

ANALYSIS OF IRRATIONAL AND TRANSCENDENTAL
TRANSFER FUNCTIONS

A Thesis
Presented to
The Faculty of the Department of Electrical Engineering
The University of Houston

In Partial Fulfillment
of the Requirements for the Degree
Master of Science in Electrical Engineering

by
Leang-San Shieh
January, 1968

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ABSTRACT

This thesis is an exposition of approximation techniques on irrational and transcendental functions. Continued fractions are used for approximating the irrational and transcendental functions. A new general matrix is then developed for the inversion of the continued fractions. Two digital computer programs for the expansion and the inversion are included.

TABLE OF CONTENTS

CHAPTER	PAGE
I. INTRODUCTION	1
II. GENERAL CONSIDERATIONS OF FUNCTION EXPANSION . .	5
III. CONTINUED FRACTION EXPANSION	9
IV. EXPANSION BY ROUTH'S ALGORITHM	15
V. CONTINUED FRACTION INVERSION	24
VI. INVERSION OF THE ROUTH ARRAY BY DIGITAL COMPUTER	36
VII. APPROXIMATIONS OF THE ERROR FUNCTION	44
VIII. APPROXIMATION OF $\frac{1}{\sqrt{s}}$	50
IX. APPROXIMATIONS OF THE DELAY FUNCTION	55
X. CONCLUSIONS	58
REFERENCES	59
APPENDIX	61

LIST OF FIGURES

FIGURE		PAGE
1.	Block diagram corresponds to continued fraction expansion	28
2.	Reduced block diagram corresponds to continued fraction expansion	28
3.	Response curves of $\frac{1}{s\sqrt{s+1}}$ (even number quotients)	46
4.	Response curves of $\frac{1}{s\sqrt{s+1}}$ (odd number quotients)	49
5.	Response curves of $\frac{1}{s\sqrt{s}}$ (even number quotients)	52
6.	Response curves of $\frac{1}{s\sqrt{s}}$ (odd number quotients)	53
7.	Response curves of closed loop system	54
8.	Response curves of $\frac{e^{-s}}{s}$	57

CHAPTER I

INTRODUCTION

Control engineers often encounter a large number of control systems which involve distributed parameters. Thermal processes, hole diffusion of transistors, electromagnetic devices and transportation lags are typical examples.^{1,2,3} Correspondingly, the mathematical descriptions in the Laplace transform domain for these elements usually contain the operator s under the radical sign or other irrational or transcendental functions. To analyze or synthesize an isolated irrational transfer function is not particularly difficult; however, when the element in question is represented by an irrational function in a closed loop system, the problem becomes very complicated. Historically, the first irrational transfer function in engineering was noticed by Heaviside.⁴ He observed that the impedance of an infinite RC cable is $\frac{1}{\sqrt{s}}$. Subsequently, many kinds of irrational or transcendental functions have been derived from physical mathematical models. Some typical ones are listed as follows:⁵

(1)	$\frac{1}{\sqrt{s}}$	$\frac{1}{\sqrt{\pi t}}$
(2)	$\frac{1}{s \sqrt{s+1}}$	$\text{erf}(\sqrt{t})$
(3)	$\frac{1}{\sqrt{s^2+1}}$	$J_0(t)$
(4)	$\frac{1}{s} \tan^{-1} \frac{1}{s}$	$S_i(t)$
(5)	e^{-s}	$u(t-1)$
(6)	$e^{-\sqrt{s}}$	$\frac{1}{2\sqrt{\pi t^3}} e^{-\frac{1}{4}t}$

If any one of these functions is contained in a closed loop system, the analysis is indeed quite tedious. One popular method of attacking the problem has been to approximate the irrational function by a rational one. A fundamental approach utilizing this technique has been to replace the irrational transfer function by a ratio of two polynomials. The historical developments of this approach will be reviewed first.

The Method Based on the Logarithmic Potential.

Lerner⁶ used a method for constructing the broad-band impedance which is similar to potential analog approximation methods. He found that a finite array of alternating poles and zeros placed along the negative real axis in the complex frequency plane produces a good approximation.

The Method Based on a Regular Newton's Process.

The main contributors are Carlson and Halijak.⁷ Their approximation is to predistort the algebraic expression $f(s) = s^n - a = 0$. The resulting approximation in real variables has the unique property of preserving upper and lower approximations to the n th root of the real number a . By using this regular Newton's process, they generate rational functions for the approximation.

The Method Based on Padé's Approximation. Stewart⁸

perhaps was the first to generalize the well known Padé's approximation by minimizing the magnitude of the complex error phasor in the steady state. He found the particular case of the Taylor approximation about zero frequency to be equivalent to the Taylor approximation in terms of the complex variable.

The Method Based on Substitution. Kilomeitseva and

Netushil⁹ conceived a novel approach to this problem. They substituted \sqrt{s} , by p . An irrational transfer function of s then becomes a rational one of p . The regular Heaviside expansion is performed on the new transfer function, and the inverse process is taken by using certain typical irrational function tables.

In reviewing the methods mentioned above, one finds that the first is for driving point impedance approximation only, while the second method is limited to fractional

capacitors. The third method has a very poor result for the transport lag case whereas the fourth method is restricted to radical functions only.

It is highly desirable that a unified approximation technique be developed by which most irrational functions can be treated in a simple manner. In addition, this approach should be digital computer-oriented.

CHAPTER II

GENERAL CONSIDERATIONS OF FUNCTION EXPANSION

To approximate an irrational function by a rational one generally involves two steps:

1. To expand the function into a convergent series.
2. To approximate the series by the first few dominant terms.

For example, one can expand the exponential function e^{-TS} into a Maclaurin series:

$$e^{-TS} = 1 - TS + \frac{T^2 S^2}{2!} - \frac{T^3 S^3}{3!} + \dots \quad (1)$$

and take the first few terms as the approximation.

One can also first rewrite the function into a fraction¹⁰

$$e^{-TS} = \frac{e^{-\frac{T}{2}S}}{e^{\frac{T}{2}S}}$$

Then expand the numerator and denominator into two series and then take the dominant parts.

$$e^{-TS} = \frac{1 - \frac{T}{2}S + \frac{\left(\frac{T}{2}\right)^2 S^2}{2!} - \frac{\left(\frac{T}{2}\right)^3 S^3}{3!} + \dots}{1 + \frac{T}{2}S + \frac{\left(\frac{T}{2}\right)^2 S^2}{2!} + \frac{\left(\frac{T}{2}\right)^3 S^3}{3!} + \dots} \quad (2)$$

Padé's method can be derived from Equation (2), and the result is somewhat better than that obtained from Equation (1).

Analogously, the method developed in this thesis can also be described by two steps:

1. To expand the function into a continued fraction.
2. To truncate the fraction and discard unimportant parts.

For example, following Auslander's approach,¹¹ one can use the differentiation technique to expand the exponential function as the following continued fraction:

$$e^{-TS} = \frac{1}{1 + \frac{TS}{1 + \frac{TS}{-2 + \frac{TS}{-3 + \frac{TS}{2 + \frac{TS}{5 + \frac{TS}{-2 + \frac{TS}{\ddots}}}}}}}} \quad (3)$$

If the parts appearing in the lower positions are discarded, one has the following familiar approximations:

$$e^{-TS} \approx \frac{1}{1 + TS}, \text{ if only one element is taken;} \quad (4)$$

$$\text{or} \quad \approx \frac{2 - TS}{2 + TS}, \text{ if only two elements are considered,}$$

etc.

The accuracy of an approximation depends on where we truncate the function, of course. In general, if more terms are taken, a better result can be obtained. In view of the availability of high speed, large capacity digital computers, one can take as many elements as desired. However, two new problems arise:

1. Expanding an irrational function becomes increasingly tedious.
2. The inversion process by which one can change the truncated continued fraction into a rational fraction becomes very laborious.

If a digital computer is used and a more accurate approximation desired, the foregoing two fundamental problems must be solved. This thesis will develop a new approach to solving these problems one by one:

1. An algorithm will be developed for the continued fraction expansion and a matrix for performing the reverse process will be given.
2. The technique will be applied to some daily encountered irrational and transcendental functions.
3. An analysis of a closed loop system with an irrational function will be shown by using the approximate transfer functions.

4. Several tables with various degrees of approximations for the general often used irrational and transcendental functions will be worked out and presented as an appendix.

CHAPTER III

CONTINUED FRACTION EXPANSION

Among the numerous forms of continued fraction expansions,^{12,13} the following one will be chosen and used as a standard form:

$$f(x) = h_0 + \frac{x}{h_1 + \frac{x}{h_2 + \frac{x}{h_3 + \frac{x}{h_4 + \frac{x}{h_5 + \ddots}}}}} \quad (5)$$

According to Lerner's theory, if the poles and zeros are alternately distributed, a good approximation can be achieved. Therefore, the continued fraction (5) usually gives a satisfactory solution.

The next problem we are facing is that if an irrational or a transcendental function is given, how do we expand it into the standard form (5)?

There are many techniques available, among which Auslander's differentiation method is a popular one. We would rather analyze the problem from a different angle.

Consider, for example, a given irrational function $\sqrt[3]{s}$. Its continued fraction form is required. Expanding the irrational function $\sqrt[3]{s}$ into Taylor series about 1, we have:

$$\begin{aligned}
\sqrt[3]{s} &= [1 + (s-1)]^{\frac{1}{3}} \\
&= 1 + \frac{s-1}{3} - \frac{(s-1)^2}{9} + \frac{5(s-1)^3}{81} - \frac{10(s-1)^4}{243} + \dots
\end{aligned} \tag{6}$$

Assume that $\sqrt[3]{s}$ can be expanded into the following continued fraction:

$$\sqrt[3]{s} = h_0 + \frac{(s-1)}{h_1 + \frac{(s-1)}{h_2 + \frac{(s-1)}{h_3 + \frac{(s-1)}{\ddots}}}} \tag{7}$$

where h_i are undetermined constants.

Equating (6) and (7), we obtain

$$1 + \frac{s-1}{3} - \frac{(s-1)^2}{9} + \frac{5(s-1)^3}{81} \dots = h_0 + \frac{(s-1)}{h_1 + \frac{(s-1)}{h_2 + \frac{(s-1)}{\ddots}}} \tag{8}$$

Let $s=1$, h_0 is immediately found; or $h_0=1$. Substituting h_0 into (8), we rewrite

$$\frac{\cancel{(s-1)}}{3} - \frac{(s-1)^2}{9} + \frac{5(s-1)^{\cancel{3}}}{81} - \dots = \frac{\cancel{(s-1)}}{h_1 + \frac{(s-1)}{h_2 + \frac{(s-1)}{\ddots}}} \tag{8a}$$

or

$$\frac{1}{\frac{1}{3} - \frac{1}{9}(s-1) + \frac{5}{81}(s-1)^2 \dots} = h_1 + \frac{(s-1)}{h_2 + \frac{(s-1)}{\ddots}} \quad (8b)$$

Again, setting $s=1$, we have $h_1=3$. Repeating these processes give

$$h_2 = 1, h_3 = \frac{9}{2}, h_4 = \frac{4}{5} \dots \quad (9)$$

Consequently, the complete expansion has been obtained.

$$\sqrt[3]{s} = 1 + \frac{(s-1)}{3 + \frac{(s-1)}{1 + \frac{9}{2} + \frac{(s-1)}{\frac{4}{5} + \frac{(s-1)}{\ddots}}}} \quad (10)$$

It is interesting to note that the procedure we just developed is very similar to that of Heaviside's partial fraction expansion. If we desire to expand a transcendental function into a continued fraction expansion, the same technique can be used. For example, assume

$$e^S = h_0 + \frac{s}{h_1 + \frac{s}{h_2 + \frac{s}{h_3 + \ddots}}} \quad (11)$$

We can easily rewrite the left hand side of (11) into a series:

$$1 + s + \frac{s^2}{2!} + \frac{s^3}{3!} + \frac{s^4}{4!} + \dots = h_0 + \frac{s}{h_1 + \frac{s}{h_2 + \frac{s}{h_3 + \dots}}} \quad (12)$$

Setting $s=0$, we have $h_0=1$. Then a similar procedure is used and the result is found as follows:

$$e^s = 1 + \frac{s}{1 + \frac{s}{-2 + \frac{s}{-3 + \frac{s}{2 + \dots}}}} \quad (13)$$

In control systems studies, some particular irrational functions are usually of interest. The following examples are well known.

$$\begin{aligned} \text{(i)} \quad & \sqrt{s} \\ \text{(ii)} \quad & \sqrt{s+1} \\ \text{(iii)} \quad & \sqrt{s} + 1 \\ \text{(iv)} \quad & \sqrt{s^2 + 1} \end{aligned} \quad (14)$$

These functions can be considered as special forms of $\sqrt{s+k}$. If we want to approximate this function by a continued fraction, a somewhat simpler technique can be applied.

$$\begin{aligned} \text{Let } \sqrt{s+k} &= B \\ \text{or } s + k &= B^2. \end{aligned} \quad (15)$$

Adding B to both sides, we have $s+k+B = B(1+B)$.

Rewriting gives

$$\begin{aligned}
 B &= \frac{s+k+B}{1+B} \\
 &= \frac{1+B+s+(k-1)}{1+B} \\
 &= 1 + \frac{s+(k-1)}{1+B}
 \end{aligned} \tag{15a}$$

or

$$\sqrt{s+k} = 1 + \frac{s+(k-1)}{1+B} \tag{15b}$$

Continually substituting (15a) into (15b), we obtain

$$\begin{aligned}
 \sqrt{s+k} &= 1 + \frac{s+(k-1)}{1+1+\frac{s+(k-1)}{1+B}} \\
 &= 1 + \frac{s+(k-1)}{2+\frac{s+(k-1)}{2+\frac{s+(k-1)}{2+\frac{s+(k-1)}{2+\frac{s+(k-1)}{\ddots}}}}}
 \end{aligned} \tag{16}$$

If $k=0$, Equation (16) becomes

$$\begin{aligned}
 \sqrt{s} &= 1 + \frac{s-1}{2+\frac{s-1}{2+\frac{s-1}{2+\frac{s-1}{\ddots}}}}
 \end{aligned} \tag{17}$$

If $k=1$, Equation (16) is reduced to

$$\sqrt{s+1} = 1 + \frac{s}{2 + \frac{s}{2 + \frac{s}{2 + \frac{s}{\ddots}}}} \quad (18)$$

The third function of (14) is easily seen to be

$$\sqrt{s} + 1 = 2 + \frac{s-1}{2 + \frac{s-1}{2 + \frac{s-1}{2 + \frac{s-1}{\ddots}}}} \quad (19)$$

Replacing s by s^2 , into (16), and letting $k=1$, we have

$$\sqrt{s^2+1} = 1 + \frac{s^2}{2 + \frac{s^2}{2 + \frac{s^2}{2 + \frac{s^2}{\ddots}}}} \quad (20)$$

The methods we developed so far are better than the existing techniques; however, they are still not suitable for digital computers. If high accuracy is desired, we have to take more elements. Then to find certain algorithms for performing the expansion process is not only highly desirable but more important. We will, based on the principles of this section, present an algorithm in the next chapter.

