AUTOCORRELATION AND SPECTRAL DENSITY FUNCTIONS FOR ASK, PSK, AND FSK WITH NON-COHERENT OR COHERENT SPLIT-PHASE CODE MODULATION

A Thesis

Presented to

the Faculty of the Department of Electrical Engineering

University of Houston

In Partial Fulfillment

of the Requirements for the Degree

Master of Science in Electrical Engineering

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Robert A. Van Cleave, Jr.

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ABSTRACT

Pulse code modulation (PCM) involves transforming analog signals into a series of digitally coded pulses. The binary code is a widely used special case of the coding theoretically possible in a PCM system. One of the code structures used in binary coding is referred to as "bi-phase-level" or "split-phase". A technique used in generating a split-phase code utilizes the binary states "10" to represent a logic one and the binary states "01" to represent a logic zero. In a typical telemetry system the split-phase code is used to modulate the amplitude, phase, or frequency of a carrier.

This thesis provides an analytical determination of the ensemble-average autocorrelation and the power spectral density of a carrier which is amplitude-shift-keyed (ASK), phase-shift-keyed (PSK), or frequency-shift-keyed (FSK) by a split-phase code. These transmission characteristics are first obtained for "non-coherent" modulation of the carrier by the split-phase code, and then obtained for "coherent" modulation of the carrier by the split-phase code. It is then shown that the time average of the autocorrelation and spectral density functions for the coherent cases reduce to the corresponding autocorrelation and spectral density functions for the non-coherent cases.

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CHAPTER I

INTRODUCTION

In the design of near optimal communications systems, it is necessary to determine as accurately as possible the transmission requirements of each subsystem. In order to determine the requirements of each subsystem it is necessary to define the characteristics of the communication signal. Some general characteristics of a communication signal might be its amplitude dynamic range, frequency spectrum, information rate, etc.

In data transmission the signal can usually be described only in a statistical manner. Some of the statistical characteristics of the signal might be its autocorrelation function, power spectral density, moments, etc.

This thesis develops the autocorrelation function and power spectral density for an amplitude-shift-keyed (ASK), phase-shift-keyed (PSK) or frequency-shift-keyed (FSK) system non-coherently modulated by a split-phase PCM code. The autocorrelation function and power spectral density are also developed for an ASK, PSK, or FSK system coherently modulated by a split-phase PCM code. It is then shown that the time averaged autocorrelation and spectral density functions for the coherent ASK, PSK, and FSK cases are identical to the corresponding autocorrelation and spectral density functions for the non-coherent cases.

CHAPTER II

HISTORICAL BACKGROUND

The background material necessary for the development of the statistical characteristics of ASK, PSK, and FSK systems where the modulation is a random PCM split-phase code is developed in this chapter.

I. PULSE CODE MODULATION

Pulse modulation exhibits many characteristics which make it particularly applicable to communications systems. One of the most important of these characteristics is the ease with which many intelligence signals can be transmitted over a single link utilizing time division multiplexing. In time division multiplexing the intelligence channels are sampled sequentially with each sample sequentially occupying an allotted time slot. The sequential sampling is then repeated after the last designated intelligence signal has been sampled. The sampling must be rapid enough so that the signal amplitude of a particular intelligence channel does not change too much between sampling instants. Theoretical studies based on an idealized case have shown that no information is lost if the sampling rate is at least twice the highest frequency component in the sampled intelligence channel. Because physically realizable systems cannot attain the ideal system characteristics, and because of a desire for the elimination of unnecessarily complex circuitry, the sampling rate in a typical telemetry system is about five times the highest frequency component in the sampled intelligence channels[1]. This series of samples might then be used to modulate the amplitude, phase, or frequency of a sinusoidal carrier. If the samples of the intelligence channels are used to modulate the amplitude of the sinusoidal carrier, the system is called a pulse amplitude modulation system (PAM), if the samples are used to modulate the phase of the sinusoidal carrier the system is called pulse phase modulation (PPM)*, and if the samples are used to modulate the frequency of the sinusoidal carrier the system is called pulse frequency modulation (PFM).

The pulse modulation technique discussed to this point uses the samples of the intelligence channels to modulate the amplitude, phase, or frequency of the sinusoidal carrier. A very important pulse modulation technique which does not use the samples of the intelligence channels to directly modulate the carrier is called pulse code modulation (PCM)[2]. PCM involves transforming the intelligence channels into a series of digitally encoded pulses. The first operation is to "sample" the intelligence channels as previously discussed. Next, the voltage amplitude of each sample is assigned to the nearest value of a set of predetermined discrete voltages. This process is known as "quantizing" and is equivalent to rounding off to the nearest whole number in mathematics. The error introduced in the quantizing process is called "quantization noise". The final step is to "encode" each discrete amplitude value into N-ary digital form. The PCM

*This should not be confused with pulse position modulation (PPM) where the modulation varies the position of a pulse with respect to some reference.

technique has the advantage of having only "N" discrete voltage levels to be recognized at the receiver. It is only necessary to determine which of the N possible voltage levels the received signal is closest to, and not the exact magnitude of the signal.

A binary code is a widely used special case of the encoding theoretically possible in a PCM system. In the binary PCM code it is only necessary to recognize two discrete voltage levels at the receiver. As mentioned in the PCM N-ary system it is only necessary to determine which of the N possible voltage levels the received signal is closest to, and not the exact magnitude of the signal. In the binary PCM system (N = 2) the minimum possible number of voltage levels needed to carry information is used. Therefore, it is only necessary to determine at the receiver which of the two possible binary voltage levels was transmitted. The binary PCM lends itself to detection by the integrate and dump detector, which is one form of the "matched filter" optimum detector for signals with a "white noise" background.

One of the code structures used in binary encoding is called "bi-phase-level" or "split-phase" coding. A technique used in generating a split-phase code utilizes the binary states "10" to represent a logic one and the binary states "01" to represent a logic zero. A characteristic of split-phase encoding is that at least one transition occurs during each bit period. Transmission of all logic ones or all logic zeroes results in two transitions per bit period, while alternating logic ones and logic zeroes result in the minimum one transition per bit period. The greater transition density for a random PCM split-phase code usually allows more efficient bit synchronization to be maintained at the receiver. A segment of a random split-phase code is shown in Figure 1.

II. ENSEMBLE AVERAGE AUTOCORRELATION

In obtaining the statistical description of a random variable, it is necessary to determine the frequency of occurrence of a given outcome in a large number of experiments repeated under similar conditions. The relative frequency of occurrence can be determined by observing a large number of identical experiments simultaneously, or by observing the same experiment repeated in time succession. In the first method, referred to as the "ensemble" method, it is assumed that there are available a large number of identical experiments and that a set of instantaneous observations are made at the same instant of time. In the second method, referred to as the "extension-in-time" method, the samples are obtained from a single system over an extended period of time.

From the discussion to this point, two types of averaging can be determined.[3] The first kind of average is the "ensemble average" and is defined as

$$E[x] = \tilde{x} = \int_{-\infty}^{\infty} x \, p(x,t) \, dx \tag{2.1}$$

In general this average is a time dependent function. The second type of average is the "time average" and is defined as

$$\overline{x} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t) \, dt \tag{2.2}$$

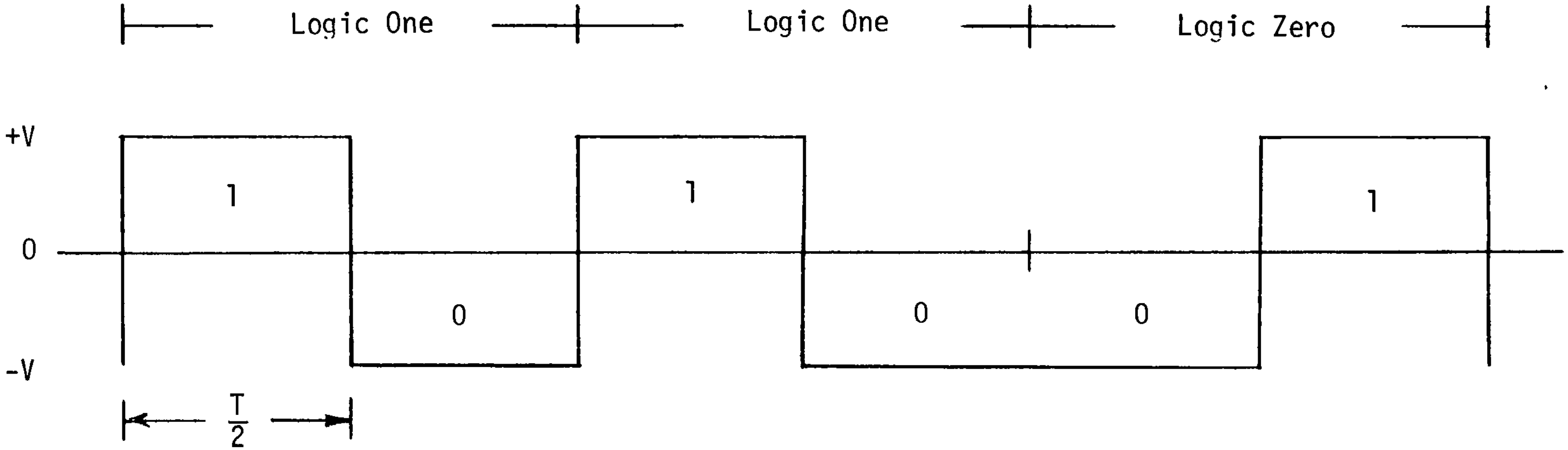

FIGURE 1

SPLIT-PHASE CODE

A random process is said to be "stationary" if all orders of the ensemble averages and all orders of the time averages are independent of the times at which the observations took place. A random variable is said to be "ergodic" if all orders of ensemble averages are equal to the corresponding order of time averages. A random process is said to be wide sense stationary if its expected value is a constant and if its autocorrelation is only a function of the time difference, $t_1 - t_2 = \tau$, between samples and not the actual time of sampling.

It should be noted that the expected value of the random PCM split-phase signal, Figure 1, is zero.

$$E[Vm(t)] = V\,E[m(t)] = 0. \tag{2.3}$$

Two types of autocorrelation functions corresponding to the two types of averages exist. The "ensemble average autocorrelation function" is defined as

$$R(t,\tau) = E\left\{f(t)\ f(t+\tau)\right\} . \tag{2.4}$$

In the form of Equation 2.1 this becomes

$$R(t,\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_1\ f_2\ p(f_1,\ f_2;\ t,\ \tau)\ df_1\ df_2\ . \tag{2.5}$$

The "time average autocorrelation function" is defined as

$$(2.6) \qquad \mathscr{R}(\tau) = \overline{f(t)\ f(t+\tau)}$$

and in the form of Equation 2.2

$$(2.7) \qquad \mathscr{R}(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(t)\ f(t+\tau)\ dt$$

In order to appreciate the importance of the autocorrelation function, the physical meaning of the concept of the autocorrelation function is considered. If the amplitude of a random variable at time t is very small, it is unlikely, for τ sufficiently small, that at time $t + \tau$ the random variable will have a large amplitude. But as τ becomes larger the random variable might possibly have any possible amplitude. Therefore the autocorrelation function is an indication of the statistical relationship existing between two samples of the random variable at t and $t + \tau$. As τ becomes sufficiently large, so that the two samples can be assumed to be statistically independent (S.I.), the mean value of the product becomes the product of the mean values.

$$(2.8) \qquad E[f(t)\ f(t+\tau)] = E[f(t)]\ E[f(t+\tau)] \quad \text{where S.I.}$$

It should be observed that the autocorrelation function is an even function, if the process is wide sense stationary.

The ensemble average autocorrelation function for a random PCM split-phase code, Figure 2,

$$(2.9) \qquad R_m(\tau) = V^2 E[m(t)\, m(t+\tau)]$$

has been found.[4] Changing the nomenclature to that of this thesis the ensemble average autocorrelation function for a random PCM split-phase code is

$$(2.10) \quad R_m(\tau) = \begin{cases} 0 & \tau \leq -T \\ \frac{-V^2}{T}(\tau + T) & -T \leq \tau \leq -\frac{T}{2} \\ \frac{V^2}{T}(3\tau + T) & -\frac{T}{2} \leq \tau \leq 0 \\ \frac{V^2}{T}(-3\tau + T) & 0 \leq \tau \leq \frac{T}{2} \\ \frac{V^2}{T}(\tau - T) & \frac{T}{2} \leq \tau \leq T \\ 0 & T \leq \tau \end{cases}$$

In the derivation of this result it was assumed that the process was at least wide sense stationary.

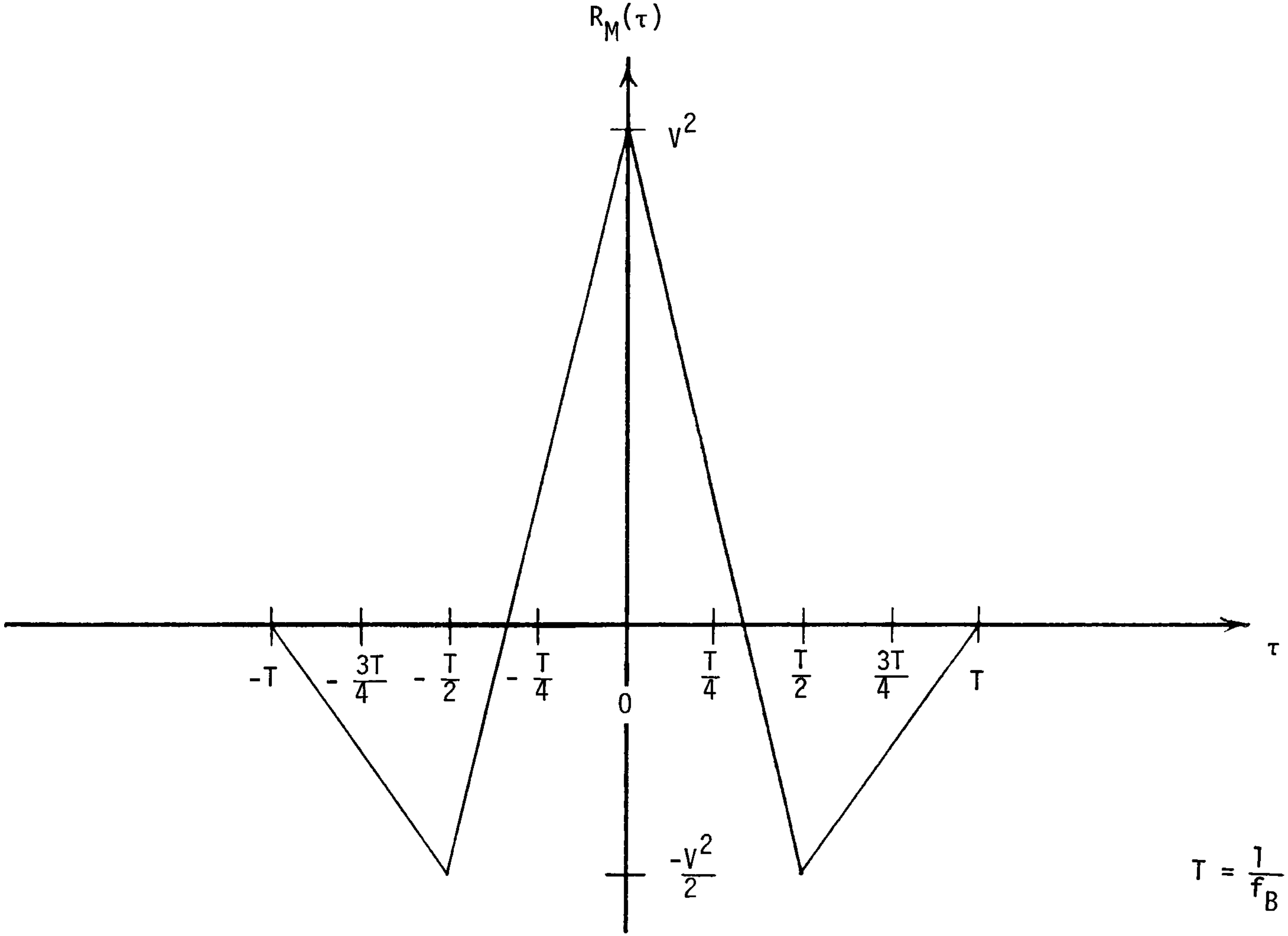

FIGURE 2

AUTOCORRELATION FUNCTION FOR
SPLIT-PHASE CODE

III. POWER SPECTRAL DENSITY

The power spectral density is defined as

$$S(\omega) = \int_{-\infty}^{\infty} R\,(\tau) e^{-j\omega\tau}\, d\tau \tag{2.11}$$

and by virtue of the Fourier transform pair

$$R(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega) e^{+j\omega\tau}\, d\omega \; . \tag{2.12}$$

The power spectral density, as the name implies, is a function describing the distribution of signal power as a function of frequency.

The power spectral density for the random PCM split-phase code

$$S_m(\omega) = \int_{-\infty}^{\infty} R_m(\tau) e^{-j\omega\tau}\, d\tau \tag{2.13}$$

has been found [5]

$$S_m(\omega) = V^2 T \, \frac{\sin^4\left(\frac{\omega T}{4}\right)}{\left(\frac{\omega T}{4}\right)^2} \tag{2.14}$$

and is pictorially represented by Figure 3.

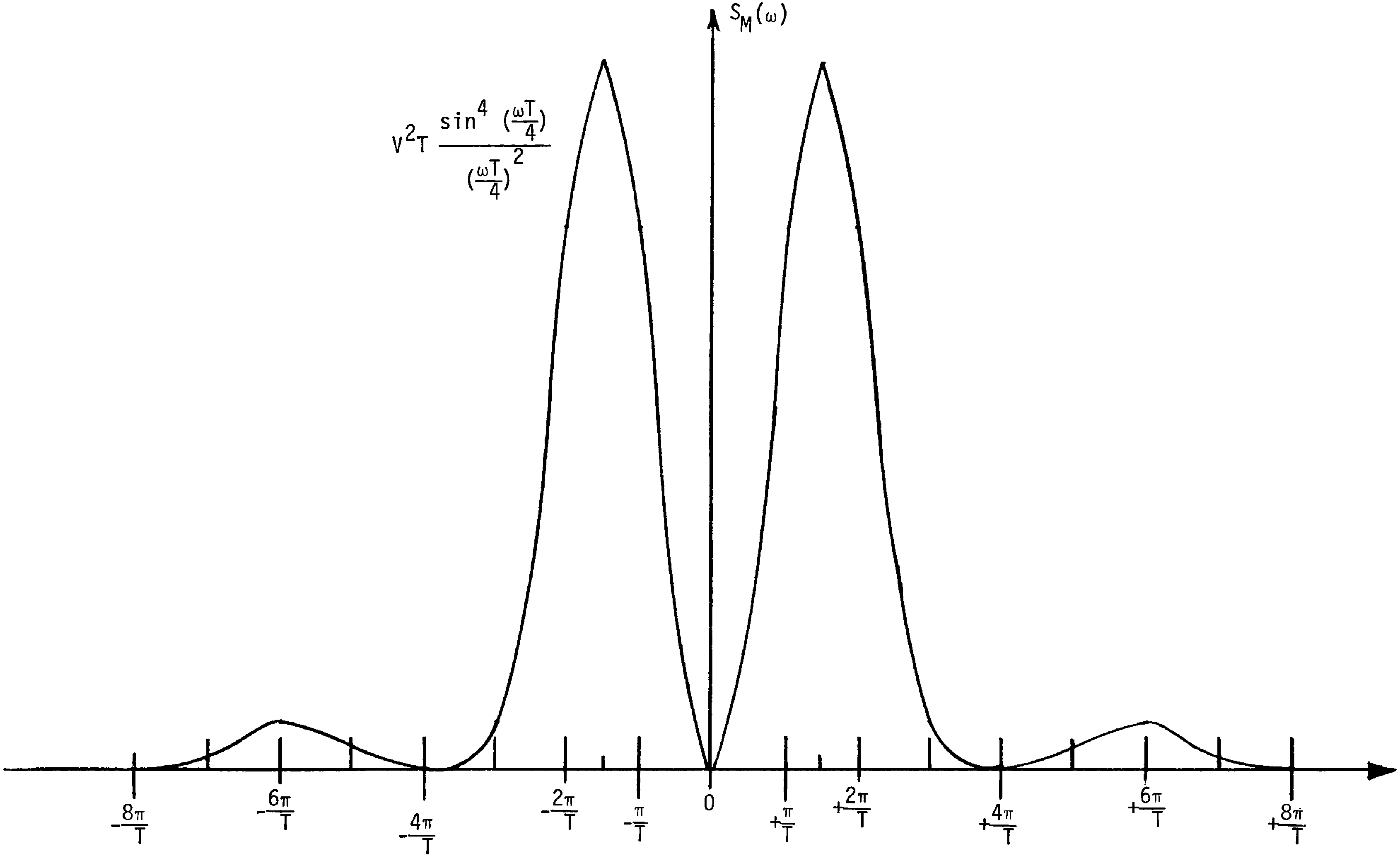

FIGURE 3

POWER SPECTRAL DENSITY OF
A SPLIT-PHASE CODE

CHAPTER III

AMPLITUDE SHIFT KEYING

An expression for a sinusoidal carrier amplitude modulated by a split-phase code is

(3.1) $$e_{ASK}(t) = A\left[1 + \beta V m(t)\right]\sin\left[\omega_c t + \theta\right]$$

where

A - carrier peak amplitude

β - modulation index

V - voltage level of split-phase code

$m(t)$ - split-phase code switching function (± 1)

ω_c - carrier angular frequency

θ - initial phase of the carrier

I. NON-COHERENT MODULATION

For non-coherent amplitude modulation the carrier frequency, f_c, is not an integral multiple of the bit rate, f_B, of the split-phase code

$$f_c \neq K f_B \qquad K = \text{integer}$$

$$f_c \neq \frac{K}{T}$$

This relationship between the carrier frequency and the bit rate of the split-phase code, requires that the initial phase of the carrier, θ, be considered a random variable uniformly distributed between 0 and 2π.

