DESIGN OF OPTIMUM FILTERS
FOR ANALOG FEEDBACK TELEMETRY SYSTEM

A Dissertation

Presented to

the Faculty of the Department of Electrical Engineering

University of Houston

In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy in Electrical Engineering

---

by

Syed Zafar Hasnain Taqvi

August 1969

To my parents

and sisters

and to all those

who gave me

inspiration

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ABSTRACT

An analog feedback telemetry system having noise in both the forward and feedback channels is analyzed. The complete system is treated as a feedback control system with disturbances in forward and feedback loops and technique of calculus of variation is applied to study the optimum performance.

Cases of both with and without delay in the channels are considered. The technique evaluates the expressions for optimum filters which minimizes the mean square error between the data sent and the data received.

Various cases of correlation between signal and noise processes are considered and expressions of optimum filter transfer functions are derived. Mean square errors are evaluated for various cases of correlation.

It is shown that for the systems where noise in the feedback channel is not correlated either with forward channel noise or the signal, a feedback channel is not required for optimum performance. Channel delays could also be adjusted to minimize the mean square error.

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NOTATIONS

$h(\tau)$	Impulse response of the filter; inverse fourier transform of the system function
$H(\omega)$	System function = $\int_{-\infty}^{\infty} h(t)\, e^{-j\omega t} dt$
$R_{xy}(\tau)$	Cross correlation of two stationary processes x(t) and y(t)
$S_{xy}(\omega)$	Cross spectral density of processes x(t) and y(t)
$R_{xx}(\tau)$	Autocorrelation of process x(t)
$S_{xx}(\omega)$	Spectral density of process x(t)

CHAPTER I

INTRODUCTION

Since early times effective communication has been responsible for mankind's progress. Examining chronologically, the earliest form of communication was by acoustic signals. The limited concept of transmission of sound waves was well utilized through the blowing of horns, the beating of drums, etc. Next visual means of communication, such as the semapher system, namely, signalling of light sources in a certain sequence was utilized to convey information. However, this system had a very low rate of transmission. Improvement in the rate of communication was made through the use of code and the development of signalling devices. The next major invention was telegraphy and its subsequent modification to the more complex multiplex system.

With the discovery of electromagnetic radiation by Maxwell in 1873 and the development of the first wireless communication in 1894, the progress in this area has been rather fast leading to inventions of newer and better techniques of transmission. These techniques which are used in the data transmission from machine to machine or from machine to a human operator have made reliability of transmission a critical factor in system design.

Mathematically, reliability is expressed through the system performance. For example, the performance may be evaluated on the basis of quality of message and the fidelity of the signal. In the present age of space technology, the concept of system performance has completely changed. We are now concerned with communication through space, where the signal is perturbed by random noise. This noise can be described statistically and to separate the required data, statistical methods are necessary and the new field of statistical theory of communication has been successfully used for system design. As we will see in the later chapters, this approach leads to simpler design equations in terms of statistical properties of signal and noise.

Digital transmission is the most common form even though all the physical quantities are in analog form. In a digital system, the analog data is sampled and then quantized. Corresponding to the quantized levels coded digits are transmitted. The advantage of digital data transmission lies in the equipment reliability, ease in data generation and smaller probability of error. However it introduces quantization error. Analog transmission, on the contrary, does not introduce any quantization error but needs more sophisticated instrumentation than digital.

Often there is a need for the receiver to confirm the message sent by the transmitter. A two way communication system with two channels, one connecting the transmitter to the receiver and the other from the receiver to the transmitter becomes necessary. This system can also be modified to be used as a negative feedback system, which will reduce the non-linear distortion due to noise in the channel and thus improve the reliability of the system. It is worthwhile to note Shannon's [1] results which show that with the availability of a feedback link, the complexity of message coding can be reduced for a given system performance or probability of error.

Previous research in the above field of investigation had been rather limited. However, there is a good possibility of improvement in system performance if we use the feedback link. In this dissertation various aspects of feedback telemetry are discussed. It is found that the design of optimum filters in the two channels, forward and feedback, play an important role in the system performance. Control system approach is followed to design optimum filters of communication system and the system performance is evaluated.

CHAPTER II

FEEDBACK TELEMETRY

In this chapter a review of the development of feed-telemetry systems is given. Feedback has already been applied to digital systems to simplify the complex coding of the message. However it is worthwhile to consider the effect of feedback on the system performance employing analog data. With the selection of analog data, the criterion of performance of the system is changed from probability of error for digital data to mean square error; a brief account of the criterion of performance is included.

The various considerations of this chapter are as follows;

(a) Analog data

(b) Criterion of performance

(c) Survey of Feedback System

Analog Data

In an analog system, all the quantities to be measured and telemetered are continuously variable. For example the temperature in the spacecraft continuously varies over a desired range. Other examples of continuous data required to be telemetered may be physical parameters such as gas flow, pressure, level, speed or physiological data which includes blood pressure, body temperature, heart beats etc. When the telemetering transmitter converts this physical quantity into electrical form, the converted quantity is an analog of the data. The purpose of the telemetry system is to send this electrical analog in any of the modulated forms. This analog is measured at the receiving end to recover the original data. An example of a simple telemetry system is given in Fig. 2-1.

It is a characteristic of analog telemetry that the final measurements of the telemetered quantity is made on the analog data at the receiving end. This is in contrast to the digital system where the continuous data is reduced by discrete steps into quantized form at the transmitter end.

While there are certain advantages in digitizing the information, it may be worthwhile to send the data in the original form if a suitable system is developed so that it satisfies the necessary performance criterion wihtout much elaborate circuitry.

Criterion of Performance

The purpose of a communication system is to make the source output available at the receiver. The channel or the transmission media introduces extra undesired random noise to the signal. This random noise is unknown but its statistical properties can be studied and specified. A system has to be devised to extract the signal from the noise. It is evident that perfect reproduction of the original signal is impossible. However, we should achieve a suitable reproduction which will be satisfactory for the specific purpose. This leads us to specify a criterion of acceptability, which will depend upon the type of problem encountered. For example, in machine to machine communication, we specify the degree of accuracy or precision of the reproduced information which is required for acceptable calculations. It is to be noted that the criterion of acceptability is not a function of the source or of the receiver alone but that of the system as a whole. The basis of system design is the performance criterion.

The design based on the performance criterion gives the optimum system performance. For a system where $s(t)$ is the desired signal to be transmitted and $s_i(t)$ is the received data ($s_i(t) = s(t) + n(t)$ where $n(t)$ is the noise introduced in the channel), the instantaneous error between the output and the desired signal is given by

$$\varepsilon(t) = s(t) - s_i(t)$$

Besides material cost and system compatibility, the major factor involved in the selection of performance criterion is a function of the error between the desired signal and the output. This can be written as

$$J = F\ [\varepsilon(t)\ ]$$

where F is the specified function of error which has been determined by the performance criterion. The various different performance criteria used in system design are the following:

a] $F[\varepsilon(t)] = \overline{\varepsilon(t)}$ Average value of error

b] $F[\varepsilon(t)] = \int |\varepsilon(t)|$ Summation of absolute value of error

c] $F[\varepsilon(t)] = \overline{\varepsilon(t)^2}$ Mean square error

The first function is not generally appropriate, for when $\varepsilon(t)$ is averaged, positive and negative errors tend to cancel each other even though for any particular value of t, the magnitude of the error may be quite large. The second and third functions do not suffer from this defect and the mean square error in particular is commonly used [2, 7, 13, 17, 18, 20, 29] in system design because it lends itself conveniently to mathematical analysis.

Survey of Feedback System

A feedback system is defined as one in which the information about the data received at the receiver end is made available to the transmitter. In a "one way" system consisting of a transmitter - receiver link as in Fig. 2-1, the transmitter has no information about the data received at the receiving end. In many cases such as Fig. 2-2, there is a return link available from the receiver to the transmitter for some other purpose and this can be combined with the forward link. The information from the feedback channel of this composite system could be exploited to give possible significant increase in the reliability of the forward transmission of information. One example of a system in which a return link is available and can be made use of is in a missle or aircraft control system. A forward link exists from the vehicle to the ground and a return or command link is available from the ground to the missile or aircraft.

The possibility of increased reliability in transmission of data has been investigated in some detail for digital systems using a feedback link and indicated in the literature [5,6,10,11,12,13,14,25,27-32], significant results have been obtained. Several operating systems have been built based on the feedback principle and improvement in the system performance has been achieved.

Early investigations [12,32,6,] have shown the possibility of improvement in the system performance by the use of feedback. The basic idea is to provide the transmitter with information about a certain state at the receiver; either the received signal or the decision made at the receiver. The so-called pre-decision feedback technique in digital communication [12] has been developed in which the transmitter is informed about the continuous data received at the receiver.

The technique exploits the feedback channel in eliminating the noise and uncertainty accumulated during the time the signal is passed through the forward channel. In Elias's [12] work a wide band forward channel and a wide band feedback channel are interconnected to make a composite feedback system. The receiver sends back the received signal through the feedback channel which is added to the input signal. Elias concludes as follows: if noise processes $n_1(t)$ and $n_2(t)$ in the two channels are non-zero and uncorrelated, then the presence of a feedback loop cannot increase the rate of information. If the feedback is noiseless, the performance is neither improved nor degraded. However if both additive noises are nonzero and correlated, then there is a possibility of increasing the rate of information. In other words, the channel capacity can be increased by feedback if there is a statistical dependence between $n_1(t)$ and $n_2(t)$ in the two channels. Similar results are reported by Hayes [34]. Simultaneous noise jamming of the two channels and noise introduced by radio stars of small angular size are examples in which noise processes are correlated.

Another feedback system, known as a post-decision feedback system has been reported for digital systems by various investigator such as Viterbi [11], Schalwijk [14], etc. In this system, the transmitter is supplied information about the decisions made by the receiver. Shannon's results show that for a noisy forward channel the capacity of the forward channel is not increased by the addition of a noiseless feedback channel provided the forward channel has no memory. However it is found that the maximum rate of "no error" information can be increased with the use of feedback. Chang's analysis of the post-decision feedback system deals mainly with the coding aspect of the signal and his results show that in systems that require elaborate coding techniques for error free transmission, the addition of even a noisy feedback link will provide the same system performance using simpler coding technique. Viterbi's model [11] is a form of post-decision feedback using a white Gaussian noise corrupted channel. The receiver computes the likelihood ratio as a function of time and makes a decision when the value of likelihood ratio crosses a pair of threshold values. The transmitter repeats the data until the receiver informs the transmitter that it has taken a decision; at this time the transmitter starts sending the next data. When higher transmission rates are used, this model claims higher reliability as compared to "one-way" systems.

Improvement over Viterbi's model has been made by Turin [10] who utilizes predecision feedback instead of the post-decision feedback of Viterbi's model. In this system the receiver computes the likelihood ratio as a function of time and sends it back to the transmitter continuously.

Most of the investigators have focussed their attention on the coding schemes utilized for digital data system models. Kailath [13] and Schalkwijk [13,14] worked out coding schemes for additive noise channels with feedback, both with and without bandwidth constraint on the tranmitted signal. Their scheme utilizes the Robbin-Monro Stochastic approximation technique and reduces the complexity in coding compared to the "one way" system. Kashyap's [28] coding scheme is also applicable for additive noise channels. His scheme is more general than Schalkwijk and Kailath and it reduces to the latter's scheme when noisy feedback is replaced by noiseless feedback. Similar work on the feedback scheme is done by Smerage [27], Goblich [30], etc.

These basic results of using the feedback channel in the digital system show that good improvement in system performance and reliability can be obtained. Similar results are possible for analog feedback telemetry systems and will be investigated in the following chapters.

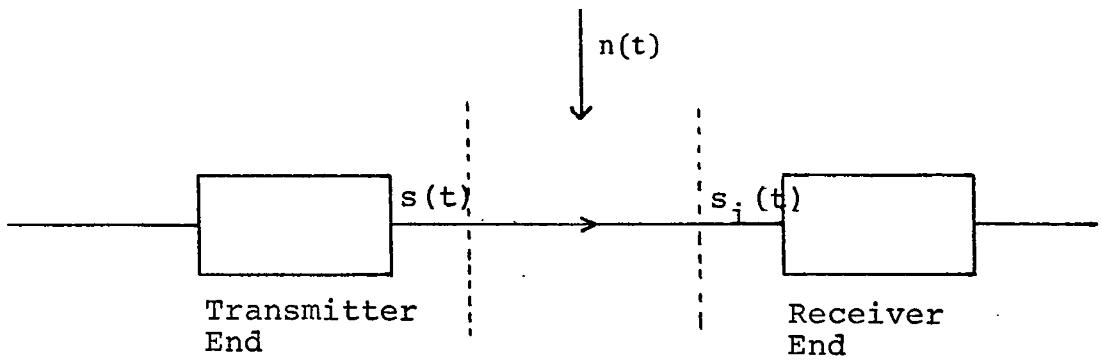


Fig. 2.1

One Way Telemtry System

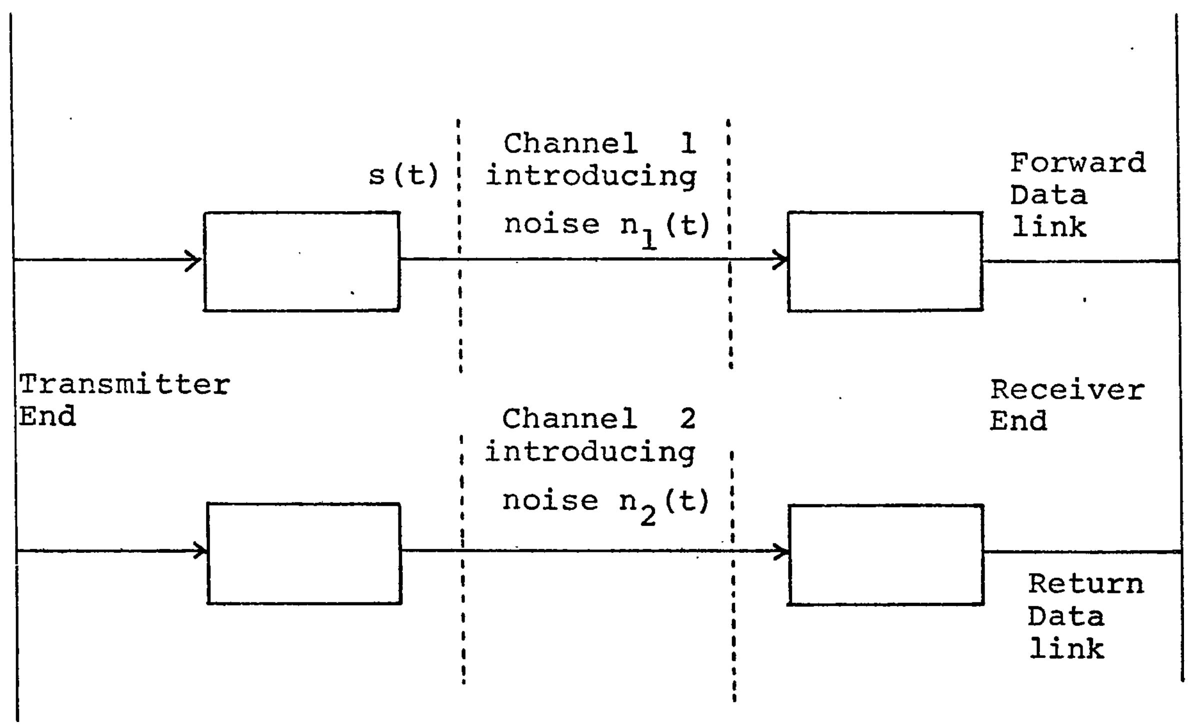


Fig. 2.2

Two Way Telemetry System

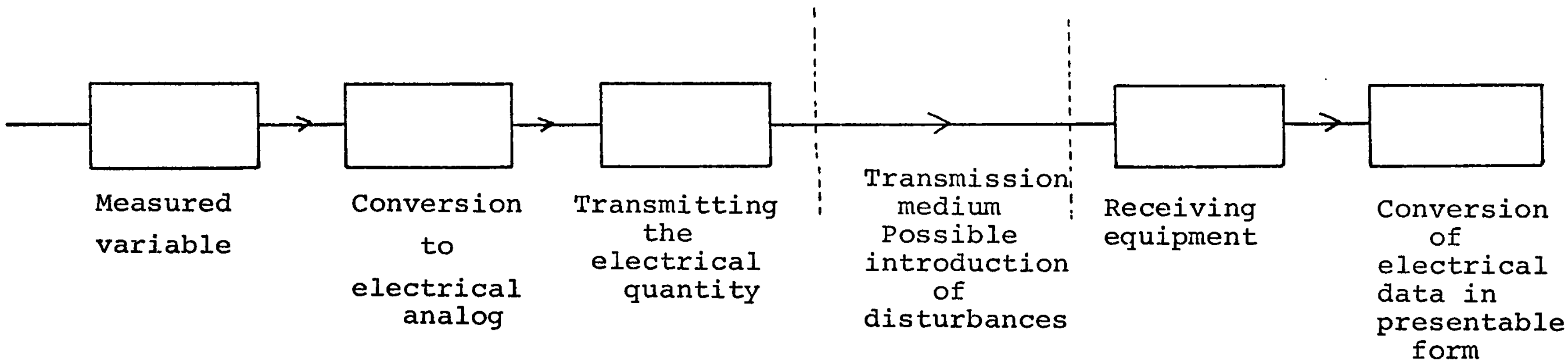


Fig. 2- 3

Explanation of One Way System

CHAPTER III

LINEAR FILTER PERFORMANCE

The process of extracting the signal from the noise is known as filtering. It is evident that complete separation of signal from noise cannot be done unless the properties of signal and noise are completely different. For example, if they have non-overlapping frequency spectra, complete separation is possible. In practice the signal and the noise spectra overlap and hence the filter performance involves some error in separating the signal from the noise. In this chapter the characteristics of various filters are considered and an expression for optimum filter design is derived.

Consider the input to the filter to be $s_i(t)$;

$$s_i(t) = s(t) + n(t)$$

where $s(t)$ = signal

$n(t)$ = noise

The function of the filter is to process the received data and separate the signal from the noise. In mathematical terms it operates on the received data and gives an output $s_0(t)$. This operation may be represented as

$$s_0(t) = H[s_i(t)]$$

where $s_0(t)$ = output of the filter and

$H[s_i(t)]$ = the operator representing the filtering action of the input data.

The selection of a particular mathematical expression for operator H depends upon the type of performance of the filter and the output desired.

The filter may be required to process an infinite or finite set of data. On the basis of the quantity of data processed the linear filters are classified as follows;

(a) Type I

The input $s_i(t)$ is stored for a certain interval of time (theoretically for $-\infty < t < \infty$) and then processed subject to performance criterion to give $s_o(t)$ at the output. This type of filter gives a more effective suppression of noise than other filters. To compute the output $s_o(t)$, Type I uses values of $s_i(t')$ for all t' such that

$$-\infty < t' < \infty$$

(b) Type II

With this filter type, the processing is continuously done and the output $s_o(t)$ is influenced only by the input $s_i(t)$ available up to that moment. To compute $s_o(t)$, Type II uses values of $s_i(t')$ for all t' such that

$$-\infty < t' < t$$

(c) Type III

This filter is similar to Type II filter and uses the input data $s_i(t')$ of duration T such that

$$t-T \leq t' \leq t$$

Linear filters perform linear operations on the input functions. In the time domain, the filter can be written in terms of the impulse response h(t) which is related to the transfer function $H(\omega)$ in the following manner

$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega)\ e^{+j\omega t} d\omega$$

$$H(\omega) = \int_{-\infty}^{\infty} h(t)\ e^{-j\omega t} dt \qquad (3.1)$$

Knowledge of h(t) or the impulse resonse in the time domain is completely equivalent to knowledge of $H(\omega)$ in the frequency domain. h(t) is defined mathematically as the filter response to a unit impulse at t = 0 as in Fig. 3.2

The output in terms of the impulse response of the filter is written in the form

$$s_0(t) = \int_{-\infty}^{\infty} h(t')\ s_i(t-t')dt'$$

or

$$s_0(t) = \int_{-\infty}^{\infty} s_i(t')\ h(t-t')dt' \qquad (3.2)$$

which is the convolution of the input with the impulse response. As seen above, for the filter of Type II, which processes only the past and the present values of the input, the impulse response will have the following restriction;

$$h(t) = 0 \qquad \text{for } t < 0$$

which gives

$$s_0(t) = \int_{-\infty}^{\infty} h(t')\ s_i(t-t')dt$$

$$= \int_{-\infty}^{t} h\ (t-t')\ s_i(t')dt' \qquad (3.3)$$

Similarly for a finite memory filter or Type III filter, we have

$$h(t) = 0 \qquad \text{for } t < 0 \text{ and } t > T$$

and

$$s_0(t) = \int_{-\infty}^{T} h(t')\ s_i(t-t')dt$$

$$= \int_{t-T}^{t} h(t-t')\ s_i(t')dt' \qquad (3.4)$$

As stated in Chapter II, the optimum linear filter will be described by the minimum mean square error $\overline{\varepsilon^2}$ where ε is the error given as

$$\varepsilon(t) = s_0(t) - s(t)$$

Solution of this optimization problem for minimizing the mean square error, reduces to an integral equation involving the impulse response which characterizes the optimum filter.

For the linear time-invariant filter as shown in Fig. (3.1), the error is given as

$$\varepsilon(t) = \int_{-\infty}^{\infty} h(t')\ s_i(t-t')\ dt' - s_d(t) \qquad (3.5)$$

The mean square value of the error can be simplified as

$$\overline{\varepsilon^2} = \int \int_{-\infty}^{\infty} h(\tau)\ h(\sigma)\ \overline{s_i(t-\tau)\ s_i(t-\sigma)}\ d\tau\ d\sigma$$

$$-2\int_{-\infty}^{\infty} h(\tau)\ \overline{s_d(\tau)\ s_i(t-\tau)}\ d\tau + \overline{s_d^{\,2}(t)} \tag{3.6}$$

However, the terms with a bar can be replaced by the correlation functions as follows:

$$R_{ii}(\sigma-\tau) = \overline{s_i(t-\tau)\ s_i(t-\sigma)}$$

$$R_{di}(\tau) = \overline{s_d(t)\ s_i(t-\tau)}$$

$$R_{dd}(0) = \overline{s_d^{\,2}(t)} \tag{3.7}$$

By the substitution of correlation functions in (3.6), we get

$$\overline{\varepsilon^2} = \int\int_{-\infty}^{\infty} h(\tau)\ h\ (\sigma)\ R_{ii}(\sigma-\tau)\ d\tau d\sigma$$

$$-2\int_{-\infty}^{\infty} h(\tau)\ R_{di}(\tau)d\tau + R_{dd}(0) \tag{3.8}$$

As we shall see, the condition of minimum $\overline{\varepsilon^2}$ is given in the integral form as

$$\int_{-\infty}^{\infty} h(\tau)\ R_{ii}(\sigma-\tau)d\tau = R_{di}(\sigma) \tag{3.9}$$

To show that the above integral holds, the method of calculus of variation is applied to the equation (3.8). Let J be the mean square error $\overline{\varepsilon^2}$ corresponding to impulse response h(t). Then J + δJ will correspond to an impulse response h(t) + δk(t) where δk(t) is the variation of h(t). Substitution and simplification leads to

$$\delta J = 2 \int_{-\infty}^{\infty} \delta k(\tau) d\tau \left[\int_{-\infty}^{\infty} h(\sigma)\ R_{ii}(\tau-\sigma) d\sigma - R_{di}(\tau) \right]$$

$$+ \int \int_{-\infty}^{\infty} \delta k(\tau)\ \delta k(\sigma)\ R_{ii}(\tau-\sigma) d\tau$$

or

$$\delta J = 2 \int_{-\infty}^{\infty} \delta k(\tau) d\tau \left[\int_{-\infty}^{\infty} h(\sigma)\ R_{ii}(\tau-\sigma) d\sigma - R_{di}(\tau) \right]$$

$$+ \left[\int_{-\infty}^{\infty} \delta k(\tau)\ s_i(t-\tau) d\tau \right]^2 \tag{3.10}$$

From (3.10), since the last term is positive, and independent of h(t) the condition of minimality leads to (3.9). To find the optimum filter, equation (3.9) is solved for the unknown h(t) or corresponding $H(\omega)$. For the filter of Type I, equation (3.9) leads to a simple solution. Writing (3.9)

$$\int_{-\infty}^{\infty} h(\tau)\ R_{ii}(\sigma-\tau) d\tau = R_{di}(\sigma)$$

Multiplying both sides of (3.9) by $e^{-j\omega\sigma}$ and integrating, we get

$$\int \int_{-\infty}^{\infty} h(\tau)\ R_{ii}(\sigma-\tau)\ e^{-j\omega\sigma}\ d\tau d\sigma = \int_{-\infty}^{\infty} R_{di}(\sigma)\ e^{-j\omega\sigma} d\sigma$$

or

$$H(\omega)\ S_{ii}(\omega) = S_{di}(\omega) \tag{3.11}$$

Where

$$S_{ii}(\omega) = \text{Spectral density of input}$$
$$= \int_{-\infty}^{\infty} R_{ii}(\tau)\ e^{-j\omega\tau} d\tau$$

and

$$S_{di}(\omega) = \text{Cross spectral density of input and desired}$$
$$= \int_{-\infty}^{\infty} R_{di}(\tau)\ e^{-j\omega\tau} d\tau$$

Hence

$$H(\omega) = \frac{S_{di}(\omega)}{S_{ii}(\omega)} \tag{3.12}$$

Also the mean square error is

$$\overline{\varepsilon^2} = R_{dd}(0) - \int \int_{-\infty}^{\infty} h(\tau)\ h(\sigma)\ R_{ii}(\tau-\sigma)d\tau d\sigma \tag{3.13}$$

Since

$$R_{dd}(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{dd}(\omega)d\omega$$

and

$$\int \int_{-\infty}^{\infty} h(\tau)\ h(\sigma)\ R_{ii}(\tau-\sigma)d\tau d\sigma$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega)\ H(-\omega)\ S_{ii}(\omega)d\omega$$

The expression for mean square error can be written as

$$\overline{\varepsilon^2} = \frac{1}{2} \int_{\infty}^{\infty} \left[S_{dd}(\omega)\ S_{ii}(\omega)-S_{di}(\omega)\ S_{di}(-\omega)\right] \Big/ S_{dd}(\omega)d\omega \tag{3.14}$$

For practical cases, realizable and constructible filters have to be derived from the optimum filters thus calculated. Given the input and output spectra, the optimum filter is given as

$$H(s) = \frac{S_{di}(s)}{S_{ii}(s)} \text{ ; where } s = \sigma + j\omega,$$

$S_{ii}(s)$ being the spectral density of the input data consisting of the useful signal s(t) and perturbing noise n(t).

This is an even quantity, the roots being symmetric with respect to the real and imaginary axis. $S_{ii}(s)$ can be written as

$$S_{ii}(s) = W_1(s)\ W_1(-s)$$

Where $W_1(s)$ has zeros and poles in the upper half plane and $W_1(-s)$ has poles and zeros in the lower half plane. We can define $H_1'(s)$ as

$$H_1'(s) = W_1(s)\ H(s)$$

$$= \frac{S_{di}(s)}{W_1(-s)}$$

such that it contains all the critical frequencies of the optimum filter transfer function and only the lower half plane zeros and poles of its denominator. This can be expanded in partial fractions as

$$H_1'(s) = \frac{a_1}{(s+b_1)} + \frac{a_2}{(s+b_2)} + \text{----------}$$

$$+ \frac{c_1}{(s-d_1)} + \frac{c_2}{(s-d_2)} + \text{-------------}$$

The a's are the residues corresponding to left-half plane poles and the c's are the residue corresponding to right half plane poles. Since the terms containing c's are not realizable, a new transfer function containing only a's are taken. So

$$H_1''(s) = \text{Realizable part of } H_1'(s)$$

The optimum realizable filter transfer function (which can be constructed) is obtained by dividing $H_1''(s)$ by $W_1(s)$.

This can be written as

$$H^*(s) = \text{Optimum realizable filter} = \frac{H_1''(s)}{W_1(s)}$$

Also the expression for H*(s) can be written simply as

$$H^*(s) = \frac{1}{W_1(s)} \left[\frac{N(s)}{W_1(-s)}\right]_+$$

Where N(s) is the numerator of the expression of the optimum filter, the product $W_1(s) \cdot W_1(-s)$ is equal to the denominator and the plus sign takes into account only the poles in the left half plane in the partial fraction expansion of the term in parenthesis. Appendix C gives the detailed theoretical evaluation of realizable filters.

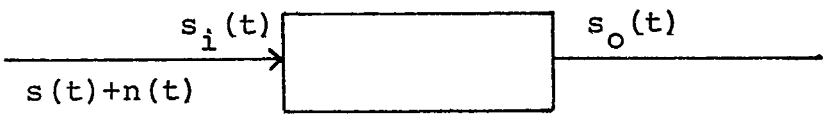


Fig. 3.1

System Block Diagram

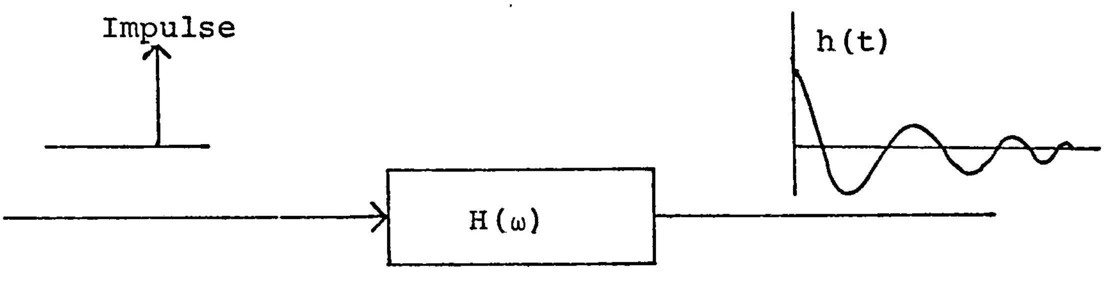


Fig. 3.2

Impulse Response

CHAPTER IV

OPTIMUM FILTER DESIGN FOR NOISY FEEDBACK SYSTEM WITHOUT DELAY

The simplest noisy feedback telemetry system is described in terms of two one way systems in such a way that the data received in the feedback channel is subtracted from the original signal and the resultant data is transmitted through the forward channel. Both channels introduce noise and have filters to help in extracting the signal out of noise. The resultant system is represented in Fig. 4.1.

In this chapter a system such as shown in Fig. 4.1 is studied. No delays are introduced in the two channels. The complete system from signal source to receiving end, including the feedback link is treated as dynamic control system perturbed by noises in the two channels. Except for the stationarity of the processes no assumptions are made regarding either signal or noise.

Referring to Fig. 4.1, the following terms are defined.

$s(t)$ - Signal being sent

$n_1(t)$ - Noise in the forward channel

$n_2(t)$ - Noise in the feedback channel

$c(t)$ - Output at the receiving end

$H_1(\omega)$ - Filter transfer function in the forward channel

$H_2(\omega)$ - Filter transfer function in the feedback channel

Applying standard control system technique Fig. 4.1 is reduced to an equivalent open loop system with inputs s(t), $n_1(t)$ and $n_2(t)$ and output c(t) as shown in Fig. 4.2. Representing the equivalent transfer functions as G's, we get

$$G_1(\omega) = \frac{H_1(\omega)}{1 + H_1(\omega)\; H_2(\omega)}$$

$$G_2(\omega) = \frac{H_1(\omega)\; H_2(\omega)}{1 + H_1(\omega)\; H_2(\omega)} \tag{4.1}$$

The corresponding impulse responses $g_1(t)$ and $g_2(t)$ are

$$g_1(t) = F^{-1} \frac{H_1(\omega)}{1 + H_1(\omega)\; H_2(\omega)}$$

$$g_2(t) = F^{-1} \frac{H_1(\omega)\; H_2(\omega)}{1 + H_1(\omega)\; H_2(\omega)} \tag{4.2}$$

where F^{-1} is the inverse Fourier transform i.e.

$$g_1(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{H_1(\omega)}{1 + H_1(\omega)\; H_2(\omega)} e^{-j\omega t} d\omega \tag{4.3}$$

The equivalent system could be written as in Fig. 4.3

The output and the mean square error can be written in the form of convolution integrals of the inputs and the effective impulse responses $g_1(t)$ and $g_2(t)$.

Hence we can write

$$\varepsilon(t) = c(t) - s(t)$$

$$= \int_{-\infty}^{\infty} s(t-\tau)\ g_1(\tau)d\ + \int_{-\infty}^{\infty} n_1(t-\tau)\ g_1(\tau)d\tau \qquad (4.4)$$

$$-\int_{-\infty}^{\infty} n_2(t-\tau)\ g_2(\tau)\ d\tau - s(t)$$

where $\varepsilon(t)$ is the error between the signal and data received.