CHAPTER IV

EXPANSION BY ROUTH'S ALGORITHM

Expanding a ratio of two polynomials into a continued fraction by using Routh's algorithm was first done by Frank¹⁵ and later developed by Wall.¹² However, they restrict themselves to Stieltjes' form and to Caue's first form. Neither formulation can be used in a low pass filter transfer function. Thus, their forms and algorithms cannot be applied in expanding a ratio of two polynomials or an irrational function into a continued fraction like (5).

We will develop a new algorithm so that not only can a ratio of two polynomials be converted, but also an irrational function can be easily expanded into a continued fraction.

For deriving the algorithm, we consider the following transfer function:

$$\begin{aligned} F(s) &= \frac{g_2(s)}{g_1(s)} \\ &= \frac{A_{2,n} s^{n-1} + A_{2,n-1} s^{n-2} + \dots + A_{21}}{A_{1,n+1} s^n + A_{1,n} s^{n-1} + \dots + A_{11}} \quad (21) \end{aligned}$$

First, rearrange the polynomial of (21) into ascending order.

$$\begin{aligned}
 F(s) &= \frac{g_2(s)}{g_1(s)} \\
 &= \frac{A_{21} + A_{22}S + A_{23}S^2 + \dots + A_{2,n}S^{n-1}}{A_{11} + A_{12}S + A_{13}S^2 + \dots + A_{1,n+1}S^n} \quad (21a)
 \end{aligned}$$

It will be shown that since most daily encountered control systems act as low pass filters, the truncated continued function obtained after this rearrangement gives excellent approximation. Performing synthetic division on (21a) gives

$$\begin{aligned}
 F(s) &= \frac{g_2(s)}{g_1(s)} \\
 &= \frac{1}{\frac{A_{11}}{A_{21}} + \frac{\frac{A_{21}A_{12} - A_{11}A_{22}}{A_{21}}S + \frac{A_{21}A_{13} - A_{11}A_{23}}{A_{21}}S^2}{A_{21} + A_{22}S + A_{23}S^2 + \dots + A_{2,n}S^{n-1}}} \\
 &= \frac{1}{\frac{A_{11}}{A_{21}} + \frac{A_{31}S + A_{32}S^2 + A_{33}S^3 + \dots}{A_{21} + A_{22}S + A_{23}S^2 + \dots}} \\
 &= \frac{1}{\frac{A_{11}}{A_{21}} + \frac{S}{\frac{A_{21}}{A_{31}} + \frac{\frac{A_{22}A_{31} - A_{32}A_{21}}{A_{31}}S + \dots}{A_{31} + A_{32}S + \dots}}} \\
 &= \frac{1}{\frac{A_{11}}{A_{21}} + \frac{S^2}{\frac{A_{21}}{A_{31}} + \frac{A_{41}S + A_{42}S^2 + \dots}{A_{31} + A_{32}S + \dots}}} \quad (21b)
 \end{aligned}$$

It is seen that the manipulation of A_{31}, A_{32}, \dots and A_{41}, A_{42}, \dots can be written in the following formula:

$$A_{j,k} = A_{j-2,k+1} - \frac{A_{j-1,1} A_{j-1,k+1}}{A_{j-1,1}} \quad \begin{array}{l} j = 3, 4, \dots, n+1 \\ k = 1, 2, \dots \end{array} \quad (22)$$

and

$$h_p = \frac{A_{p,1}}{A_{(p+1),1}} \quad p = 1, 2, 3, \dots \quad (23)$$

If the coefficients of $g_1(s)$ and $g_2(s)$ in (21a) are arranged in the first row and the second row respectively of the Routh array Formula (22) coincides with the traditional method for evaluating the elements of the Routh Array.¹⁶ Therefore, the continued fraction expansion can be simplified by the Routh Algorithm as follows:

$$\begin{aligned} h_1 &= \frac{A_{11}}{A_{21}}, \begin{array}{ccc} A_{11} & A_{12} & A_{13} & A_{14} \dots \\ A_{21} & A_{22} & A_{23} & A_{24} \dots \end{array} \\ h_2 &= \frac{A_{21}}{A_{31}}, A_{31} = \frac{A_{21}A_{12} - A_{11}A_{22}}{A_{21}}, A_{32} = \frac{A_{21}A_{13} - A_{11}A_{23}}{A_{21}} \dots \\ h_3 &= \frac{A_{31}}{A_{41}}, A_{41} = \frac{A_{22}A_{31} - A_{32}A_{21}}{A_{31}}, A_{42} = \frac{A_{23}A_{31} - A_{21}A_{33}}{A_{31}} \dots \quad (24) \\ &\dots \end{aligned}$$

Clearly, the elements of continued fraction (5), h_i , may be obtained by the quotients of members of the first column in Routh's array.

or

$$\begin{aligned}
 F(s) &= \frac{1}{\frac{A_{11}}{A_{21}} + \frac{\frac{A_{21}}{A_{31}} + \frac{A_{31}}{A_{41}} + \dots}{s}} \\
 &= \frac{1}{h_1 + \frac{s}{h_2 + \frac{s}{h_3 + \dots}}}
 \end{aligned} \tag{25}$$

An alternate algorithm which is somewhat simpler than the regular one is developed as follows. First, identify the rows of Routh's array with

$$\begin{array}{cccccl}
 A_{11} & A_{12} & A_{13} & A_{14} \cdots & \leftarrow g_1(s) \\
 A_{21} & A_{22} & A_{23} & A_{24} \cdots & \leftarrow g_2(s) \\
 A_{31} & A_{32} & A_{33} & A_{34} \cdots & \leftarrow g_3(s)
 \end{array} \tag{26}$$

$g_1(s)$, $g_2(s)$, \cdots and $g_{k+1}(s)$, respectively, as indicated.

Expanding gives

$$\begin{aligned}
 F(s) &= \frac{g_2(s)}{g_1(s)} \\
 &= \frac{1}{h_1 + \frac{s g_3(s)}{g_2(s)}} \\
 &= \frac{1}{h_1 + \frac{s}{\frac{s g_4(s)}{g_3(s)}}}
 \end{aligned}$$

then

$$\begin{aligned}
 g_1(s) &= h_1 g_2(s) + s g_3(s) \\
 g_2(s) &= h_2 g_3(s) + s g_4(s) \\
 g_3(s) &= h_3 g_4(s) + s g_5(s)
 \end{aligned}
 \tag{27}$$

Two general formulas can be written as

$$\begin{aligned}
 g_m(s) &= h_m g_{m+1}(s) + s g_{m+2}(s) \\
 \text{and } h_m &= \frac{A_{m,1}}{A_{m+1,1}}
 \end{aligned}
 \tag{28}$$

$m=1,2,3, \dots$

where m is the n th row of the Routh Array. The details of applying the algorithm are:

1. Find $h_1 = \frac{A_{11}}{A_{21}}$.
2. Each element in row 2 of the Routh array is multiplied by h_1 to form a new row (2a).
3. Subtract the new row (2a) from row 1 forming row (2b).
4. Shift all elements of row (2b) one column left, dropping the zero first element to form row 3.

A numerical example is given for illustrating the above steps.

Row 1	720	702	71	1	$\leftarrow g_1(s)$
$h_1 = \frac{720}{360} = 2$					
Row 2	360	171	10	0	$\leftarrow g_2(s)$
Row (2a)	720	342	20	0	
Row (2b)	0	360	51	1	
Row 3	360	51	1		$\leftarrow g_3(s)$

How does one apply this algorithm in a practical problem? Consider the following transfer function:

$$F(s) = \frac{360 + 171s + 10s^2}{720 + 702s + 71s^2 + s^3} \quad (29)$$

and find its continued fraction.

Recognize the coefficients and substitute into (26), and we have

$h_1 = \frac{720}{360} = 2$	720	702	71	1	$\leftarrow g_1(s)$
	360	171	10		$\leftarrow g_2(s)$

Using (28) yields

$$g_3(s) = \frac{1}{s} \left[g_1(s) - h_1 g_2(s) \right] = 360 + 51s + s^2$$

$$h_2 = \frac{360}{360} = 1$$

$$g_4(s) = \frac{1}{s} \left[g_2(s) - h_2 g_3(s) \right] = 120 + 9s$$

$$h_3 = \frac{360}{120} = 3$$

$$g_5(s) = \frac{1}{s} \left[g_3(s) - h_3 g_4(s) \right] = 24 + s$$

$$h_4 = \frac{120}{24} = 5$$

$$g_6(s) = \frac{1}{s} \left[g_4(s) - h_4 g_5(s) \right] = 4$$

$$h_5 = \frac{24}{4} = 6$$

$$g_7(s) = \frac{1}{s} \left[g_5(s) - h_5 g_6(s) \right] = 1$$

$$h_6 = \frac{4}{1} = 4$$

Finally, the required continued fraction is obtained.

$$F(s) = \frac{1}{2 + \frac{s}{1 + \frac{s}{3 + \frac{s}{5 + \frac{s}{6 + \frac{s}{4}}}}}}$$

Based on the new algorithm, a digital computer program has been written. The input nomenclature are listed as follows:

ND = the total terms of the first row of the Routh Array.

A(1)---A(ND) = the numbers of the first row of the Routh Array.

NN = the total terms of the second row of the Routh Array, where NN is equal to ND or ND-1.

B(1)---A(NN) = the numbers of the second row of the Routh Array.

We read the results h(1)---h(ND+NN-1) from the output of the computer. These values are the required quotients of the continued fraction. The program is printed on the next page.

```

C             MAIN PROGRAM BEGINS HERE
      DIMENSION A(200),B(200),H(500)
1000 READ(5,501) ND,(A(I),I=1,ND)
      READ(5,501) NN,(B(I),I=1,NN)
      501 FORMAT(I5/(5F16.6))
      WRITE(6,601)
      601 FORMAT(/5X,12HA(1)...A(ND))
      WRITE(6,602) (A(I),I=1,ND)
      602 FORMAT(//((15X,(6F16.6)))
      WRITE(6,603)
      603 FORMAT(/5X,12HB(1)...B(NN))
      WRITE(6,602) (B(I),I=1,NN)
      L=0
      IF(ND.EQ.NN) GO TO 1
      IF(ND.NE.NN+1) GO TO 1000
      2 L=L+1
      H(L)=A(1)/B(1)
      WRITE(6,602) (A(I),I=1,ND)
      WRITE(6,604) L,H(L)
      604 FORMAT(/2X,2HH(,I2,3H) =,E16.6)
      B(ND)=0.
      ND=ND-1
      DO 10 I=1,ND
      A(I)=B(I)
      I1=I+1
      10 B(I)=A(I1)-B(I1)*H(L)
      1 L=L+1
      H(L)=A(1)/B(1)
      WRITE(6,602) (A(I),I=1,ND)
      WRITE(6,604) L,H(L)
      IF(ND.EQ.1) GO TO 3
      ND=ND-1
      DO 20 I=1,ND
      A(I)=B(I)
      I1=I+1
      20 B(I)=A(I1)-B(I1)*H(L)
      ND=ND+1
      A(ND)=B(ND)
      GO TO 2
      3 WRITE(6,602) B(1)
      GO TO 1000
      END

```

CHAPTER V

CONTINUED FRACTION INVERSION

It is known that a rational function, when expanded, becomes a finite continued fraction, while an irrational function would become an infinite one. For convenience, write the infinite continued fraction

$$\begin{array}{rcl}
 h_0 + \frac{x}{h_1 + \frac{x}{h_2 + \frac{x}{h_3 + \frac{x}{\ddots}}}} & & (30)
 \end{array}$$

in the following form,

$$h_0 + x[h_1, h_2, h_3, \dots, h_k, \dots] \quad (30a)$$

and a finite continued fraction

$$\begin{array}{rcl}
 h_0 + \frac{x}{h_1 + \frac{x}{h_2 + \frac{x}{\ddots}}}} & & (31) \\
 & & \vdots \\
 & & h_{k-1} + \frac{x}{h_k}
 \end{array}$$

in the following expression:

$$h_0 + x[h_1, h_2, \dots, h_k] \quad (31a)$$

The general approximation technique is to truncate an infinite continued fraction into a finite one. Then change the latter into a ratio of two polynomials:

$$f(x) = \frac{g_2(x, h_0, h_1, h_2, \dots, h_k)}{g_1(x, h_0, h_1, h_2, \dots, h_k)} \quad (32)$$

To convert a truncated continued fraction (31) into a ratio of two polynomials, or transfer function (32) is called the inversion of continued fraction.

The inversion process seems to be very simple and straightforward; however, when k is high, the operations become very laborious. In this section, we will particularly develop a matrix and some related algorithms to simplify this tedious problem.

