# A PROBLEM IN OPTIMAL REORIENTATION
# OF ASYMMETRIC SPACE VEHICLES

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A Thesis

Presented To

the Faculty of the Department of Mechanical Engineering

University of Houston

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In Partial Fulfillment

of the Requirements for the Degree

Master of Science in Mechanical Engineering

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by

Charles Landess Morefield

August, 1969

## ACKNOWLEDGEMENTS

The author is indebted to his thesis advisor Dr. Bart Childs of the University of Houston both for his many suggestions and for the computer program used to produce the necessary numerical results. Mr. Char-An Chao of TRW Systems Group provided valuable criticism in the initial formulation of the problem. Linda Nettle deserves special credit for her work in typing the manuscript.

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## ABSTRACT

The problem of large-angle, three-dimensional reorientation of asymmetric space vehicles in a fuel optimal manner is considered. A general problem is posed and solved numerically. The resultant numerical data is compared to a feedback configuration derived from a subset of the general problem. In both instances, Pontryagin's Minimum Principle is utilized to determine the set of differential equations which describe the optimal trajectory. The control is presumed to be unconstrained, resulting in a continuous optimal control.

TABLE OF CONTENTS

## SYMBOLS AND NOTATION

| SYMBOL | DEFINITION | DIMENSION |
|---|---|---|
| $\left[A(t)\right]$ | The nine element transformation matrix relating the body system at time t to a desired inertial orientation. | n.d. |
| $\underline{b}_x, \underline{b}_y, \underline{b}_z$ | Body-fixed unit vectors colinear with the principal moment of inertia axes, with origin at the center-of-mass of V. | n.d. |
| $\underline{d}_x, \underline{d}_y, \underline{d}_z$ | The orthogonal unit vectors defining a desired orientation of V. | n.d. |
| $\left[E\right]$ | The unit matrix. | n.d. |
| $\underline{f}$ | The r.h.s. of equation (3.10). | rad/sec$^2$ |
| $g_5, g_6, g_7$ | The coefficients of the inertial cross-coupling terms of Euler's Equations: $2(I_y-I_z)/I_x$, $2(I_z-I_x)/I_y$, $2(I_x-I_y)/I_z$ respectively. | n.d. |
| H | The state function of Pontryagin. | n.a. |
| $\left[I\right]$ | The diagonal inertia tensor, with components $I_x$, $I_y$, $I_z$. | slug-ft$^2$ |
| $I_n$ | The moment of inertia about $\underline{n}$. | slug-ft$^2$ |
| J | The functional to be minimized as an index of performance. | n.a. |
| m(t) | The control torque about $\underline{n}$ for the restricted problem. | rad/sec$^2$ |
| $\underline{n}(t)$ | The unit vector with components $n_x$, $n_y$, $n_z$ in body axes, about which a single rotation may be performed to attain $\left[A(t_f)\right] = \left[E\right]$. | n.d. |
| $\underline{p}$ | The r.h.s. of (3.11). | Various |
| $t, t_f$ | The independent variable time, and final trajectory time respectively. | sec |
| $\underline{u}(t)$ | The system control acceleration. | rad/sec$^2$ |

| SYMBOL | DEFINITION | DIMENSION |
|---|---|---|
| V | The vehicle. | n.a. |
| $\underline{x}$, $\underline{y}$ | The state vectors for the restricted and general problems respectively. | Various |
| $\alpha(t)$ | The component of cross-coupling torque along $u_r\underline{n}$. | rad/sec$^2$ |
| $\beta(t)$ | The input to plant defined by the restricted problem. | rad/sec$^2$ |
| $\left[\Gamma\right]$ | The Jacobian matrix used in the Modified Newton-Raphson algorithm. | Various |
| $\delta_{ij}, \epsilon_{ijk}$ | The Kronecker Delta and Levi-Civita Epsilon. | n.d. |
| $\theta(t)$ | The angle through which V must rotate about $\underline{n}$ to attain $\left[A(t_f)\right] = \left[E\right]$. | rad |
| $\underline{\lambda}(t)$ | The costate variables defined by Pontryagin's Minimum Principle; associated with the general problem. | Various |
| $\underline{\rho}(t)$ | The costate variables defined by Pontryagin's Minimum Principle; associated with the restricted problem. | rad/sec$^2$ |
| $\omega^2$ | The weighting coefficient selected for $x_2^2$ in $J_r$. | n.d. |

## NOTATION

Brackets indicate matrix or interval.
Dot indicates differentiation w.r.t. time.
Underscored variables indicate vectors.
Vertical bars indicate magnitude.

## SUBSCRIPTS

c refers to cross-coupling acceleration.
N refers to the nth iteration of the numerical scheme.
r refers to the restricted problem.
x, y, z refer to vector components along $\underline{b}_x$, $\underline{b}_y$, $\underline{b}_z$.

## SUPERSCRIPTS

o refers to the optimal path or control.
T indicates transpose.

## CHAPTER I

## INTRODUCTION

The purpose of this work is two-fold. First, to examine a problem in three-dimensional attitude control in a general (albeit numerical) manner. Given a rigid space vehicle of constant mass in a torque-free environment, the problem may be stated as follows: what unconstrained control torque exerted on the interval $[0,t_f]$ will rotate the vehicle to some arbitrary attitude in a fuel optimal manner? The assumptions of vehicular mass symmetry and small-angle maneuver have been purposely avoided. Numerical results from the general nonlinear problem provide a framework for the second purpose of the paper, which is to demonstrate a simple system whose fuel usage falls close to the general (open-loop) optimal.

The method of solution is based upon standard implementation of Pontryagin's Minimum Principle (1). In the general case, an open-loop optimal control $\underline{u}^o(t)$ results. For the restricted problem, $\underline{u}^o(t)$ may be expressed in part as a function of the system state variables, so that a feedback configuration is feasible.

Chapter II contains the differential equations describing the system, and poses both the general and restricted problems under consideration. In Chapter III the minimum principle is used to derive the constraining differential equations for the general problem, and a closed-form solution for fuel-optimal control in the restricted problem. Chapter IV provides numerical results for the general problem, given some specific sets of initial conditions for the system. Additionally, a comparison is made to the restricted problem. Chapter V presents some conclusions resulting from the work.

## CHAPTER II

## DEFINITION OF THE PROBLEM

Consider a rigid, asymmetric vehicle V in three-dimensional space. Let $\underline{b}_x$, $\underline{b}_y$, $\underline{b}_z$ denote the body-fixed principal axes of inertia through its center of mass. These axes form body-fixed vector bases whose relation to some inertial bases $\underline{d}_x$, $\underline{d}_y$, $\underline{d}_z$ is given uniquely at any time t by

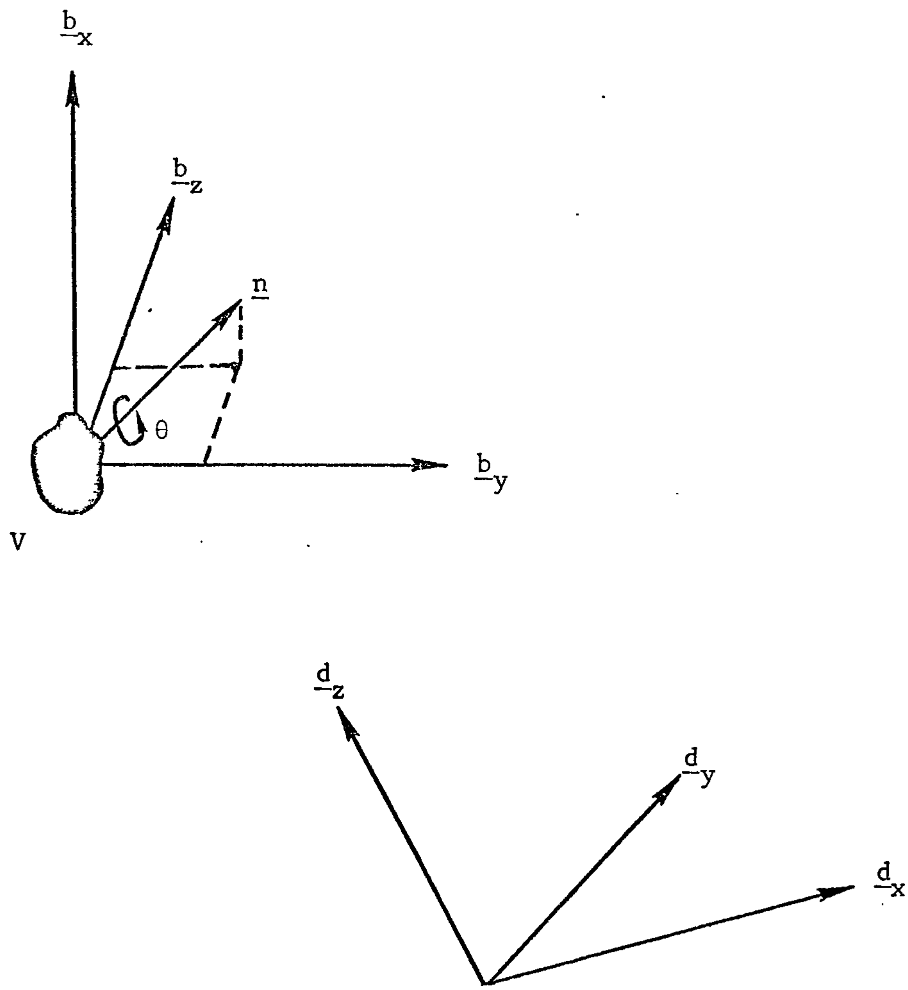
$$\left[\underline{d}_x \quad \underline{d}_y \quad \underline{d}_z\right] = \left[A(t)\right]^T \left[\underline{b}_x \quad \underline{b}_y \quad \underline{b}_z\right] \tag{2.1}$$