Also at $(t+p)$, $\varepsilon(t+p)$ is given as

$$\varepsilon(t+p) = \int s(t+p-\tau)\ g_1(\tau)d\tau + \int n_1(t+p-\tau)\ g_1(\tau)d\tau \qquad (4.5)$$

$$-\int n_2(t+p-\tau)\ g_2(\tau)d\tau - s(t+p)$$

The auto-correlation function (3) of error signal can be calculated as

$$R_{\varepsilon\varepsilon}(p) = \text{Expected value of } [\varepsilon(t+p)\varepsilon(t)]$$

$$= \overline{\varepsilon(t + p)\ \varepsilon(t)}$$

Substitution from [4.4] and [4.5] results in

$$R_{\varepsilon\varepsilon}(p) = \overline{\varepsilon(t + p)\ \varepsilon(t)}$$

$$= \iint \overline{s(t+p-\tau)\ s(t-\tau)}\ g_1(\tau)\ g_1(\sigma)d\tau d\sigma$$

$$+ \iint \overline{s(t+p-\tau)\ n_1(t-\sigma)}\ g_1(\tau)\ g_1(\sigma)\ d\tau\ d\sigma$$

$$- \iint \overline{s(t+p-\tau)\ n_2(t-\sigma)}\ g_1(\tau)\ g_2(\sigma)\ d\tau\ d\sigma$$

$$- \int \overline{s(t+p-\tau)\ s(t)}\ g_1(\tau)\ d\tau$$

$$+ \iint \overline{s(t-\tau)\ n_1(t+p-\sigma)}\ g_1(\tau)\ g_1(\sigma)\ d\tau\ d\sigma$$

$$+ \iint \overline{n_1(t+p-\tau)\ n_1(t-\sigma)}\ g_1(\tau)\ g_1(\sigma)\ d\tau\ d\sigma$$

$$- \iint \overline{n_1(t+p-\tau)\ n_2(t-\sigma)}\ g_1(\tau)\ g_2(\sigma)\ d\tau\ d\sigma$$

$$- \int \overline{n_1(t+p-\tau)\ s(t)}\ g_1(\tau)\ d\tau$$

$$- \iint \overline{s(t-\tau)\ n_2(t+p-\sigma)}\ g_1(\tau)\ g_2(\sigma)\ d\tau\ d\sigma$$

$$- \iint \overline{n_1(t-\tau)\ n_2(t+p-\sigma)}\ g_1(\tau)\ g_2(\sigma)\ d\tau\ d\sigma$$

$$+ \iint \overline{n_2(t-\tau)\ n_2(t+p-\sigma)}\ g_2(\tau)\ g_2(\sigma)\ d\tau\ d\sigma$$

$$+ \int \overline{s(t)\ n_2(t+p-\tau)}\ g_2(\tau)\ d\tau$$

$$- \int \overline{s(t-\tau)\ s(t+p)}\ g_1(\tau)\ d\tau - \int \overline{s(t+p)\ n_1(t-\tau)}\, g_1(\tau)\, d\tau$$

$$+ \int \overline{s(t+p)\ n_2(t-\tau)}\ g_2(\tau)\ d\tau + \overline{s(t+p)\ s(t)}$$

where the bar represents the expected value.

As $\overline{S(t+p-\tau)\ n_1(t-\sigma)} = R_{sn_1}(\sigma+p-\tau)$ and so on, we can represent these expected values in te ms of correlation functions. Hence the auto-correlation of error is

$$
\begin{aligned}
R_{\varepsilon\varepsilon}(p) = & \iint R_{ss}(\sigma-\tau+p)\ g_1(\tau)\ g_1(\sigma)\ d\tau\ d\sigma \\
& + \iint R_{sn_1}(\sigma-\tau+p)\ g_1(\tau)\ g_1(\sigma)\ d\tau\ d\sigma \\
& - \iint R_{sn_2}(\sigma-\tau+p)\ g_1(\tau)\ g_2(\sigma)\ d\tau\ d\sigma \\
& - \int R_{ss}(p-\tau)\ g_1(\tau)\ d\tau \\
& + \iint R_{sn_1}(\sigma-\tau-p)\ g_1(\tau)\ g_1(\sigma)\ d\tau\ d\sigma \\
& + \iint R_{n_1n_1}(\sigma+p-\tau)\ g_1(\tau)\ g_1(\sigma)\ d\tau\ d\sigma \\
& - \iint R_{n_1n_2}(\sigma-\tau+p)\ g_1(\tau)\ g_2(\sigma)\ d\tau\ d\sigma \\
& - \int R_{sn_1}(\tau-p)\ g_1(\tau)\ d\tau \\
& - \iint R_{sn_2}(\sigma-\tau-p)\ g_1(\tau)\ g_2(\sigma)\ d\sigma\ d\tau \\
& - \iint R_{n_1n_2}(\sigma-\tau-p)\ g_1(\tau)\ g_2(\sigma)\ d\sigma\ d\tau \\
& + \iint R_{n_2n_2}(\sigma-\tau-p)\ g_2(\sigma)\ g_2(\tau)\ d\sigma\ d\tau \\
& + \int R_{sn_2}(\tau-p)\ g_2(\tau)\ d\tau - \int R_{ss}(p+\tau)\ g_1(\tau)\ d\tau \\
& - \int R_{sn_1}(p+\tau)\ g_1(\tau)\ d\tau + \int R_{sn_2}(p+\tau)\ g_2(\tau)\ d\tau + R_{ss}(p)
\end{aligned}
\qquad (4.6)
$$

Where R_{ss}, R_{sn_1}, R_{sn_2}, $R_{n_1n_2}$, $R_{n_1n_1}$, $R_{n_2n_2}$, etc., are the correlation functions. The error spectral density is given by the following expression [3]:

$$S_{\varepsilon\varepsilon}(\omega) = \int_{-\infty}^{\infty} R_{\varepsilon\varepsilon}(p)\ e^{-j\omega p}\ dp \tag{4.7}$$

since

$$\begin{aligned} R_{sn_1}(\sigma-p-\tau) &= R_{n_1s}(p+\tau-\sigma) \\ R_{sn_1}(\tau-p) &= R_{n_1s}(p-\tau) \\ R_{n_1n_2}(\sigma-p-\tau) &= R_{n_2n_1}(p+\tau-\sigma) \end{aligned} \tag{4.8}$$

By substituting p = 0 in equation (4.7) we can write the expression of mean square error as follows

$$\overline{\varepsilon^2(0)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{\varepsilon\varepsilon}(\omega)\ d\omega$$

Mathematical simplification results in the following:

$$\begin{aligned} \overline{\varepsilon^2} = & \int g_1(\tau)d\tau \int [R_{ss}(\sigma-\tau)+R_{sn_1}(\sigma-\tau)+R_{n_1s}(\sigma-\tau)+R_{n_1n_1}(\sigma-\tau)] \\ & g_1(\sigma)\ d\sigma + \int g_1(\tau)\ d\tau \int [- R_{sn_2}(\sigma-\tau) - R_{n_1n_2}(\sigma-\tau)-R_{n_2s}(\tau-\sigma) \\ & - R_{n_2n_1}(\tau-\sigma)]\ g_2(\sigma)\ d\sigma + \int g_1(\tau)\ d\tau\ [- R_{ss}(-\tau)- R_{n_1s}(-\tau) \\ & - R_{ss}(\tau) - R_{sn_1}(\tau)] + \int [R_{n_2s}(-\tau) + R_{sn_2}(\tau)]\ g_2(\tau)\ d\tau \\ & + \int g_2(\tau)\ d\tau \int g_2(\sigma)\ R_{n_2n_2}(\tau-\sigma)\ d\sigma + R_{ss}(0) \end{aligned} \tag{4.9}$$

where the limits are from $-\infty$ to ∞ and R's are the correlation function. Letting

$$
\begin{aligned}
E_1 &= R_{ss}(\sigma-\tau) + R_{sn_1}(\sigma-\tau) + R_{n_1s}(\sigma-\tau) + R_{n_1n_1}(\sigma-\tau) \\
E_2 &= -R_{sn_2}(\sigma-\tau) - R_{n_1n_2}(\sigma-\tau) - R_{n_2s}(\tau-\sigma) - R_{n_2n_2}(\tau-\sigma) \\
E_3 &= -R_{ss}(-\tau) - R_{n_1s}(-\tau) - R_{ss}(\tau) - R_{sn_1}(\tau) \\
E_4 &= R_{n_2s}(-\tau) + R_{sn_2}(\tau) \qquad (4.10) \\
E_5 &= R_{n_2n_2}(\tau-\sigma)
\end{aligned}
$$

We get the following expression of mean square error

$$
\overline{\varepsilon^2} = \int g_1(\tau)\, d\tau \int E_1 g_1(\sigma)\, d\sigma + \int g_1(\tau)\, d\tau \int E_2 g_2(\sigma)\, d\sigma + \int E_3 g_1(\tau) d\tau + \int E_4 g_2(\tau) d\tau + R_{ss}(0) + \int g_2(\tau) d\tau \int E_5 g_2(\sigma) d\sigma \qquad (4.11)
$$

Our aim in this analysis is to find the optimum values of $H_1(\omega)$ and $H_2(\omega)$ which minimize the mean square error. These can be found by applying the variational techniques to the effective impulse responses $g_1(t)$ and $g_2(t)$ [4]. Since we have two variables, we can take variation of $g_1(t)$ and keep $g_2(t)$ constant. This will give one set of expression in $g_1(t)$ and $g_2(t)$. Then we take the variation of $g_2(t)$ keeping $g_1(t)$

constant. This will give another set of expression in $g_1(t)$ and $g_2(t)$. The optimum filter could be found from the two expressions.

Let us take variation of the impulse response $g_1(t)$, keeping $g_2(t)$ constant. Replacing $g_1(t)$ by $g_1(t) + \delta h(t)$ where $h(t)$ vanishes at the boundaries as in Appendix A. Hence equation (4.11) becomes

$$\overline{\varepsilon^2}\ [g_1(\tau) + \delta h(\tau)] = \iint d\tau\ d\sigma\ E_1[g_1(\tau)\ g_1(\sigma) + \delta g_1(\tau)\ h(\sigma) + \delta g_1(\sigma)\ h(\tau) + \delta^2 h(\tau)\ h(\sigma)] + \int d\tau\ d\sigma\ E_2\ g_2(\sigma)\ [g_1(\tau) + \delta h(\tau)] + \int E_3\ d\tau\ [g_1(\tau) + \delta h(\tau)] + R_{ss}(0) + \int E_4\ g_2(\tau)\ d\tau + \int E_5\ g_2(\tau)\ g_2(\sigma)\ d\tau\ d\sigma$$

Taking the derivative of $\overline{\varepsilon^2}$ with respect to δ and letting $\delta \to 0$, we get

$$0 = \frac{d}{d\delta}\ \delta\overline{\varepsilon^2}\ \Big|_{\delta = 0} = \int g_1(\tau) d\tau \int E_1 h(\sigma) d\sigma + \int g_1(\sigma) d\sigma \int E_1 h(\tau) d\tau + \int g_2(\sigma) d\sigma \int E_2 h(\tau) d\tau + \int E_3 h(\tau) d\tau \quad (4.12)$$

where the limits of integration are from $-\infty$ to ∞. Since

$$\int g_1(\tau)\ d\tau \int E_1 h(\sigma) d\sigma = \int g_1(\sigma) d\sigma \int E_1 h(\tau) d\tau$$

and E_1 being even in argument; we get

$$\int [2 \int E_1 g_1(\sigma) d\sigma + \int E_2 g_2(\sigma) d\sigma + E_3]\ h(\tau) d\tau = 0 \quad (4.13)$$

Since this is valid for all value of $h(\tau)$, the term in the parenthesis is identically equal to zero. Hence we get

$$2 \int E_1 g_1(\sigma)d\sigma + \int E_2 g_2(\sigma)d\sigma + E_3 = 0 \tag{4.14}$$

Where E_1, E_2, and E_3 are defined in Egn. (4.10)

Taking the variation of $g_2(t)$, keeping $g_1(t)$ constant and replacing $g_2(t)$ by $g_2(t) + \delta' f(t)$ where $f(t)$ vanishes at the boundaries. Equation (4.11) leads to the following expression:

$$\begin{aligned}
&\overline{\varepsilon^2}\,[g_2(\tau) + \delta' f(\tau)] \\
&\quad = \iint d\tau\, d\sigma\, E_1 g_1(\tau)\, g_1(\sigma) \\
&\quad + \iint d\tau\, d\sigma\, E_2 g_1(\tau)\, [g_2(\sigma) + \delta' f(\sigma)] \\
&\quad + \iint E_3 g_1(\tau)d\tau + \int E_4\, [g_2(\tau) + \delta' f(\tau)]d\tau + R_{ss}[0] \\
&\quad + \iint E_5 [g_2(\tau) + \delta' f(\tau)]\, [g_2(\sigma) + \delta' f(\sigma)]d\tau\, d\sigma
\end{aligned} \tag{4.15}$$

Taking derivative of equation (4.15) with respect to δ' and letting $\delta' \to 0$, results

$$\begin{aligned}
0 = \frac{d}{d\delta'}\, \overline{\delta\varepsilon^2}\,\Big|_{\delta' = 0} &= \int g_1(\tau)d\tau \int E_2\, f(\sigma)d\sigma + \int E_4\, f(\sigma)d\sigma \\
&+ \int g_2(\tau)d\tau \int E_5\, f(\sigma)d\sigma + \int g_2(\sigma)d\sigma \int E_5\, f(\tau)d\tau
\end{aligned} \tag{4.16}$$

Since E_5 is even in argument, i.e. $R_{n_2 n_2}(\tau-\sigma) = R_{n_2 n_2}(\sigma-\tau)$, we get

$$\int [2 \int E_5 g_2(\tau)d\tau + \int E_2 g_1(\tau)d\tau + E_4] \, f(\sigma)d\sigma = 0 \tag{4.17}$$

where limits of integration are from $-\infty$ to ∞. Since the above expression is valid for all values of $f(\sigma)$, the term in the parenthesis is identically equal to zero. Hence we have

$$2 \int E_5 g_2(\tau)d\tau + \int E_2 g_1(\tau)d\tau + E_4 = 0. \tag{4.18}$$

Substituting for E_1, E_2, E_3, E_4, and E_5 in the above expression and taking the Fourier transform of (4.14) and (4.18) we get

$$\begin{aligned} 2\, G_1(\omega) \, [S_{ss}(\omega)+S_{sn_1}(\omega)+S_{n_1 s}(\omega)+S_{n_1 n_1}(\omega)] - G_2(\omega) \, [S_{sn_2}(\omega) + S_{n_2 n_1}(\omega) \\ +S_{n_1 n_2}(\omega)+S_{n_2 s}(\omega)] - 2\, S_{ss}(\omega) - 2\, S_{sn_1}(\omega) = 0 \end{aligned} \tag{4.19}$$

$$\begin{aligned} 2\, G_2(\omega) \, [S_{n_2 n_2}(\omega)] - 2\, G_1(\omega) \, [S_{sn_2}(\omega) + S_{n_1 n_2}(\omega)] \\ + 2\, S_{sn_2}(\omega) = 0 \end{aligned} \tag{4.20}$$

where S's are the spectral densities of various processes. Equations (4.19) and (4.20) could be written in matrix form and solved for $G_1(\omega)$ and $G_2(\omega)$. Thus we get the following expression

$$\begin{bmatrix} 2\left[S_{ss}(\omega)+S_{n_1s}(\omega)+S_{sn_1}(\omega)+S_{n_1n_1}(\omega)\right] - \left[S_{sn_2}(\omega)+S_{n_2s}(\omega)+S_{n_2n_1}(\omega)+S_{n_1n_2}(\omega)\right] \\ -\left[S_{sn_2}(\omega) + S_{n_1n_2}(\omega)\right] \qquad S_{n_2n_2}(\omega) \end{bmatrix}$$

$$\begin{bmatrix} G_1(\omega) \\ G_2(\omega) \end{bmatrix} = \begin{bmatrix} 2\left[S_{ss}(\omega)+S_{sn_1}(\omega)\right] \\ -S_{sn_2}(\omega) \end{bmatrix} \tag{4.21}$$

Then

$$G_1(\omega) = \frac{1}{\Delta}\begin{vmatrix} 2\,[S_{ss}(\omega)+S_{sn_1}(\omega)] - [S_{sn_2}(\omega)+S_{n_2s}(\omega)+S_{n_2n_1}(\omega)+S_{n_1n_2}(\omega)] \\ -S_{sn_2}(\omega) \qquad S_{n_2n_2}(\omega) \end{vmatrix} \tag{4.22}$$

and

$$G_2(\omega) = \frac{1}{\Delta}\begin{vmatrix} 2\,[S_{ss}(\omega)+S_{n_1s}(\omega)+S_{sn_1}(\omega)+S_{n_1n_1}(\omega)] & 2\,[S_{ss}(\omega)+S_{sn_1}(\omega)] \\ -\,[S_{sn_2}(\omega) + S_{n_1n_2}(\omega)] & -\,S_{sn_2}(\omega) \end{vmatrix} \tag{4.23}$$

Where Δ is the determinant of the square matrix on the left of expression (4.21) or

$$\Delta = 2\,S_{n_2n_2}(\omega)\;[S_{ss}(\omega)+S_{n_1s}(\omega)+S_{sn_1}(\omega)+S_{n_1n_1}(\omega)]$$
$$-\,[S_{sn_2}(\omega)+S_{n_1n_2}(\omega)]\;[S_{sn_2}(\omega)+S_{n_2s}(\omega)+S_{n_1n_2}(\omega)+S_{n_2n_1}(\omega)] \tag{4.24}$$

Substituting for $G_1(\omega)$ and $G_2(\omega)$ in terms of $H_1(\omega)$ and $H_2(\omega)$ as

$$G_1(\omega) = \frac{H_1(\omega)}{1 + H_1(\omega)\; H_2(\omega)}$$

$$G_2(\omega) = \frac{H_1(\omega)\; H_2(\omega)}{1 + H_1(\omega)\; H_2(\omega)}$$

the optimum values of $H_1(\omega)$ and $H_2(\omega)$ for the general cases of stationary signals and noises are calculated.

Various cases of correlation between signal for noises may exist in a practical case. In the following section, the optimum filters for these specific cases of correlation will be found. It is to be observed that these filters are optimum as far as minimum mean square error criterian is concerned, but they may not be physically realizable.

General Case

$s(t)$, $n_1(t)$, $n_2(t)$ all correlated, all the processes real. From (4.22), (4.23), and (4.24), we get

$$H_1(\omega) = \frac{S_{sn_2}(\omega)\;[S_{n_2n_1}(\omega)+S_{n_2s}(\omega)] - S_{n_2n_2}(\omega)\;[S_{ss}(\omega)+S_{sn_1}(\omega)]}{\begin{aligned}\Big[&S_{sn_2}(\omega)\;[S_{n_2n_1}(\omega)+S_{n_2s}(\omega) - 2S_{n_1s}(\omega)+S_{sn_1}(\omega) - S_{n_1n_1}(\omega)] \\ &+S_{n_1n_2}(\omega)\;[S_{n_2n_1}(\omega)+S_{n_2s}(\omega)+S_{ss}(\omega)+S_{sn_1}(\omega)] \\ &-S_{n_2n_2}(\omega)\;[S_{ss}(\omega)+S_{n_1n_1}(\omega)+2S_{n_1s}(\omega)]\Big]\end{aligned}} \tag{4.25}$$

and

$$H_2(\omega) = \frac{\Big[S_{sn_2}(\omega)\ [S_{ss}(\omega)+S_{n_1n_1}(\omega)+S_{n_1s}(\omega)+S_{sn_1}(\omega)]-[S_{ss}(\omega)+S_{sn_1}(\omega)]\,[S_{sn_2}(\omega)+S_{n_1n_2}(\omega)]\Big]}{\Big[S_{sn_2}(\omega)\ [S_{n_2n_1}(\omega)+S_{n_2s}(\omega)]\ -S_{n_2n_2}(\omega)\ [S_{ss}(\omega)+S_{sn_1}(\omega)]\Big]} \tag{4.26}$$

Case I

$s(t)$, $n_1(t)$, $n_2(t)$ all uncorrelated, all processes real and noise means zero.

i.e. $$S_{sn_1}(\omega) = S_{n_1s}(\omega) = 0$$

$$S_{sn_2}(\omega) = S_{n_2s}(\omega) = 0$$

$$S_{n_1n_2}(\omega) = S_{n_2n_1}(\omega) = 0$$

Substitution of above values in (4.25) and (4.26) results

$$H_1(\omega) = \frac{S_{ss}(\omega)}{S_{ss}(\omega)+S_{n_1n_1}(\omega)}$$

$$H_2(\omega) = 0 \tag{4.27}$$

This is a very interesting result. It shows that for a system where signal and noise are uncorrelated it does not pay to use the feedback link. Similar result is derived for one way system in reference [3].

Case II

$n_1(t)$ and $n_2(t)$ correlated, all processes real and noise means zero. i.e.

$$S_{sn_1}(\omega) = S_{n_1s}(\omega) = 0$$

$$S_{sn_2}(\omega) = S_{n_2s}(\omega) = 0$$

This leads to following expressions of $H_1(\omega)$ and $H_2(\omega)$

$$H_1(\omega) = \frac{S_{ss}(\omega)}{S_{ss}(\omega)\left[1 - \frac{S_{n_1n_2}(\omega)}{S_{n_2n_2}(\omega)}\right] + S_{n_1n_1}(\omega) - \frac{S_{n_1n_2}(\omega)S_{n_2n_1}(\omega)}{S_{n_2n_2}(\omega)}} \tag{4.28}$$

and

$$H_2(\omega) = \frac{S_{n_1n_2}(\omega)}{S_{n_2n_2}(\omega)} \tag{4.29}$$

Case III

$s(t)$ and $n_1(t)$ correlated, all processes real and noise means zero i.e.

$$S_{sn_2}(\omega) = S_{n_2s}(\omega) = 0$$

$$S_{n_1n_2}(\omega) = S_{n_2n_1}(\omega) = 0$$

This leads to

$$H_1(\omega) = \frac{S_{ss}(\omega) + S_{sn_1}(\omega)}{S_{ss}(\omega) + S_{n_1n_1}(\omega) + 2S_{n_1s}(\omega)}$$

$$H_2(\omega) = 0 \tag{4.30}$$

Case IV

$s(t)$, and $n_2(t)$ correlated, all processes real and noise means zero;

$$\text{i.e.}\quad S_{sn_1}(\omega) = S_{n_1s}(\omega) = 0$$

$$S_{n_1n_2}(\omega) = S_{n_2n_1}(\omega) = 0$$

This leads to

$$H_1(\omega) = \frac{S_{ss}(\omega) - \dfrac{S_{sn_2}(\omega)S_{n_2s}(\omega)}{S_{n_2n_2}(\omega)}}{S_{ss}(\omega)+S_{n_1n_1}\left[1+\dfrac{S_{sn_2}(\omega)}{S_{n_2n_2}(\omega)}\right] - \dfrac{S_{sn_2}(\omega)S_{n_2s}(\omega)}{S_{n_2n_2}(\omega)}} \tag{4.31}$$

$$H_2(\omega) = \frac{S_{n_1n_1}(\omega)}{S_{n_2s}(\omega) - \dfrac{S_{n_2n_2}(\omega)\ S_{ss}(\omega)}{S_{sn_2}(\omega)}} \tag{4.32}$$

The above results are tabulated in Table 4.1

TABLE 4 - 1

OPTIMUM FILTERS FOR FEEDBACK SYSTEM

WITHOUT DELAY

Correlation	$H_1(\omega)$	$H_2(\omega)$
S, n_1, n_2 Uncorrelated	$\dfrac{S_{ss}(\omega)}{S_{ss}(\omega) + S_{n_1n_1}(\omega)}$	0
n_1, n_2 Correlated	$\dfrac{S_{ss}(\omega)}{\left[S_{ss}(\omega)\left[1-\dfrac{S_{n_1n_2}(\omega)}{S_{n_2n_2}(\omega)}\right]+ S_{n_1n_1}(\omega)-\dfrac{S_{n_1n_2}(\omega)S_{n_2n_1}(\omega)}{S_{n_2n_2}(\omega)}\right]}$	$\dfrac{S_{n_1n_2}(\omega)}{S_{n_2n_2}(\omega)}$
S, n_1 Correlated	$\dfrac{S_{ss}(\omega) + S_{sn_1}(\omega)}{S_{ss}(\omega) + S_{n_1n_1}(\omega) + 2S_{n_1s}(\omega)}$	0
S, n_2 Correlated	$\dfrac{S_{ss}(\omega) - \dfrac{S_{sn_2}(\omega)\ S_{n_2s}(\omega)}{S_{n_2n_2}(\omega)}}{\left[S_{ss}(\omega)+S_{n_1n_1}(\omega)\left[1+\dfrac{S_{sn_2}(\omega)}{S_{n_2n_2}(\omega)}\right]- \dfrac{S_{sn_2}S_{n_2s}(\omega)}{S_{n_2n_2}(\omega)}\right]}$	$\dfrac{S_{n_1n_1}(\omega)}{\left[S_{n_2s}(\omega) - \dfrac{S_{n_2n_2}(\omega)S_{ss}(\omega)}{S_{sn_2}(\omega)}\right]}$

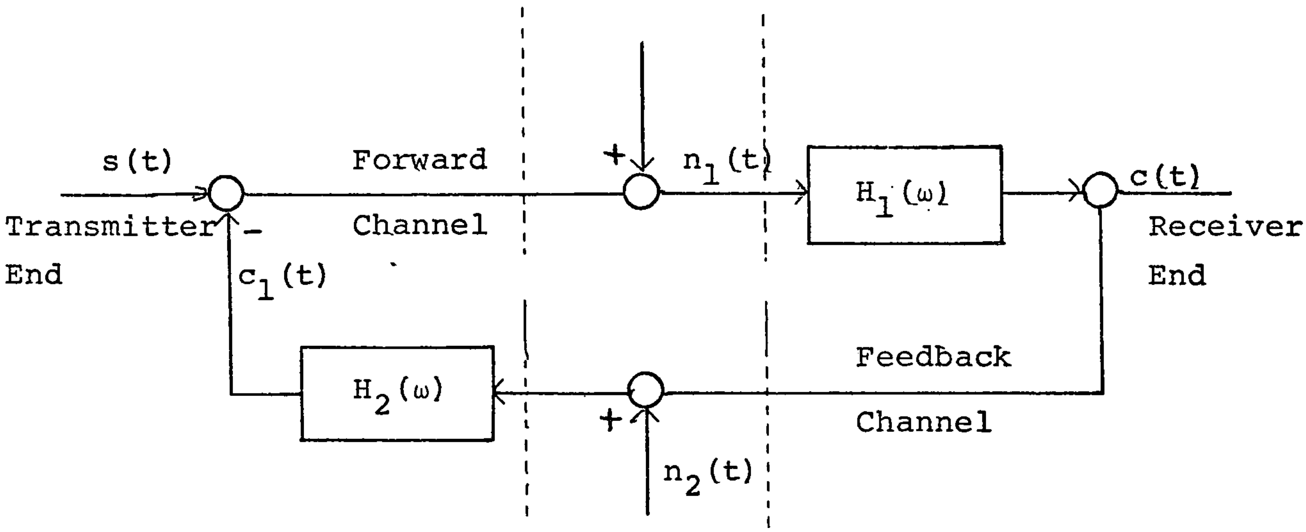


Fig. 4.1

Feedback Telemetry System

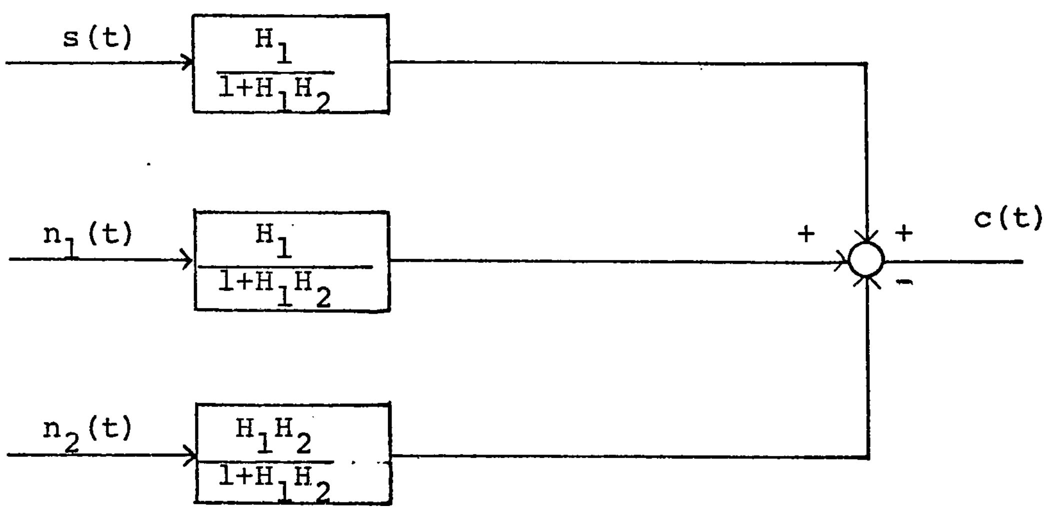


Fig. 4.2

Equivalent Open-loop System of Fig 4.1

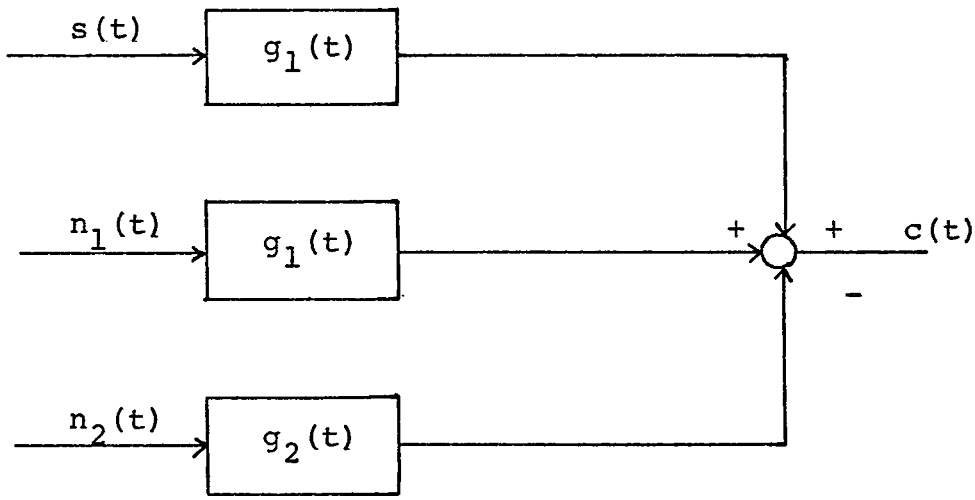


Fig. 4.3

Equivalent Open-loop system of Fig. 4.1

CHAPTER V

OPTIMUM FILTER DESIGN FOR NOISY FEEDBACK SYSTEM-DELAYS IN CHANNELS

In practical cases of feedback telemetry, inherent delays exist in the system. The delays may include transmission delays in the channel or delays at the two terminals or some forced delay to help in optimizing the system. Assuming the delays in the two channels as t_1 and t_2 respectively and the rest of the parameters as defined in Chapter IV, the system may be represented as in Fig. 5.1 . The delays t_1 and t_2 may be represented by transfer functions $e^{-t_1 s}$ and $e^{-t_2 s}$. The resultant block diagram is given in Fig. 5.2. Writing the function in S-domain, we may write

$$C(s) = n_1(s)\ H_1(s) + C_2(s)\ H_1(s)\ e^{-t_1 s}$$

and
$$C_2(s) = S(s) - n_2(s)\ H_2(s) - C(s)\ H_2(s)\ e^{-t_2 s}$$

On simplification, this reduces to

$$C(s) = n_1(s)\frac{H_1(s)}{1 + H_1(s)\ H_2(s)\ e^{-(t_1 + t_2)s}} + \frac{S(s)\ .H_1(s)\ e^{-t_1 s}}{1 + H_1(s)\ H_2(s)\ e^{-(t_1 + t_2)s}}$$

$$-n_2(s) \frac{H_1(s)\ H_2(s)}{1 + H_1(s)\ H_2(s)\ e^{-(t_1 + t_2)s}}$$

The block diagram may be written in term of equivalent open-loop transfer function with the resulting block diagram as shown in Fig. 5.3, where $G_1(s)$, $G_2(s)$ and $G_3(s)$ are effective system functions for $n_1(t)$, $n_2(t)$ and $s(t)$ respectively.

$$G_1(s) = \frac{H_1(s)}{1 + H_1(s)\ H_2(s)\ e^{-(t_1 + t_2)s}}$$

$$G_2(s) = \frac{H_1(s)\ H_2(s)\ e^{-t_1 s}}{1 + H_1(s)\ H_2(s)\ e^{-(t_1 + t_2)s}} \tag{5.1}$$

$$G_3(s) = \frac{H_1(s)\ e^{-t_1 s}}{1 + H_1(s)\ H_2(s)\ e^{-(t_1 + t_2)s}}$$

In terms of impulse response, the system could be represented as in Fig. 5.4, where

$$g_1(t) = F^{-1} \frac{H_1(\omega)}{1 + H_1(\omega)\ H_2(\omega)\ e^{-(t_1 + t_2)j\omega}}$$

$$g_2(t) = F^{-1} \frac{H_1(\omega)\ e^{-t_1 \cdot j\omega}\ H_2(\omega)}{1 + H_1(\omega)\ H_2(\omega)\ e^{-(t_1 + t_2)j\omega}} \tag{5.2}$$

$$g_1(t-t_1) = F^{-1} \frac{H_1(\omega)\ e^{-t_1 \cdot j\omega}}{1 + H_1(\omega)\ H_2(\omega)\ e^{-(t_1 + t_2)j\omega}}$$

Where F^{-1} is the inverse Fourier transform.