Rewrite (21)

$$f(s) = \frac{A_{2,1} + A_{22}s + A_{23}s^2 + \dots + A_{2,n}s^{n-1}}{A_{1,1} + A_{12}s + A_{13}s^2 + \dots + A_{1,n+1}s^n} \quad (33)$$

and then expand it

$$f(s) = \frac{1}{h_1 + \frac{s}{h_2 + \frac{s}{h_3 + \frac{s}{\ddots \frac{s}{h_{2n}}}}}} \quad (33a)$$

An alternate form is easily written as,

$$f(s) = \frac{1}{h_1 + \frac{1}{\frac{h_2}{s} + \frac{1}{h_3 + \frac{1}{\ddots \frac{h_{2n}}{s}}}}} \quad (33b)$$

Equation (33b) can be represented by the block diagram of a physical system shown in Figure 1. If a state space formulation is used, the equations in the time domain are written as follows:

$$\begin{aligned} \dot{z}_1 &= -h_2 h_1 z_1 - h_4 h_1 z_2 - h_6 h_1 z_3 \dots - h_{2n} h_1 z_n + r \\ \dot{z}_2 &= -h_2 h_1 z_1 - h_4 (h_1 + h_3) z_2 - h_6 (h_1 + h_3) z_3 \dots - h_{2n} (h_1 + h_3) z_n + r \\ \dot{z}_3 &= -h_2 h_1 z_1 - h_4 (h_1 + h_3) z_2 - h_6 (h_1 + h_3 + h_5) z_3 \dots - h_{2n} (h_1 + h_3 + h_5) z_n + r \\ &\dots \\ \dot{z}_n &= -h_2 h_1 z_1 - h_4 (h_1 + h_3) z_2 - h_6 (h_1 + h_3 + h_5) z_3 \dots \\ &\quad - h_{2n} (h_1 + h_3 + \dots + h_{2n-1}) z_n + r \end{aligned}$$

where z_i are state variables and r is the input.

A corresponding matrix form is

$$\dot{[z]}_1 = [H]_1 [z] + [D]_1 r \quad (34)$$

where

$$[H]_1 = - \begin{bmatrix} h_2 h_1 & h_4 h_1 & h_6 h_1 & \dots h_{2n} h_1 \\ h_2 h_1 & h_4 (h_1 + h_3) & h_6 (h_1 + h_3) & \dots h_{2n} (h_1 + h_3) \\ h_2 h_1 & h_4 (h_1 + h_3) & h_6 (h_1 + h_3 + h_5) & \dots h_{2n} (h_1 + h_3 + h_5) \\ \cdot & \cdot & h_6 (h_1 + h_3 + h_5) & \dots \cdot \\ \cdot & \cdot & \cdot & \cdot \\ h_2 h_1 & h_4 (h_1 + h_3) & h_6 (h_1 + h_3 + h_5) & \dots h_{2n} (h_1 + h_3 + \dots + h_{2n-1}) \end{bmatrix}$$

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \cdot \\ \cdot \\ \dot{z}_n \end{bmatrix} \quad \text{and} \quad [D]_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ \cdot \\ \cdot \\ 1 \end{bmatrix} \quad (35)$$

Figure 1 can be interpreted in two ways: (1) It is an equivalent block diagram of $\frac{g_2(s)}{g_1(s)}$ in the frequency domain; and (2) It is a state diagram of (35) in the time domain.

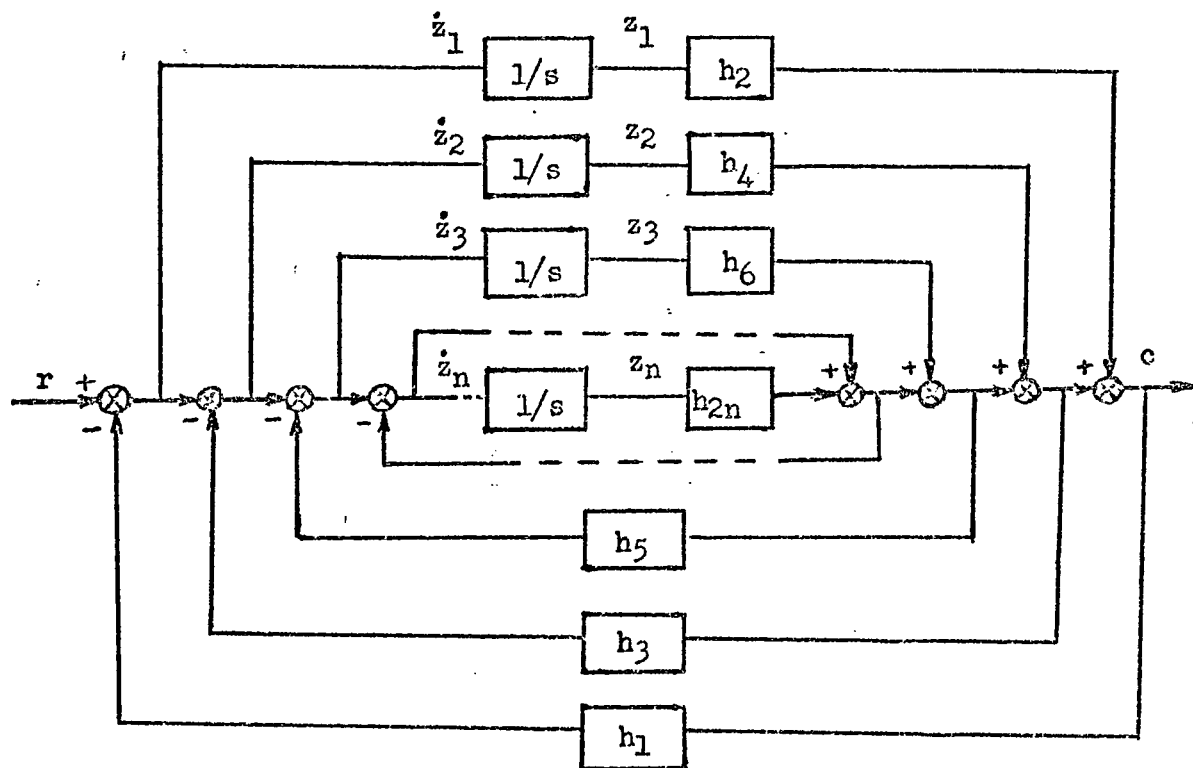


Fig. 1. Block diagram corresponds to continued fraction expansion.

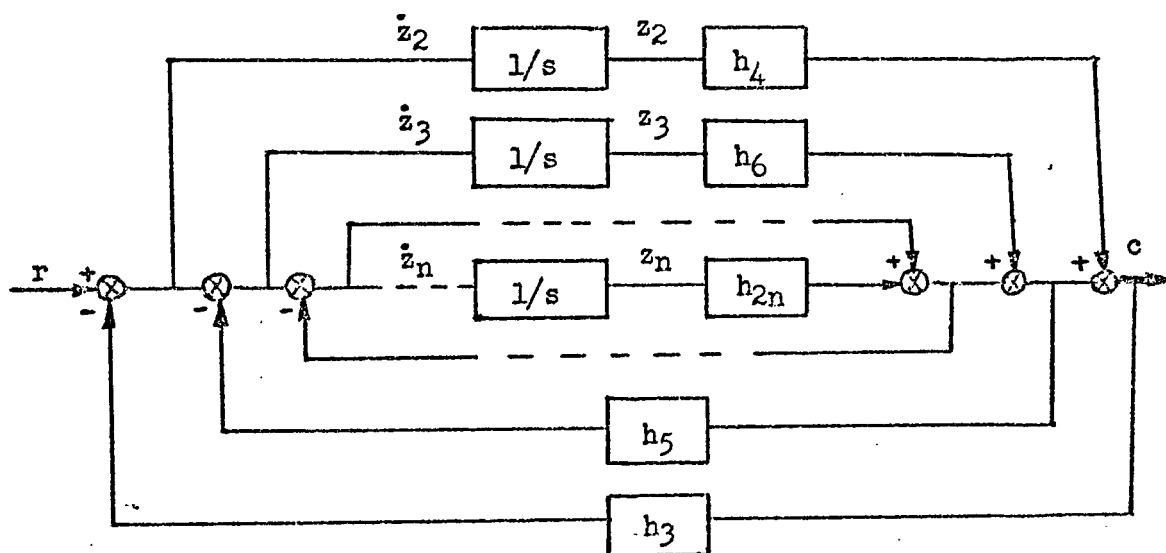


Fig. 2. Reduced block diagram corresponds to continued fraction expansion.

Of course, the determinant of (35), or the following expression

$$\det \{ s[I] - [H]_1 \} \quad (36)$$

is the denominator of the transfer function, or the characteristic equation of the system, $g_1(s)$.

If we omit the outer feedback and feed forward loops in Figure 1, we would obtain a configuration as shown in Figure 2. A corresponding state equation can be written

$$[\dot{z}]_3 = [H]_3 [z]_3 + [D]_3 r$$

where

$$[H]_3 = - \begin{bmatrix} h_4 h_3 & h_6 h_3 & h_8 h_3 & \dots & h_{2n} h_3 \\ h_4 h_3 & h_6 (h_3 + h_5) & h_8 (h_3 + h_5) & \dots & h_{2n} (h_3 + h_5) \\ h_4 h_3 & h_6 (h_3 + h_5) & h_8 (h_3 + h_5 + h_7) & \dots & h_{2n} (h_3 + h_5 + h_7) \\ \cdot & \cdot & \cdot & \cdot \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \cdot & \cdot \\ h_4 h_3 & h_6 (h_3 + h_5) & h_8 (h_3 + h_5 + h_7) & \dots & h_{2n} (h_3 + h_5 + \dots + h_{2n-1}) \end{bmatrix} \quad (37)$$

$$[\dot{z}]_3 = \begin{bmatrix} \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \\ \cdot \\ \cdot \\ \dot{z}_n \end{bmatrix} \quad \text{and} \quad [D]_3 = \begin{bmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ 1 \end{bmatrix} \quad (38)$$

The determinant of (37) would give us the elements of the third row of Routh's Array, or $g_3(s)$.

After $g_1(s)$ and $g_3(s)$ are found, $g_2(s)$ can be evaluated by substituting the values of elements of $g_1(s)$ and $g_3(s)$ into (28).

If $[H]_1, [H]_3, \dots$ are extended further, the following general formula is obtained:

$$[H]_m = - \begin{bmatrix} h_{m+1}h_m & h_{m+3}h_m & \dots & h_{l-1}h_m \\ h_{m+1}h_m & h_{m+3}(h_m+h_{m+2}) & \dots & h_{l-1}(h_m+h_{m+2}) \\ h_{m+1}h_m & h_{m+3}(h_m+h_{m+2}) & \dots & h_{l-1}(h_m+h_{m+2}+h_{m+4}) \\ \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \dots & \cdot \\ h_{m+1}h_m & h_{m+3}(h_m+h_{m+2}) & \dots & h_{l-1}(h_m+h_{m+2}+\dots+h_{l-2}) \end{bmatrix} \quad (38)$$

where m is the required m th row of the Routh Array.

l is the total number of rows of the Routh Array, or $k+1$.

Several observations may be made:

- (1) The elements in the state matrix $[H]_m$ are simple combinations of the quotients obtained from the continued fraction expansion.
- (2) The element appearing in a position lower than the diagonal has the same value as the element at the diagonal.

(3) The element which appeared in the upper diagonal can be expressed in terms of quotients in a very regular way.

(4) $[H]_m$ matrix is a negative square matrix. The dimension of $[H]_m$ is $\frac{l-m}{2} \times \frac{l-m}{2}$. (When l is odd, m should be odd also; when l is even, m should be even).

For illustrating the procedure of inversion, consider the examples.

Given: the first column of Routh's Array:

$$\begin{aligned}
 A_{11} &= 720 \\
 A_{21} &= 360 \\
 A_{31} &= 360 \\
 A_{41} &= 120 \\
 A_{51} &= 24 \\
 A_{61} &= 4 \\
 A_{71} &= 1
 \end{aligned}
 \tag{39}$$

Find: the first two rows of Routh's Array.

From the given information, we have

$$\begin{aligned}
 l &= 7 \\
 \text{and } h_1 &= 2, \\
 h_2 &= 1, \\
 h_3 &= 3, \\
 h_4 &= 5, \\
 h_5 &= 6, \\
 h_6 &= 4.
 \end{aligned}
 \tag{40}$$

If the first row is required, taking $m=1$, and substituting (40) into (38), we have

$$\begin{aligned}
 [H]_1 &= - \begin{bmatrix} h_2 h_1 & h_4 h_1 & h_6 h_1 \\ h_2 h_1 & h_4 (h_1 + h_3) & h_6 (h_1 + h_3) \\ h_2 h_1 & h_4 (h_1 + h_3) & h_6 (h_1 + h_3 + h_5) \end{bmatrix} \\
 &= - \begin{bmatrix} 2 & 10 & 8 \\ 2 & 25 & 20 \\ 2 & 25 & 44 \end{bmatrix}
 \end{aligned}$$

Then $g_1(s)$ is found by evaluating the characteristic equation.

$$g_1(s) = \det \{ s[I] - [H]_1 \} = 720 + 702s + 71s^2 + s^3 \quad (41)$$

The third row is found in a similar way, or

$$\begin{aligned}
 [H]_3 &= - \begin{bmatrix} h_4 h_3 & h_6 h_3 \\ h_4 h_3 & h_6 (h_3 + h_5) \end{bmatrix} \\
 &= - \begin{bmatrix} 15 & 12 \\ 15 & 36 \end{bmatrix}
 \end{aligned}$$

$$\text{The determinant of } \{ s[I] - [H]_3 \} = 360 + 51s + s^2 \quad (42)$$

By substituting (41) and (42) into (28), we find

$$g_2(s) = 360 + 171s + 10s^2.$$

$g_1(s)$ and $g_2(s)$ are the required first two rows of the Routh Array.

Example 2

Given: the first column of Routh's Array.

$$A_{11} = 720$$

$$A_{21} = 720$$

$$A_{31} = 360$$

$$A_{41} = 360$$

$$A_{51} = 120$$

$$A_{61} = 24$$

$$A_{71} = 4$$

$$A_{81} = 1$$

(43)

Find: the first two rows of Routh's Array.