where $\left[A(t)\right]$ is just the usual orthogonal direction cosine matrix whose determinant is +1. If $\underline{d}_x$, $\underline{d}_y$, $\underline{d}_z$ describe a desired inertial orientation of V, then we may consider the following problem: given $\left[A(0)\right]$, what is the unconstrained fuel-optimal control on $[0,t_f]$ such that

$$\left[A(t_f)\right] = \left[E\right] \tag{2.2}$$

Many portions of this general problem have been attacked by various authors. At present, no analytic solution has been found (2). A numerical solution is presented in this paper. Two points should be noted concerning this solution: the control is unconstrained (with the result that $\underline{u}^o(t)$ is continuous), and normalized quaternions are used as attitude variables. This particular configuration is chosen because of current interest in (continuous moment) control moment gyros (3) and quaternion techniques for onboard systems (4).

FIGURE 1. A Sketch of the Reorientation Problem.



The vehicular orientation $\left[A(t)\right]$ may be parameterized in rational algebraic form by four normalized quaternions (5), $y_i$ (i=i,4):

$$A_{ij} = 2y_i y_j + \delta_{ij}(y_4^2 - y_k y_k) + 2\epsilon_{ijk} y_k y_4 \qquad (2.3)^*$$

Accordingly, the initial attitude of V corresponding to $\left[A(0)\right]$ is given by

$$\begin{bmatrix} y_1(0) \\ y_2(0) \\ y_3(0) \\ y_4(0) \end{bmatrix} = \begin{bmatrix} y_{10} \\ y_{20} \\ y_{30} \\ y_{40} \end{bmatrix} \qquad (2.4)$$

A final condition on the quaternions equivalent to Equation (2.2) is (4):

$$\begin{bmatrix} y_1(t_f) \\ y_2(t_f) \\ y_3(t_f) \\ y_4(t_f) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad (2.5)$$

*The summation convention for repeated indices has been utilized, with i,j,k allowed to range from one to three.

If $y_5$, $y_6$, $y_7$ are the components of vehicular angular velocity about $\underline{b}_x$, $\underline{b}_y$, $\underline{b}_z$ respectively, then the kinematical equations for the attitude variables are given by (5):

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \\ \dot{y}_4 \end{bmatrix} = 1/2 \begin{bmatrix} y_4 & -y_3 & y_2 \\ y_3 & y_4 & -y_1 \\ -y_2 & y_1 & y_4 \\ -y_1 & -y_2 & -y_3 \end{bmatrix} \begin{bmatrix} y_5 \\ y_6 \\ y_7 \end{bmatrix} \tag{2.6}$$

or equivalently

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \\ \dot{y}_4 \end{bmatrix} = 1/2 \begin{bmatrix} 0 & y_7 & -y_6 & y_5 \\ -y_7 & 0 & y_5 & y_6 \\ y_6 & -y_5 & 0 & y_7 \\ -y_5 & -y_6 & -y_7 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} \tag{2.7}$$

The differential equations for the body rates are Euler's Equations (6):

$$\begin{bmatrix} \dot{y}_5 \\ \dot{y}_6 \\ \dot{y}_7 \end{bmatrix} = 1/2 \begin{bmatrix} g_5 y_6 y_7 \\ g_6 y_7 y_5 \\ g_7 y_5 y_6 \end{bmatrix} + \begin{bmatrix} u_5 \\ u_6 \\ u_7 \end{bmatrix} \tag{2.8}$$

Without loss of generality, let the initial and final angular velocities on $[0,t_f]$ be

$$\begin{bmatrix} y_5(0) \\ y_6(0) \\ y_7(0) \end{bmatrix} = \begin{bmatrix} y_5(t_f) \\ y_6(t_f) \\ y_7(t_f) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{2.9}$$

A general problem may now be posed: given the plant defined by equations (2.7) and (2.8) with boundary conditions (2.4), (2.5), and (2.9), find the unconstrained control $\underline{u}^o(t)$ on $[0,t_f]$ such that the following cost functional is minimized:

$$J = \int_0^{t_f} ( \left[ I \right] \underline{u})^T ( \left[ I \right] \underline{u}) dt \tag{2.10}$$

Since $\left[ I \right] \underline{u}$ (control torque) is proportional to fuel expenditure, minimization of J should provide fuel optimal control.

* * * * *

A restricted problem may also be posed as an outgrowth of Euler's Theorem (7). The theorem states that given $\left[A(0)\right]$ there exists some body-fixed (and inertially-fixed) axis $\underline{n}$ and some angle $\theta(0)$ such that a single rotation of magnitude $\theta(0)$ about $\underline{n}$ will result in $\left[A(t_f)\right] = \left[E\right]$. There are an infinity of trajectories other than the Euler trajectory which will achieve this result. These trajectories will in general be characterized by the time histories $\underline{n} = \underline{n}\,(t)$, $\theta = \theta(t)$. The axis $\underline{n}(t)$

and angle $\theta(t)$ are directly related to the quaternion parameters (4):

$$
\begin{aligned}
y_1(t) &= n_x(t)\ \sin\theta(t)/2 \\
y_2(t) &= n_y(t)\ \sin\theta(t)/2 \\
y_3(t) &= n_z(t)\ \sin\theta(t)/2 \\
y_4(t) &= \cos\theta(t)/2
\end{aligned}
\tag{2.11}
$$

Euler's Theorem states that $\left[A(t_f)\right] = \left[E\right]$ is achievable with a constant axis of rotation ($\underline{n}(t) = \underline{n}(0)$). If $\underline{n}$ is not a function of time, then the attitude of V is a function of $\theta(t)$ only. Additionally, if V is constrained to rotate just about $\underline{n}$, then since

$$
\begin{bmatrix} \dot{y}_5 \\ \dot{y}_6 \\ \dot{y}_7 \end{bmatrix} = \ddot{\theta}\underline{n} \tag{2.12}
$$

equation (2.8) reduces to

$$
\ddot{\theta}(t) = m(t)/I_n \tag{2.13}
$$

where (6)

$$
I_n = \underline{n}^T \left[ I \right] \underline{n} \tag{2.14}
$$

If indeed $\underline{n}$ is constant, then since V is asymmetric the total control must be given by

$$
\underline{u}(t) = u_r(t)\underline{n} + \underline{u}_c(t) \tag{2.15}
$$

where

$$
u_r(t) = m(t)/I_n \tag{2.16}
$$

$$
\underline{u}_c(t) = -\ 1/2 \begin{bmatrix} g_5 y_6 y_7 \\ g_6 y_7 y_5 \\ g_7 y_5 y_6 \end{bmatrix} \tag{2.16a}
$$

I.e. $\underline{u}_c(t)$ is just that constraint torque which negates inertial cross-coupling. The component of $\underline{u}_c(t)$ along $u_r\underline{n}$ is given by

$$\alpha(t) = -\,1/2 \begin{bmatrix} g_5y_6y_7 \\ g_6y_7y_5 \\ g_7y_5y_6 \end{bmatrix}^T (\text{sign } u_r)\underline{n}$$

$$\alpha(t) = -3 \text{ sign } (u_r)\, \dot{\theta}^2\, n_x n_y n_z (g_5+g_6+g_7)/2 \tag{2.17}$$

Note that in two important cases, $\alpha = 0$: V has at least one axis of symmetry ($g_5+g_6+g_7 = 0$); the rotation $\left[A(t)\right]$ doesn't involve all body axes ($n_x n_y n_z = 0$). In the general case of asymmetric V and rotations where $\alpha(t) \neq 0$, there is no a priori knowledge of $\alpha(t)$; it may be a help or a hindrance. Consequently, a reasonable control philosophy would be to minimize $\left|\underline{u}_c(t)\right|$.