θ_N - random variable uniformly distributed between 0 and 2π

Figure 4 pictorially represents non-coherent ASK.

Autocorrelation Function

The ensemble average autocorrelation function for the non-coherent ASK signal is

$$(3.2) \quad R_{ASKN}(t_1,\tau) = E\left[e_{ASKN}(t_1)e_{ASKN}(t_1+\tau)\right]$$

$$(3.3) \quad R_{ASKN}(t_1,\tau) = E\left[A\left\{1+\beta Vm(t_1)\right\}\sin\left\{\omega_c t_1+\theta_N\right\} A\left\{1+\beta Vm(t_1+\tau)\right\}\sin\left\{\omega_c(t_1+\tau)+\theta_N\right\}\right]$$

$$(3.4) \quad R_{ASKN}(t_1,\tau) = A^2E\left[\left\{1+\beta Vm(t_1)\right\}\left\{1+\beta Vm(t_1+\tau)\right\} \sin\left\{\omega_c t_1+\theta_N\right\}\sin\left\{\omega_c(t_1+\tau)+\theta_N\right\}\right]$$

Multiplying and expanding trigonometrically $R_{ASKN}(t_1,\tau)$ becomes

$$(3.5) \quad R_{ASKN}(t_1,\tau) = \frac{A^2}{2} E\left\{\left[1+\beta Vm(t_1)+\beta Vm(t_1+\tau)+\beta^2V^2m(t_1)m(t_1+\tau)\right] \left[\cos\omega_c\tau+\cos\left\{\omega_c(2t_1+\tau)+2\theta_N\right\}\right]\right\}$$

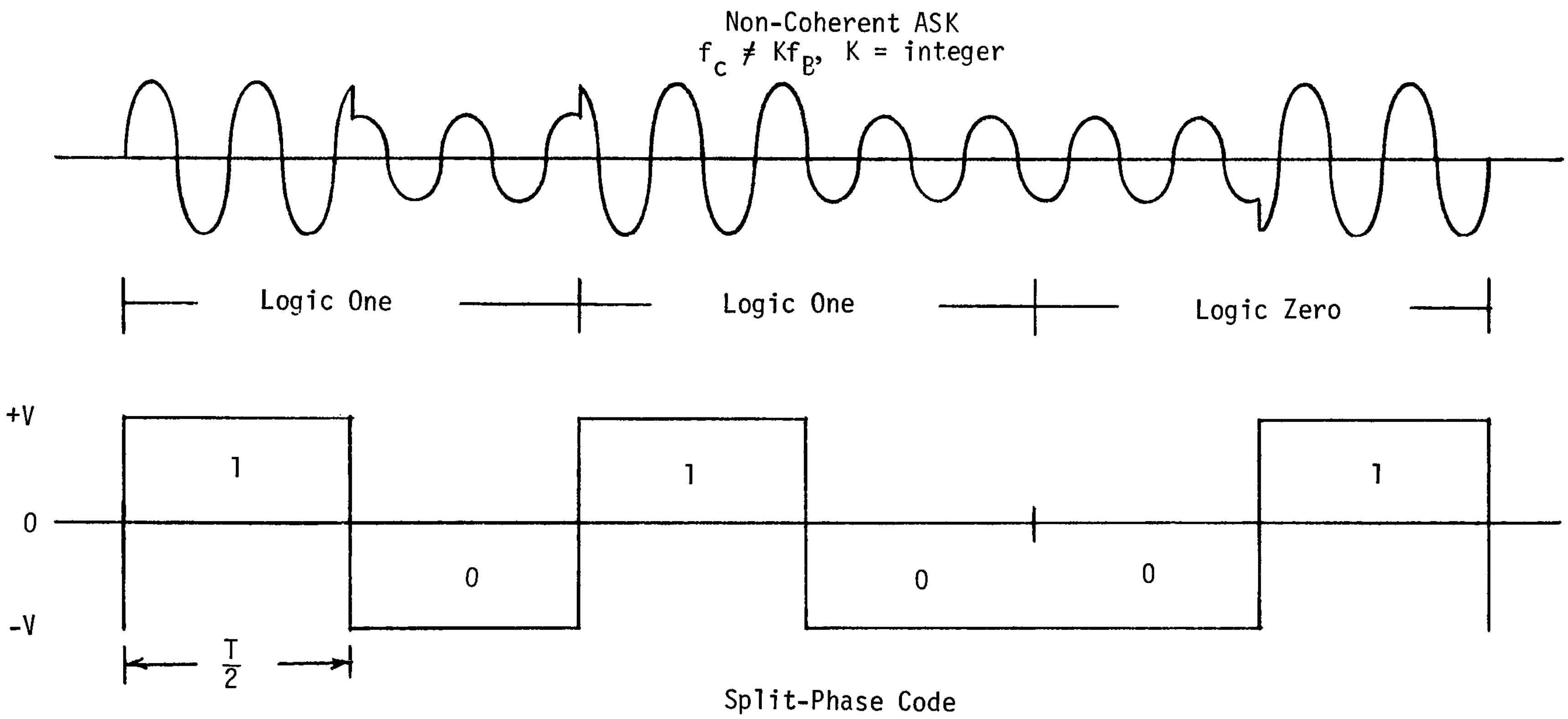

FIGURE 4

NON-COHERENT ASK WITH SPLIT-PHASE
CODE MODULATION

Since the modulation process is non-coherent, the carrier and split-phase modulating signal can be assumed to be statistically independent. Recognizing that the expected value of a sum is equal to the sum of the expected values, $R_{ASKN}(t_1,\tau)$ becomes

$$(3.6) \qquad R_{ASKN}(t_1,\tau) = \frac{A^2}{2}\left\{1 + \beta V E\left[m(t_1)\right] + \beta V E\left[m(t_1 + \tau)\right] + \beta^2 V^2 E\left[m(t_1)m(t_1 + \tau)\right]\right\}\left\{E\left[\cos \omega_C \tau\right] + E\left[\cos\left\{\omega_C(2t_1 + \tau) + 2\theta_N\right\}\right]\right\}$$

But it has previously been shown that

$$(2.3) \qquad E\left[m(t)\right] = E\left[m(t_1)\right] = E\left[m(t_1 + \tau)\right] = 0$$

and noting that the expected value of a sinusoidal with a random phase uniformly distributed between 0 and 2π is equal to zero.

$$(3.7) \qquad E\left[\cos\left\{\omega_C(2t_1 + \tau) + 2\theta_N\right\}\right] = 0$$

Now

$$(3.8) \qquad R_{ASKN}(t_1,\tau) = R_{ASKN}(\tau) = \left[1 + \beta^2 R_M(\tau)\right] R_{Carrier}(\tau)$$

where

$$(2.9) \qquad R_M(\tau) = V^2 E\left[m(t_1)m(t_1 + \tau)\right]$$

and

$$(3.9) \qquad R_{Carrier}(\tau) = \frac{A^2}{2} E\left[\cos \omega_c \tau\right] = \frac{A^2}{2} \cos \omega_c \tau$$

The autocorrelation function, $R_M(\tau)$, for the split-phase code, Vm(t), changing the nomenclature to that of this thesis, is

$$(2.10) \qquad R_M(\tau) = \begin{cases} 0 & \tau \leq -T \\ -\frac{V^2}{T}(\tau + T) & -T \leq \tau \leq -\frac{T}{2} \\ \frac{V^2}{T}(3\tau + T) & -\frac{T}{2} \leq \tau \leq 0 \\ \frac{V^2}{T}(-3\tau + T) & 0 \leq \tau \leq \frac{T}{2} \\ \frac{V^2}{T}(\tau - T) & \frac{T}{2} \leq \tau \leq T \\ 0 & \tau > T \end{cases}$$

where T is the inverse of the bit-rate $\left(T = \frac{1}{f_B}\right)$.

<u>Power Spectral Density</u>

The non-coherent ASK autocorrelation function, Equation 3.8, can be written as

$$(3.10) \qquad R_{ASKN}(\tau) = R'_{ASKN}(\tau) + \beta^2 R''_{ASKN}(\tau)$$

. with a power spectral density

(3.11) $$S_{ASKN}(\omega) = S'_{ASKN}(\omega) + \beta^2 S''_{ASKN}(\omega)$$

where $S'_{ASKN}(\omega)$ and $S''_{ASKN}(\omega)$ are the Fourier transforms of the first and second terms respectively of Equation 3.10. The first term of Equation 3.11 is the power spectral density of the carrier

(3.12) $$S'_{ASKN}(\omega) = \int_{-\infty}^{\infty} R_{Carrier}(\tau) e^{-j\omega\tau} d\tau$$

(3.13) $$S'_{ASKN}(\omega) = \frac{A^2}{2} \int_{-\infty}^{\infty} \cos \omega_c \tau e^{-j\omega\tau} d\tau$$

But

(3.14) $$\frac{1}{2\pi} \int_{-\infty}^{\infty} \cos ux_0 e^{jux} du = \frac{1}{2}\left[\delta(x - x_0) + \delta(x + x_0)\right]$$

Making a change of variable and putting Equation 3.13 in the form of Equation 3.14,

(3.15) $$S'_{ASKN}(\omega) = \frac{A^2\pi}{2}\left[\delta(\omega + \omega_c) + \delta(\omega - \omega_c)\right]$$

The second term of Equation 3.11 can be obtained by realizing that the Fourier transform of the product of two autocorrelation functions is equivalent to the convolution of the two power spectra. Therefore

(3.16) $$S''_{ASKN}(\omega) = S_M(\omega) * S_{Carrier}(\omega)$$

where

$$(2.14) \qquad S_M(\omega) = F\left[R_M(\tau)\right] = V^2T\left[\frac{\sin^4\left(\frac{\omega T}{4}\right)}{\left(\frac{\omega T}{4}\right)^2}\right]$$

$$(3.17) \qquad S_{Carrier}(\omega) = S'_{ASKN}(\omega) = \frac{A^2\pi}{2}\left[\delta(\omega + \omega_c) + \delta(\omega - \omega_c)\right]$$

Implementing Equation 3.16

$$(3.18) \qquad S''_{ASKN}(\omega) = \int_{-\infty}^{\infty} S_M(\omega_0)S_{Carrier}(\omega - \omega_0)d\omega_0$$

$$(3.19) \qquad S''_{ASKN}(\omega) = \int_{-\infty}^{\infty}\left[V^2T\,\frac{\sin^4\left(\frac{\omega_0 T}{4}\right)}{\left(\frac{\omega_0 T}{4}\right)^2}\right]$$

$$\left[\frac{A^2\pi}{2}\left\{\delta(\omega + \omega_c - \omega_0) + \delta(\omega - \omega_c - \omega_0)\right\}\right]d\omega_0$$

$$(3.20) \qquad S''_{ASKN}(\omega) = \frac{A^2V^2\pi T}{2}\left[\int_{-\infty}^{\infty}\frac{\sin^4\left(\frac{\omega_0 T}{4}\right)}{\left(\frac{\omega_0 T}{4}\right)^2}\,\delta(\omega + \omega_c - \omega_0)d\omega_0\right.$$

$$\left. + \int_{-\infty}^{\infty}\frac{\sin^4\left(\frac{\omega_0 T}{4}\right)}{\left(\frac{\omega_0 T}{4}\right)^2}\,\delta(\omega - \omega_c - \omega_0)d\omega_0\right]$$

$$(3.21) \qquad S''_{ASKN}(\omega) = \frac{A^2V^2\pi T}{2}\left\{\frac{\sin^4\left[\frac{(\omega + \omega_c)T}{4}\right]}{\left[\frac{(\omega + \omega_c)T}{4}\right]^2} + \frac{\sin^4\left[\frac{(\omega - \omega_c)T}{4}\right]}{\left[\frac{(\omega - \omega_c)T}{4}\right]^2}\right\}$$

Therefore, utilizing Equation 3.11, the power spectral density for the sinusoidal carrier non-coherently amplitude modulated by the split-phase code is

$$S_{ASKN}(\omega) = \frac{A^2\pi}{2}\left[\delta(\omega + \omega_c) + \delta(\omega - \omega_c)\right] + \frac{A^2\beta^2V^2\pi T}{2}\left\{\frac{\sin^4\left[\frac{(\omega + \omega_c)T}{4}\right]}{\left[\frac{(\omega + \omega_c)T}{4}\right]^2} + \frac{\sin^4\left[\frac{(\omega - \omega_c)T}{4}\right]}{\left[\frac{(\omega - \omega_c)T}{4}\right]^2}\right\} \quad (3.22)$$

Figure 5 pictorially represents $S_{ASKN}(\omega)$ of Equation 3.22.

As observed from Equation 3.22 and Figure 5, the power spectral density for the non-coherent ASK signal consists of discrete carrier components, and sidebands resulting from the split-phase spectrum being translated to appear about plus and minus the carrier frequency. It should be noted that maximum sideband power occurs when the product βV equals one ($\beta V = 1$), which corresponds to "on-off" keying of the carrier by the split-phase code.

II. COHERENT MODULATION

For coherent amplitude modulation the carrier frequency, f_c, is an integral multiple of the bit rate, f_B, of the split-phase code.

$$f_c = Kf_B \qquad K = \text{integer}$$

$$f_c = \frac{K}{T}$$

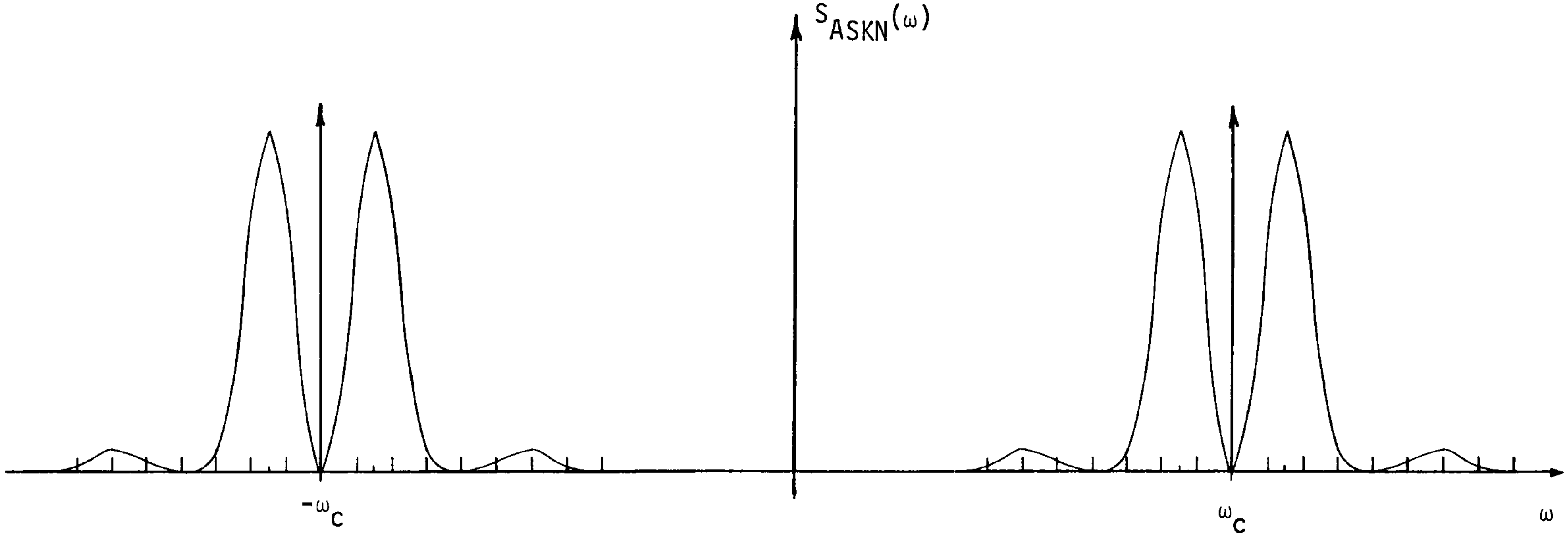

FIGURE 5

POWER SPECTRAL DENSITY FOR NON-COHERENT ASK
WITH SPLIT-PHASE CODE MODULATION

This condition has the effect of making the θ of Equation 3.1 a constant

$$\theta = \theta_c = \text{constant}$$

Figure 6 pictorially represents coherent ASK. It should be noted from Figure 6 that the minimum system bandwidth requirement occurs when θ_c equals zero ($\theta_c = 0$) or some integral multiple of π ($\theta_c = K\pi$, K = integer). Under these conditions no instantaneous amplitude change is required when a transition occurs in the split-phase code. The system bandwidth requirement increases with increased θ_c from the minimum bandwidth conditions to a maximum when θ_c equals an odd integral multiple of $\frac{\pi}{2}$ ($\theta_c = Q\,\frac{\pi}{2}$, Q = odd integer), where the maximum possible instantaneous amplitude change is required.