In this example, we have

$$l = 8$$

$$\text{and } h_1 = 1$$

$$h_2 = 2$$

$$h_3 = 1$$

$$h_4 = 3$$

$$h_5 = 5$$

$$h_6 = 6$$

$$h_7 = 4$$

(44)

If the first row is required, taking $m=2$, and substituting (44) into (38), we have

$$\begin{aligned}
 [H]_2 &= - \begin{bmatrix} h_3 h_2 & h_5 h_2 & h_7 h_2 \\ h_3 h_2 & h_5 (h_2 + h_4) & h_7 (h_2 + h_4) \\ h_3 h_2 & h_5 (h_2 + h_4) & h_7 (h_2 + h_4 + h_6) \end{bmatrix} \\
 &= - \begin{bmatrix} 2 & 10 & 8 \\ 2 & 25 & 20 \\ 2 & 25 & 44 \end{bmatrix}
 \end{aligned}$$

Then $g_2(s)$ is found by evaluating the characteristic equation

$$\begin{aligned}
 g_2(s) &= \det \{ s[I] - [H]_2 \} \\
 &= 720 + 702s + 71s^2 + s^3
 \end{aligned} \tag{45}$$

Secondly, let $m=4$, we have

$$\begin{aligned}
 [H]_4 &= - \begin{vmatrix} h_5 h_4 & h_7 h_4 \\ h_5 h_4 & h_7 (h_4 + h_6) \end{vmatrix} \\
 &= - \begin{bmatrix} 15 & 12 \\ 15 & 36 \end{bmatrix}
 \end{aligned}$$

$$\begin{aligned}
 g_4(s) &= \det \{ s[I] - [H]_4 \} \\
 &= 360 + 51s + s^2
 \end{aligned} \tag{46}$$

By substituting (45) and (46) into (28), we find

$$g_1(s) = 720 + 1062s + 242s^2 + 11s^3$$

Functions $g_1(s)$ and $g_2(s)$ are the required rows of the Routh Array.

CHAPTER VI

INVERSION OF THE ROUTH ARRAY BY DIGITAL COMPUTER

Let us summarize the nature of $[H]_m$ from the calculation viewpoint.

- (1) $[H]_m$ matrix is a negative square matrix.

The dimension of $[H]_m$ is $\frac{l-m}{2} \times \frac{l-m}{2}$.

When l is odd, m should be odd; and when l is even, m is even.

- (2) Each element of $[H]_m$ is the product of two elements, A and B. We define A and B:

A (K,P) is an element at K row and P column.

B (K,P) is an element at K row and P column also.

$$A (K,P) = h(m+2 \cdot p-1)$$

$$B (K,P) = B (P,K) \text{ and}$$

$$(a) B (K,K) = \sum_{j=1}^k h(m+2 \cdot (j-1))$$

$$= B (P,P)$$

$$= \sum_{j=1}^P h(m+2 \cdot (j-1)), \text{ if } k=p \quad (47)$$

$$(b) B (K,P) = B (P,P) \text{ if } k > p$$

$$(c) B (K,P) = B (K,K) \text{ if } k < p$$

In order to set up a computer program, the general formula (28) should be translated into a computer oriented form.

Let us define $\Delta_m = \det \{ S[I] - [H]_m \}$ (48)

$$\Delta_{m+2} = \det \{ S[I] - [H]_{m+2} \} \quad (49)$$

$$L = A_{\ell,1} = \frac{A_{1,1}}{h_1 h_2 \cdots h_{\ell-1}} \quad (50)$$

$$h_p = \frac{A_{p,1}}{A_{(p+1),1}}, \quad p = 1, 2, \dots, (\ell-1), \text{ and } h_p \neq 0 \quad (51)$$

where m is the m th row of the Routh Array

ℓ is the total rows of the Routh Array

or the total given quotients of the continued fraction plus one.

L is a normalization constant.

The general formulas are written accordingly:

(1) when ℓ is odd, $m = \text{odd}; 1, 3, 5, \dots, \ell-2$.

$$\text{Then } g_m(s) = L \cdot \Delta_m \quad (52)$$

$$g_{m+1}(s) = \frac{L}{h_m} \cdot [\Delta_m - s \cdot \Delta_{m+2}] \quad (53)$$

(2) when ℓ is even, $m = \text{even}; 2, 4, 6, \dots, \ell-2$.

$$\text{Then } g_m(s) = L \cdot \Delta_m \quad (54)$$

$$g_{m-1}(s) = \frac{L}{h_m} \cdot [(h_{m-1} h_m + s) \Delta_m - s^2 \Delta_{m+2}]$$

(3) $\Delta_\ell = 1$, $g_\ell(s) = L$, and $g_{\ell-1}(s) = h_{\ell-1} \cdot g_\ell(s)$ (55)

If the inverse process is wanted, we simply substitute

$m=1$ in (52), (53) when the number of quotients is even.

When the number of quotients is odd, substitute $m=2$ into

(54) and (55).

The required function is then

$$f(s) = \frac{g_2(s)}{g_1(s)}$$

In performing (48) or (49), we find that the calculation is tedious when the order is high. In order to obtain the characteristic equation in a systematic way, we form the Newton formula into a matrix form. For clarification, consider the characteristic equation

$$\det \{ s[I] - [H]_m \} = s^n + A_1 s^{n-1} + A_2 s^{n-2} + \dots + A_{n-1} s + A_n \quad (56)$$

The Newton formulas are written as follows:

$$\begin{aligned} A_1 &= -T_1 \\ A_2 &= -\frac{1}{2}(A_1 T_1 + T_2) \\ &\cdot \quad \cdot \quad \cdot \end{aligned} \quad (57)$$

$$A_n = -\frac{1}{n}(A_{n-1} T_1 + A_{n-2} T_2 + \dots + A_1 T_{n-1} + T_n)$$

where $T_j = \text{Trace } [H]_m^j \quad j = 1, 2, \dots, n$

$$= \sum_{i=1}^n H_{ii}^j$$

Equation (57) in matrix form is:

$$\begin{bmatrix} A_1 \\ 2A_2 \\ 3A_3 \\ \vdots \\ nA_n \end{bmatrix} = \begin{bmatrix} 1 & & & & & & 0 \\ A_1 & 1 & & & & & \\ A_2 & A_1 & \cdot & & & & \\ \cdot & \cdot & A_2 & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\ A_{n-1} & A_{n-2} & \cdot & A_2 & A_1 & 1 \end{bmatrix} \begin{bmatrix} -T_1 \\ -T_2 \\ -T_3 \\ \cdot \\ \cdot \\ -T_n \end{bmatrix} \quad (58)$$

Equation (58) can be considered as a series of algorithmic equations and can be solved by computer easily.

Basing on Equations (48)-(58), we write a digital computer program for finding the inversion of the continued fraction.

The input symbols are:

NH = Numbers of total quotients.

H(1)---H(NH) = quotients of the continued fraction
or the ratio of $\frac{A_{p,1}}{A_{(p+1),1}}$ $p=1,2---$

NR = How many rows of the Routh Array are required.

M(1)---M(NR) = Any required rows of the Routh Array.

KS = 1 for inverse of continued fraction.

KS = 0 for inverse of the Routh Array.

AO = 1 if KS = 1.

AO = A(1,1) if KS = 0.

The program is printed in the following pages.

```

C      THE INVERSION OF THE ROUTH ARRAY OR CONTINUED FRACTION
C      H(1)...H(NH) ... QUOTIENTS OF THE CONTINUED FRACTION
C      .OR.  $H(J)=R(J,1)/R(J+1,1)$ 
C      WHERE  $R(J,K)$ =THE ROUTH ARRAY
C      NH... NO. OF TOTAL QUOTIENTS.
C      NR... HOW MANY ROWS OF THE ROUTH ARE REQUIRED.
C      KS... =1 FOR CONTINUED FRACTION.
C      KS... =0 FOR ROUTH ARRAY
C      M(1)...M(NR)... ANY REQUIRED ROWS.
C      AO=1. IF KS=1
C      AO=R(1,1) IF KS=0
C      MAIN PROGRAM BEGINS HERE
      DIMENSION H(100),M(100),F(100),HH(40,40)
      DOUBLE PRECISION DA(50),DB(50),DC(50),DD(50)
1000 READ(5,501) NH,NR,KS,AO
      501 FORMAT(3I5,F16.9)
      READ(5,502) (H(I),I=1,NH)
      502 FORMAT(5F16.9)
      READ(5,503) (M(I),I=1,NR)
      503 FORMAT(16I5)
      WRITE(6,601) NH,NR,AO
      601 FORMAT( //5X,21HNH...NO. OF QUOTIENTS,I3,18H NR...NO. OF ROWS,I3
1,7H KS...,39H AO...COEFF. AT 1ST. ROW AND 1ST. COL.,F16.6/)
      WRITE(6,602)
      602 FORMAT(/9X,34HGIVEN QUOTIENTS H(1),H(2),...H(NH)//)
      WRITE(6,603) (H(I),I=1,NH)
      603 FORMAT(5X,6F16.6)
      WRITE(6,604)
      604 FORMAT(/9X,36HTHE REQUIRED ROWS OF THE ROUTH ARRAY//)
      WRITE(6,605) (M(I),I=1,NR)
      605 FORMAT(3X,10I10)
      N=NH/2
      IF (KS.EQ.1) DAO=AO
      IF (KS.EQ.1) GO TO 11
      S=1.
      DO 10 I=1,NH
10 S=H(I)*S
      DAO=AO/S
11 WRITE(6,610)
      610 FORMAT(/20X,43H**...***.....$.*****//)
      DO 100 IM=1,NR
      IF (M(IM).EQ.(NH+1) ) WRITE(6,606) M(IM),DAO
      606 FORMAT( /10X,26HTHE REQUIRED COEFF. AT ROW,I5,5X,21HOF THE ROUTH
1ARRAY IS// 5X,E20.12)
      IF (M(IM).GE.(NH+1)) GO TO 100
      IF (2*N.NE.NH) GO TO 1
      IF (2*(M(IM)/2).EQ.M(IM)) GO TO 2
      NM=N-(M(IM)-1)/2
      MM=M(IM)

```

```

      CALL HMAT(NM,MM,H,HH)
      CALL CHAR(NM,HH,DA)
      NM1=NM+1
      DO 20 I=1,NM1
20    DA(I)=DAO*DA(I)
      GO TO 3
      2 MM=M(IM)-1
      NM=N-(MM-1)/2
      CALL HMAT(NM,MM,H,HH)
      CALL CHAR(NM,HH,DB)
      MM=M(IM)+1
      NM=N-(MM-1)/2
      CALL HMAT(NM,MM,H,HH)
      CALL CHAR(NM,HH,DC)
      NM1=NM+2
      DC(NM1)=0.
      DO 30 I=2,NM1
      I1=I-1
      M1=M(IM)-1
30    DA(I1)=DAO*(DB(I)-DC(I))/H(M1)
      NM1=NM1-1
      GO TO 3
      1 IF(2*(M(IM)/2).NE.M(IM)) GO TO 4
      NM=N-(M(IM)-2)/2
      MM=M(IM)
      CALL HMAT(NM,MM,H,HH)
      CALL CHAR(NM,HH,DA)
      NM1=NM+1
      DO 40 I=1,NM1
40    DA(I)=DAO*DA(I)
      GO TO 3
      4 MM=M(IM)+1
      NM=N-(MM-1)/2
      CALL HMAT(NM,MM,H,HH)
      CALL CHAR(NM,HH,DB)
      MM=M(IM)+3
      NM=N-(MM-1)/2
      CALL HMAT(NM,MM,H,HH)
      CALL CHAR(NM,HH,DD)
      NM1=NM+2
      DD(NM1)=0.
      DO 50 I=2,NM1
      I1=I-1
      M1=M(IM)+1
50    DC(I1)=(DB(I)-DD(I))/H(M1)
      NM1=NM+2
      DC(NM1)=0.
      DO 60 I=1,NM1
60    DA(I)=DAO*(DB(I)*H(M(IM))+DC(I))
      3 DO 70 I=1,NM1
      I1=NM1-I+1
      DB(I)=DA(I1)
70    F(I)=SNGL(DA(I))

```

```

        WRITE(6,607) M(IM),(DB(I),I=1,NM1)
607 FORMAT( /10X,26H THE REQUIRED COEFF. AT ROW,I5,5X,22H OF THE ROUTH
        1ARRAY ARE// (5X,5E20.12))
        WRITE(6,610)
100 CONTINUE
        GO TO 1000
        END

```

```

        SUBROUTINE HMAT(NM,MM,H,HH)
        DIMENSION H(100),HH(40,40)
        DO 10 I=1,NM
            I2=MM+2*I-1
            DO 10 J=1,NM
                S=0.
                IF(J.GT.I) HH(J,I)=G
                IF(J.GT.I) GO TO 10
                DO 20 K=1,J
                    K2=MM+2*(K-1)
20 S=S+H(K2)
                HH(J,I)=-H(I2)*S
                G=HH(J,I)
10 CONTINUE
        RETURN
        END

```

```

        SUBROUTINE CHAR(NM,HH,DA)
        DIMENSION HH(40,40)
        DOUBLE PRECISION DA(50),TR(40),AA(40,40),CC(40,40),S,SS
        DO 10 I=1,NM
            DO 10 J=1,NM
10 AA(I,J)=HH(I,J)
            S=0.
            DO 20 I=1,NM
20 S=AA(I,I)+S
                TR(1)=-S
                DO 30 L=2,NM
                    DO 40 I=1,NM
                        DO 40 J=1,NM
                            CC(I,J)=0.
                            DO 40 K=1,NM
40 CC(I,J)=AA(I,K)*HH(K,J)+CC(I,J)
                                S=0.
                                DO 50 I=1,NM
50 S=CC(I,I)+S
                                    TR(L)=-S
                                    DO 60 I=1,NM
                                        DO 60 J=1,NM
60 AA(I,J)=CC(I,J)
70 CONTINUE
                                DO 70 I=1,NM

```



```
DO 70 J=1,NM
  IF(I.EQ.J) AA(I,J)=1.
  IF(I.EQ.J) GO TO 70
  AA(I,J)=0.
70 CONTINUE
DO 80 I=1,NM
  S=0.
  DO 90 J=1,NM
90  S=AA(I,J)*TR(J)+S
    SS=S/FLOAT(I)
    I1=I+1
    DA(I1)=SS
    IF(I.EQ.NM) GO TO 81
    NM1=NM-I
    DO 100 K=1,NM1
      L=K+I
100  AA(L,K)=SS
80  CONTINUE
81  DA(1)=1.
    RETURN
  END
```

CHAPTER VII

APPROXIMATIONS OF THE ERROR FUNCTION

Many physical phenomena are the result of the motion of material particles due to density gradients. The mathematical description of these phenomena requires the use of an equation of continuity which accounts for the generation, decay, and transport of the material particles under consideration. Heat flow, the changing of a transmission line, the diffusion of liquids and the flow of mobile charges in semiconductors are examples of this type of process.

Consider the non-inductive transmission line, the propagation function γ , and the characteristic impedance Z_0 are

$$r = \sqrt{R(G + CS)}$$

and

$$Z_0 = \sqrt{\frac{R}{G + CS}} \text{ respectively.}$$

In the transient evaluation of this type of problem, an error function or a gamma function are usually involved. We can apply the techniques developed in this thesis to approach the problem.