Select the following counterpart to the cost functional (2.10):

$$J_r = \int_0^{t_f} \left\{ u_r^2 + 1/2 \left( (g_5y_6y_7)^2 + (g_6y_7y_5)^2 + (g_7y_5y_6)^2 \right)^{1/2} \right\} dt \tag{2.18}$$

This will minimize both $u_r(t)$ and $\underline{u}_c(t)$, although a less stringent (non-quadratic) minimization requirement is placed upon $\underline{u}_c(t)$. The reason for this will become apparent later. Since V is now constrained to one axis of rotation, the system dynamics are given completely by (2.13). Let $x_1 = \theta$, $x_2 = \dot{\theta}$. Then a restricted problem may be posed: given

$$\dot{\underline{x}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ u_r \end{bmatrix} \tag{2.19}$$

and the boundary conditions

$$\begin{aligned} x_1(0) &= x_{10} \;(= 2\cos^{-1} y_{40}) \\ x_2(0) &= x_1(t_f) = x_2(t_f) = 0 \end{aligned} \tag{2.20}$$

find $u_r(t)$ such that the functional

$$J_r = \int_0^{t_f} \left\{ u_r^2 + \omega^2 x_2^2 \right\} dt \tag{2.21}$$

$$\left(\text{where } \omega^2 = 1/2 \left( (g_5 n_y n_z)^2 + (g_6 n_z n_x)^2 + (g_7 n_x n_y)^2 \right)^{1/2} \right)$$

is minimized. The reason for selecting Equation (2.18) is now apparent: a quadratic performance index has been obtained.

## CHAPTER III

## THE GENERAL OPTIMAL TRAJECTORY AND A RESTRICTED CONTROL

Pontryagin's Minimum Principle (1) will be utilized to obtain solutions for both the general and restricted problems. In the general case, a set of fourteen first-order nonlinear differential equations will result. Since these equations are not easily integrable, numerical solutions for specific cases will be obtained in Chapter IV. The restricted problem will result in a feedback configuration.

For convenience, combine the plant equations (2.7), (2.8) to obtain the standard form $\dot{\underline{y}} = \underline{f}\ (\underline{y},\underline{u},t)$:

$$\dot{\underline{y}} = 1/2 \begin{bmatrix} 0 & y_7 & -y_6 & y_5 & 0 & 0 & 0 \\ -y_7 & 0 & y_5 & y_6 & 0 & 0 & 0 \\ y_6 & -y_5 & 0 & y_7 & 0 & 0 & 0 \\ -y_5 & -y_6 & -y_7 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & g_5 y_7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & g_6 y_5 \\ 0 & 0 & 0 & 0 & g_7 y_6 & 0 & 0 \end{bmatrix} \underline{y} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ u_5 \\ u_6 \\ u_7 \end{bmatrix} \tag{3.1}$$

Equation (3.1) together with the cost functional (2.10) may be used to develop the state function of Pontryagin:

$$H(\underline{y},\underline{u},\underline{\lambda},t) = \underline{\lambda}^T \underline{f}(\underline{y},\underline{u},t) + \left( \begin{bmatrix} I \end{bmatrix} \underline{u}\right)^T \left( \begin{bmatrix} I \end{bmatrix} \underline{u}\right) \tag{3.2}$$

By substitution, H is given by

$$
\begin{aligned}
H = {} & 1/2\ \lambda_1(y_7y_2 - y_6y_3 + y_5y_4) \\
& + 1/2\ \lambda_2(-y_7y_1 + y_5y_3 + y_6y_4) \\
& + 1/2\ \lambda_3(y_6y_1 - y_5y_2 + y_7y_4) \\
& + 1/2\ \lambda_4(-y_5y_1 - y_6y_2 - y_7y_3) \\
& + 1/2\ \lambda_5(g_5y_7y_6) + \lambda_5u_5 \\
& + 1/2\ \lambda_6(g_6y_5y_7) + \lambda_6u_6 \\
& + 1/2\ \lambda_7(g_7y_6y_5) + \lambda_7u_7 \\
& + I_x^2\ u_5^2 + I_y^2\ u_6^2 + I_z^2\ u_7^2
\end{aligned}
\tag{3.3}
$$

The optimal control $\underline{u}^o(t)$ must satisfy

$$\frac{\partial H(\underline{y},\underline{u},\underline{\lambda},t)}{\partial \underline{u}} = 0 \tag{3.4}$$

From (3.3), this condition is met if

$$
\begin{aligned}
u_5^o &= -\ 1/2\ \lambda_5/I_x^2 \\
u_6^o &= -\ 1/2\ \lambda_6/I_y^2 \\
u_7^o &= -\ 1/2\ \lambda_7/I_z^2
\end{aligned}
\tag{3.5}
$$

The optimal Pontryagin state function may now be obtained:

$$
\begin{aligned}
H^o(\underline{y},\underline{\lambda},t) = {} & \underline{\lambda}^T\underline{f}\ \ [\underline{y},\underline{u}^o(\underline{y},\underline{\lambda},t),t] \\
& + (\left[I\right]\underline{u}^o)^T(\left[I\right]\underline{u}^o)
\end{aligned}
\tag{3.6}
$$

$$
\begin{aligned}
H^o = {} & 1/2\ \lambda_1\ (y_7y_2 - y_6y_3 + y_5y_4) \\
& + 1/2\ \lambda_2\ (-y_7y_1 + y_5y_3 + y_6y_4) \\
& + 1/2\ \lambda_3\ (y_6y_1 - y_5y_2 + y_7y_4) \\
& + 1/2\ \lambda_4\ (-y_5y_1 - y_6y_2 - y_7y_3) \\
& + 1/2\ \lambda_5\ (g_5y_7y_6) + 1/2\ \lambda_6\ (g_6y_5y_7) + 1/2\ \lambda_7\ (g_7y_6y_5) \\
& - 1/4\ (\lambda_5^2/I_x^2 + \lambda_6^2/I_y^2 + \lambda_7^2/I_z^2)
\end{aligned}
\tag{3.7}
$$

The optimal trajectory $\underline{y}^o(t)$ on $[0,t_f]$ must satisfy the differential equations

$$\dot{\underline{y}} = \frac{\partial H^o}{\partial \underline{\lambda}} \tag{3.8}$$

$$\dot{\underline{\lambda}} = -\frac{\partial H^o}{\partial \underline{y}} \tag{3.9}$$

Then

$$
\dot{\underline{y}} = 1/2
\begin{bmatrix}
0 & y_7 & -y_6 & y_5 & 0 & 0 & 0 \\
-y_7 & 0 & y_5 & y_6 & 0 & 0 & 0 \\
y_6 & -y_5 & 0 & y_7 & 0 & 0 & 0 \\
-y_5 & -y_6 & -y_7 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & g_5y_7 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & g_6y_5 \\
0 & 0 & 0 & 0 & g_7y_6 & 0 & 0
\end{bmatrix}
\underline{y} +
\begin{bmatrix}
0 \\ 0 \\ 0 \\ 0 \\ -1/2\lambda_5/I_x^2 \\ -1/2\lambda_6/I_y^2 \\ -1/2\lambda_7/I_z^2
\end{bmatrix}
\tag{3.10}
$$

$$\dot{\underline{\lambda}} = 1/2 \begin{bmatrix} 0 & -y_7 & y_6 & -y_5 & 0 & 0 & 0 \\ y_7 & 0 & -y_5 & -y_6 & 0 & 0 & 0 \\ -y_6 & y_5 & 0 & -y_7 & 0 & 0 & 0 \\ y_5 & y_6 & y_7 & 0 & 0 & 0 & 0 \\ y_4 & y_3 & -y_2 & -y_1 & 0 & g_6 y_7 & g_7 y_6 \\ -y_3 & y_4 & y_1 & -y_2 & g_5 y_7 & 0 & g_7 y_5 \\ y_2 & -y_1 & y_4 & -y_3 & g_5 y_6 & g_6 y_5 & 0 \end{bmatrix} \underline{\lambda} \quad (3.11)$$

Equations (2.4), (2.5), (2.9) provide the boundary conditions

$$\begin{aligned} \underline{y}(0) &= [y_{10}, y_{20}, y_{30}, y_{40}, 0, 0, 0]^T \\ \underline{y}(t_f) &= [0, 0, 0, 1, 0, 0, 0]^T \end{aligned} \quad (3.12)$$

Equations (3.10) - (3.12) completely specify the solution to the general problem. A numerical solution to this system will be developed in Section IV.