The output signal and error could be written as convolution integrals of inputs and the impulse responses $g_1(t)$ and $g_2(t)$. The error is given as

$$\varepsilon(t) \doteq c(t) - s(t)$$

$$= \int s(t - \tau)\, g_1(\tau - t_1)d\tau + \int n_1(t - \tau)g_1(\tau)d\tau$$

$$- \int n_2(t - \tau)g_2(\tau)d\tau - s(t) \tag{5.3}$$

and the mean square error is given as

$$\overline{\varepsilon^2} = \int g_1(\tau-t_1)d\tau \int g_1(\sigma-t_1)\, R_{ss}(\sigma-\tau)d\sigma$$

$$+ 2 \int g_1(\tau-t_1)d\tau \int g_1(\sigma)\, R_{sn_1}(\sigma-\tau)d\sigma$$

$$- \int 2g_1(\tau-t_1)\, d\tau \int g_2(\sigma)\, R_{sn_2}(\sigma-\tau)d\sigma$$

$$+ \int g_1(\tau)d\tau \int g_1(\sigma)\, R_{n_1n_1}(\sigma-\tau)d\sigma$$

$$+ \int g_2(\tau)d\tau \int g_2(\sigma)\, R_{n_2n_2}(\sigma-\tau)d\sigma$$

$$- 2 \int g_1(\tau)d\tau \int g_2(\sigma)\, R_{n_1n_2}(\sigma-\tau)d\sigma$$

$$- 2 \int g_1(\tau-t_1)\, R_{ss}(\tau)d\tau - 2 \int g_1(\tau)\, R_{sn_1}(\tau)d\tau$$

$$+ 2 \int g_1(\tau)\, R_{sn_2}(\tau)d\tau$$

$$+ R_{ss}(0) \tag{5.4}$$

The limits are from $-\infty$ to ∞ and R's are the correlation functions of various signal and noise processes.

We are interested in the expressions of $g_1(t)$ and $g_2(t)$ which minimize the mean square error. The procedure is the same as in Chapter IV and variations of $g_1(t)$ and $g_2(t)$ are taken one at a time and keeping the other constant.

Taking the variation of $g_2(t)$, keeping $g_1(t)$ constant and applying the condition of minimality as in Appendix B, we get

$$\frac{d}{de}\,\overline{\delta\varepsilon^2}\Big|_{e=0} = -2 \int g_1(\tau-t_1)d\tau \int h(\sigma)\, R_{sn_2}(\sigma-\tau)d\sigma$$

$$+ \int g_2(\tau)d\tau \int h(\sigma)\, R_{n_2n_2}(\sigma-\tau)d\sigma$$

$$+ \int g_2(\sigma)d\sigma \int h(\tau)\, R_{n_2n_2}(\sigma-\tau)d\tau$$

$$- 2 \int g_1(\tau)d\tau \int h(\sigma)\, R_{n_1n_2}(\sigma-\tau)d\sigma$$

$$+ 2 \int R_{sn_2}(\tau)\, h(\tau)d\tau = 0 \qquad (5.5)$$

where $eh(t)$ is the variation of $g_2(t)$ and $h(t)$ satisfies the boundary conditions as given in Appendix A. Since

$$\int g_2(\tau)d\tau \int R_{n_2n_2}(\sigma-\tau)\, h(\sigma)d\sigma = \int g_2(\sigma)d\sigma \int R_{n_2n_2}(\sigma-\tau)h(\tau)\, d\tau$$

$$\qquad (5.6)$$

i.e. $R_{n_2n_2}(\tau)$ being even, we can write

$$\int h(\sigma)\ [\int -\ g_1(\tau-t_1)\ R_{sn_2}\ (\sigma-\tau)d\tau\ +\ \int R_{n_2n_2}(\sigma-\tau)g_2(\tau)d\tau$$

$$-\int R_{n_1n_2}(\sigma-\tau)\ g_1(\tau)d\tau + R_{sn_2}(\sigma)]\ d\sigma = 0 \qquad (5.7)$$

For this expression to be valid for all values of $h(\sigma)$, the term in parenthesis is identically equal to zero; i.e.

$$-\int R_{sn_2}(\sigma-\tau)\ g_1(\tau-t_1)d\tau + \int R_{n_2n_2}(\sigma-\tau)g_2(\tau)d\tau$$

$$-\int R_{n_1n_2}(\sigma-\tau)\ g_1(\tau)d\tau + R_{sn_2}(\sigma) = 0 \qquad (5.8)$$

The limits of integration are from $-\infty$ to ∞.

Before taking the variation of $g_1(t)$, we have to change the variable suitably since the expression for mean square error contains terms in $g_1(\tau-t_1)$.

Changing the variables such that

$$\tau-t_1 = \tau_1$$

and

$$\sigma-t_1 = \sigma_1 \qquad (5.9)$$

and substituting (5.9) in (5.4), we get the expression of mean square error as

$$\overline{\varepsilon^2} = \int g_1(\tau_1)d\tau_1 \int g_1(\sigma_1)\ R_{ss}(\sigma_1-\tau_1)d\sigma_1$$

$$+\ 2\int g_1(\tau)\ d\tau \int g_1(\sigma)\ R_{sn_1}(\sigma-\tau_1-t_1)\ d\sigma$$

$$-2 \int g_1(\tau_1)\, d\tau_1 \int g_2(\sigma)\, R_{sn_2}(\sigma-\tau_1-t_1)\, d\sigma$$

$$+ \int g_1(\tau) d\tau \int g_1(\sigma)\, R_{n_1n_1}(\sigma-\tau) d\sigma$$

$$+ \int g_2(\tau) d\tau \int g_2(\sigma)\, R_{n_2n_2}(\sigma-\tau) d\sigma$$

$$- 2\int g_1(\tau) d\tau \int g_2(\sigma)\, R_{n_1n_2}(\sigma-\tau) d\sigma$$

$$- 2\int g_2(\tau_1) d\tau_1\, R_{ss}(\tau_1+t_1) - 2 \int g_1(\tau)\, R_{sn_1}(\tau) d\tau$$

$$+ \int 2\, g_2(\tau)\, R_{sn_2}(\tau) d\tau + R_{ss}(0) \tag{5.10}$$

Taking the variation of $g_1(t)$, keeping $g_2(t)$ constant and applying the condition of minimality as in Appendix B; we get

$$\frac{d}{de'} \delta\varepsilon^2\Big|_{e'=0} = \int h(\tau_1)\, [\int g_1(\sigma_1)\, R_{ss}(\sigma_1-\tau_1) d\sigma_1$$

$$+ \int g_1(\sigma)\, R_{sn_1}(\sigma-\tau_1-t_1) d\sigma + \int g_1(\sigma)\, R_{sn_1}(\tau_1-\sigma-t_1) d\sigma$$

$$- \int g_2(\sigma)\, R_{sn_2}(\sigma-\tau_1-t_1) d\sigma - R_{ss}(\tau_1+t_1)]\, d\tau_1$$

$$+ \int [\int g_1(\sigma)\, R_{n_1n_1}(\sigma-\tau) d\sigma - \int g_2(\sigma)\, R_{n_1n_2}(\sigma-\tau) d\sigma$$

$$- R_{sn_1}(\tau)]\, h(\tau) d\tau = 0 \tag{5.11}$$

where $e'h(t)$ is the variation of $g_1(t)$ satisfying the boundary conditions. In the first part of (5.11) τ_1 is a dummy variable and we can replace it by τ. With this change, (5.11) becomes

$$\int h(\tau)\Big[\int g_1(\sigma_1)\, R_{ss}(\sigma_1-\tau)d\sigma_1 + \int g_1(\sigma)\, R_{sn_1}(\sigma-\tau-t_1)d\sigma$$

$$+ \int g_1(\sigma)\, R_{sn_1}(\tau-\sigma-t_1)d\sigma - \int g_2(\sigma)\, R_{sn_2}(\sigma-\tau-t_1)d\sigma - R_{ss}(\tau+t_1)$$

$$+ \int g_1(\sigma)\, R_{n_1n_1}(\sigma-\tau)d\sigma - \int g_2(\sigma)\, R_{n_1n_2}(\sigma-\tau)d\sigma$$

$$- R_{sn_1}(\tau)\Big]\, d\tau = 0 \tag{5.12}$$

For (5.12) to be valid for all $h(\tau)$, we have

$$\int g_1(\sigma_1)\, R_{ss}(\sigma_1-\tau)d\sigma_1 + \int g_1(\sigma)\, R_{sn_1}(\sigma-\tau-t_1)d\sigma$$

$$+ \int g_1(\sigma)\, R_{sn_1}(\tau-\sigma-t_1)d\sigma - \int g_2(\sigma)\, R_{sn_2}(\sigma-\tau-t_1)d\sigma$$

$$+ \int g_1(\sigma)\, R_{n_1n_1}(\sigma-\tau)d\sigma - \int g_2(\sigma)\, R_{n_1n_2}(\sigma-\tau)d\sigma$$

$$- R_{ss}(\tau+t_1) - R_{sn_1}(\tau) = 0 \tag{5.13}$$

Taking the Fourier transform of (5.8) and (5.13), we get two simultaneous equations in $G_1(\omega)$ and $G_2(\omega)$ as follows;

$$-G_1(\omega)[S_{n_1n_2}(\omega)+S_{sn_2}(\omega)e^{j\omega t_1}]+G_2(\omega)\, S_{n_2n_2}(\omega) = -S_{sn_2}(\omega) \tag{5.14}$$

$$G_1(\omega)[S_{ss}(\omega)+S_{n_1n_1}(\omega)+S_{sn_1}(\omega)e^{-j\omega t_1} + S_{n_1s}(\omega)e^{j\omega t_1}]$$

$$-G_2(\omega)[S_{n_2n_1}(\omega) + S_{n_2s}(\omega)\, e^{j\omega t_1}] = S_{sn_1}(\omega)+S_{ss}(\omega)e^{-j\omega t_1} \tag{5.15}$$

(5.14) and (5.15) can be solved for $G_1(\omega)$ and $G_2(\omega)$ which in turn will give the optimum values of $H_1(\omega)$ and $H_2(\omega)$. Writing (5.14) and (5.15) in matrix form

$$\begin{bmatrix} -\left[S_{n_1n_2}(\omega) + S_{sn_2}(\omega)\, e^{j\omega t_1}\right] & S_{n_2n_2}(\omega) \\ \left[S_{ss}(\omega)+S_{n_1n_1}(\omega)+S_{sn_1}(\omega)e^{-j\omega t_1}+S_{n_1s}(\omega)e^{j\omega t_1}\right] & -\left[S_{n_2n_1}+S_{n_2s}(\omega)e^{j\omega t_1}\right] \end{bmatrix}$$

$$\times \begin{bmatrix} G_1(\omega) \\ G_2(\omega) \end{bmatrix} = \begin{bmatrix} -S_{sn_2}(\omega) \\ S_{sn_1}(\omega) + S_{ss}(\omega)e^{-j\omega t_1} \end{bmatrix}$$

This leads to values of $G_1(\omega)$ and $G_2(\omega)$ as follows

$$G_1(\omega) = \frac{1}{\Delta}\left(S_{sn_2}(\omega)\left[S_{n_2n_1}(\omega)+S_{n_2s}(\omega)e^{j\omega t_1}\right]-S_{n_2n_2}(\omega)\left[S_{sn_1}(\omega) + S_{ss}(\omega)e^{-j\omega t_1}\right]\right)$$

$$G_2(\omega) = \frac{1}{\Delta}\left(S_{sn_2}(\omega)\left[S_{ss}(\omega)+S_{n_1n_1}(\omega)+S_{sn_1}(\omega)e^{-j\omega t_1}+S_{n_1s}(\omega)e^{j\omega t_1}\right] - \left[S_{sn_1}(\omega)+S_{ss}(\omega)e^{-j\omega t_1}\right]\left[S_{n_1n_2}(\omega)+S_{sn_2}(\omega)e^{j\omega t_1}\right]\right)$$

Where $\Delta = \left[S_{n_1n_2}(\omega)+S_{sn_2}(\omega)e^{j\omega t_1}\right]\left[S_{n_2n_1}(\omega)+S_{n_2s}(\omega)e^{j\omega t_1}\right] - S_{n_2n_2}(\omega)\left[S_{ss}(\omega)+S_{n_1n_1}(\omega)+S_{sn_1}(\omega)e^{-j\omega t_1}+S_{n_1s}(\omega)e^{j\omega t_1}\right]$

From equation (5.1)

$$H_2(\omega) = e^{j\omega t_1}\,\frac{G_2(\omega)}{G_1(\omega)}$$

$$H_1(\omega) = \frac{G_1(\omega)}{1-G_2(\omega)e^{-j\omega t_2}}$$

or

$$H_2 = \frac{e^{+j\omega t}\left(S_{sn_2}(\omega)\left[S_{ss}(\omega)+S_{n_1n_1}(\omega)+S_{sn_1}(\omega)e^{-j\omega t_1}+S_{n_1s}e^{j\omega t_1}\right] - \left[S_{sn_1}(\omega)+S_{ss}(\omega)e^{-j\omega t_1}\right]\left[S_{n_1n_2}(\omega)+S_{sn_2}(\omega)e^{j\omega t_1}\right]\right)}{\left(S_{sn_2}(\omega)\left[S_{n_2n_1}(\omega)+S_{n_2s}(\omega)e^{j\omega t_1}\right] - S_{n_2n_2}(\omega)\left[S_{sn_1}(\omega)+S_{ss}(\omega)e^{-j\omega t_1}\right]\right)}$$

and (5.16)

$$H_1(\omega) = \frac{S_{sn_2}(\omega)\left[S_{n_2n_1}(\omega)+S_{n_2s}(\omega)e^{j\omega t_1}\right] - S_{n_2n_2}(\omega)\left[S_{sn_1}(\omega)+S_{ss}(\omega)e^{-j\omega t_1}\right]}{\left(\left[S_{n_1n_2}(\omega)+S_{sn_2}(\omega)e^{j\omega t_1}\right]\left[S_{n_2n_1}(\omega)+S_{n_2s}(\omega)e^{+j\omega t_1}\right] - S_{n_2n_2}(\omega)\left[S_{ss}(\omega)+S_{n_1n_1}(\omega)+S_{sn_1}(\omega)e^{-j\omega t_1}+S_{n_1s}(\omega)e^{j\omega t_1}\right] - e^{-j\omega t_2}\left\{\left[S_{sn_2}(\omega)\left(S_{ss}(\omega)+S_{n_1n_1}(\omega)+S_{sn_1}(\omega)e^{-j\omega t_1}+S_{n_1s}(\omega)e^{j\omega t_1}\right)\right] - \left[S_{sn_1}(\omega)+S_{ss}(\omega)e^{-j\omega t_1}\right]\left[S_{n_1n_2}(\omega)+S_{sn_2}(\omega)e^{j\omega t_1}\right]\right\}\right)}$$

$$= \frac{\left[S_{n_2n_2}(\omega)\left\{S_{sn_1}(\omega)+S_{ss}(\omega)e^{-j\omega t_1}\right\}\right] - S_{sn_2}(\omega)\left[S_{n_2n_1}(\omega)+S_{n_2s}(\omega)e^{j\omega t_1}\right]}{\left(S_{ss}(\omega)\left[S_{n_2n_2}(\omega)-S_{n_1n_2}(\omega)e^{-j\omega(t_1+t_2)}\right] + \left[S_{n_2n_2}(\omega)+S_{sn_2}(\omega)e^{-j\omega t_2}\right] \times S_{n_1n_1}(\omega) + \right.}$$

$$+ S_{sn_1}(\omega)\Big[S_{n_2n_2}(\omega)e^{-j\omega t_1} - S_{sn_2}(\omega)e^{-j\omega(t_1+t_2)} - S_{sn_2}(\omega)e^{j\omega(t_1-t_2)} - S_{n_1n_2}(\omega)\Big]$$

$$- S_{n_1n_2}(\omega)\Big[S_{n_2n_1}(\omega)+S_{n_2s}(\omega)e^{j\omega t_1}\Big]+S_{n_2n_2}(\omega)S_{n_1s}e^{j\omega t_1}$$

$$- S_{sn_2}(\omega)\Big[S_{n_2n_1}(\omega)e^{j\omega t_1} - S_{n_1s}(\omega)e^{j\omega(t_1-t_2)}+S_{n_2s}(\omega)e^{2j\omega t_1}\Big]\Big) \tag{5.17}$$

<u>Case I</u>

$s(t)$, $n_1(t)$, $n_2(t)$ uncorrelated, noise means zero. All processes real i.e.

$$S_{sn_1}(\omega) = S_{n_1s}(\omega) = 0$$

$$S_{sn_2}(\omega) = S_{n_2s}(\omega) = 0$$

$$S_{n_1n_2}(\omega) = S_{n_2n_1}(\omega) = 0$$

This gives

$$H_1(\omega) = \frac{S_{ss}(\omega)e^{-j\omega t_1}}{S_{ss}(\omega) + S_{n_1n_1}(\omega)} \tag{5.18}$$

and

$$H_2(\omega) = 0$$

Hence for uncorrelated case the optimum system is obtained without feedback. (5.18) has also been derived in reference [33].

Case II

$n_1(t)$ and $n_2(t)$ correlated, all processes real, noise means zero.

i.e. $$S_{sn_1}(\omega) = S_{n_1 s}(\omega) = 0$$

$$S_{sn_2}(\omega) = S_{n_2 s}(\omega) = 0$$

The optimum filters for this case are,

$$H_2(\omega) = \frac{S_{n_1n_2}(\omega)}{S_{n_2n_2}(\omega)} e^{j\omega t_1} \tag{5.19}$$

$$H_1(\omega) = \frac{S_{ss}(\omega) e^{-j\omega t_1}}{\left(S_{ss}(\omega)\left[1 - \frac{S_{n_1n_2}(\omega)}{S_{n_2n_2}(\omega)} e^{-j\omega(t_1+t_2)} \right] + S_{n_1n_1}(\omega) - \frac{S_{n_1n_2}(\omega)\ S_{n_2n_1}(\omega)}{S_{n_2n_2}(\omega)} \right)} \tag{5.20}$$

Case III

$s(t)$ and $n_1(t)$ correlated, all processes real, noise means zero. i.e.

$$S_{n_1n_2}(\omega) = S_{n_2n_1}(\omega) = 0$$

$$S_{sn_2}(\omega) = S_{n_2 s}(\omega) = 0$$

The optimum filters for this case are

$$H_2(\omega) = 0$$

and

$$H_1(\omega) = \frac{S_{sn_1}(\omega) + S_{ss}(\omega)e^{-j\omega t_1}}{S_{ss}(\omega)+S_{n_1n_1}(\omega)+S_{sn_1}(\omega)e^{-j\omega t_1}+S_{n_1s}(\omega)e^{j\omega t_1}} \tag{5,21}$$

Case IV

s(t) and $n_2(t)$ correlated, all processes real, noise means zero, i.e.

$$S_{sn_1}(\omega) = S_{n_1s}(\omega) = 0$$

$$S_{n_1n_2}(\omega) = S_{n_2n_1}(\omega) = 0$$

The optimum filters for this case are

$$H_2(\omega) = \frac{S_{n_1n_1}(\omega)S_{sn_2}(\omega)}{S_{sn_2}(\omega)\ S_{n_2s}(\omega)-S_{n_2n_2}(\omega)\ S_{ss}(\omega)e^{-2j\omega t_1}} \tag{5.22}$$

and

$$H_1(\omega) = e^{-j\omega t_1} \frac{S_{ss}(\omega) - \dfrac{S_{sn_2}(\omega)\ S_{n_2s}(\omega)e^{2j\omega t_1}}{S_{n_2n_2}(\omega)}}{\left(S_{ss}(\omega)+S_{n_1n_1}(\omega)\left[1+\dfrac{S_{sn_2}(\omega)e^{-j\omega t_2}}{S_{n_2n_2}(\omega)}\right] - \dfrac{S_{sn_2}(\omega)\ S_{n_2s}(\omega)}{S_{n_2n_2}(\omega)}\ e^{2j\omega t_1}\right)} \tag{5.23}$$

The results of this chapter are summarized in Table 5.1

TABLE 5.1

OPTIMUM FILTERS FOR FEEDBACK SYSTEM WITH DELAY

	$H_1(\omega)$	$H_2(\omega)$
$s(t)$, $n_1(t)$, $n_2(t)$ Uncorrelated	$\dfrac{S_{ss}(\omega)\ e^{-j\omega t_1}}{S_{ss}(\omega)+S_{n_1n_1}(\omega)}$	0
$n_1(t)$, $n_2(t)$ Correlated	$\dfrac{S_{ss}(\omega)\ e^{-j\omega t_1}}{\left[S_{ss}(\omega)\left[1-\dfrac{S_{n_1n_2}(\omega)}{S_{n_2n_2}(\omega)}\ e^{-j\omega(t_1+t_2)}\right]+S_{n_1n_1}(\omega) - \dfrac{S_{n_1n_2}(\omega)\ S_{n_2n_1}(\omega)}{S_{n_2n_2}(\omega)}\right]}$	$\dfrac{S_{n_1n_2}(\omega)\ e^{j\omega t_1}}{S_{n_2n_2}(\omega)}$
$s(t)$, $n_1(t)$ Correlated	$\dfrac{S_{sn_1}(\omega)+S_{ss}(\omega)\ e^{-j\omega t_1}}{S_{ss}(\omega)+S_{n_1n_1}(\omega)+S_{sn_1}(\omega)\ e^{-j\omega t_1}+S_{n_1s}(\omega)e^{j\omega t_1}}$	0

S(t), $n_2(t)$ Correlated	$e^{-j\omega t_1} \dfrac{\left[S_{ss}(\omega) - \dfrac{S_{sn_2}(\omega)\ S_{n_2s}(\omega)\ e^{2j\omega t_1}}{S_{n_2n_2}(\omega)}\right]}{\left[S_{ss}(\omega)+S_{n_1n_1}(\omega)\left[1+\dfrac{S_{sn_2}(\omega)\ e^{-j\omega t_2}}{S_{n_2n_2}(\omega)}\right] - \dfrac{S_{sn_2}(\omega)\ S_{n_2s}(\omega)\ e^{2j\omega t_1}}{S_{n_2n_2}(\omega)}\right]}$	$\dfrac{S_{n_1n_1}(\omega)\ S_{ns_2}(\omega)}{\left[S_{sn_2}(\omega)\ S_{n_2s}(\omega) - S_{n_2n_2}(\omega)\ S_{ss}(\omega)e^{-2j\omega t_1}\right]}$

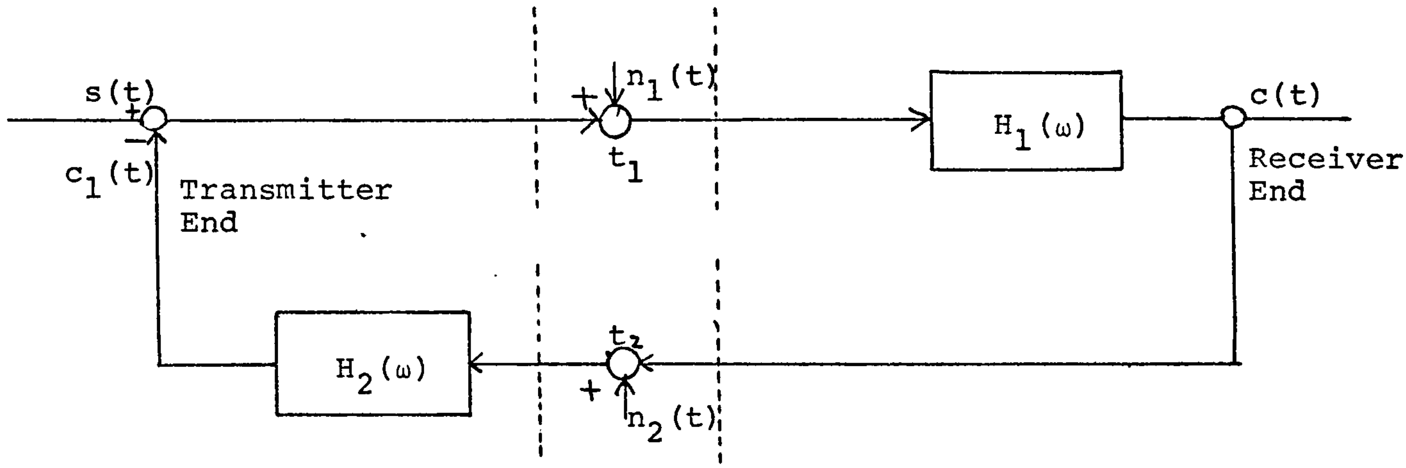


Fig. 5.1

Feedback Telemetry System with Delay

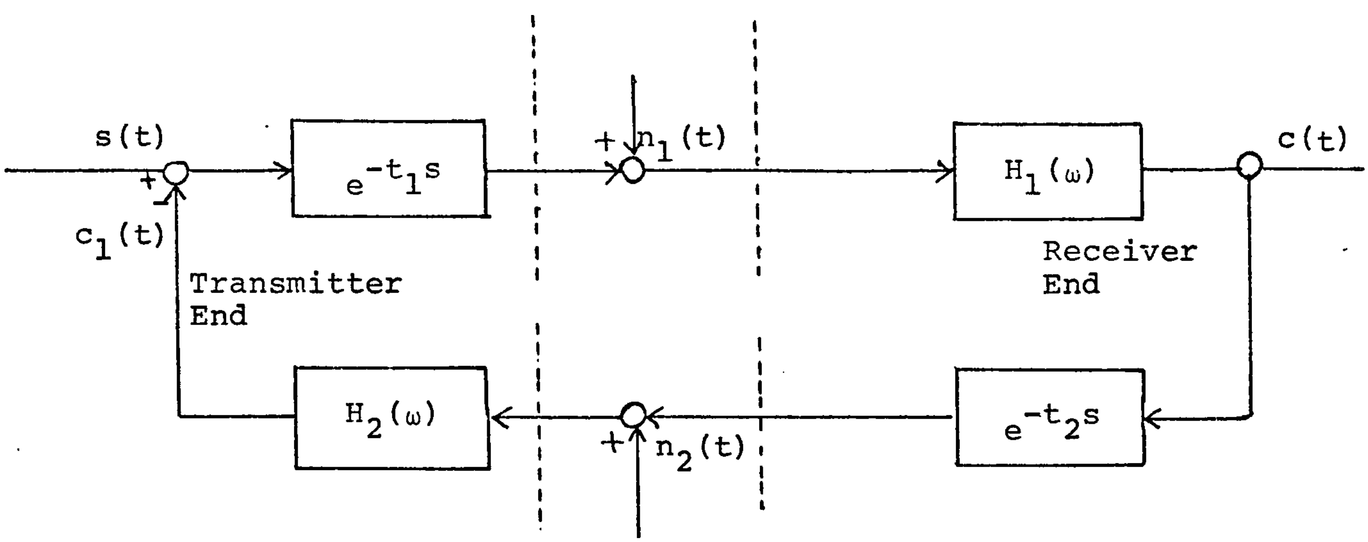


Fig. 5.2

Feedback Telemetry System
Delays represented by blocks

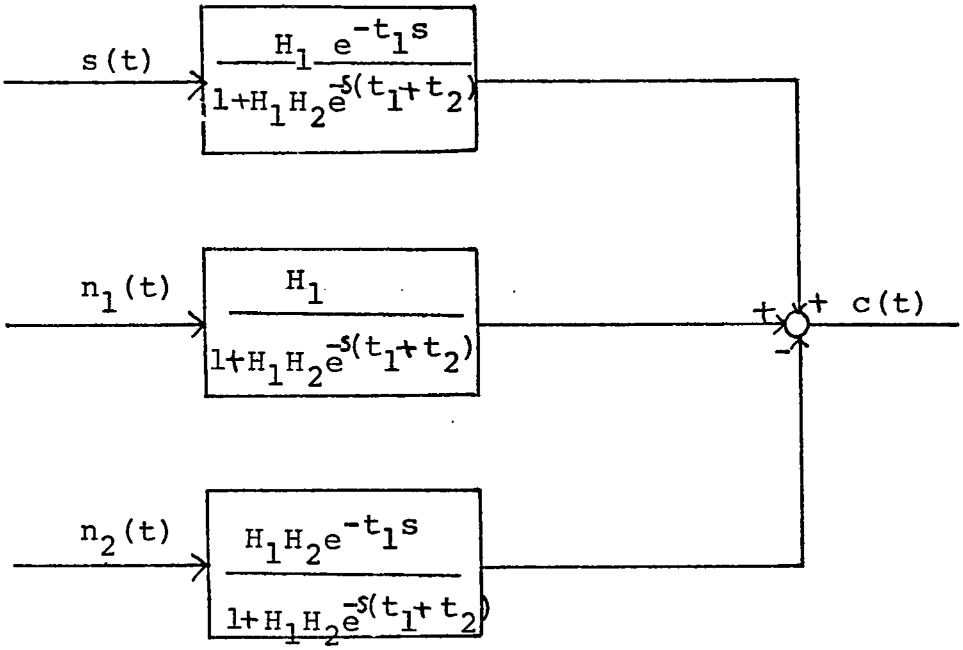


Fig. 5.3

Open loop equivalent of Fig. 5.2

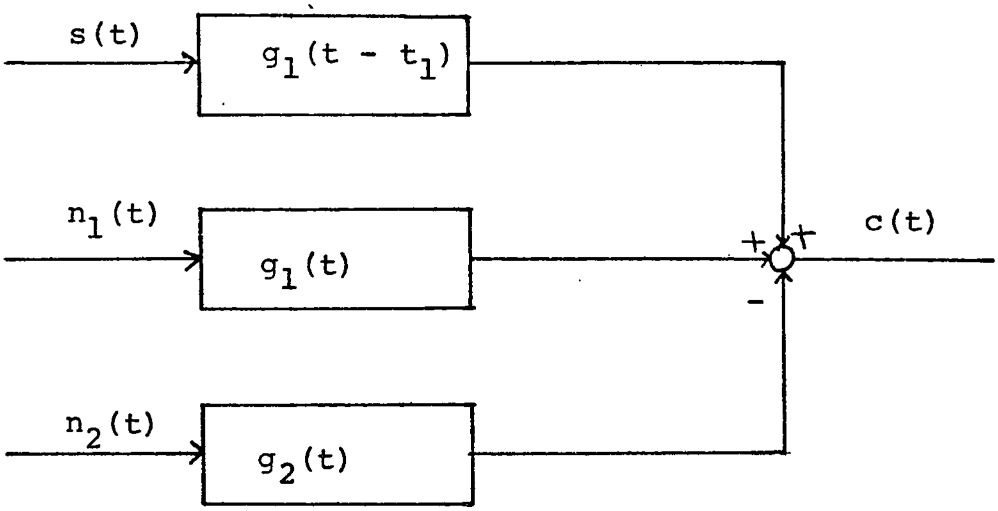


Fig. 5.4

Open loop equivalent of Fig. 5.2
Blocks show effective impulse response

CHAPTER VI

OPTIMUM REALIZABLE FILTERS FOR THE SYSTEM WITH NO DELAYS

Optimum filters have been derived in Chapter IV for the feedback telemetry system with no delays in channels. In this chapter optimum realizable filters are calculated for practical cases. The signal and noise processes are taken stationary and noise mean zero.

The following two signal spectra are considered:

a) $$S_{ss}(\omega) = \frac{4\lambda}{\omega^2+4\lambda^2}$$

b) $$S_{ss}(\omega) = \frac{1}{\omega^4+1}$$

The noises are taken White Gaussian with following spectral densities:

$$S_{n_1n_1}(\omega) = N_1$$

$$S_{n_2n_2}(\omega) = N_2$$

The expression of optimum filters from Table 4.1 are the starting point of this calculation. For the case where two processes are correlated, a general expression for the cross-correlation and cross-spectral density is assumed to cover all the possible cases. For example, for the case of correlation between noise $n_1(t)$ and noise $n_2(t)$,

$$R_{n_1n_2}(\tau) = r_1 \, e^{-a_1|\tau|}$$

so that

$$S_{n_1n_2}(\omega) = \frac{2a_1r_1}{\omega^2+a_1^2}$$

where

$$0 < r_1 < 1 \text{ and } a_1 > 0$$

Similarly for $s(t)$ and $n_1(t)$ correlated

$$R_{sn_1}(\tau) = r_2 \cdot e^{-a_2|\tau|}$$

$$S_{sn_1}(\omega) = \frac{2a_2r_2}{\omega^2+a_2^2}$$

and for $s(t)$ and $n_2(t)$ correlated

$$R_{sn_2}(\tau) = r_3 \cdot e^{a_3|\tau|}$$

$$S_{sn_2}(\omega) = \frac{2a_3r_3}{\omega^2+a_3^2}$$

where $0 < r_2 < 1$, $a_2 > 0$, $0 < r_3 < 1$ and $a_3 > 0$

By taking various values of r's and a's, we can achieve various degrees of correlation between the two processes. Representative plots of cross spectral density with various values of a's and r's are given in Appendix D.