<u>Autocorrelation Function</u>

The ensemble average autocorrelation function for the coherent ASK signal is

$$(3.23)\quad R_{ASKC}(t_1,\tau) = E\Big[e_{ASKC}(t_1)e_{ASKC}(t_1+\tau)\Big]$$

$$(3.24)\quad R_{ASKC}(t_1,\tau) = E\Big[A\Big\{1+\beta Vm(t_1)\Big\}\sin\Big\{\omega_c t_1+\theta_c\Big\}$$
$$A\Big\{1+\beta Vm(t_1+\tau)\Big\}\sin\Big\{\omega_c(t_1+\tau)+\theta_c\Big\}\Big]$$

$$(3.25)\quad R_{ASKC}(t_1,\tau) = A^2E\Big[\Big\{1+\beta Vm(t_1)\Big\}\Big\{1+\beta Vm(t_1+\tau)\Big\}$$
$$\sin\Big\{\omega_c t_1+\theta_c\Big\}\sin\Big\{\omega_c(t_1+\tau)+\theta_c\Big\}\Big]$$

$$(3.26)\quad R_{ASKC}(t_1,\tau) = A^2E\Big[\Big\{1+\beta Vm(t_1)+\beta Vm(t_1+\tau)+\beta^2V^2m(t_1)m(t_1+\tau)\Big\}$$
$$\sin\Big\{\omega_c t_1+\theta_c\Big\}\sin\Big\{\omega_c(t_1+\tau)+\theta_c\Big\}\Big]$$

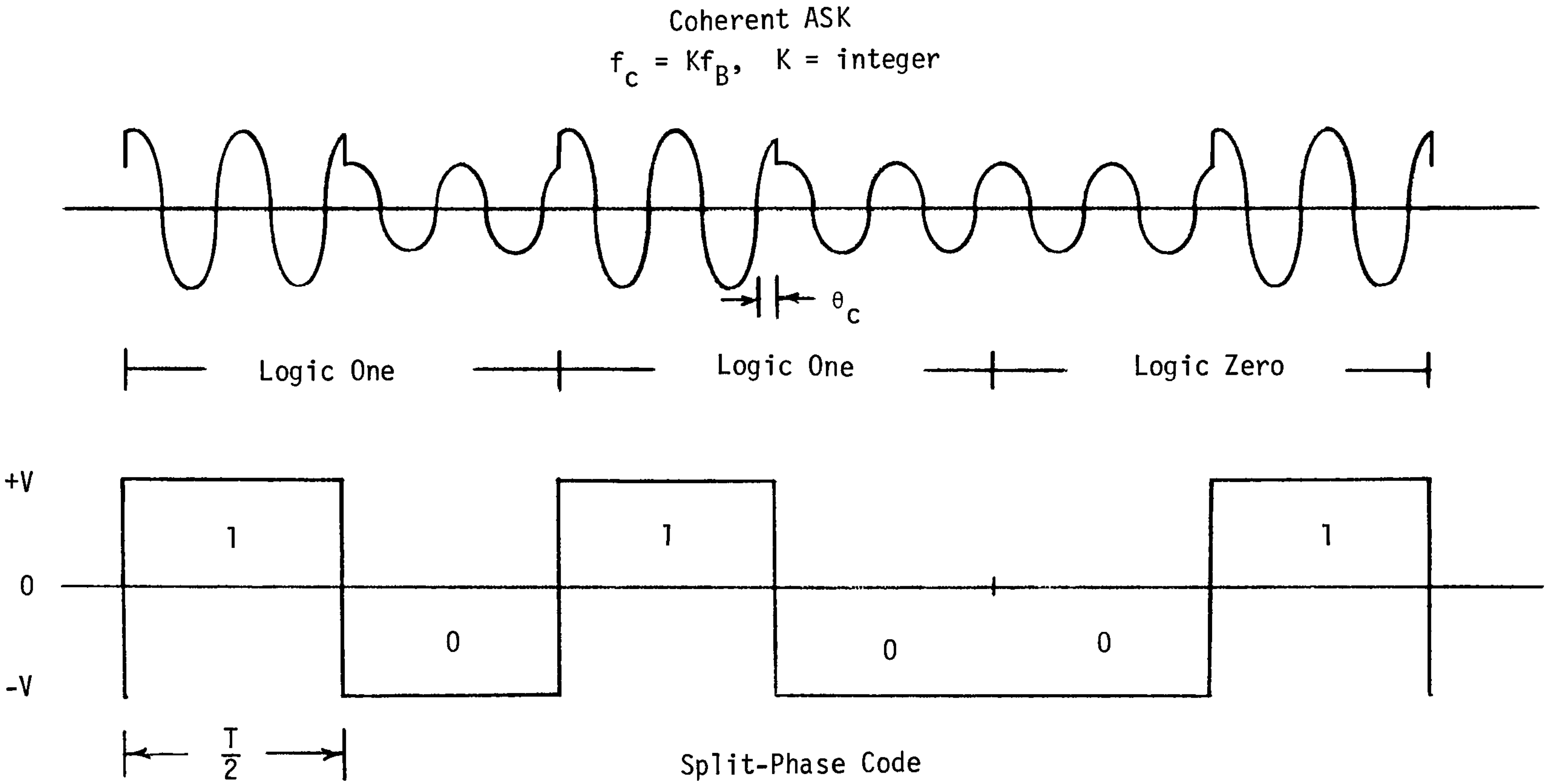

FIGURE 6

COHERENT ASK WITH SPLIT-PHASE
CODE MODULATION

The random variables here are $m(t_1)$ and $m(t_1 + \tau)$, and from the definition of ensemble average autocorrelation function, Equation 2.5,

(3.27) $R_{ASKC}(t_1,\tau) =$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A^2\left[1 + \beta V m(t_1) + \beta V m(t_1 + \tau) + \beta^2 V^2 m(t_1) m(t_1 + \tau)\right]$$

$$\sin\left[\omega_c t_1 + \theta_c\right]\sin\left[\omega_c(t_1 + \tau) + \theta_c\right] p\left[m(t_1)m(t_1 + \tau)\right] dm(t_1)dm(t_2)$$

(3.28) $R_{ASKC}(t_1,\tau) =$

$$A^2\sin\left[\omega_c t_1 + \theta_c\right]\sin\left[\omega_c(t_1 + \tau) + \theta_c\right]$$

$$\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p\left[m(t_1)m(t_1 + \tau)\right] dm(t_1)dm(t_1 + \tau)\right.$$

$$+ \beta V\int_{-\infty}^{\infty} m(t_1)p\left[m(t_1)\right]dm(t_1) + \beta V\int_{-\infty}^{\infty} m(t_1 + \tau)p\left[m(t_1 + \tau)\right]dm(t_1 + \tau)$$

$$\left. + \beta^2 V^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} m(t_1)m(t_1 + \tau)p\left[m(t_1)m(t_1 + \tau)\right]dm(t_1)dm(t_1 + \tau)\right\}$$

where

(3.29) $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} m(t_1)p\left[m(t_1)m(t_1 + \tau)\right]dm(t_1)dm(t_1 + \tau)$$

$$= \int_{-\infty}^{\infty} m(t_1)p\left[m(t_1)\right]dm(t_1)$$

(3.30) $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} m(t_1 + \tau)p\left[m(t_1)m(t_1 + \tau)\right]dm(t_1)dm(t_1 + \tau)$$

$$= \int_{-\infty}^{\infty} m(t_1 + \tau)p\left[m(t_1 + \tau)\right]dm(t_1 + \tau)$$

But

$$(3.31)\quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p\Big[m(t_1)m(t_1+\tau)\Big]dm(t_1)dm(t_1+\tau) = 1$$

$$(3.32)\quad \int_{-\infty}^{\infty} m(t_1)p\Big[m(t_1)\Big]dm(t_1) = E\Big[m(t_1)\Big] = 0$$

$$(3.33)\quad \int_{-\infty}^{\infty} m(t_1+\tau)p\Big[m(t_1+\tau)\Big]dm(t_1) = E\Big[m(t_1+\tau)\Big] = 0$$

$$(3.34)\quad R_M(\tau) = V^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} m(t_1)m(t_1+\tau)p\Big[m(t_1)m(t_1+\tau)\Big]dm(t_1)dm(t_1+\tau)$$

Utilizing Equations 3.29 through 3.34, $R_{ASKC}(t_1,\tau)$ of Equation 3.28 becomes

$$(3.35)\quad R_{ASKC}(t_1,\tau) = A^2\Big[1+\beta^2R_M(\tau)\Big]\sin\Big[\omega_C t_1+\theta_C\Big]\sin\Big[\omega_C(t_1+\tau)+\theta_C\Big]$$

Noting that

$$(3.36)\quad R_{Carrier}(t_1,\tau) = A^2\sin\Big[\omega_C t_1+\theta_C\Big]\sin\Big[\omega_C(t_1+\tau)+\theta_C\Big]$$

$R_{ASKC}(t_1,\tau)$ becomes

$$(3.37)\quad R_{ASKC}(t_1,\tau) = \Big[1+\beta^2R_M(\tau)\Big]R_{Carrier}(t_1,\tau)$$

<u>Time averaged autocorrelation function</u>. The time averaged autocorrelation function for coherent ASK is

$$(3.38)\quad \overline{R_{ASKC}(t_1,\tau)} = \frac{1}{T}\int_0^T R_{ASKC}(t_1,\tau)dt_1$$

$$(3.39)\qquad \overline{R_{ASKC}(t_1,\tau)} = \frac{1}{T}\int_0^T \left[1 + \beta^2 R_M(\tau)\right] R_{Carrier}(t_1,\tau)dt_1$$

$$(3.40)\qquad \overline{R_{ASKC}(t_1,\tau)} = \frac{\left[1 + \beta^2 R_M(\tau)\right]}{T}\int_0^T R_{Carrier}(t_1,\tau)dt_1$$

Before the integration is performed it should be noted that $R_{Carrier}(t_1,\tau)$ can be trigonometrically changed to

$$(3.41)\ R_{Carrier}(t_1,\tau) = \frac{A^2}{2}\left\{\cos \omega_c\tau - \cos\left[2\omega_c t_1 + \omega_c\tau + 2\theta_c\right]\right\}$$

$$(3.42)\ \overline{R_{Carrier}(t_1,\tau)} = \frac{1}{T}\int_0^T R_{Carrier}(t_1,\tau)dt_1$$

$$(3.43)\ \overline{R_{Carrier}(t_1,\tau)} = \frac{A^2}{2T}\left\{\int_0^T \cos \omega_c\tau dt_1 - \int_0^T \cos\left[2\omega_c t_1 + \omega_c\tau + 2\theta_c\right]dt_1\right\}$$

Expanding the second term trigonometrically

$$(3.44)\ \overline{R_{Carrier}(t_1,\tau)} = \frac{A^2}{2}\left\{\cos \omega_c\tau - \frac{1}{T}\cos(\omega_c\tau + 2\theta)\int_0^T \cos 2\omega_c t_1 dt_1 + \frac{1}{T}\sin(\omega_c\tau + 2\theta)\int_0^T \sin 2\omega_c t_1 dt_1\right\}$$

$$(3.45)\ \overline{R_{Carrier}(t_1,\tau)} = \frac{A^2}{2}\left\{\cos \omega_c\tau - \frac{\cos(\omega_c\tau + 2\theta)}{T}\left[\frac{\sin 2\omega_c t_1}{2\omega_c}\right]_0^T - \frac{\sin(\omega_c\tau + 2\theta)}{T}\left[\frac{\cos 2\omega_c t_1}{2\omega_c}\right]_0^T\right\}$$

(3.46) $$\overline{R_{Carrier}(t_1,\tau)} = \frac{A^2}{2}\left\{\cos\omega_c\tau - \frac{\cos(\omega_c\tau + 2\theta)}{2\omega_c T}\left[\sin 2\omega_c T - 0\right] - \frac{\sin(\omega_c\tau + 2\theta)}{2\omega_c T}\left[\cos 2\omega_c T - 1\right]\right\}$$

But from the definition of the coherent ASK signal

$$\omega_c = \frac{2\pi K}{T} \qquad K = \text{integer}$$

Utilizing this relationship

(3.47) $$\overline{R_{Carrier}(t_1,\tau)} = \frac{A^2}{2}\left\{\cos\omega_c\tau - \frac{\cos(\omega_c\tau + 2\theta)}{4\pi K}\left[\sin 4\pi K - 0\right] - \frac{\sin(\omega_c\tau + 2\theta)}{4\pi K}\left[\cos 4\pi K - 1\right]\right\}$$

Noting that

$$\sin 4\pi K = 0 \qquad \text{where } K = \text{integer}$$

$$\cos 4\pi K = 1 \qquad \text{where } K = \text{integer}$$

The time averaged autocorrelation function for the carrier becomes

(3.48) $$\overline{R_{Carrier}(t_1,\tau)} = \frac{A^2}{2}\cos\omega_c\tau$$

(3.49) $$\overline{R_{Carrier}(t_1,\tau)} = R_{Carrier}(\tau)$$

Thus, the time averaged autocorrelation function for the carrier with the constant phase θ_c is equal to the autocorrelation function of the carrier with a uniformly distributed random phase.

The time averaged autocorrelation function of coherent ASK becomes

(3.50) $$\overline{R_{ASKC}(t_1,\tau)} = \left[1 + \beta^2 R_M(\tau)\right] R_{Carrier}(\tau)$$

which is equal to the autocorrelation function for the non-coherent ASK signal.

(3.51) $$\overline{R_{ASKC}(t_1,\tau)} = R_{ASKN}(\tau)$$

Power Spectral Density

The coherent ASK autocorrelation function, Equation 3.37 can be written as

(3.52) $$R_{ASKC}(t_1,\tau) = R'_{ASKC}(t_1,\tau) + \beta^2 R''_{ASKC}(t_1,\tau)$$

with corresponding Fourier transforms

(3.53) $$F'_{ASKC}(t_1,\omega) = F'_{ASKC}(t_1,\omega) + \beta^2 F''_{ASKC}(t_1,\omega)$$

where $F'_{ASKC}(t_1,\omega)$ and $F''_{ASKC}(t_1,\omega)$ are the Fourier transforms of the first and second terms respectively of Equation 3.52.

The Fourier transform of Equation 3.53 can be interpreted as a time varying power spectral density.

(3.54) $$S_{ASKC}(t_1,\omega) = S'_{ASKC}(t_1,\omega) + \beta^2 S''_{ASKC}(t_1,\omega)$$

The first term of Equation 3.54 is the Fourier transform of the autocorrelation function, $R_{Carrier}(t_1,\tau)$, of the carrier.

(3.55) $$S'_{ASKC}(t_1,\omega) = \int_{-\infty}^{\infty} R'_{ASKC}(t_1,\tau)e^{-j\omega\tau}d\tau$$

$$S'_{ASKC}(t_1,\tau) = \int_{-\infty}^{\infty} R_{Carrier}(t_1,\tau)\ e^{-j\omega\tau}d\tau \tag{3.56}$$

The expression for $R_{Carrier}(t_1,\tau)$, Equation 3.41, is

$$R_{ASKC}(t_1,\tau) = \frac{A^2}{2}\left[\cos\ \omega_c\tau\ -\ \cos\left\{2\omega_c t_1 + \omega_c\tau + 2\theta_c\right\}\right] \tag{3.41}$$

which can trigonometrically be reduced to

$$\begin{aligned} R_{ASKC}(t_1,\tau) = \frac{A^2}{2}\Big[&\cos\ \omega_c\tau\ -\ \cos\ 2(\omega_c t_1 + \theta_c)\cos\ \omega_c\tau \\ &+ \sin\ 2(\omega_c t_1 + \theta_c)\sin\ \omega_c\tau\Big] \end{aligned} \tag{3.57}$$

Utilizing Equation 3.56 , $S'_{ASKC}(t_1,\omega)$ is

$$\begin{aligned} S'_{ASKC}(t_1,\omega) = \frac{A^2}{2}\Big\{&\int_{-\infty}^{\infty}\cos\ \omega_c\tau e^{-j\omega\tau}d\tau \\ &-\ \cos\ 2(\omega_c t_1 + \theta_c)\int_{-\infty}^{\infty}\cos\ \omega_c\tau e^{-j\omega\tau}d\tau \\ &+\ \sin\ 2(\omega_c t_1 + \theta_c)\int_{-\infty}^{\infty}\sin\ \omega_c\tau e^{-j\omega\tau}d\tau\Big\} \end{aligned} \tag{3.58}$$

Noting Equation 3.13 and Equation 3.15 and that

$$(3.59)\qquad \frac{1}{2\pi}\int_{-\infty}^{\infty} \sin ux_o \, e^{jux} \, du = \frac{1}{2j}\left[\delta(X + X_o) - \delta(X - X_o)\right]$$

$$(3.60)\qquad S'_{ASKC}(t_1,\omega) =$$

$$\frac{A^2\pi}{2}\left[\delta(\omega + \omega_c) + \delta(\omega - \omega_c)\right]\left[1 - \cos 2(\omega_c t_1 + \theta_c)\right]$$

$$+ \frac{A^2\pi}{2j}\left[\delta(\omega + \omega_c) - \delta(\omega - \omega_c)\right] \sin 2(\omega_c t_1 + \theta_c)$$

The second term of Equation 3.54 can be obtained by realizing that the Fourier transform of the product of two autocorrelation functions is equal to the convolution of the two power spectra.

$$(3.61)\qquad S''_{ASKC}(t_1,\omega) = S_M(\omega) * S_{Carrier}(t_1,\omega)$$

Implementing Equation 3.61

$$(3.62)\qquad S''_{ASKC}(t_1,\omega) = \int_{-\infty}^{\infty} S_M(\omega_0) \, S_{Carrier}(t_1,\omega - \omega_0) d\omega_0$$

(3.63) $$S''_{ASKC}(t_1,\omega) = \int_{\infty}^{\infty} \left[\frac{V^2T \sin^4\left(\frac{\omega_0 T}{4}\right)}{\left(\frac{\omega_0 T}{4}\right)^2} \right] \left[\frac{A^2\pi}{2} \left\{ \delta(\omega + \omega_c - \omega_0) + \delta(\omega-\omega_c-\omega_0) \right\} \left\{ 1 - \cos 2(\omega_c t_1 + \Theta_c) \right\} + \frac{A^2\pi}{2j} \left\{ \delta(\omega+\omega_c-\omega_0) - \delta(\omega-\omega_c-\omega_0) \right\} \left\{ \sin 2(\omega_c t_1 + \omega_c) \right\} \right] d\omega_0$$

(3.64) $$S''_{ASKC}(t_1,\omega) = \frac{A^2V^2\pi T}{2} \left| 1 - \cos 2(\omega_c t_1 + \Theta_c) \right| \int_{-\infty}^{\infty} \frac{\sin^4\left(\frac{\omega_0 T}{4}\right)}{\left(\frac{\omega_0 T}{4}\right)^2} \left| \delta(\omega+\omega_c-\omega_0) + \delta(\omega-\omega_c-\omega_0) \right| d\omega_0$$

$$- j \frac{A^2V^2\pi T}{2} \sin 2(\omega_c t_1 + \theta_c) \int_{-\infty}^{\infty} \frac{\sin^4\left(\frac{\omega_0 T}{4}\right)}{\left(\frac{\omega_0 T}{4}\right)^2} \left| \delta(\omega+\omega_c-\omega_0) - \delta(\omega-\omega_c-\omega_0) \right| d\omega_0$$

(3.65) $$S''_{ASKC}(t_1,\omega) = \frac{A^2V^2\pi T}{2} \left| 1 - \cos 2(\omega_c t + \Theta_c) \right| \left[\frac{\sin^4\left\{\frac{(\omega+\omega_c)T}{4}\right\}}{\left\{\frac{(\omega+\omega_c)T}{4}\right\}^2} + \frac{\sin^4\left\{\frac{(\omega-\omega_c)T}{4}\right\}}{\left\{\frac{(\omega-\omega_c)T}{4}\right\}^2} \right] - j \frac{A^2V^2\pi T}{2} \sin 2(\omega_c t_1 + \theta_c) \left[\frac{\sin^4\left\{\frac{(\omega+\omega_c)T}{4}\right\}}{\left\{\frac{(\omega+\omega_c)T}{4}\right\}^2} - \frac{\sin^4\left\{\frac{(\omega-\omega_c)T}{4}\right\}}{\left\{\frac{(\omega-\omega_c)T}{4}\right\}^2} \right]$$

Substitution of Equation 3.60 and Equation 3.65 into Equation 3.54 yeilds the expression for the time varying power spectral density of the coherent ASK case.

(3.66) $S_{ASKC}(t_1,\omega) =$

$$\left[\frac{A^2\pi}{2}\left[\delta(\omega + \omega_c) + \delta(\omega - \omega_c)\right] + \frac{A^2\beta^2V^2\pi T}{2}\left\{\frac{\sin^4\left[\frac{(\omega + \omega_c)T}{4}\right]}{\left[\frac{(\omega + \omega_c)T}{4}\right]^2}\right.\right.$$

$$\left.\left. + \frac{\sin^4\left[\frac{(\omega - \omega_c)T}{4}\right]}{\left[\frac{(\omega - \omega_c)T}{4}\right]^2}\right\}\right]\left[1 - \cos 2(\omega_c t_1 + \theta_c)\right]$$

$$- j\left[\frac{A^2\pi}{2}\left|\delta(\omega+\omega_c) - \delta(\omega-\omega_c)\right| + \frac{A^2V^2\pi T}{2}\left\{\frac{\sin^4\left[\frac{(\omega+\omega_c)T}{4}\right]}{\left[\frac{(\omega+\omega_c)T}{4}\right]^2}\right.\right.$$

$$\left.\left.\frac{-\sin\left[\frac{(\omega-\omega_c)T}{4}\right]}{\left[\frac{(\omega-\omega_c)T}{4}\right]^2}\right\}\right] \sin 2(\omega_c t_1 + \Theta_c)$$

As observed from Equation 3.66, the power spectral density for the coherent ASK signal consists of discrete time varying carrier components, and time varying sidebands resulting from the split-phase spectrum being translated to appear about plus and minus the carrier frequency. The imaginary term appears because $R_{ASKC}(t_1,\tau)$ is not an even function of τ . It should be noted that maximum sideband power occurs when the product βV equals one ($\beta V = 1$), which corresponds to "on-off" keying of the carrier by the split-phase code.