* * * * *

The (linear) restricted problem given by (2.19) - (2.21) is much simpler, and by application of the same technique will yield a closed-form solution. Proceeding in the same manner, form $H_r$:

$$H\ (\underline{x},\ u_r,\ \underline{\rho},\ t) = \rho_1 x_2 + \rho_2 u_r + u_r^2 + \omega^2 x_2^2 \quad (3.13)$$

The optimal control again must satisfy (3.4):

$$u_r^o = -\ \rho_2/2 \quad (3.14)$$

Then

$$H_r^o(\underline{x},\underline{\rho},t) = \rho_1 x_2 - \rho_2^2/4 + \omega^2 x_2^2 \tag{3.15}$$

and from (3.8), (3.9) the optimal trajectory $\underline{x}^o(t)$ is specified by

$$\dot{\underline{x}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ -\rho_2/2 \end{bmatrix} \tag{3.16}$$

$$\dot{\underline{\rho}} = - \begin{bmatrix} 0 \\ \rho_1 + 2\omega^2 x_2 \end{bmatrix} \tag{3.17}$$

together with the boundary conditions (2.20).

Consider the case where $\omega^2 = 0$ (V is fully symmetric or $\underline{n}$ is colinear with one of the body axes $\underline{b}_x$, $\underline{b}_y$, $\underline{b}_z$). Then:

$$u_r^o(t) = 12x_{10}t/t_f^3 - 6x_{10}/t_f^2 \tag{3.18}$$

or the optimal control is just a ramp, symmetric about $t_f/2$.

When $\omega^2 \neq 0$,

$$\rho_1(t) = \rho_{10} \tag{3.19}$$

$$\dot{x}_2(t) = \rho_{10}/2 + \omega^2 x_2(t) \tag{3.20}$$

Applying the Laplace Transform results in

$$X_2(s) = \frac{\rho_{10}}{2s(s^2-\omega^2)} - \frac{\rho_{20}}{2(s^2-\omega^2)} \tag{3.21}$$

The inverse transformation gives

$$x_2(t) = \frac{\rho_{10}}{2\omega^2} \cosh\omega t - \frac{\rho_{20}}{2\omega} \sinh\omega t - \frac{\rho_{10}}{2\omega^2} \tag{3.22}$$

where

$$\rho_{10} = \frac{2\omega^3 \, x_{10} \, \sinh\omega t_f}{\omega t_f \, \sinh\omega t_f - 2\,(\cosh\omega t_f - 1)} \tag{3.23}$$

$$\rho_{20} = \frac{2\omega^2 \, x_{10} \, (\cosh\omega t_f - 1)}{\omega t_f \, \sinh\omega t_f - 2\,(\cosh\omega t_f - 1)} \tag{3.24}$$

Then, since $x_1 = \int \dot{x}_2 dt$,

$$x_1(t) = \frac{\rho_{10}}{2\omega^3} \sinh\omega t - \frac{\rho_{20}}{2\omega^2} \cosh\omega t - \frac{\rho_{10}}{2\omega^2} t + \frac{\rho_{20}}{2\omega^2} + x_{10} \tag{3.25}$$

Note that from (3.17),

$$\rho_2(t) = -2\dot{x}_2(t) = -\frac{\rho_{10}}{\omega}\sinh\omega t + \rho_{20}\cosh\omega t \tag{3.26}$$

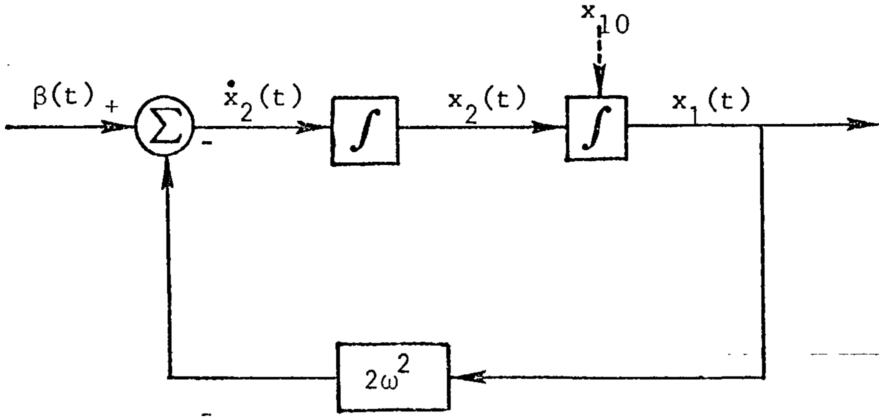
Then from (3.25)

$$\rho_2(t) = -2\omega^2 x_1(t) + \beta(t) \tag{3.27}$$

where

$$\beta(t) = -\rho_{10} t + 2\omega^2 x_{10} + \rho_{20} \tag{3.28}$$

This is equivalent to the following configuration:



(The necessary corrective acceleration for cross-coupling has been neglected. It takes the form of a nonlinear feedback element). In Chapter IV, the control history associated with this configuration (including $\underline{u}_c$ ) will be compared to that achieved by the general optimal control.

## CHAPTER IV

## NUMERICAL RESULTS

The solution to the general problem is now stated in the form of a nonlinear differential equation with initial and terminal conditions on seven of the (state) variables specified. Since initial conditions on the remaining (costate) variables are unknown, a numerical solution is called for. Childs (8) has developed a computer program which permits iterative solution of this type of problem. The technique is that of the generalized Newton-Raphson operator (9), wherein the nonlinear problem is transformed to a sequence of linear problems. If $\underline{\lambda}(0)_N$ is the Nth approximation of the correct initial conditions ($\underline{y}(0)$ is of course known), then successive improvements are obtained from the following algorithm (10):

$$(\dot{\underline{y}}, \dot{\underline{\lambda}})^T_{N+1} = \left[\Gamma\right]_N \left\{ \begin{bmatrix} \underline{y} \\ \underline{\lambda} \end{bmatrix}_{N+1} - \begin{bmatrix} \underline{y} \\ \underline{\lambda} \end{bmatrix}_N \right\} + (\underline{f}, \underline{p})^T_N \qquad (4.1)$$

where $\underline{p}$ is the r.h.s. of equation (3.11) and

$$\left[\Gamma\right]_N = \left. \frac{\partial(\underline{f}, \underline{p})^T}{\partial(\underline{y}, \underline{\lambda})^T} \right|_N \qquad (4.2)$$

The elements of $\left[\Gamma\right]$ are detailed in Appendix A. The linear set (4.1) is solved by superposition of particular solutions (8). Necessary conditions for the convergence of the sequence $\left\{ \underline{\lambda}(0)_N \right\}$ are not known (10). It is interesting to note that by inspection at least one known sufficiency condition (10) is not met:

$$\left( \Gamma_{ij} \right)_N > 0, \quad i \neq j \tag{4.3}$$

Regardless of (4.3), convergence was obtained for the example cases below.

The three cases chosen to illustrate the application of this approach (and to obtain a comparison with the restricted problem) are summarized in Appendix B. Appendix C contains tabulated time histories of $\underline{y}$ for each case.

Case I is a physically uninteresting case chosen to illustrate the accuracy of the program for this particular problem. No real attempt to achieve ultimate accuracy was made; the run was simply terminated after ten iterations. Case I represents a fully symmetric V. For this case, the general optimal control coincides with the restricted control (a single rotation about $\underline{n}$), since no cross-coupling is present. The closed-form solution for this case is given by (3.18). Figure 2 below illustrates the fact that the closed-form solution coincides almost exactly with computed values of control acceleration. In the absence of cross-coupling, then, excellent convergence is obtained.

In Case II, V has one axis of inertial symmetry ($\underline{b}_x$). Since the cross-coupling accelerations are symmetric about $t_f/2$ for the restricted problem, as would be expected the controls are symmetric about $t_f/2$. Note that the restricted control acceleration $\dot{x}_2^o\underline{n}+\underline{u}_c$ has maximum degradation at the endpoints of the trajectory. Also, greater differences with the general solution $\underline{u}^o$ occur for the $\underline{b}_y$, $\underline{b}_z$ axes, an undesirable situation because of the larger inertias associated with these axes. (We have presumed in the development that for the ith body axis fuel consumption $\sim I_i u_i$).