For illustration, we consider the function $\frac{1}{s\sqrt{s+1}}$ first. Based on the technique which is similar to Equation (18) of Chapter III, we have the continued fraction expansion

$$\frac{1}{s\sqrt{s+1}} = \frac{1}{1 + \frac{s}{2 + \frac{s}{2 + \frac{s}{\ddots}}}} \cdot \frac{1}{s} \quad (59)$$

The first, second and third --- quotients of the continued fraction are $h_1=1$, $h_2=2$, $h_3=2$,--- respectively.

When we truncate the continued fraction, it means that we approximate the infinite one by a finite one, of course. For $k=2$ as an approximation, we have

$$\begin{aligned} \frac{1}{s\sqrt{s+1}} &\approx \frac{1}{1 + \frac{s}{2}} \cdot \frac{1}{s} \\ &\approx \frac{2}{s+2} \cdot \frac{1}{s} \\ &\approx \frac{2}{s^2 + 2s} \end{aligned} \quad (59a)$$

The time domain curve of (59) is shown in Figure 3. (The curve which is indicated by $n=2$). We see that even $k=2$, we have had a very good approximation to the original standard curve.

If $k=3$, 4 or more, we will obtain different approximations, of course, and if more quotients are taken, better results will be obtained.

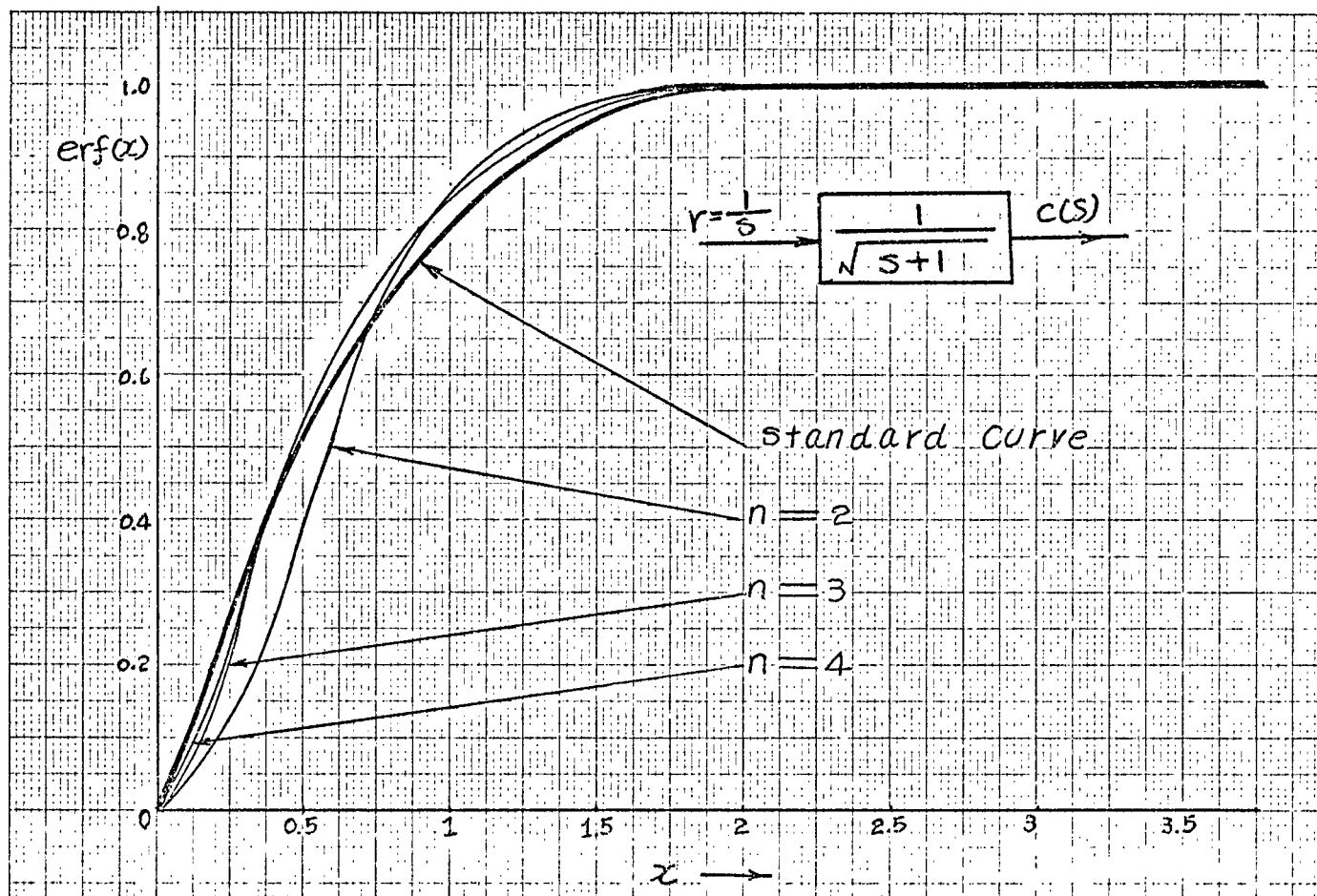


Fig. 3. Response Curves of $\frac{1}{s\sqrt{s+1}}$
(even number quotients)

Figure 3 shows the different approximation when $k=2,4,6---$. (These correspond to $n=\frac{k}{2} + 1 = 2,3,4---$). The comparison of numerical data is listed in Table 1.

On the other hand, if $k=3,5,7---$ (these correspond to $n=\frac{k+1}{2} = 2,3,4---$), we would have the inverse Laplace transforms shown in Figure 4.

TABLE I
COMPARISON OF DATA

		EXACT	APPROXIMATIONS				
x	$t=x^2$	$\text{erf}(\sqrt{t})$	n=2	n=3	n=4	n=5	n=6
0	0	0	0	0	0	0	0
0.02	0.0004	0.0226	0.0008	0.0016	0.0024	0.0032	0.0039
0.04	0.0016	0.0451	0.0032	0.0064	0.0095	0.0126	0.0156
0.06	0.0036	0.0676	0.0072	0.0142	0.0211	0.0277	0.0339
0.08	0.0064	0.0901	0.0127	0.0251	0.0369	0.0478	0.0576
0.1	0.01	0.1125	0.0198	0.0388	0.0564	0.0719	0.0851
0.2	0.04	0.2227	0.0769	0.1424	0.1895	0.2172	0.2295
0.3	0.09	0.3286	0.1647	0.2796	0.3319	0.3426	0.3377
0.4	0.16	0.4284	0.2739	0.4178	0.4466	0.4374	0.4292
0.5	0.25	0.5205	0.3935	0.5363	0.5349	0.5215	0.5185
0.6	0.36	0.6039	0.5132	0.6293	0.6096	0.6016	0.6024
0.7	0.49	0.6778	0.6247	0.7007	0.6775	0.6756	0.6775
0.8	0.64	0.7421	0.7219	0.7575	0.7394	0.7409	0.7423
0.9	0.81	0.7969	0.8021	0.8045	0.7941	0.7967	0.7972
1.0	1.0	0.8427	0.8647	0.8445	0.8408	0.8429	0.8428
1.2	1.44	0.9103	0.9439	0.9074	0.9101	0.9105	0.9103
1.3	1.69	0.9340	0.9659	0.9310	0.9341	0.9341	0.9339
1.4	1.96	0.9523	0.9802	0.9497	0.9526	0.9523	0.9523
1.5	2.25	0.9661	0.9889	0.9642	0.9664	0.9661	0.9661
2	4	0.9953	0.9996	0.9954	0.9953	0.9953	0.9953
3	9	0.9999	1.0000	0.9999	0.9999	0.9999	1.0000

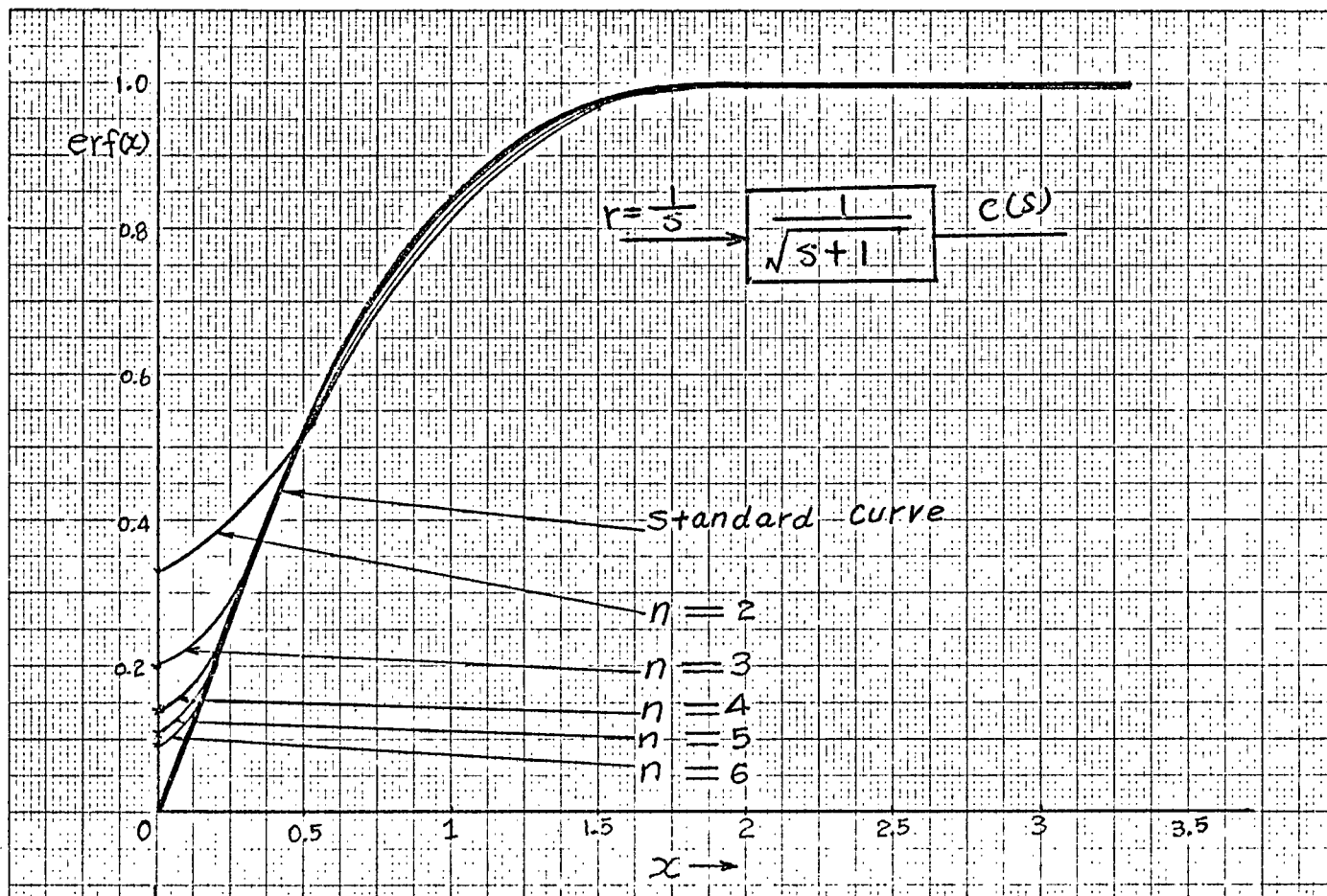


Fig. 3. Response Curves of $\frac{1}{s\sqrt{s+1}}$ (odd number quotients).

CHAPTER VIII

APPROXIMATION OF $\frac{1}{\sqrt{s}}$

For automatic control systems which, in addition to the components with lumped parameters, also contain one or more elements with distributed parameters, the transfer function is written in the following form:

$$W_3 = \frac{W_1 W_2}{1 + W_1 W_2} \quad (60)$$

where W_1 is a transfer function containing distributed parameters. The forms could be $\frac{1}{\sqrt{s}}$ and W_2 is the element involving lumped parameters.

In this chapter, we will approximate this irrational function, $\frac{1}{\sqrt{s}}$ by rational functions with different degrees (see Figures 5 and 6) and then apply the approximation functions to the feedback system.

Now we study the given feedback system of (60):
The transfer functions are defined as

$$W_1 = \frac{100}{1 + 0.63 \sqrt{s}}$$

$$W_2 = \frac{1}{1 + s}$$

Assume that we can approximate \sqrt{s} by

$$\sqrt{s} = \frac{g_2(s)}{g_1(s)}$$

Then

$$W_3 = \frac{\frac{100}{(1+s)(1+0.63\frac{g_2(s)}{g_1(s)})}}{1 + \frac{100}{(1+s)(1+0.63\frac{g_2(s)}{g_1(s)})}}$$

$$= \frac{100 g_1(s)}{(101+s)g_1(s) + 0.63(1+s)g_2(s)}$$

And when a unit step input is applied, we easily obtain the output

$$C(s) \cong \frac{100s^2 + 1000s + 500}{4.15s^4 + 120.45s^3 + 1021.93s^2 + 505.63s}$$

If a better approximation is used, or let

$$\sqrt{s} = \frac{g_2(s)}{g_1(s)} \cong \frac{9s^4 + 84s^3 + 126s^2 + 36s + 1}{s^4 + 36s^3 + 126s^2 + 84s + 9}$$

the corresponding output will be as follows:

$$C(s) \cong \frac{100s^4 + 3600s^3 + 12600s^2 + 8400s + 900}{6.67s^6 + 195.59s^5 + 3894.3s^4 + 12912.06s^3 + 8516.31s^2 + 909.63s}$$

Figure 7 shows the two approximation curves in time domain.

It is noted that when $n=6$, the approximation curve almost coincides with the original standard curve.