Time averaged power spectral density. It should be observed that the expression for the time varying power spectral density of coherent ASK when time averaged with respect to t_1 reduces to the expression for the power spectral density of non-coherent ASK, as follows. Note that Equation 3.66 can be written as

$$S_{ASKC}(t_1,\omega) = \left[1 - \cos 2(\omega_c t_1 + \theta_c)\right] S_{ASKN}(\omega) - Q(\omega) \sin 2(\omega_c t_1 + \Theta_c) \tag{3.67}$$

Taking the average with respect to t_1

$$\overline{S_{ASKC}(t_1,\omega)} = \frac{1}{T}\int_0^T S_{ASKC}(t_1,\omega)dt_1 \tag{3.68}$$

$$\overline{S_{ASKC}(t_1,\omega)} = S_{ASKN}(\omega)\left\{\frac{1}{T}\int_0^T dt - \frac{1}{T}\int_0^T \cos 2(\omega_c t_1 + \theta_c)dt_1\right\} - j\,\frac{Q(\omega)}{T}\int_0^T \sin 2(\omega_c t_1 + \Theta_c)\, dt \tag{3.69}$$

Expanding $\cos 2(\omega_c t_1 + \theta_c)$ and $\sin 2(\omega_c t_1 + \theta_c)$ trigonometrically

$$\cos 2(\omega_c t_1 + \theta_c) = \cos 2\omega_c t_1 \cos 2\theta_c - \sin 2\omega_c t_1 \sin 2\theta_c$$

$$\sin 2(\omega_c t_1 + \theta_c) = \sin 2\omega_c t_1 \cos 2\theta_c + \cos 2\omega_c t_1 \sin 2\theta_c$$

(3.70) $$\overline{S_{ASKC}(t_1,\omega)} = S_{ASKN}(\omega)\left\{ 1 - \frac{1}{T} \cos 2\theta_c \int_0^T \cos 2\omega_c t_1 dt_1 \right.$$
$$\left. + \frac{1}{T} \sin 2\theta_c \int_0^T \sin 2\omega_c t_1 dt_1 \right\}$$
$$- j\,\frac{Q(\omega)}{T} \left\{ \cos 2\theta_c \int_0^T \sin 2\omega_c t_1 dt_1 \right.$$
$$\left. + \sin 2\theta_c \int_0^T \cos 2\omega_c t_1 dt_1 \right\}$$

(3.71) $$\overline{S_{ASKC}(t_1,\omega)} = S_{ASKN}(\omega)\left\{ 1 - \frac{1}{T} \cos 2\theta_c \left[\frac{1}{2\omega_c} \sin x\right]_0^{2\omega_c T} \right.$$
$$\left. - \frac{1}{T} \sin 2\theta_c \left[\frac{1}{2\omega_c} \cos x\right]_0^{2\omega_c T} \right\} - j\,Q(\omega)$$
$$\left\{ \frac{1}{T} \cos 2\theta_c \left[- \frac{1}{2\omega_c} \cos x\right]_0^{2\omega_c T} + \frac{1}{T} \sin 2\theta_c \left[\frac{1}{2\omega_c} \sin x\right]_0^{2\omega_c T} \right\}$$

$$(3.72)\qquad \overline{S_{ASKC}(t_1,\omega)} = S_{ASKN}(\omega)\left\{1 - \frac{1}{2\omega_c T}\cos 2\theta_c \sin 2\omega_c T - \frac{1}{2\omega_c T}\sin 2\theta_c\left[\cos 2\omega_c T - 1\right]\right\} - j\,Q(\omega)\left\{\frac{1}{2\omega_c T}\cos 2\theta_c\left[1 - \cos 2\omega_c T\right] + \frac{1}{2\omega_c T}\sin 2\theta_c\left[\sin 2\omega_c T\right]\right\}$$

From the definition of coherent ASK

$$f_c = Kf_B = \frac{K}{T} \qquad K = \text{integer}$$

$$\omega_c = \frac{2\pi K}{T}$$

$$(3.73)\qquad \overline{S_{ASKC}(t_1,\omega)} = S_{ASKN}(\omega)\left\{1 - \frac{1}{4\pi K}\cos 2\theta_c \cancelto{0}{\sin 4\pi K} - \frac{1}{4\pi K}\sin 2\theta_c \cancelto{0}{\left[\cos 4\pi K - 1\right]}\right\} - j\,Q(\omega)\left\{\frac{1}{4\pi K}\cos 2\theta_c \cancelto{0}{\left[1 - \cos 4\pi K\right]} + \frac{1}{4\pi K}\sin 2\theta_c \cancelto{0}{\left[\sin 4\pi K\right]}\right\}$$

$$(3.74)\qquad \overline{S_{ASKC}(t_1,\omega)} = S_{ASKN}(\omega)$$

Showing that the time average of the coherent ASK time varying power spectral density is equal to the power spectral density of non-coherent ASK.

CHAPTER IV

PHASE SHIFT KEYING

An expression for a sinusoidal carrier, phase modulated by a split-phase code is

(4.1) $$e_{PSK}(t) = A \sin\left[\omega_c t + \theta + \beta V m(t)\right]$$

where

A - carrier peak amplitude

β - modulation index

V - voltage level of split phase code

m(t) - split-phase code switching function (±1)

ω_c - carrier angular frequency

θ - initial phase of the carrier

Expanding Equation 4.1 trigonometrically

(4.2) $$e_{PSK}(t) = A\left[\sin(\omega_c t + \theta)\cos \beta V m(t) + \cos(\omega_c t + \theta)\sin \beta V m(t)\right]$$

(4.3) $$e_{PSK}(t) = A \sin(\omega_c t + \theta)\cos \beta V m(t) + A \cos(\omega_c t + \theta)\sin \beta V m(t)$$

Noting that

$$\cos\left[\pm x\right] = \cos x$$

$$\sin\left[\pm x\right] = \pm \sin x$$

then

(4.4) $$e_{PSK}(t) = A\left[\cos \beta V\right]\sin(\omega_c t + \theta) + A\, m(t)\left[\sin \beta V\right]\cos(\omega_c t + \theta)$$

Since βV is a constant, the first term represents a discrete carrier component, and the second term represents double sideband suppressed-carrier modulation by the split-phase code.

I. NON-COHERENT MODULATION

For non-coherent phase modulation the carrier frequency, f_c, is not an integral multiple of the bit rate of the split-phase code.

$$f_c \neq K f_B \qquad K = \text{integer}$$

$$f_c \neq \frac{K}{T}$$

This relationship between the carrier frequency and the bit rate of the split-phase code, requires that the initial phase of the carrier, θ, be considered a random variable uniformly distributed between 0 and 2π

θ_N - random variable uniformly distributed between 0 and 2π

Figure 7 pictorially represents non-coherent PSK.

<u>Autocorrelation Function</u>

The ensemble average autocorrelation function is

$$(4.5)\quad R_{PSKN}(t_1,\tau) = E\left[e_{PSKN}(t_1)e_{PSKN}(t_1+\tau)\right]$$

$$(4.6)\quad R_{PSKN}(t_1,\tau) =$$

$$E\left[\left\{A\cos\beta V\sin(\omega_c t_1+\theta_N) + A\sin\beta V m(t_1)\cos(\omega_c t_1+\theta_N)\right\}\right.$$

$$\left\{A\cos\beta V\sin(\omega_c(t_1+\tau)+\theta_N)\right.$$

$$\left.\left.+ A\sin\beta V m(t_1+\tau)\cos(\omega_c(t_1+\tau)+\theta_N)\right\}\right]$$

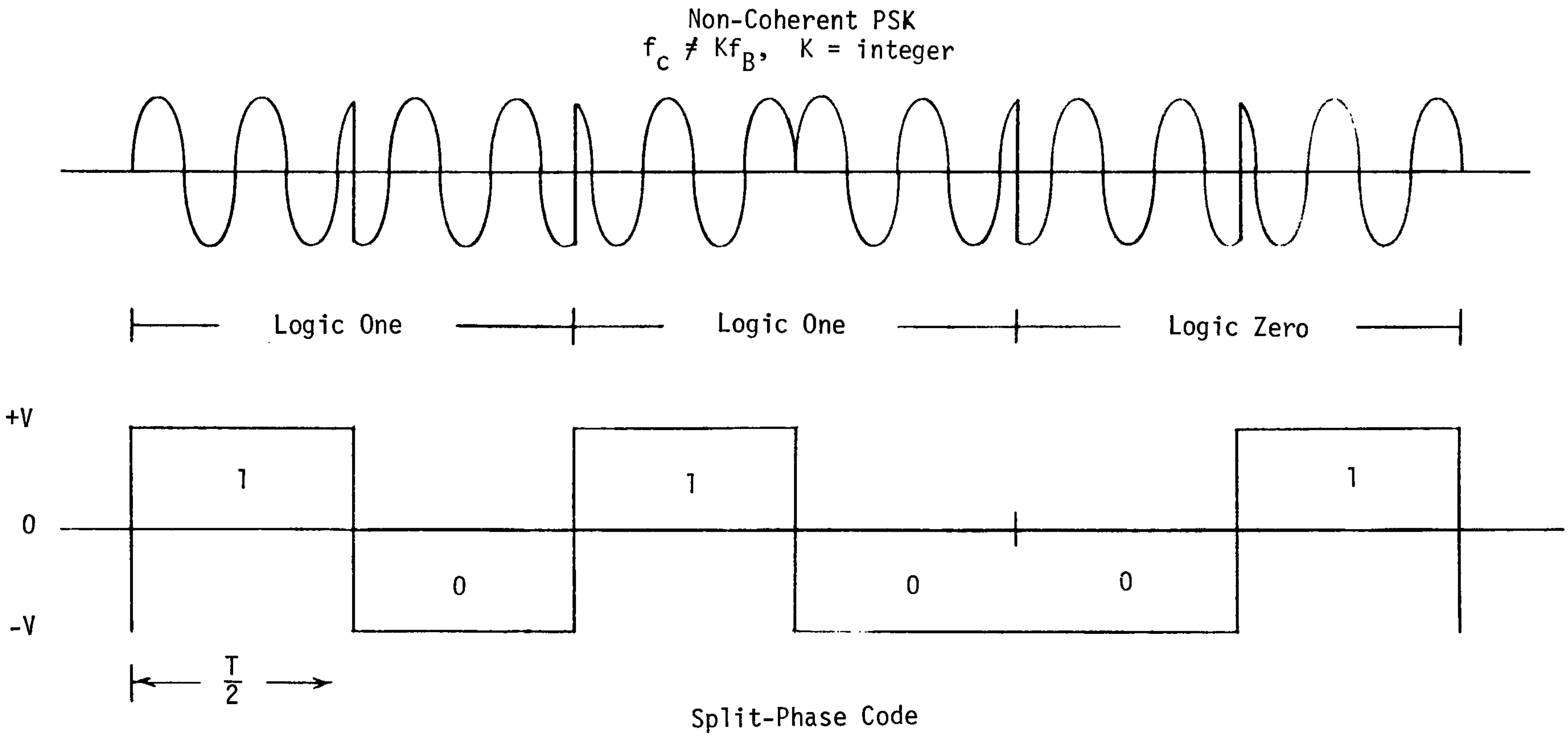

FIGURE 7

NON-COHERENT PSK WITH SPLIT-PHASE
CODE MODULATION

Noting that the expected value of a sum is equal to the sum of the expected values, and that since the carrier and modulation are non-coherent, the expected value of the product is equal to the product of the expected values.

(4.7) $R_{PSKN}(t_1,\tau) =$

$$A^2 \cos^2 \beta V E\left[\sin(\omega_c t_1 + \theta_N)\sin\left\{\omega_c(t_1 + \tau) + \theta_N\right\}\right]$$

$$+ A^2 \sin \beta V \cos \beta V E\left[m(t_1)\right] E\left[\sin\left\{\omega_c(t_1 + \tau) + \theta_N\right\}\cos(\omega_c t_1 + \theta_N)\right]$$

$$+ A^2 \sin \beta V \cos \beta V E\left[m(t_1 + \tau)\right] E\left[\sin(\omega_c t_1 + \theta_N)\cos\left\{\omega_c(t_1 + \tau) + \theta_N\right\}\right]$$

$$+ A^2 \sin^2 \beta V E\left[m(t_1)m(t_1 + \tau)\right] E\left[\cos(\omega_c t_1 + \theta_N)\cos\left\{\omega_c(t_1 + \tau) + \theta_N\right\}\right]$$

Expanding trigonometrically

(4.8) $R_{PSKN}(t_1,\tau) =$

$$\frac{A^2 \cos^2 \beta V}{2}\left\{\left[E \cos \omega_c \tau\right] - E\left[\cos\left\{\omega_c(2t_1 + \tau) + 2\theta_N\right\}\right]\right\}$$

$$+ \frac{A^2}{2} \sin \beta V \cos \beta V E\left[m(t_1)\right]\left\{E\left[\sin\left\{\omega_c(2t_1 + \tau) + 2\theta_N\right\}\right] + E\left[\sin \omega_c \tau\right]\right\}$$

$$+ \frac{A^2}{2} \sin \beta V \cos \beta V E\left[m(t_1 + \tau)\right]\left\{E\left[\sin\left\{\omega_c(2t_1 + \tau) + 2\theta_N\right\}\right]\right.$$

$$\left. - E\left[\sin \omega_c \tau\right]\right\} + \frac{A^2}{2} \sin^2 \beta V E\left[m(t_1)m(t_1 + \tau)\right]$$

$$\left\{E\left[\cos\left\{\omega_c \tau + \cos \omega_c(2t_1 + \tau) + 2\theta_N\right\}\right]\right\}$$

Again, recognizing that the expected value of a sum is equal to the sum of the expected values, and noting that

$$R_{Carrier}(\tau) = \frac{A^2}{2} E\left[\cos \omega_c\tau\right] = \frac{A^2}{2} \cos \omega_c\tau \tag{3.9}$$

$$R_M(\tau) = V^2 E\left[m(t_1)m(t_1 + \tau)\right] \tag{2.9}$$

$$E\left[m(t)\right] = E\left[m(t_1)\right] = E\left[m(t_1 + \tau)\right] = 0 \tag{2.3}$$

$$E\left[\cos \omega_c(2t_1 + \tau) + 2\theta\right] = E\left[\sin \omega_c(2t_1 + \tau) + 2\theta\right] = 0$$

The expected value of a sinusoid with uniformly distributed random phase is zero. Therefore

$$R_{PSKN}(t_1,\tau) = R_{PSK}(\tau) = \cos^2 \beta V R_{Carrier}(\tau) + \frac{\sin^2 \beta V}{V^2} R_M(\tau)R_{Carrier}(\tau) \tag{4.9}$$

Power Spectral Density

The non-coherent PSK autocorrelation function, Equation 4.9, can be written as

$$R_{PSKN}(\tau) = \cos^2 \beta V R'_{PSKN}(\tau) + \frac{\sin^2 \beta V}{V^2} R''_{PSKN}(\tau) \tag{4.10}$$

with corresponding power spectral density

$$S_{PSKN}(\omega) = \cos^2 \beta V S'_{PSKN}(\omega) + \frac{\sin^2 \beta V}{V^2} S''_{PSKN}(\omega) \tag{4.11}$$

where $S'_{PSKN}(\omega)$ and $S''_{PSKN}(\omega)$ are the Fourier transforms of the first and second terms respectively of Equation 4.10.

$$(4.12) \qquad S'_{PSKN}(\omega) = \int_{-\infty}^{\infty} R'_{PSKN}(\tau)e^{-j\omega\tau}d\tau = \int_{-\infty}^{\infty} R_{Carrier}(\tau)e^{-j\omega\tau}d\tau$$

This integral has been evaluated previously, Equation 3.12, and found to be, from Equation 3.15,

$$(4.13) \qquad S'_{PSKN}(\omega) = \frac{A^2\pi}{2}\Big[\delta(\omega + \omega_c) + \delta(\omega - \omega_c)\Big]$$

The second term of Equation 4.11 can be obtained by noting that the Fourier transform of the product of two autocorrelation functions is equivalent to the convolution of the two power spectra.

$$(4.14) \qquad S''_{PSKN}(\omega) = S_M(\omega) * S_{Carrier}(\omega)$$

This convolution has been previously implemented, Equation 3.16, and found to be, from Equation 3.21,

$$(4.15) \qquad S''_{PSKN}(\omega) = \frac{A^2V^2\pi T}{2}\left\{\frac{\sin^4\left[\frac{(\omega + \omega_c)T}{4}\right]}{\left[\frac{(\omega + \omega_c)T}{4}\right]^2} + \frac{\sin^4\left[\frac{(\omega - \omega_c)T}{4}\right]}{\left[\frac{(\omega - \omega_c)T}{4}\right]^2}\right\}$$

Utilizing Equation 4.11, the power spectral density for the sinusoidal carrier non-coherently phase modulated by the split-phase code is

$$(4.16)\quad S_{PSKN}(\omega) = \frac{A^2\pi}{2}\left[\cos^2 \beta V\right]\left[\delta(\omega + \omega_c) + \delta(\omega - \omega_c)\right] + \frac{A^2\beta^2\pi T}{2}\left[\sin^2 \beta V\right]\left\{\frac{\sin^4\left[\frac{(\omega + \omega_c)T}{4}\right]}{\left[\frac{(\omega + \omega_c)T}{4}\right]^2} + \frac{\sin^4\left[\frac{(\omega + \omega_c)T}{4}\right]}{\left[\frac{(\omega + \omega_c)T}{4}\right]^2}\right\}$$

Figure 8 pictorially represents $S_{PSKN}(\omega)$, the power spectral density for the non-coherent PSK case.

In general, as observed from Equation 4.16 and Figure 8, the power spectral density for the non-coherent PSK signal consists of discrete carrier components, and sidebands resulting from the split-phase spectrum being translated to appear about plus or minus the carrier frequency. It should be noted that as the sideband power is maximized, the carrier components tend to vanish, and indeed at $\beta V = Q\,\frac{\pi}{2}$, Q = odd integer, the carrier does vanish and the sideband power is a maximum.