Figure 2. Case I Control Acceleration vs. Time (All Axes)

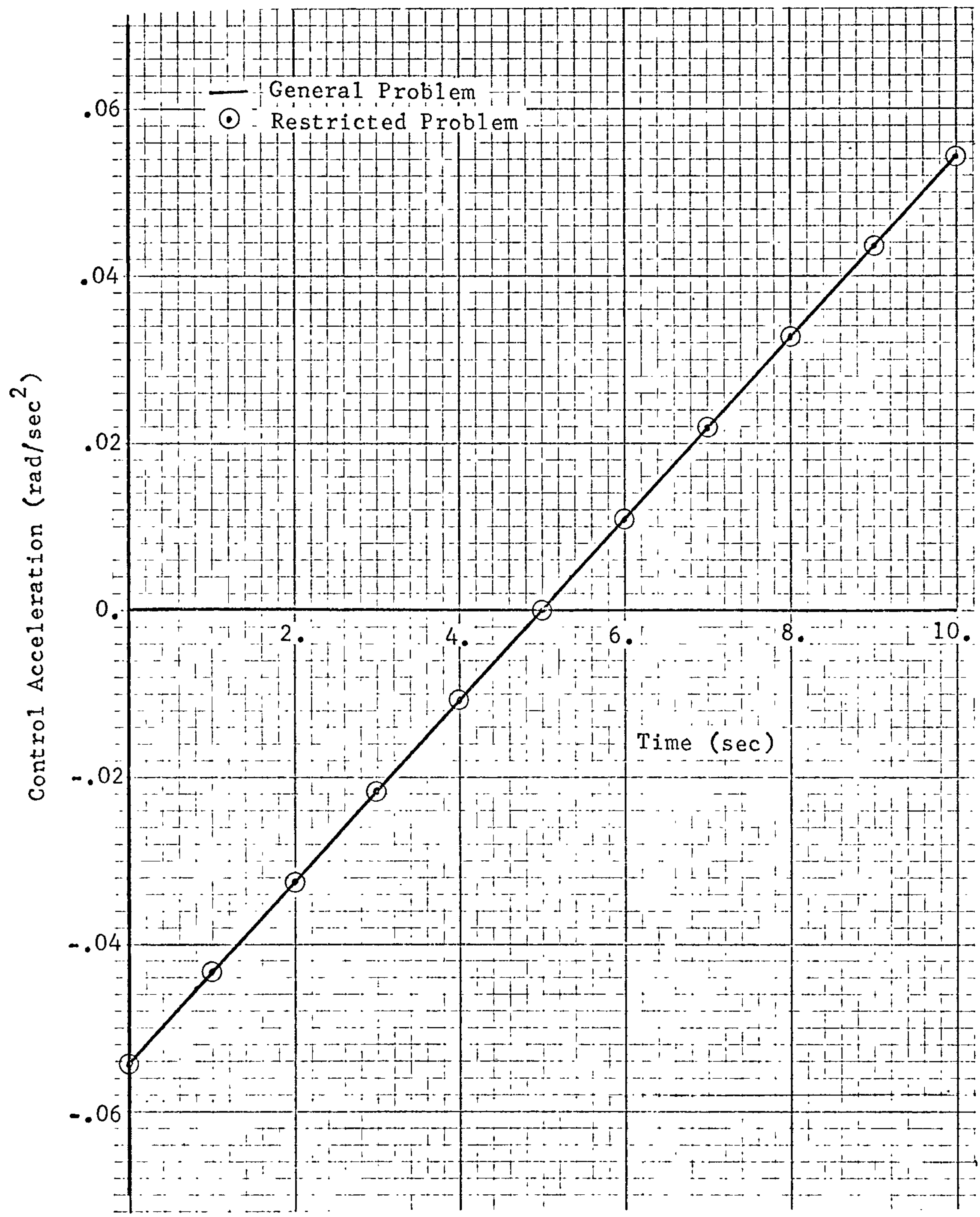


Figure 3. Case II Control Acceleration vs. Time ($\underline{b}_x$ Axis)

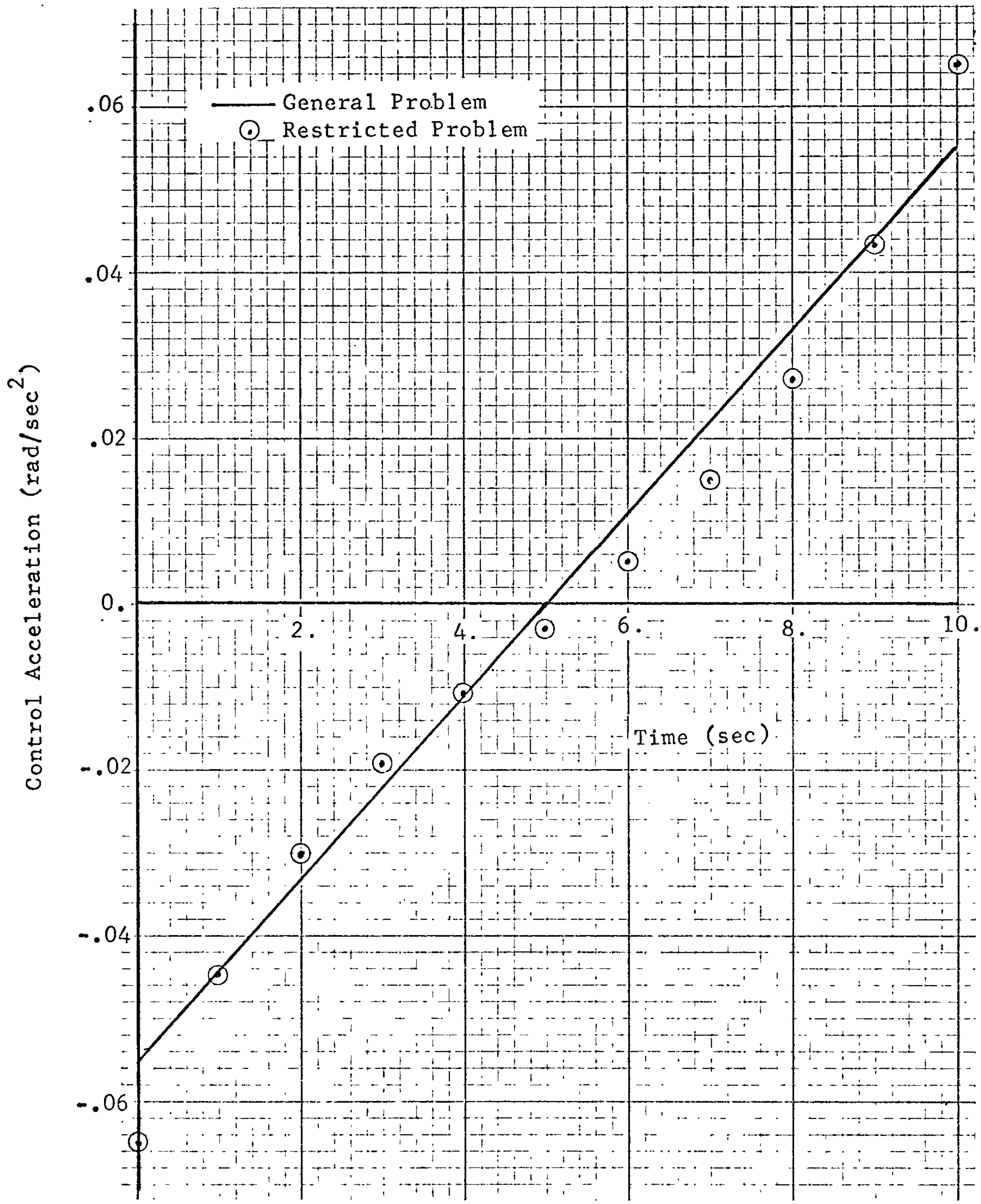


Figure 4. Case II Control Acceleration vs. Time ($\underline{b}_y$ Axis)