Case I

When the signal and noise are uncorrelated:

In this case, the two optimum filters are

$$H_1(\omega) = \frac{S_{ss}(\omega)}{S_{ss}(\omega) + S_{n_1n_1}(\omega)}$$

and

$$H_2(\omega) = 0$$

Example 1

$$S_{ss}(\omega) = \frac{4\lambda}{\omega^2 + 4\lambda^2}$$

$$S_{n_1n_1}(\omega) = N_1$$

This gives

$$H_1(\omega) = \frac{4\lambda/(\omega^2 + 4\lambda^2)}{N_1 + 4\lambda/(\omega^2 + 4\lambda^2)}$$

Following the procedure as described in Chapter III and Appendix C,

$$|w(\omega)|^2 = S_{ss}(\omega) + S_{n_1n_1}(\omega)$$

$$= \frac{N_1\omega^2 + 4N_1\lambda^2 + 4\lambda}{\omega^2 + 4\lambda^2}$$

so that

$$w(\omega) = \frac{(\sqrt{N_1}\, j\omega + \sqrt{4\lambda + 4N_1\lambda^2})}{(2\lambda + j\omega)}$$

Then

$$H_1'(\omega) = W(\omega) \cdot H_1(\omega)$$

$$= \frac{4\lambda}{(2\lambda + j\omega)[\sqrt{4\lambda + 4\lambda^2 N_1} - j\omega\sqrt{N_1}]}$$

$$H_1''(\omega) = \text{Realizable part of } H_1'(\omega)$$

$$= \frac{4\lambda}{(\sqrt{4\lambda + 4\lambda^2 N_1} + 2\lambda\sqrt{N_1})\ (j\omega + 2\lambda)}$$

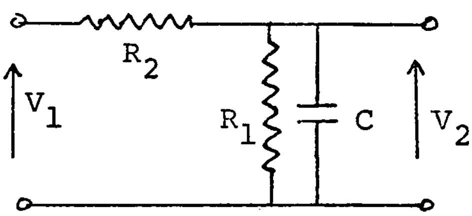
Hence the optimum realizable filter is

$$H_1^*(\omega) = H_3(\omega)/W(\omega)$$

$$= \frac{4\lambda/[\sqrt{4\lambda + 4\lambda^2 N_1} + 2\lambda\sqrt{N_1}]}{[\sqrt{4\lambda + 4\lambda^2 N_1} + j\omega\sqrt{N_1}]}$$

This can be realized by the following RC network

$$H_1 = V_2 / V_1$$



where

$$R_1 = \frac{4\lambda}{[\sqrt{4\lambda + 4\lambda^2 N_1} + 2\lambda\sqrt{N_1}]}$$

$$R_2 = \sqrt{4\lambda + 4\lambda^2 N_1} - \frac{4\lambda}{\sqrt{4\lambda + 4\lambda^2 N_1} + 2\lambda\sqrt{N_1}}$$

and $$C = \frac{1}{8\lambda^2} \quad [2\lambda\sqrt{N_1} + \sqrt{4\lambda + 4\lambda^2 N_1}]$$

Example 2

The signal spectral density for this case is taken as

$$S_{ss}(\omega) = \frac{1}{1 + \omega^4}$$

This gives

$$H_1(\omega) = \frac{1/(1 + \omega^4)}{N_1 + 1/(1 + \omega^4)}$$

$$H_2(\omega) = 0$$

$$|w(\omega)|^2 = N_1 + \frac{1}{1 + \omega^4}$$

or $$w(\omega) = \frac{(1 + N_1)^{1/2} + \sqrt{2}N_1^{1/4}(1 + N_1)^{1/4} j\omega - \sqrt{N_1}\,\omega^2}{1 - \omega^2 + \sqrt{2}\, j\omega}$$

Then $H_1'(\omega) = w(\omega) \cdot H_1(\omega)$

$$= \frac{\qquad}{(1-\omega^2+2j\omega)\left[j\omega - \frac{(1+N_1)^{1/4}(1+j)}{\sqrt{2}N_1^{1/4}}\right]\left[j\omega - \frac{(1+N_1)^{1/4}(1-j)}{\sqrt{2}N_1^{1/4}}\right]}$$

$$H_1'' = \frac{(1+N_1)^{1/4} - j\omega\sqrt{2}N_1^{1/4} - N_1^{1/4}}{[N_1^{1/4} + (1+N_1)^{1/4}]\ [1-\omega^2+\sqrt{2}\, j\omega]}$$

Hence the optimum realizable filter is

$$H_1^*(\omega) = \frac{(1+N_1)^{1/4} - N_1^{1/4} - j\omega\sqrt{2}\, N_1^{1/4}}{\left[(1+N_1)^{1/2} + \sqrt{2}\, j\omega N_1^{1/4}(1+N_1)^{1/4} - \sqrt{N_1}\,\omega^2\right][N_1^{1/4} + (1+N_1)^{1/4}]}$$

where

$$d_1 = (1 + N_1)^{1/4} - N_1^{1/4}$$

$$d_2 = \frac{(\sqrt{2} + 1)}{\sqrt{2}} \qquad (1 + N_1)^{1/4}$$

$$d_3 = \frac{(\sqrt{2} - 1)}{\sqrt{2}} \qquad (1 + N_1)^{1/4}$$

$$b_1 = -\sqrt{2}\, N_1^{1/4}$$

$$c = N_1^{1/4}$$

This can be realized by a Lattice network using passive elements. For example , for the value of $N_1 = 1$

$$H_1''(\omega) = \frac{.145 - 1.41\, j\omega}{(2.45)[1 + \sqrt{2}\, j\omega - \omega^2]}$$

So that the optimum realizable filter is

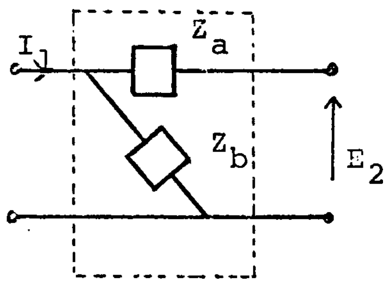
$$H_1^*(\omega) = \frac{.145 - 1.41\, j\omega}{(2.45)[1.315 + 3.6\, j\omega - \omega^2]}$$

$$= \frac{.0926}{j\omega + .414} - \frac{.6828}{j\omega + 3.186}$$

If we treat the filter transfer function as transfer impedance of network

$$H_1^*(\omega) = Z_{12}(\omega)$$

where $Z_{12}(\omega)$ is given as

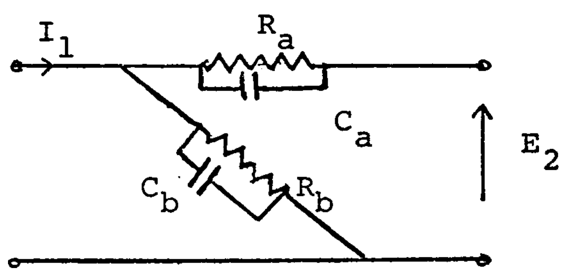


$$Z_{12}(\omega) = \frac{E_2(\omega)}{I_1(\omega)} = \frac{1}{2}[Z_b - Z_a] = H_1^*(\omega)$$

Hence for our network

$$Z_a = \frac{1.13656}{j\omega+3.186} \qquad \text{and } Z_b = \frac{.1852}{j\omega+.414}$$

This reduces to the R-C lattice network as shown below



Case II

In this case

$$H_1(\omega) = \frac{S_{ss}(\omega)}{\left(S_{ss}(\omega)\left[1-\frac{S_{n_1n_2}(\omega)}{S_{n_2n_2}(\omega)}\right]+S_{n_1n_1}(\omega)-\frac{S_{n_1n_2}(\omega)S_{n_2n_1}(\omega)}{S_{n_2n_2}(\omega)}\right)}$$

and

$$H_2(\omega) = \frac{S_{n_1n_2}(\omega)}{S_{n_2n_2}(\omega)}$$

Also

$$S_{n_1n_2}(\omega) = \frac{2a_1r_1}{\omega^2+a_1^2}$$

Example 1

$$S_{n_1n_1}(\omega) = N_1$$

$$S_{n_2n_2}(\omega) = N_2$$

$$S_{ss}(\omega) = \frac{4\lambda}{\omega^2+4\lambda^2}$$

In this case

$$H_2^*(\omega) = \frac{r_1}{N_2(j\omega+a_1)}$$

which is realized by a simple R-C network. Similarly

$$H_1(\omega) = \frac{4\lambda/(\omega^2+4\lambda^2)}{\frac{4\lambda}{(\omega^2+4\lambda^2)}\left[1-\frac{2r_1a_1}{N_2(\omega^2+a_1^2)}\right] + N_1 - \frac{(2r_1a_1)^2}{N_2(\omega^2+a_1^2)^2}}$$

This gives

$$\left|W(\omega)\right|^2 = \frac{4\lambda}{(\omega^2+4\lambda^2)}\left[1-\frac{2r_1a_1}{N_2(\omega^2+a_1^2)}\right] + \left[N_1 - \frac{(2r_1a_1)^2}{N_2(\omega^2+a_1)^2}\right]$$

$$= \frac{\left(\begin{gathered}N_1N_2\omega^6+\omega^4(4\lambda^2 N_1N_2 + 2a_1^2 N_1N_2 + 4\lambda N_2) \\ +\omega^2[N_1N_2a_1^4 - 4r_1^2a_1^2 + 8\lambda^2a_1^2 + 8\lambda N_2a_1^2 - 8\lambda r_1a_1] \\ +4\lambda a_1^3(N_2a_1-2r_1) + 4\lambda^2(N_1N_2a_1^4 - 4r_1^2a_1^2)\end{gathered}\right)}{N_2(\omega^2+a_1^2)^2\ (\omega^2+4\lambda^2)}$$

For the r. h.s. of the above equation to equal amplitude square; all the coeffecients of polynomial have to be positive. This means that there are restrictions on the parameters which give the optimum realizable filter. In this case the restrictions are

$$N_1N_2a_1^4 - 4r_1^2a_1^2 + 8\lambda^2a_1^2 + 8\lambda N_2a_1^2 - 8\lambda r_1a_1 > 0$$

and

$$a_1(N_2a_1-2r_1) + \lambda\ (N_1N_2a_1^2-4r_1^2) > 0$$

For the case, satisfying the above restrictions, we could write $|W(\omega)|^2$ as

$$|W(\omega)|^2 = \frac{(c_1^2\omega^2+d_1^2)\ (c_2^2\omega^2+d_2^2)^2}{(\omega^2+4\lambda)\ (\omega^2+a_1^2)^2}$$

where c's and d's are functions of a_1, r_1, N_1, N_2 and λ. The expression for optimum filter is reduced to

$$H_1^*(\omega) = \frac{4\lambda(a_1+2\lambda)^2}{(d_1+2\lambda c_1)\ (d_2+2\lambda c_2)^2}\ \frac{(j\omega+a_1)^2}{(j\omega c_1+d_1)\ (j\omega c_2+d_2)^2}$$

which can be realized by a ladder network.

For $N_1 = N_2 = a_1 = \lambda = 1$, $r_1 < \frac{1}{2}$ satisfies the above restrictions. Let $r_1 = .1$. This gives

$$|W(\omega)|^2 = \frac{4}{(\omega^2+4)}\left[1-\frac{.2}{\omega^2+1}\right]+\left[1-\frac{.04}{(\omega^2+1)^2}\right]$$

$$= \frac{4(\omega^2+.8)\ (\omega^2+1) + (\omega^2+4)\ [\omega^4+2\omega^2+.96]}{(\omega^2+4)\ (\omega^2+1)^2}$$

$$= \frac{\omega^6+10\omega^4+16.16\omega^2+7.04}{(\omega^2+4)\ (\omega^2+1)^2}$$

By factorizing we get

$$W(\omega) = \frac{(j\omega+1.04)\ (j\omega+.894)\ (j\omega+2.85)}{(j\omega+2)\ (j\omega+1)^2}$$

So
$$H_1'(\omega) = \frac{4(-j\omega+1)^2}{(j\omega+2)\ (-j\omega+1.04)\ (-j\omega+.896)\ (-j\omega+2.85)}$$

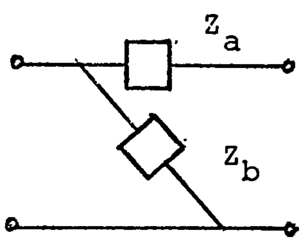
and
$$H_1''(\omega) = \frac{.844}{(j\omega+2)}$$

Hence the optimum realizable filter is

$$H_1^*(\omega) = \frac{.844(j\omega+1)^2}{(j\omega+1.04)\ (j\omega+.894)\ (j\omega+2.85)}$$

$$= \frac{1.01\ j\omega+.98}{(j\omega+.894)\ (j\omega+2.85)} - \frac{.005}{(j\omega+1.0^4)}$$

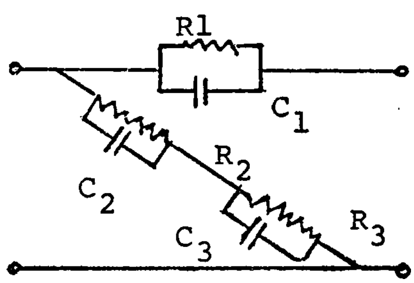
This can be realized by a lattice network



where
$$Z_a = \frac{.01}{j\ \ +.04}$$

and
$$Z_b = \frac{2.02\ j\ \omega+1.96}{(j\omega+.894)(j\omega+2.85)}$$

or



Example 2

$$S_{ss}(\omega) = \frac{1}{\omega^4+1}$$

In this case

$$H_2(\omega) = \frac{2a_1r_1/(\omega^2+a_1^2)}{N_2}$$

Hence

$$H_2^*(\omega) = \frac{r_1}{N_2(j\omega+a_1)}$$

which is the same as in example 1.

Also

$$H_1(\omega) = \frac{1/1 + \omega^4}{\frac{1}{(1+\omega^4)}\left[1-\frac{2a_1r_1}{N_2(\omega^2+a_1^2)}\right] + \left[N_1 - \frac{(2r_1a_1)^2}{(\omega^2+a_1^2)^2}\ \frac{1}{N_2}\right]}$$

$$|w(\omega)| = \frac{1}{(1+\omega^4)}\left[1-\frac{2ar_1}{N_2(\omega^2+a_1^2)}\right] + N_1 - \frac{(2r_1a_1)^2}{N_2(\omega^2+a_1^2)^2}$$

$$= \frac{\left(N_1N_2\omega^8 + 2N_1N_2a_1^2\omega^6 + \omega^4(N_2 + N_1N_2 + N_1N_2a_1^4 - 4r_1^2a_1^2) + \omega^2(2N_2a_1^2 + 2N_1N_2a_1^2 - 2a_1r_1) + N_2a_1^4 + N_1N_2a_1^4 - 4r_1^2a_1^2\right)}{N_2(1 + \omega^4)\ (\omega^2 + a_1^2)^2}$$

For the realizable filter, the restrictions are

$$N_2 + N_1N_2 + N_1N_2\, a_1^4 - 4r_1^2\, a_1^2 > 0$$

$$2N_2\, a_1^2 + 2N_1N_2\, a_1^2 - 2a_1r_1 > 0$$

and $$N_2a_1^4 + N_1N_2\, a_1^4 - 2a^2r_1 - 4r_1^2\, a_1^2 > 0$$

Satisfying the above restrictions, we obtain

$$|w(\omega)|^2 = \frac{(b_1{}^2\, \omega^2 + c_1{}^2)^2\ (b_2{}^2\omega^2 + c_2{}^2)^2}{(1 + \omega^4)\quad (\omega^2 + a_1^2)^2}$$

where

$$b_1{}^4\, b_2{}^4 = N_1$$

$$c_1^4\, c_1^4 = \frac{1}{N_2}\, [a_1^2(N_2a_1^2 - 2ra_1) + N_1N_2\, a_1 4 - 4r_1^2\, a_1^2]$$

$$2b_1^2\, b_2^2(c_1^2\, b_2^2 + b^2c_1^2) = 2a_1^2\, N_1$$

$$(4b_1^2\, b_2^2\, c_1^2\, c_2^2 + b_1^4\, c_2^4 + b_2^4\, c_1^4) = \frac{1}{N_2}\Big[N_2 + N_1N_2a_1^4 - 4\, r_1{}^2a_1{}^2 + N_1N_2\Big]$$

$$c_1^2\, c_2^2\, [c_1^2\, b_2^2 + b_1^2\, c_2^2] = \frac{1}{N_2}\quad [a_1^2\, N_2 - r_1a_1 + a_1^2N_1N_2]$$

So

$$w(\omega) = \frac{(c_1 + j\omega b_1)^2\ (c_2 + j\omega b_2)^2}{(a_1 + j\omega)^2\ [1-\omega^2 + j\omega\sqrt{2}]}$$

Then

$$H_1'(\omega) = \frac{(a_1-j\omega)^2}{(c_1-j\omega b_1)^2\ (c_2-j\omega b_2)^2\ [1-\omega^2+j\omega\sqrt{2}]}$$

$$H_1''(\omega) = \text{Realizable part of } H_2(\omega)$$

$$= \frac{K_1}{j\omega+\frac{1}{\sqrt{2}}+j/\sqrt{2}} + \frac{K_2}{j\omega+\frac{1}{\sqrt{2}}-j\ /\sqrt{2}}$$

where K_1 and K_2 are the residues in partial fraction expansion of $H_2(\omega)$, K_2 = complex conjugate of K_1

If $K_1 = p + j\ q$

$$H_1''(\omega) = \frac{j\omega p+\sqrt{2}(p+q)}{1-\omega^2 + j\omega\sqrt{2}}$$

and the realizable filter is

$$H_1^*(\omega) = \frac{(a_1+j\omega)^2\left[j\omega p+\sqrt{2}(p+q)\right]}{(c_1+j\omega b_1)^2\ (c_2+j\omega b_2)^2}$$

which can be realized by a ladder network.

For example

for $N_1 = N_2 = a_1 = 1$, $r_1 < 1/2$ satisfies the restriction. For $r_1 = .1$

$$|w(\omega)|^2 = \frac{\omega^8 + 2\omega^6 + 2.96\omega^4 + 3.8\omega^2 + 1.76}{(\omega^4 + 1)\ (\omega^2 + 1)^2}$$

So

$$w(\omega) = \frac{(j\omega+.895)(j\omega+1.067)(j\omega+.843+j.824)(j\omega+.843-j.824)}{(j\omega+1)^2\ (1-\omega^2+j\omega\sqrt{2})}$$

Hence

$$H_1'(\omega) = \frac{(-j\omega + 1)^2}{\left[(1-\omega^2+j\omega\sqrt{2})\ (-j\omega+.895)(-j\omega+1.067)(+j\omega-.843+j.824)\cdot (j\omega-.843-j.824)\right]}$$

Then

$$H_1''(\omega) = \frac{j\omega p + \sqrt{2}(p+q)}{1 - \omega^2 + j\omega\sqrt{2}}$$

where p and q are defined as above

$$K_1 = p + jq = \text{Residue of } (j\omega + \frac{1}{\sqrt{2}} + j/\sqrt{2})$$

$$= \frac{(.707 + j.707 + 1)^2}{\left[(j1.414)\ (.707+j.707+.895)(.707+j.707+1.067)\cdot (-.707-8.707-.843+j.824)\cdot (-.707-j.707-.843-j.824)\right]}$$

$$= \frac{(1.707+j.707)^2}{\left[(j1.414)(1.602+j.707)(1.774+j.707)(-1.55+j.117)\cdot (-1.55-j1.531)\right]}$$

$$= -.139-j.161$$

So

$$H_1''(\omega) = \frac{-.139j\omega-.212}{1-\omega^2+j\omega\sqrt{2}}$$

and optimum realizable filter is

$$= \frac{(-.139j\omega-.212)\ (j\omega+1)^2}{(j\omega+.895)\ (j\omega+1.067)\ (1.59+1.686j\omega-\omega^2)}$$

This can be realized by Lattice network.

Case III

When s(t) and forward channel noise $n_1(t)$ are correlated

In this case

$$H_1(\omega) = \frac{S_{ss}(\omega) + S_{sn_1}(\omega)}{S_{ss}(\omega) + S_{n_1n_1}(\omega) + 2S_{n_1s}(\omega)}$$

and

$$H_2(\omega) = 0$$

Example 1

$$S_{ss}(\omega) = \frac{4\lambda}{\omega^2+4\lambda^2}$$

$$H_1(\omega) = \frac{\dfrac{4\lambda}{\omega^2+4\lambda^2} + \dfrac{2a_2r_2}{\omega^2+a2^2}}{N_1+\dfrac{4\lambda}{\omega^2+4\lambda^2} + \dfrac{4a_2r_2}{\omega^2+a_2^2}}$$

$$= \frac{(c_1^2\omega^2+d_1^2)\ (c_1^2\omega_2^2+d_2^2)}{(\omega^2+4\lambda^2)\ (\omega^2+a_2^2)}$$

where

$$c_1^4 = N_1$$

$$c_1{}^2(d_1^2 + d_2^2) = N_1 a_2^2 + N_1 4\lambda^2 + 4\lambda + 4a_2 r_2$$

$$d_1^2\, d_2^2 = 4\lambda^2 a_2^2\, N_1 + 16 a_2 r_2 \lambda^2 + 4a_2^2 \lambda$$

So

$$W(\omega) = \frac{(d_1+j\omega c_1)\ (d_2+j\omega c_1)}{(2\lambda+j\omega)\ (a_2+j\omega)}$$

$$H_1'(\omega) = H_1(\omega)\, H(\omega)$$

$$= \frac{\omega^2(4\lambda+2a_2r_2) + 4a_2^2 + 8a_2r_2\lambda^2}{(j\omega+2\lambda)\ (j\omega+a_2)\ (d_1-j\omega c_1)\ (d_2-j\omega c_1)}$$

$$H_1''(\omega) = \text{Realizable part of } H_2(\omega)$$

$$= \frac{K_1}{j\omega+2\lambda} + \frac{K_2}{j\omega+a_2}$$

Where

$$K_1 = \frac{4\lambda(a_2+2\lambda)}{(d_1+2\lambda c_1)\ (d_2+2\lambda c_1)}$$

and

$$K_2 = \frac{4\lambda(a_2+2\lambda)}{(d_1+a_2c_1)\ (d_2+2\lambda c_1)}$$

Hence the realizable filter is

$$H_1^*(\omega) = \frac{j\omega(k_1+k_2) + a_2K_1 + 2\lambda K_2}{(d_1+j\omega c_1)\ (d_2+j\omega c_1)}$$

which is a ladder network,

For example for $N_1 = \lambda = a_2 = 1, r_2 = .1$

$$|w(\omega)|^2 = \frac{(j\omega+3.282)\ (j\omega+1.493)}{(j\omega+1)\ (j\omega+2)}$$

So that

$$H_1'(\omega) = \frac{6(\omega^2+2)}{(j\omega+1)\ (j\omega+2)\ (-j\omega+3.282)\ (-j\omega+1.493)}$$

This gives

$$H_1^*(\omega) = \frac{.887\ j\ \ +\ 1.449}{(j\omega+1)\ (j\omega+2)}$$

The optimum realizable filter for this case is

$$H_1^*(\omega) = \frac{(.887\ j\omega\ +\ 1.449)}{(j\omega+3.282)\ (j\omega+1.493)}$$

Assuming $H_1^*(\omega)$ as voltage transfer ratio $= \frac{z_{12}}{z_{11}}$, this can be realized as R-C ladder network in more than one way.

For example for $N_1 = a_2 = 1$ and $r_2 = .1$ we have

$$|w(\omega)|^2 = \frac{\omega^6+1.4\omega^4+2\omega^2+2.4}{(\omega^4+1)\ (\omega^2+1)}$$

or

$$w(\omega) = \frac{(j\omega+1.136)\ (j\omega+.842+j.809)\ (j\omega+.862-j.809)}{(j\omega+1)\ (1-\omega^2+j\omega\sqrt{2})}$$

Hence

$$H_1'(\omega) = \frac{.2\omega^4 + \omega^2+1.2}{\left[(j\omega+1)\cdot(1-\omega^2+j\omega\sqrt{2})\cdot(-j\ \omega+1.136)\cdot (j\omega+.842+j.809)\cdot(j\omega+.842-j.809)\right]}$$

and

$$H_1''(\omega) = \frac{.168\omega^2 + 2.444\ j\omega+7.9}{(j\omega+1)\ (1-\omega^2+j\omega\sqrt{2})}$$

Hence the optimum realizable filter is given as

$$H_1^*(\omega) = \frac{-.168\omega^2 + 2.444j\omega+7.9}{(j\omega+1.136)\ (-\omega^2+1.363+1.684j\omega)}$$

which can be realized by lattice network.

<u>Example 2</u>

$$S_{ss}(\omega) = \frac{1}{1 + \omega^4}$$

Then

$$H_1(\omega) = \frac{\dfrac{2r_2a_2}{\omega^2+a_2^2} + \dfrac{1}{1+\omega^4}}{N_1+ \dfrac{1}{1+\omega^4} + \dfrac{4r_2a_2}{\omega^2+a_2{}^2}}$$

This gives

$$|w(\omega)|^2 = \frac{N_1\omega^6 +\omega^4(N_1a_2^2+4r_2a_2)+\omega^2(N_1+1)+N_1a_2^2+a_2^2+4r_2a_2}{(1+\omega^4)\ (\omega^2+a_2^2)}$$

$$= \frac{(c_1^2\omega^2 + d_1^2)\ (c_2^2\omega^2 + d_2^2)^2}{(1 + \omega^4)\ (\omega^2 + a_2^2)}$$

where

$$c_1^2\ \ c_2^4 = N_1$$

$$c_2^2(d_1^2\ c_2^2 + 2c_1^2\ d_2^2) = N_1\ a_2^2 + 4r_2\ a_2$$

$$d_2^2(c_1^2\ d_2^2 + 2c_2^2\ d_1^2) = N_1 + 1$$

$$d_1^2\ d_2^4 = N_1 a_2^2 + a_2^2 + 4r_2 a_2$$

So

$$w(\omega) = \frac{(d_1 + j\omega c_1)\ (d_2 + j\omega c_2)^2}{(a_2 + j\omega)\ (1 - \omega^2 + j\omega\sqrt{2})}$$

Then

$$H_1'(\omega) = \frac{2r_2\ a_2\ \omega^4 + \omega^2 + a_2^2 + 2a_2r_2}{(a_2 + j\omega)\ (d_1 - j\omega c_1)\ (d_2 - j\omega c_2)^2(1-\omega^2 + j\omega\sqrt{2})}$$

$$H_1''(\omega) = \frac{K_1}{j\omega + a_2} + \frac{K_2}{j\omega + \frac{1}{\sqrt{2}} + \frac{j}{\sqrt{2}}} + \frac{K_3}{j\omega + \frac{1}{\sqrt{2}} - j/\sqrt{2}}$$

where K's are the residues in the partial fraction expansion of $H_2(\omega)$.

$$K_1 = \frac{2r_2\ a_2(1 + a_2{}^4)}{(d_1 + a_2\ c_1)\ (d_2 + a_2\ c_2)^2(1 + a_2^2 - a_2\sqrt{2})}$$

If $K_2 = p + j\,q$, $K_3 = -j\,q + p$

Then

$$\frac{K_2}{j\omega + \frac{1}{\sqrt{2}} + \frac{j}{\sqrt{2}}} + \frac{K_3}{j\omega + \frac{1}{\sqrt{2}} - \frac{j}{\sqrt{2}}} = \frac{j\,p + 2\,(p+q)}{1 - \omega^2 + j\omega\sqrt{2}}$$

$$K_2 = \frac{(1 + ja_2^2)}{\sqrt{2}(a_2 - \frac{1}{\sqrt{2}} - \frac{j}{\sqrt{2}})\,[d_1 + c_1(\frac{1}{\sqrt{2}} + \frac{j}{\sqrt{2}})]\cdot[d_2 + c_2(\frac{1}{\sqrt{2}} + \frac{j}{\sqrt{2}})]^2}$$

$$K_3 = \frac{(1 - ja_2^2)}{\sqrt{2}(a_2 + \frac{1}{\sqrt{2}} + \frac{j}{\sqrt{2}})\,[d_1 + c_1(\frac{1}{\sqrt{2}} - \frac{j}{\sqrt{2}})]\cdot[d_2 + c_2(\frac{1}{\sqrt{2}} - \frac{j}{\sqrt{2}})]^2}.$$

So

$$H_1''(\omega) = \frac{K_1 + \sqrt{2}\,a_2(p+q) + j\omega[\sqrt{2}K_1 + \sqrt{2}(p+q) + pa_2] - \omega^2(K_1+P)}{(j\omega + a_2)\,(1 - \omega^2 + j\omega\sqrt{2})}$$

where all the constants are real.

Hence

$$H_1^*(\omega) = \frac{K_1 + \sqrt{2}\,a_2(p+q) + j\omega[\sqrt{2}K_1 + \sqrt{2}(p+q) + pa_2] - \omega^2(K_1+p)}{(d_1 + j\omega c_1)\,(d_2 + j\omega c_2)^2}$$

This is also a ladder network.

Case IV

When the signal s(t) and the noise in the feedback channel $n_2(t)$ are correlated.

In this case the optimum filters are as follows;

$$H_2(\omega) = \frac{S_{n_1n_1}(\omega)}{S_{n_2s}(\omega) - \dfrac{S_{n_2n_2}(\omega)\ S_{ss}(\omega)}{S_{n_2n_2}(\omega)}}$$

and

$$H_1(\omega) = \frac{S_{ss}(\omega) - \dfrac{S_{sn_2}(\omega)\ S_{n_2s}(\omega)}{S_{n_2n_2}(\omega)}}{S_{ss}(\omega) + S_{n_1n_1}(\omega)\left[1 + \dfrac{S_{sn_2}(\omega)}{S_{n_2n_2}(\omega)}\right] - \dfrac{S_{sn_2}(\omega)\ S_{n_2s}(\omega)}{S_{n_2n_2}(\omega)}}$$

Example 1

$$S_{ss}(\omega) = \frac{4\lambda}{\omega^2 + 4\lambda^2}$$

Then

$$H_2(\omega) = \frac{N_1}{\dfrac{2a_3\ r_3}{\omega^2 + a_3^2} - \dfrac{N_2\ 4\lambda/(\omega^2 + 4\lambda^2)}{2a_3\ r_3/(\omega^2 + a_3^2)}}$$

$$= \frac{-2a_3\, r_3\, N_1 (\omega^2 + a_3^2)\ (\omega^2 + 4\lambda^2)}{\left[4\lambda N_2 \omega^4 + \omega^2 (8\lambda N_2\, a_3^2 - 4a_3^2\, r_3^2) + 4\lambda N_2 a_3^4\ -16\, a_3^2\, r_3^2\, \lambda^2\right]}$$

This suggests positive feedback. The restrictions on the parameters are as follows ;

$$8\lambda N_2\, a_3^2 - 4a_3^2\, r_3^2 > 0$$

and

$$4\lambda N_2\, a_3^2 - 16a_3^2\, r_3^2\, \lambda^2 > 0$$

For $a_3 = N_1 = \lambda = 1$ and $r_3 = .1$, we have

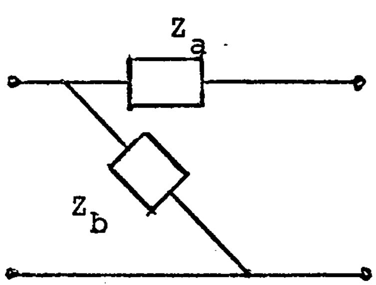
$$H_2(\omega) = \frac{2(\omega^2 + 1)\ (\omega^2 + 4)}{\omega^4 + 7.96\ \omega^2 + 3.84}$$

$$= \frac{2(\omega^2 + 1)\ (\omega^2 + 4)}{(j\omega + .718)\ (j\omega - .718)\ (j\omega + 2.728)\ (j\omega - 2.728)}$$

Hence the realizable filter is

$$H_2^*(\omega) = \frac{.34}{j\omega + .718} - \frac{1.17}{j\omega + 2.728}$$

This can be realized by a R-C lattice network



where

$$Z_a = \frac{2.34}{j\,\omega + 2.728}$$

and $$Z_b = \frac{.68}{j\,\omega + .718}$$

Also

$$H_1(\omega) = \frac{\dfrac{4\lambda}{\omega^2 + 4\lambda^2} - \dfrac{4a_3^2\, r_3^2}{N_2(\omega^2 + a_3^2)^2}}{\dfrac{4}{\omega^2 + 4\lambda^2} + N_1\left[1 + \dfrac{2a_3 r_3}{N_2(\omega^2 + a_3^2)}\right] - \dfrac{1}{N_2}\,\dfrac{4a_3^2\, r_3^2}{(\omega^2 + a_3^2)^2}}$$

$$|W(\omega)|^2 = \frac{4\lambda}{\omega^2 + 4\lambda^2} + \frac{N_1(N_2{}^2 + N_2 a_3^2 + 2a_3\, r_3)}{N_2(\omega^2 + a_3^2)} - \frac{4a_3^2\, r_3^2}{N_2(\omega^2 + a_3^2)^2}$$

$$= \frac{\begin{aligned}&\Big(N_1 N_2 \omega^6 + \omega^4\,[4\lambda N_2 + 2N_1 N_2\, a_3^2 + 4\lambda N_1 N_2 + 2N_1\, a_3\, r_3] \\ &+ \omega^2\,[8a_3^2\,\lambda N_2 + 4\lambda^2\, N_1 N_2\, a_3^2 + N_1(a_3^2 + 4\lambda^2)\,(N_2\, a_3^2 + 2a_3\, r_3) \\ &- 4a_3^2\, r_3^2\,] + 4\, N_2 a_3{}^4 - 16\, a_3{}^2\, \lambda^2\, r_3{}^2\Big)\end{aligned}}{N_2\,(\omega^2 + 4\lambda^2)\,(\omega^2 + a_3{}^2)^2}$$

For the above to represent the amplitude square

$$4\,\lambda N_2\, a_3{}^4 - 16\, a_3{}^2\, \lambda^2\, r_3{}^2 > 0$$

Satisfying the above restriction we can write

$$|W(\omega)|^2 = \frac{(\omega^2 c_1^2 + d_1^2)\,(c_2^2\omega^2 + d_2^2)^2}{(\omega^2 + 4\lambda^2)\,(\omega^2 + a_3^2)^2}$$

where $$C_1{}^2 C_2{}^4 = N_1$$

$$c_2^2(d_1^2 c_2^2 + 2c_1^2 d_2^2) = \frac{1}{N_2}\,[N_1(2a_3\, r_3 + 2a_3^2 N_2 + 4\lambda^2 N_2) + 4\lambda N_2]$$

$$d_2^2(c_1^2d_2^2 + 2c_2^2d_1^2) = \frac{1}{N_2}\Big[(a_3^2 + 4\lambda^2)\ (a_3^2N_2 + 2a_3\ r_3)N_1 + 4a_3^2\lambda^2N_1N_2 + 8\lambda N_2a_3^2 - 4a_3^2r_3^2\Big]$$

$$d_1^2d_2^4 = \frac{1}{N_2}\Big[N_1 4a_3^2\lambda^2(a_3^2N_2 + 2a_3r_3) + 4\ N_2a_3^4 - 16a_3^2r_3^2\lambda^2\Big]$$

Hence

$$w(\omega) = \frac{(j\omega c_1 + d_1)\ (j\omega c_2 + d_2)^2}{(a_3 + j\omega)^2\ (2\lambda + j\omega)}$$

and

$$H_1'(\omega) = \frac{4}{N_2}\ \frac{\left[\lambda N_2(\omega^2+a_3^2)^2 - a_3^2r_3^2(\omega^2+4\lambda^2)\right]}{(d_1-j\omega c_1)\ (d_2-j\omega c_2)^2\ (j\omega+2\lambda)\ (j\omega+a_3)^2}$$

Then

$$H_1''(\omega) = \frac{K_1}{2\lambda + j\omega} + \frac{K_2}{j\omega + a_3} + \frac{K_3}{(j\omega + a_3)^2}$$

where K's are the residues in the partial fraction expansion of $H_2(\omega)$.

