APPLICATIONS OF FRACTIONAL CALCULUS IN LAPLACE TRANSFORM THEORY

A Thesis

Presented to

the Faculty of the Department of Electrical Engineering

University of Houston

In Partial Fulfillment

of the Requirements for the Degree

Master of Science in Electrical Engineering

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by

Richard B. Whipple

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ABSTRACT

In the study of Laplace transforms, the need sometimes arises to calculate a transform or inverse transform involving variables of non-integer order. The integral definitions of the Laplace transform can always be used directly but the required integration is sometimes rather difficult to carry out. When the variables involved are of integer order, a number of convenient operational techniques are available to aid in calculation. The purpose of this thesis is to generalize certain of these techniques so that application is possible to non-integer order variables.

Before extending the methods of handling Laplace transforms, it is necessary to introduce the concept of fractional calculus. Since the subject is not widely known, a part of this thesis is concerned with providing the background material necessary to develop the Laplace transform extensions. Included is a historical review of fractional calculus along with a presentation of certain methods which can be used to calculate fractional derivatives and integrals for several specific functions.

Finally, certain theorems of Laplace transform theory are modified to encompass the concepts of fractional calculus. These theorems are then applied to several example problems to demonstrate their usefulness.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

CHAPTER I

INTRODUCTION

The basic definitions of the Laplace transform and its inverse for a function f whose domain is $[0,\infty]$ are given by eqs. I-1 and I-2 below.

$$F(s) = \int_{0}^{\infty} f(t)e^{-st}dt \qquad \text{I-1}$$

$$f(t) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} F(s)e^{st}ds \qquad \text{I-2}$$

The variable "t" is usually time or some spatial variable while "s" is called the frequency variable.[1] From these relations, it is possible to deduce a number of useful properties, some of which are given in Table 1. Note that the symbol "<———>" is used to indicate a transform pair, ie., if f(t)<———>F(s), F(s) is said to be the Laplace transform of f(t) and f(t) is said to be the inverse transform of F(s). The true advantage of these properties is realized when they are used to determine a transform or inverse transform without the direct application of eqs. I-1 and I-2. In fact, it is possible to start with a very limited number of transform pairs and build an extensive list without resorting to the original definitions.

In their present form, the properties of Table 1 have certain limitations. For instance, it is generally not

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[1]For convenience in the discussions of Chapters I and IV, reference will be made to the time domain and the corresponding variable "t"; but the results are in no way restricted to that particular domain or variable.

TABLE I

PROPERTIES OF THE LAPLACE TRANSFORM[2]

	OPERATION	TIME DOMAIN		FREQUENCY DOMAIN
1.	Linearity	$a_1 f_1(t)+a_2 f_2(t)$	$\longleftrightarrow$	$a_1 F_1(s)+a_2 F_2(s)$
2.	Time differentiation	$\frac{df(t)}{dt}$	$\longleftrightarrow$	$sF(s)-f(0^-)$
3.	Time integration	$\int f(t)dt$	$\longleftrightarrow$	$\frac{F(s)}{s} + \frac{f^{-1}(0^-)}{s}$
4.	Frequency differentiation	$-tf(t)$	$\longleftrightarrow$	$\frac{dF(s)}{ds}$
5.	Frequency integration	$\frac{f(t)}{t}$	$\longleftrightarrow$	$\int_s^\infty F(s)ds$
6.	Time shift $t_0>0$	$f(t-t_0)$	$\longleftrightarrow$	$F(s)e^{-st_0}$
7.	Frequency shift	$f(t)e^{s_0 t}$	$\longleftrightarrow$	$F(s-s_0)$
8.	Scaling	(at)	$\longleftrightarrow$	$\frac{1}{a}F(\frac{s}{a})$
9.	Time convolution	$f_1(t)*f_2(t)$	$\longleftrightarrow$	$F_1(s)F_2(s)$
10.	Frequency convolution	$f_1(t)f_2(t)$	$\longleftrightarrow$	$\frac{1}{2\pi i}[F_1(s)*F_2(s)]$

[2] B.P. Lathi, Signals, Systems and Communication (New York: John Wiley & Sons, Inc., 1967), p. 186.

possible to utilize these properties to derive transform pairs that involve non-integer orders of "t" or "s". The transform pair of eq. I-3 is such an example:

$$\frac{1}{\sqrt{\pi t}} \longleftrightarrow \frac{1}{\sqrt{s}} \; . \qquad \text{I-3}$$

The error function, Bessel function, and numerous others involve non-integer powers in the time domain and/or the frequency domain. The essential purpose of this thesis is to develop a method of extending the properties of Table 1 so that transform pairs involving non-integral powers of "t" or "s" can be derived.

To achieve this stated purpose, properties two through five of Table 1 are of primary concern. Note that these properties allow for multiplication and division of integer powers of "t" and "s" only. A casual inspection yields the conclusion that this integer limitation is somehow related to the fact that these same properties involve only integer order differentiation and integration. It seems at least reasonable that if non-integer differentiation and integration were possible, then these same properties might produce non-integer powers of "t" and "s". Subsequently, this conjecture will be shown to be correct.

A major limitation involving the generalization of properties two through five is the lack of general knowledge concerning Calculus of non-integer order. The concept of a "fractional" derivative may seem difficult to accept intuitively; yet, it is a perfectly valid idea provided it is properly defined. A body of knowledge does exist on the subject though it is highly disorganized and in some cases quite obscure. A fairly large proportion of this thesis will be needed to establish the basic concepts of this so-called generalized

Calculus. Once the basic ideas of generalized Calculus are clearly put forth, it is rather a simple matter to deal with the Laplace transform applications.

Chapter II is devoted mainly to a presentation of historical background on generalized Calculus. Surprisingly, it is as old as ordinary Calculus, having been originally suggested by Leibnitz in the late 1700's. Searches through the literature have failed to produce any definitive presentation of the rather extensive historical development generalized Calculus has undergone. Chapter II attempts to alleviate this situation and at the same time present a basic introduction to the subject. The concentration of interest will be mainly toward contributors to generalized Calculus before 1900. This period is of particular interest since most of the truly practical contributions are found therein. Writers since 1900 have generally dealt with details of a purely theoretical nature achieving sometimes important results on relatively small points. These papers have often ignored the more practical aspects of generalized Calculus such as actually calculating the non-integer derivatives and integrals of the most simple functions. In short, the needed working knowledge of generalized Calculus is best achieved by a detailed study of early writers who emphasized practical details.

The generalized derivative concept is further developed in Chapter III and a method of obtaining non-integer derivatives and integrals is given. Pitfalls in the interpretation of the generalized derivative and the implementation of the methods are discussed. Finally, the generalized derivatives of several simple functions are calculated.

The concepts of generalized derivatives are utilized in Chapter IV to extend the properties of the Laplace transform. Example problems are worked to illustrate the various methods developed. A physical problem dealing with a distributed parameter system is discussed to show how non-integer powers of "t" and "s" arise.

The findings of the work are given in the concluding Chapter V.

CHAPTER II

HISTORICAL BACKGROUND

The state of generalized Calculus today is much the result of inconclusive treatments that have been given it over the past two hundred fifty years. As will be seen in this chapter, some of the world's foremost mathematicians have made passing references to the subject, but only a very few have made significant contributions. In most cases, the efforts have been directed at providing a basic definition of generalized differentiation. Several writers have provided applications to illustrate the usefulness of their respective definitions. The various accounts that follow attempt to illuminate the more important contributions.

Recent mathematical historians credit Gottfried Leibnitz and Isaac Newton as co-inventors of the Calculus. But credit for the idea of a generalized Calculus must be given to Leibnitz alone. In Leibnitz's memoirs there are three letters written to his contemporaries which give his basic thoughts on the idea of a differential of non-integer order. The letter to John Bernoulli dated December 28, 1695, seems to be the earliest and most complete.[3] By the time of this letter, Leibnitz had formulated several basic concepts of the Calculus for first order differentials. He had also invented a very convenient symbolism similar to that used today. Although he had a relatively firm understanding of first order differentials, Leibnitz had considerable difficulty with those of higher order. He relied heavily on geometric example which often caused him confusion. Prompted by a

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[3]C.I. Gerhardt, Leibnizens Mathematische Schriften. (Vol. 21. Haile: H.W. Schmidt, 1859), p. 226.

question from Bernoulli concerning his rule for the derivative of a product of functions, Leibnitz gave the following explanation of the generalized differential. First, he considered the second order differential. Using the concept of ratios, he wrote

$$d^2x : dx = dx : x$$

or

$$\frac{d^2x}{dx} = \frac{dx}{x} .$$

He further noted that if "a" were a constant and "dh" were a constant differential, then

$$\frac{dx}{x} = \frac{dh}{a}$$

or

$$dx = \frac{dh}{a} x.$$

A simple substitution yielded

$$d^2x = \left(\frac{dh}{a}\right)^2 x.$$

Leibnitz then made the simple generalization that

$$d^ex = \left(\frac{dh}{a}\right)^e x$$

where "e" could take on any value whatsoever. For instance, he wrote

$$d^{\frac{1}{2}}x = \left(\frac{dh}{a}\right)^{\frac{1}{2}} x \ .$$

The reader may be somewhat puzzled as to how this definition could be applied since Leibnitz did not clearly define the values "a" and "dh". In later works, he completely abandoned this approach in preference to a geometrical interpretation very similar to that given today in courses of elementary Calculus. The concept of a slope was used to generate each higher order derivative from the graph of the previous derivative. Such a procedure did not give any physical interpretation to non-integer derivatives. As a result, Leibnitz made no further attempt to define a generalized derivative. It should be noted that the lack of physical interpretation in terms of a geometrical slope does present a significant obstacle in gaining an intuitive feeling for a generalized derivative. Although his direct contributions to the subject were meager, Leibnitz had established the idea of non-integer ordered differentials from which many to follow would take interest.

Examination of the next two hundred years following Leibnitz's work reveals three fairly definite periods in the development of generalized Calculus. The distinction between periods can be based primarily on the differences in approach taken to define general ordered differentiation and integration. The periods fall basically in chronological order. Although the techniques became more sophisticated in the later years, all the methods from the very earliest have at least some appeal from the practical sense.

The first period of study extends from the early 1700's through 1840. This era was a productive one for all of mathematics and especially for ordinary Calculus. Several prominent mathematicians of the time had occasion to at least conjecture on the generalization of the Calculus. This first period was characterized by the attempt to base the definition of non-integer ordered differentiation on the generalization of integer ordered differentiation for certain simple and well-known functions. As an example, suppose the expression for the n^{th} integer ordered derivative of a function were known. To define the non-integer order derivative for that function, the order parameter n would be allowed to take on all positive real values. Such a procedure is not always possible as will be demonstrated shortly for the function x^m (m constant). Nonetheless, the guiding principle for this technique is this: the general ordered derivative of a function must reduce to the correct integer ordered derivative for integer values of the order parameter n.

The earliest example of this first period technique was displayed by Leonard Euler (1707-1783) in 1730.[4] The basic function he chose was $f(x) = x^m$, m being a constant. The multiple derivative of x^m was well established at the time, and in this respect the choice was a logical one. But generalizing the derivative was far from straightforward. Before dealing explicitly with Euler's method, consider how a solution might be attempted. The n^{th} order derivative of x^m for n an integer and $m + 1 > n$ is

$$\frac{d^n x^m}{dx^n} = m(m-1)\ .\ .\ .\ (m-n+1)x^{m-n}. \qquad \text{II-1}$$

The simplest way to generalize the expression is to allow n to take on positive real values. Denoting this generalized

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[4] Leonhard Euler, Commentatones Analyticae (in Series 1, Vol. XIV of Leonhardi Euleri Opera Omnia. Berlin: B.G. Teubneri, 1924), pp. 1-24.

parameter by v, eq. II-1 becomes

$$\frac{d^v x^m}{dx^v} = m(m-1)\ .\ .\ .\ (m-v+1)x^{m-v}\ . \qquad \text{II-2}$$

This result is virtually meaningless since the coefficient product is calculatable only for integer values of v. In 1806, Joseph Lagrange (1736-1813) actually suggested this form as the definition of the general derivative of x^m.[5] He did not elaborate on the obvious flaw it contained. It should be clear that merely generalizing the order parameter is a rather weak technique.

Euler apparently realized that such a difficulty would be encountered and proceeded to develop a different form of generalized derivative for x^m. In the Commentationes Analyticae [6] of 1730, Euler presented a paper in which he investigated the properties of the progression,

1, 2, 6, 24, 120, 720, etc.

The n^{th} term of the progression was n!. Euler subsequently demonstrated that for integer values of N, the following equation could be verified:

$$n! = \int_o^1 (-\text{Ln } x)^n dx \qquad n \geq 0\ .$$

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[5]Joseph Lagrange, Lecons sur le Calcul des Fonctions. (Vol. X of Oeuvres de Lagrange. Paris: Gauthier-Villars, 1884), pp.95-96.

[6]Euler, loc. cit.

The n^{th} derivative of x^m he then wrote as

$$\frac{d^n x^m}{dx^n} = \frac{m!}{(m-n)!} \; x^{m-n}. \qquad \text{II-3}$$

Using the formula previously developed for n!, he wrote eq. II-3 as

$$\frac{d^n x^m}{dx^n} = \frac{\int_0^1 (-\text{Ln } x)^m dx}{\int_0^1 (-\text{Ln } x)^{m-n} dx} \cdot x^{m-n}. \qquad \text{II-4}$$

Euler made the further observation that there was no reason to limit n to integer values since the integral was valid in any case. He then concluded that eq. II-4 could be used to define the generalized derivative for x^m. Euler's result can be verified under certain limitations on m and n. Chapter III will deal with this question more fully.