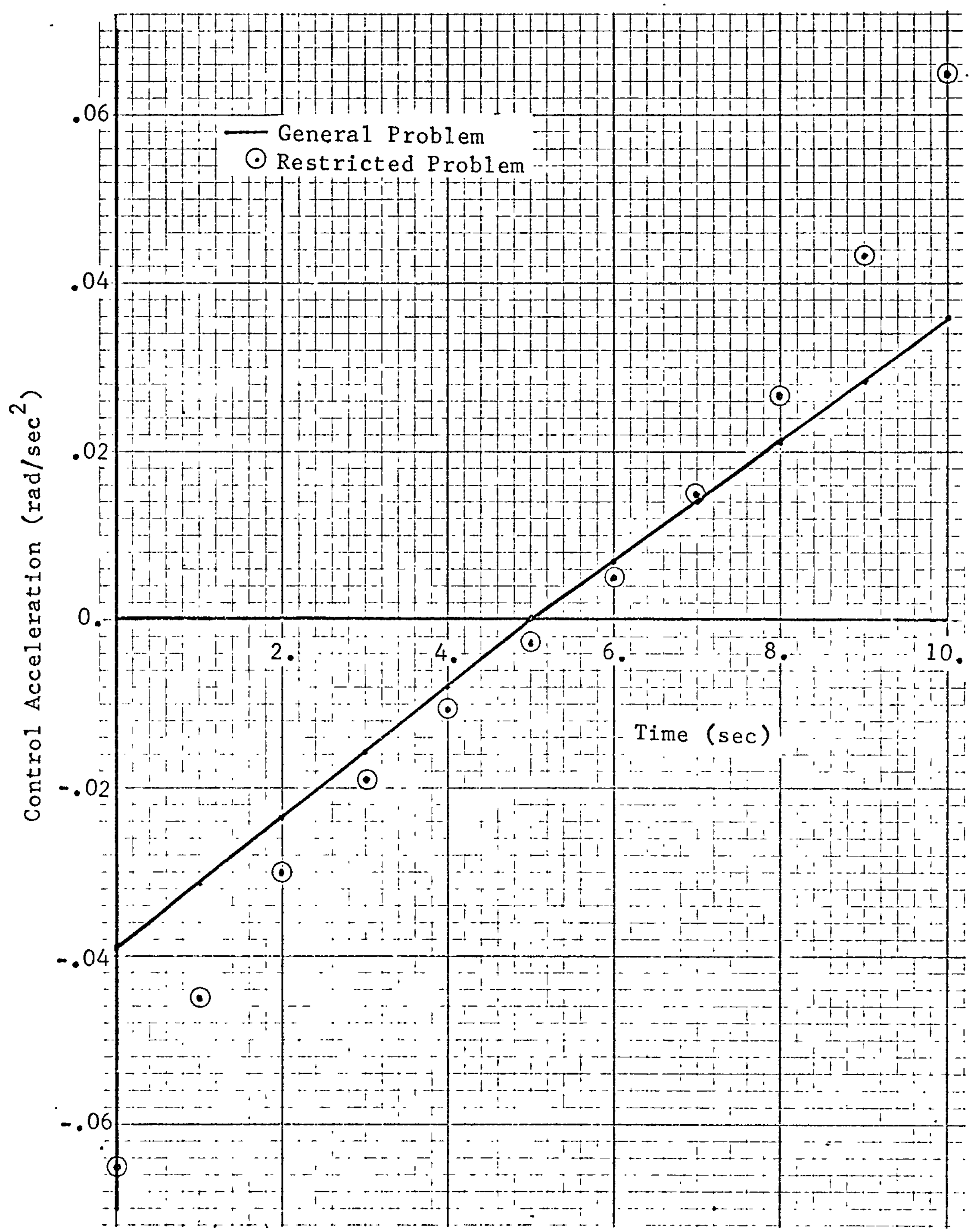
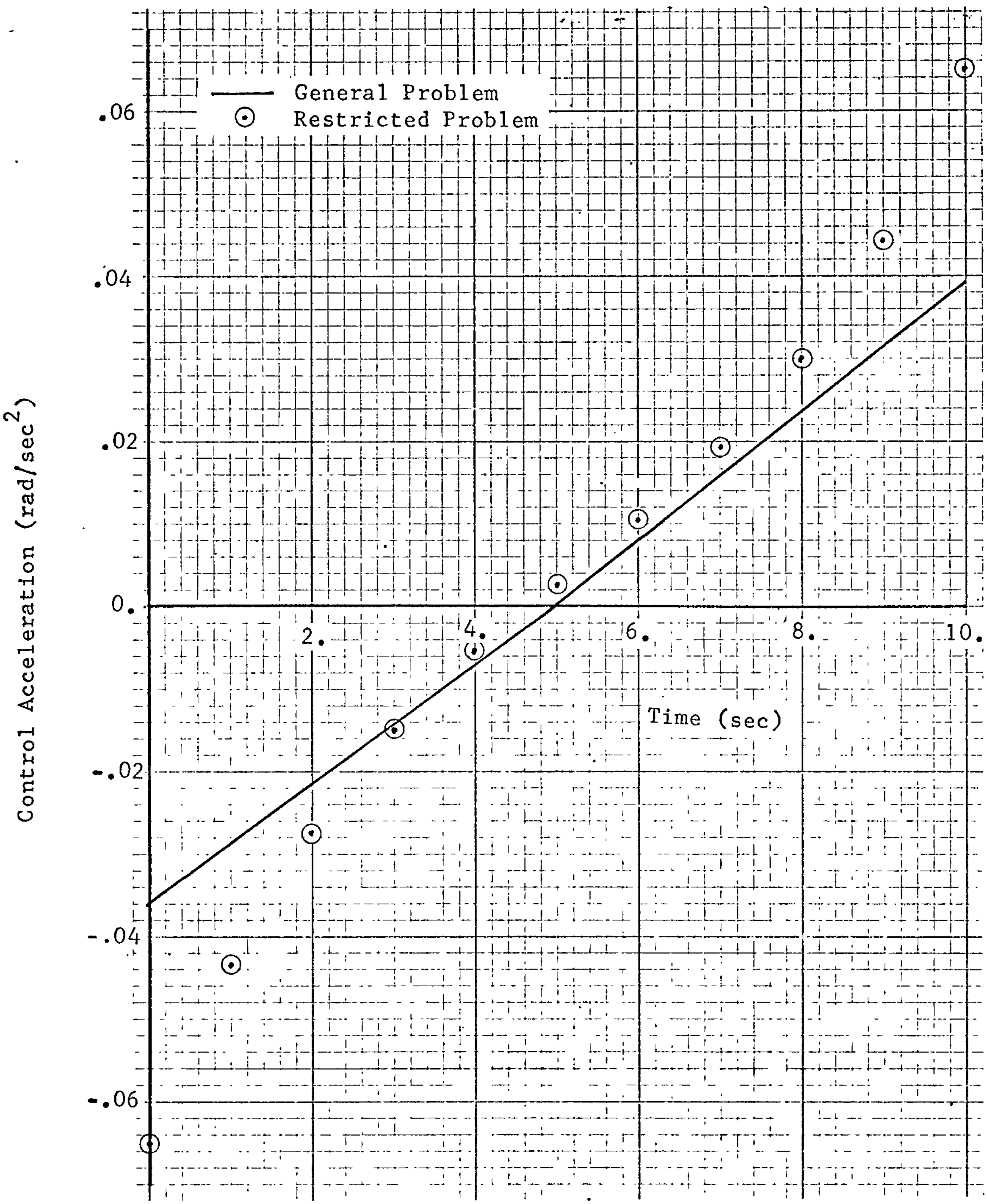


Figure 5. Case II Control Accleleration vs. Time ($\underline{b}_z$ Axis)



The $\underline{b}_x$ control for the restricted problem $(\dot{x}_2^o n_x + u_{cx})$ falls closer to optimal $(u_5^o)$, since cross-coupling is not present in this axis.

The most general case, Case III, involves a completely asymmetric V. We see that again, the restricted problem yields reasonable results when compared to the general problem. As in Case II, maximum degradation occurs at the endpoints of the trajectory.

Figure 6. Case III Control Acceleration vs. Time ($\underline{b}_x$ Axis)