Hence

$$H_1^*(\omega) = \frac{K_1(a_3 + j\omega)^2 + (j\omega + \lambda)\ \ j\omega K_2 + bK_2 + K_3}{(d_1 + j\omega c_1)\ (d_2 + j\omega c_2)^2}$$

which can be realized by a ladder network.

For $N_1 = N_2 = a_3 = \lambda = 1$, $r_3 = 0.1$, we have

$$|w(\omega)|^2 = \frac{\omega^6 + 10.2\ \omega^4 + 17.96\ \omega^2 + 3.86}{(\omega^2 + 4)\ (\omega^2 + 1)^2}$$

or

$$W(\omega) = \frac{(j\omega + .4978)\ (j\omega + 2.832)\ (j\omega + 1.3898)}{(j\omega + 1)^2\ (\omega^2 + 2)}$$

$$H_1'(\omega) = \frac{4(\omega^2+1)^2 - .04(\omega^2+4)}{(j\omega+2)\ (j\omega+1)^2\ (-j\omega+.4978)\ (-j\omega+2.832)\ (-j\omega+1.3898)}$$

Hence

$$H_1''(\omega) = \frac{A}{j\omega + 2} + \frac{j\ B + c}{(j\omega + 1)^2}$$

$$= \frac{(j\omega)^2\ A + j\omega(2A + B) + A + C}{(j\omega + 2)\ (j\omega + 1)^2}$$

The optimum realizable filter is

$$H_1^*(\omega) = \frac{(j\omega)^2\ A + (2A + B)\ j\omega + A + C}{(j\omega + .498)\ (j\omega + 2.832)\ (j\omega + 1.39)} .$$

which can be realized by a lattice or a ladder network

CHAPTER VIII

OPTIMUM REALIZABLE FILTERS FOR THE SYSTEMS WITH DELAYS

The expressions for filters which give optimum system performance were derived in Chapter V. Optimum realizable part of such filters will be calculated in this chapter. Some special cases which help in simplifying the calculations are considered.

The system delay introduces terms involving exponential in $j\omega t$, where t is the time delay. The synthesis of the filters having such exponential terms involves great difficulty in evaluating realizable components and hence an approximation of this term is necessary. Pade's approximation of exponential terms $e^{j\omega t}$ in terms of ratio of polynomials of various degrees is given by Takahashi [26] and Truxal [36]. Some of the approximations of $e^{j\omega t}$ are given in Table 7.1 and only simple cases of lower order will be used in the analysis of this chapter. Depending on the value of delay t and the number of terms included in the approximation, it introduces some error. For large values of t, a good many terms are to be taken to obtain a satisfactory approximation.

The assumption of Chapter VI for the stationarity of signal and noise are also valid for this chapter. The following two signal spectra are considered

a) $S_{ss}(\omega) = \dfrac{4\lambda}{\omega^2 + 4\lambda^2}$

b) $S_{ss}(\omega) = \dfrac{1}{1 + \omega^4}$

<u>Case I</u>

<u>When signal and noise are uncorrelated</u>

In this case, the optimum filters are

$$H_1(\omega) = \frac{S_{ss}(\omega)e^{-j\omega t_1}}{S_{ss}(\omega) + S_{n_1n_1}(\omega)}$$

and

$$H_2(\omega) = 0$$

<u>Example 1</u>

$$S_{ss}(\omega) = \frac{4\lambda}{\omega^2 + 4\lambda^2}$$

$$S_{n_1n_1}(\omega) = N_1$$

$$H_1(\omega) = \frac{4\lambda/(\omega^2 + 4\lambda^2)}{N_1 + \dfrac{4\lambda}{\omega^2 + 4\lambda^2}} e^{-j\omega t_1}$$

so

$$w(\omega) = \frac{(\sqrt{N_1}\; j\omega + \sqrt{4\lambda + 4N_1\lambda^2})}{(j\omega + 2\lambda)}$$

$$H_1'(\omega) = \frac{4\lambda}{2\lambda + j\omega} \cdot \frac{e^{-j\omega t_1}}{(\sqrt{4\lambda + 4\lambda^2 N_1} - j\omega\sqrt{N_1})}$$

writing $$e^{-j\omega t_1} = \frac{1 - j\omega t_{1/2}}{1 + j\omega t_{1/2}}.$$

so

$$H_1'(\omega) = \frac{4\lambda(1 - j\omega t_{1/2})}{(2\lambda+j\omega)(1+j\omega t_{1/2})\left[\sqrt{4 + 4\lambda^2 N_1} - j\omega\sqrt{N_1}\right]}$$

So

$$H_1''(\omega) = \frac{K_1}{2\lambda + j\omega} + \frac{K_2}{(1 + j\omega t_{1/2})}$$

$$= \frac{j\omega\left[K_1 t_{1/2} + K_2 + K_1 + 2\lambda K_2\right]}{(2\lambda+j\omega)\;(1+j\omega t_{1/2})}$$

$$K_1 = \frac{4\lambda(1 + \lambda t_1)}{(1 - \lambda t_1)\left[\sqrt{4\lambda + 4\lambda^2 N_1} + 2\lambda\sqrt{N_1}\right]}$$

$$K_2 = \frac{8\lambda}{(2\lambda - \frac{2}{t_1})\left[\sqrt{4\lambda + 4\lambda^2 N_1} + \frac{2\sqrt{N_1}}{t_1}\right]}$$

Hence

$$H_1^*(\omega) = \frac{j\omega\left[\frac{K_1\ t_1}{2} + K_2\right] + K_1 + 2\lambda K_2}{(1 + j\omega\frac{t_1}{2})\left[\sqrt{N_1}\ j\omega + \sqrt{4\lambda + 4N_1\lambda^2}\right]}$$

which is realized by a lattice network.

<u>Example 2</u>

$$S_{ss}(\omega)\ \frac{1}{1 + \omega^4}$$

$$S_{n_1 n_1}(\omega) = N_1$$

As in Chapter VI

$$w(\omega) = \frac{(1 + N_1)^{1/2} + \sqrt{2}\ N_1^{1/4}\ (1 + N_1)^{1/4}\ j\omega - \sqrt{N_1}\omega^2}{1 - \omega^2 + \sqrt{2}\ j\omega}$$

and

$$H_1'(\omega) = \frac{(1 - j\omega t_{1/2})}{\left((1 + \frac{j\omega t_1}{2})(1 - \omega^2 + \sqrt{2}j\omega)\left[(1 + N_1)^{1/2} - \sqrt{N_1}\omega^2 - \sqrt{2}j\omega N_1^{1/4}\ (1 + N_1)^{1/4}\right]\right)}$$

Hence

$$H_1''(\omega) = \frac{K_1}{j\omega + 1/\sqrt{2} + j/\sqrt{2}} + \frac{K_2}{j\omega + \frac{1}{\sqrt{2}} - j/\sqrt{2}} + \frac{K_3}{1 + j\omega \frac{t_1}{2}}$$

$$= \frac{j\omega p + \sqrt{2}(p+q)}{1 - \omega^2 + \sqrt{2}j\omega} + \frac{K_3}{1 + j\omega t_{1/2}}$$

where $K_1 = p + jq$ and $K_2 = K_1^* =$ conjugate of K_1

and

$$K_1 = \frac{1 + \frac{t_1}{2\sqrt{2}}(1 + j)}{(-2j)\left[1 - \frac{t_1}{2\sqrt{2}}(1 + j)\right]\left[N_1^{1/4} + (N_1+1)^{1/4}\right]\left[(1+N_1)^{1/4} + jN_1^{1/4}\right]}$$

and

$$K_3 = \frac{2}{\left(\left[1 - \frac{2\sqrt{2}}{t_1} + \frac{4}{t_1^2}\right]\left[(1 + N_1)^{1/2} + \frac{2\sqrt{2}}{t_1}(N_1)^{1/4}(1+N_1)^{1/4} + \frac{4\sqrt{N_1}}{t_1^2}\right]\right)}$$

so

$$H_1''(\omega) = \frac{K_3 + \sqrt{2}(p+q) - \omega^2(K_3 + \frac{pt_1}{2}) + j\omega\left[p + \sqrt{2}K_3 + \frac{t_1}{2}(p+q)\right]}{(1 + \frac{j\omega t_1}{2})\ (1 - \omega^2 + \sqrt{2}j\omega)}$$

Hence the realizable filter is

$$H_1^*(\omega) = \frac{K_3 + \sqrt{2}\ (p+q) - \omega^2(K_3 + \frac{pt_1}{2}) + j\omega\left[p + \sqrt{2}K_3 + \frac{t_1}{2}(p+q)\right]}{(1 + \frac{j\omega t_1}{2})\left[(1+N_1)^{1/2} - \sqrt{N_1}\omega^2 + \sqrt{2}N_1^{1/4}(1+N_1)^{1/4}j\omega\right]}$$

which is realized by a lattice network.

Case II

When the noise processes $n_1(t)$ and $n_2(t)$ are correlated

In this case the expressions of optimum filters are as follows;

$$H_1(\omega) = \frac{S_{ss}(\omega)\ e^{-j\omega t_1}}{\left(S_{ss}(\omega)\left[1-\frac{S_{n_1n_1}(\omega)\ e^{-j\omega(t_1+t_2)}}{S_{n_2n_2}(\omega)}\right]+S_{n_1n_1}(\omega)-\frac{S_{n_1n_2}(\omega)S_{n_2n_1}(\omega)}{S_{n_2n_2}(\omega)}\right)}$$

and

$$H_2(\omega) = \frac{S_{n_1n_2}(\omega)\ e^{j\omega t_1}}{S_{n_2n_2}(\omega)}$$

The expressions having $e^{j\omega t}$ as a multiplier can be synthesized for relizable case as in previous section. However the exponential term in the denominator which ends up as quotient of the terms in various powers of ω creates trouble in the synthesis. This is encountered in all the three cases of correlation between signal and noise.

For
$$S_{ss}(\omega) = \frac{4\lambda}{\omega^2+4\lambda^2}$$
$$S_{n_1n_1}(\omega) = \frac{2r_1a_1}{\omega^2+a_1{}^2}$$

$$H_2(\omega) = \frac{2r_1a_1/(\omega^2+a_1{}^2)}{N_2} \cdot e^{j\omega t_1}$$

This results in optimum realizable filter as

$$H_2^*(\omega) = \frac{r_1}{N_2(j\omega+a_1)} \qquad \frac{1-\frac{a_1t_1}{2}}{(1+\frac{a_1t_1}{2})}$$

Which is a low pass filter.

Also
$$H_1(\omega) = \frac{\frac{4}{(\omega^2+4\lambda^2)} \qquad e^{-j\omega t_1}}{\frac{4\lambda}{(\omega^2+4\lambda^2)}\left[1 - \frac{2r_1a_1}{N_2(\omega^2+a_1^2)} \quad e^{-j\omega(t_1+t_2)}\right] + N_1 - \frac{4r_1{}^2a_1{}^2}{N_2(\omega^2+a_1^2)^2}}$$

Expanding $e^{-j\omega(t_1+t_2)}$ by Padé approximation, factoring the denominator in terms of $\pm j\omega$ and taking the $j\omega$ terms leads to optimum realizable filters.

Another method of constructing a realizable filter by employing the distributed parameter approach is discussed by Ghausi [37]. The method is generally known as equivalent dominant pole and excess phase approximation technique. The exponential term is expanded in truncated series and only dominant realizable poles are taken. The resulting filter thus obtained can be realized by distributed parameters or a combination of distributed and lumped parameters. In many cases the circuit may include active elements too.

Filters for both case III and case IV can be realized as for case II.

TABLE 7-1

PADE' APPROXIMATION FUNCTIONS OF $e^{j\omega t}$

Denominator / Numerator	Degree 0	Degree 1	Degree 2	Degree 3
Degree 0	$\frac{1}{1}$	$\frac{1+j\omega t}{1}$	$\frac{1+j\omega t-\frac{\omega^2t^2}{2!}}{1}$	$\frac{1+j\omega t-\frac{\omega^2t^2}{2!}-\frac{j\omega^3t^3}{3!}}{1}$
Degree 1	$\frac{1}{1-j\omega t}$	$\frac{1+\frac{j\omega t}{2}}{1-\frac{j\omega t}{2}}$	$\frac{1+\frac{2}{3}j\omega t-\frac{\omega^2t^2}{3.2!}}{1-\frac{j\omega t}{3}}$	$\frac{1+\frac{3}{4}j\omega t-\frac{2}{4}\frac{\omega^2t^2}{2!}-\frac{j\omega^3t^3}{4.3!}}{1-\frac{j\omega t}{4}}$
Degree 2	$\frac{1}{1-j\omega t-\frac{\omega^2t^2}{2!}}$	$\frac{1+\frac{j\omega t}{3}}{1-\frac{2}{3}j\omega t-\frac{\omega^2t^2}{3.2!}}$	$\frac{1+\frac{j\omega t}{2}-\frac{1}{6}\frac{\omega^2t^2}{2!}}{1-\frac{j\omega t}{2}\frac{1}{6}\frac{\omega^2t^2}{2!}}$	$\frac{1+\frac{3}{5}j\omega t-\frac{3}{10}\frac{\omega^2t^2}{2!}-\frac{1}{10}\frac{j\omega^3t^3}{3!}}{1-\frac{2}{5}j\omega t-\frac{1}{10}\frac{\omega^2t^2}{2!}}$
Degree 3	$\frac{1}{\left[1-j\omega t-\frac{\omega^2t^2}{2!}+\frac{1}{3!}j\omega^3t^3\right]}$	$\frac{1+\frac{j\omega t}{4}}{1-\frac{3}{4}j\omega t-\frac{2}{4}\frac{\omega^2t^2}{2!}+\frac{1}{4}j\frac{\omega^3t^3}{3!}}$	$\frac{1+\frac{2}{5}j\omega t-\frac{1}{10}\frac{\omega^2t^2}{2!}}{1-\frac{3}{5}j\omega t-\frac{3}{10}\frac{\omega^2t^2}{2!}+j\frac{\omega^3t^3}{10.3!}}$	$\frac{1+\frac{j\omega t}{2}-\frac{1}{5}\frac{\omega^2t^2}{2!}-\frac{1}{20}\frac{j\omega^3t^3}{3!}}{1-\frac{j\omega t}{2}-\frac{1}{5}\frac{\omega^2t^2}{2!}+\frac{j\omega^3t^3}{20\ 3!}}$

TABLE 7-1 (continued)

Denominator / Numerator	Degree 0	Degree 1	Degree 2	Degree 3
Degree 4	$\dfrac{1}{\left[1-j\omega t-\frac{\omega^2t^2}{2!}+\frac{j\omega^3t^3}{3!}+\frac{\omega^4t^4}{4!}\right]}$	$\dfrac{1+\frac{j\omega t}{5}}{\left[1-\frac{4j\omega t}{5}-\frac{3}{5}\frac{\omega^2t^2}{2!}+\frac{2}{5}\frac{j\omega^3t^3}{3!}+\frac{\omega^4t^4}{5.\ 4!}\right]}$	$\dfrac{\left[1+\frac{j\omega t}{3}-\frac{\omega^2t^2}{15.\ 2!}\right]}{\left[1-\frac{2}{3}j\omega t-\frac{2}{5}\frac{\omega^2t^2}{2!}+\frac{j\omega^3t^3}{5.\ 3!}+\frac{\omega^4t^4}{15.\ 4!}\right]}$	$\dfrac{\left[1+\frac{3j\omega t}{7}-\frac{\omega^2t^2}{7.\ 2!}-\frac{j\omega^3t^3}{35.\ 3!}\right]}{\left[1-\frac{4}{3}j\omega t-\frac{2}{7}\frac{\omega^2t^2}{2!}+\frac{j\omega^3t^3}{35.\ 3!}+\frac{\omega^4t^4}{35.\ 4!}\right]}$

CHAPTER VIII

MEAN SQUARE ERROR IN VARIOUS CASES OF FEEDBACK TELEMETRY

In this chapter, various expressions of optimum filters derived in Chapter V for the optimum feedback telemetry system are utilized to calculate the mean square error of the system as a whole. Mean square errors for various degrees of correlation between various signal and noise processes are calculated. The realizability of filters are not taken into account. It will be noted that the mean square errors of the actual system using the realizable filters are different from the ones calculated in this chapter.

a) No Delay Case

The expression for the mean square error is written in terms of error spectral density as follows;

$$\text{mse} = R_{ee}(0) = \frac{1}{2\pi} \int_{\infty}^{\infty} S_{ee}(\omega)\,d\omega \qquad (8.1)$$

where $S_{ee}(\omega)$ is the error spectral density. For the system without delay, referring to Eqn. (4.5) in Chapter IV and taking the Fourier transform, we obtain

$$\begin{aligned} S_{ee}(\omega) = {} & |G_1(\omega)|^2 \left[S_{ss}(\omega) + S_{n_1n_1}(\omega) + S_{n_1s}(\omega) + S_{sn_1}(\omega)\right] \\ & - 2G_1(\omega)G_2^*(\omega)\left[S_{sn_2}(\omega) + S_{n_1n_2}(\omega)\right] + \left[G_2(\omega)S_{n_2s}(\omega) + G_2^*(\omega)S_{sn_2}(\omega)\right] \\ & - \left[S_{n_1s}(\omega)G_1(\omega) + S_{sn_1}(\omega)G_1^*(\omega)\right] - S_{ss}(\omega)\left[G_1(\omega) + G_1^*(\omega)\right] \end{aligned}$$

$$+ S_{ss}(\omega)+|G_2(\omega)|^2 S_{n_2n_2}(\omega) \tag{8.2}$$

Also since we are considering real processes with even cross spectral densities such that

$$S_{sn_1}(\omega) = S_{n_1s}(\omega)$$

We can write from Equations 4.16 and 4.17

$$G_1(\omega) = \frac{S_{n_2n_2}(\omega)\left[S_{ss}(\omega) + S_{sn_1}(\omega)\right] - S_{sn_2}(\omega)\left[S_{sn_2}(\omega) + S_{n_1n_2}(\omega)\right]}{S_{n_2n_2}(\omega)\left[S_{ss}(\omega)+S_{n_1n_1}(\omega)+2S_{sn_1}(\omega)\right] - \left[S_{sn_2}(\omega)+S_{n_1n_2}(\omega)\right]^2}$$

and

$$G_2(\omega) = \frac{\left[S_{ss}(\omega) + S_{sn_1}(\omega)\right]\left[S_{sn_2}(\omega) + S_{n_1n_2}(\omega)\right] - S_{sn_2}(\omega)\left[S_{ss}(\omega) + S_{n_1n_1} + 2S_{sn_1}(\omega)\right]}{\left[S_{n_2n_2}(\omega)S_{ss}(\omega) + S_{n_1n_1}(\omega) + 2S_{sn_1}(\omega)\right] - \left[S_{sn_2}(\omega) + S_{n_1n_2}(\omega)\right]^2} \tag{8.3}$$

Under the various conditions of correlation as considered in Chapter IV, the results are summarized as follows;

<u>Case I</u>

$s(t)$, $n_1(t)$ and $n_2(t)$ uncorrelated

$$G_1(\omega) = \frac{S_{ss}(\omega)}{S_{ss}(\omega) + S_{n_1n_1}(\omega)}$$

$$G_2(\omega) = 0$$

Case II

$n_1(t)$ and $n_2(t)$ correlated

$$G_1(\omega) = \frac{S_{ss}(\omega)\ S_{n_2n_2}(\omega)}{S_{n_2n_2}(\omega)\left[S_{ss}(\omega) + S_{n_1n_1}(\omega)\right] - S^2_{n_1n_2}(\omega)}$$

$$G_2(\omega) = \frac{S_{ss}(\omega)\ S_{n_1n_2}}{S_{n_2n_2}(\omega)\left[S_{ss}(\omega) + S_{n_1n_1}(\omega)\right] - S^2_{n_1n_2}(\omega)}$$

Case III

$s(t)$ and $n_1(t)$ correlated

$$G_1(\omega) = \frac{S_{ss}(\omega) + S_{sn_1}(\omega)}{S_{ss}(\omega) + S_{n_1n_1}(\omega) + 2S_{sn_1}(\omega)}$$

$$G_2(\omega) = 0$$

Case IV

$s(t)$ and $n_2(t)$ correlated

$$G_1(\omega) = \frac{S_{n_2n_2}(\omega)\ S_{ss}(\omega) - S_{sn_2}^{\ 2}(\omega)}{S_{n_2n_2}(\omega)\ [S_{ss}(\omega) + S_{n_1n_1}(\omega)] - S_{sn_2}^{\ 2}(\omega)}$$

$$G_2(\omega) = \frac{S_{n_1n_1}(\omega)\ S_{sn_2}(\omega)}{-S_{n_2n_2}(\omega)\ [S_{ss}(\omega) + S_{n_1n_1}(\omega)] + S_{sn_2}^{\ 2}(\omega)}$$

Substituting the above values of $G_1(\omega)$ and $G_2(\omega)$ in the expression (8.2) for error spectral density we get the following relations;

Case I

s(t), $n_1(t)$ and $n_2(t)$ uncorrelated

$$S_{ee}(\omega) = \frac{S_{ss}(\omega)\ S_{n_1n_1}(\omega)}{S_{ss}(\omega) + S_{n_1n_1}(\omega)}$$

Case II

$n_1(t)$ and $n_2(t)$ correlated

$$S_{ee}(\omega) = \frac{S_{ss}(\omega)[S_{n_1n_1}(\omega)\ S_{n_2n_2}(\omega) - S^2_{n_1n_2}(\omega)]}{S_{n_2n_2}(\omega)[S_{ss}(\omega) + S_{n_1n_1}(\omega)] - S^2_{n_1n_2}(\omega)}$$

Case III

s(t) and $n_1(t)$ correlated

$$S_{ee}(\omega) = \frac{S_{ss}(\omega)\ S_{n_1n_1}(\omega) - S^2_{sn_1}(\omega)}{S_{ss}(\omega) + S_{n_1n_1}(\omega) + 2S_{sn_1}(\omega)}$$

Case IV

s(t) and $n_2(t)$ correlated

$$S_{ee}(\omega) = \frac{S_{n_1n_1}(\omega)[S_{ss}(\omega)\ S_{n_2n_2}(\omega) - S^2_{sn_2}]}{S_{n_2n_2}(\omega)[S_{ss}(\omega) + S_{n_1n_1}(\omega)] - S^2_{sn_2}(\omega)}$$

The resultant mean square error is given in terms of integral equation of Equation 8.1 as

$$\text{mse} = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{ee}(\omega)\,d\omega$$

Where $S_{ee}(\omega)$ has the above mentioned form for the given correlation case. The above expression of effective optimum transfer functions $G_1(\omega)$ and $G_2(\omega)$ and error spectral density for the various cases of correlation between signal and noise are given in Table 8.1 and 8.2.

In the following section mean square error for the two representative signals as considered in previous chapters are evaluated.

Example 1

$$S_{ss}(\omega) = \frac{4\lambda}{\omega^2 + 4\lambda^2}$$

$$S_{n_1n_1}(\omega) = N_1, \; S_{n_2n_2}(\omega) = N_2$$

$$S_{n_1n_2}(\omega) = 2a_1r_1/(\omega^2 + a_1^2)$$

$$S_{sn_1}(\omega) = 2a_2r_2/(\omega^2 + a_2^2)$$

$$S_{sn_2}(\omega) = 2a_3r_3/(\omega^2 + a_3^2)$$

Case I

$s(t)$, $n_1(t)$ and $n_2(t)$ uncorrelated

$$\text{mse} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{S_{ss}(\omega)\, S_{n_1n_1}(\omega)}{S_{ss}(\omega) + S_{n_1n_1}(\omega)}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{4\lambda N_1}{N_1\omega^2 + 4\lambda + 4\lambda^2 N_1}\, d\omega$$

For a specific case when $\lambda = 1$, $N_1 = 1$

$$\text{mse} = .694093$$

Case II

$n_1(t)$ and $n_2(t)$ correlated

$$\text{mse} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{S_{ss}(\omega)[S_{n_1n_1}(\omega)\; S_{n_2n_2}(\omega) - S_{n_1n_2}{}^2(\omega)]}{\left(S_{n_2n_2}(\omega)[S_{ss}(\omega) + S_{n_1n_1}(\omega)] - S_{n_1n_2}(\omega)^2\right)}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\left[N_1N_2\omega^4 + 2N_1N_2a_1^2\omega^2 + N_1N_2a_1^4 - 4a_1^2r_1^2\right]}{\left(\begin{aligned}&N_1N_2\omega^6 + \omega^4N_2(4\lambda + 4\lambda^2N_1 + 2a_1^2N_1)\\ &+ \omega^2(N_1N_2a_1^4 + 8\lambda N_2a_1^2 + 8\lambda N_1N_2\lambda^2a_1^2 - 4a_1^2r_1^2)\\ &+ a_1^4N_2(4\lambda + 4N_1\lambda^2) - 16\lambda^2a_1^2r_1^2\end{aligned}\right)}\, d\omega$$

For the specific case when

$$\lambda = 1,\ N_1 = 1,\ N_2 = 1,\ a_1 = 1,\ r_1 = .1$$

mse = .691852

Case III

s(t), n_1(t) correlated

$$\text{mse} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{S_{ss}(\omega)\ S_{n_1n_1}(\omega) - S_{sn_1}^{2}(\omega)}{S_{ss}(\omega) + S_{n_1n_1}(\omega) + 2S_{sn_1}(\omega)}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{4}{(\omega^2+a_2^2)} \frac{[N_1\omega^4+\omega^2(2\lambda N_1a_2^2-a_2^2r_2^2) + \lambda N_1a_2^4 - 4a_2^2r_2^2\lambda^2]}{[N_1\omega^4 + \omega^2(N_1a_2^2 + 4\lambda + 4\lambda^2N_1 + 4a_2r_2) + a_2^2(4\lambda+4\lambda^2N_1)+16a_2r_2\lambda^2]}\, d\omega$$

For the specific case when $\lambda = 1$, $N_1 = 1$, $a_2 = 1$, and $r_2 = .1$

mse = .652188

Case IV

s(t) and n_2(t) correlated

$$\text{mse} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{S_{n_1n_1}(\omega)\,[S_{ss}(\omega)\,S_{n_2n_2}(\omega)-S_{sn_2}^{\,2}(\omega)]}{S_{n_2n_2}(\omega)[S_{ss}(\omega)+S_{n_1n_1}(\omega)]-S_{sn_2}^{\,2}(\omega)}\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{4\lambda\,[N_1N_2\omega^4+2N_1N_2a_1^2\omega^2+N_1N_2a_1^4-4a_1^2r_1^2]}{\begin{aligned}\Big[&N_1N_2\omega^6+\omega^6(4\lambda N_2+4\lambda^2N_1N_2+2a_3^2N_2)\\ &+\omega^2(N_1N_2a_3^4+8\lambda N_2a_3^2+8\lambda^2N_1N_2a_3^2-4a_3^2r_3^2)\\ &+N_2a_3^4(4\lambda+4\lambda^2N_1)\ -16a_3^2r_3^2\lambda^2\Big]\end{aligned}}\,d\omega$$

For the specific case when $N_1 = 1$, $N_2 = 1$, $a_3 = 1$, $\lambda = 1$ and $r_3 = .1$

$$\text{mse} = .691169$$

Example 2

$$S_{ss}(\omega) = \frac{1}{\omega^4 + 1}$$

The rest of the spectral densities are the same as in example 1.

Case I

$s(t)$, $n_1(t)$ and $n_2(t)$ uncorrelated

$$\text{mse} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{N_1\dfrac{1}{1+\omega^4}}{N_1+\dfrac{1}{1+\omega^4}}\,d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{N_1\omega^4 + N_1+1} d\omega$$

For the specific case when

$N_1 = 1$

mse = .1709

<u>Case II</u>

$n_1(t)$ and $n_2(t)$ correlated

$$\text{mse} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\frac{1}{(1+\omega^4)} N_1N_2 - \frac{4a_1^2\, r_1^2}{(\omega^2+a_1^2)^2}}{N_2\left[N_1+\frac{1}{1+\omega^4}\right] - \frac{4a_1^2\, r_1^2}{(\omega^2+a_1^2)^2}} \cdot d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{N_1N_2\omega^4+2N_1N_2a_1^2\omega^2+a_1^4N_1N_2-4a_1^2r_1^2}{[N_1N_2\omega^8+2N_1N_2a_1^2\omega^6+\omega^4(N_2+N_1N_2+N_1N_2a_1^4-4a_1^2r_1^2) + [2N_2a_1^2+2N_1N_2a_1^2]\omega^2+N_2a_1^4(1+N_1)-4a_1^2r_1^2]} d\omega$$

For the case when

$N_1 = 1$, $N_2 = 1$, $a_1 = 1$ and $r_1 = .1$

mse = .1693

Case III

s(t) and n_1(t) correlated

$$\text{mse} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\dfrac{N_1}{1+\omega^4} - \dfrac{4a_2^2r_2^2}{(\omega^2+a_2^2)^2}}{N_1+\dfrac{1}{1+\omega^4} + \dfrac{4a_2r_2}{\omega^2+a_2^2}}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{[(N_1-4a_2^2r_2^2)\omega^4+2N_1a_2^2\omega^2+N_1a_2^4-4a_2^2r_2^2]}{[(\omega^2+a_2^2)[N_1\omega^6+\omega^4(N_1a_2^2+4a_2r_2)+\omega^2(N_1+1) +N_1a_2^2+a_2^2+4a_2r_2]}\, d\omega$$

For the case when

$N_1 = 1$, $a_2 = 1$ and $r_2 = .1$

mse = .1452

Case IV

s(t) and n_2(t) correlated

$$\text{mse} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\left[N_1 \cdot \dfrac{N_2}{1+\omega^4} - \dfrac{4a_3^2r_3^2}{(\omega^2+a_3^2)^2}\right]}{N_2\left[N_1+\dfrac{1}{\omega^4+1}\right] - \dfrac{4a_3^2r_3^2}{(\omega^2+a_3^2)^2}}\, d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\left[N_1 \quad \omega^4 (N_2 - 4a_3^2 r_3^2) + 2N_2 a_3^2 \omega^2 + N_2 a_3^4 - 4r_3^2 a_3^2\right]}{\left[N_1 N_2 \omega^8 + 2N_1 N_2 a_3^2 \omega^6 + \omega^4 (N_1 N_2 + N_2 + N_1 N_2 a_3^4\right.} \, d\omega$$

$$-4a_3^2 r_3^2) + \omega^2 [2N_2 a_3^2 + 2N_1 N_2 a_3^2]$$

$$\left. + a_3^2 N_2 (N_1 + 1) - 4a_3^2 r_3^2 \right]$$

For the specific case when

$N_1 = 1$, $N_2 = 1$, $a_3 = 1$ and $r_3 = .1$

mse = .1673

b) Delay Case

From equation 5.4 of Chapter 5 for the case of delays in the channels, we derive the expression of mean square error spectral density as follows:

$$S_{ee}(\omega) = |G_1(\omega)^2 \; [S_{ss}(\omega) + S_{n_1 n_1}(\omega) + S_{sn_1}(\omega) e^{-j\omega t_1} + S_{n_1 s}(\omega) e^{j\omega t_1}]$$

$$+ |G_2(\omega)|^2 \; S_{n_2 n_2}(\omega) + S_{ss}(\omega) - G_1(\omega) G_2^*(\omega) \; [S_{n_1 n_2}(\omega)$$

$$+ S_{sn_2}(\omega) e^{-j\omega t_1}] - G_2(\omega) G_1^*(\omega) \; [S_{n_2 n_1}(\omega) + S_{n_2 s}(\omega) e^{j\omega t_1}]$$

$$+ [G_2(\omega) S_{n_2 s}(\omega) + G_2^*(\omega) S_{sn_2}(\omega)]$$

$$- [S_{n_1 s}(\omega)G_1(\omega)+S_{sn_1}(\omega)G_1^*(\omega)]$$

$$- S_{ss}(\omega)\ [G_1(\omega)e^{-j\omega t_1} + G_1^*(\omega)e^{j\omega t_1}]$$

