mission. As previously mentioned, for all the cases tested, the delta velocity change required was nearly the same by both methods. For that reason it would be advisable to make the initial design studies for mission analysis by using the optimal impulsive analysis. After the feasible missions were decided upon and the task of actually determining the real-time control to be used was completed, the parameterization method for obtaining optimal finite-burn solutions would be advised. The method has excellent convergence properties and could be used by a mission design group without the aid of an expert in the field of trajectory optimization.

5.2 Recommendations

An area of interest which has been examined but has not produced useful information at this time is the examination of the switch function on the optimal trajectory. The necessary conditions specify that $\mathcal{S}$ be positive over the burn arcs, zero at the junction points, and negative on the coast arcs. Impulsive optimization uses the primer vector to determine where an impulse should be added or deleted to improve the solution. In a similar manner the switch function ($\mathcal{S}$) could be used for the determination of the necessity of adding or deleting burns in order to improve the solution. Preliminary studies have been made, but no significant results have been obtained. It appears that an error of 10^{-9} in the constraints at the switching times is still not close enough to obtain starting values for the mass multiplier, w. The initial conditions for w would have to be guessed at t_o; however, since at the first switch point, t_1, the value of the switch function is known, it should be possible to use the equation for $\mathcal{S}$ (Eq. 3-14) and the value of $\mathcal{S}$ at t_1 to obtain a value for w at t_1 and then integrate backwards to obtain w at t_o. Recall that

$$\mathcal{S}\left(t_1\right) = 0 \tag{5-1}$$

and

$$\mathcal{S}\left(t_j\right) = +1 - w\left(t_j\right) + P\frac{c}{m} \tag{5-2}$$

or

$$w\left(t_j\right) = +1 + P\frac{c}{m} \tag{5-3}$$

Further study and a new integration package designed especially for evaluating $\mathcal{S}(t)$ for $t_o \leq t \leq t_f$ are needed in order to be able to use the switch function to determine if the assumed N-burn trajectory is optimal or if it could be improved by the addition of another burn.

However, the ability to calculate accurately the switch function ($\mathcal{S}$) should be added. The function should be studied in an attempt to forecast the need for an additional burn or coast arc. This capability would allow the possibility of building from two or three burns up to the multiple burns required for low thrust vehicles. Presently low thrust nuclear engine vehicle transfers are solved only by very educated guessing at the solution with an iteration scheme to obtain convergence in meeting the boundary conditions (Funk and McAdoo).[32]

In general the method of parameterization, using a Davidon, or conjugate gradient-type convergence package, yields a powerful method for obtaining the extremal solutions for the control of the multistage systems. Due to the long execution times in the problems tested (2 to 10 minutes), it would seem advisable to use some approximation technique, such as optimal impulsive trajectory analysis, for initial studies followed by studies of the dynamic model using parameterization when the areas of interest are defined.

By examining the amount of work which has been done in the area of optimal impulsive trajectory analysis during the past few years, it is clear that the power of the multiburn trajectory package presented opens the door to a similar analysis for the finite-burn trajectory problem. Many areas of interest are immediately apparent. The rendezvous capability would be extremely important. Rendezvous would not be an extremely difficult capability to add to the present model. However, it would require some detailed analysis and some added program capability.

To use the technique for mission design and analysis, the model needs to have atmospheric forces added. Given a good atmosphere approximation along with a rendezvous capability, it would be possible to design a package to perform launch to rendezvous mission analysis.

Convergence speed is a definite problem and could be greatly improved by including some improved initial approximations such as those suggested by Gill and Wambold.[33] The integration package is excellent, but some analysis into numerical integration step sizes might allow the determination of an optimal step size to be used in the Runge-Kutta technique which would improve accuracy and hence the speed of convergence.

The method of parameterization is a powerful tool to be used in the optimization of continuous dynamic systems. The application to the difficult multiburn transfer trajectories has been done with excellent results. The technique is relatively easy to understand, especially when compared to second and third generation indirect optimization schemes. The most serious problem is convergence speed. The technique could definitely be used in a hybrid approach to develop a set of very accurate initial conditions for a second-order convergence scheme, such as the method of perturbation functions.

The preliminary analysis of parameterization points to the fact that the technique is applicable to any dynamic model. The positive convergence of the method is the characteristic which makes it desirable. Long execution times are not as critical as nonconvergence. The positive convergence of parameterization should make it a useful tool.

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