The next contributor to the first period of generalized Calculus was Joseph Fourier (1768-1830). His reference to generalized Calculus was briefly stated in the latter portions of his comprehensive treatise, The Analytical Theory of Heat.[7] The results Fourier obtained were of little consequence to him and thus, he did not elaborate beyond a mere definition. Fourier's method of definition relied upon a conclusion drawn from the integral representation of a function which now bears his name, the Fourier Integral. The integral is usually given

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[7]Joseph Fourier, The Analytical Theory of Heat (New York: G.E. Stechert & Co., 1945), pp. 434, 437.

as

$$F(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} F(x')\cos k(x-x')dx'dk. \qquad \text{II-5}$$

Fourier noted that under this special representation, the function F was affected by the dummy variable x', not the variable x. Therefore, he could take a derivative of F(x) with respect to x by taking a derivative of the cosine function under the integral signs with respect to x. Thus, he wrote

$$\frac{d^{2i}F(x)}{dx^{2i}} = \frac{(-1)^{i}}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} F(x')k^{2i}\cos k(x-x')dx'dk \qquad \text{II-6}$$

or $$\frac{d^{2i+1}F(x)}{dx^{2i+1}} = \frac{(-1)^{i+1}}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} F(x')k^{2i+1}\sin k(x-x')dx'dk \qquad \text{II-7}$$

for i = 0, 1, 2, . . . in each case. He then generalized the derivative of cos kx so that both expressions could be written as one. He first assumed that

$$\frac{d^n}{dx^n}\cos(kx) = k^n\cos(kx+\frac{n\pi}{2}) \qquad \text{II-8}$$

for n = 0, 1, 2, ... Rewriting the pair of equations above, he produced a single expression for the n^{th} derivative of F:

$$\frac{d^nF(x)}{dx^n} = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} F(x')k^n\cos(k\{x-x'\}+\frac{n\pi}{2})dx'dk. \qquad \text{II-9}$$

He reasoned that there was no need to restrict n to integer values. Thus for n real and positive, eq. II-9 could be taken as the definition of a generalized derivative. Involved in Fourier's result was the assumption that the generalized derivative of the cosine function was correctly given in eq. II-8. Although the choice was arbitrary for the most part, eq. II-8 did satisfy the characteristic principle of the first period, ie. for integer values of n, it reduced to the known multiple derivatives of the cosine function.

Before leaving Fourier's work, it should be pointed out that he was rather vague regarding the definition just presented. Although he implied that he sought a relation for the derivative of general order, he never explicitly stated that n could be any more than a positive integer. Writers generally credit Fourier with the definition of non-integer ordered derivatives apparently overlooking his hesitation on the point. It is quite probable that Fourier was purposely vague so as not to be put into a position of having to defend the then obscure notion of a generalized Calculus.

At about the time of Fourier's publication, Pierre-Simon, Marquis de Laplace (1749-1827) published his important work, Theorie Analytique des Probabilities. Within this rather lengthy volume is found a suggestion by Laplace of a generalized derivative.[8] While Euler and Fourier had attempted to generalize the ordinary derivatives of x^m and cos kx respectively, Laplace chose to use a function which is commonly associated with him, the exponential e^{as}. His argument proceeded in the following way: Suppose a function f could be written as an integral of the form

$$f(s) = \int_0^\infty \phi(x)e^{-sx}dx.$$

[8]Pierre Simon Laplace, Theorie Analytique des Probabilities. (Vol. VII of Oeuvres de Laplace. Paris: Gauthier-Villars, 1886), pp. 159-160.

This is easily recognized as the Laplace transform already discussed. $\phi(x)$ would be chosen such that the integral existed and was equal to the function f for all $s \varepsilon D_f$. Since within the integral, ϕ did not depend on s directly, the problem of finding the n^{th} derivative reduced to knowing the derivative of e^{-sx} with respect to s. For n an integer, he wrote that

$$\frac{d^n}{ds^n} e^{-sx} = (-x)^n e^{-sx}.$$

There would appear to be no difficulty in permitting n to take on real positive values. Hence, Laplace deduced that the generalized derivative of f could be written as

$$\frac{d^v f(s)}{ds^v} = \int_0^\infty \phi(x)(-x)^v e^{-sx} dx \qquad \text{II-10}$$

where v was a real, positive number. Laplace made no further contributions on the subject.

Both Fourier and Laplace gave definitions of non-integer derivatives for arbitrary functions that depended on the correctness of the generalized derivatives of cos kx and e^{as}, respectively. It will be demonstrated subsequently that under the appropriate conditions, their assumptions were correct. Thus, the definitions of generalized derivative given in eqs. II-9 and II-10 have merit and could be applied. A more general definition will be given presently that has certain advantages over both of these.

Another famous mathematician made a unique contribution to generalized Calculus which is not commonly recognized by writers on the subject. He was Niels Henrik Abel (1802-1829). Abel's work belongs to this first period because he sought his definition by assuming a generalized form for the derivative of x^m. His results are of particular interest because included was a primitive form of the integral definition of the generalized derivative that has become the basis for modern approaches to the subject.

In 1823, a short article appeared in "Magazin for Natiroidenskaberne" containing Abel's elegant solution to the tautochrone problem.[9] His solution is of no concern to the present subject, but at the end of the article he made some very interesting comments which are summarized below.

Abel assumed that a function ψ could be given as a power series of x. There then existed constants a_m such that

$$\psi(x) = \sum_{\text{all } m} a_m \cdot x^m \, . \qquad \text{II-11}$$

For n an integer, the derivative of ψ was simply

$$\frac{d^n\psi(x)}{dx^n} = \sum_{\text{all } m} \frac{m!}{(m-k)!} x^{m-k} a_m \, . \qquad \text{II-12}$$

Abel generalized this result for a real number, v, in a manner similar to Euler's. The form of the generalized factorial function (now called Gamma function) used by Abel was

$$\Gamma(a) = \int_0^1 (\text{Ln } \tfrac{1}{x})^{a-1} dx. \qquad \text{II-13}$$

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[9]Niels Henrik Abel, "Solution de Quelques Problemes A' L'aide D'Integrales Definies," Oeuvres de Niels Henrik Abel (Christiana: Imprimerie de Grondahl & Son, 1881), pp. 11-27.

For a an integer and non-negative, it was true that

$$\Gamma(a) = (a-1)!. \qquad \text{II-14}$$

Abel concluded that for v real and positive,

$$\frac{d^v\psi(x)}{dx^v} = \sum_{\text{all } m} \frac{\Gamma(m+1)}{\Gamma(m-v+1)} x^{m-v} a_m. \qquad \text{II-15}$$

Abel then derived the expression given as eq. II-16.

$$\frac{\Gamma(m+1)}{\Gamma(m-v+1)} = \frac{1}{\Gamma(-k)} \int_o^1 \frac{t^m}{(1-t)^{1-v}}\, dt \qquad \text{II-16}$$

Substituting the result of eq. II-16 into II-15 led to

$$\frac{d^v\psi(x)}{dx^v} = \sum_{\text{all } m} \frac{1}{x^v\Gamma(-v)} \int_o^1 \frac{t^m}{(1-t)^{1+v}}\, dt \cdot x^m a_m \qquad \text{II-17}$$

or

$$\frac{d^v\psi(x)}{dx^v} = \frac{1}{x^v\Gamma(-v)} \int_o^1 \frac{\sum_{\text{all } m} a_m(xt)^m}{(1-t)^{1+v}}\, dt \qquad \text{II-18}$$

with the assumption that the order of summation and integration could be changed. But clearly,

$$\psi(xt) = \sum_{\text{all } m} a_m (xt)^m. \qquad \text{II-19}$$

Hence, he concluded that

$$\frac{d^v \psi(x)}{dx^v} = \frac{1}{x^v \Gamma(-v)} \int_0^1 \frac{\psi(xt)}{(1-t)^{1+v}}\, dt. \qquad \text{II-20}$$

Abel thus arrived at a definition of the generalized derivative based on an ordinary first order integral. His result was restricted to functions expandable in a power series. As will be shown later, this restriction can be removed. In fact, a simple change of variable leads to a special case of the Liouville-Riemann integral definition which is the basis of most modern theory on the generalized differentiation.

A very interesting application of generalized derivatives discussed by Abel concerned his tautochrone problem. Although he solved the problem with an integral equation, Abel showed that if one were willing to accept the idea of non-integer ordered differentials, the solution to the tautochrone problem could be written as a differential equation of one-half order,

$$\phi(s) = \Gamma(\tfrac{1}{2}) \frac{d^{\frac{1}{2}} \psi(s)}{ds^{\frac{1}{2}}} \qquad \text{II-21}$$

where s is the solution curve and ϕ is the time required for the point mass to slide down the curve s as a function of the height x. This is simply a variables-separable differential equation whose solution is obtained by performing a one-half order integration. If generalized Calculus were available, the solution would be relatively simple compared to the solution of an integral equation. If the physical laws of nature were appropriately generalized to accept this expanded concept of Calculus, it is very possible that many now mathematically tedious problems might yield much simpler solutions.

The first period was brought to a close by a French mathematician Joseph Liouville (1809-1882). In 1832, several long articles by Liouville appeared in the Journal de L'Ecole Polytechnique regarding the Calculus of general order.[10] Although his treatment of the subject was more extensive than given by his predecessors, it generally lacked rigorous arguments. Included in the articles were a number of interesting applications to the theory.

Liouville's basic approach involved the same assumption made by Laplace; that is,

$$\frac{d^v e^{ax}}{dx^v} = a^v e^{ax} \qquad \text{II-22}$$

where v is real. He then proceeded to write various functions as discrete sums of e^{mx} as in eq. II-23 below:

$$f(x) = \sum_{\text{all } m} A_m e^{mx} \qquad \text{II-23}$$

[10] Joseph Liouville, "Questions de Geometrie et de Mechanique," Journal de L'Ecole Polytechnique, Vol. 13, (1832), pp. 1-69.

where A_m is constant. Using eq. II-22, he then wrote that

$$\frac{d^v f(x)}{dx^v} = \sum_{\text{all } m} A_m m^v e^{mx}. \qquad \text{II-24}$$

He also considered the case where eq. II-23 might include an integration with respect to m. To illustrate the method, he gave the following short example: Find the v^{th} derivative of x^{-1}.

It is true that

$$x^{-1} = -\int_0^\infty e^{ax} da \qquad \text{for } x < 0.$$

Then

$$\frac{d^v x^{-1}}{dx^v} = -\int_0^\infty a^v e^{ax} da.$$

And finally,

$$\frac{d^v x^{-1}}{dx^v} = \frac{(-1)^v \Gamma(v+1)}{x^{v+1}} . \qquad \text{II-25}$$

Under proper conditions on v, this result reduced to the correct form for integer order derivatives. Liouville used this same technique to obtain generalized derivatives for other functions.

Liouville's work extended to an integral definition similar to Abel's. Recall that Abel assumed that the given function could be expanded into a power series. Similarly, Liouville assumed the function, say f, could be expanded in a series of e^{mx}. Following an argument very much like Abel's, Liouville deduced the following integral definition:

$$\frac{d^v f(x)}{dx^v} = \frac{(-1)^v}{\Gamma(-v)} \int_0^\infty f(x+t) t^{-(v+1)} dt \qquad v \leq -1 \cdot \qquad \text{II-26}$$

Note that with $v \leq -1$, he had actually defined a generalized integral. This important connection between generalized derivatives and integrals was formalized by Anton Krug and will be discussed later in this chapter.

Liouville's work was criticized by his contemporaries on the grounds that the expansion of functions in exponential series was not formally established. In addition, objections were raised regarding the fact that his generalized integration process did not reflect the then popular concept of the indefinite integral. Whereas a multiple indefinite integration yielded the so-called anti-derivative plus a power series in the independent variable with arbitrary constants, Liouville's formula yielded only the anti-derivative. This important point will be considered in more detail in Chapter III. Recent critics, such as Ferrar[11] have found other weaknesses in Liouville's treatment. Nonetheless, his work was the first serious attempt to deal directly with the concept of generalized Calculus. A slightly different form of the integral definition of eq. II-26 now bears Liouville's name.

The second period in the development of the generalized Calculus was characterized by the concentration of efforts to develop a rigorous integral definition while maintaining

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[11] W.L. Ferrar, "Generalised Derivatives and Integrals," Proceedings of the Royal Society of Edinburgh, Vol. 48 (1927), pp. 18.

maximum generality. Although there were several contributors to this period, the two most important were Bernhard Riemann (1826-1866) and Anton Karl Grunwald. Although Riemann's paper is most often sighted by modern writers, Grunwald's treatment is more rigorous and certainly more easily understood. Riemann's arguments are difficult to follow and in some cases reside on insecure ground mathematically.[12] Full elaboration on Riemann's paper does not seem justified, and only a brief description of what he attempted to do will be given. A more detailed explanation of Grunwald's method is included since it appears the more satisfactory of the two. In Appendix A, an intuitively convincing derivation of the Riemann-Liouville formula is presented from the rather basic concept of generalized areas. Arguments of this type were used by a number of authors of the period, most notably the Russian mathematicians.[13]

Although written during his student days in 1847, Riemann's paper on generalized Calculus was not published until after his death.[14] His basic approach involved producing a doubly infinite series analogous to the Taylor series expansion of a function. This generalized Taylor series contained non-integer order derivatives. Riemann then deduced the coefficients of the series necessary to produce the desired functional expansion. After considerable manipulation, he was able to recognize the necessary form

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[12] Arthur Cayley, "Note on Riemann's Paper," Collected Mathematical Papers of Cayley, (Vol. XI, Cambridge: University Press, 1896), pp. 235-236.