Also from equations (5.14) and (5.15), we obtain

$$G_1(\omega) = \frac{S_{sn_2}(\omega)[S_{n_2n_1}(\omega)+S_{n_2s}(\omega)e^{j\omega t_1}] - S_{n_2n_2}(\omega)[S_{sn_1}(\omega)+S_{ss}(\omega)e^{-j\omega t_1}]}{\left([S_{n_1n_2}(\omega)+S_{sn_2}(\omega)e^{j\omega t_1}][S_{n_2n_1}(\omega)+S_{n_2s}(\omega)e^{j\omega t_1}] - S_{n_2n_2}(\omega)[S_{ss}(\omega)+S_{n_1n_1}(\omega)+S_{sn_1}(\omega)e^{-j\omega t_1}+S_{n_1s}(\omega)e^{j\omega t_1}]\right)}$$

$$G_2(\omega) = \frac{\left[S_{sn_2}(\omega)[S_{ss}(\omega)+S_{n_1n_1}(\omega)+S_{sn_1}(\omega)e^{-j\omega t_1}+S_{n_1s}(\omega)e^{j\omega t_1}] - [S_{sn_1}(\omega)+S_{ss}(\omega)e^{-j\omega t_1}][S_{n_1n_2}(\omega)+S_{sn_2}(\omega)e^{j\omega t_1}]\right]}{\left[[S_{n_1n_2}(\omega)+S_{sn_2}(\omega)e^{j\omega t_1}][S_{n_2n_1}(\omega)+S_{n_2s}(\omega)e^{j\omega t_1}] - S_{n_2n_2}(\omega)[S_{ss}(\omega)+S_{n_1n_1}(\omega)+S_{sn_1}(\omega)e^{-j\omega t_1}+S_{n_1s}(\omega)e^{j\omega t_1}]\right]}$$

For various cases of correlation between signal and noise, the effective transfer functions will have the following values;

Case I

$s(t)$, $n_1(t)$ and $n_2(t)$ uncorrelated

$$G_1(\omega) = \frac{S_{ss}(\omega)e^{-j\omega t_1}}{S_{ss}(\omega) + S_{n_1n_1}(\omega)}$$

$$G_2(\omega) = 0$$

Case II

$n_1(t)$ and $n_2(t)$ correlated

$$G_1(\omega) = \frac{S_{ss}(\omega)\, S_{n_2n_2}(\omega)e^{-j\omega t_1}}{[S_{ss}(\omega)+S_{n_1n_1}(\omega)]S_{n_2n_2}(\omega)-S_{n_1n_2}(\omega)S_{n_2n_1}(\omega)}$$

$$G_2(\omega) = \frac{S_{ss}(\omega)\, S_{n_1n_2}(\omega)e^{-j\omega t_1}}{\left[S_{ss}(\omega)+S_{n_1n_1}(\omega)\right]S_{n_2n_2}(\omega)-S_{n_1n_2}(\omega)S_{n_2n_1}(\omega)}$$

Case III

$s(t)$ and $n_1(t)$ correlated

$$G_1(\omega) = \frac{S_{sn_1}(\omega) + S_{ss}(\omega)e^{-j\omega t_1}}{S_{ss}(\omega)+S_{n_1n_1}(\omega)+S_{sn_1}(\omega)e^{-j\omega t_1} + S_{n_1s}(\omega)e^{j\omega t_1}}$$

$$G_2(\omega) = 0$$

Case IV

$s(t)$ and $n_2(t)$ correlated

$$G_1(\omega) = \frac{S_{sn_2}(\omega)S_{n_2s}(\omega)e^{j\omega t_1} - S_{n_2n_2}(\omega)S_{ss}(\omega)e^{-j\omega t_1}}{S_{sn_2}(\omega)S_{n_2s}(\omega)e^{2j\omega t_1} - S_{n_2n_2}(\omega)\ [S_{ss}(\omega)+S_{n_1n_1}(\omega)]}$$

$$G_2(\omega) = \frac{S_{sn_2}(\omega)\, S_{n_1n_1}(\omega)}{S_{sn_2}(\omega) S_{n_2s}(\omega) e^{2j\omega t_1} - S_{n_2n_2}(\omega)\, [S_{ss}(\omega) + S_{n_1n_1}(\omega)]}$$

The spectral densities for the respective cases are found by substituting the corresponding values of G's in the general expression of error spectral density. The following results are obtained

Case I

$s(t)$, $n_1(t)$, $n_2(t)$ correlated

$$S_{ee}(\omega) = \frac{S_{ss}(\omega)}{S_{ss}(\omega) + S_{n_1n_1}(\omega)} \, [2S_{ss}(\omega) + S_{n_1n_1}(\omega) - 2S_{ss}(\omega) \mathrm{Cos}\, 2\omega t_1]$$

Case II

$n_1(t)$ and $n_2(t)$ correlated

$$\begin{aligned} S_{ee}(\omega) &= |G_1(\omega)|^2 (S_{ss}(\omega) + S_{n_1n_1}(\omega)) + |G_2(\omega)|^2\, S_{n_2n_2}(\omega) + S_{ss}(\omega) \\ &\quad - G_1(\omega) G_2^*(\omega) S_{n_1n_2}(\omega) - G_2(\omega) G_1^*(\omega) S_{n_2n_1}(\omega) \\ &\quad - S_{ss}(\omega)\, [G_1(\omega) e^{-j\omega t_1} + G_1^*(\omega) e^{j\omega t_1}] \\ &= S_{ss}(\omega) - \frac{S_{ss}^{\,2}(\omega)\; S_{n_2n_2}(\omega)\, [2\mathrm{Cos}\; 2\omega t_1 - 1]}{S_{n_2n_2}(\omega) \left[S_{ss}(\omega) + S_{n_1n_1}(\omega)\right] - S_{n_1n_2}(\omega) S_{n_2n_1}(\omega)} \end{aligned}$$

Case III

s(t) and n_1(t) correlated

$$S_{ee}(\omega) = \frac{S_{ss}(\omega)S_{n_1n_1}(\omega)-S_{sn_1}^{\ 2}(\omega)+S_{ss}^{\ 2}(\omega)[2-2\text{Cos } 2\omega t_1]}{\left[S_{ss}(\omega)+S_{n_1n_1}(\omega)+2S_{sn_1}(\omega)\text{ Cos } 2\omega t_1\right]}$$

Case IV

s(t) and n_2(t) correlated

$$\begin{aligned} S_{ee}(\omega) = \Big(& S_{n_1n_1}(\omega)S_{sn_2}^{\ 4}(\omega)[2\text{Cos } 2\omega t_1-1]+S_{ss}(\omega)S_{sn_2}^{\ 4}(\omega)[2-2\text{Cos } 2\omega t_1] \\ & - S_{ss}(\omega)S_{n_1n_1}(\omega)S_{n_2n_2}(\omega)S_{sn_2}^{\ 2}\ [2\text{Cos } 2\omega t_1] \\ & + S_{ss}^{\ 3}(\omega)S_{n_2n_2}^{\ 2}(\omega)\ [2-2\text{Cos } 2\omega t_1] \\ & + S_{ss}(\omega)S_{n_1n_1}^{\ 2}(\omega)S_{n_2n_2}^{\ 2}(\omega) \\ & + S_{ss}^{\ 2}(\omega)S_{n_1n_1}(\omega)S_{n_2n_2}^{\ 2}(\omega)[3-2\text{Cos } 2\omega t_1] \\ & + S_{n_2n_2}(\omega)S_{ss}^{\ 2}(\omega)S_{sn_2}^{\ 2}(\omega)[2-4\text{Cos } 2\omega t_1\ +2\text{Cos } 4\omega t_1] \\ & - S_{n_2n_2}(\omega)S_{n_1n_1}^{\ 2}(\omega)S_{sn_2}^{\ 2}(\omega)]\Big) \Big/ \\ & \left[S_{sn_2}^{\ 2}(\omega)-S_{n_2n_2}(\omega)(S_{ss}(\omega)+S_{n_1n_1}(\omega)e^{2j\omega t_1}][S_{sn_2}^{\ 2}(\omega)\right. \\ & \left.- S_{n_2n_2}(\omega)(S_{ss}(\omega)+S_{n_1n_1}(\omega))e^{-2j\omega t_1}\right] \end{aligned}$$

The above expressions of optimum effective transfer functions and error spectral densities for the system with delay are given in Table 8-3 and Table 8-4.

In the following section mean square error in calculated for two examples considered in the previous section.

Example 1

$$S_{ss}(\omega) = \frac{4\lambda}{\omega^2 + 4\lambda^2}$$

Case I

$s(t)$, $n_1(t)$ and $n_2(t)$ uncorrelated

Substitution from Table 8-4 results in

$$\text{mse} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{ee}(\omega)d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{S_{ss}(\omega)}{S_{ss}(\omega)+S_{n_1n_1}(\omega)} [S_{n_1n_1}(\omega)+S_{ss}(\omega)\ 4\text{Sin}^2\ \omega t_1]\ d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{4\lambda}{N_1\ {}^2+4\lambda^2N_1+4\lambda}[N_1+\frac{16\lambda}{\omega^2+4\lambda^2}\ \text{Sin}^2\ \omega t_1]d\omega$$

For the specific case, when $\lambda = 1$, $N = 1$

$$\text{mse} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{4}{\omega^2 + 8}[1+\frac{16}{\omega^2 + 4}\ \text{Sin}^2\ \omega t_1]d\omega$$

For various values of t_1, the mse is tabulated in Table 9-5

Case II

$n_1(t)$ and $n_2(t)$ correlated

From Table 8-4

$$mse = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{ee}(\omega)d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[S_{ss}(\omega) - \frac{S_{ss}^{2}(\omega)S_{n_2n_2}(\omega)[2\text{Cos } 2\omega t_1 - 1]}{S_{n_2n_2}(\omega)[S_{ss}(\omega)+S_{n_1n_1}(\omega)]-S_{n_1n_2}^{2}(\omega)}\right]d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{4\lambda}{(\omega^2+4\lambda^2)} - \frac{16\lambda^2N_2(2\text{Cos } 2\omega t_1-1)(\omega^2+a_1^2)^2}{(\omega^2+4\lambda^2)\left[N_2(\omega^2N_1+4\lambda^2N_1+4\lambda)(\omega^2+a_1^2)^2 - 4a_1^2r_1^2(\omega^2+4\lambda^2)^2\right]}\right]d\omega$$

For the case when $\lambda = 1$, $N_1 = 1$, $N_2 = 1$, $a_1 = 1$ and $r_1 = .1$

$$mse = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{4}{\omega^2+4} - \frac{16(2\text{Cos } 2\omega t_1-1)(\omega^2+1)^2}{(\omega^2+4)[(\omega^2+8)(\omega^2+1)^2-.04(\omega^2+4)^2]}\right]d\omega$$

The values of mse for various values of t_1 are tabulated in Table 9-5 .

Case III

$s(t)$ and $n_1(t)$ correlated

From Table 8-4

$$mse = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{S_{ss}(\omega)S_{n_1n_1}(\omega) - S_{sn_1}^{2}(\omega)+4S_{ss}^{2}(\omega)\text{Sin}^2\omega t_1}{S_{ss}(\omega)+S_{n_1n_1}(\omega)+2S_{sn_1}(\omega)\text{Cos}\omega t_1}\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{\left[4\lambda N_1(\omega^2+4\lambda^2)(\omega^2+a_2^2)^2-4a_2^2r_2^2(\omega^2+4\lambda^2)^2+64\lambda^2\sin^2\omega t_1 \times(\omega^2+a_2^2)^2\right]}{(\omega^2+a_2^2)(\omega^2+4\lambda^2)\left[N_1(\omega^2+4\lambda^2)(\omega^2+a_2^2)+4\lambda(\omega^2+a_2^2)+4a_2r_2\cos\omega t_1(\omega^2+4\lambda^2)\right]}\right]d\omega$$

For the case when $\lambda = 1$, N_1 and $r_2 = .1$

$$mse = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{(\omega^2+4)(\omega^2+1)^2-.01(\omega^2+4)^2+16(\omega^2+1)^2\sin^2\omega t_1}{\left[(\omega^2+1)(\omega^2+4)\ (\omega^2+4)(\omega^2+1)+4(\omega^2+1)+.4(\omega^2+4)\cos\omega t_1\right]}.d\omega$$

The results are tabulated in Table 9-5 .

Case IV

s(t) and $n_2(t)$ correlated

From Table 8-4

$$mse = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Big[S_{n_1n_1}(\omega)S_{sn_2}{}^4(\omega)\left[2\cos 2\omega t_1-1\right]+4S_{ss}(\omega)S_{sn_2}{}^4(\omega)\sin^2\omega t_1$$
$$-2S_{ss}(\omega)S_{n_1n_1}(\omega)S_{n_2n_2}(\omega)S_{sn_2}{}^2(\omega)\cos 2\omega t_1$$
$$+4S_{ss}{}^3(\omega)S_{n_2n_2}{}^2(\omega)\sin^2\omega t_1$$
$$+S_{ss}(\omega)S_{n_1n_1}{}^2(\omega)S_{n_2n_2}{}^2(\omega)$$
$$+S_{ss}{}^2(\omega)\ S_{n_1n_1}(\omega)S_{n_2n_2}{}^2(\omega)\ [3\ -2\cos 2\omega t_1]$$
$$-8\ S_{n_2n_2}(\omega)S_{ss}{}^2(\omega)S_{sn_2}{}^2(\omega)\sin^2\omega t_1\ \ \cos 2\omega t_1\Big]d\omega$$

$$\Big[S_{sn_2}{}^4(\omega)+S_{n_2n_2}{}^2(\omega)\,[S_{ss}(\omega)+S_{n_1n_1}(\omega)]^2-2S_{n_2n_2}(\omega)S_{sn_2}{}^2(\omega)\,[S_{ss}(\omega)$$

$$+S_{n_1n_1}(\omega)]\,\mathrm{Cos}\;2\omega t_1\Big]$$

For various values of parameters the values of mean square are tabulated in Table 9-5.

Example 2

$$S_{ss}(\omega) = \frac{1}{\omega^4 + 1}$$

Case I

$s(t)$, $n_1(t)$ and $n_2(t)$ correlated

From Table 8-4

$$\text{mse} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{1}{(N_1\omega^4 + N_1 + 1)}\quad N_1 + \frac{4\mathrm{Sin}^2\,\omega t_1}{1+\omega^4}\right]d\omega$$

For $N_1 = 1$

$$\text{mse} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\omega^4 + 1 + 4\mathrm{Sin}^2\,\omega t_1}{(\omega^4+2)(\omega^4+1)}\,d\omega$$

Case II

$n_1(t)$ and $n_2(t)$ correlated

From Table 8-4

$$mse = \frac{1}{2\pi}\int_{\infty}^{\infty}\left[\frac{1}{\omega^4+1} - \frac{\frac{N_2}{(1+\omega^4)^2}(2\mathrm{Cos}\ 2\omega t_1-1)}{N_2\left[N_1+\frac{1}{\omega^4+1}\right]-\frac{2a_1r_1}{\omega^2+a_1^2}}\right]d\omega$$

For $N_1 = 1$, $N_2 = 1$, $a_1 = 1$ and $r_1 = .1$

$$mse = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{1}{\omega^4+1} - \frac{(\omega^2+1)^2(2\mathrm{Cos}\ 2\omega t_1-1)}{(\omega^4+1)[(\omega^4+2)(\omega^2+1)^2 - .04(\omega^4+1)]}\right]d\omega$$

Case III

s(t) and $n_1(t)$ correlated

From Table 8-4

$$mse = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{\frac{N_1}{1+\omega^4} - \frac{2a_2r_2}{\omega^2+a_2^2} + \frac{4\mathrm{Sin}^2\ \omega t_1}{(\omega^4+1)^2}}{\frac{1}{1+\omega^4} \qquad \frac{4a_2r_2\ \mathrm{Cos}\omega t_1}{\omega^2+a_2^2}}\right]d\omega$$

For the case when $N_1 = 1$, $a_2 = 1$, and $r_2 = .1$

$$mse = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{\frac{1}{1+\omega^4} - \frac{.04}{(\omega^2+1)^2} + \frac{4\mathrm{Sin}^2\ \omega t_1}{(\omega^4+1)^2}}{\frac{\omega^4+2}{\omega^4+1} + \frac{.4\mathrm{Cos}\ \omega t_1}{\omega^2+1}}\right]d\omega$$

Case IV

s(t) and $n_2(t)$ correlated

From Table 8-4

$$mse = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{A(\omega)}{B(\omega)}\ d\omega$$

$$A(\omega) = (2\cos 2\omega t_1 - 1)\, N_1 \left[\frac{2a_3 r_3}{\omega^2 + a_3^2}\right]^4 + \frac{4\sin^2 \omega t_1}{(1+\omega^4)} \left[\frac{2a_3 r_3}{\omega^2 + a_3^2}\right]^4$$

$$- \frac{2N_1 N_2 \cos 2\omega t_1}{(\omega^4 + 1)} \left[\frac{2a_3 r_3}{\omega^2 + a_3^2}\right]^2 + \frac{4N_2^2 \sin^2 \omega t_1}{(1+\omega^4)^3}$$

$$+ \frac{N_1^2 N_2^2}{1+\omega^4} + \frac{N_1 N_2^2 \quad (3 - 2\cos 2\omega t_1)}{(1+\omega^4)^2}$$

$$- \frac{8N_2 \sin^2 \omega t_1 \cos 2\omega t_1}{(1+\omega^4)^2} \left[\frac{2a_3 r_3}{\omega^2 + a_3^2}\right]^2$$

$$- N_1^2 N_2 \left[\frac{2a_3 r_3}{\omega^2 + a_3^2}\right]^2$$

$$B(\omega) = \left[\frac{2a_3 r_3}{\omega^2 + a_3^2}\right]^4 + N_2^2 \left[N_1 + \frac{1}{\omega^4 + 1}\right]^2$$

$$- 2N_2 \left[\frac{2a_3 r_3}{\omega^2 + a_3^2}\right]^2 \left[N_1 + \frac{1}{\omega^4 + 1}\right] \cos 2\omega t_1$$

TABLE 8-1

OPTIMUM EFFECTIVE TRANSFER FUNCTIONS

NO DELAY

	$G_1(\omega)$	$G_2(\omega)$
$s(t)$, $n_1(t)$, $n_2(t)$ Uncorrelated	$S_{ss}(\omega)/[S_{ss}(\omega) + S_{n_1n_1}(\omega)]$	0
$n_1(t)$, $n_2(t)$ Correlated	$\dfrac{S_{ss}(\omega)S_{n_2n_2}(\omega)}{S_{n_2n_2}(\omega)[S_{ss}(\omega)+S_{n_1n_1}(\omega)]-S_{n_1n_2}^2(\omega)}$	$\dfrac{S_{ss}(\omega)S_{n_1n_2}(\omega)}{\left[S_{n_2n_2}(\omega)[S_{ss}(\omega)+S_{n_1n_1}(\omega)] -S_{n_1n_2}^2(\omega)\right]}$
$s(t)$ $n_1(t)$ Correlated	$\dfrac{S_{ss}(\omega)+S_{sn_1}(\omega)}{S_{ss}(\omega)+S_{n_1n_1}(\omega)+2S_{sn_1}(\omega)}$	0

TABLE 8-1 continued

s(t), n_2(t) Correlated	$$\frac{S_{n_2n_2}(\omega)S_{ss}(\omega)-S_{sn_2}^{2}(\omega)}{S_{n_2n_2}(\omega)\left[S_{ss}(\omega)+S_{n_1n_1}(\omega)\right]-S_{sn_2}^{2}(\omega)}$$	$$\frac{S_{n_1n_1}(\omega)S_{sn_2}(\omega)}{S_{sn_2}^{2}(\omega)-S_{n_2n_2}(\omega)\left[S_{ss}(\omega)+S_{n_1n_1}(\omega)\right]}$$

TABLE 8-2

ERROR SPECTRAL DENSITIES FOR OPTIMUM SYSTEM

	$S_{ee}(\omega)$
$s(t)\ n_1(t),\ n_2(t)$ Uncorrelated	$\dfrac{S_{ss}(\omega)S_{n_1n_1}(\omega)}{S_{ss}(\omega)+S_{n_1n_1}(\omega)}$
$n_1(t),\ n_2(t)$ Correlated	$\dfrac{S_{ss}(\omega)\left[S_{n_1n_1}(\omega)S_{n_2n_2}(\omega)-S_{n_1n_2}^2(\omega)\right]}{S_{n_2n_2}(\omega)\left[S_{ss}(\omega)+S_{n_1n_1}(\omega)\right]-S_{n_1n_2}^2(\omega)}$
$s(t),\ n_1(t)$ Correlated	$\dfrac{S_{ss}(\omega)S_{n_1n_1}(\omega)-S_{sn_1}^2(\omega)}{S_{ss}(\omega)+S_{n_1n_1}(\omega)+2S_{sn_1}(\omega)}$
$s(t),\ n_2(t)$ Correlated	$\dfrac{S_{n_1n_1}(\omega)\left[S_{ss}(\omega)S_{n_2n_2}(\omega)-S_{sn_2}^2(\omega)\right]}{S_{n_2n_2}(\omega)\left[S_{ss}(\omega)+S_{n_1n_1}(\omega)\right]-S_{sn_2}^2(\omega)}$

TABLE 8-3

OPTIMUM EFFECTIVE TRANSFER FUNCTIONS

FOR SYSTEMS WITH DELAY

	$G_1(\omega)$	$G_2(\omega)$
$S(t)$, $n_1(t)$, $n_2(t)$ Uncorrelated	$\dfrac{S_{ss}(\omega)\ e^{-j\omega t_1}}{S_{ss}(\omega) + S_{n_1n_1}(\omega)}$	0
$n_1(t)$, $n_2(t)$ Correlated	$\dfrac{S_{ss}(\omega)S_{n_2n_2}(\omega)\ e^{-j\omega t_1}}{S_{n_2n_2}(\omega)\left[S_{ss}(\omega)+S_{n_1n_1}(\omega)\right]-S_{n_2n_1}(\omega)S_{n_1n_2}(\omega)}$	$\dfrac{S_{ss}(\omega)S_{n_1n_2}(\omega)\ e^{-j\omega t_1}}{\left(S_{n_2n_2}(\omega)\left[S_{ss}(\omega)+S_{n_1n_1}(\omega)\right]-S_{n_1n_2}\ S_{n_2n_1}(\omega)\right)}$
$S(t)$, $n_1(t)$ Correlated	$\dfrac{S_{sn_1}(\omega) + S_{ss}(\omega)e^{-j\omega t_1}}{S_{ss}(\omega)+S_{n_1n_1}(\omega)+S_{n_1s}(\omega)e^{j\omega t_1}+S_{sn_1}(\omega)e^{-j\omega t_1}}$	0
$S(t)$, $n_2(t)$ Correlated	$\dfrac{S_{n_2n_2}(\omega)S_{ss}(\omega)e^{-j\omega t_1}-S_{sn_2}(\omega)S_{n_2s}(\omega)e^{+j\omega t_1}}{\left(S_{n_2n_2}(\omega)\left[S_{ss}(\omega)+S_{n_1n_1}(\omega)\right]-S_{sn_2}(\omega)S_{n_2s}(\omega)\ e^{+2j\omega t_1}\right)}$	$\dfrac{S_{n_1n_1}(\omega)\ S_{sn_2}(\omega)}{\left(S_{sn_2}(\omega)S_{n_2s}(\omega)e^{+2j\omega t_1}-S_{n_2n_2}(\omega)\left[S_{ss}(\omega)+S_{n_1n_1}(\omega)\right]\right)}$

TABLE 8-4

ERROR SPECTRAL DENSITY FOR OPTIMUM

FEEDBACK SYSTEM WITH DELAY

$s(t)$, $n_1(t)$ and $n_2(t)$ Uncorrelated	$\dfrac{S_{ss}(\omega)}{S_{ss}(\omega) + S_{n_1 n_1}(\omega)} \cdot \left[S_{n_1 n_1}(\omega) + 4\, S_{ss}(\omega)\, \mathrm{Sin}^2\, \omega t_1\right]$
$n_1(t)$, $n_2(t)$ Correlated	$S_{ss}(\omega) - \dfrac{S_{ss}{}^2(\omega)\, S_{n_2 n_2}(\omega)\, (2\, \mathrm{Cos}\, 2\omega t_1 - 1)}{S_{n_2 n_2}(\omega)\left[S_{ss}(\omega) + S_{n_1 n_1}(\omega)\right] - S_{n_1 n_2}(\omega)\, S_{n_2 n_1}(\omega)}$
$s(t)$, $n_1(t)$ Correlated	$\dfrac{S_{ss}(\omega)\, S_{n_1 n_1}(\omega) - S_{sn_1}{}^2(\omega) + 4\, S_{ss}{}^2(\omega)\, \mathrm{Sin}^2\, \omega t_1}{S_{ss}(\omega) + S_{n_1 n_1}(\omega) + S_{sn_1}(\omega)\; 2\, \mathrm{Cos}\, \omega t_1}$

s(t), n_2(t) Correlated	$\dfrac{\begin{array}{l}\Big[S_{n_1n_1}(\omega)\ S_{sn_2}{}^{4}(\omega)\big[2\ \mathrm{Cos}\ 2\ \omega t_1 - 1\big] + S_{ss}(\omega)\ S_{sn_2}{}^{4}(\omega)\big[2 - 2\ \mathrm{Cos}\ 2\ \omega t_1\big] \\ - S_{ss}(\omega)\ S_{n_1n_1}(\omega)\ S_{n_2n_2}(\omega)\ S_{sn_2}{}^{2}(\omega) \cdot 2 \cdot \mathrm{Cos}\ 2\ \omega t_1 + S_{ss}{}^{3}(\omega)\ S_{n_2n_2}{}^{2}(\omega)\ \big[2 - \\ 2\ \mathrm{Cos}\ 2\omega t_1\big] + S_{ss}(\omega)\ S_{n_1n_1}{}^{2}(\omega)\ S_{n_2n_2}{}^{2}(\omega) + S_{ss}{}^{2}(\omega)\ S_{n_1n_1}(\omega)\ S_{n_2n_2}{}^{2}(\omega) \times \\ [3 - 2\ \mathrm{Cos}\ 2\omega t_1] + S_{n_2n_2}(\omega)\ S_{ss}{}^{2}(\omega)\ S_{sn_2}{}^{2}(\omega)\ [2 - 4\ \mathrm{Cos}\ 2\omega t_1 + 2\ \mathrm{Cos}\ 4\omega t_1] \\ - S_{n_2n_2}(\omega)\ S_{n_1n_1}{}^{2}(\omega)\ S_{sn_2}{}^{2}(\omega)\ \Big]\end{array}}{\begin{array}{l}\Big(S_{sn_2}{}^{4}(\omega) + S_{n_2n_2}{}^{2}(\omega)\big[S_{ss}(\omega) + S_{n_1n_1}(\omega)\big]^{2} - S_{n_2n_2}(\omega)\ S_{sn_2}{}^{2}(\omega)\big[S_{ss}(\omega) + \\ S_{n_1n_1}(\omega)\big]\ 2\ \mathrm{Cos}\ 2\omega t_1\Big)\end{array}}$

TABLE 8-5

$$S(\omega) = \frac{2\,a\,r}{\omega^2 + a^2}$$

CROSS SPECTRAL DENSITY

a = 1

Frequency	r =.25	r = .50	r =.75	r =1.0
-9.5000	0.0055	0.0110	0.0164	0.0219
-9.0000	0.0061	0.0122	0.0183	0.0244
-8.5000	0.0068	0.0137	0.0205	0.0273
-8.0000	0.0077	0.0154	0.0231	0.0308
-7.5000	0.0087	0.0175	0.0262	0.0349
-7.0000	0.0.00	0.0200	0.0300	0.0400
-6.5000	0.0116	0.0231	0.0347	0.0462
-6.0000	0.0135	0.0270	0.0405	0.0541
-5.5000	0.0160	0.0320	0.0480	0.0640
-5.0000	0.0192	0.0385	0.0577	0.0769
-4.5000	0.0235	0.0471	0.0706	0.0941
-4.0000	0.0294	0.0588	0.0882	0.1176
-3.5000	0.0377	0.0755	0.1132	0.1509
-3.0000	0.0500	0.1000	0.1500	0.2000
-2.5000	0.0690	0.1379	0.2069	0.2759
-2.0000	0.1000	0.2000	0.3000	0.4000
-1.5000	0.1539	0.3077	0.4616	0.6154
-1.0000	0.2500	0.5000	0.7500	1.0000
-0.5000	0.4000	0.8000	1.2000	1.6000
0.0000	0.5000	1.0000	1.5000	2.0000
0.5000	0.4000	0.8000	1.2000	1.6000
1.0000	0.2500	0.5000	0.7500	1.0000
1.5000	0.1538	0.3077	0.4615	0.6154
2.0000	0.1000	0.2000	0.3000	0.4000
2.5000	0.0690	0.1379	0.2069	0.2759
3.0000	0.0500	0.1000	0.1500	0.2000
3.5000	0.0377	0.0755	0.1132	0.1509
4.0000	0.0294	0.0588	0.0882	0.1176
4.5000	0.0235	0.0471	0.0706	0.0941
5.0000	0.0192	0.0385	0.0577	0.0769
5.5000	0.0160	0.0320	0.0480	0.0640
6.0000	0.0135	0.0270	0.0405	0.0541
6.5000	0.0116	0.0231	0.0347	0.0462
7.0000	0.0100	0.0200	0.0300	0.0400
7.5000	0.0087	0.0175	0.0262	0.0349
8.0000	0.0077	0.0154	0.0231	0.0308
8.5000	0.0068	0.0137	0.0205	0.0273
9.0000	0.0061	0.0122	0.0183	0.0244
9.5000	0.0055	0.0110	0.0164	0.0219

CHAPTER IX

System Performance

An idea of the relative system performance is needed before selecting any particular system design. In the previous chapters, the expressions of mean square error which is used as the criterion of systems performance have been derived. The expressions thus derived are functions of signal and noise spectra and the correlation betweeen them. For a general idea of the expected performance of the system, signal and noise spectra could be approximated and performance evaluated. It is shown in Popoulis [3] that any correlation function can be approximated by exponentials and hence an approximation of signal and noise spectral densities and cross-spectral densities can be found. For our calculations the following approximations are made for the spectral densities.

$$S_{ss}(\omega) = \text{Signal spectral density}$$
$$= 2ar/(\omega^2 + a^2)$$

If the power is normalized by setting r equal to one then

$$S_{ss}(\omega) = \frac{2a}{\omega^2 + a^2}$$

The noise spectral densities could either be considered to be constant for the white Gaussian noise case or could be approximated as for the signal. Similarly the cross spectral densities could be approximated as

$$S(\omega) = \frac{2a'r'}{\omega^2 + a'^2}$$

and, by varying the values of a' and r', the correlation could be varied.

Mean square error for various values of parameters of the signal and noise spectral densities has been calculated by Romberg's method of numerical integration by the Sigma 7 Computer and are tabulated in Table 9-1 for the case where the delay in the channel is zero.

It may be noted that a small value of "a" results in sharper spectral density and large "a" results in broader spectral density.

For the case with no delay and with white Gaussian noise the following observations are made from Table 9-1.

a). <u>Uncorrelated case</u>

Mean square error increases for broader signal spectra (large a).

b). <u>n_1, n_2 Correlated case</u>

For a fixed correlation between noise n_1 and n_2, the error is smaller for sharper signal spectra. Also for any signal spectral shape, mse increases for broader cross spectral densities.

c). <u>s, n_1 Correlated</u>

For a fixed correlation between signal s(t) and noise n_1 (t), mse decreases as the signal spectral density becomes sharper. For cross-correlation factor r_2 of .1, mse increases for broader cross spectral densities for any given signal spectra. However, for broader signal spectral densities and cross correlation factor of .4, mse has minimum at $a_1 = 1$.

d). <u>s, n_2 Correlated Case</u>

The mse is low for the case of sharper signal spectra. The mse follows the same trend as in s, n_1 correlated case.