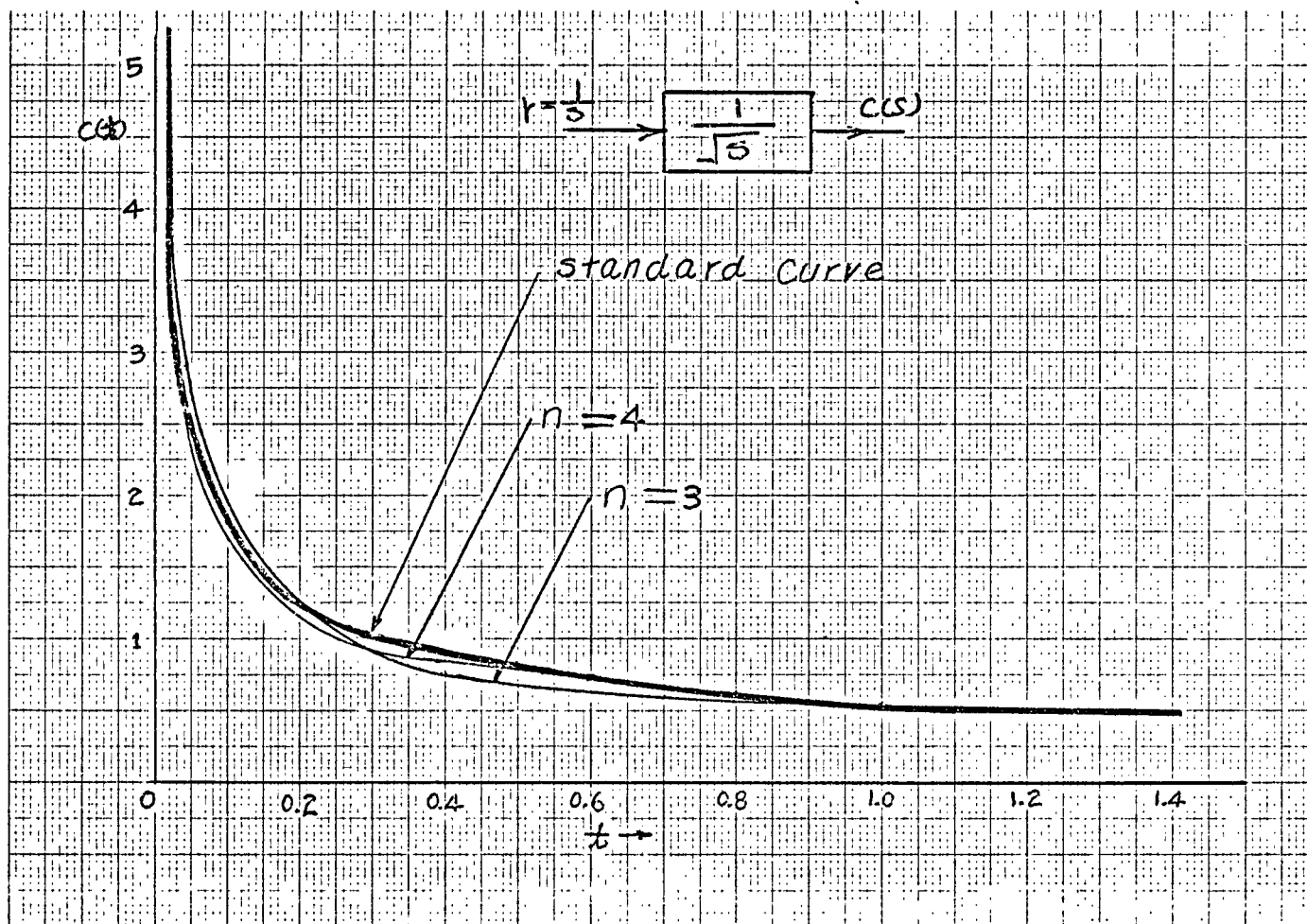


Fig. 5. Response Curves of $\frac{1}{s\sqrt{s}}$ (even number quotients).

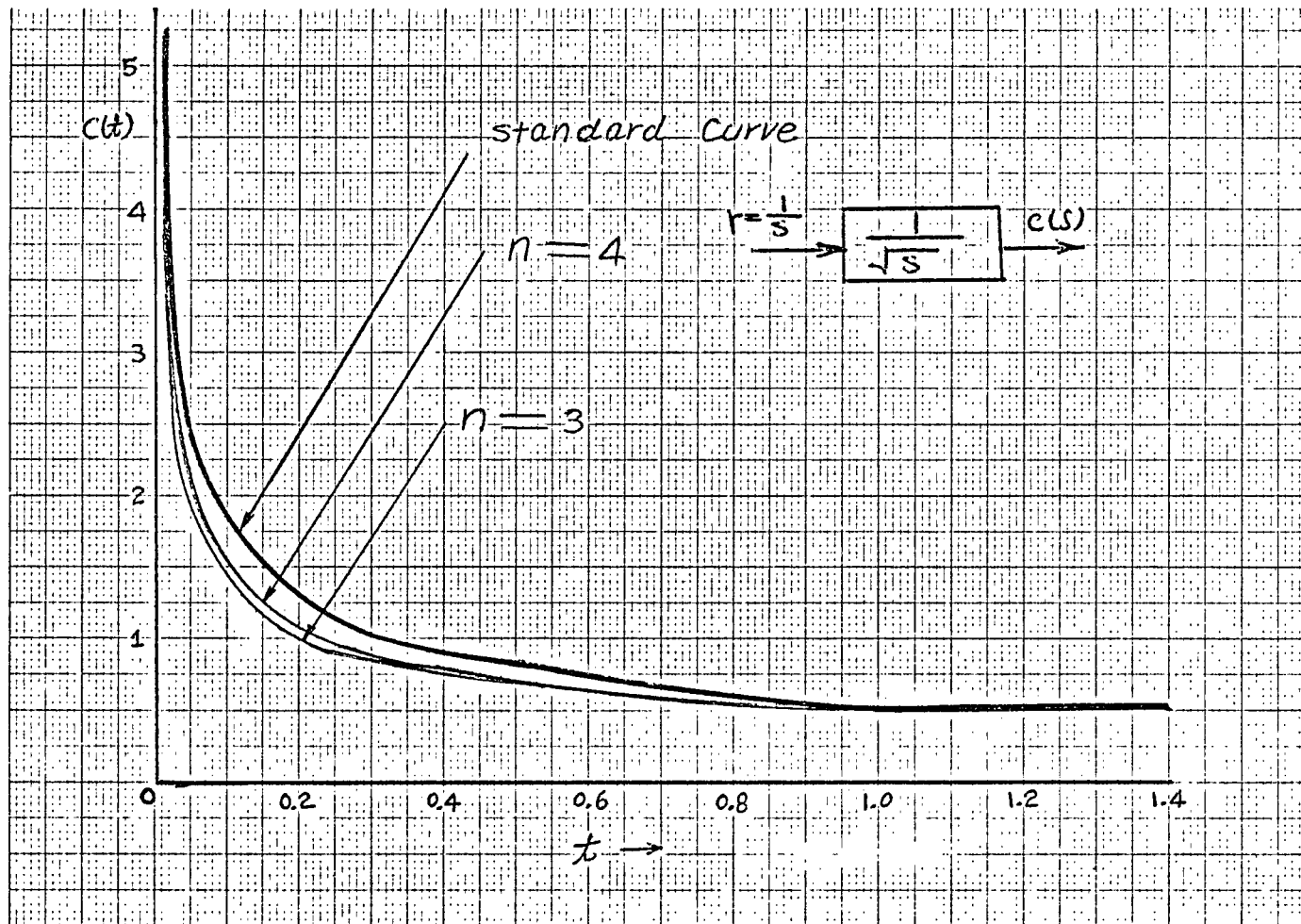


Fig. 6. Response Curves of $\frac{1}{s\sqrt{s}}$ (odd number quotients).

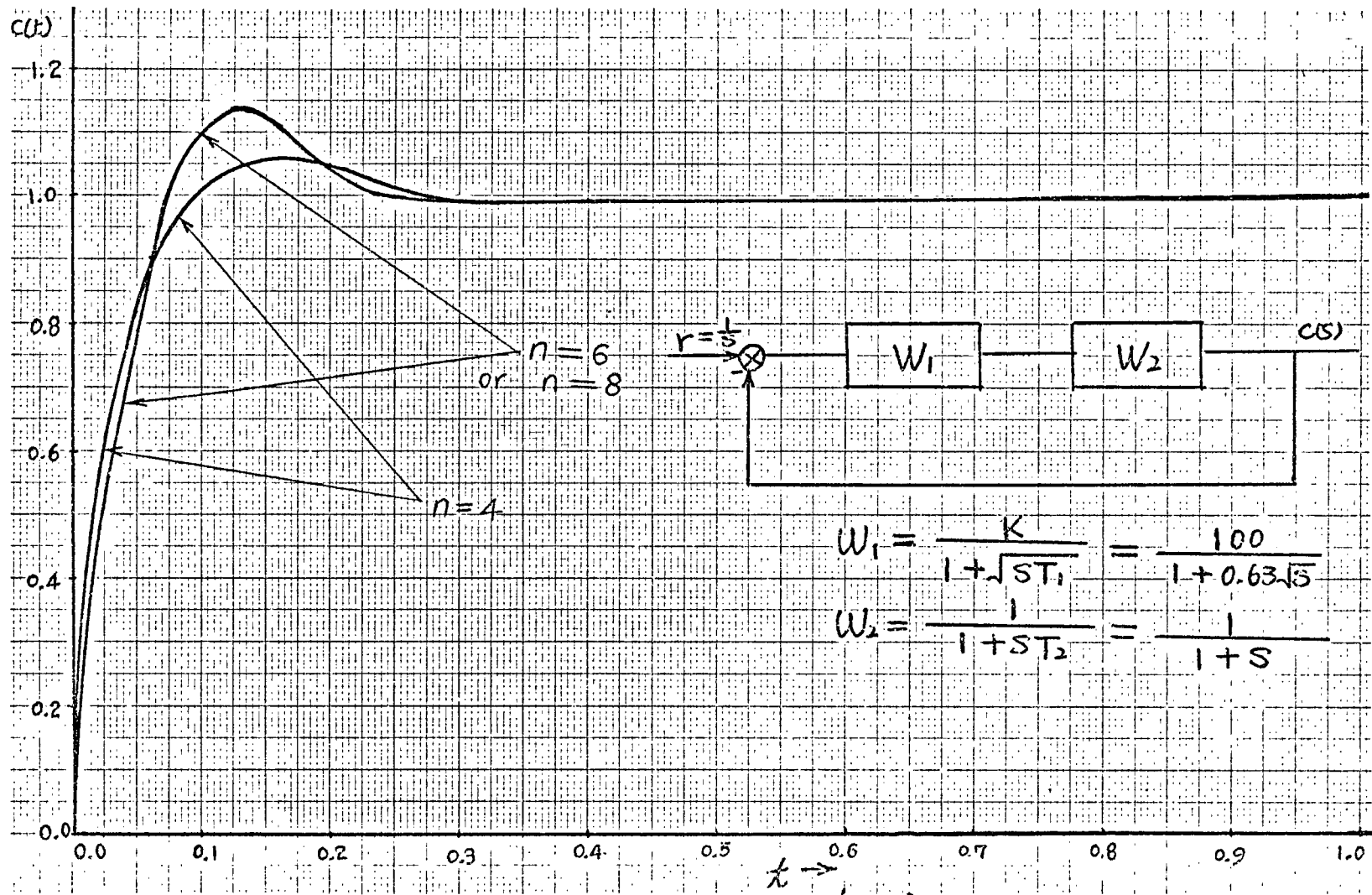


Fig. 7. Response curves of closed loop system.

CHAPTER IX

APPROXIMATIONS OF THE DELAY FUNCTION

Consider a unit step input being applied to a pure time delay system as shown in Figure 8. The Laplace transform of the delay function is the transcendental function, e^{-s} . We expand e^s as a Taylor series about $s=0$ first.

$$e^s = 1 + s + \frac{s^2}{2!} + \frac{s^3}{3!} + \frac{s^4}{4!} + \frac{s^5}{5!} + \dots \quad (61)$$

Equation (61) is assumed to be expandable into a continued fraction as follows:

$$1 + s + \frac{s^2}{2!} + \frac{s^3}{3!} + \frac{s^4}{4!} + \frac{s^5}{5!} + \dots = h_0 + \frac{s}{h_1 + \frac{s}{h_2 + \frac{s}{h_3 + \dots}}} \quad (62)$$

where h_i are the constants to be determined. We set that $s=0$, it is seen that $h_0=1$.

Both sides are divided by s ,

$$1 + \frac{s}{2!} + \frac{s^2}{3!} + \frac{s^3}{4!} + \frac{s^4}{5!} + \dots = \frac{1}{h_1 + \frac{s}{h_2 + \frac{s}{h_3 + \dots}}} \quad (63)$$

The left hand side of Equation (63) can be considered as a ratio of two polynomial

$$f(s) = \frac{g_2(s)}{g_1(s)} = \frac{1 + \frac{s}{2!} + \frac{s^2}{3!} + \frac{s^3}{4!} + \frac{s^4}{5!} + \dots}{1 + 0s + 0s^2 + 0s^3 + 0s^4 + \dots} \quad (64)$$

We can rearrange Equation (64) into the following Routh Array:

$$\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & \dots \leftarrow g_1(s) \\ 1 & \frac{1}{2!} & \frac{1}{3!} & \frac{1}{4!} & \frac{1}{5!} & \dots \leftarrow g_2(s) \end{array}$$

Applying Equation (28), repeatedly we can easily obtain $h_1=1$, $h_2=-2$, $h_3=-3$, $h_4=2$, $h_5=5$, $h_6=-2$, $h_7=-7, \dots$. Thus, the continued expansion of the transcendental function, e^{-s} , is written:

$$e^{-s} = \frac{1}{1 + \frac{s}{1 + \frac{s}{-2 + \frac{s}{-3 + \frac{s}{\dots}}}}} \quad (65)$$

After truncating and inverting the continued fraction, we have two approximation formulas for e^{-s} . The time curves are shown in Figure 8. It is noted that the results of the approximating of the delay function by continued fractions are not as good as before. This is because our technique is particularly suitable for low pass filter systems. Unfortunately, time delay function does not belong to this class.