[13] A.V. Letnikov, "Theory of Differentiation of Fractional Order," Moscow Mathematicheskii Sbernik, Vol. III (1868), pp. 1-68.

[14] Heinrich Weber, Collected Works of Bernhard Riemann (New York: Dover Publications, Inc., 1953), p. 353.

of the derivative as

$$\frac{d^v f(x)}{dx^v} = \frac{1}{\Gamma(-v)} \int_c^x (x-t)^{-v-1} f(t)dt + \sum_{i=1}^{\infty} K_i \frac{x^{-v-i}}{\Gamma(-i-v+1)} \qquad \text{II-27}$$

where v is real and $v \leq -1$. The integral on the r.h.s. has been given the name Liouville-Riemann integral, or simply the L-R integral formula. The lower limit of integration c, is usually called the origin of the derivative. Note that the limitation $v \leq -1$ implies again that eq. II-27 represents the definition of an integral of arbitrary order. K_i is one of a possibly infinite collection of arbitrary constants. The sum on the r.h.s. is called the generalized complementary function. It is clear that this definition is extremely ambiguous depending as it does on arbitrary constants K_i and c. In Chapter III, some of this ambiguity will be removed.

In Grunwald's 1867 paper, he employed a generalization of the definition of the integer order derivative to derive the L-R integral formula.[15] Before presenting a brief synopsis of his derivation, a few words are necessary regarding the notation to be used in the remainder of this thesis. Rather than writing the derivative in the conventional manner, ie. $\frac{d^n f}{dx^n}$, the notation often seen in operational Calculus will be used. Thus, $D^n f$ will be taken as meaning the n^{th} derivative of f with respect to x. Later, the D notation will be modified

[15] Anton Grunwald, "Ueber begrenzte Derivationem und deren Anwendung," Zeitschrift for Mathematik und Physik, Vol. 12 (1867), pp. 441-480.

further.

To simplify Grunwald's arguments, assume that the function f is continuous on the interval of definition of f. The first derivative of f at all points on the interval of definition is given by

$$D^1 f(x) = \lim_{d \to o} \frac{f(x) - f(x-d)}{d} .$$

The second derivative is

$$D^2 f(x) = \lim_{d \to o} \frac{D^1 f(x) - D^1 f(x-d)}{d}$$

or

$$D^2 f(x) = \lim_{d \to o} \frac{f(x) - 2f(x-d) + f(x-2d)}{d^2} .$$

Similarly,

$$D^3 f(x) = \lim_{d \to o} \frac{f(x) - 3f(x-d) + 3(x-2d) - f(x-3d)}{d^3}$$

. . . . .

$$D^n f(x) = \operatorname{Lim}_{d \to o} \frac{f(x) + \sum_{i=1}^{n} (-1)^i \binom{n}{i} \cdot f(x-id)}{d^n} . \qquad \text{II-28}$$

where n is an integer and $\begin{Bmatrix} n \\ 1 \end{Bmatrix}$ is the coefficient given by

$$\begin{Bmatrix} n \\ 1 \end{Bmatrix} = \frac{n(n+1)\cdot\cdot\cdot\cdot(n+m-1)}{m!} \quad . \qquad \text{II-29}$$

Consider a new function F such that

$$F[u,x,e,d] = \lim_{d\to o} \frac{f(x)+\sum_{i=1}^{n}(-1)^{i}\begin{Bmatrix} e \\ i \end{Bmatrix}f(x-id)}{d^{e}} \qquad \text{II-30}$$

where $n = \frac{x-u}{d}$. For e negative and real,

$$D^{e}f(x)\Big|_{x=u}^{x=x} = \lim_{d\to o} F(u,x,e,d) \qquad \text{II-31}$$

where $D^{e}f(x)\Big|_{x=u}^{x=x}$ is defined as the e^{th} order integral of f(x) over the limits of integration u to x. If e is a negative integer, eq. II-31 does indeed reduce to the definition of the definite integral in the Riemann sense. Since u + d = x - (n-1)d, then

$$F[u+d,x,e,d] = \lim_{d\to o} \frac{f(x)+\sum_{i=1}^{n-1}(-1)^{i}\begin{Bmatrix} e \\ i \end{Bmatrix}f(x-id)}{d^{e}} \quad . \qquad \text{II-32}$$

Consider

$$\frac{F[u+d,x,e,d] - F[u,x,e,d]}{d} = (-1)^{n+1} \binom{e}{n} \frac{f(x-nd)}{d^{e+1}}$$

$$= (-1)^{n+1} \binom{e}{n} \frac{n^{e+1} f(u)}{(x-u)^{e+1}} \quad . \qquad \text{II-33}$$

Taking the limit as $d \to o$, the l.h.s. is merely the partial derivative of $\lim\limits_{d \to o} F$ with respect to u or

$$\frac{\partial}{\partial u}\left(D^e f(x) \Big|_{x=u}^{x=x} \right) = \lim_{d \to o} (-1)^{n+1} \binom{e}{n} \frac{n^{e+1} f(u)}{(x-u)^{e+1}} \quad . \qquad \text{II-34}$$

Since $n \to \infty$ as $d \to o$ and because

$$\lim_{n\to\infty} (-1)^n \binom{e}{n} n^{e+1} = \frac{1}{\Gamma(-e)} \quad ,$$

eq. II-34 becomes

$$\frac{\partial}{\partial u}\left(D^e f(x) \Big|_{x=u}^{x=x} \right) = - \frac{1}{\Gamma(-e)} \frac{f(u)}{(x-u)^{e+1}} \quad . \qquad \text{II-35}$$

Integrating eq. II-35 with respect to u

$$\int_x^u \frac{\partial}{\partial u}\left[D^e f(x)\Big|_{x=u}^{x=x}\right] = -\frac{1}{\Gamma(-e)} \int_x^u \frac{f(u)du}{(x-u)^{e+1}}$$

or

$$D^e f(x)\Big|_{x=u}^{x=x} - D^e f(x)\Big|_{x=x}^{x=x} = \frac{1}{\Gamma(-e)} \int_u^x \frac{f(u)du}{(x-u)^{e+1}} \qquad \text{II-36}$$

Since f is continuous, an integral of zero width must be zero. With a simple change of variable, the final result is obtained.

$$D^e f(x)\Big|_{x=u}^{x=x} = \frac{1}{\Gamma(-e)} \int_u^x \frac{f(\phi)d\phi}{(x-\phi)^{e+1}} \qquad \text{II-37}$$

for $e < 0$. Eq. II-37 is the Liouville-Riemann integral defining integration of arbitrary order. Note that the u of eq. II-37 is the origin of the general integral and that the complementary function of Riemann's result does not appear. A similar derivation yields a slightly different result which is given eq. II-38 below.

$$D^e f(x)\Big|_{x=u}^{x=x} = \frac{1}{\Gamma(-e)} \int_x^u \frac{f(\phi)d\phi}{(x-\phi)^{e+1}} \qquad \text{II-38}$$

Eq. II-37 is termed the left-hand L-R integral with origin u while eq. II-38 is termed the right-hand L-R integral with origin u. Both forms will be found useful. During the second period in the study of a generalized Calculus, the L-R integral became firmly established as the basic definition for integration of arbitrary order. Several important questions remained unanswered, however. For instance, there appeared to be a limitation on the use of the L-R integral to define differentiation of arbitrary order. Grunwald's derivation had led to the most general result in this regard, yet it limited the order to be less than zero which provided for integration of arbitrary order only. The question still remained: Could the L-R formula be used to define general differentiation and integration with the common parameter v? The nature of the origin was likewise unsettled. No procedure was available for finding the correct origin to be used with any given function. So long as the origin was seen to be arbitrary, the results of the L-R formula could not be of much use. The uncertainty with regard to the origin led to disagreements as to the correct form of the generalized integral of various functions. For instance, Liouville and Riemann obtained two completely different results for the integral of x^m. Although it was not immediately obvious to the mathematicians of the time, the differences were due to different choices of origin. All these difficulties remained to be resolved in the third period.

Another and more general method of attack was to come to the forefront in the third period of study. During the 1880's interest in generalized Calculus was revitalized by the application of certain results from the theory of complex variables. In particular, mathematicians discovered that Cauchy's integral formula for the derivative was merely a special case of a more general formula for the derivative of any order. Cauchy's work with his integral had taken place in the 1820's; so it is a little surprising that it took more than sixty years for the connection to be made. At any rate, the use of Cauchy's formula characterized the third period in the development of generalized Calculus.

Cauchy's formula for the derivative can be stated in the following way: Given a function of a complex variable z denoted by f, then

$$\frac{d^n f(z)}{dz^n}\bigg|_{z=z_o} = \frac{n!}{2\pi i} \oint_C \frac{f(z)dz}{(z-z_o)^{n+1}} \qquad \text{II-39}$$

for n = 0, 1, 2, . . . where f is analytic everywhere within and on a closed contour C and z_o is any point interior to C. In order to generalize this definition, it was necessary to replace the factorial with the Gamma function, $\Gamma(n+1)$. The new definition of the derivative of general order was

$$\frac{d^v f(z)}{dz^v}\bigg|_{z=z_o} = \frac{\Gamma(v+1)}{2\pi i} \oint_C \frac{f(z)dz}{(z-z_o)^{v+1}} \qquad \text{II-40}$$

where v could be a complex number. The use of eq. II-40 was pioneered by Grunwald and a fellow German, R. Most.[16] Their efforts were primarily concerned with justification of the L-R formula by the use of eq. II-40. While the work of Grunwald and Most is at least worthy of note, a more excellent

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[16] R. Most, "Ueber die Anwendung der Differentialquotienten mit allgemeinem Index zum Integrirem von Differentialgleichungen," Zeitschrift for Mathematik und Physik, Vol. 16 (1871), pp. 190-210.

and complete analysis was published in 1889 by another German mathematician, Anton Krug.[17] A discussion of Krug's approach and his principal conclusions will be most beneficial for present purposes.

Krug chose the form of eq. II-40 as the basis for a definition of generalized differentiation. He then proceeded to show that the integral existed for all $v \in C$. The case of v a negative integer would have seemed particularly questionable since the gamma function was not defined for these values. Krug was able to show that the integral did exist in the limit as v approached any negative integer. He then proceeded to verify the very important conclusion that differentiation and integration could be considered continuous operations of the same index. Thus, eq. II-40 defined

(1) derivatives of arbitrary order for $\mathrm{Re}(v) > 0$

(2) integrals of arbitrary order for $\mathrm{Re}(v) < 0$

and (3) the function f itself for $v = 0$.

It is as a result of this that the term, "generalized differentiation" is accepted as being all-inclusive in its meaning.

Krug's next major point was to demonstrate that the L-R integral formula was a natural consequence of Cauchy's formula provided the contour C was chosen properly. Thus, the L-R formula could be thought of as the special case of the Cauchy formula when only real variables were involved. In a sense, the choice of contour determined the value of the origin of the derivative for a given function. Krug then proceeded to find the derivatives of general order for several elementary functions. In each case, the choice of contour was the critical

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[17] Anton Krug, "Theorie der Derivationem," <u>Denkschriften der Mathematisch Naturwissenschaftliche Klasse</u>, Vol. 57, (1890), pp. 151-228.

element.

Another significant result found in Krug's paper greatly eliminated the ambiguity in the choice of origin for a particular function. Before presenting this, it is necessary to modify the D operator notation. Since the origin of the derivative for a function can have one or more values depending on various parameters that might be involved, the D notation should give the particular origin involved in any calculation. Therefore, the v^{th} derivative of f with respect to x and with origin a is defined as

$$\underset{a}{D}^{v} f(x).$$

Krug investigated the necessary requirements for the generalized derivative to possess the semi-group property. This property held for integer ordered derivatives so it seemed reasonable to determine the conditions under which the generalized derivative defined by the L-R formula possessed this property. He reached the following important conclusion: in order for

$$\underset{a}{D}^{m} \underset{a}{D}^{n} f(x) = \underset{a}{D}^{m+n} f(x) \qquad \text{II-41}$$

it was necessary that

$$\lim_{t \to a} [(t-a)^{1-n} f(t)] = 0. \qquad \text{II-42}$$

It should be noted that the condition of eq. II-42 does not necessarily imply that the generalized differential operator possesses the commutative property. At any rate, a variation of this condition will be used to great advantage in Chapter III to determine origins for several elementary functions.

The remainder of Krug's paper was concerned with a wide variety of other topics concerning the use of Cauchy's formula in its generalized form. He touched upon the concept of the complementary function and discussed the form of derivatives for certain of the higher functions. Elaboration on these remaining points does not seem required in light of the material yet to be presented in Chapter III.

Krug's paper is lengthy and covers the subject completely from the standpoint of contour integration. Dealing with the subject on that level is not necessarily required for the purposes of understanding the concepts of generalized Calculus for functions of a real variable. Therefore, in the remainder of this thesis, contour integration techniques will be avoided when possible. Krug's results with regard to the semi-group property will be indispensable, however.