It should be noted that for some combinations of a, r', a', mse has not been evaluated as these combinations are not physically realizable. The conditions of physical realizability has been already treated in chapter VII. For the following approximation of this chapter,

$$S_{ss}(\omega) = 2a/(\omega^2 + a^2)$$

$$S_{n_1n_1}(\omega) = N_1$$

$$S_{n_2n_2}(\omega) = N_2$$

$$S_{n_1n_2}(\omega) = 2a_1r_1/(\omega^2 + a_1^2)$$

$$S_{sn_1}(\omega) = 2a_2r_2/(\omega^2 + a_2^2)$$

$$S_{sn_2}(\omega) = 2a_3r_3/(\omega^2 + a_3^2)$$

for the various realizability conditions and the expressions of realizable filters are as follows,

1. <u>Uncorrelated Case</u>

   No restriction on parameters,

   Realizable filter =

$$\frac{2a}{(j\omega\sqrt{N_1} + \sqrt{2a + N_1 a^2})(a\sqrt{N_1} + \sqrt{2a + N_1 a^2})}$$

2. <u>n_1, n_2 Correlated</u>

   Realizability restriction are

$$N_1(a^2 + 2a_1^2) + 2a > o$$

$$N_1 N_2 a_1(2a^2 + a_1^2) + 4aa_1 N_2 - 4r_1 (a + a_1 r_1) \geq o$$

$$N_2 a_1^2(2 + aN_1) - 4r_1(ar_1 + a_1) > o$$

and d_1, d_2 and d_3 are positive, where

$$d_1 = A + B - C_{1/3}$$

$$d_2 = -\frac{A+B}{2} + \frac{A-B}{2}\sqrt{-3}-\frac{C_1}{3}$$

$$d_3 = -\frac{A+B}{2} - \frac{A-B}{2}\sqrt{-3}-\frac{C_1}{3}$$

and $$A = \left[\frac{-e_2}{2} + \sqrt{\frac{e_2^2}{4} + \frac{e_1^3}{27}}\right]^{1/3}$$

$$B = \left[\frac{-e_2}{2} - \sqrt{\frac{e_2^2}{4} + \frac{e_1^3}{27}}\right]^{1/3}$$

and $$e_1 = \frac{1}{3}\left[3C_2 - C_1^{\,2}\right]$$

$$e^2 = \frac{1}{27}\left[2C_1^{\,3} - 9C_1C_2 + 27C_3\right]$$

where C's are

$$C_1 = a^2 + 2a_1^{\,2} + 2a/N_1$$

$$C_2 = a_1^{\,4} + 2a^2a_1^{\,2} + 4aa_1^{\,2}/N_1$$

$$- 4a_1r_1(a_1r_1+a)/N_1N_2$$

$$C_3 = a^2a_1^{\,4} + 2aa_1^{\,4}/N_1$$

$$-4a_1^{\,2}r_1a(ar_1 + a_1)/N_1N_2$$

The realizable filters are

$$H_1 = \frac{2a(a+a_1)\quad (j\omega+a_1)^2}{(a+\sqrt{d_1})\quad (a+\sqrt{d_2})\quad (a+\sqrt{d_3})\quad (j\omega+\sqrt{d_1})\quad (j\omega+\sqrt{d_2})\quad (j\omega+\sqrt{d_3})}$$

and

$$H_2 = \frac{r_1}{N_2(j\omega + a_1)}$$

3. <u>s, n_1 Correlated</u>

The realizability condition is

$$\left[N_1(a^2+a_2^{\,2}) + 2a+4a_2\ r_2\right]^2 - 4N_1(N_1a^2a_2^{\,2} + 2a\ a_2^{\,2} + 4a_2a^2r_2) > 0$$

and the realizable filter is

$$H_1 = \frac{j\omega(A + B) + Aa_2 + aB}{\sqrt{N_1}\ (j\omega+\sqrt{C_1})\ (j\omega+\sqrt{C_2})}$$

where

$$C_1, C_2 = \frac{(N_1 a^2 + N_1 a_2^{\ 2} + 2a + 4a_2 r_2) \pm \sqrt{\{N_1(a^2+a_2^{\ 2}) + 2a+4a_2r_2\}^2 - 4N_1\{N_1 a^2 a_2^{\ 2} + 2aa_2^{\ 2} + 4a_2 a^2 r_2\}}}{2N_1}$$

and are positive if above inequality holds and where

$$A = \frac{2a(a+a_2)}{\sqrt{N_1}\ (a + \sqrt{C_1})\ \ (a + \sqrt{C_2})}$$

$$B = \frac{2a_2r_2(a + a_2)}{\sqrt{N_1}\ (a_2 + \sqrt{C_1})\ \ (a_2 + \sqrt{C_2})}$$

4. <u>s, n_2 Correlated</u>

For the feedback loop, the restrictions are

$$aN_2 - r_3^{\ 2} > 0$$

$$aN_2 a_3^{\ 4} - 2a_3^{\ 2}\ r_3^{\ 2}\ a^2 > 0$$

and the realizable filter is

$$H_2 = \frac{j\omega(d_1 + d_2) + d\sqrt{C_2} + d_2\sqrt{C_1}}{(j\omega + \sqrt{C_1})\ \ (j\omega + \sqrt{C_2})}$$

where d_1, d_2 are functions of C_1, C_2, a_1, a_3, r_3, N_1 and N_2 and C_1, C_2 are positive if the above inequalities hold.

For the forward loop, the restrictions are

$$N_2 a_3^{\ 2} - 2r_3^{\ 2} a > 0$$

$$N_1 N_2 a_3 (a_3^{\ 2} + 2_a^{\ 2}) + 2r_3 N_1 (a^2 + a_3^{\ 2}) + 4aa_3^3 N_2 - 4a_3 r_3^{\ 2} > 0$$

$$N_2 a_3^{\ 2}\ (2 + aN_1) + 2aa_3 r_3 N_1 - 4\ ar_3 > 0$$

d_1 and d_2 are given by the same formulas as in case (b) except that C's are as follows

$$C_1 = a^2 + 2a_3^2 + 2a/N_1 + 2a_3\dot{r}_3/N_2$$

$$C_2 = a_3^4 + 2a^2a_3^2 + 2a_3r_3(a_3^2 + 2a^2)/N_2$$
$$+ 4a_3^2a/N_1 - 4a_3^2r_3^2/N_1N_2$$

$$C_3 = a^2a_3^4 + 2aa_3^4/N_1 + 2a^2a_3^3r_3/N_2$$
$$- 4a^2a_3^2r_3^2/N_1N_2$$

The realizable filter for this case is

$$H_1 = \frac{\left[(j\omega)^2 (p_1 + p_2) + j\omega (2a_3p_1 + ap_2 + p_2a_3 + p_3) + p_1a_3^2 + ap_3 + ap_2a_3\right]}{\sqrt{N_1} \quad (j\omega + \sqrt{d_1}) (j\omega + \sqrt{d_2}) (j\omega + \sqrt{d_3})}$$

where p_1, p_2, and p_3 are functions of a, a_3, r_3, N_1, N_2, d_1, d_2, d_3, and r_3.

For the combination of signal and cross spectral density parameters satisfying the realizability conditions, mean square error has been calculated and tabulated in Table 9-2.

For the realizable case with no delay in the channel the following observations are made

a). Uncorrelated case

Mean square error is minimum for sharp signal spectra and increases as the spectrum becomes broad.

b). n_1, n_2 Correlated

For a fixed cross-correlation, mse increases as the signal spectra becomes broad. For cross-correlation factor of .1, mse increases as the signal becomes broader and broader. However, for cross-correlation factor of .4, mse is minimum for $a_3 = 1$.

c). s, n_1 Correlated

The mse increases as the signal spectra becomes broader for a given cross-spectral density. Except for the broad signal spectra case for $r_3 = .1$, and not a very broad signal spectra case for $r_3 = .4$, mse increases as the cross-spectral density becomes broader. For the above two cases minimum occurs at $a_3 = 1$.

d). s, n_2 Correlated

Here the condition of realizability is satisified for very few combinations of the parameters. For the data obtained for broad signal case and for $r_3 = .1$, mse has a maximum at $a_3 = 1$ whereas for $r_3 = .4$, it decreases as the noise becomes broad.

While comparing the trend of the variation of mse in the case of optimum filters and in the case of optimum realizable filter, both behave almost in the same manner. However, mse is higher in the case of realizable filter than the optimum filters which is to be expected.

When a correlated signal as given in case (b) through (d) is applied to an optimum open loop filter, the manner in which this system compares with the feedback system is shown in Table 9-3. This table lists the mse as percentage of the mse obtained with open loop. Table 9-3 gives the comparisons for optimum case while Table 9-4 gives the corresponding percentages for realizable case with the realizable open loop filter used for the comparison.

For the realizable case as shown in Tables 9-2 and 9-4, there is improvement in the performance for only a few combinations of signal and cross spectral densities. For the correlation factor of r = .1, the case for which n_1 and n_2 are correlated does not show any improvement while the case for which s and n_2 are correlated gives some improvement for sharp cross spectra and broad signal. Similarly for r = .4, the case for which n_1 and n_2 are correlated shows improvement for combinations of a = 1., a' = 1. and a = 2, a' = 1., whereas the case for which s and n_2 are correlated shows improvement for combinations of a = 1., a' = 1. and a = 2, a' = 2. It could be observed that realizable case figures show no improvement in many cases and very little improvement for remaining cases when compared to optimum case.

For four combinations of signal and noise parameters, the variation of mse has been calculated for optimum systems with delay and tabulated in Table 9-5.

It could be noted that for all the cases considered, the mse for the uncorrelated case is larger than the rest of the cases. Also, there is more than one minimum value of mse for all the four cases. For example, the three combinations of a_1, r^1 and a^1 as in Table 9-5 (b), (c), and (d), all the four cases have minimum at t = o, .5 and 1 sec. For a = 1, $r^1 = 1$ and $a^1 = .5$, mse is minimum at t = o and 1. and maximum at t = .5. From the above tables, we could also observe the following trend of the performance for various cases of correlation. For all the cases, minimum mean square error is lowest for sharp cross-spectral density and a sharp signal spectrum, while for broader signal, minimum mse is lowest for sharp cross-spectrum and hence wherever a choice of signal is possible, it could be made accordingly. The results of this chapter are plotted in Fig. 9-a through Fig. 9-g.

For the expected performance of any particular system for a given signal and noise characteristics, the following procedures are recommended.

a). Approximate the given correlation by exponentials. [3]

b). Evaluate the expressions of optimum filter or realizable filter from Table [4-1] and [5-1]. Substitute them in expressions of mean square error as in Table [8-2] and [8-4] and integrate. The System with minimum square error is obviously the best system.

Table 9-1

Mean Square Error for Zero Delay in Channels
r^1, a^1 refers to r and a of Cross Spectral Densities

a	r^1	a^1	Uncorrelated Case	n_1, n_2 Correlated	s, n_1 Correlated	s, n_2 Correlated
	.1	.5	.420676	.404584	.388250	XX
.5	.1	.1	.420676	.416588	.392091	XX
	.1	2.	.420676	.419498	.394847	XX
	.1	.5	.570768	.508131	.471547	.568012
1	.1	1.	.570768	.567248	.531931	.568831
	.1	2.	.570768	.569479	.538674	XX
	.1	.5	.694093	.689344	.523402	.688864
2	.1	1.	.694093	.691852	.652188	.691169
	.1	2.	.694093	.693180	.658487	.692322
	.4	.5	.420676	XX	.225223	.379671
	.4	.5	.570768	XX	.128948	XX
1.	.4	1.	.570768	.421176	.385420	.535876
	.4	2.	.570768	.549019	.629048	XX
	.4	.5	.699093	XX	.523066	XX
2	.4	1.	.699093	.503218	.393418	.502218
	.4	2.	.699093	.678552	.548894	.664463

XX - Non-realizable cases

Table 9 - 2
Mean Square Error for Realizable Filters
Zero Delay in Channel and White Gaussian Noise

a	r^1	a^1	Uncorrelated Case	n_1, n_2 Correlated	s, n_1 Correlated	s, n_2 Correlated
	.1	.5	.581360	.598293	.552057	XX
.5	.1	1.	.581360	.633506	.555908	XX
	.1	2.	.581360	.710839	.578416	XX
	.1	.5	.723705	.737821	.691920	.725108
1.	.1	1.	.723705	.750633	.692236	.724325
	.1	2.	.723705	.799266	.649304	XX
	.1	.5	.813153	.821612	.788254	.807588
2.	.1	.1	.813153	.840661	.783434	.814632
	.1	2.	.813153	.860064	.788210	.813389
.5	.4	.5	.581360	XX	.394249	.645485
	.4	.5	.723705	XX	.623327	XX
1.	.4	1.	.723705	.586421	.557392	.718610
	.4	2.	.723705	.780084	.630720	XX
	.4	.5	.813853	XX	.603621	XX
2.	.4	.1	.813153	.796961	.615559	.869747
	.4	2.	.813153	.852194	.769237	.789501

XX Non-realizable case

Table 9 - 3

Relative Performance of Closed Loop and Open Loop Optimum Systems, zero delay and white guassian noise. Mean Square Error Calculated as Percentage of Open Loop Case.

a	r^1	a^1	n_1, n_2 Correlated	s, n_2 Correlated
	.1	.5	96.17	XX
.5	.1	1.	99.03	XX
	.1	2.	99.72	XX
	.1	.5	89.03	99.52
1.	.1	1.	99.38	99.66
	.1	2.	98.77	XX
	.1	.5	99.32	99.25
2	.1	1.	99.68	99.58
	.1	2.	99.87	99.74
.5	.4	.5	XX	90.25
1.	.4	1.	73.79	93.89
	.4	2.	96.19	XX
2.	.4	1.	72.50	72.36
	.4	2.	97.76	95.73

XX Non-realizable case

Table 9 - 4

Relative Performance of filters as in Table 9-3, but for

Realizable Case

a	r^1	a^1	n_1, n_2' Correlated	s, n_2 Correlated
.5	.1	.5	102.91	XX
	.1	1.	108.97	XX
	.1	2.	115.45	XX
	.1	.5	101.95	100.09
1.	.1	1.	103.72	100.25
	.1	2.	110.44	XX
	.1	.5	101.09	99.32
2.	.1	1.	103.38	100.18
	.1	2.	105.77	100.03
.5	.4	.5	XX	111.03
	.4	1.	81.03	99.30
1.	.4	2.	107.79	XX
2.	.4	1.	97.95	106.96
	.4	2.	103.01	99.08

XX Non-realizable case

Table 9-5(a)

Mean Square Error Variation with Delay in Channel and White Gaussian Noise

Parameters a=1. , r^1=.1, a^1=.5

t	Uncorrelated	n_1,n_2 Correlated	s,n_1 Correlated	s,n_2 Correlated
0	.570768	.508131	.471547	.568012
.1	.595198	.528834	.491743	.592442
.2	.595468	.572719	.536259	.589673
.3	.645192	.622447	.586608	.639397
.4	.685375	.662634	.628139	.679581
.5	.702401	.696981	.674745	.700662
.6	.686979	.664238	.633558	.681184
.7	.647696	.624946	.595612	.641896
.8	.598113	.575364	.546578	.592318
.9	.553029	.530277	.501532	.547235
1.	.530954	.508200	.479401	.525159

Table 9-5(b)

Mean Square Error Variation with Delay in Channels and White Gaussian Noise

Parameters a=2. , r^1=.1, a^1=1.

t	Uncorrelated Case	n_1, n_2 Correlated	s, n_1 Correlated	s, n_2 Correlated
0	.694093	.691852	.652188	.691169
.1	.742447	.740246	.617891	.739523
.2	.729470	.723289	.667946	.723226
.3	.730340	.724159	.670607	.724094
.4	.682017	.675834	.623379	.675771
.5	.645023	.639640	.587407	.639578
.6	.679153	.672970	.620562	.672907
.7	.728565	.722384	.668918	.722320
.8	.731173	.724992	.669738	.724928
.9	.728455	.725350	.620806	.725046
1.	.672773	.669657	.651605	.669369

Table 9-5(c)

Mean Square Error Variation with Delay in Channel and White Gaussian Noise

Parameters $a=2.$, $r^1=.1$, $a^1=.5$

t	Uncorrelated	n_1,n_2 Correlated	s, n_1 Correlated	s, n_2 Correlated
0	.694093	.689344	.523402	.688864
.1	.742447	.737731	.558346	.737218
.2	.729470	.703127	.607833	.703112
.3	.730340	.703997	.609616	.703981
.4	.682017	.655674	.561848	.655658
.5	.645823	.619481	.525765	.619464
.6	.679153	.652810	.558998	.652794
.7	.728565	.702223	.607884	.702206
.8	.731173	.704830	.609581	.704815
.9	.728455	.715283	.561244	.715195
1.	.672773	.659598	.523546	.659514

Table 9-5(d)

Mean Square Error Variation with Delay in Channel and White Gaussian Noise

Parameters $a=2.$, $r^1=.1$, $a^1=2.$

t	Uncorrelated Case	n_1,n_2 Correlated	s, n_1 Correlated	s, n_2 Correlated
0	.694093	.693180	.658487	.692322
.1	.742447	.741573	.706311	.740676
.2	.729470	.727953	.697336	.727729
.3	.730340	.728820	.701531	.708599
.4	.682017	.680194	.655259	.680275
.5	.645823	.644300	.619489	.644082
.6	.679153	.677630	.652447	.677411
.7	.728565	.727047	.699917	.722824
.8	.731173	.729656	.699205	.729433
.9	.728455	.727658	.728135	.726891
1.	.672772	.671950	.671737	.671210