II. COHERENT MODULATION

For coherent phase modulation the carrier frequency, f_c, is an integral multiple of the split-phase code bit rate, f_B.

$$f_c = Kf_B \qquad K = \text{integer}$$

$$f_c = \frac{K}{T}$$

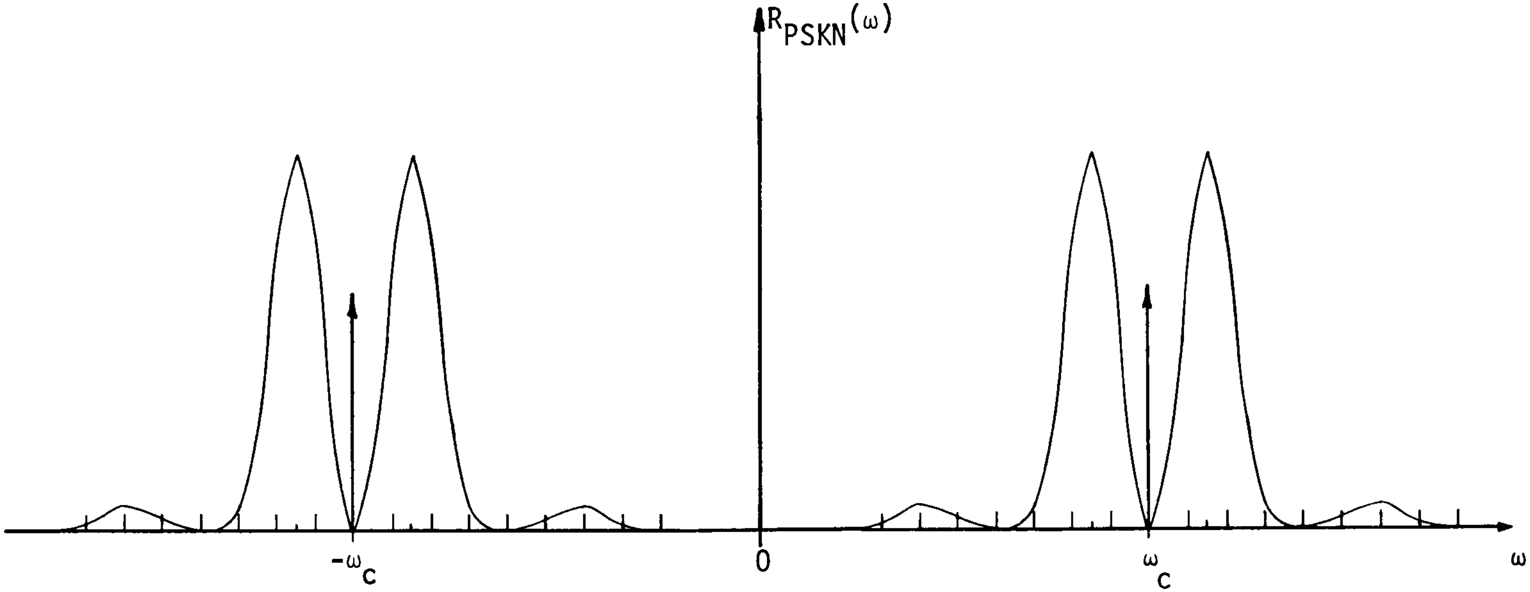

FIGURE 8

POWER SPECTRAL DENSITY FOR NON-COHERENT PSK
WITH SPLIT-PHASE CODE MOUDLATION

This condition has the effect of making the θ of Equation 3.1 a constant

$$\theta = \theta_c = \text{constant}$$

Figure 9 pictorially represents coherent PSK. It should be noted from Figure 9 that the minimum system bandwidth requirement occurs when θ_c equals zero ($\theta_c = 0$) or some integral multiple of π ($\theta_c = K\pi$, K = integer). Under these conditions no instantaneous amplitude change is required when a transition occurs in the split-phase code. The system bandwidth requirement increases with increased θ_c from the minimum bandwidth conditions to a maximum when θ_c equals an odd integral multiple of $\frac{\pi}{2}$ ($\theta_c = Q\,\frac{\pi}{2}$, Q = odd integer), where the maximum possible instantaneous amplitude change is required.

Autocorrelation Function

The ensemble average autocorrelation function for coherent PSK is

$$R_{PSKC}(t_1,\tau) = E\left[e_{PSKC}(t_1)e_{PSKC}(t_1 + \tau)\right] \tag{4.17}$$

Utilizing Equation 4.4 with $\theta = \theta_c$

$$\begin{aligned}R_{PSKC}(t_1,\tau) = {} & E\Big\{\left[A \cos \beta V \sin(\omega_c t_1 + \theta_c) + A \sin \beta V m(t_1)\cos(\omega_c t_1 + \theta_c)\right] \\ & \Big[A \cos \beta V \sin\left\{\omega_c(t_1 + \tau) + \theta_c\right\} \\ & + A \sin \beta V m(t_1 + \tau)\cos\left\{\omega_c(t_1 + \tau) + \theta_c\right\}\Big]\Big\}\end{aligned} \tag{4.18}$$

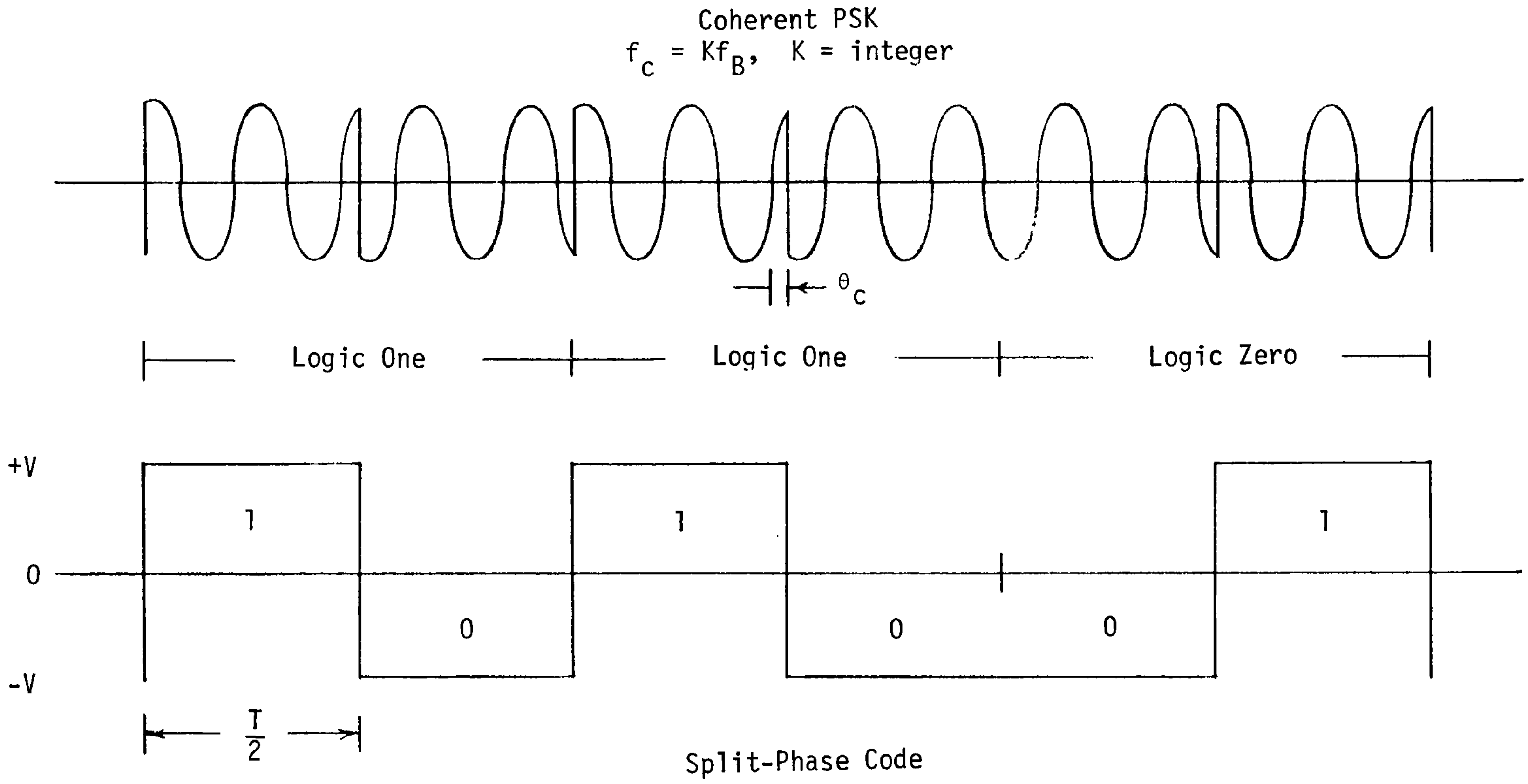

FIGURE 9

COHERENT PSK WITH SPLIT-PHASE
CODE MODULATION

(4.19) $R_{PSKC}(t_1,\tau) =$

$$E\Big\{A^2 \cos^2 \beta V \sin(\omega_c t_1 + \theta_c)\sin\left[\omega_c(t_1 + \tau) + \theta_c\right]$$

$$+ A^2 \sin \beta V \cos \beta V m(t_1)\cos(\omega_c t_1 + \theta_c)\sin\left[\omega_c(t_1 + \tau) + \theta_c\right]$$

$$+ A^2 \cos \beta V \sin \beta V m(t_1 + \tau)\sin(\omega_c t_1 + \theta_c)\cos\left[\omega_c(t_1 + \tau) + \theta_c\right]$$

$$+ A^2 \sin^2 \beta V m(t_1)m(t_1 + \tau)\cos(\omega_c t_1 + \theta_c)\cos\left[\omega_c(t_1 + \tau) + \theta_c\right]\Big\}$$

The random variables here are $m(t_1)$ and $m(t_1 + \tau)$, and from the definition of ensemble average autocorrelation function, Equation 2.5.

(4.20) $R_{PSKC}(t_1,\tau) =$

$$A^2 \cos^2 \beta V \sin(\omega_c t_1 + \theta_c)\sin\left[\omega_c(t_1 + \tau) + \theta_c\right]$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p\left[m(t_1)m(t_1 + \tau)\right]dm(t_1)dm(t_1 + \tau)$$

$$+ A^2 \sin \beta V \cos \beta V \cos(\omega_c t_1 + \theta_c)\sin\left[\omega_c(t_1 + \tau) + \theta_c\right]$$

$$\int_{-\infty}^{\infty} m(t_1)p\left[m(t_1)\right]dm(t_1)$$

$$+ A^2 \sin \beta V \cos \beta V \sin(\omega_c t_1 + \theta_c)\cos\left[\omega_c(t_1 + \tau) + \theta_c\right]$$

$$\int_{-\infty}^{\infty} m(t_1 + \tau)p\left[m(t_1 + \tau)\right]dm(t_1 + \tau)$$

$$+ A^2 \sin^2 \beta V \cos(\omega_c t_1 + \theta_c)\cos\left[\omega_c(t_1 + \tau) + \theta_c\right]$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} m(t_1)m(t_1 + \tau)p\left[m(t_1)m(t_1 + \tau)\right]dm(t_1)dm(t_1 + \tau)$$

where

$$(3.29)\quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} m(t_1)p\Big[m(t_1)m(t_1 + \tau)\Big]dm(t_1)dm(t_1 + \tau)$$

$$= \int_{-\infty}^{\infty} m(t_1)p\Big[m(t_1)\Big]dm(t_1)$$

$$(3.30)\quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} m(t_1 + \tau)p\Big[m(t_1)m(t_1 + \tau)\Big]dm(t_1)dm(t_1 + \tau)$$

$$= \int_{-\infty}^{\infty} m(t_1 + \tau)p\Big[m(t_1 + \tau)\Big]dm(t_1 + \tau)$$

But

$$(3.31)\quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p\Big[m(t_1)m(t_1 + \tau)\Big]dm(t_1)dm(t_1 + \tau) = 1$$

$$(3.32)\quad \int_{-\infty}^{\infty} m(t_1)p\Big[m(t_1)\Big]dm(t_1) = E\Big[m(t_1)\Big] = 0$$

$$(3.33)\quad \int_{-\infty}^{\infty} m(t_1 + \tau)p\Big[m(t_1 + \tau)\Big]dm(t_1) = E\Big[m(t_1 + \tau)\Big] = 0$$

$$(4.21)\quad \frac{R_M(\tau)}{V^2} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} m(t_1)m(t_1 + \tau)p\Big[m(t_1)m(t_1 + \tau)\Big]dm(t_1)dm(t_1 + \tau)$$

Utilizing Equation 3.29 through Equation 3.33, and Equation 4.21, $R_{PSKC}(t_1,\tau)$ of Equation 4.20 becomes

$$(4.22)\quad R_{PSKC}(t_1,\tau) = A^2 \cos^2 \beta V \sin(\omega_c t_1 + \theta_c)\sin\left[\omega_c(t_1 + \tau) + \theta_c\right]$$

$$+ \frac{A^2 \sin^2 \beta V}{V^2} \cos(\omega_c t_1 + \theta_c)\cos\left[\omega_c(t_1 + \tau) + \theta_c\right]R_M(\tau)$$

Noting that

$$(3.36)\quad R_{Carrier}(t_1,\tau) = A^2 \sin\left[\omega_c t_1 + \theta_c\right]\sin\left[\omega_c(t_1 + \tau) + \theta_c\right]$$

$$(4.23)\quad R_{q\text{-}Carrier}(t_1,\tau) = A^2 \cos\left[\omega_c t_1 + \theta_c\right]\cos\left[\omega_c(t_1 + \tau) + \theta_c\right]$$

where $R_{q\text{-}Carrier}(t_1,\tau)$ is the autocorrelation function for the quadrature component of the carrier. Then $R_{PSKC}(t_1,\tau)$ becomes

$$(4.24)\quad R_{PSKC}(t_1,\tau) = \cos^2 \beta V R_{Carrier}(t_1,\tau)$$

$$+ \frac{\sin^2 \beta V}{V^2} R_M(\tau) R_{q\text{-}Carrier}(t_1,\tau)$$

<u>Time Averaged Autocorrelation Function</u>

The time averaged autocorrelation function for coherent PSK is

$$(4.25)\quad \overline{R_{PSKC}(t_1,\tau)} = \frac{1}{T}\int_0^T R_{PSKC}(t_1,\tau)dt_1$$

$$(4.26)\quad \overline{R_{PSKC}(t_1,\tau)} = \frac{1}{T}\left\{\cos^2 \beta V \int_0^T R_{Carrier}(t_1,\tau)dt_1\right.$$

$$\left. + \frac{\sin^2 \beta V}{V^2} R_M(\tau) \int_0^T R_{q\text{-}Carrier}(t_1,\tau)dt_1\right\}$$

The first integral has previously been evaluated

(3.42) $$\overline{R_{Carrier}(t_1,\tau)} = \frac{1}{T}\int_0^T R_{Carrier}(t_1,\tau)dt_1$$

and found to be, from Equation 3.48 and Equation 3.49

(3.48) $$\overline{R_{Carrier}(t_1,\tau)} = \frac{A^2}{2} \cos \omega_c\tau$$

(3.49) $$\overline{R_{Carrier}(t_1,\tau)} = R_{Carrier}(\tau)$$

Similarly, the time average of the autocorrelation function of the quadrature component $\overline{R_{q\text{-}Carrier}(t_1,\tau)}$ is

(4.27) $$\overline{R_{q\text{-}Carrier}(t_1,\tau)} = \frac{1}{T}\int_0^T R_{q\text{-}Carrier}dt_1$$

(4.28) $$\overline{R_{q\text{-}Carrier}(t_1,\tau)} = \frac{A^2}{2} \cos \omega_c\tau$$

(4.29) $$\overline{R_{q\text{-}Carrier}(t_1,\tau)} = R_{Carrier}(\tau)$$

Substitution of Equation 3.49 and Equation 4.29 into Equation 4.26 yeilds the time averaged autocorrelation function of the coherent PSK

(4.30) $$\overline{R_{PSKC}(t_1,\tau)} = \cos^2 \beta V R_{Carrier}(\tau) + \frac{\sin^2 \beta V}{V^2} R_M(\tau)R_{Carrier}(\tau)$$

which is equal to the autocorrelation function for the non-coherent PSK signal.

(4.31) $$\overline{R_{PSKC}(t_1,\tau)} = R_{PSKN}(\tau)$$

Power Spectral Density

The coherent PSK autocorrelation function, Equation 4.24, can be written as

$$R_{PSKC}(t_1,\tau) = \cos^2 \beta V R'_{PSKC}(t_1,\tau) + \frac{\sin^2 \beta V}{V^2} R''_{PSKC}(t_1,\tau) \tag{4.32}$$

with corresponding Fourier transforms

$$F_{PSKC}(t_1,\tau) = \cos^2 \beta V F'_{PSKC}(t_1,\omega) + \frac{\sin^2 \beta V}{V^2} F''_{PSKC}(t_1,\omega) \tag{4.33}$$

where $F'_{PSKC}(t_1,\omega)$ and $F''_{PSKC}(t_1,\omega)$ are the Fourier transforms of the first and second terms respectively of Equation 4.33. The Fourier transform of Equation 4.33 can be interpreted as a time varying power spectral density

$$S_{PSKC}(t_1,\omega) = \cos^2 \beta V S'_{PSKC}(t_1,\omega) + \frac{\sin^2 \beta V}{V^2} S''_{PSKC}(t_1,\omega) \tag{4.34}$$

The first term of Equation 4.33 is the Fourier transform of the autocorrelation function, $R_{Carrier}(t_1,\tau)$, of the carrier

$$S'_{PSKC}(t_1,\omega) = \int_{-\infty}^{\infty} R'_{PSKC}(t_1,\tau)e^{-j\omega\tau}d\tau \tag{4.35}$$

This integral has been previously evaluated, Equation 3.56, and found to be, from Equation 3.60,

$$S'_{PSKC}(t_1,\omega) = \frac{A^2\pi}{2}\left[\delta(\omega + \omega_c) + \delta(\omega - \omega_c)\right]\left[1 - \cos 2(\omega_c t_1 + \theta_c)\right] \tag{4.36}$$

$$- j\,\frac{A^2\pi}{2} \sin 2(\omega_c t_1 + \theta_c)\left[\delta(\omega + \omega_c) - \delta(\omega - \omega_c)\right]$$

$S''_{PSKC}(t_1,\omega)$ of Equation 4.34 can be obtained by noting that the Fourier transform of the product of two autocorrelation functions is equal to the convolution of the two power spectra.

$$(4.37) \qquad S''_{PSKC}(t_1,\omega) = S_M(\omega) * S_{q\text{-Carrier}}(t_1,\omega) \quad .$$

Obtaining an expression for $S_{q\text{-Carrier}}(t_1,\omega)$,

$$(4.38) \qquad S_{q\text{-Carrier}}(t_1,\omega) = \int_{-\infty}^{\infty} R_{q\text{-Carrier}}(t_1,\tau)\, e^{-j\omega\tau}\, d\tau$$

Utilizing Equation 4.23,

$$(4.39) \qquad S_{q\text{-Carrier}}(t_1,\omega) = A^2\int_{-\infty}^{\infty} \cos\left[\omega_c t_1 + \Theta_c\right]\cos\left[\omega_c(t_1+\tau) + \Theta_c\right] e^{-j\omega\tau}\, d\tau$$

Expanding trigonometrically

$$(4.40) \quad S_{q\text{-Carrier}}(t_1,\omega) = \frac{A^2}{2}\int_{-\infty}^{\infty}\left\{\cos\left[2\omega_c t_1 + \omega_c\tau + 2\Theta_c\right] + \cos\omega_c\tau\right\} e^{-j\omega\tau}\, d\tau$$

$$(4.41) \quad S_{q\text{-Carrier}}(t_1,\omega) = \frac{A^2}{2}\int_{-\infty}^{\infty} \cos\omega_c\tau\, e^{-j\omega\tau}\, d\tau + \frac{A^2}{2}\int_{-\infty}^{\infty} \cos\left[2(\omega_c t_1 + \Theta_c) + \omega_c\tau\right] e^{-j\omega\tau}\, d\tau$$

Noting Equation 3.13 and Equation 3.15, the first integral of Equation 4.41 can be obtained. Expanding the second integral trigonometrically

$$(4.42)\quad S_{q\text{-Carrier}}(t_1, \omega) = \frac{A^2\pi}{2}\left[\delta(\omega + \omega_c) + \delta(\omega - \omega_c)\right]$$

$$+ \frac{A^2}{2}\int_{-\infty}^{\infty}\Big\{\cos 2(\omega_c t_1 + \theta_c)\cos \omega_c\tau$$

$$- \sin 2(\omega_c t_1 + \theta_c)\sin \omega_c\tau\Big\}e^{-j\omega\tau}\, d\tau$$

$$(4.43)\quad S_{q\text{-Carrier}}(t_1, \omega) = \frac{A^2\pi}{2}\left[\delta(\omega + \omega_c) + \delta(\omega - \omega_c)\right]$$

$$+ \frac{A^2}{2}\cos 2(\omega_c t_1 + \theta_c)\int_{-\infty}^{\infty}\cos \omega_c\tau\, e^{-j\omega\tau}\, d\tau$$

$$- \frac{A^2}{2}\sin 2(\omega_c t_1 + \theta_c)\int_{-\infty}^{\infty}\sin \omega_c\tau\, e^{-j\omega\tau}\, d\tau$$