The interest aroused in the third period wained rather quickly, and from 1900 to the present, little has been contributed concerning the generalization of Cauchy's formula. To close this chapter, it seems appropriate to mention some of the most important work that has been done since 1900. The L-R integral has been the center of interest for this latest period. Its properties have been well established in papers by G.H. Hardy and J.E. Littlewood.[18] In most papers, the origin has been assumed arbitrarily zero and little attempt made to deal with the specific cases of individual functions. In an excellent paper by E. Post, the idea of a generalized

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[18] G.H. Hardy and J.E. Littlewood, "Some Properties of Fractional Integrals," Mathematische Zeitschrift, Vol. 27 (1928), pp. 565-606.

operator was dealt with very completely, but again specific examples were avoided.[19] Several articles have appeared giving applications of fractional ordered integration to certain potential theory problems.[20] One rather interesting article by B. Spain treated the problem of finding non-integer order derivatives by applying interpolation techniques.[21] In his analysis, the generalized derivatives were obtained by interpolating between known integer order derivatives and integrals using the cardinal interpolation formula. His results are particularly impressive since he showed that the L-R formula followed from an arbitrary function when his technique was applied. The reader is referred to the bibliography for other recent papers on the subject of generalized Calculus.

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[19]Emil L. Post, "Generalized Differentiation," Transactions of the American Mathematical Society, Vol. 32 (1930), pp. 723-781

[20]A. Erdelyi, "Axially Symmetric Potentials and Fractional Integration," Journal of the Society for Industrial and Applied Mathematics, Vol. 13 (1965), pp. 216-228.

[21]B. Spain, "Interpolative Derivatives," Proceedings of the Royal Society of Edinburgh, Vol. 60 (1940), pp. 134-140.

CHAPTER III

GENERALIZED DERIVATIVES

The purpose of this chapter is twofold. First, an attempt will be made to properly establish the role of the L-R integral definition in computing the generalized derivative. Second, the generalized derivatives of several elementary functions will be calculated. The results obtained in this chapter will be preparatory to the derivations concerning the Laplace transform in the next chapter.

In the previous chapter, two questions arose which must be answered before any intelligent use can be made of the L-R formula. These were as follows:

(1) What is the sigificance of the complementary function?

and (2) How does one choose a value for the origin to be used in the calculation of the generalized derivative for a given function?

The former has been the subject of much controversy in the literature while the latter has been avoided almost totally. The following resolution of these questions has a strong intuitive appeal and is as mathematically rigorous as practically possible.

In the case of the complementary function, resolving the question is most readily accomplished by interpreting the general case in terms of what is already known about the integer order integration process. When one discusses the indefinite integral of first order, he usually assumes that it arises in the following way: Suppose the derivative of a function is given and it is required to find the function. Or restated, find a function F such that when it is differentiated once, the result will be a given function f. It is in this sense that a desirable definition of the integral or antiderivative as the inverse of differentiation seems called upon. Thus, a loose definition of integral could be the following:

> If $f(x)$ is a given function and if $F(x)$
>
> is a particular solution to $\frac{dF(x)}{dx} = f(x)$
>
> over a given interval $[a,b]$, then $F(x)$ is called the <u>integral</u> or <u>antiderivative</u> of $f(x)$.

The solution to the stated problem requires a more general form than the integral as just defined. For instance, one could add a constant C to $F(x)$ and still have a proper solution. Thus, it is necessary to define the indefinite integral in the following way:

> The most general integral of a function $f(x)$ is called the <u>indefinite integral</u> of $f(x)$ and it is denoted by the symbol
>
> $$\int f(x)dx$$
>
> so that $$\int f(x)dx = F(x) + C$$
>
> where $F(x)$ is the integral of $f(x)$ as previously defined and C is an arbitrary constant.

In other words, $\int f(x)dx$ is the most general solution of the differential equation

$$\frac{dF(x)}{dx} = f(x). \qquad \text{III-1}$$

Suppose the problem had been posed in a slightly more general way: Find a function F such that when it is differentiated n times (n, an integer), the result will be a given function f. In this case, F is the most general solution of the differential equation

$$\frac{d^n F(x)}{dx^n} = f(x). \qquad \text{III-2}$$

By applying the definition of indefinite integral n times, the desired solution of eq. III-2 is the n^{th} order integral of f(x) plus a power series in x with n arbitrary constants. This power series is sometimes called the complementary function. For an n^{th} order indefinite integral it is given by

$$\psi(x) = \sum_{i=1}^{n} \frac{C_i x^{n-i}}{(n-i)!} \; . \qquad \text{III-3}$$

Then suppose $D^{-n}f(x)$ denotes the n^{th} order integral of f(x); F can be written as

$$F(x) = D^{-n}f(x) + \sum_{i=1}^{n} \frac{C_i x^{n-i}}{(n-i)!} \; . \qquad \text{III-4}$$

Eq. III-4 is the most general solution of the differential equation given in eq.III-2.

Before making the connection between eq. III-4 and the generalized derivative, the concept of the definite integral must be introduced. Using the fundamental theorem of integral

Calculus, the definite integral below is given by

$$\int_{c_1}^{x} f(x)dx = F(x) - F(c_1) \qquad \text{III-5}$$

where F(x) is the integral of f(x). One would be tempted to conclude that this result was identical to that obtained by an indefinite integration of f(x). But this is not necessarily the case since $F(c_1)$ is not necessarily an arbitrary constant. To illustrate this point, consider the following problem:

> The function x is the derivative of some unknown function F(x), and the value F(0) = 1. Find F(x).

The solution is the indefinite integral given by eq. III-4; thus,

$$F(x) = \int x dx = \frac{x^2}{2} + C.$$

Applying the boundary condition it is clear that

$$F(x) = \frac{x^2}{2} + 1.$$

Now, attempt a solution using the definite integral. Write

$$F(x) \overset{?}{=} \int_{c_1}^{x} x dx = \frac{x^2}{2} - \frac{c_1}{2} .$$

It is immediately evident that no real value of c_1 exists that will cause the boundary condition to be satisfied. Hence, the definite integral lacks sufficient generality to be equivalent to an indefinite integral. It is equally wrong to consider the multiple definite integral below equivalent to a multiple indefinite integral of equal order. Thus,

$$\underbrace{\int\int\int\ldots\int}_{n\text{ times}} f(x)(dx)^n = \int_{c_n}^{x}\int_{c_{n-1}}^{x}\ldots\int_{c_2}^{x}\int_{c_1}^{x} f(x)(dx)^n. \qquad \text{III-6}$$

Putting this conclusion aside for the moment, consider the form of the L-R integral where the order v is an integer n and n > 0.

$$\underset{c}{D}^{-n}f(x) = \frac{1}{(n+1)!}\int_{c}^{x}\frac{f(u)du}{(x-u)^{-n+1}} \; . \qquad \text{III-7}$$

Inherent in Grunwald's method and especially in the derivation given in Appendix A is the fact that the L-R integral of eq. III-7 represents an n^{th} order definite integral like the r.h.s. of eq. III-6 with all constants c_1, c_2, . . ., c_n equal to the same constant c. Therefore, the L-R integral is not in general equivalent to an indefinite integral of equal order. To generalize $\underset{c}{D}^{-n}f(x)$ sufficiently so as to obtain a true multiple indefinite integral, the following approach could be taken. First, a choice of c could be made such that the L-R integral formula would produce only the integral of f(x). With the addition of an appropriate complementary function, $\underset{c}{D}^{-n}f(x)$ could be made an indefinite integral of order n, ie.,

$$\underbrace{\int\int\int\ldots\int}_{n\text{ times}} f(x)(dx)^n = \underset{c_n}{D}^{-n}f(x) + \sum_{i=1}^{n}\frac{C_i x^{n-i}}{(n-i)!} \qquad \text{III-8}$$

where c_n is the special choice of origin c described above. Eq. III-8 is now a perfectly general solution to

$$\frac{d^n F(x)}{dx^n} = f(x)$$

provided c_n exists.

Armed with this conclusion and the one previously established, it is now possible to interpret the meaning of the L-R integral formula and the complementary function for arbitrary order v. Recall Riemann's result of eq. II-27 where $v \geq 1$:

$$\frac{d^{-v} f(x)}{dx^{-v}} = \frac{1}{\Gamma(+v)} \int_c^x (x-t)^{+v-1} f(t)dt$$
$$+ \sum_{i=1}^{\infty} K_i \frac{x^{+v-i}}{\Gamma(-i+v+1)} .$$

Suppose one chooses the origin c such that the L-R integral (the first term on the r.h.s.), produces the integral of f(x). Then Riemann's complementary function (the second term on the r.h.s.) serves the purpose of generalizing the L-R integral so that eq. III-9 defines the indefinite integral of arbitrary order. Eq. III-9 is then the solution of the general order differential equation given below,

$$\frac{d^v F(x)}{dx^v} = f(x).$$

The fact that Riemann's complementary function involves an infinite number of arbitrary constants may seem odd at first glance. To gain reassurance in the reasonableness of his result, one could check to see if this generalized complementary function reduces to the complementary function for integer order given in eq. III-3. If v is an integer n, Riemann's form is

$$\psi(x) = \sum_{i=1}^{\infty} K_i \frac{x^{n-i}}{\Gamma(n-i+1)} . \qquad \text{III-10}$$

Note that for all values of the index i greater than or equal to n+1, the argument of the Gamma function is a negative integer or zero, respectively. From the theory of the Gamma function, it is found that

$$\frac{1}{\Gamma(n)} = 0 \qquad \text{for } n = 0, -1, -2, \ldots .$$

Using this fact, eq. III-10 reduces to

$$\psi(x) = \sum_{i=1}^{n} K_i \frac{x^{n-i}}{\Gamma(n-i+1)}$$

from which it immediately follows that

$$\psi(x) = \sum_{i=1}^{n} K_i \frac{x^{n-i}}{(n-i)!} . \qquad \text{III-11}$$

Eq. III-11 is exactly the form of the complementary function for integer order integrations of n^{th} order. This agreement lends credibility to Riemann's generalized complementary function.

As has been noted previously, when v is non-integer, Riemann's complementary function yields an infinite number of arbitrary constants. Any attempt to dispute this result from an intuitive point of view is destined to fail. Although one arbitrary constant is clearly needed for a single indefinite integration, how could one possibly decide the specific number for a one-half order indefinite integration? One's intuition fails miserably to solve this problem. Leibnitz's difficulty with fractional derivatives and the slope concept is somehow better appreciated at this point. Although Riemann's result seems plausible enough, his methods of derivation are still in question. It is hoped that mathematicians will take an interest in verifying his work.

In summary, the complementary function for the general order v represents the function which must be added to the integral of f(x) to form the general solution of the differential equation

$$\frac{d^v F(x)}{dx^v} = f(x). \qquad \text{III-12}$$

In all the previous discussion, v has been regarded as being positive so that the solution of eq. III-12 would be an indefinite integral of order v. If v is less than zero, the general complementary function given by Riemann agrees with expected results for integer values, ie., $\psi(x) \equiv 0$. For non-integer v less than zero, intuition fails again to justify the result.

Fortunately, the somewhat ambiguous quality of the complementary function need not offer much difficulty so long as the origin is chosen so that the L-R formula gives the integral of the function. In operational Calculus, an integral operator applied to a function is expected to give the integral or equivalently the indefinite integral with all arbitrary constants set to zero. The L-R formula seems to offer a way of producing a generalized integral operator provided the origin is chosen properly. If the resulting operator possesses the semi-group property, then derivatives of arbitrary order can be obtained by taking the appropriate integral first and then taking whole order derivatives of the result.

The next necessary step in this analysis is to describe methods one might use to determine the origin that will produce this generalized integral operator. There are two methods that have proved useful in providing a suitable value for the origin c. Before these are discussed, a statement of caution is necessary. A trap exists for the unwary in the use of the L-R formula given again below.

$$ {}_{c}D^{v}f(x) = \frac{1}{\Gamma(-v)} \int_{c}^{x} \frac{f(t)dt}{(x-t)^{v+1}} \; . \qquad \text{III-13} $$

Let G(z) be the integrand of eq. III-13. One might be tempted to choose a value of c that would make the integral of G(z) evaluated at c equal to zero. In that case, the integral of G(z) evaluated at x would give the desired generalized derivative of f. In general, such a procedure does not yield the correct result. The difficulty lies in the fact that G(z) is actually a function of z and x. To illustrate this point more clearly, consider the following example: Find the third integral (not indefinite integral) of the constant 1.

Applying the L-R formula,

$$ {}_{c}D^{-3}(1) = \frac{1}{\Gamma(3)} \int_{c}^{x} \frac{dt}{(x-t)^{-3+1}} . \qquad \text{III-14} $$

Solving the integral,

$$ {}_{c}D^{-3}(1) = \frac{1}{3!} [(x-t)^{3}]\Big|_{c}^{x} . $$

No constant c exists which makes the term in brackets zero. Further, evaluation at x produces zero regardless of c. The desired result is known to be $\frac{x^3}{3!}$. By inspection, this can be obtained only for c = 0 which is clearly not the value of origin which makes the integral of the integrand in eq. III-14 equal to zero!

A method which does give results in certain instances involves a principle introduced in the first historic period of generalized Calculus. Restated it is this: The non-integer ordered derivative or integral must produce the correct form for integer values of the order index. To use this method, the general index v in the L-R formula is assumed to be an integer n. The indicated integration is performed and the result is evaluated at both x and the origin c (as yet unspecified). The n^{th} order differentiation or integration of the function can generally be found by applying conventional techniques and the result compared to that already arrived at by means of the L-R formula. A value of c is then chosen that will cause both to be the same. The L-R integral is reevaluated with this fixed value of c and a general order index v. This method has two prominent disadvantages. In the original integration of the L-R formula for integer index n, it is not always possible to obtain a form that lends itself to easy determination of a value for c. In such cases, recognizing a value of c may well be impossible. The second disadvantage lies in the fact that functions sometimes require more than one value of origin as parameters of the function and the order index v are varied. The first method can give ambiguous results in such cases.