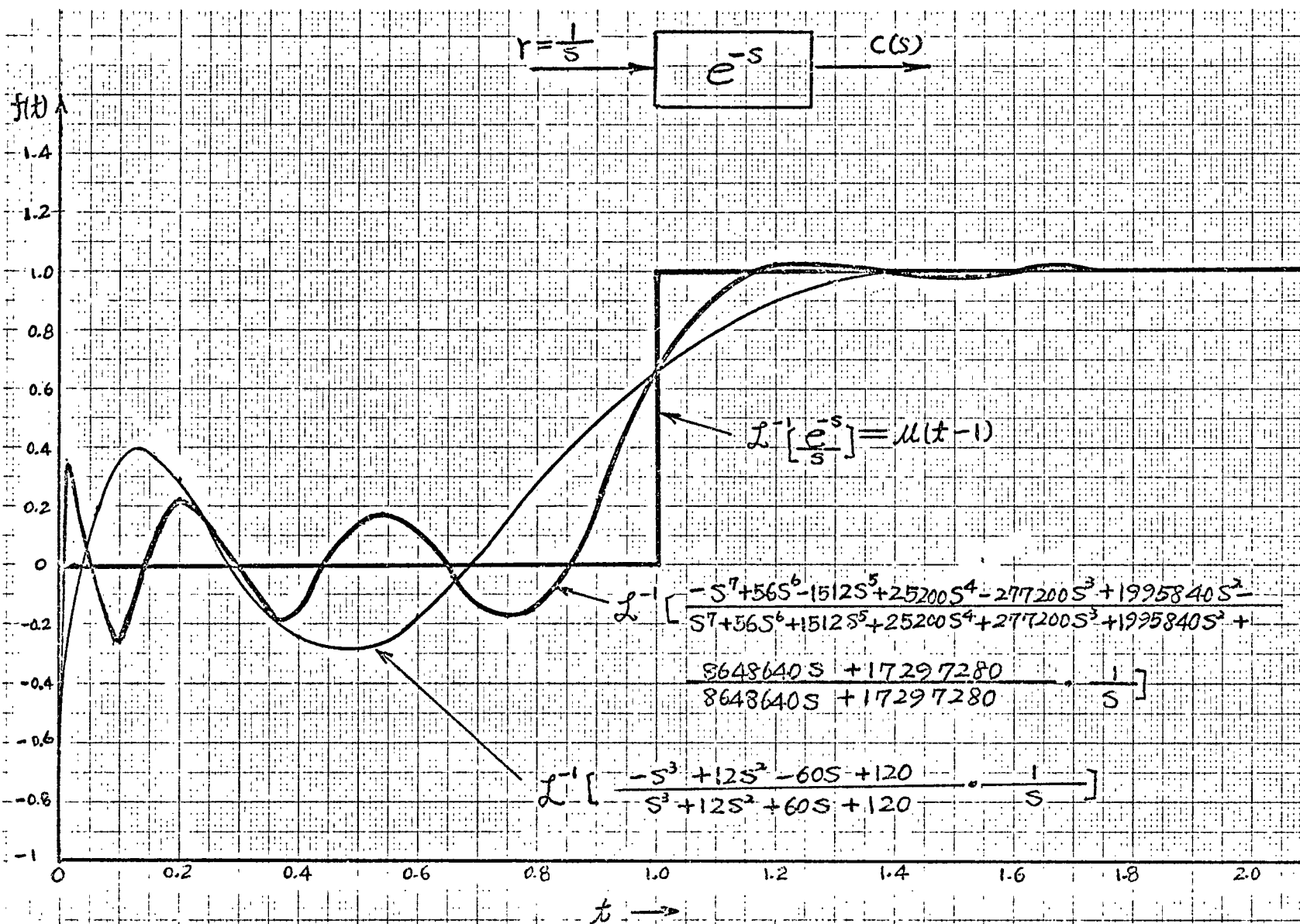


Fig. 8. Response curves of $\frac{e^{-s}}{s}$.

CHAPTER X

CONCLUSIONS

Techniques for expanding certain irrational and transcendental functions into a particular continued fraction and for inverting the truncated continued fraction into an approximate rational transfer function are established.

The continued fraction expanding method is similar to the Heaviside partial fraction expansion method. It is very easy for using while the inverting method is based on the state variable analysis and a new matrix for the inversion has been developed.

The two techniques are converted into digital computer programs through using Routh's algorithm in the general sense.

The error function and time delay function approximations by using the techniques developed are illustrated as applications. Finally, a feedback system which involves lumped and distributed parameters components is investigated.

It is believed that this digital computer oriented approach to the irrational transfer function problem is simple and useful.

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$$e^s = 1 + s(1, -2, -3, 2, 5, -2, -7, \dots, 2, 2n-1, -2(2n+1), \dots)$$

$$|s| < \infty$$

$$e^s = \frac{s+1}{s-2}$$

$$= \frac{s^2 + 4s + 6}{-2s + 6}$$

$$= \frac{s^2 + 6s + 12}{s^2 - 6s + 12}$$

$$= \frac{s^3 + 9s^2 + 36s + 60}{3s^3 - 24s + 60}$$

$$= \frac{-s^3 - 12s^2 - 60s - 120}{s^3 - 12s^2 + 60s - 120}$$

$$= \frac{s^4 + 16s^3 + 120s^2 + 480s + 840}{-4s^4 + 60s^3 - 360s^2 + 840s}$$

$$= \frac{s^4 + 20s^3 + 180s^2 + 840s + 1680}{s^4 - 20s^3 + 180s^2 - 840s + 1680}$$

$$= \frac{s^5 + 25s^4 + 300s^3 + 2100s^2 + 8400s + 15120}{5s^5 - 120s^4 + 1260s^3 - 6720s^2 + 15120s}$$

$$= \frac{-s^5 - 30s^4 - 420s^3 - 3360s^2 - 15120s - 30240}{s^5 - 30s^4 + 420s^3 - 3360s^2 + 15120s - 30240}$$

$$= \frac{s^6 + 36s^5 + 630s^4 + 6720s^3 + 45360s^2 + 181440s + 332640}{-6s^6 + 210s^5 - 3360s^4 + 30240s^3 - 151200s^2 + 332640s}$$

$$= \frac{s^6 + 42s^5 + 840s^4 + 10080s^3 + 75600s^2 + 332640s + 665280}{s^6 - 42s^5 + 840s^4 - 10080s^3 + 75600s^2 - 332640s + 665280}$$

$$= \frac{s^7 + 49s^6 + 1176s^5 + 17640s^4 + 176400s^3 + 1164240s^2 + 4656960s + 8648640}{7s^7 - 336s^6 + 7560s^5 - 100800s^4 + 831600s^3 - 3991680s^2 + 8648640s}$$

$$= \frac{-s^7 - 56s^6 - 1512s^5 - 25200s^4 - 277200s^3 - 1995840s^2 - 8648640s - 17297280}{s^7 - 56s^6 + 1512s^5 - 25200s^4 + 277200s^3 - 1995840s^2 + 8648640s - 17297280}$$

$$\begin{aligned}
&= \frac{s^8 + 64s^7 + 2016s^6 + 40320s^5 + 554400s^4 + 5322240s^3 + 34594560s^2 + 138378240s + 259459200}{-8s^7 + 504s^6 - 15120s^5 + 277200s^4 - 3326400s^3 + 25945920s^2 - 121080960s + 259459200} \\
&= \frac{s^8 + 72s^7 + 2520s^6 + 55440s^5 + 831600s^4 + 8648640s^3 + 60540480s^2 + 259459200s + 518918400}{s^8 - 72s^7 + 2520s^6 - 55440s^5 + 831600s^4 - 8648640s^3 + 60540480s^2 - 259459200s + 518918400} \\
&= \frac{s^9 + 81s^8 + 3240s^7 + 83160s^6 + 1496880s^5 + 19459440s^4 + 181621440s^3 + 1167566336s^2 + 4670263296s + 8821612544}{9s^8 - 720s^7 + 27720s^6 - 665280s^5 + 10810800s^4 - 121080960s^3 + 908107008s^2 - 4151347200s + 8821612544} \\
&= \frac{-s^9 - 90s^8 - 3960s^7 - 110880s^6 - 2162160s^5 - 30270240s^4 - 302702592s^3 - 2075673600s^2 - 8821616640s - 17643229184}{s^9 - 90s^8 + 3960s^7 - 110880s^6 + 2162160s^5 - 30270240s^4 + 302702592s^3 - 2075673600s^2 + 8821616640s - 17643229184}
\end{aligned}$$

$$\sqrt{s+1} = 1+s(2,2,2,\dots,2,2,\dots)$$

$$|s| < \infty$$

$$\sqrt{s+1} = \frac{s+2}{2}$$

$$= \frac{3s+4}{s+4}$$

$$= \frac{s^2 + 8s + 8}{4s + 8}$$

$$= \frac{5s^2 + 20s + 16}{s^2 + 12s + 16}$$

$$= \frac{s^3 + 18s^2 + 48s + 32}{6s^2 + 32s + 32}$$

$$= \frac{7s^3 + 56s^2 + 112s + 64}{s^3 + 24s^2 + 80s + 64}$$

$$= \frac{s^4 + 32s^3 + 160s^2 + 256s + 128}{8s^3 + 80s^2 + 192s + 128}$$

$$= \frac{9s^4 + 120s^3 + 432s^2 + 576s + 256}{s^4 + 40s^3 + 240s^2 + 448s + 256}$$

$$= \frac{s^5 + 50s^4 + 400s^3 + 1120s^2 + 1280s + 512}{10s^4 + 160s^3 + 672s^2 + 1024s + 512}$$

$$= \frac{11s^5 + 220s^4 + 1232s^3 + 2816s^2 + 2816s + 1024}{s^5 + 60s^4 + 560s^3 + 1792s^2 + 2304s + 1024}$$

$$= \frac{s^6 + 72s^5 + 840s^4 + 3584s^3 + 6912s^2 + 6144s + 2048}{12s^5 + 280s^4 + 1792s^3 + 4608s^2 + 5120s + 2048}$$

$$= \frac{13s^6 + 364s^5 + 2912s^4 + 9984s^3 + 16640s^2 + 13312s + 4096}{s^6 + 84s^5 + 1120s^4 + 5376s^3 + 11520s^2 + 11264s + 4096}$$

$$= \frac{s^7 + 98s^6 + 1568s^5 + 9408s^4 + 26880s^3 + 39424s^2 + 28672s + 8192}{14s^6 + 448s^5 + 4032s^4 + 15360s^3 + 28160s^2 + 24576s + 8192}$$

$$= \frac{15s^7 + 560s^6 + 6048s^5 + 28800s^4 + 70400s^3 + 92160s^2 + 61440s + 16384}{s^7 + 112s^6 + 2016s^5 + 13440s^4 + 42240s^3 + 67584s^2 + 53248s + 16384}$$

$$= \frac{s^8 + 128s^7 + 2688s^6 + 21504s^5 + 84480s^4 + 180224s^3 + 212992s^2 + 131072s + 32768}{16s^7 + 672s^6 + 8064s^5 + 42240s^4 + 112640s^3 + 159744s^2 + 114688s + 32768}$$

$$= \frac{17s^8 + 816s^7 + 11424s^6 + 71808s^5 + 239360s^4 + 452608s^3 + 487424s^2 + 278528s + 65536}{s^8 + 144s^7 + 3360s^6 + 29568s^5 + 126720s^4 + 292864s^3 + 372736s^2 + 245760s + 65536}$$

$$= \frac{s^9 + 162s^8 + 4320s^7 + 44352s^6 + 228096s^5 + 658944s^4 + 1118208s^3 + 1105920s^2 + 589824s + 131072}{18s^8 + 960s^7 + 14784s^6 + 101376s^5 + 366080s^4 + 745472s^3 + 860160s^2 + 524288s + 131072}$$

$$= \frac{19s^9 + 1140s^8 + 20064s^7 + 160512s^6 + 695552s^5 + 1770496s^4 + 2723840s^3 + 2490368s^2 + 1245184s + 262144}{s^9 + 180s^8 + 5280s^7 + 59136s^6 + 329472s^5 + 1025024s^4 + 1863680s^3 + 1966080s^2 + 1114112s + 262144}$$

$$\sqrt{s} = 1 + (s-1) (2, 2, 2, \dots, 2, 2, \dots)$$

$$|s| < \infty$$

$$\sqrt{s} = \frac{s+1}{2}$$

$$= \frac{3s+1}{s+3}$$

$$= \frac{s^2+6s+1}{4s+4}$$

$$= \frac{5s^2+10s+1}{s^2+10s+5}$$

$$= \frac{s^3+15s^2+15s+1}{6s^2+20s+6}$$

$$= \frac{7s^3+35s^2+21s+1}{s^3+21s^2+35s+7}$$

$$= \frac{s^4+28s^3+70s^2+28s+1}{8s^3+56s^2+56s+8}$$

$$= \frac{9s^4+84s^3+126s^2+36s+1}{s^4+36s^3+126s^2+84s+9}$$

$$= \frac{s^5+45s^4+210s^3+210s^2+45s+1}{10s^4+120s^3+252s^2+120s+10}$$

$$= \frac{11s^5 + 165s^4 + 462s^3 + 330s^2 + 55s + 1}{s^5 + 55s^4 + 330s^3 + 462s^2 + 165s + 11}$$

$$= \frac{s^6 + 66s^5 + 495s^4 + 924s^3 + 495s^2 + 66s + 1}{12s^5 + 220s^4 + 792s^3 + 792s^2 + 220s + 12}$$

$$= \frac{13s^6 + 286s^5 + 1287s^4 + 1716s^3 + 715s^2 + 78s + 1}{s^6 + 78s^5 + 715s^4 + 1716s^3 + 1287s^2 + 286s + 13}$$

$$= \frac{s^7 + 91s^6 + 1001s^5 + 3003s^4 + 3003s^3 + 1001s^2 + 91s + 1}{14s^6 + 364s^5 + 2002s^4 + 3432s^3 + 2002s^2 + 364s + 14}$$

$$= \frac{15s^7 + 455s^6 + 3003s^5 + 6435s^4 + 5005s^3 + 1365s^2 + 105s + 1}{s^7 + 105s^6 + 1365s^5 + 5005s^4 + 6435s^3 + 3003s^2 + 455s + 15}$$

$$= \frac{s^8 + 120s^7 + 1820s^6 + 8008s^5 + 12870s^4 + 8008s^3 + 1820s^2 + 120s + 1}{16s^7 + 560s^6 + 4368s^5 + 11440s^4 + 11440s^3 + 4368s^2 + 560s + 16}$$