Again noting Equation 3.13 and Equation 3.15, and using the identity of Equation 3.59,

$$(4.44)\quad S_{q\text{-Carrier}}(t_1,\omega) = \frac{A^2\pi}{2}\left[\delta(\omega + \omega_c) + \delta(\omega - \omega_c)\right]$$

$$\left[1 + \cos 2(\omega_c t_1 + \theta_c)\right]$$

$$- j\,\frac{A^2\pi}{2}\sin 2(\omega_c t_1 + \theta_c)\left[\delta(\omega + \omega_c) - \delta(\omega - \omega_c)\right]$$

Now implementing Equation 4.37

$$(4.37)\quad S''_{PSKC}(t_1, \omega) = S_M(\omega) * S_{q\text{-Carrier}}(t_1, \omega)$$

$$(4.45)\quad S''_{PSKC}(t_1,\ \omega) = \int_{-\infty}^{\infty} S_M(\omega_o)\ S_{q\text{-Carrier}}(t_1, \omega - \omega_o)\ d\omega_o$$

$$(4.46)\quad S''_{PSKC}(t_1,\ \omega) = \int_{-\infty}^{\infty} \left[\frac{V^2T\ \sin^4\left(\frac{\omega_o T}{4}\right)}{\left[\frac{\omega_o T}{4}\right]^2}\right]\left[\frac{A^2\pi}{2}\left\{\delta(\omega + \omega_c - \omega_o) + \delta(\omega - \omega_c - \omega_o)\right\}\left\{1 + \cos 2(\omega_c t_1 + \theta_c)\right\} - j\,\frac{A^2\pi}{2}\ \sin 2(\omega_c t_1 + \theta_c)\left\{\delta(\omega + \omega_c - \omega_o) - \delta(\omega - \omega_c - \omega_o)\right\}\right] d\omega_o$$

$$(4.47)\quad S''_{PSKC}(t_1,\omega) = \frac{A^2\pi}{2}\left\{1 + \cos 2(\omega_c t_1 + \theta_c)\right\}\int_{-\infty}^{\infty}\left[\frac{V^2T\ \sin^4\left(\frac{\omega_o T}{4}\right)}{\left(\frac{\omega_o T}{4}\right)^2}\right]\left[\delta(\omega + \omega_c - \omega_o) + \delta(\omega - \omega_c - \omega_o)\right] d\omega_o$$

$$- j\,\frac{A^2\pi}{2}\ \sin 2(\omega_c t_1 + \theta_c)\int_{-\infty}^{\infty}\left[\frac{V^2T\ \sin^4\left(\frac{\omega_o T}{4}\right)}{\left(\frac{\omega_o T}{4}\right)^2}\right]\left[\delta(\omega + \omega_c - \omega_o) - \delta(\omega - \omega_c - \omega_o)\right] d\omega_o$$

$$(4.48) \qquad S''_{PSKC}(t_1,\omega) = \frac{A^2\pi}{2}\left\{1 + \cos 2(\omega_c t_1 + \theta_c)\right\}$$

$$\left\{\frac{V^2T \sin^4\left[\frac{(\omega + \omega_c)T}{4}\right]}{\left[\frac{(\omega + \omega_c)T}{4}\right]^2} + \frac{V^2T \sin^4\left[\frac{(\omega - \omega_c)T}{4}\right]}{\left[\frac{(\omega - \omega_c)T}{4}\right]^2}\right\}$$

$$- j\,\frac{A^2\pi}{2}\ \sin 2(\omega_c t_1 + \theta_c)\left\{\frac{V^2T \sin^4\left[\frac{(\omega + \omega_c)T}{4}\right]}{\left[\frac{(\omega + \omega_c)T}{4}\right]^2}\right.$$

$$\left. - \frac{V^2T \sin^4\left[\frac{(\omega - \omega_c)T}{4}\right]}{\left[\frac{(\omega - \omega_c)T}{4}\right]^2}\right\}$$

Substitution of Equation 4.36 and Equation 4.48 into Equation 4.34 yields the expression for the time varying power spectral density of the coherent PSK case.

$$(4.49)\quad S_{PSKC}(t_1,\omega) = \frac{A^2\pi}{2}\cos^2\beta V\Bigg\{\Big[\delta(\omega+\omega_c)+\delta(\omega-\omega_c)\Big]$$

$$\Big[1-\cos 2(\omega_c t_1+\theta_c)\Big] - j\sin 2(\omega_c t_1+\theta_c)\Big[\delta(\omega+\omega_c)-\delta(\omega-\omega_c)\Big]\Bigg\}$$

$$+\frac{A^2\pi T}{2}\sin^2\beta V\Bigg\{\Big[1+\cos 2(\omega_c t_1+\theta_c)\Big]\left[\frac{\sin^4\left\{\frac{(\omega+\omega_c)T}{4}\right\}}{\left\{\frac{(\omega+\omega_c)T}{4}\right\}^2}\right.$$

$$\left.+\frac{\sin^4\left\{\frac{(\omega-\omega_c)T}{4}\right\}}{\left\{\frac{(\omega-\omega_c)T}{4}\right\}^2}\right] - j\sin 2(\omega_c t_1+\theta_c)\left[\frac{\sin^4\left\{\frac{(\omega+\omega_c)T}{4}\right\}}{\left\{\frac{(\omega+\omega_c)T}{4}\right\}^2}-\frac{\sin^4\left\{\frac{(\omega-\omega_c)T}{4}\right\}}{\left\{\frac{(\omega-\omega_c)T}{4}\right\}^2}\right]\Bigg\}$$

In general, as observed from Equation 4.49 the power spectral density for the coherent PSK signal consists of discrete time varying carrier components, and time varying sidebands resulting from the split-phase spectrum being translated to appear about plus or minus the carrier frequency. The imaginary term appears because $R_{PSKC}(t_1,\tau)$ is not an even function of τ. It should be observed that as the sideband power is maximized the carrier components tend to vanish, and indeed at $\beta V = Q\,\pi/2$, Q = odd integer, the carrier does vanish and the sideband power is a maximum.

Time averaged power spectral density. It should be observed that the expression for the time varying power spectral density of coherent PSK when time averaged with respect to t_1 reduces to the expression for the power spectral density of non-coherent PSK, as follows.

Inserting $S_{PSKN}(\omega)$ and taking the time average with respect to t_1

$$(4.50)\quad \overline{S_{PSKC}(t_1,\omega)} = S_{PSKN}(\omega) - \frac{A^2\pi}{2T}\cos^2\beta V$$

$$\left\{\left[\delta(\omega + \omega_c) + \delta(\omega - \omega_c)\right]\int_0^T \cos 2(\omega_c t_1 + \theta_c)dt_1\right.$$

$$\left. + j\left[\delta(\omega + \omega_c) - \delta(\omega - \omega_c)\right]\int_0^T \cos 2(\omega_c t_1 + \theta_c)dt_1\right\}$$

$$+ \frac{A^2\pi}{2}\sin^2\beta V\left\{\left[\frac{\sin^4\left\{\frac{(\omega + \omega_c)T}{4}\right\}}{\left\{\frac{(\omega + \omega_c)T}{4}\right\}^2} + \frac{\sin^4\left\{\frac{(\omega - \omega_c)T}{4}\right\}}{\left\{\frac{(\omega - \omega_c)T}{4}\right\}^2}\right]\int_0^T \cos 2(\omega_c t_1 + \theta_c)dt_1\right.$$

$$\left. - j\left[\frac{\sin^4\left\{\frac{(\omega + \omega_c)T}{4}\right\}}{\left\{\frac{(\omega + \omega_c)T}{4}\right\}^2} - \frac{\sin^4\left\{\frac{(\omega - \omega_c)T}{4}\right\}}{\left\{\frac{(\omega - \omega_c)T}{4}\right\}^2}\right]\int_0^T \sin 2(\omega_c t_1 + \theta_c)dt_1\right\}$$

Noting the evaluation of the above integrals in Equation 3.69 through Equation 3.74,

$$(4.53)\quad \overline{S_{PSKC}(t_1,\omega)} = \frac{A^2\pi}{2}\left[\cos^2\beta V\right]\left[\delta(\omega + \omega_c) + \delta(\omega - \omega_c)\right]$$

$$+ \frac{A^2\beta^2\pi T}{2}\left[\sin^2\beta V\right]\left\{\frac{\sin^4\left[\frac{(\omega + \omega_c)T}{4}\right]}{\left[\frac{(\omega + \omega_c)T}{4}\right]^2} + \frac{\sin^4\left[\frac{(\omega + \omega_c)T}{4}\right]}{\left[\frac{(\omega + \omega_c)T}{4}\right]^2}\right\}$$

A comparison of Equation 4.53 with Equation 4.16 yields

(4.54) $$\overline{S_{PSKC}(t_1,\omega)} = S_{PSKN}(\omega)$$

Showing that the time average of the coherent PSK time varying power spectral density is equal to the power spectral density of non-coherent PSK.

CHAPTER V

FREQUENCY SHIFT KEYING

The case of a sinusoidal carrier frequency modulated by a split-phase code is equivalent to switching between two oscillators separated in frequency. When the split phase code assumes a +V voltage level, the modulated signal will be of the form

$$e_{FSK}(t) = A\cos(\omega_1 t + \theta_1) \tag{5.1}$$

where

ω_1 - oscillator angular frequency

θ_1 - oscillator initial phase angle

and when the modulation assumes a -V voltage level, the output signal will be of the form

$$e_{FSK}(t) = A\cos(\omega_2 t + \theta_2) \tag{5.2}$$

where

ω_2 - oscillator angular frequency

θ_2 - oscillator initial phase angle

Physically, the FSK case consists of gating the first oscillator on and the second oscillator off when the split-phase code assumes a +V voltage level and gating the first oscillator off and the second oscillator on when the split-phase code assumes a -V voltage level. One scheme which will allow this switching pattern to be accomplished requires that the split-phase code and its inverse be converted into unipolar codes, Figure 10.

The split-phase code can be expressed as

(5.3) $$e_m(t) = Vm(t)$$

where

$m(t) = \pm 1$ is the split-phase switching function

the inverse of the split-phase code can be expressed as

(5.4) $$e'_m(t) = Vm'(t)$$

where

$m'(t) = \mp 1$ is the inverse of the split-phase switching function.

The unipolar modulation can be expressed as

(5.5) $$e_{m_1}(t) = m_1(t)V$$

where

$$m_1(t) = \begin{cases} +1 & \text{random unipolar gating function} \\ 0 & \text{corresponding to split-phase code} \end{cases}$$

and

(5.6) $$e_{m_1'}(t) = m_1'(t)V$$

where

$$m_1'(t) = \begin{cases} 0 \text{ when } m_1(t) = +1 \\ 1 \text{ when } m_1(t) = 0 \end{cases}$$

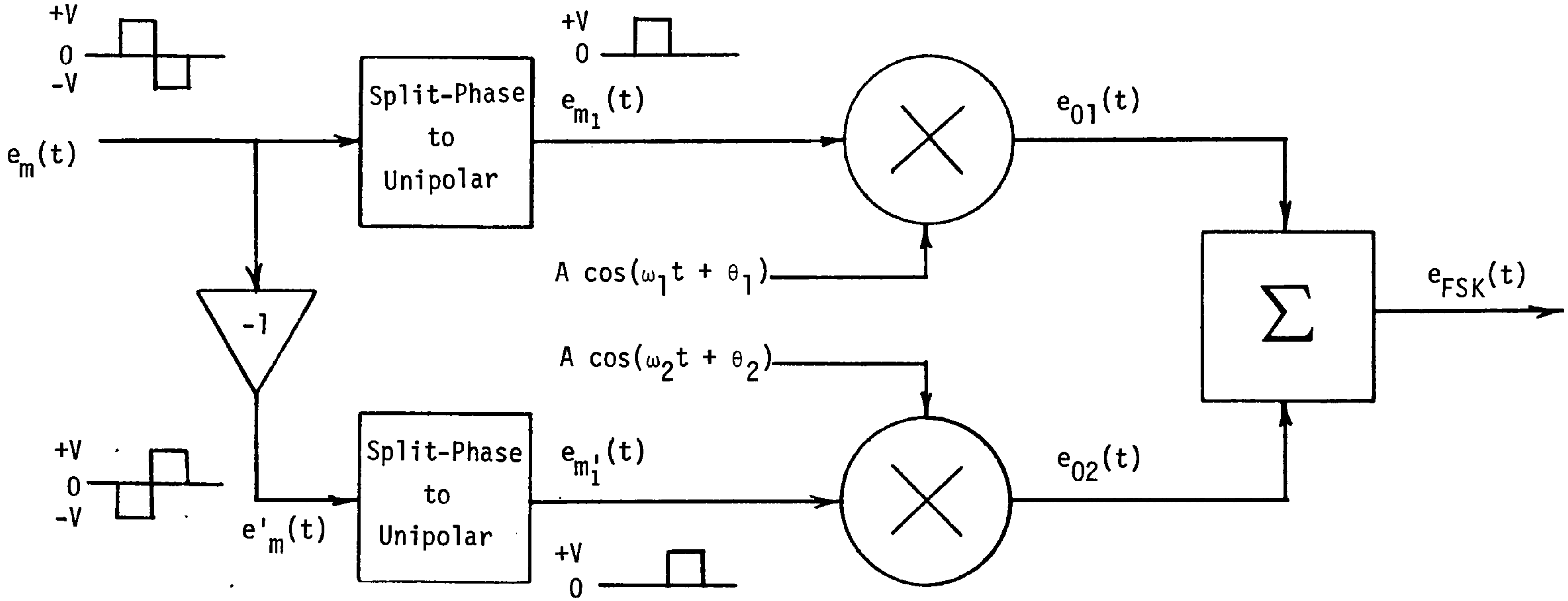

FIGURE 10

A METHOD OF GENERATING AN FSK SIGNAL WHEN THE MODULATION IS A SPLIT-PHASE CODE

I. NON-COHERENT MODULATION

For non-coherent FSK the two oscillator frequencies f_1 and f_2 are not integral multiples of the split-phase code bit rate f_B

$$f_1 \neq Kf_B \qquad K = \text{integer}$$

$$f_1 \neq \frac{K}{T}$$

and

$$f_2 \neq Df_B \qquad D = \text{integer}$$

$$f_2 \neq \frac{D}{T}$$

where

$$D \neq K$$

These relationships between the two oscillator frequencies and the split-phase code bit rate require that the initial phase angles of the two oscillators be considered as random variables uniformly distributed between 0 and 2π.

θ_1 - random variable uniformly distributed between 0 and 2π.

θ_2 - random variable uniformly distributed between 0 and 2π.

Figure 11 pictorially represents non-coherent FSK.

As observed from Figure 11, the output signal from the first multiplier is

$$e_{01}(t) = Ae_{m_1}(t)\cos(\omega_1 t + \theta_1) \tag{5.7}$$

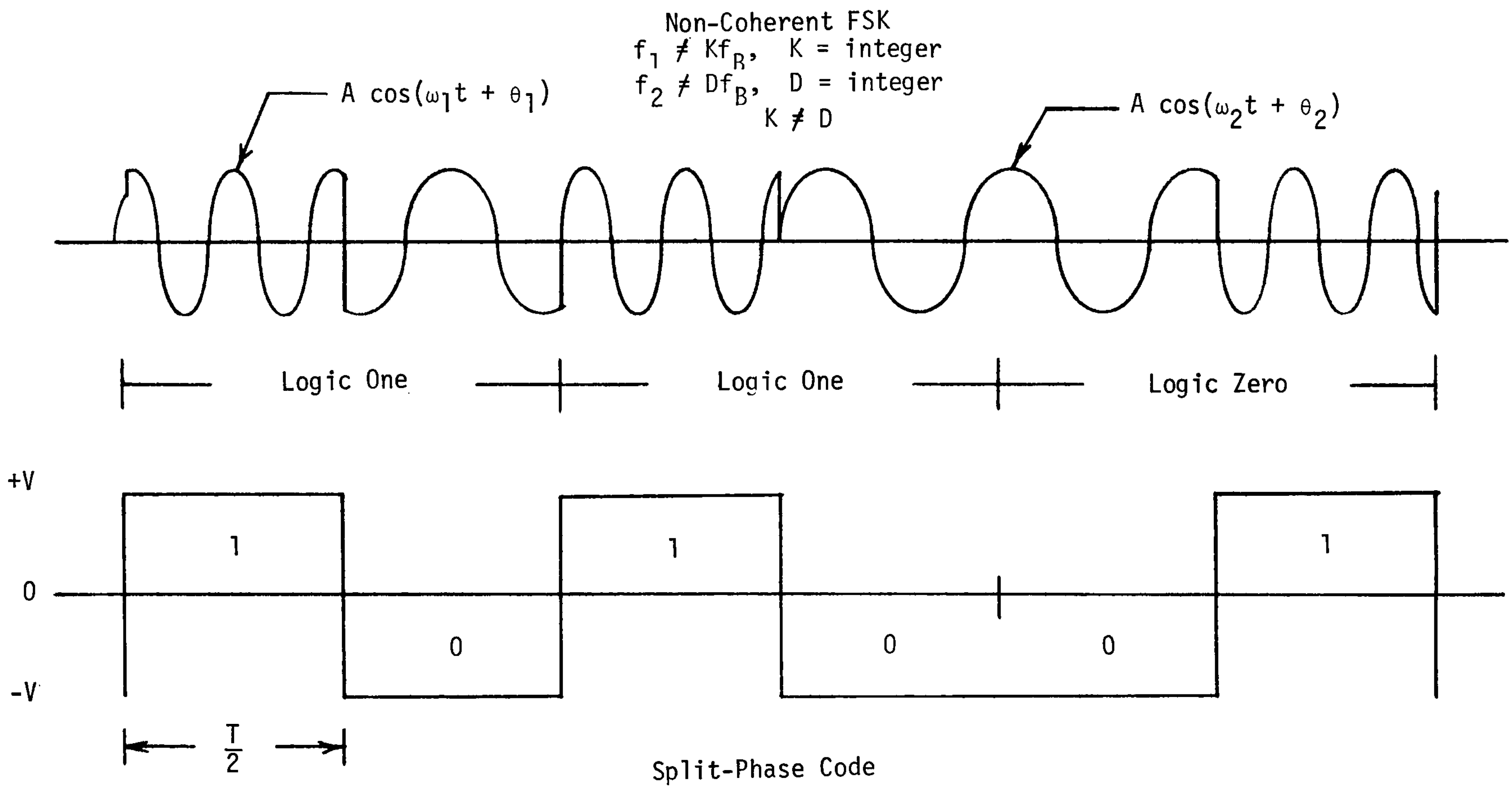

FIGURE 11

A NON-COHERENT FSK SIGNAL WITH SPLIT-PHASE CODE MODULATION

(5.8) $$e_{01}(t) = AVm_1(t) \cos(\omega_1 t + \theta_1)$$

and the output signal from the second multiplier is

(5.9) $$e_{02}(t) = Ae'_{m1}(t) \cos(\omega_2 t + \theta_2)$$

(5.10) $$e_{02}(t) = AVm'_1(t) \cos(\omega_2 t + \theta_2)$$

The complete output signal is given by the sum of $e_{01}(t)$ and $e_{02}(t)$

(5.11) $$e_{FSKN}(t) = e_{01}(t) + e_{02}(t)$$

(5.12) $$e_{FSKN}(t) = AV[m_1(t) \cos(\omega_1 t + \theta_1) + m'_1(t) \cos(\omega_2 t + \theta_2)]$$

Equation 5.12 is the expression for the non-coherent FSK signal.