The second method appears to offer a more general approach and one that is decidedly easier to use. The method to be described had its conception in a result obtained by Krug. It seems desirable that the L-R integral define integral and differential operators that possess the semi-group property. This property is expecially important since it will be used to allow non-integer order derivatives to be defined in terms of non-integer order integrals. For example, suppose the three-halves order derivative of f is desired. If the semi-group property holds, a convenient way to get this result is to first compute the one-half order derivative. Then the three-halves derivative is given by

$$\underset{c}{D}^{2}\underset{c}{D}^{-\frac{1}{2}}f(x) = \underset{c}{D}^{2-\frac{1}{2}}f(x) = \underset{c}{D}^{\frac{3}{2}}f(x) \quad .$$

To generalize this technique, one merely uses the formula given in eq. III-15 below.

$$\underset{c}{D}^{\beta}f(x) = \frac{d}{dx}\,[\underset{c}{D}^{\beta-1}f(x)] \qquad 0<\beta<1 \quad . \qquad \text{III-15}$$

Higher order derivatives can be found by taking subsequent whole derivatives of eq. III-15.

In connection with eq. III-1, Krug derived a formula which showed that possession of the semi-group property hinged directly upon the choice of origin. He did not use the result to determine origins since his contour integration techniques avoided the question altogether. But for the real variable case and hence, the L-R formula, requiring the semi-group property to hold gives a convenient way to choose the origin. The so-called Krug condition is given in eq. III-16.

$$\underset{c}{D}^{n}\underset{c}{D}^{v}f(x) = \underset{c}{D}^{n+v}f(x) - \frac{1}{\Gamma(1-n-v)}\frac{f(c)}{(x-c)^{v+n}} - \frac{1}{\Gamma(2-v-n)}\frac{f^{(1)}(c)}{(x-c)^{v+n-1}} - \cdots - \frac{1}{\Gamma(-v)}\frac{f^{(n-1)}(c)}{(x-c)^{v+n}} \qquad \text{III-16}$$

where n is any positive integer and v is arbitrary. The choice of origin c is made so that all of the terms on the r.h.s. are zero except $\underset{c}{D}^{n+v} f(x)$. For that c, the semi-group property will hold. The examples to follow shortly will adequately illustrate the use of the Krug condition.

It seems appropriate at this juncture to summarize the conclusions of the preceeding analysis. The L-R integral formula is a satisfactory definition of the generalized derivative provided the origin for a particular function is properly chosen. Two methods are available to aid in choosing the correct origin. Use of the Krug condition of eq.III-16 offers the best approach of the two. A properly chosen origin has two major effects:

(1) For a negative order index, the L-R formula gives the integral of the function, ie., the complementary function of the indefinite integral is identically zero,

and (2) Within limits established for the particular function, the semi-group property will hold for the generalized derivative.

Once the origin is determined, the L-R formula, eq. III-13, can be used to calculate the generalized derivative. If an origin for a function cannot be found, it is still possible to find the generalized derivative. The techniques necessary to circumvent this difficulty will be illustrated shortly in regard to the function x^m.

Several generalized derivatives will now be calculated to aid in the understanding of the procedures just outlined. Attention will be directed to functions that have potential use in solving Laplace transform problems. Among those to be considered are: x^m ("m" constant); $x^m U(x)$ where $U(x)$ is the unit step function; e^{ax} ("a" constant); $e^{ax}U(x)$; sin ax and cos ax ("a" constant); also sinh ax and cosh ax ("a" constant).

The function x^m is chosen first because its solution requires the greatest diversity of techniques. In addition, through the use of power series expansion, the generalized derivative of a broad collection of additional functions can be found. Applying the Krug condition to x^m, the following terms must be identically zero for any positive integer n:

$$\frac{c^m}{(x-c)^v}, \frac{mc^{m-1}}{(x-c)^{v-1}}, \ldots, \frac{m(m-1)\ldots(m-n+2)c^{m-n+1}}{(x-c)^{v+1-n}}$$

where v is the order of the generalized derivative and c is the origin of the derivative. The choice of c is made so that each term is zero for all x within the domain of x^m. It can be shown that if the condition is satisfied for n=1, then the condition is satisfied for all remaining values of n. For $v \geq 0$, the following values of c are indicated:

$$c=0 \text{ for } \left\{{m-v\geq 0 \atop v\geq 0}\right\} \text{ since } \lim_{c\to 0} \frac{c^{m-v}}{(\frac{x}{c}-1)^{v}} = 0$$

and $$c=\pm\infty \text{ for } \left\{{m-v\leq 0 \atop v\geq 0}\right\} \text{ since } \lim_{c\to \pm\infty} \frac{c^{m-v}}{(\frac{x}{c}-1)^{v}} = 0 \,.$$

In the second instance, $c = -\infty$ is selected in order to preserve the sense of the L-R integral. Substitution in eq.III-13 produces

$$\frac{d^v(x^m)}{dx^v} = {}_0D^v(x^m) = \frac{1}{\Gamma(1-v)} \frac{d}{dx}\left(\int_0^x (x-t)^{-v} t^m dt \right)$$

and $$\frac{d^v(x^m)}{dx^v} = {}_{-\infty}D^v(x^m) = \frac{1}{\Gamma(1-v)} \frac{d}{dx}\left(\int_{-\infty}^x (x-t)^{-v} t^m dt \right) \,.$$

Completing the integration, the derivative of order v becomes

$$\frac{d^v(x^m)}{dx^v} = \frac{\Gamma(m+1)}{\Gamma(m-v+1)} x^{m-v} \quad \text{for } \left\{{v\geq 0 \atop m\geq v}\right\} \qquad \text{III-17}$$

and $$\frac{d^v(x^m)}{dx^v} = (-1)^v \frac{\Gamma(v-m)}{\Gamma(-m)} x^{m-v} \text{ for } \left\{{v\geq 0 \atop m<v}\right\} \,. \qquad \text{III-18}$$

To calculate the integral of arbitrary order, the same Krug condition must hold for $v < 0$. Since for $v < 0$ and $m < v$,

$$\lim_{c\to -\infty} \frac{c^{m-v}}{(\frac{x}{c}-1)^{v}} = 0$$

then an origin of $-\infty$ is proper. The situation where $v < 0$ and $m > v$ has an origin of $c = 0$ provided $m > 0$ because

$$\lim_{c \to 0} \frac{c^m}{(x-c)^v} = 0 \quad \text{for } \left\{ {m>0 \atop v<0} \right\} .$$

Thus, the generalized derivatives of x^m for $v < 0$ are as follows:

$$\frac{d^v(x^m)}{dx^v} = {}_{-\infty}D^v(x^m) = (-1)^v \frac{\Gamma(-m+v)}{\Gamma(-m)} x^{m-v} \text{ for } \left\{ {v<0 \atop m<v} \right\} \qquad \text{III-19}$$

and

$$\frac{d^v(x^m)}{dx^v} = {}_{0}D^v(x^m) = \frac{\Gamma(m+1)}{\Gamma(m-v+1)} x^{m-v} \text{ for } \left\{ {v<0 \atop m>0} \right\} . \qquad \text{III-20}$$

In the case where $v < 0$ and $v = m$, there is no value of c which causes the Krug condition to be satisfied. Interestingly enough, this is similar to the case which produces the odd situation in ordinary Calculus when one integrates x^{-1}. In that case, the result is $\ln|x|$. One might expect that for arbitrary index, the integral might produce a like result. The following derivation confirms that this is indeed the case and at the same time illustrates how to avoid the L-R formula when no proper origin exists. First, assume that the semi-group property holds. For $-1 < v < 0$, it is true that

$$\frac{d}{dx}\left(\frac{d^v(x^v)}{dx^v}\right) = \frac{d^{v+1}(x^v)}{dx^{v+1}} . \qquad \text{III-21}$$

Since v+1 > 0 and v+1 > v, eq. III-19 yields

$$\frac{d}{dx}\left(\frac{d^v(x^v)}{dx^v}\right) = (-1)^{v+1} \frac{x^{-1}}{\Gamma(-v)} \quad . \qquad \text{III-22}$$

By direct integration, it follows that

$$\frac{d^v(x^v)}{dx^v} = \frac{(-1)^{v+1}}{\Gamma(-v)} \operatorname{Ln}|x| \qquad \text{III-23}$$

for -1 < v < 0 only. A similar approach for the n^{th} interval, -n < v < -n+1, shows that eq. III-23 holds for all negative non-integer values of v. In this case, it is true that

$$\frac{d^n}{dx^n}\left(\frac{d^v(x^v)}{dx^v}\right) = \frac{d^{n+v}(x^v)}{dx^{n+v}} \quad . \qquad \text{III-24}$$

Since n+v > 0 and n+v > v, eq. III-19 can be used to obtain

$$\frac{d^n}{dx^n}\left(\frac{d^v(x^v)}{dx^v}\right) = (-1)^{v+n} \frac{\Gamma(n)}{\Gamma(-v)} x^{-n} \quad . \qquad \text{III-25}$$

Finding the n^{th} ordinary integral of eq. III-25 yields the final result:

$$\frac{d^v(x^v)}{dx^v} = \frac{(-1)^{v+1}}{\Gamma(-v)} \operatorname{Ln}|x| \quad . \qquad \text{III-26}$$

Ordinary integration also verifies that eq. III-26 is valid for v a negative integer. Therefore, eq. III-26 is correct for all $v < 0$.

The region where $v < 0$, $m < 0$, and $m > v$ has not yet been considered. Here again, the Krug condition fails to give a meaningful value of origin. To interpret this result, consider what this type of generalized derivative means. In the region defined, one is taking the $(-v)^{th}$ integral of x^m where $m > v$. Therefore, the semi-group property provides that

$$\frac{d^v(x^m)}{dx^v} = \frac{d^{v-m}}{dx^{v-m}}\left(\frac{d^m(x^m)}{dx^m}\right) \quad . \qquad \text{III-27}$$

The terms in brackets is given by eq. III-26 since $m < 0$; thus, eq. III-27 reduces to

$$\frac{d^v(x^m)}{dx^v} = \frac{d^{v-m}}{dx^{v-m}}\left(\frac{(-1)^{m+1}}{\Gamma(-m)} \operatorname{Ln}|x| \right) \quad . \qquad \text{III-28}$$

Since $v-m < 0$, it is clear that for this region one needs to take the $-(v-m)^{th}$ integral of $\ln|x|$. The integral of $\ln|x|$ for arbitrary order does not appear soluble with any of the methods described in this paper. Fortunately, leaving the generalized derivative of x^m undefined in this region is of no great consequence since applications to Laplace transform theory do not usually involve such conditions.

A vastly more important region is the case for $m = 0$. This is the generalized derivative of the constant 1. Since the L-R formula involves an ordinary integral, the generalized derivative of a constant C is simply C times the generalized derivative of 1. For $v > 0$, it is clear from eq. III-17 that

$$\frac{d^v(1)}{dx^v} = {}_{-\infty}D^v 1 = (-1)^v \frac{\Gamma(v)}{\Gamma(0)} x^{-v} = 0$$

or

$$\frac{d^v(C)}{dx^v} = 0 \quad \text{for } v > 0 .$$

In looking through the literature, one is amazed to find the statement that the v^{th} derivative of a constant is some function of x. If such were true, then derivatives of non-integer order greater than one would be a function of x while the first derivative would be zero. This clearly violates the semi-group property and seems quite unacceptable. This inconsistency arises from the tacit assumptions of some authors that the origin for all functions is zero. Shortly it will be shown that such an assumption is valid in some cases when the function is multiplied by the unit step, U(x). The non-zero derivative of a constant turns out to be the derivative of CU(x) instead.

For $m = 0$ and $v < 0$, the Krug condition does not provide an origin. The solution is obtained by

(1) assuming the semi-group property holds,

(2) finding the $(1-v)^{th}$ derivative of the first integral of 1 for $0 > v > -1$,

and (3) recognizing the result as the $(-v)$th integral of 1.