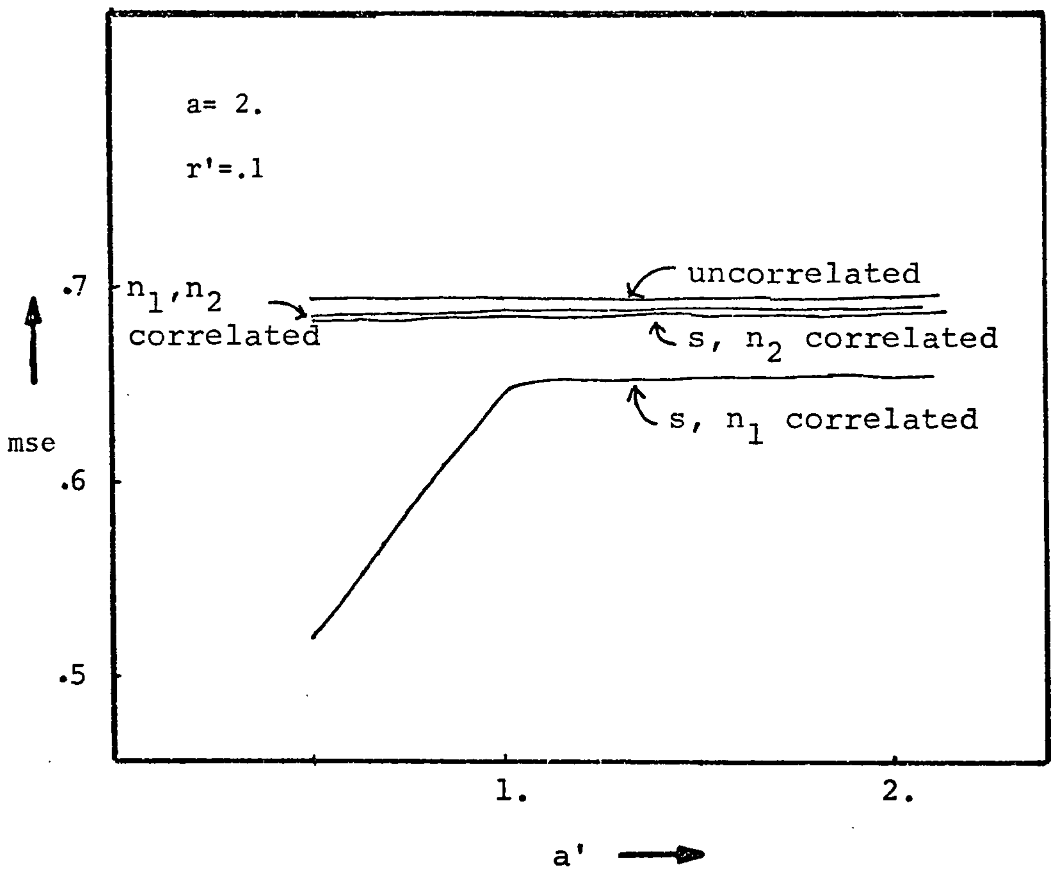


Fig. 9-a

Mean Square Error
Zero delay
White Gaussian noise

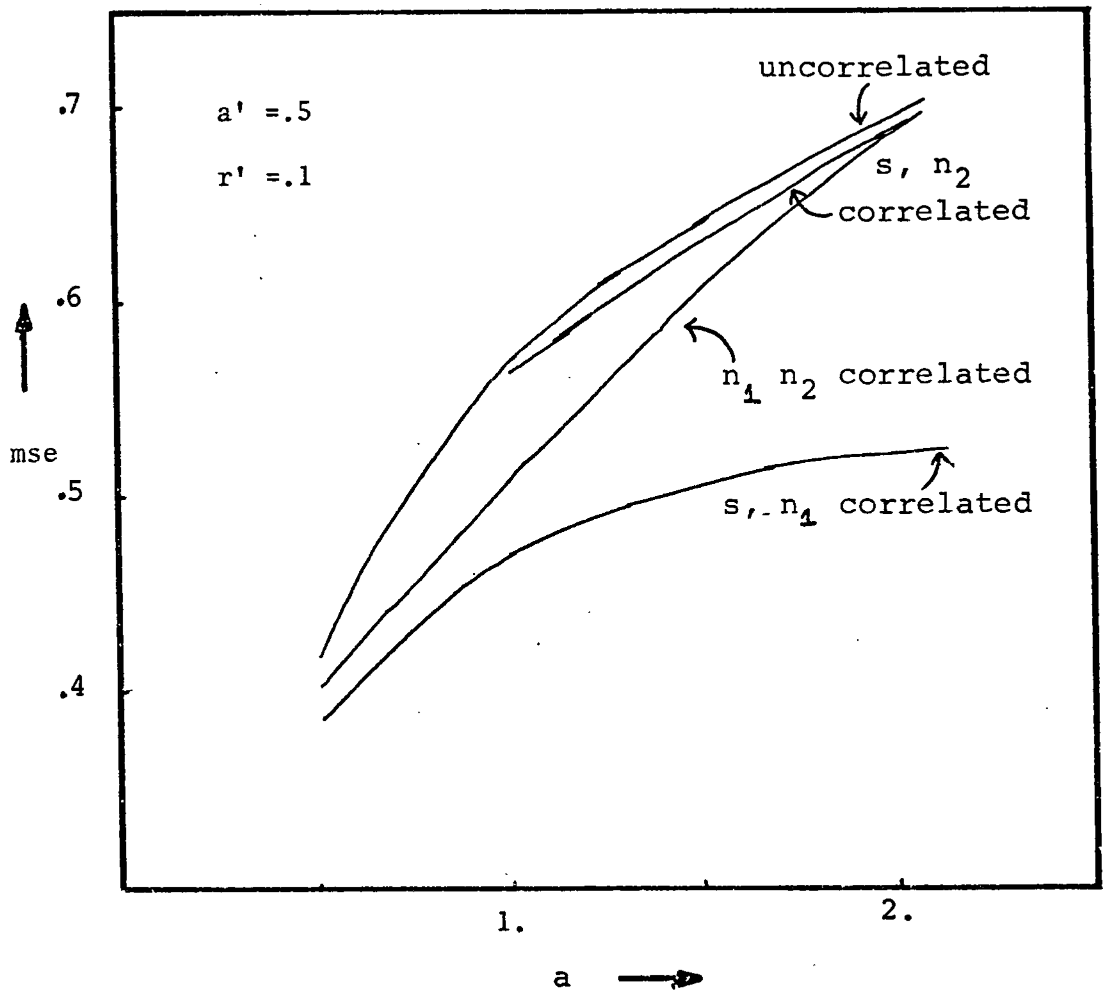


Fig. 9-b

Mean Square Error
Zero delay
White Gaussian noise

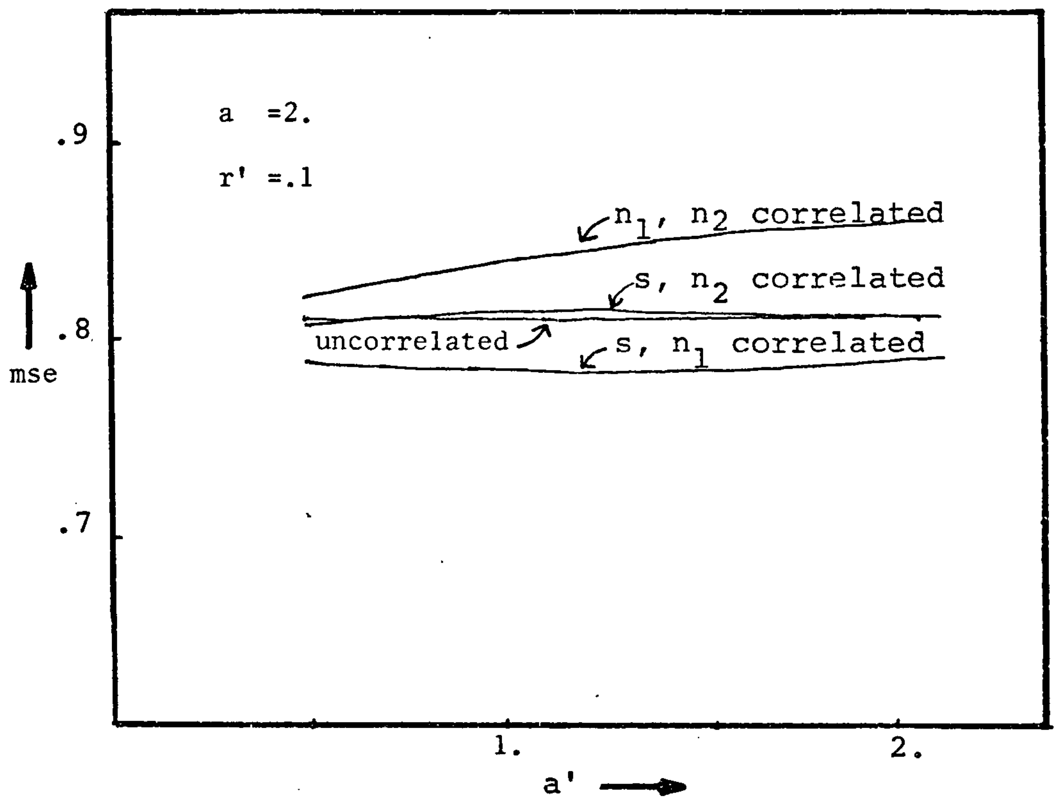


Fig. 9-c

Mean Square Error
Realizable Case
Zero Delay
White Gaussian Noise

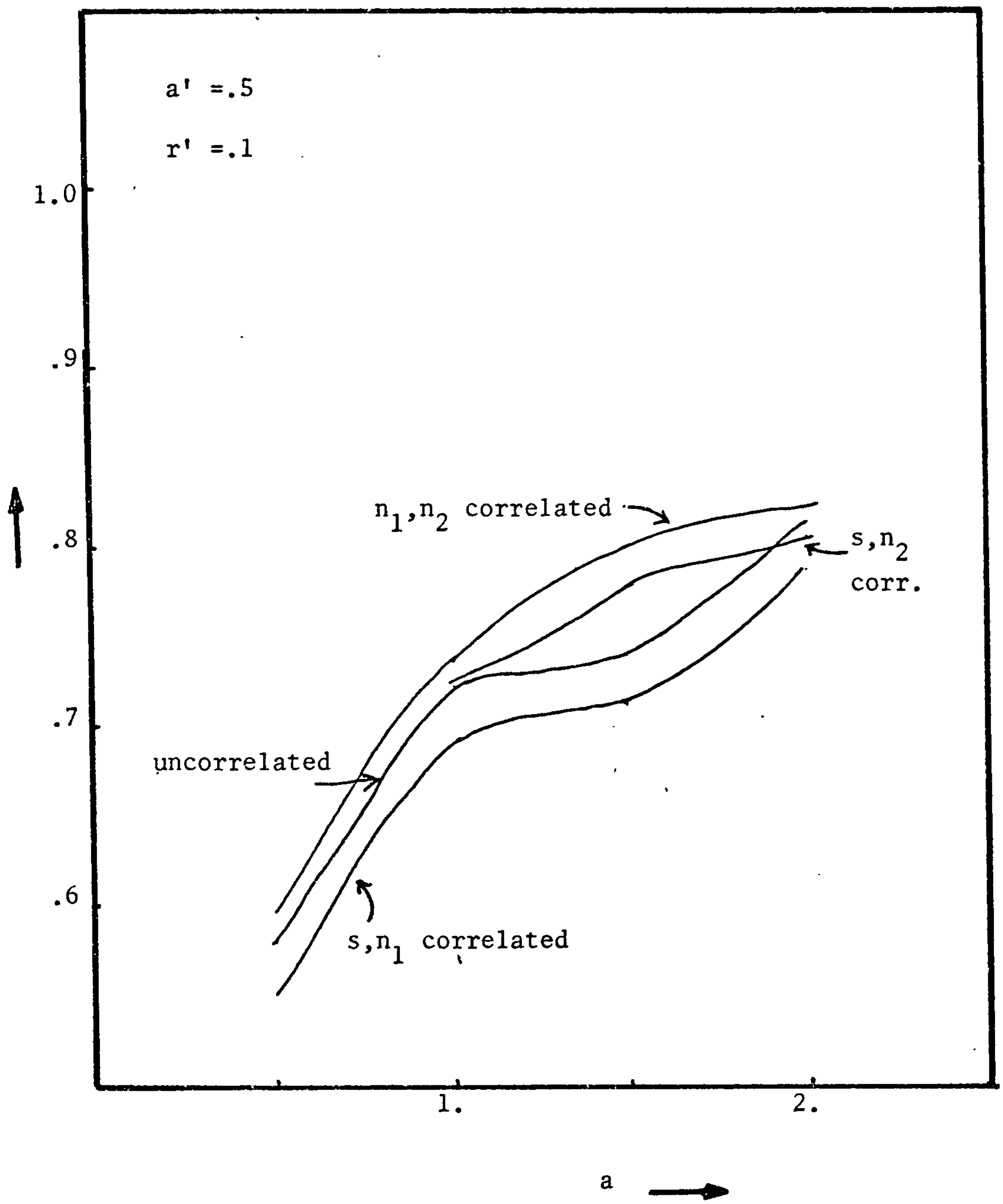


Fig. 9-d
Mean square error
Realizable case
Zero Delay
White Gaussian Noise

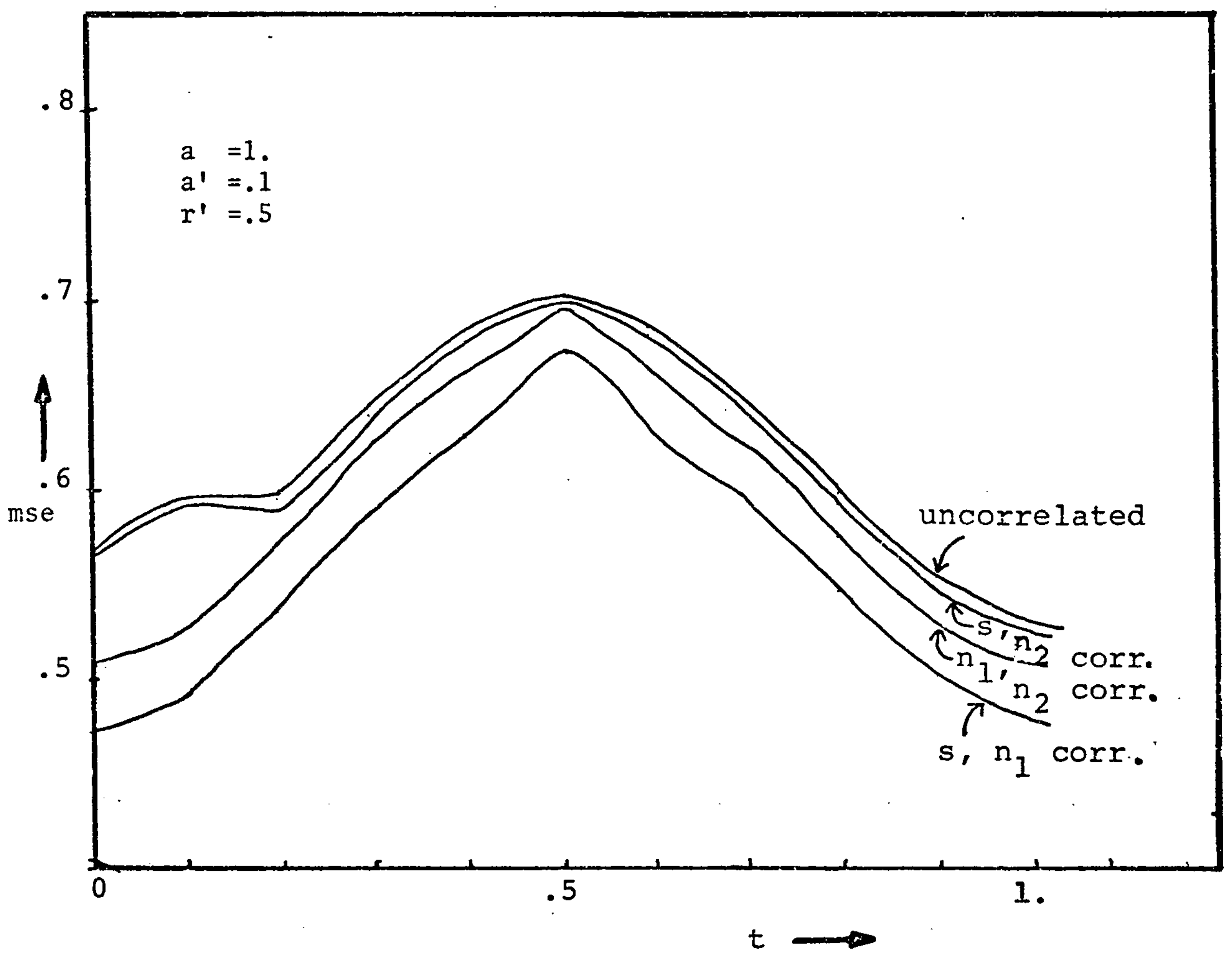


Fig. 9-e

Mean Square Error
Variation with delay
Optimum filter
White Gaussian Noise

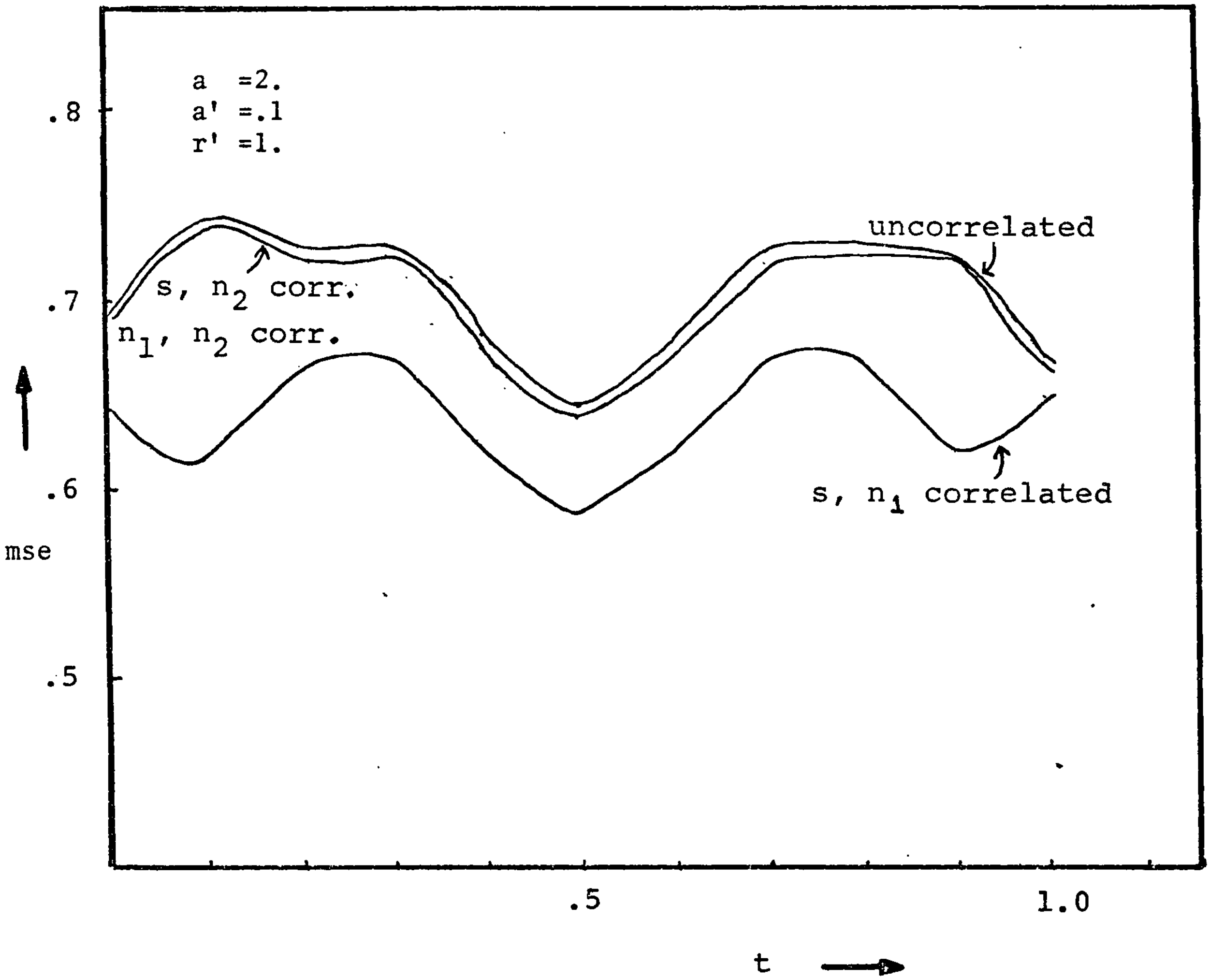


Fig. 9-f

Mean Square Error
Variation with delay
Optimum filter
White Gaussian Noise

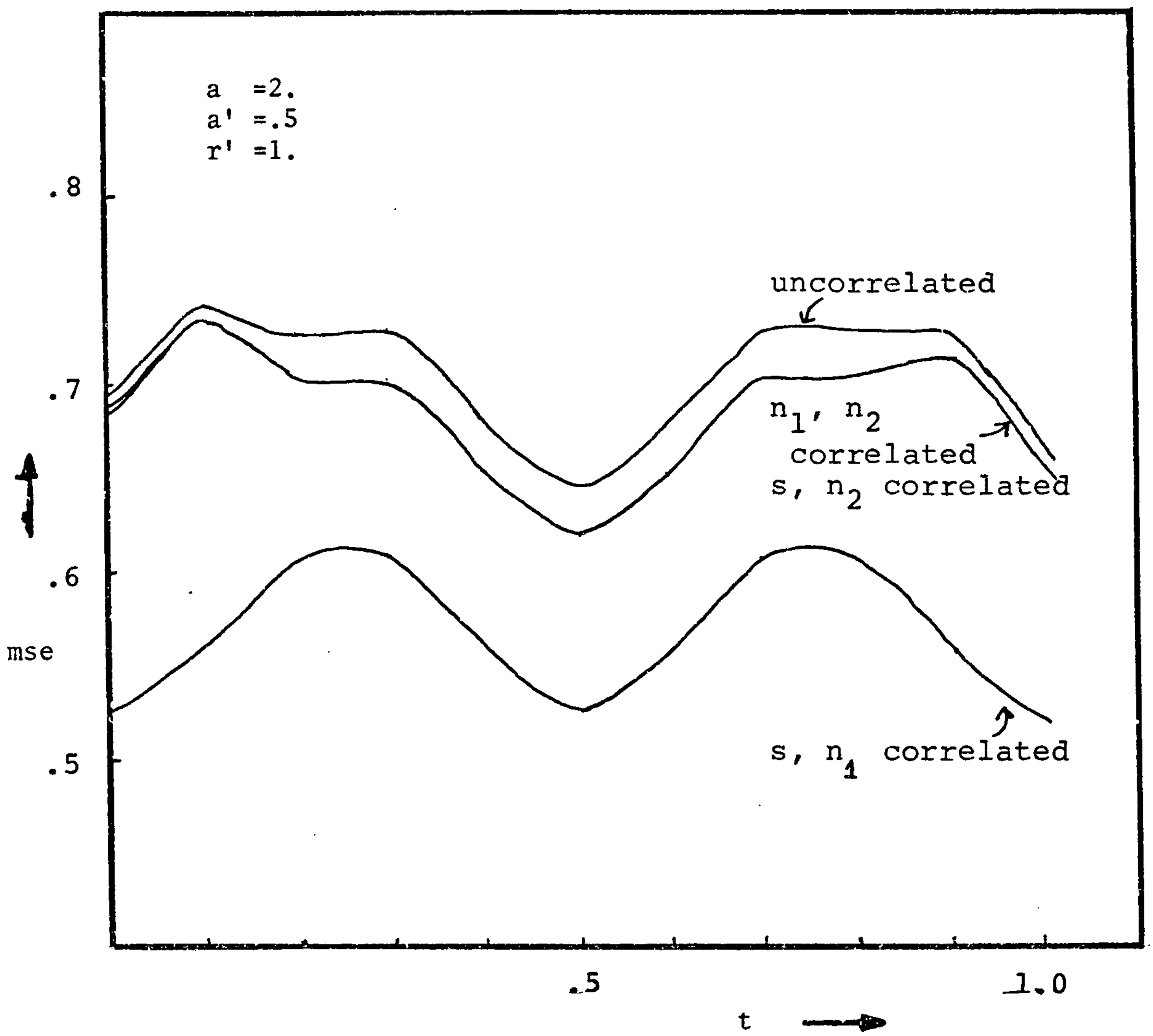


Fig. 9-g

Mean Square Error
Variation with delay
Optimum filter
White Gaussian Noise

CHAPTER X

CONCLUSION

A detailed analysis of feedback telemetry system models has been made. From the study it is found that any communication system can be reduced to an open-loop model by using control system techniques and based on that model an optimum performance of the system can be obtained.

For the case of transmission through the channels without any delay it is observed that when signal and noise processes are uncorrelated and when signal and noise in forward channel are correlated, the expression for the optimum filter transfer function in the feedback link is zero. It is a very significant result. It shows that it does not pay to use the feedback link for the above two cases.

For no delay case, optimum realizable filters could be synthesized by passive network without much difficulty. However, for the case of delay, exponential terms in $j\omega t$ suggest synthesis either by distributed parameter passive network or by a combination of distributed parameter network and active network [37].

The mean square error has been calculated by Sigma 7 Computer for various delays and various degrees of correlation between signal and noise. The outcome of the calculation is discussed in the previous chapter and it is found that for a sharp cross spectral density, mean square error is at its lowest for a sharp signal spectrum.

Future Study

Although the mean square error has been computed for optimum system, further study of the variation of mean square error for optimum systems employing optimum realizable filters and various shapes of noise spectra would give more insight into the use of feedback systems. Also in general most of the systems are limited by either peak or average transmitter power and this factor could also be taken into account. Lagrangian multiplier method of optimization is recommended for the analysis of this case.

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APPENDIX A

OPTIMIZATION BY CALCULUS OF VARIATION

Calculus of variation is a mathematical tool to optimize a function involving a certain parameter by taking its variation along a certain path. The simplest problem is the maximization or minimization of the functional

$$I = \int_a^b f\,(\,t,\, x,\, x'\,)\; dt$$

where f is the given function, $x'=dx/dt$, and the integral is taken along the curve $x(t)$ with $x(a)=A$ and $x(b)=B$. Under the given terminal conditions, we have to find the curve $x(t)$ which optimizes I.

To proceed with optimization, we take the variation of curve $x(t)$ denoted by $\delta x(t)$. Hence the new curve becomes

$$x(t) + \delta x(t)$$

$$\text{or } x+\delta x$$

This variation can also be written as $en(t)$ or simply en where n is any differentiable function of t, vanishing at the terminals as

$$n(a) = 0$$

$$n(b) = 0$$

and e is any real parameter. For a particular form of n, satisfying the above terminal conditions, the new curve x + en will take different paths depending on the value of n as shown in curve A-1. With the above variation of x(t), the functional I takes a new value I + δI given as follows

$$I + \delta I = \int_a^b f(t, x + en, x' + en')\, dt$$

The functional I will be maximum or minimum if its derivative with respect to e is zero, i.e.

$$d(I + \delta I)de=0$$

Also for the optimum path x(t) giving maximum or minimum I, the variation en will be zero no matter what the value of n is and hence e is taken as zero. Hence the condition of optimality is given as

$$\left.\frac{d\ (I + \delta I)}{de}\right|_{e=0} = 0$$

The application of this method leads to the famous Euler-la grange equation

$$\frac{\partial f}{\partial x} - \frac{d}{\partial t}\left[\frac{\partial f}{\partial x}\right] = 0$$

and is given in detail by LEE (16). The above equation leads to value of x(t) maximizing or minimizing the functional I.

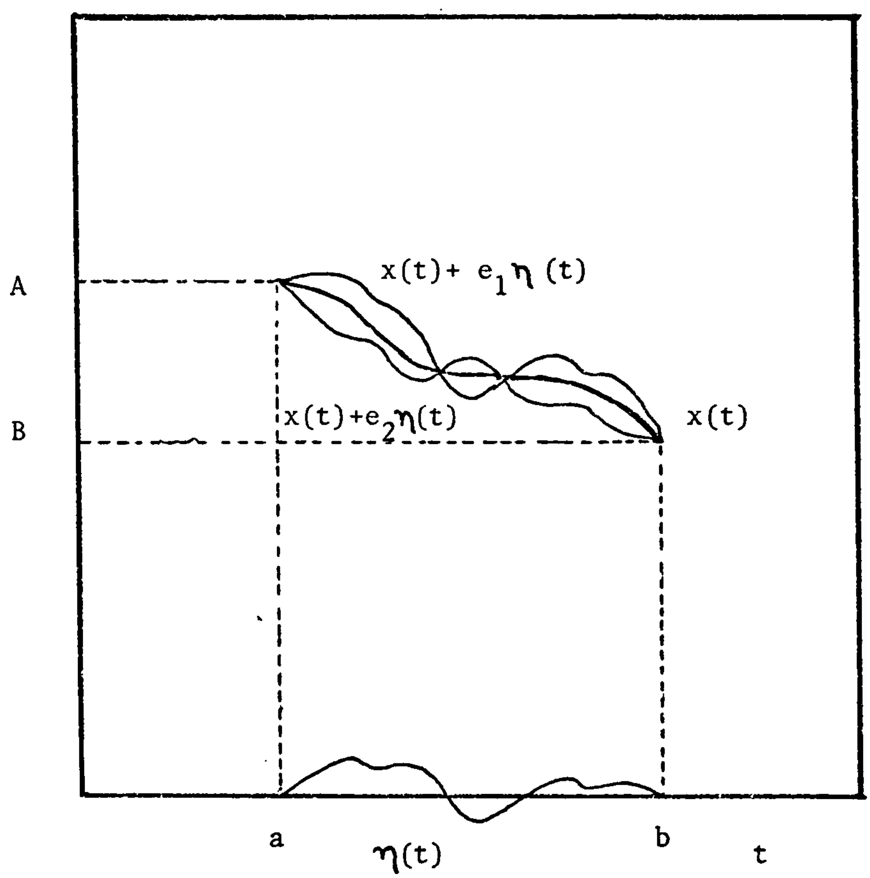

Fig. A-1

APPENDIX B

EVALUATION OF OPTIMUM LINEAR FILTER BY CALCULUS OF VARIATION

The linear system which satisfies the minimum mean square error criterion is called optimum linear system. The calculation of optimum linear filters is the core of the problem in the design of feedback telemetry system as described in the text and hence the following procedure is described for a simple linear system. For a given simple linear system the mean square error is given as

$$\overline{\varepsilon^2}(t) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} [s_o(t) - s_d(t)]^2 \, dt$$

The linear filter is required to extract the message from the corrupted message s_i, according to the minimum mean square criterion between the data received and the data desired s_d. The filter is to be designed with the above criterion of performance. It is either specified by system function $H(\omega)$ or by the unit-impulse response $h(t)$, their realizability and synthesis could be studied after they are evaluated. $s_o(t)$ could be written as convolution of $h(t)$ with $s_i(t)$ and hence the mean square error could be written as

$$\overline{\varepsilon^{2}}(t) = \underset{T\to\infty}{\mathrm{Lim}} \frac{1}{2T} \int_{-T}^{T} \left[\int_{\infty}^{\infty} h(\tau) s_i(t-\tau) d\tau - s_d(t) \right]^2 dt$$

This is simplified as

$$\overline{\varepsilon^{2}}(t) = \int_{-\infty}^{\infty} h(\tau) d\tau \int_{-\infty}^{\infty} h(\sigma) d\sigma R_{ii}(\tau-\sigma) - 2\int_{\infty}^{\infty} h(\tau) d\tau R_{id}(\tau)$$

$$+ R_{dd}(0)$$

where R's are the correlation functions as follows

$$R_{dd}(0) = \underset{T\to\infty}{\mathrm{Lim}} \frac{1}{2T} \int_{-T}^{T} s_d^{2}(t)\, dt$$

$$R_{di}(\tau) = \underset{T\to\infty}{\mathrm{Lim}} \frac{1}{2T} \int_{-T}^{T} s_d(t) \cdot s_i(t-\tau) dt$$

$$R_{ii}(\tau-\sigma) = \underset{T\to\infty}{\mathrm{Lim}} \frac{1}{2T} \int_{-T}^{T} s_i(t-\tau)\, s_i(t-\sigma) dt$$

and $s_i(t) = s(t) + n(t)$

where $s(t)$ is the original message and $n(t)$ is the noise contaminating the message, $s_d(t)$ may be any desired form of output as $s(t)$, $s(t + t_1)$, $ds(t)/dt$ etc. The filter system function will have the shape corresponding to the desired output.

In the practical case, the impulse response has the following restriction

$$h(t) = o; \; t<o$$

as the response of the system cannot precede the excitation and hence for realizable filters, the above restriction should hold.

Either we can constraint the expression for mean square error and apply calculus of variation to get optimum realizable filters or we can apply calculus of variation without the above constraint and calculate the realizable part from the optimum filter thus obtained. We will follow the second procedure. Writing the mean square error again as

$$\overline{\varepsilon^2}(t) = \int_{-\infty}^{\infty} h(\tau)d\tau \int_{-\infty}^{\infty} h(\sigma)d\sigma \, R_{ii}(\tau-\sigma)$$
$$-2\int_{-\infty}^{\infty} h(\tau)d\tau \, R_{id}(\tau) + R_{dd}(0)$$

Taking the variation of h(t), the above expression becomes

$$\overline{\varepsilon^2}(t)+\delta\varepsilon^2(t) = \int_{-\infty}^{\infty} \Big[h(\tau)+e\eta(\tau)\Big]d\tau \int_{-\infty}^{\infty} \Big[h(\sigma)+e\eta(\sigma)\Big] d\sigma R_{ii}(\tau-\sigma)$$
$$-2\int_{-\infty}^{\infty}\Big[h(\tau) + e\eta(\tau)\Big] R_{id}(\tau)d\tau + R_{dd}(0)$$

By comparing this expression of $\overline{\varepsilon^2}(\tau)$ we get

$$\overline{\delta\varepsilon^2}(t) = 2e\int_{-\infty}^{\infty} \eta(\tau)d\tau \int_{-\infty}^{\infty} h(\sigma) \, d\sigma \, R_{ii}(\tau-\sigma)$$
$$+e^2\int_{-\infty}^{\infty} \eta(\tau) \, d\tau \int_{-\infty}^{\infty} \eta(\sigma) \, d\sigma \, R_{ii}(\tau-\sigma)$$
$$-2e\int_{-\infty}^{\infty} \eta(\tau) \, d\tau \, R_{id}(\tau)$$

The condition for minimum mean square error, as discussed in Appendix A is,

$$\frac{d}{de} \left. \overline{\delta \varepsilon^{2}} (\tau) \right|_{e=o} = 0$$

for all possible η's

This gives

$$\int_{-\infty}^{\infty} \eta(\tau) \left[\int_{-\infty}^{\infty} h(\sigma) \; R_{ii}(\tau-\sigma) d\sigma - R_{id}(\tau) \right] d\tau = 0$$

for all possible η's

For a physically realizable function

$$\eta(t) = 0 \text{ for } t < 0$$

and hence

$$\int_{-\infty}^{\infty} h(\sigma) \; R_{ii}(\tau-\sigma) d = R_{id}(\tau)$$

$$\text{for } \tau > 0$$

Since at this stage we are not considering the realizability, for optimum filter we can write

$$R_{id}(\tau) = \int_{-\infty}^{\infty} h(\sigma) \; R_{ii} \; (\tau-\sigma) \; d\sigma$$

The solution of this expression will give the desired filter minimizing the mean square error. This may not be a realizable function and the later could be extracted using standard techniques.

APPENDIX C

REALIZABLE FILTER CALCULATION

Equation (3.9) is satisfied for the optimum system leading to minimum mean square error. When the realizability of the filter is not taken into account, we can rewrite it as

$$R_{di}(\sigma) = \int_{-\infty}^{\infty} h(\tau)\ R_{ii}(\sigma-\tau)\ d\tau \qquad \text{(C-1)}$$

$$\text{for } -\infty<\sigma<\infty$$

However for realizable filters, since

$$h(t) = 0\ ;\ t < 0,$$

(C-1) is modified as

$$R_{di}(\sigma) = \int_{-\infty}^{\infty} h(\tau)\ R_{ii}(\sigma-\tau)\ d\tau \qquad \text{(C-2)}$$

$$\text{for } \sigma > 0$$

Here $h(\tau) = 0$ for $\tau < 0$ but $R_{ii}(\tau)$ may not be zero for $\tau < 0$ and we have to find a new impulse response which satisfies the above condition.

Assuming two functions $c_1(t)$ and $c_2(t)$ such that

$$c_1(t) = 0 \qquad \text{for } t < 0$$

and

$$c_2(t) = 0 \qquad \text{for } t > 0 \qquad \text{(C-3)}$$

and their fourier transforms as

$$C_1(\omega) = \int_{-\infty}^{\infty} c_1(t)\ e^{-j\omega t}\ dt = \int_{0}^{\infty} c(t)\ e^{-j\omega t}\ dt \qquad \text{(C-4)}$$

and

$$C_2(\omega) = \int_{-\infty}^{\infty} c_2(t) e^{-j\omega t}\ dt = \int_{-\infty}^{0} c(t)\ e^{-j\omega t}\ dt \qquad \text{(C-5)}$$

Further assume that $c_1(t)$ when convolved with $c_2(t)$ gives $R_{ii}(\tau)$ as

$$R_{ii}(\tau) = \int_{-\infty}^{\infty} c_1(\tau-\sigma)\ c_2(\sigma) d\sigma \qquad \text{(C-6)}$$

Substituting in (C-2) we get

$$R_{di}(\tau) = \int_{0}^{\infty} h(\sigma)\ d\sigma \int_{-\infty}^{\infty} c_1(\tau-\sigma-\eta)\ c_2(\eta)\ d\eta$$

$$= \int_{-\infty}^{0} c_2(\eta) d\eta \int_{0}^{\infty} c_1(\tau-\sigma-\eta)\ h(\sigma)\ d\sigma \qquad \text{(C-7)}$$

(C-7) is obtained by interchanging $c_2(t)$ and $h(t)$, the limits of $c_2(t)$ being $-\infty$ to o

Assuming a function $\beta_{(t)}$ such that

$$R_{di}(\tau) = \int_{-\infty}^{0} c_2(\eta)\ \beta(\tau-\eta)\ d\eta \qquad -\infty < \tau < \infty \qquad \text{(C-8)}$$

This gives

$$\int_{-\infty}^{0} c_2(\eta)\ d\eta \left[\beta(\tau-\eta) - \int_{0}^{\infty} c_1(\tau-\eta-\sigma)\ h(\sigma)\ d\sigma\right] = 0$$

$$\text{for } \tau \geqslant 0 \qquad \text{(C-9)}$$

This holds true if

$$\beta(\tau) = \int_{0}^{\infty} c_1(\tau-\sigma)\ h(\sigma)\ d\sigma \qquad \tau \geqslant 0 \qquad \text{(C-10)}$$

In expression (C-10) both the functions $c_1(t)$ and $h(t)$ vanish for $t<0$.

Multiplying (C-10) by $e^{-j\omega\tau}$ and integrating between 0 and ∞ we get

$$\int_0^\infty \beta(\tau)\ e^{-j\omega\tau}\ d\tau = \int_0^\infty h(\sigma)\ e^{-j\omega\sigma}\ d\sigma \int_0^\infty c_1(\tau-\sigma)e^{-j\omega(\tau-\sigma)}d\tau$$

This gives

$$H(\omega) = \int_0^\infty h(\sigma)e^{-j\omega\sigma}d\sigma = \frac{\int_0^\infty \beta(\tau)e^{-j\omega\tau}d\tau}{\int_0^\infty c_1(\tau-\sigma)e^{-j\omega(\tau-\sigma)}d\tau}$$

or

$$H(\omega) = \frac{\int_0^\infty \beta(\tau)e^{-j\omega\tau}\ d\tau}{C_1\ (\omega)} \qquad \text{(C-11)}$$

Also multiplying (C-8) by $e^{-j\omega\tau}$ and integrating between $-\infty$ and ∞, we get

$$\int_{-\infty}^{\infty} R_{di}(\tau)e^{-j\omega\tau}d\tau = \int_{-\infty}^{0} c_2(\eta)e^{-j\omega\eta}d\ \int_{-\infty}^{\infty} \beta(\tau')e^{-j\omega\tau'}d\tau'$$

or

$$S_{di}(\omega) = C_2(\omega) \int_{-\infty}^{\infty} \beta(\tau)e^{-j\omega\tau}d\tau \qquad \text{(C-12)}$$

The expression for $\beta(\tau)$ could be written as

$$\beta(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{S_{di}(\omega)\ d\omega}{C_2(\omega)} \qquad \text{(C-13)}$$

Hence from (C-11), the expression for optimum realizable transfer function is

$$H^*(\omega) = \frac{1}{C_1(\omega)} \int_0^\infty e^{-j\omega\tau} d\tau \cdot \frac{1}{2\pi} \int_{-\infty}^\infty \frac{S_{di}(\omega)d\omega}{C_2(\omega)}$$

$$= \frac{1}{2\pi\ C_1(\omega)} \int_0^\infty e^{-j\omega\tau} d\tau \int_{-\infty}^\infty \frac{S_{di}(\omega)d\omega}{C_2(\omega)} \qquad \text{(C-14)}$$

From (C-6) by taking the fourier transform we can write

$$S_{ii}(\omega) = C_1(\omega)\ C_2(\omega) \qquad \text{(C-15)}$$

Where $C_1(\omega)$ is fourier transform of $c_1(t)$ which vanishes for $t<0$ and $C_2(\omega)$ is fourier transform of $c_2(t)$ which vanishes for $t>0$. This shows that $C_1(\omega)$ and $C_2(\omega)$ have singularities in left half and right half **s** planes respectively. Since $S_{ii}(\omega)$ is even

$$C_1(\omega) = C_2^*(\omega) = C(\omega) \qquad \text{(C-16)}$$

and

$$S_{ii}(\omega) = |C(\omega)|^2$$

This gives the final expression of optimum realizable filter as

$$H(\omega) = \frac{1}{2\pi C(\omega)} \int_0^\infty e^{-j\omega\tau} d\tau \int_{-\infty}^\infty \left[\frac{S_{di}(\omega)}{C^*(\omega)} \right] e^{j\omega\tau} d\omega$$

In other words the expression for optimum realizable filter can be written as

$$H(\omega) = \frac{1}{C(\omega)} \left[\frac{S_{di}(\omega)}{C^*(\omega)} \right]_+$$

Where $\left[\quad \right]_+$ means the part of $\frac{S_{di}(\omega)}{C^*(\omega)}$ on the upper half plane and $C(\omega)$ is as defined above.

The step by step procedure of calculating the optimum realizable filter is given in chapter III.

APPENDIX D

CROSS SPECTRAL DENSITY

It is found [3] that correlation function for any signal could be approximated by any exponential term. For example if we approximate the correlation function between signal and noise as

$$R(\tau) = r \quad e^{-a|\tau|}$$

this will give various shapes to the plot of $R(\tau)$ for various values of 'r' and 'a'. The corresponding spectral density $S(\omega)$, where

$$S(\omega) = \frac{2\ a\ r}{\omega^2 + a^2}$$

could approximate any signal spectra by the adjustment of its two parameters 'a' and 'r'. We may have to approximate the given signal or noise spectra by such exponentials. The use of this form of cross spectral density is based on the above approximation.

The plots of Cross Spectral density for various values of a's and r's as evaluated by Sigma 7 computer are given in Fig. D-1 and Fig. D-2. It is observed that for smaller values of a's, the spectrum is centered around zero frequency. For larger a's, it spreads to higher frequencies. The variation of 'a' changes the slope of the spectral density plot whereas any variation of r changes the corresponding value at zero frequency.

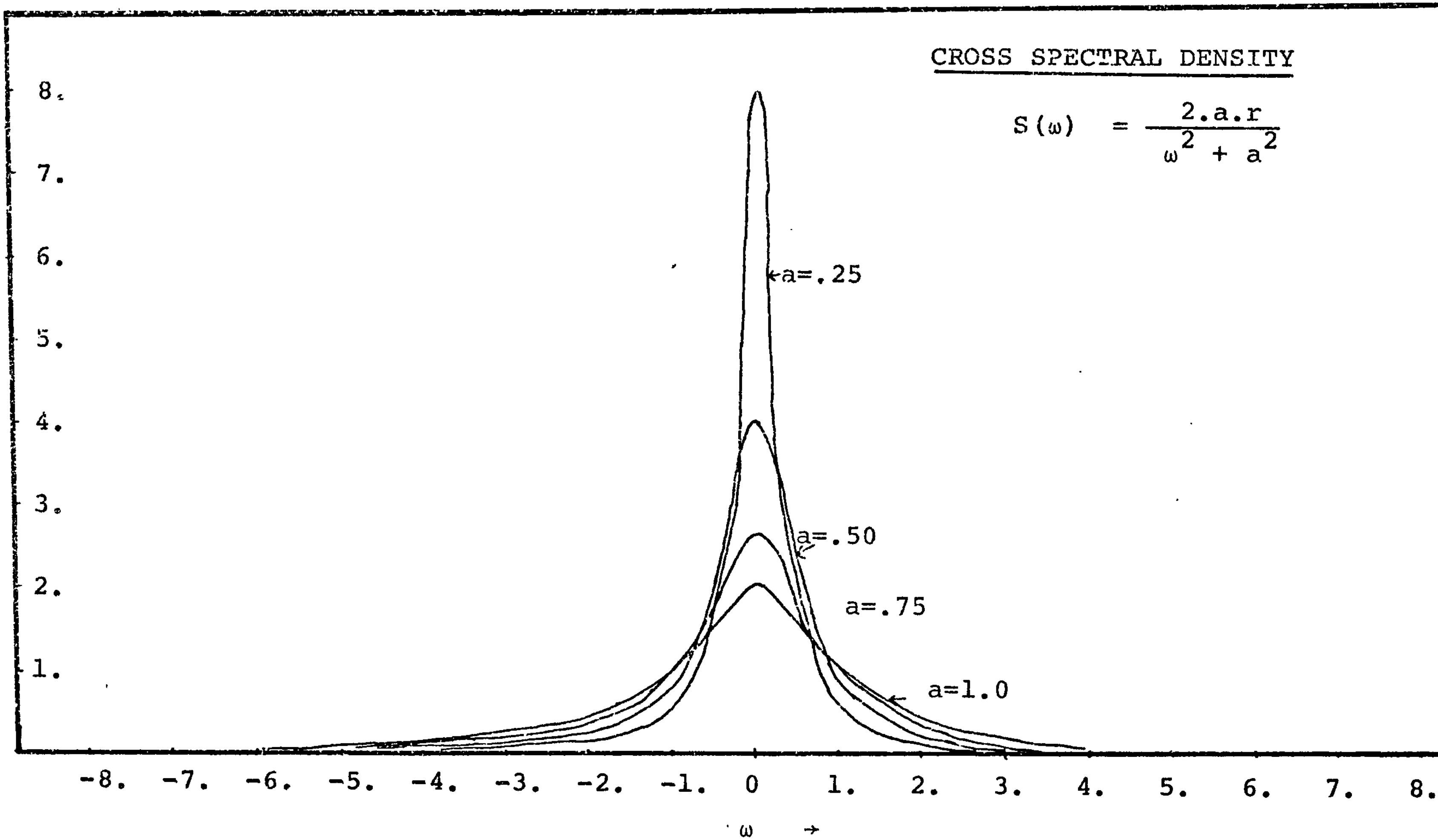
CROSS SPECTRAL DENSITY
S (ω) = 2.a.r / (ω² + a²)
8.
7.
6.
5.
4.
3.
2.
1.
a=.25
a=.50
a=.75
a=1.0
-8. -7. -6. -5. -4. -3. -2. -1. 0 1. 2. 3. 4. 5. 6. 7. 8.
ω →

Fig. D-1

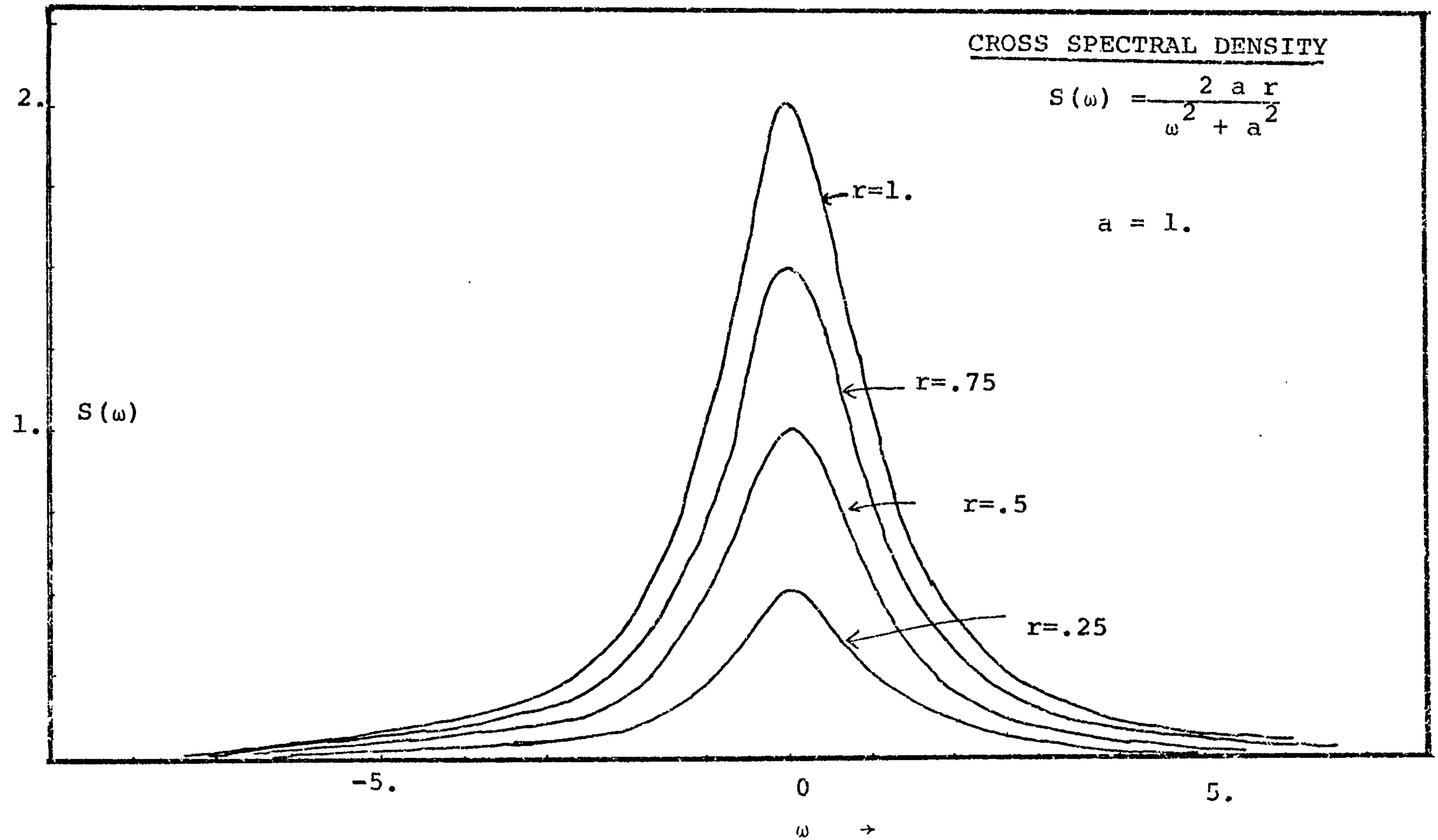

Fig. D-2