Autocorrelation Function

The autocorrelation function of the non-coherent FSK signal is

(5.13) $$R_{FSKN}(t_1, \tau) = E\left[e_{FSKN}(t_1)\, e_{FSKN}(t_1 + \tau)\right]$$

(5.14) $$R_{FSKN}(t_1, \tau) = E\Big[\Big\{AV[m_1(t_1) \cos(\omega_1 t_1 + \theta_1) + m'_1(t_1) \cos(\omega_2 t_1 + \theta_2)]\Big\} \Big\{AV[m_1(t_1 + \tau) \cos\{\omega_1(t_1 + \tau) + \theta_1\} + m'_1(t_1 + \tau) \cos\{\omega_2(t_1 + \tau) + \theta_2\}]\Big\}\Big]$$

$$(5.15) \quad R_{FSKN}(t_1, \tau) = A^2V^2E\Big[m_1(t_1)\, m_1(t_1+\tau)$$
$$\cos(\omega_1 t_1 + \theta_1)\, \cos\{\omega_1(t_1+\tau) + \theta_1\}$$
$$+ m_1'(t_1)\, m_1(t_1+\tau)$$
$$\cos(\omega_2 t_1 + \theta_2)\, \cos\{\omega_1(t_1+\tau) + \theta_1\}$$
$$+ m_1(t_1)\, m_1'(t_1+\tau)$$
$$\cos(\omega_1 t_1 + \theta_1)\, \cos\{\omega_2(t_1+\tau) + \theta_2\}$$
$$+ m_1'(t_1)\, m_1'(t_1+\tau)$$
$$\cos(\omega_2 t_1 + \theta_2)\, \cos\{\omega_2(t_1+\tau) + \theta_2\}\Big]$$

Noting that the expected value of a sum is equal to the sum of the expected values, $R_{FSKN}(t_1, \tau)$ can be written

$$(5.16) \quad R_{FSKN}(t_1, \tau) = A^2V^2\Big\{E[m_1(t_1)\, m_1(t_1+\tau)]$$
$$E[\cos(\omega_1 t_1 + \theta_1)\, \cos\{\omega_1(t_1+\tau) + \theta_1\}]$$
$$+ E[m_1'(t_1)\, m_1(t_1+\tau)]$$
$$E[\cos(\omega_2 t_1 + \theta_2)]\, E[\cos\{\omega_1(t_1+\tau) + \theta_1\}]$$
$$+ E[m_1(t_1)\, m_1'(t_1+\tau)]$$
$$E[\cos(\omega_1 t_1 + \theta_1)]\, E[\cos\{\omega_2(t_1+\tau) + \theta_2\}]$$
$$+ E[m_1'(t_1)\, m_1'(t_1+\tau)]$$
$$E[\cos(\omega_2 t_1 + \theta_2)\, \cos\{\omega_2(t_1+\tau) + \theta_2\}]\Big\}$$

where the fact that the two oscillators and the unipolar gating function are statistically independent of each other has been utilized.

Noting that the ensemble average of a sinusoid with random phase is zero

(5.17) $$E[\cos(\omega_2 t_1 + \theta_2)] = E[\cos(\omega_1 t_1 + \theta_1)]$$
$$= E[\cos\{\omega_2(t_1 + \tau) + \theta_2\}]$$
$$= E[\cos\{\omega_1(t_1 + \tau) + \theta_1\}] = 0$$

and utilizing the fact that

(5.18) $$R_{m1}(\tau) = V^2 E[m_1(t_1)\ m_1(t_1 + \tau)]$$

(5.19) $$R'_{m1}(\tau) = V^2 E[m'(t_1)\ m'_1(t_1 + \tau)]$$

(5.20) $$R_{FSKN}(t_1, \tau) = A^2 \Big\{ R_{m1}(\tau)\ E[\cos(\omega_1 t_1 + \theta_1) \cos\{\omega_1(t_1 + \tau) + \theta_1\}] + R'_{m1}(\tau)\ E[\cos(\omega_2 t_1 + \theta_2) \cos\{\omega_2(t_1 + \tau) + \theta_2\}] \Big\}$$

Expanding the cosine products trigonometrically

(5.21) $$\cos(\omega_1 t_1 + \theta_1)\ \cos\{\omega_1(t_1 + \tau) + \theta_1\}$$
$$= 1/2[\cos\omega_1\tau + \cos\{\omega_1(2t_1 + \tau) + 2\theta_1\}]$$

(5.22) $$\cos(\omega_2 t_1 + \theta_2)\ \cos\{\omega_2(t_1 + \tau) + \theta_2\}$$
$$= 1/2[\cos\omega_2\tau + \cos\{\omega_2(2t_1 + \tau) + 2\theta_2\}]$$

Noting again that the ensemble average of a sinusoid with random phase is zero

(5.23) $$E[\cos\{\omega_1(2t_1 + \tau) + 2\theta_1\}]$$
$$= E[\cos\{\omega_2(2t_1 + \tau) + 2\theta_2\}] = 0$$

(5.24) $$R_{FSKN}(t_1, \tau) = R_{FSKN}(\tau)$$

$$= R_{m_1}(\tau)\, R_{carrier(1)}(\tau) + R_{m_1}'(\tau)\, R_{carrier(2)}(\tau)$$

where

(5.25) $$R_{carrier(1)}(\tau) = \frac{A^2}{2} \cos\omega_1\tau$$

(5.26) $$R_{carrier(2)}(\tau) = \frac{A^2}{2} \cos\omega_2\tau$$

But it can be shown that

(5.27) $$R_{m_1}(\tau) = R_{m_1}'(\tau) = \frac{V^2}{4} + \frac{1}{4} R_m(\tau)$$

(5.28) $$R_{FSKN}(\tau) = \frac{V^2}{4} R_{carrier(1)}(\tau) + \frac{1}{4} R_m(\tau)\, R_{carrier(1)}(\tau)$$

$$+ \frac{V^2}{4} R_{carrier(2)}(\tau) + \frac{1}{4} R_m(\tau)\, R_{carrier(2)}(\tau)$$

<u>Power Spectral Density</u>

The non-coherent FSK autocorrelation function, Equation 5.28, can be written as

(5.29) $$R_{FSKN}(\tau) = \frac{V^2}{4} R_{carrier(1)}(\tau) + \frac{1}{4} R_{FSKN}'(\tau)$$

$$+ \frac{V^2}{4} R_{carrier(2)}(\tau) + \frac{1}{4} R_{FSKN}''(\tau)$$

with corresponding power spectral density

(5.30) $$S_{FSKN}(\omega) = \frac{V^2}{4} S_{carrier(1)}(\omega) + \frac{1}{4} S_{FSKN}'(\omega)$$

$$+ \frac{V^2}{4} S_{carrier(1)}(\omega) + \frac{1}{4} S_{FSKN}''(\omega)$$

$S'_{FSKN}(\omega)$ and $S''_{FSKN}(\omega)$ can be evaluated by recognizing that the product of two autocorrelation functions is equivalent to the convolution of the two corresponding power spectra.

(5.31) $$S'_{FSKN}(\omega) = S_M(\omega) * S_{carrier(1)}(\omega)$$

(5.32) $$S''_{FSKN}(\omega) = S_M(\omega) * S_{carrier(2)}(\omega)$$

To evaluate Equation 5.31 and Equation 5.32 it is necessary to have the expressions for $S_M(\omega)$, $S_{carrier(1)}(\omega)$, and $S_{carrier(2)}(\omega)$. Noting Equation 3.12 and its results, Equation 3.15

(5.33) $$S_{carrier(1)}(\omega) = \frac{A^2\pi}{2}\ [\delta(\omega + \omega_1) + \delta(\omega - \omega_1)]$$

(5.34) $$S_{carrier(2)}(\omega) = \frac{A^2\pi}{2}\ [\delta(\omega + \omega_1) + \delta(\omega - \omega_1)]$$

Noting that $S_M(\omega)$ is

(2.15) $$S_M(\omega) = V^2T\left[\frac{\sin^4\left(\frac{\omega T}{4}\right)}{\left(\frac{\omega T}{4}\right)^2}\right]$$

Noting Equation 3.16 and its results, Equation 3.21

(5.35) $$S'_{FSKN}(\omega) = \frac{A^2V^2\pi T}{2}\left\{\frac{\sin^4\left[\frac{(\omega + \omega_1)T}{4}\right]}{\left[\frac{(\omega + \omega_1)T}{4}\right]^2} + \frac{\sin^4\left[\frac{(\omega - \omega_1)T}{4}\right]}{\left[\frac{(\omega - \omega_1)T}{4}\right]^2}\right\}$$

$$(5.36) \quad S''_{FSKN}(\omega) = \frac{A^2V^2\pi T}{2} \left\{ \frac{\sin^4\left[\frac{(\omega + \omega_2)T}{4}\right]}{\left[\frac{(\omega + \omega_2)T}{4}\right]^2} + \frac{\sin^4\left[\frac{(\omega - \omega_2)T}{4}\right]}{\left[\frac{(\omega - \omega_2)T}{4}\right]^2} \right\}$$

Utilizing Equation 5.30

$$(5.37) \quad S_{FSKN}(\omega) = \frac{A^2V^2\pi}{8} [\delta(\omega + \omega_1) + \delta(\omega - \omega_1)]$$

$$+ \frac{A^2V^2\pi T}{8} \left\{ \frac{\sin^4\left[\frac{(\omega + \omega_1)T}{4}\right]}{\left[\frac{(\omega + \omega_1)T}{4}\right]^2} + \frac{\sin^4\left[\frac{(\omega - \omega_1)T}{4}\right]}{\left[\frac{(\omega - \omega_1)T}{4}\right]^2} \right\}$$

$$+ \frac{A^2V^2\pi}{8} [\delta(\omega + \omega_2) + \delta(\omega - \omega_2)]$$

$$+ \frac{A^2V^2\pi T}{8} \left\{ \frac{\sin^4\left[\frac{(\omega + \omega_2)T}{4}\right]}{\left[\frac{(\omega + \omega_2)T}{4}\right]^2} + \frac{\sin^4\left[\frac{(\omega - \omega_2)T}{4}\right]}{\left[\frac{(\omega - \omega_2)T}{4}\right]^2} \right\}$$

The power spectral density for non-coherent FSK, Equation 5.37, is pictorially represented in Figure 12. The similiarity between non-coherent FSK and non-coherent ASK should be observed. Intuitively this similiarity results from the fact that the non-coherent FSK was shown to be the sum of two "on-off" keyed carriers of different frequency.

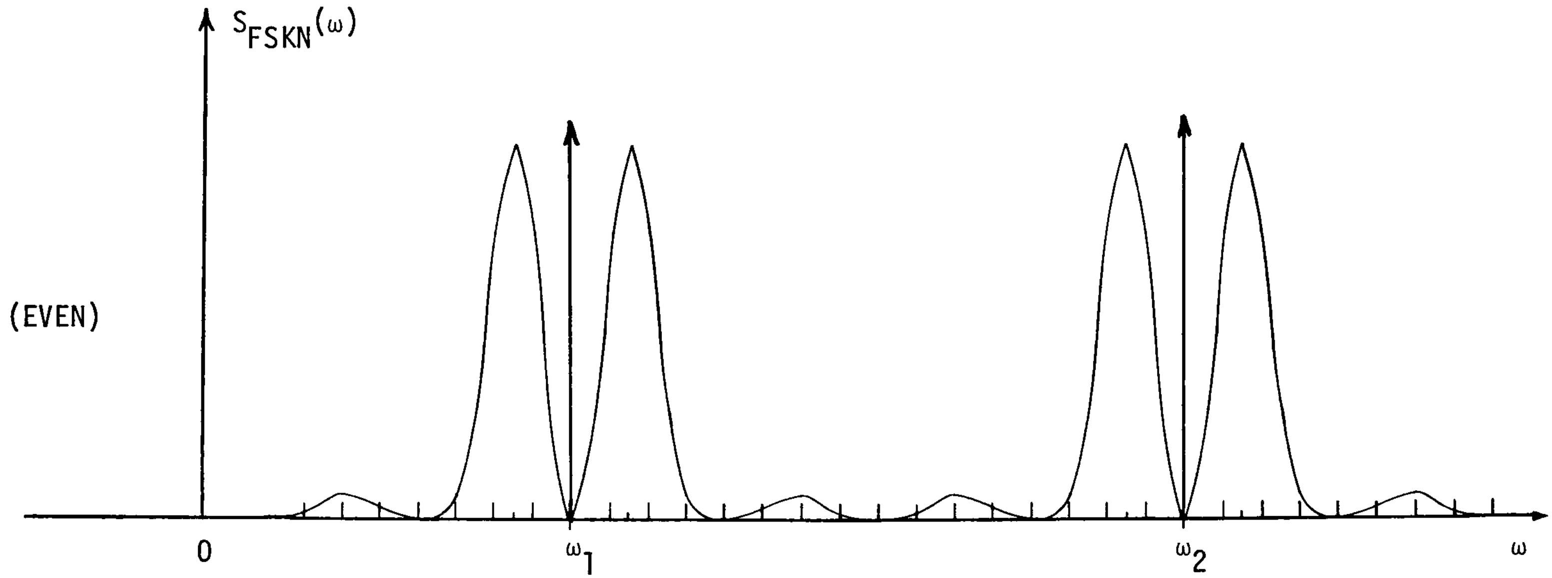

FIGURE 12

POWER SPECTRAL DENSITY OF NON-COHERENT FSK
WITH SPLIT-PHASE CODE MODULATION

II. COHERENT MODULATION

For coherent FSK the two oscillator frequencies f_1 and f_2 are integral multiples of the split-phase code bit rate f_B

$$f_1 = Kf_B \qquad K = \text{integer}$$

$$f_1 = \frac{K}{T}$$

and

$$f_2 = Df_B \qquad D = \text{integer}$$

$$f_2 = \frac{D}{T}$$

where

$$D \neq K$$

These relationships between the two oscillator frequencies and the split phase code bit rate make the initial phase angles of the two oscillators constants.

$$\theta_1 = \theta_{1C} = \text{constant}$$

$$\theta_2 = \theta_{2C} = \text{constant}$$

Figure 13 pictorially represents coherent FSK.

It should be observed that the minimum system bandwidth requirement occurs when the initial phase angles θ_{1C} and θ_{2C} are equal to zero or some integral multiple of π. The system bandwidth requirement increases as either θ_{1C} or θ_{2C} increases toward an odd integral multiple of $\pi/2$ ($Q\ \pi/2$, Q = odd integer) and reaches a maximum when θ_{1C} and θ_{2C} are odd multiples of $\pi/2$ and opposite in sign ($\theta_{1C} = \pi/2$, $\theta_{2C} = 3\ \pi/2 = -\pi/2$).

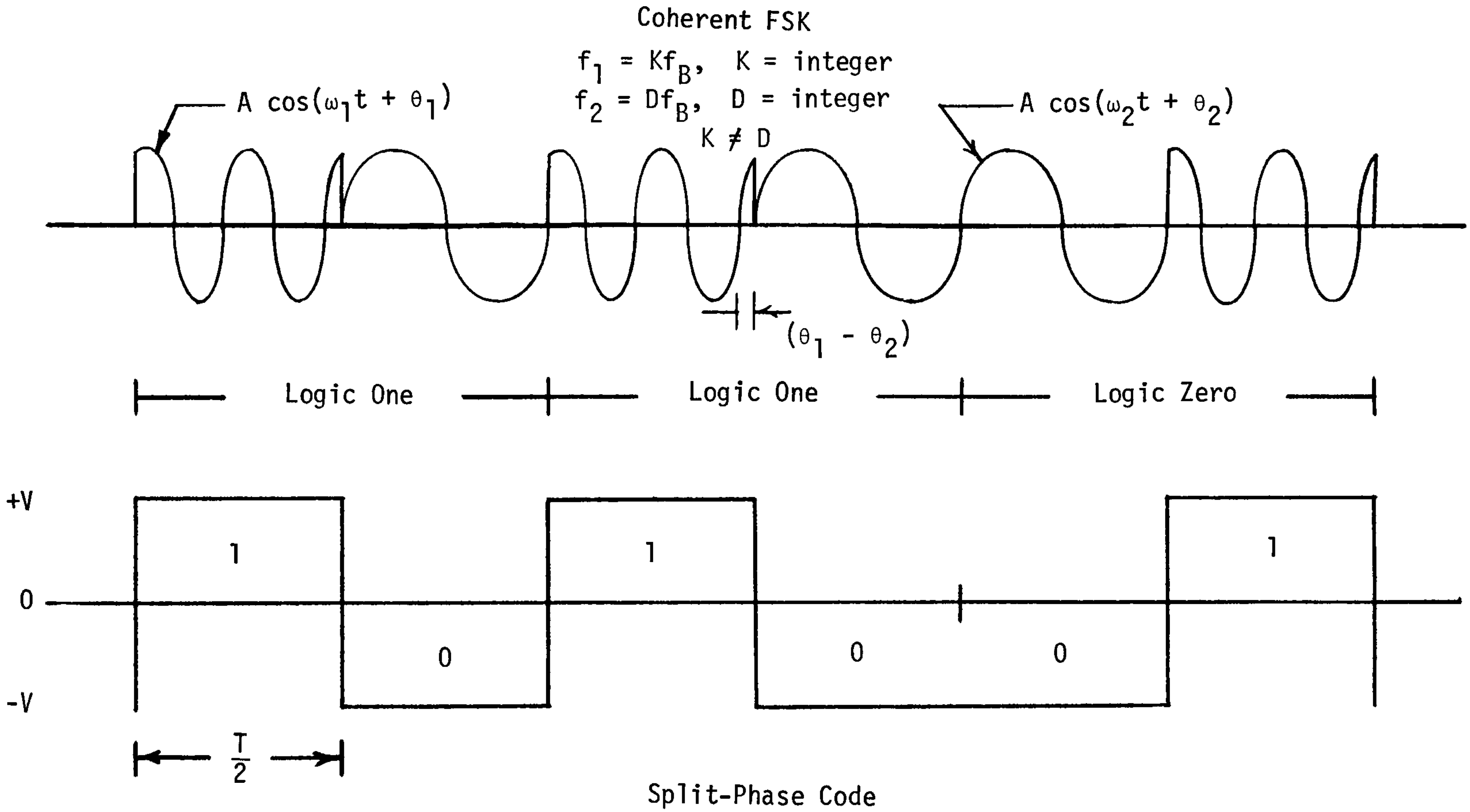

FIGURE 13

A COHERENT FSK SIGNAL WITH SPLIT-PHASE CODE MODULATION

The output signal from the first multiplier of Figure 10 is

(5.38) $$e_{01}(t) = Ae_{m_1}(t)\cos(\omega_1 t + \theta_{1c})$$

(5.39) $$e_{01}(t) = AVm_1(t)\cos(\omega_1 t + \theta_{1c})$$

and the output signal from the second multiplier is

(5.40) $$e_{02}(t) = Ae'_{m_1}(t)\cos(\omega_2 t + \theta_{2c})$$

(5.41) $$e_{02}(t) = AVm'_1(t)\cos(\omega_2 t + \theta_{2c})$$

the complete output signal is the sum of $e_{01}(t)$ and $e_{02}(t)$

(5.42) $$e_{FSKC}(t) = e_{01}(t) + e_{02}(t)$$

(5.43) $$e_{FSKC}(t) = AV[m_1(t)\cos(\omega_1 t + \theta_{1c}) + m'_1(t)\cos(\omega_2 t + \theta_{2c})]$$

Equation 5.43 is the expression for the coherent FSK signal.