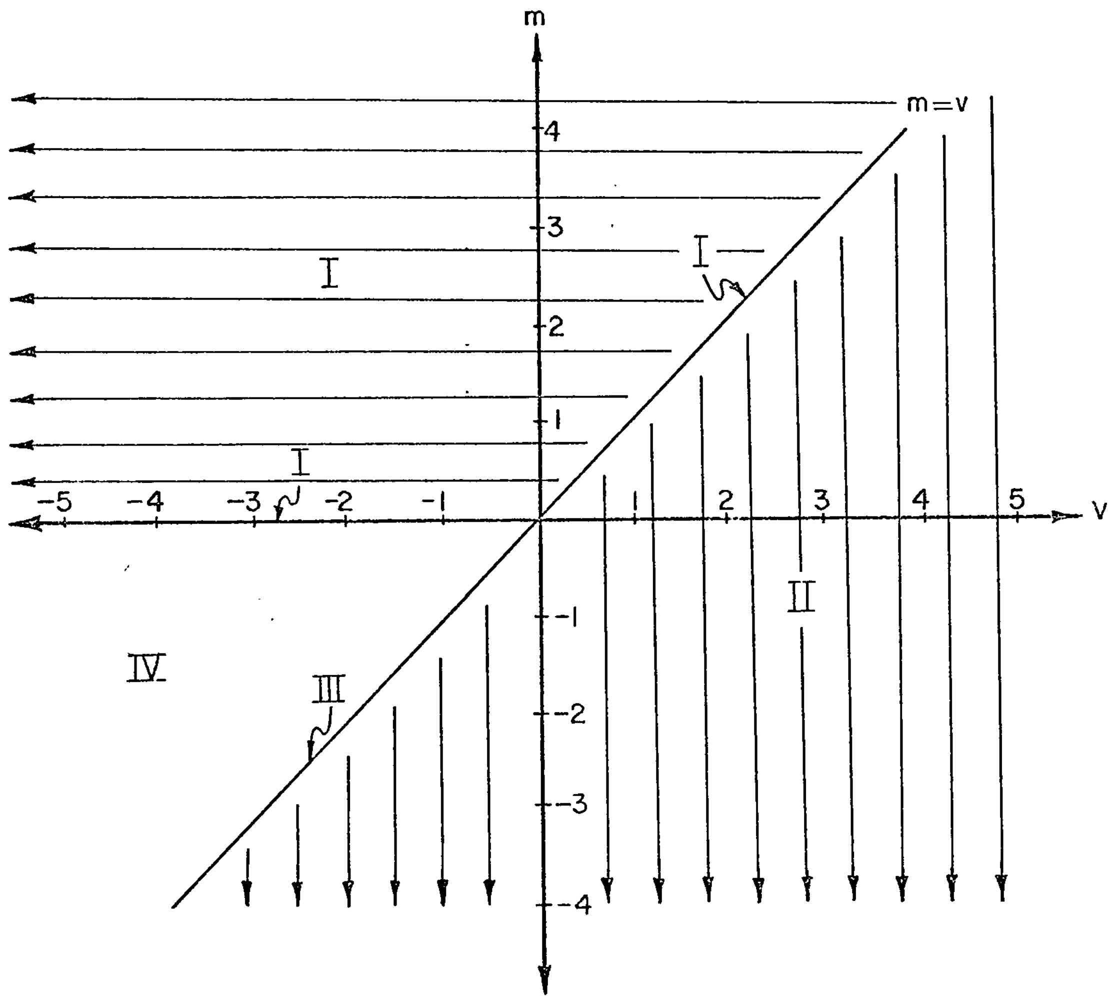
Thus,

$$\frac{d^v(1)}{dx^v} = \frac{x^{-v}}{\Gamma(1-v)} \quad \text{for} \quad -1 < v < 0 \ . \qquad \text{III-29}$$

Higher order integrals can be obtained by taking a sufficient number of whole order integrals of eq. III-29.

This completes the analysis of the generalized derivative of x^m. To aid in getting an overall view of the results, a special graphical representation is presented in Figure 1. The vertical axis corresponds to the exponent m while the horizontal axis represents the derivative order parameter v. The various regions are coded to indicate which formula is to be used to calculate the generalized derivative of x^m.

For applications to problems in Laplace transform theory, it is necessary to calculate the generalized derivative of $x^mU(x)$. The special technique used in Chapter IV to solve such problems requires only that the integral of arbitrary order be known. Thus, the region where $v < 0$ is the only one of interest. Since $U(x)$ is zero for all $x < 0$, the regions of origin $c = 0$ in figure 1 for which $v < 0$ are unaffected and the value of generalized integral for $x^mU(x)$ is the same as that for x^m alone. The region with origin $c = -\infty$ is affected,



Formula No.

I. $$\frac{d^v(x^m)}{dx^v} = \frac{\Gamma(m+1)}{\Gamma(m-v+1)} x^{m-v}$$

II. $$\frac{d^v(x^m)}{dx^v} = \frac{\Gamma(v-m)}{\Gamma(-m)} x^{m-v}$$

III. $$\frac{d^v(x^m)}{dx^v} = \frac{(-1)^{v+1}}{\Gamma(-v)} \operatorname{Ln} x$$

IV. $$\frac{d^v(x^m)}{dx^v} \text{ unspecified}$$

Figure 1

Generalized Derivatives of x^m

however. In this case, the L-R integral is broken into two separate integrals as in the equation below:

$$ {}_{-\infty}D^{v}(x^{m}U(x)) = \frac{1}{\Gamma(-v)}\int_{-\infty}^{0}\frac{t^{m}U(t)dt}{(x-t)^{v+1}} + \frac{1}{\Gamma(-v)}\int_{0}^{x}\frac{t^{m}U(t)dt}{(x-t)^{v+1}} \quad . $$

Because $U(t) = 0$ for $t < 0$, the first integral on the r.h.s. is zero. With $U(t) = 1$ for $t \geq 0$, the second integral is simply ${}_{0}D^{v}(x^{m})$. In the region $v < 0$ and $m < v$, the generalized integral is given by eq. III-19. The generalized integral along the line $m = v$ for $v < 0$ and throughout the wedge-shaped region where $v < m < 0$ for $v < 0$ need not be considered since no use will be made of these regions in subsequent applications. Finally, it is easily shown that the generalized integral of $U(x)$ is given by

$$ \frac{d^{v}(U(x))}{dx^{v}} = \frac{1}{\Gamma(1-v)}\, x^{-v} $$

for all $v < 0$. Summarizing all of the above for $x^{m}U(x)$, it is true that

$$ \frac{d^{v}(x^{m}U(x))}{dx^{v}} = \frac{\Gamma(m+1)}{\Gamma(m-v+1)}\, x^{m-v} \qquad \text{III-30} $$

so long as $v \leq m < 0$.

The next function to be considered is the exponential e^{ax} where a is a constant. The Krug condition is found to be satisfied by origins $c = \pm\infty$ since

$$\lim_{c\to\pm\infty} \frac{e^{ac}}{(x-c)^{v}} = 0$$

for $a < 0$ and $a > 0$ respectively. The v derivative of e^{ax} is therefore

$$\frac{d^{v}(e^{ax})}{dx^{v}} = {}_{-\infty}D^{v}(e^{ax}) = \frac{1}{\Gamma(-v)} \int_{-\infty}^{x} \frac{e^{at}dt}{(x-t)^{v+1}}$$

which, if solved, yields

$$\frac{d^{v}(e^{ax})}{dx^{v}} = a^{v}e^{ax} \quad . \qquad \text{III-31}$$

By using the concept of left hand derivative introduced briefly in Chapter II, the generalized derivative for $a < 0$ is found by taking the origin $+\infty$ and solving

$$\frac{d^{v}(e^{ax})}{dx^{v}} = \frac{1}{\Gamma(-v)} \int_{x}^{+\infty} \frac{e^{at}dt}{(x-t)^{v+1}} \qquad \text{III-32}$$

from which it is found that

$$\frac{d^{v}(e^{ax})}{dx^{v}} = a^{v}e^{ax} \quad .$$

Thus for all $a \neq 0$, eq. III-31 is the generalized derivative of e^{ax}.

For applications in Laplace transform problems, the integral of general order for $e^{ax}U(x)$ is found. This is accomplished most easily by writing $e^{ax}U(x)$ as a power series. This series is known to be uniformly convergent and can be integrated term by term. Since the L-R formula is an ordinary integral, term by term integration of arbitrary order is readily justified. Thus, write

$$e^{ax}U(x) = [\ 1 + ax + \frac{(ax)^2}{2!} + \ldots + \frac{(ax)^n}{n!} + \ldots]U(x) \ .$$

For $v < 0$, term by term general differentiation produces,

$$\frac{d^v(e^{ax}U(x))}{dx^v} = \frac{x^{-v}}{\Gamma(1-v)} M[1,1-v,ax]\ U(x) \qquad \text{III-33}$$

where M is the confluent hypergeometric function called Kummer's function. M is given by the equation below:

$$M[a,b,z] = 1 + \frac{(a)_1 z}{(b)_1} + \frac{(a)_2 z^2}{(b)_2} + \ldots + \frac{(a)_n z^n}{(b)_n} + \ldots$$

where $(a)_n = a(a+1)(a+2)\ .\ .\ .\ (a+n-1)$ and $(a)_o = 1$. The Kummer function reduces to any one of a number of functions depending on its arguments. A list of thirty-nine such reductions is given in the references.[22]

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[22]Milton Abramowitz and Irene Stegun (ed.), Handbook of Mathematical Functions (New York: Dover Publications, Inc., 1965), p. 509.

Knowing the form of the generalized derivative of e^{ax} now permits the derivation of several additional forms. For instance, the expression for sin(ax) in terms of exponentials is

$$\sin(ax) = \frac{e^{+iax} - e^{-iax}}{2i} \quad . \qquad \text{III-34}$$

Taking the generized derivative of eq. III-34 gives

$$\frac{d^v[\sin(ax)]}{dx^v} = \frac{(ia)^v e^{+iax} - (-ia)^v e^{-iax}}{2i} \quad . \qquad \text{III-35}$$

Recognizing the fact that $(ia)^v = a^v e^{iv(\pi/2)}$ and that $(-ia)^v e^{-iv(\pi/2)}$ (using the principal roots only), it is possible to write eq. III-35 as

$$\frac{d^v[\sin(ax)]}{dx^v} = a^v \sin(ax+v\pi/2) \quad . \qquad \text{III-36}$$

A similar technique produces the following result for cos(ax):

$$\frac{d^v[\cos(ax)]}{dx^v} = a^v \cos(ax+v\pi/2) \quad . \qquad \text{III-37}$$

As a sidelight, notice that the generalized derivative of e^{ax} and cos(ax) agree with the assumptions made by Laplace and Fourier respectively. Their methods of defining generalized derivatives are quite valid and worthy of further investigations.

The generalized derivatives of cosh(ax) and sinh(ax) are easily found by noting the relationship between the hyperbolic and circular functions, ie.,

$$\sinh(ax) = -i \sin(iax)$$

$$\cosh(ax) = \cos(iax) .$$

Hence,

$$\frac{d^v(\cosh(ax))}{dx^v} = (ia)^v \cos(iax+v\pi/2)$$

and

$$\frac{d^v(\sinh(ax))}{dx^v} = -(i)^{v+1} a^v \sin(iax+v\pi/2) \quad .$$

This completes the derivations of generalized derivatives.

CHAPTER IV

LAPLACE TRANSFORM TECHNIQUES

Now that the basic ideas of generalized differentiation have been discussed, it is possible to consider applications to Laplace transform theory. The valuable use of the properties of Table 1 in determining transforms and inverse transforms was alluded to in Chapter I by use of a simple example. The purpose of this chapter is to broaden the use of these properties to encompass the concepts of generalized differentiation. In addition, examples will be used to illustrate the techniques developed.

From Table 1, properties two through five will be of primary concern. Consider first the frequency integration and frequency differentiation properties. It will be shown that under the concepts of generalized differentiation these two properties can be written as one. Recall that the definition of Laplace transform is

$$L\{f(t)\} = F(s) = \int_0^\infty f(t)e^{-st}dt .$$

The generalized derivative of order v is then given by

$$\frac{d^v F(s)}{ds^v} = \frac{d^v}{ds^v}\left[\int_0^\infty f(t)e^{-st}dt \right] .$$

It is now necessary to interchange the order of general differentiation and ordinary integration. Theorems developed in the literature are not sufficiently general to justify this step.[23]

[23]N.W. McLachlan, Laplace Transforms (New York: Dover Publications, Inc., 1954), p. 172.

When dealing with the Laplace transform, it is often easier to make the interchange and later check the results by performing the inverse transformation. This method will be considered acceptable here. Therefore, assuming the order of general differentiation and ordinary integration can be changed, gives

$$\frac{d^vF(s)}{ds^v} = \int_0^\infty f(t)\frac{d^v e^{-st}}{ds^v}\, dt \ .$$

Since $t > 0$, a left-hand derivative of e^{-st} with origin $+\infty$ is required. This result has already been given in eq. III-32. Thus,

$$\frac{d^vF(s)}{ds^v} = \int_0^\infty f(t)(-t)^v e^{-st}dt \qquad \text{IV-1}$$

for all real values of order parameter v. Properly interpreting eq. IV-1, the general frequency differentiation property is deduced as given below:

$$\text{If } f(t) \longleftrightarrow F(s)$$

$$\text{Then } (-t)^v f(t) \longleftrightarrow \frac{d^vF(s)}{ds^v} \ . \qquad \text{IV-2}$$

It is clear from eq. IV-2 that multiplying in the time domain by $(-t)^v$ where v is any real number corresponds to taking the generalized derivative of v^{th} order.

Next, one might want to consider the effect of multiplying in the frequency domain by s^v where v is any real number. In this case, a generalized form of the time differentiation and time integration properties of Table 1 appears necessary. While generalizing the time integration property is straightforward, the generalization of the time differentiation property reveals certain mathematical difficulties which are yet unsolved. Fortunately, an alternate approach is available that allows one to avoid these problems. Before discussing this approach, it is necessary to develop the generalized time integration property.