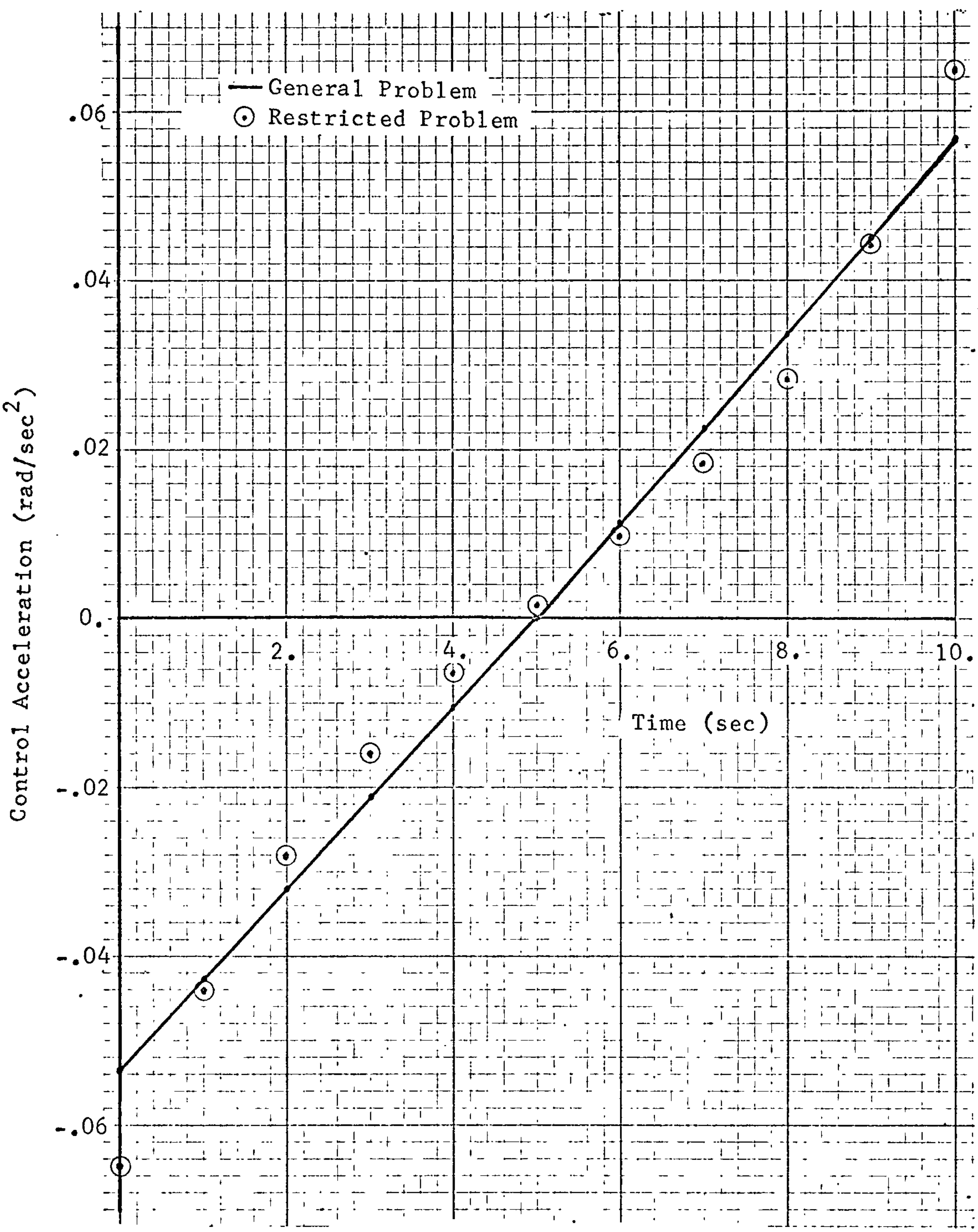


Figure 7. Case III Control Acceleration vs. Time ($\underline{b}_y$ Axis)

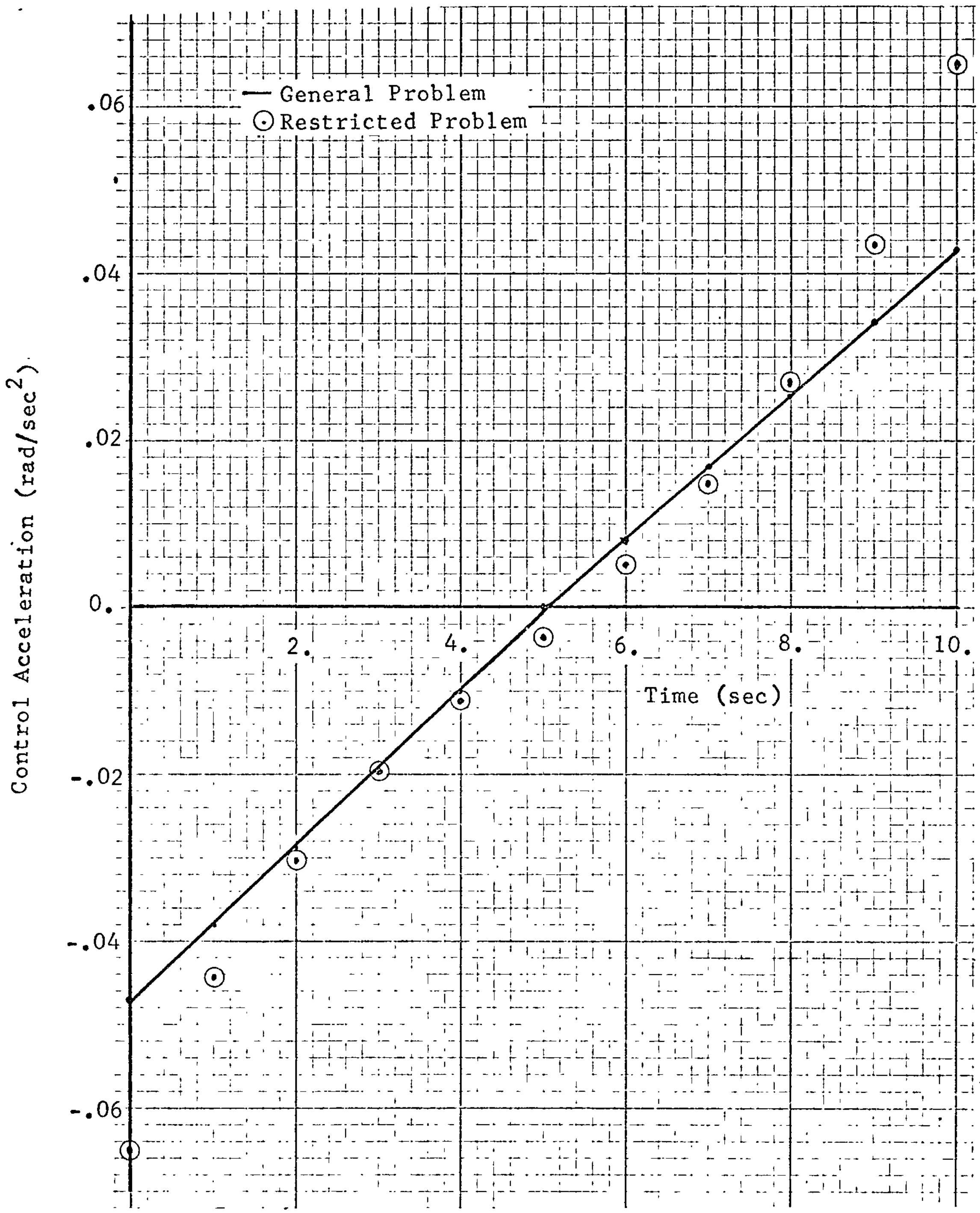
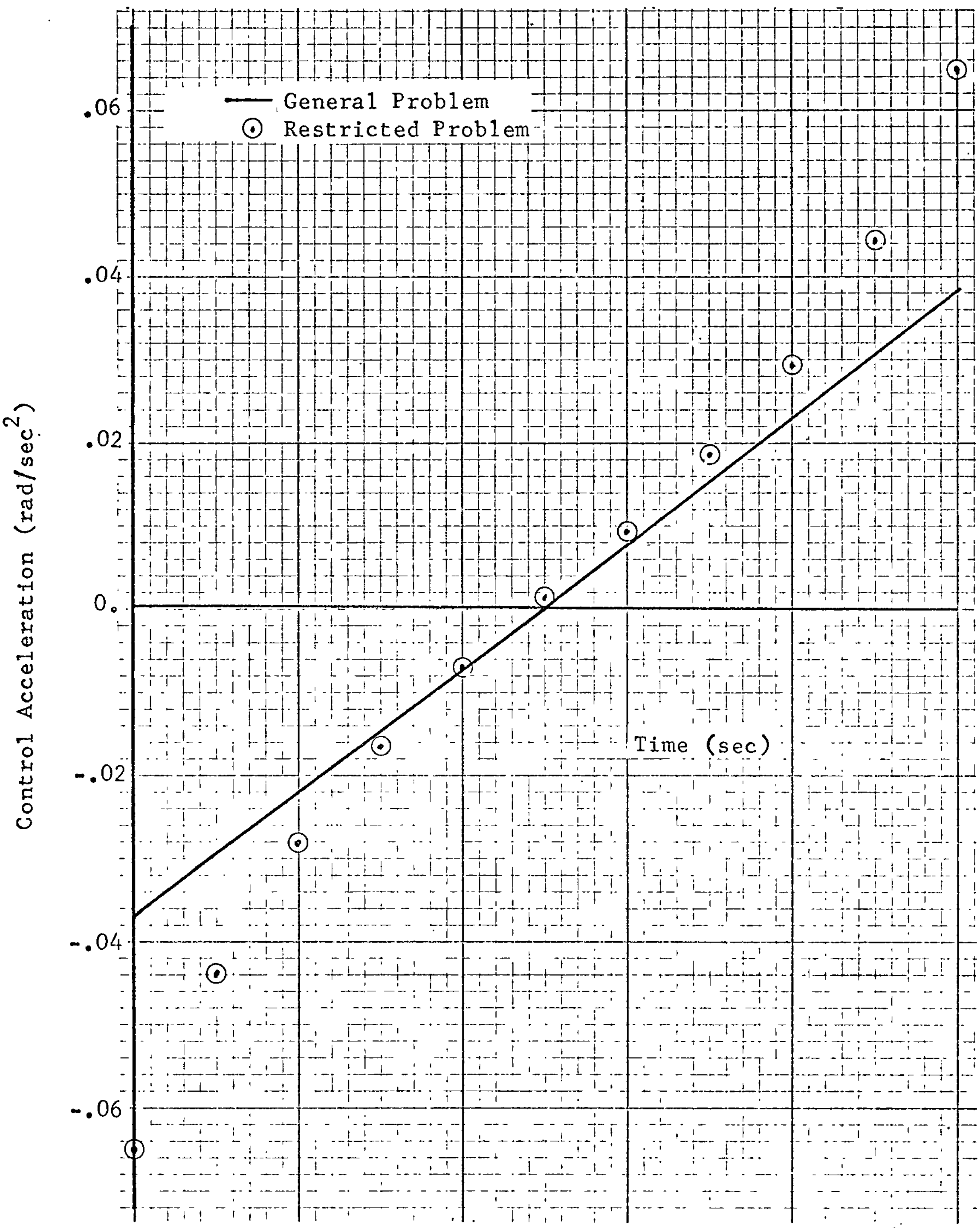