$$= \frac{17s^8 + 680s^7 + 6188s^6 + 19448s^5 + 24310s^4 + 12376s^3 + 2380s^2 + 136s + 1}{s^8 + 136s^7 + 2380s^6 + 12376s^5 + 24310s^4 + 19448s^3 + 6188s^2 + 680s + 17}$$

$$= \frac{s^9 + 153s^8 + 3060s^7 + 18564s^6 + 43758s^5 + 43758s^4 + 18564s^3 + 3060s^2 + 153s + 1}{18s^8 + 816s^7 + 8568s^6 + 31824s^5 + 48620s^4 + 31824s^3 + 8568s^2 + 816s + 18}$$

$$= \frac{19s^9 + 969s^8 + 11628s^7 + 50388s^6 + 92378s^5 + 75582s^4 + 27132s^3 + 3876s^2 + 171s + 1}{s^9 + 171s^8 + 3876s^7 + 27132s^6 + 75582s^5 + 92378s^4 + 50388s^3 + 11628s^2 + 969s + 19}$$

$$\sqrt{s}-1/\sqrt{s} = s(1,2,2,2,\dots,2,2\dots)$$

$$|s| < \infty$$

$$\sqrt{s}-1/\sqrt{s} = s$$

$$= \frac{2s}{s+2}$$

$$= \frac{s^2+4s}{3s+4}$$

$$= \frac{4s^2+8s}{s^2+8s+8}$$

$$= \frac{s^3+12s^2+16s}{5s^2+20s+16}$$

$$= \frac{6s^3+32s^2+32s}{s^3+18s^2+48s+32}$$

$$= \frac{s^4+24s^3+80s^2+64s}{7s^3+56s^2+112s+64}$$

$$= \frac{8s^4+80s^3+192s^2+128s}{s^4+32s^3+160s^2+256s+128}$$

$$= \frac{s^5+40s^4+240s^3+448s^2+256s}{s^4+120s^3+432s^2+576s+256}$$

$$\begin{aligned}
&= \frac{10s^5+160s^4+672s^3+1024s^2+512s}{s^5+50s^4+400s^3+1120s^2+1280s+512} \\
&= \frac{s^6+60s^5+560s^4+1792s^3+2304s^2+1024s}{11s^5+220s^4+1232s^3+2816s^2+2816s+1024} \\
&= \frac{12s^6+280s^5+1792s^4+4608s^3+5120s^2+2048s}{s^6+72s^5+840s^4+3584s^3+6912s^2+6144s+2048} \\
&= \frac{s^7+84s^6+1120s^5+5376s^4+11520s^3+11264s^2+4096s}{13s^6+364s^5+2912s^4+9984s^3+16640s^2+13312s+4096} \\
&= \frac{14s^7+448s^6+4032s^5+15360s^4+28160s^3+24576s^2+8192s}{s^7+98s^6+1568s^5+9408s^4+26880s^3+39424s^2+28672s+8192} \\
&= \frac{s^8+112s^7+2016s^6+13440s^5+42240s^4+67584s^3+53248s^2+16384s}{15s^7+560s^6+6048s^5+28800s^4+70400s^3+92160s^2+61440s+16384} \\
&= \frac{16s^8+672s^7+8064s^6+42240s^5+112640s^4+159744s^3+114688s^2+32768s}{s^8+128s^7+2688s^6+21504s^5+84480s^4+180224s^3+212992s^2+131072s+32768} \\
&= \frac{s^9+144s^8+3360s^7+29568s^6+126720s^5+292864s^4+372736s^3+245760s^2+65536s}{17s^8+816s^7+11424s^6+71808s^5+239360s^4+452608s^3+437424s^2+278528s+65536} \\
&= \frac{18s^9+960s^8+14784s^7+101376s^6+366080s^5+745472s^4+860160s^3+524288s^2+131072s}{s^9+162s^8+4320s^7+44352s^6+228096s^5+658944s^4+1118208s^3+1105920s^2+589824s+131072}
\end{aligned}$$

$$e^{Js} = 1+s(-J, -2, 3J, 2, -5J, -2, 7J, \dots)$$

$$e^{Js} = Js+1$$

$$= \frac{-Js-2}{Js-2}$$

$$= \frac{-s^2+4Js+6}{-2Js+6}$$

$$= \frac{-s^2+6Js+12}{-s^2-6Js+12}$$

$$= \frac{-Js^3-9s^2+36Js+60}{-3s^2-24Js+60}$$

$$= \frac{Js^3+12s^2-60Js-120}{-Js^3+12s^2+60Js-120}$$

$$= \frac{s^4-16Js^3-120s^2+480Js+840}{4Js^3-60s^2-360Js+840}$$

$$= \frac{s^4-20Js^3-180s^2+840Js+1680}{s^4+20Js^3-180s^2-840Js+1680}$$

$$= \frac{Js^5+25s^4-300Js^3-2100s^2+8400Js+15120}{5s^4+120Js^3-1260s^2-6720Js+15120}$$

$$= \frac{-Js^5 - 30s^4 + 420Js^3 + 3360s^2 - 15120Js - 30240}{Js^5 - 30s^4 - 420Js^3 + 3360s^2 + 15120Js - 30240}$$

$$= \frac{-s^6 + 36Js^5 + 630s^4 - 6720Js^3 - 45360s^2 + 181440Js + 332640}{-6Js^5 + 210s^4 + 3360Js^3 - 30240s^2 - 151200Js + 332640}$$

$$= \frac{-s^6 + 42Js^5 + 840s^4 - 10080Js^3 - 75600s^2 + 332640Js + 665280}{-s^6 - 42Js^5 + 840s^4 + 10080Js^3 - 75600s^2 - 332640Js + 665280}$$

$$= \frac{-Js^7 - 49s^6 + 1176Js^5 + 17640s^4 - 17640Js^3 - 1164240s^2 + 4656960Js + 8648640}{-7s^6 - 336Js^5 + 7560s^4 + 100800Js^3 - 831600s^2 - 3991680Js + 8648640}$$

$$= \frac{Js^7 + 56s^6 - 1512Js^5 - 25200s^4 + 277200Js^3 + 1995840s^2 - 8648640Js - 17297280}{-Js^7 + 56s^6 + 1512Js^5 - 25200s^4 - 277200Js^3 + 1995840s^2 + 8648640Js - 17297280}$$

$$= \frac{s^8 - 64Js^7 - 2016s^6 + 40320Js^5 + 554400s^4 - 5322240Js^3 - 34594560s^2 + 138378240Js + 2594559200}{8Js^7 - 504s^6 - 15120Js^5 + 277200s^4 + 3326400Js^3 - 25945920s^2 - 121080960Js + 259459200}$$

$$= \frac{s^8 - 72Js^7 - 2520s^6 + 55440Js^5 + 831600s^4 - 8648640Js^3 - 60540480s^2 + 259459200Js + 518918400}{s^8 + 72Js^7 - 2520s^6 + 55440Js^5 + 831600s^4 + 8648640Js^3 - 60540480s^2 - 259459200Js + 518918400}$$

$$\frac{1}{\sqrt{s+1}} = \frac{2}{s+3}$$

$$= \frac{s+3}{4s+4}$$

$$= \frac{4s+4}{s^2+10s+5}$$

$$= \frac{s^2+10s+5}{6s^2+20s+6}$$

$$= \frac{6s^2+20s+6}{s^3+21s^2+35s+7}$$

$$= \frac{s^3+21s^2+35s+7}{8s^3+56s^2+56s+8}$$

$$= \frac{8s^3+56s^2+56s+8}{s^4+36s^3+126s^2+84s+9}$$

$$= \frac{s^4+36s^3+126s^2+84s+9}{10s^4+120s^3+252s^2+120s+10}$$

$$= \frac{10s^4+120s^3+252s^2+120s+10}{s^5+55s^4+330s^3+462s^2+165s+11}$$

$$= \frac{s^5+55s^4+330s^3+462s^2+165s+11}{12s^5+220s^4+792s^3+792s^2+220s+12}$$

$$= \frac{12s^5 + 220s^4 + 792s^3 + 792s^2 + 220s + 12}{s^6 + 78s^5 + 715s^4 + 1716s^3 + 1287s^2 + 286s + 13}$$

$$= \frac{s^6 + 78s^5 + 715s^4 + 1716s^3 + 1287s^2 + 286s + 13}{14s^6 + 364s^5 + 2002s^4 + 3432s^3 + 2002s^2 + 364s + 14}$$

$$= \frac{14s^6 + 364s^5 + 2002s^4 + 3432s^3 + 2002s^2 + 364s + 14}{s^7 + 105s^6 + 1365s^5 + 5005s^4 + 6435s^3 + 3003s^2 + 455s + 15}$$

$$= \frac{s^7 + 105s^6 + 1365s^5 + 5005s^4 + 6435s^3 + 3003s^2 + 455s + 15}{16s^7 + 560s^6 + 4368s^5 + 11440s^4 + 11440s^3 + 4368s^2 + 560s + 16}$$

$$= \frac{16s^7 + 560s^6 + 4368s^5 + 11440s^4 + 11440s^3 + 4368s^2 + 560s + 16}{s^8 + 136s^7 + 2380s^6 + 12376s^5 + 24310s^4 + 19448s^3 + 6188s^2 + 680s + 17}$$

$$\begin{aligned}
\frac{1}{\sqrt{s}(s-1)} &= \frac{2}{s^2-1} \\
&= \frac{4s+4}{s^3+5s^2-5s-1} \\
&= \frac{6s^2+20s+6}{s^4+14s^3-14s-1} \\
&= \frac{8s^3+56s^2+56s+8}{s^5+27s^4+42s^3-42s^2-27s-1} \\
&= \frac{10s^4+120s^3+252s^2+120s+10}{s^6+44s^5+165s^4-165s^2-44s-1} \\
&= \frac{12s^5+220s^4+792s^3+792s^2+220s+12}{s^7+65s^6+429s^5+429s^4-429s^3-429s^2-65s-1} \\
&= \frac{14s^6+364s^5+2002s^4+3432s^3+2002s^2+364s+14}{s^8+90s^7+910s^6+2002s^5-2002s^3-910s^2-90s-1} \\
&= \frac{16s^7+560s^6+4368s^5+11440s^4+11440s^3+4368s^2+560s+16}{s^9+119s^8+1700s^7+6188s^6+4862s^5-4862s^4-6188s^3-1700s^2-119s-1}
\end{aligned}$$

$$\sqrt[3]{s} = 1+(s-1) \quad (3, 1, 4.5, 0.8, \dots)$$

$$\sqrt[3]{s} = \frac{s+2}{3}$$

$$= \frac{2s+1}{s+2}$$

$$= \frac{s^2+10s+2.5}{7.5s+6}$$

$$= \frac{2.8s^2+7s+1}{s^2+7s+2.8}$$

$$\ln \cos s = (\ln 2)/2 + (s - \frac{\pi}{4}) \quad (1, -1, -3, -0.5, 2, -2.5, 3 \dots)$$

$$0 \leq s \leq \frac{\pi}{2}$$

$$\ln \cos s = s - 0.439$$

$$= \frac{-0.653s - 0.167}{s - 1.785}$$

$$= \frac{s^2 + 0.736s - 0.155}{-2s + 4.571}$$

$$= \frac{-1.153s^2 + 0.312s - 0.053}{s^2 - 1.571s - 0.883}$$

$$= \frac{s^3 - 2.356s^2 - 0.11s + 0.015}{3s - 5.356}$$

$$s^S = 1+(s-1) (1,-1,-2,-3,-0.25,12.308\dots)$$

$$|s| < \infty$$

$$s^S = s$$

$$= \frac{-1}{s-2}$$

$$= \frac{s^2 - s + 2}{-s + 3}$$

$$= \frac{-3s^2 + 2s - 5}{s^2 - 7}$$

$$= \frac{s^3 - 1.25s^2 + 2.5s - 0.75}{-1.25s^2 + 4s - 1.25}$$

$$= \frac{9.308s^3 - 10.385s^2 + 23.77s - 4.231}{s^3 - 16.385s^2 + 42.232s - 3.385}$$

$$\ln(e^s+1) = \ln 2 + s(2, -1, -2, 3, 2, -1.6666\dots)$$

$$|s| < \infty$$

$$\ln(e^s+1) = \frac{s+1.386}{2}$$

$$= \frac{-0.307s-1.386}{s-2}$$

$$= \frac{s^2+2s+2.773}{4}$$

$$= \frac{2.693s^2+4.614s+8.318}{s^2-2s+12}$$

$$= \frac{s^3+7.386s^2+12s+16.636}{2s^2+24}$$

$$= \frac{1.026s^3-7.697s^2-11.682s-27.726}{s^3-5.333s^2+12s-40}$$

$$e^{\sin s} = 1 + s(1, -2, -1, 2, -15, \dots)$$

$$0 \leq s \leq \frac{\pi}{2}$$

$$e^{\sin s} = s+1$$

$$= \frac{-s-2}{s-2}$$

$$= \frac{s^2+2s+2}{2}$$

$$= \frac{s^2+2s+4}{s^2-2s+4}$$

$$= \frac{s^3-13s^2-28s-60}{-15s^2+32s-60}$$

$$\cosh s = 1 + s(-1, -2, 1, -6, -1, 10/3, \dots)$$

$$\cosh s = -s+1$$

$$= \frac{-s+2}{s+2}$$

$$= \frac{s^2-2s+2}{2}$$

$$= \frac{-7s^2+14s-12}{s^2+2s-12}$$

$$\text{Tanh}s = \frac{1}{s} \{s^2(1,3,5,7,9,11,13,15,\dots,2n+1,\dots)\}$$

$$\text{Tanh}s = s$$

$$= \frac{3s}{s^2+3}$$

$$= \frac{s^3+15s}{6s^2+15}$$

$$= \frac{10s^3+105s}{s^4+45s^2+105}$$

$$= \frac{s^5+105s^3+945s}{15s^4+420s^2+945}$$

$$= \frac{21s^5+1260s^3+10395s}{s^6+210s^4+4725s^2+10395}$$

$$= \frac{s^7+378s^5+17325s^3+135135s}{28s^6+3150s^4+62370s^2+135135}$$

$$= \frac{36s^7+6930s^5+270770s^3+2027025s}{s^8+630s^6+51975s^4+945945s^2+2027025}$$

$$= \frac{s^9+990s^7+135135s^5+4729725s^3+34459424s}{45s^8+13860s^6+945945s^4+16216200s^2+34459424}$$