As given in Table 1, Laplace transforms satisfy the time convolution property, ie.,

$$\text{If } f_1(t) \longleftrightarrow F_1(s) \text{ and } f_2(t) \longleftrightarrow F_2(s)$$

$$\text{Then } \int_0^t f_1(u) f_2(t-u)du \longleftrightarrow F_1(s)\cdot F_2(s) \; .$$

Suppose $F_2(s) = \frac{1}{s^v}$ and $F_1(s)$ is the transform of some function $f_1(t)$ whose form is subject to certain limitations shortly to be imposed. It is known that

$$\frac{t^{-v-1}}{\Gamma(-v)} \longleftrightarrow \frac{1}{s^v}$$

where $v < 0$. Applying the time convolution property yields

$$\frac{1}{\Gamma(-v)} \int_0^t \frac{f_1(u)du}{(t-u)^{v+1}} \qquad \text{IV-3}$$

where v is real and less than zero. For functions with a right-hand generalized derivative of origin zero, the l.h.s. of relation IV-3 is simply the L-R definition of the general order integral of $f_1(t)$. At this point, it is an easy matter to deduce the generalized integration property:

$$\text{If } f(t) \longleftrightarrow F(s)$$

$$\text{Then } \frac{d^v f(t)}{dt^v} \longleftrightarrow \frac{F(s)}{s^{-v}}$$

provided (1) $v \in$ Reals and $v < 0$ and (2) $f(t)$ is a function whose integral of real order is right-handed with origin zero. The added restriction on $f(t)$ is not particularly severe. The generalized derivatives of t^m and e^{at} meet this condition when multiplied by $U(t)$ as is always done in time domain analysis using Laplace transforms. All functions that can be written as uniformly convergent series in either $t^m U(t)$ or $e^{at}U(t)$ would also qualify under these restrictions.

The generalized time integration property allows the division in the frequency domain by positive real powers of s when the appropriate generalized integral is taken in the time domain. Use will now be made of this fact to perform multiplication in the frequency domain by positive real powers of s. In this case, the following procedure could be used: Suppose it is desired to multiply by s^{k+v} in the frequency domain. Here, k is a positive integer or zero, and v is any real number between zero and one. Two steps are necessary. First, the derivative of order (k+1) is taken in the time domain. By the ordinary time differentiation property, the transform in the frequency domain is multiplied by s^{k+1}. This step may introduce additional terms involving the initial value of the time function and its first k derivatives. These terms can be removed by selecting an appropriate, new time function. See example 3. In the second step, the generalized time integration property is used to divide in the frequency domain by s^{1-v}. Multiplication by s^{k+v} in the frequency domain has been accomplished.

With the generalized properties and techniques just described, it is possible to multiply or divide in the time or frequency domain by any real power of the respective domain variable. To illustrate how each of these results can be used to calculate either the transform or inverse transform of a particular function, several examples will be given. For all the examples, the only transform pair that will be assumed is

$$U(t) \longleftrightarrow \frac{1}{s} .$$

The properties of Table 1 and the techniques just described will be used to obtain the desired results.

Example 1: Find the inverse transform of $\frac{1}{s^v}$ where v is real and positive.

It is true that

$$U(t) \longleftrightarrow \frac{1}{s} . \qquad \text{IV-4}$$

Using the generalized time integration property,

$$\frac{d^{-v+1}U(t)}{dt^{-v+1}} \longleftrightarrow \frac{1}{s^{v-1}} \cdot \frac{1}{s}$$

where $v \geq 1$. (The case for $0 < v < 1$ will be dealt with separately). The $(v-1)^{th}$ general integral of U(t) was derived in Chapter III and is given by eq. III-30. Using this result,

$$\frac{t^{v-1}U(t)}{\Gamma(v)} \longleftrightarrow \frac{1}{s^v} \qquad \text{IV-5}$$

for $v \geq 1$. In the case that $0 < v < 1$, take the first derivative of U(t) and apply the ordinary time differentiation property. This gives

$$\frac{dU(t)}{dt} \longleftrightarrow 1.$$

$\frac{dU(t)}{dt}$ is the unit impulse function, $\delta(t)$. Where $0 < u < 1$, use the generalized time integration property to write

$$\frac{d^{u-1}\delta(t)}{dt^{u-1}} \longleftrightarrow 1 \cdot \frac{1}{s^{1-u}} \quad .$$

Since $u-1 < 0$, it is necessary to find the general order integral of $\delta(t)$. It can be shown quite simply that an origin of $-\infty$ used in the L-R formula yields the following result:

$$\frac{d^{u-1}\delta(t)}{dt^{u-1}} = \frac{1}{\Gamma(1-u)} \cdot \frac{U(t)}{t^u} \longleftrightarrow \frac{1}{s^{1-u}} \quad .$$

Letting $v = 1-u$ where v now is valid for $0 < v < 1$,

$$\frac{t^{v-1}U(t)}{\Gamma(v)} \longleftrightarrow \frac{1}{s^v} \quad . \qquad \text{IV-6}$$

Eq. IV-6 is identical to eq. IV-5; therefore, the solution for all $v > 0$ is given by the l.h.s. of either IV-5 or IV-6.

Example 2: Find the Laplace transform of

$$\frac{1}{\sqrt{\pi t}} e^{-at}(1-2at)U(t) \quad .$$

Using eq. IV-4 and the ordinary time differentiation property, one obtains

$$U(t) \longleftrightarrow \frac{1}{s}$$

and

$$tU(t) \longleftrightarrow \frac{1}{s^2} \quad .$$

Applying the generalized frequency differentiation property to multiply in the time domain by $\frac{1}{\sqrt{-t}}$, the relations above become

$$\frac{1}{\sqrt{-t}}\, U(t) \longleftrightarrow \frac{d^{-1/2}(s^{-1})}{ds^{-1/2}} = (-1)^{1/2}\, \frac{\Gamma(1/2)}{\Gamma(1)}\, s^{-1/2}$$

and

$$\frac{t}{\sqrt{-t}}\, U(t) \longleftrightarrow \frac{d^{-1/2}(s^{-2})}{ds^{-1/2}} = (-1)^{1/2}\, \frac{\Gamma(3/2)}{\Gamma(2)}\, s^{-3/2} \quad .$$

After reduction of the Gamma functions, the frequency shift property of Table 1 can be used to write

$$\frac{1}{\sqrt{-t}}\, e^{-at}U(t) \longleftrightarrow (-1)^{1/2}\sqrt{\pi}\; s^{-1/2}$$

and

$$\frac{t}{\sqrt{-t}}\, e^{-at}U(t) \longleftrightarrow (-1)^{1/2}(1/2)\sqrt{\pi}\; s^{-3/2} \quad .$$

Finally, the linearity property (entry 1 in Table 1) is applied to produce the solution:

$$\frac{1}{\sqrt{\pi t}}\, e^{-at}(1-2at)U(t) \longleftrightarrow \frac{s}{(s+a)^{3/2}} \quad .$$

Example 3: Find the inverse transform of

$$\frac{\sqrt{s}}{s-a^2}\ .$$

From the frequency translation property and eq. IV-4, it is found that

$$e^{a^2t}U(t) \longleftrightarrow \frac{1}{s-a^2}\ .$$

All that is needed is to multiply in the frequency domain by $\sqrt{s}$. To arrive at this, first take a full derivative and apply the ordinary time derivative property. Thus,

$$a^2e^{a^2t}U(t) \longleftrightarrow \frac{s}{s-a^2} - 1$$

or

$$a^2e^{a^2t}U(t) + \delta(t) \longleftrightarrow \frac{s}{s-a^2}\ .$$

A one-half order integration in the time domain will divide in the frequency domain by $\sqrt{s}$. The one-half order integral of $\delta(t)$ has already been computed. To find the one-half order integral of $e^{a^2t}U(t)$, use is made of the more general result, eq. III-33. Thus,

$$\frac{d^{-1/2}[e^{a^2t}U(t)]}{dt^{-1/2}} = \frac{t^{1/2}}{\Gamma(3/2)} M[1,3/2,a^2t]U(t)\ . \qquad \text{IV-7}$$

Using the relation $M[a,b,z] = e^z M[b-a,b,-z]$ and consulting Appendix B, it is found that

$$\frac{d^{-1/2}[e^{a^2t}U(t)]}{dt^{-1/2}} = \frac{e^{a^2t}}{a} \operatorname{erf}(a\sqrt{t})U(t) \quad . \qquad \text{IV-8}$$

It is a simple matter to write down the answer:

$$\frac{1}{\sqrt{\pi t}} + ae^{a^2t}\operatorname{erf}(a\sqrt{t})U(t) \longleftrightarrow \frac{\sqrt{s}}{s-a^2} \quad .$$

Example 4: In this final example, a practical problem will be presented to illustrate how non-integer powers of s and t are introduced.

Given a semi-infinite transmission line driven by a constant voltage source E_o through a source resistance R_o. Assume that the initial current throughout the line is zero and that the charge in the line is also initially zero. Also assume that for all time and all positions on the line, the voltage remains finite. The transmission line parameters are as follows:

$$R = \text{constant}$$
$$C = \text{constant}$$
$$L = 0$$
$$G = 0$$

Find the current flowing in the source resistance, R_o.

Beginning with the partial differential equations describing this particular transmission line, one can write

$$C \frac{\partial V(x,t)}{\partial t} = - \frac{\partial I(x,t)}{\partial x}$$

IV-9

$$R\, I(x,t) = - \frac{\partial V(x,t)}{\partial x}$$

where V(x,t) is the voltage along the line and I(x,t) is the current flowing in the line. Taking the Laplace transform with respect to t produces

$$C\, V(x,0) + Cs\overline{V}(x) = - \frac{d\overline{I}(x)}{dx}$$

IV-10

$$R\overline{I}(x) = - \frac{d\overline{V}(x)}{dx} .$$

Since the charge on the line is zero at t = 0, so also must be the voltage. Therefore, V(x,0) = 0. Combining eqs. IV-10 a and b, the second order equation below is obtained.

$$\frac{d^2\overline{V}(x)}{dx^2} - RCs\overline{V}(x) = 0 \quad . \qquad \text{IV-11}$$

The solution to IV-11 is given as

$$\overline{V}(x) = A(s)e^{-\sqrt{RCs}x} + B(s)e^{+\sqrt{RCs}x} . \qquad \text{IV-12}$$

Since V(x,t) is finite for large x, B(s) = 0. Note that the following equation must hold by simple circuit theory:

$$R_o \bar{I}(0) = \frac{E_o}{s} - \bar{V}(0) \quad . \qquad \text{IV-13}$$

By eqs. IV-10a and IV-12, it is possible to show that

$$\bar{I}(0) = \frac{A(s)}{R} \sqrt{RCs}$$

and

$$\bar{V}(0) = A(s) \quad .$$

Substitution of these relations in eq. IV-13 produces a value of A(s); thus,

$$A(s) = E_o \frac{1}{s[1 + (R_o/R)\sqrt{RCs}]} \quad .$$

Using eq. IV-13 and letting $K = \frac{R}{R_o{}^2C}$ it is possible to write I(0) as

$$\bar{I}(0) = \frac{E_o}{R_o}\left(\frac{1}{s-k} - \frac{\sqrt{k}}{\sqrt{s}(s-k)}\right) \quad . \qquad \text{IV-14}$$

Which is the Laplace transform of the required result. Inversion of the first term of eq. IV-14 is simply made by applying the frequency shift property to eq. IV-4. The inverse of the second term is $\sqrt{k}$ times the one-half order integral of the inverse of the first term. Thus,

$$e^{kt}U(t) \longleftrightarrow \frac{1}{s-k}$$

and IV-15

$$\sqrt{k}\,\frac{d^{-1/2}(e^{kt}U(t))}{dt^{-1/2}} \longleftrightarrow \frac{\sqrt{k}}{\sqrt{s}(s-k)} \quad .$$

Using eq. IV-8, the final result is

$$I(0,t) = \frac{E_o}{R_o}\, e^{kt}[1 - \mathrm{erf}(\sqrt{kt})] \quad .$$

This completes the presentation of illustrative examples.

CHAPTER V

CONCLUSION

The previous chapters of this thesis have dealt ultimately with the development of a technique to calculate Laplace transforms and inverses involving non-integer powers of s and t. The eventual necessity to use generalized Calculus to achieve this end required the presentation of an introduction to the subject. Chapter II provided a historical treatment of generalized Calculus in which a number of basic approaches were elaborated upon. Chapter III proceeded to interpret certain of the basic ideas presented in Chapter II and finally established the form of generalized derivatives for certain elementary functions. In Chapter IV, the basic principles of generalized Calculus were used to broaden certain of the properties of the Laplace transfrom. Finally, several examples were presented to illustrate the use of these properties in calculating Laplace transforms and inverses.

Although this thesis has emphasized the application of generalized Calculus in working Laplace transform problems, it should be noted that the potential use of similar techniques with other operational transforms is yet to be fully researched. Fourier and Hankel transforms offer definite possibilities in this regard. Uses of generalized Calculus appear not to be restricted to operational mathematics. Various branches of science stand to benefit if additional work in the area is forthcoming. The use of fractional order differential equations to describe physical phenomena has already been alluded to in a previous chapter.

An important consideration for the employment of generalized Calculus in any further practical usage depends heavily on the firmness with which the tenets of the subject are developed. The presentation of Chapters II and III of this thesis served only to illuminate the basic ideas and was not meant to be rigorously complete in itself. The inconclusive nature of that treatment is most readily apparent in the limited application of the L-R definition of the generalized derivative. The heavy dependence on selection of a proper origin makes the L-R definition somewhat restricted. Selecting an origin by applying the Krug condition worked reasonably well for the few elementary functions treated in Chapter III, but application to more complex functions seems at best uncertain. Of course, if one limits himself to the class of functions expandable in a power series, then the results of Chapter III can adequately define the generalized derivative. This limitation should not be needed. It is not altogether implausible that an entirely different definition than the L-R integral can be found that will avoid the difficulties expressed. This possibility remains to be explored.