Figure 8. Case III Control Acceleration vs. Time ($\underline{b}_z$ Axis)



# CHAPTER V

# CONCLUSIONS

Two main results are presented in this paper. First, a technique is developed which permits numerical determination of the fuel optimal control for reorientation of an asymmetric vehicle. This technique could prove useful in the design of attitude control systems. Since it provides realistic information as to the actual fuel optimal control, comparisons can be made to implementable systems as an integral part of the design process. The second result of the paper is to demonstrate a simple system whose control will produce fuel usage close to that obtained by the optimal control. For three specific sets of initial conditions and mass properties, the resultant fuel consumption was shown to be reasonably close to the general optimal. Better results might be possible if a more judicious selection of the weighting parameter $\omega^2$ is made.

It should be noted that the convergence characteristics of the Newton-Raphson approach are limited for this problem. For example, the author has attempted to attain a convergent result for the physically interesting case of a dumbbell inertia configuration ($g_5 = 0.$, $g_6 = \pm 2.$, $g_7 = \mp 2.$) with no success. It appears that as the inertial cross-coupling terms become significant in the Jacobian matrix $\left[\Gamma\right]$, instability results. It should be emphasized that this is only a tentative judgement. The author is continuing with work in this area, and some result may yet be obtained. It is possible that an indirect method such as the gradient technique would yield better results, since a fairly good estimate of the general fuel optimal control is available from the restricted problem.

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# APPENDIX A

## ELEMENTS OF THE JACOBIAN MATRIX

$$
\frac{1}{2}\begin{bmatrix}
0 & y_7 & -y_6 & y_5 & y_4 & -y_3 & y_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-y_7 & 0 & y_5 & y_6 & y_3 & y_4 & -y_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
y_6 & -y_5 & 0 & y_7 & -y_2 & y_1 & y_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-y_5 & -y_6 & -y_7 & 0 & -y_1 & -y_2 & -y_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & g_5 y_7 & g_5 \lambda_6 & 0 & 0 & 0 & 0 & -1/(2I_x^2) & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1/(2I_y^2) & 0 \\
0 & 0 & 0 & 0 & g_7 y_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1/(2I_z^2) \\
0 & 0 & 0 & 0 & \lambda_4 & -\lambda_3 & \lambda_2 & 0 & y_7 & -y_6 & y_5 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \lambda_3 & \lambda_4 & -\lambda_1 & -y_7 & 0 & y_5 & y_6 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\lambda_2 & \lambda_1 & \lambda_4 & y_6 & -y_5 & 0 & y_7 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\lambda_1 & -\lambda_2 & -\lambda_3 & -y_5 & -y_6 & -y_7 & 0 & 0 & 0 & 0 \\
\lambda_4 & \lambda_3 & -\lambda_2 & -\lambda_1 & 0 & -g_7\lambda_7 & -g_6\lambda_6 & -y_4 & -y_3 & y_2 & y_1 & 0 & -g_7 y_7 & -g_7 y_6 \\
-\lambda_3 & \lambda_4 & \lambda_1 & -\lambda_2 & -g_7\lambda_7 & 0 & -g_5\lambda_5 & y_3 & -y_4 & -y_1 & y_2 & -g_5 y_7 & 0 & -g_7 y_5 \\
\lambda_2 & -\lambda_1 & \lambda_4 & -\lambda_3 & -g_6\lambda_6 & -g_5\lambda_5 & 0 & -y_2 & y_1 & -y_4 & y_3 & -g_5 y_6 & -g_6 y_5 & 0
\end{bmatrix}
$$

# APPENDIX B

## SUMMARY OF CASES

The initial and terminal conditions on the state vectors were the same for all cases:

$$\underline{y}(0) = [.4085,\ .4085,\ .4085,\ .70711,\ 0.,\ 0.,\ 0.]^T$$

$$\underline{y}(t_f) = [0.,\ 0.,\ 0.,\ 1.,\ 0.,\ 0.,\ 0.]^T$$

$$\underline{x}(0) = [\pi/2,\ 0.]^T$$

$$\underline{x}(t_f) = [0.,\ 0.]^T$$

Also, for the restricted solutions in all cases

$$\underline{n} = 1/\sqrt{3}\ \ [1.,\ 1.,\ 1.]^T$$

The remaining parameters of interest are summarized below:

| Case No. | $I_x$<br>$I_y$<br>$I_z$ | $g_5$<br>$g_6$<br>$g_7$ | $\omega^2$ | $I_n$ |
|---|---|---|---|---|
| I | 1.0<br>1.0<br>1.0 | 0.<br>0.<br>0. | 0. | 1.0000 |
| II | 1.0<br>1.2<br>1.2 | 0.<br>.33333<br>-.33333 | .13608 | 1.1333 |
| III | 1.0<br>1.1<br>1.2 | -.20000<br>.36364<br>-.16667 | .12910 | 1.1000 |

## APPENDIX C

### TABULATED NUMERICAL RESULTS

| | Time | $y_1$ | $y_2$ | $y_3$ | $y_4$ | $y_5$ | $y_6$ | $y_7$ |
|---|---|---|---|---|---|---|---|---|
| Case I: | 0. | .40825 | .40825 | .40825 | .70711 | 0. | 0. | 0. |
| | 2. | .37365 | .37365 | .37365 | .76235 | -.08732 | -.08732 | -.08732 |
| | 4. | .28132 | .28132 | .28132 | .87326 | -.13097 | -.13097 | -.13097 |
| | 6. | .15749 | .15749 | .15749 | .96208 | -.13097 | -.13097 | -.13097 |
| | 8. | .04694 | .04694 | .04694 | .99669 | -.08730 | -.08730 | -.08730 |
| | 10. | -.00005 | -.00005 | -.00005 | 1.00000 | .00004 | .00004 | .00004 |
| Case II: | 0. | .40825 | .40825 | .40825 | .70711 | 0. | 0. | 0. |
| | 2. | .37486 | .37120 | .37491 | .76233 | -.08860 | -.09133 | -.08317 |
| | 4. | .28399 | .27509 | .28491 | .87322 | -.13309 | -.13195 | -.12791 |
| | 6. | .15963 | .15131 | .16146 | .96206 | -.13307 | -.12790 | -.02372 |
| | 8. | .04763 | .04456 | .04858 | .99669 | -.08855 | -.08314 | -.06850 |
| | 10. | -.00001 | .00002 | -.00015 | 1.00000 | .00007 | .00005 | .00000 |
| Case III: | 0. | .40825 | .40825 | .40825 | .70711 | 0. | 0. | 0. |
| | 2. | .37569 | .37173 | .37354 | .76233 | -.08625 | -.09113 | -.08432 |
| | 4. | .28632 | .27614 | .28152 | .87322 | -.13147 | -.13332 | -.12816 |
| | 6. | .16217 | .15201 | .15822 | .96206 | -.13386 | -.12890 | -.13017 |
| | 8. | .04867 | .04473 | .04737 | .99669 | -.09035 | -.08349 | -.08799 |
| | 10. | -.00003 | .00001 | -.00013 | 1.00000 | .00005 | .00006 | .00000 |