Whether this improved state of generalized Calculus will ever be realized is somewhat doubtful based on the past performance of mathematicians. In addition to the inherent difficulties, there seems to be a perpetual lack of interest in developing the subject. The renowned Scottish mathematician, Professor Kelland had a taste of this extreme disinterest. Kelland presented an introductory paper in a Royal Society of Edinburgh journal on the generalized derivative of x^n.[24] He seemed certain that his colleagues in Great Britain would find the new subject engrossing and that considerable interest would be generated by his article. Several years passed after publication and virtually nothing was forthcoming from contributors to the journal. Kelland finally wrote an additional article in which he upbraided his colleagues for not pursuing the subject further. The total lack of interest was very disappointing to him. The situation Kelland contended with has not much improved today. Until it is, the great practical utility of generalized Calculus may never be fully realized.

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[24] P. Kelland, "On General Differentiation," Transactions of the Royal Society of Edinburgh, Vol. 16 (1840), pp. 567-618.

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APPENDIX A

In elementary Calculus, the appeal is often made to the geometric intuition of the student. While a degree of intellectual satisfaction is often achieved, it is generally at the expense of mathematical rigor. The usual demonstration given for the definite integral is an example of this type of argument. In this case, the area under the graphed curve of a function is used to define the definite integral. In the following discussion, the same technique is utilized to produce the definite integral of higher order from which the L-R formula follows as a simple generalization.

Consider Figure 2a. Assume the function f is continuous and bounded on the interval [a,b]. Divide this interval into n subintervals each with width Δx. In each subinterval, choose a point E_i, $i = 1, 2, \ldots, n$. Define

$$\alpha_i = E_i - i \cdot \Delta x \ . \qquad \text{A-1}$$

for $i = 1, 2, \ldots, n$. Note also that

$$\Delta x = \frac{b-a}{n} \ . \qquad \text{A-2}$$

Then form rectangles of width Δx and height equal to the value of the function at E_i. The area under the curve for the i^{th} interval is approximately $f(E_i)\Delta x$. The area under the curve from a to b can be approximated by

$$A_{total} \approx \sum_{i=1}^{n} f(E_i)\Delta x \ . \qquad \text{A-3}$$

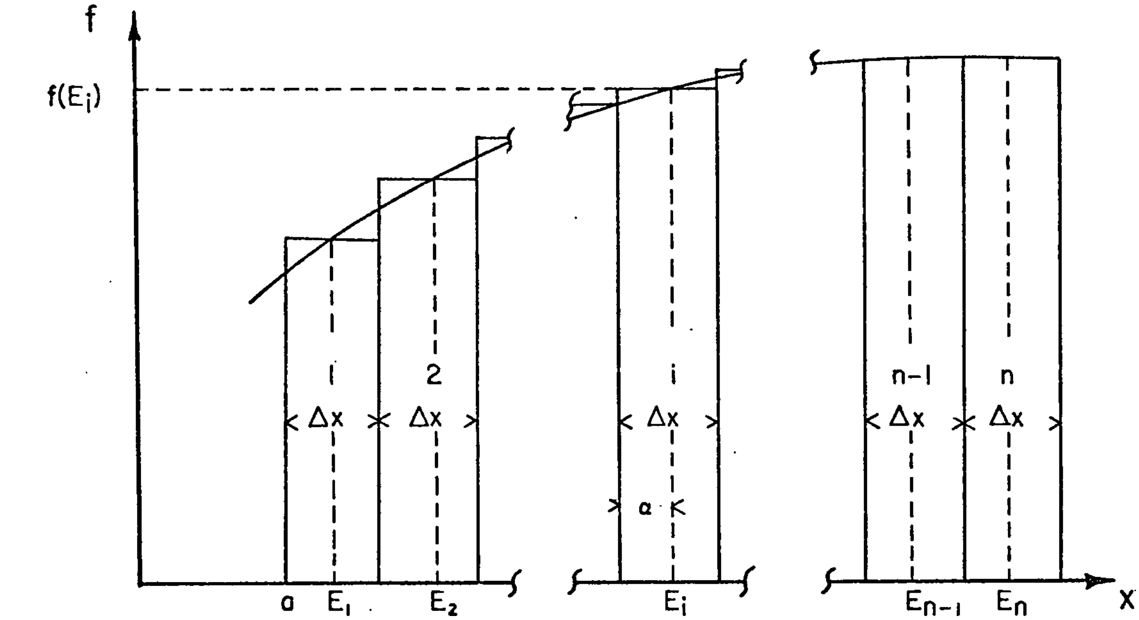

Figure 2a

Approximation of First-Order Integral by Area

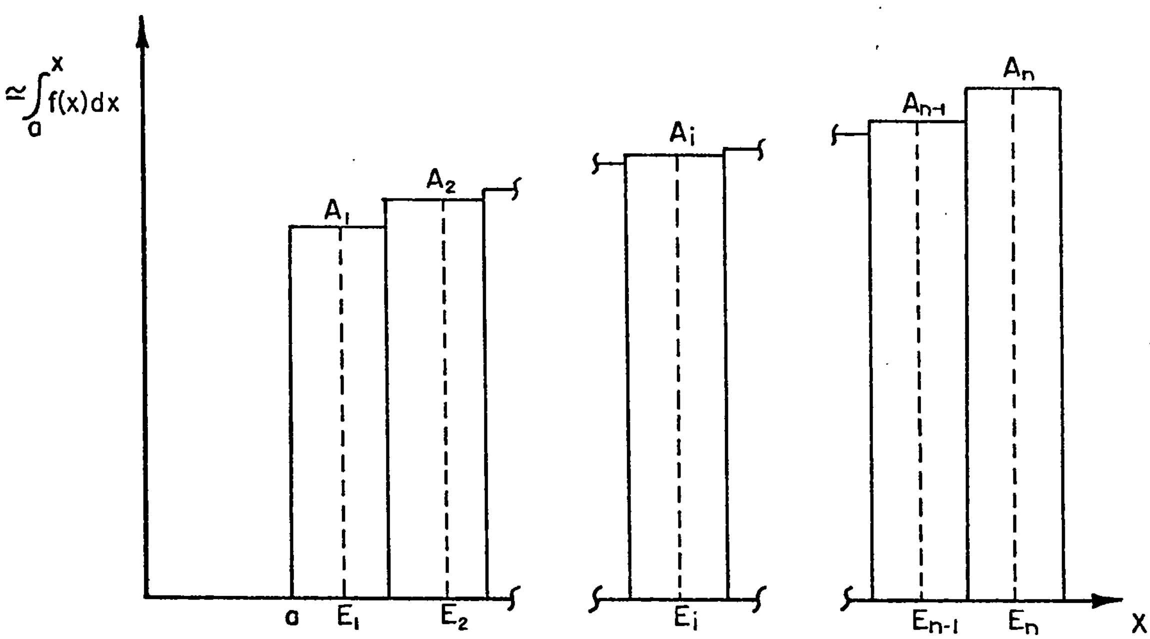

Figure 2b

Approximation of Second-Order Integral by Area

The definite integral is then defined by a limiting process as

$$\int_a^b f(x)dx = \lim_{\substack{n\to\infty \\ \Delta x\to 0}} \sum_{i=1}^{n} f(E_i)\Delta x \quad . \qquad \text{A-4}$$

Next consider the procedure for obtaining the definite integral of second order over the same interval. In this case, a curve could be plotted representing the approximate area under the curve of f from a to each point x_i on [a,b] for i = 1,2,....n. The second definite integral is then obtained by finding the area under this new curve as $n \to \infty$ and $\Delta x \to 0$. Consider Figure 2a again. The area under the graph of f for a particular x_i can be approximated by using the summation of areas for the appropriate rectangles. Thus,

$$A_1 = \int_a^{x_1} f(x)dx \approx f(E_i)\Delta x$$

$$A_2 = \int_a^{x_2} f(x)dx \approx f(E_i)\Delta x + f(E_2)\Delta x$$

$$\vdots$$

$$A_{n-1} = \int_a^{x_{n-1}} f(x)dx \approx f(E_i)\Delta x + f(E_2)\Delta x + \ldots + f(E_{n-1})\Delta x$$

$$A_n = \int_a^b f(x)dx \approx f(E_1)\Delta x + f(E_2)\Delta x + \ldots + f(E_n)\Delta x \; .$$

These discrete values are plotted in Figure 2b. If rectangles are formed for 2b as they were for 2a, the area under this new curve is given approximately by

$$\int_a^b\!\!\int_a^b f(x)(dx)^2 \approx \sum_{i=1}^{n} A_i \cdot \Delta x \quad . \qquad \text{A-5}$$

Using the fact that

$$\begin{aligned}
\Delta x &= b - (E_n - \alpha_n) \\
2\Delta x &= b - (E_{n-1} - \alpha_{n-1}) \\
&\;\;\vdots \\
(n-1)\Delta x &= b - (E_2 + \alpha_2) \\
n\Delta x &= b - (E_1 + \alpha_1)
\end{aligned} \qquad \text{A-6}$$

it is possible to reduce the second definite integral approximation of eq. A-5 to

$$\int_a^b\!\!\int_a^b f(x)(dx)^2 \approx \sum_{i=1}^{n} \Delta x(b - E_i - \alpha_i)\ f(E_i) \quad . \qquad \text{A-7}$$

Again taking the limit as $n \to \infty$ and $\Delta x \to 0$ yields

$$\int_a^b\!\!\int_a^b f(x)(dx)^2 = \int_a^b (b-x)\ f(x)dx. \qquad \text{A-8}$$

The definite integral of order three is generated from the approximate form of the second order (eq. A-7) in a very similar way. Its approximate value is given by

$$\int_a^b\int_a^b\int_a^b f(x)(dx)^3 \approx [(1+2+\ldots+n)(\Delta x)^2 f(E_i) \qquad \text{A-9}$$

$$+ (1+2+\ldots+\{n-1\})(\Delta x)^2 f(E_2) + \ldots$$

$$+ (1+2)(\Delta x)^2 f(E_{n-1}) + (1)(\Delta x)^2 f(E_n)] \; .$$

Using the fact that

$$1 + 2 + \ldots + n = \frac{n(n+1)}{2} \qquad \text{A-10}$$

and the relations of eqs. A-6, eq. A-9 above reduces to

$$\int_a^b\int_a^b\int_a^b f(x)(dx)^3 \approx \frac{1}{1\cdot 2}\sum_{i=1}^{n}(b-E_i-\alpha_i)^2 f(E_i)\Delta x$$

$$+ \Delta x \cdot \frac{1}{1\cdot 2}\sum_{i=1}^{n}(b-E_i-\alpha_i)f(E_i)\Delta x \; . \qquad \text{A-11}$$

Passing to the limit as $n \to \infty$ and $\Delta x \to 0$, eq. A-11 becomes

$$\int_a^b\int_a^b\int_a^b f(x)(dx)^3 = \frac{1}{1\cdot 2}\int_a^b (b-x)^2 f(x)dx \; . \qquad \text{A-12}$$

which is the definite integral of order three.

Following this same procedure m times and passing to the same limits produces the m^{th} order definite integral of f, ie.,

$$\underbrace{\int_a^b\int_a^b\int_a^b \cdots \int_a^b}_{m\ \text{times}} f(x)(dx)^m = \frac{1}{(m-1)!}\int_a^b (b-x)^{m-1} f(x)dx. \qquad \text{A-13}$$

To generalize this relation to non-integer values of m, it is only necessary to replace the factorial with the appropriate Gamma function equivalent. It is possible to perform this step in a fairly rigorous manner before the limit is taken to produce eq. A-13. The arguments follow closely those of Grunwald and will not be discussed here. Finally, if b is permitted to be any value of x in [a,b], eq. A-13 becomes

$$\frac{d^{-v}f(x)}{dx^{-v}} = \frac{1}{\Gamma(v)}\int_a^x (x-u)^{v-1} f(u)du \qquad \text{A-14}$$

Eq. A-14 is easily recognized as the L-R integral formula discussed in the main text.

APPENDIX B

SPECIAL CASES OF KUMMER FUNCTIONS*

	M(a,b,z) a	b	z	Relation	Function
1	$v+\frac{1}{2}$	$2v+1$	$2iz$	$\Gamma(1+v)e^{iz}(\frac{1}{2}z)^{-v}J_v(z)$	Bessel
2	$v+\frac{1}{2}$	$2v+1$	$2z$	$\Gamma(1+v)e^{z}(\frac{1}{2}z)^{-v}I_v(z)$	Modified Bessel
3	$n+1$	$2n+2$	$2iz$	$\Gamma(\frac{3}{2}+n)e^{iz}(\frac{1}{2}z)^{-n-\frac{1}{2}}J_{n+\frac{1}{2}}(z)$	Spherical Bessel
4	$-n$	$\alpha+1$	x	$\frac{n!}{(\alpha+1)_n} L_n^{(\alpha)}(x)$	Laguerre
5	a	$a+1$	$-x$	$ax^{-\alpha}\gamma(a,x)$	Incomplete Gamma
6	a	a	z	e^z	Exponential
7	1	2	$-2iz$	$\frac{e^{-iz}}{z} \sin z$	Trigonometric
8	1	2	$2z$	$\frac{e^z}{z} \sinh z$	Hyperbolic
9	$\frac{1}{2}$	$\frac{3}{2}$	$-x^2$	$\frac{\pi^{\frac{1}{2}}}{2x} \operatorname{erf} x$	Error Integral
10	$-n$	$\frac{1}{2}$	$\frac{1}{2}x^3$	$\frac{n!}{(2n)!} (-\frac{1}{2})^{-n}He_{2n}(x)$	Hermite

*Milton Abramowitz and Irene Stegun (ed.), *Handbook of Mathematical Functions* (New York: Dover Publications, Inc., 1965), p. 509.