Autocorrelation Function

The autocorrelation function of the coherent FSK signal

(5.44) $$R_{FSKC}(t_1, \tau) = E[e_{FSKC}(t_1)\, e_{FSKC}(t_1 + \tau)]$$

(5.45) $$R_{FSKC}(t_1, \tau) = E\Big[\{AV[m_1(t_1)\cos(\omega_1 t_1 + \theta_{1c}) + m'_1(t_1)\cos(\omega_2 t_1 + \theta_{2c})]\} \{AV[m_1(t_1 + \tau)\cos\{\omega_1(t_1 + \tau) + \theta_{1c}\} + m'_1(t_1 + \tau)\cos\{\omega_2(t_1 + \tau) + \theta_{2c}\}]\}\Big]$$

(5.46) $$R_{FSKC}(t_1, \tau) = A^2V^2E\Big[m_1(t_1)m_1(t_1 + \tau)\cos(\omega_1 t_1 + \theta_{1c})\cos\{\omega_1(t_1 + \tau) + \theta_{1c}\} + m'_1(t_1)m_1(t_1 + \tau)\cos(\omega_2 t_1 + \theta_{2c})\cos\{\omega_1(t_1 + \tau) + \theta_{1c}\} + m_1(t_1)m'_1(t_1 + \tau)\cos(\omega_1 t_1 + \theta_{1c})\cos\{\omega_2(t_1 + \tau) + \theta_{2c}\} + m'_1(t_1)m'_1(t_1 + \tau)\cos(\omega_2 t_1 + \theta_{2c})\cos\{\omega_2(t_1 + \tau) + \theta_{2c}\}\Big]$$

Noting that the expected value of a sum is equal to the sum of the expected values $R_{FSKC}(t_1,\tau)$ can be written

$$(5.47)\quad R_{FSKC}(t_1,\tau) = A^2V^2\Big\{\cos(\omega_1 t_1+\Theta_{1c})\ \cos\{\omega_1(t_1+\tau)+\Theta_{1c}\}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} m_1(t_1)m_1(t_1+\tau)p\big[m_1(t_1)m_1(t_1+\tau)\big]dm_1(t_1)dm_1(t_1+\tau)$$

$$+\ \cos(\omega_2 t_1+\Theta_{2c})\ \cos\{\omega_1(t_1+\tau)+\Theta_{1c}\}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} m_1'(t_1)m_1(t_1+\tau)p\big[m_1'(t_1)m_1(t_1+\tau)\big]dm_1'(t_1)dm_1(t_1+\tau)$$

$$+\ \cos(\omega_1 t_1+\Theta_{1c})\ \cos\{\omega_2(t_1+\tau)+\Theta_{2c}\}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} m_1(t_1)m_1'(t_1+\tau)p\big[m_1(t_1)m_1'(t_1+\tau)\big]dm_1(t_1)dm_1'(t_1+\tau)$$

$$+\ \cos(\omega_2 t_1+\Theta_{2c})\ \cos\{\omega_2(t_1+\tau)+\Theta_{2c}\}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} m_1'(t_1)m_1'(t_1+\tau)p\big[m_1'(t_1)m_1'(t_1+\tau)\big]dm_1'(t_1)dm_1'(t_1+\tau)\Big\}$$

where the random variables are $m_1(t_1)$, $m_1(t_1+\tau)$, $m_1'(t_1)$, and $m_1'(t_1+\tau)$.

(5.48) $$R_{FSKC}(t_1,\tau) = R_{carrier(1)}(t_1,\tau)R_{m1}(\tau) + A^2\left\{\frac{V^2}{4} - \frac{1}{4}R_m(\tau)\right\}$$
$$\left\{\cos(\omega_2 t_1+\Theta_{2c})\ \cos\left[\omega_1(t_1+\tau)+\Theta_{1c}\right] + \cos(\omega_1 t_1+\Theta_{1c})\ \cos\left[\omega_2(t_1+\tau)+\Theta_{2c}\right]\right\}$$
$$+ R_{carrier(2)}(t_1,\tau)R_{m_1'}(\tau)$$

where $V^2E\left[m_1'(t_1)m_1(t_1+\tau)\right] = V^2E\left[m_1(t_1)m_1'(t_1+\tau)\right] = \frac{V^2}{4} - R_m(\tau)$

(5.18) $$R_{m1}(\tau) = V^2E\left[m_1(t_1)m_1(t_1+\tau)\right]$$

(5.19) $$R_{m_1'}(\tau) = V^2E\left[m_1'(t_1)m_1'(t_1+\tau)\right]$$

(5.49) $$R_{carrier(1)}(t_1,\tau) = A^2\cos(\omega_1 t_1+\Theta_{1c})\ \cos\left\{\omega_1(t_1+\tau)+\Theta_{1c}\right\}$$

(5.50) $$R_{carrier(2)}(t_1,\tau) = A^2\cos(\omega_2 t_1+\Theta_{2c})\ \cos\left\{\omega_2(t_1+\tau)+\Theta_{2c}\right\}$$

Substituting the expression for $R_{m1}(\tau)$ and $R_{m_1'}(\tau)$, Equation 5.27,

(5.27) $$R_{m1}(\tau) = R_{m_1'}(\tau) = \frac{V^2}{4} + \frac{1}{4}R_m(\tau)$$

$R_{FSKC}(t_1,\tau)$ becomes

(5.51) $$R_{FSKC}(t_1,\tau) = \frac{V^2}{4}R_{carrier(1)}(t_1,\tau) + \frac{1}{4}R_m(\tau)R_{carrier(1)}(t_1,\tau)$$
$$+ \frac{V^2}{4}R_{carrier(2)}(t_1,\tau) + \frac{1}{4}R_m(\tau)R_{carrier(2)}(t_1,\tau) + A^2\left\{\frac{V^2}{4} - \frac{1}{4}R_m(\tau)\right\}$$
$$\left\{\cos(\omega_2 t_1+\Theta_{2c})\cos\left[\omega_1(t_1+\tau)+\Theta_{1c}\right] + \cos(\omega_1 t_1+\Theta_{1c})\cos\left[\omega_2(t_1+\tau)+\Theta_{2c}\right]\right\}$$

<u>Time averaged autocorrelation function.</u> The time averaged autocorrelation function for coherent FSK is

$$\overline{R_{FSKC}(t_1,\tau)} = \frac{1}{T}\int_0^T R_{FSKC}(t_1,\tau)\,dt_1 \tag{5.52}$$

$$\begin{aligned}\overline{R_{FSKC}(t_1,\tau)} &= \left\{\frac{V^2}{4} + \frac{R_m(\tau)}{4}\right\}\frac{1}{T}\int_0^T R_{carrier(1)}(t_1,\tau)\,dt_1 \\ &+ \left\{\frac{V^2}{4} + \frac{R_m(\tau)}{4}\right\}\frac{1}{T}\int_0^T R_{carrier(2)}(t_1,\tau)\,dt_1 + \frac{A^2}{T}\left\{\frac{V^2}{4} - \frac{1}{4}R_m(\tau)\right\} \\ &\left\{\int_0^T \cos(\omega_2 t_1+\Theta_{2c})\cos\left[\omega_1(t_1+\tau)+\Theta_{1c}\right]dt_1 \right.\\ &\left. + \int_0^T \cos(\omega_1 t_1+\Theta_{1c})\cos\left[\omega_2(t_1+\tau)+\Theta_{2c}\right]dt_1\right\}\end{aligned} \tag{5.53}$$

Noting Equation 3.42

$$\overline{R_{carrier}(t_1,\tau)} = \frac{1}{T}\int_0^T R_{carrier}(t_1,\tau)\,dt_1 \tag{3.42}$$

and its results Equation 3.48

$$\overline{R_{carrier}(t_1,\tau)} = \frac{A^2}{2}\cos\omega_c\tau \tag{3.48}$$

utilizing these results

$$\overline{R_{carrier(1)}(t_1,\tau)} = \frac{A^2}{2}\cos\omega_1\tau \tag{5.54}$$

$$\overline{R_{carrier(1)}(t_1,\tau)} = R_{carrier(1)}(\tau) \tag{5.55}$$

and

$$(5.56) \qquad \overline{R_{carrier(2)}(t_1,\tau)} = \frac{A^2}{2} \cos \omega_2 \tau$$

$$(5.57) \qquad \overline{R_{carrier(2)}(t_1,\tau)} = R_{carrier(2)}(\tau)$$

Utilizing the defining conditions for coherent FSK

$$f_1 = Kf_B \qquad K = \text{integer}$$

$$f_1 = \frac{K}{T}$$

and

$$f_2 = Df_B \qquad D = \text{integer}$$

$$f_2 = \frac{D}{T}$$

where $\quad D \neq K$

It can be shown in a manner analogous to that of Equation 3.41 through Equation 3.47 that the last two integrals of Equation 5.53 are equal to zero. Substitution of Equation 5.55 and Equation 5.57 into Equation 5.53 yields

$$(5.58) \qquad \overline{R_{FSKC}(t_1,\tau)} = \frac{V^2}{4} R_{carrier(1)}(\tau) + \frac{R_m(\tau)}{4} R_{carrier(1)}(\tau) + \frac{V^2}{4} R_{carrier(2)}(\tau) + \frac{R_m(\tau)}{4} R_{carrier(2)}(\tau)$$

A comparison of Equation 5.58 and Equation 5.28 will show that the time averaged autocorrelation function for coherent FSK is equal to the autocorrelation function for non-coherent FSK.

$$(5.59) \qquad \overline{R_{FSKC}(t_1,\tau)} = R_{FSKN}(\tau)$$

Power Spectral Density

The coherent FSK autocorrelation function, Equation 5.51, can be written as

$$(5.60)\quad R_{FSKC}(t_1,\tau) = \frac{V^2}{4} R_{carrier(1)}(t_1,\tau) + \frac{1}{4} R'_{FSKC}(t_1,\tau) + \frac{V^2}{4} R_{carrier(2)}(t_1,\tau) + \frac{1}{4} R''_{FSKC}(t_1,\tau) + A^2\left\{\frac{V^2}{4} - \frac{1}{4} R_m(\tau)\right\}$$
$$\left\{\cos(\omega_2 t_1+\Theta_{2c})\left[\cos(\omega_1 t_1+\Theta_{1c})\cos\omega_1\tau - \sin(\omega_1 t_1+\Theta_{1c})\sin\omega_1\tau\right]\right.$$
$$\left. + \cos(\omega_1 t_1+\Theta_{1c})\left[\cos(\omega_2 t_1+\Theta_{2c})\cos\omega_2\tau - \sin(\omega_2 t_1+\Theta_{2c})\sin\omega_2\tau\right]\right\}$$

with a corresponding Fourier transform $F_{FSKC}(t_1,\omega)$, which can be interpreted as a time varying power spectral density $S_{FSKC}(t_1,\omega)$.

$$(5.61)\quad S_{FSKC}(t_1,\omega) = \frac{V^2}{4} S_{carrier(1)}(t_1,\omega) + \frac{1}{4} S'_{FSKC}(t_1,\omega) + \frac{V^2}{4} S_{carrier(2)}(t_1,\omega)$$
$$+ \frac{1}{4} S''_{FSKC}(t_1,\omega) + \frac{A^2}{4}\cos(\omega_2 t_1+\Theta_{2c})\cos(\omega_1 t_1+\Theta_{1c})$$
$$\left\{ V^2\int_{-\infty}^{\infty}\cos\omega_1\tau\, e^{-j\omega\tau}\,d\tau - \int_{-\infty}^{\infty} R_m(\tau)\cos\omega_1\tau\, e^{-j\omega\tau}\,d\tau\right\}$$
$$- \frac{A^2}{4}\cos(\omega_2 t_1+\Theta_{2c})\sin(\omega_1 t_1+\Theta_{1c})\left\{V^2\int_{-\infty}^{\infty}\sin\omega_1\tau\, e^{-j\omega\tau}\,d\tau\right.$$
$$\left.- \int_{-\infty}^{\infty} R_m(\tau)\sin\omega_1\tau\, e^{-j\omega\tau}\,d\tau\right\} + \frac{A^2}{4}\cos(\omega_1 t_1+\Theta_{1c})\cos(\omega_2 t_1+\Theta_{2c})$$
$$\left\{ V^2\int_{-\infty}^{\infty}\cos\omega_2\tau\, e^{-j\omega\tau}\,d\tau - \int_{-\infty}^{\infty} R_m(\tau)\cos\omega_2\tau\, e^{-j\omega\tau}\,d\tau\right\}$$
$$- \frac{A^2}{4}\cos(\omega_1 t_1+\Theta_{1c})\sin(\omega_2 t_1+\Theta_{2c})\left\{V^2\int_{-\infty}^{\infty}\sin\omega_2\tau\, e^{-j\omega\tau}\,d\tau\right.$$
$$\left.- \int_{-\infty}^{\infty} R_m(\tau)\sin\omega_2\tau\, e^{-j\omega\tau}\,d\tau\right\}$$

The first term of Equation 5.61 is the Fourier transform of the autocorrelation of carrier number one, $R_{carrier(1)}(t_1,\tau)$,

(5.62) $$S_{carrier(1)}(t_1,\omega) = \int_{-\infty}^{\infty} R_{carrier(1)}(t_1,\tau)\, e^{-j\omega\tau} d\tau$$

(5.63) $$S_{carrier(1)}(t_1,\omega) = \int_{-\infty}^{\infty} A^2\cos(\omega_1 t_1+\Theta_{1C})\cos\left\{\omega_1(t_1+\tau)+\theta_{1C}\right\}e^{-j\omega\tau} d\tau$$

Noting Equation 4.39 and its results Equation 4.44

(5.64) $$S_{carrier(1)}(t_1,\omega) = \frac{A^2\pi}{2}\left[\delta(\omega+\omega_1) + \delta(\omega-\omega_1)\right]\left[1 + \cos 2(\omega_1 t_1+\Theta_{1C})\right]$$
$$- j\,\frac{A^2\pi}{2} \sin 2(\omega_1 t_1+\Theta_{1C})\left[\delta(\omega+\omega_1) - \delta(\omega-\omega_1)\right]$$

Sinilarly

(5.65) $$S_{carrier(2)}(t_1,\omega) = \frac{A^2\pi}{2}\left[\delta(\omega+\omega_2) + \delta(\omega-\omega_2)\right]\left[1 + \cos 2(\omega_2 t_1+\Theta_{2C})\right]$$
$$- j\,\frac{A^2\pi}{2} \sin 2(\omega_2 t_1+\Theta_{2C})\left[\delta(\omega+\omega_2) - \delta(\omega-\omega_2)\right]$$

$S'_{FSKC}(t_1,\omega)$ and $S''_{FSKC}(t_1,\omega)$ can be evaluated by recognizing that the Fourier transform of the product of two autocorrelation functions is equivalent to the convolution of the corresponding power spectra.

(5.66) $$S'_{FSKC}(t_1,\omega) = S_m(\omega) * S_{carrier(1)}(t_1,\omega)$$

(5.67) $$S''_{FSKC}(t_1,\omega) = S_m(\omega) * S_{carrier(2)}(t_1,\omega)$$

(5.68) $$S'_{FSKC}(t_1,\omega) = \int_{-\infty}^{\infty} S_m(\omega_o)\, S_{carrier(1)}(t_1,\omega - \omega_o)\, d\omega_o$$

$$(5.69) \quad S'_{FSKC}(t_1,\omega) = \int_{-\infty}^{\infty} \left[\frac{V^2T \sin^4\left(\frac{\omega_o T}{4}\right)}{\left(\frac{\omega_o T}{4}\right)^2} \right] \left[\frac{A^2\pi}{2} \left\{ \delta(\omega+\omega_1-\omega_o)+\delta(\omega-\omega_1-\omega_o) \right\} \right.$$

$$\left. \left\{ 1+\cos 2(\omega_1 t_1+\Theta_{1c}) \right\} - j\, \frac{A^2\pi}{2} \sin 2(\omega_1 t_1+\Theta_{1c}) \left\{ \delta(\omega+\omega_1-\omega_0)-\delta(\omega-\omega_1-\omega_0) \right\} \right] d\omega_0$$

$$(5.70) \quad S'_{FSKC}(t_1,\omega) = \frac{A^2V^2\pi T}{2} \left\{ 1 + \cos 2\,(\omega_1 t_1+\Theta_{1c}) \right\}$$

$$\left\{ \frac{\sin^4\left[\frac{(\omega+\omega_1)T}{4}\right]}{\left[\frac{(\omega+\omega_1)T}{4}\right]^2} + \frac{\sin^4\left[\frac{(\omega-\omega_1)T}{4}\right]}{\left[\frac{(\omega-\omega_1)T}{4}\right]^2} \right\}$$

$$- j\, \frac{A^2V^2\pi T}{2} \sin 2(\omega_1 t_1+\Theta_{1c}) \left\{ \frac{\sin^4\left[\frac{(\omega+\omega_1)T}{4}\right]}{\left[\frac{(\omega+\omega_1)T}{4}\right]^2} - \frac{\sin^4\left[\frac{(\omega-\omega_1)T}{4}\right]}{\left[\frac{(\omega-\omega_1)T}{4}\right]^2} \right\}$$

Similarly

$$(5.71) \quad S''_{FSKC}(t_1,\omega) = \frac{A^2V^2\pi T}{2} \left\{ 1 + \cos 2\,(\omega_2 t_1+\Theta_{2c}) \right\}$$

$$\left\{ \frac{\sin^4\left[\frac{(\omega+\omega_2)T}{4}\right]}{\left[\frac{(\omega+\omega_2)T}{4}\right]^2} + \frac{\sin^4\left[\frac{(\omega-\omega_2)T}{4}\right]}{\left[\frac{(\omega-\omega_2)T}{4}\right]^2} \right\}$$

$$- j\, \frac{A^2V^2\pi T}{2} \sin 2(\omega_2 t_1+\Theta_{2c}) \left\{ \frac{\sin^4\left[\frac{(\omega+\omega_2)T}{4}\right]}{\left[\frac{(\omega+\omega_2)T}{4}\right]^2} - \frac{\sin^4\left[\frac{(\omega-\omega_2)T}{4}\right]}{\left[\frac{(\omega-\omega_2)T}{4}\right]^2} \right\}$$

The other terms of Equation 5.61 can be evaluated utilizing the fact that the Fourier transform of the product of two functions is equivalent to the convolution of the individual power spectra. Using Equation 3.14

$$(5.72)\quad \int_{-\infty}^{\infty} R_m(\tau)\cos \omega_1\tau\, e^{-j\omega\tau}\, d\tau = S_m(\omega) * \pi\left[\delta(\omega-\omega_1)+\delta(\omega+\omega_1)\right]$$

$$(5.73)\quad \int_{-\infty}^{\infty} R_m(\tau)\cos \omega_1\tau\, e^{-j\omega\tau}\, d\tau = \int_{-\infty}^{\infty} S_m(\omega_0)\ \pi\left[\delta(\omega-\omega_1-\omega_0)+\delta(\omega+\omega_1-\omega_0)\right] d\omega_0$$

$$(5.74)\quad \int_{-\infty}^{\infty} R_m(\tau)\cos \omega_1\tau\, e^{-j\omega\tau}\, d\tau = V^2\pi T\left\{\frac{\sin^4\left[\frac{(\omega-\omega_1)T}{4}\right]}{\left[\frac{(\omega-\omega_1)T}{4}\right]^2} + \frac{\sin^4\left[\frac{(\omega+\omega_1)T}{4}\right]}{\left[\frac{(\omega+\omega_1)T}{4}\right]^2}\right\}$$

Similarly

$$(5.75)\quad \int_{-\infty}^{\infty} R_m(\tau)\cos \omega_2\tau\, e^{-j\omega\tau}\, d\tau = V^2\pi T\left\{\frac{\sin^4\left[\frac{(\omega-\omega_2)T}{4}\right]}{\left[\frac{(\omega-\omega_2)T}{4}\right]^2} + \frac{\sin^4\left[\frac{(\omega+\omega_2)T}{4}\right]}{\left[\frac{(\omega+\omega_2)T}{4}\right]^2}\right\}$$

and

$$(5.76)\quad \int_{-\infty}^{\infty} \cos \omega_1\tau\, e^{-j\omega\tau}\, d\tau = \pi\left[\delta(\omega-\omega_1) + \delta(\omega+\omega_1)\right]$$

$$(5.77)\quad \int_{-\infty}^{\infty} \cos \omega_2\tau\, e^{-j\omega\tau}\, d\tau = \pi\left[\delta(\omega-\omega_2) + \delta(\omega+\omega_2)\right]$$

Utilizing Equation 3.59 in a like manner

$$(5.78)\quad \int_{-\infty}^{\infty} R_m(\tau)\sin \omega_1\tau\, e^{-j\omega\tau}\, d\tau = -jV^2\pi T\left\{\frac{\sin^4\left[\frac{(\omega+\omega_1)T}{4}\right]}{\left[\frac{(\omega+\omega_1)T}{4}\right]^2} - \frac{\sin^4\left[\frac{(\omega-\omega_1)T}{4}\right]}{\left[\frac{(\omega-\omega_1)T}{4}\right]^2}\right\}$$

$$(5.79)\quad \int_{-\infty}^{\infty} R_m(\tau)\sin \omega_1\tau\, e^{-j\omega\tau}\, d\tau = -jV^2\pi T\left\{\frac{\sin^4\left[\frac{(\omega+\omega_1)T}{4}\right]}{\left[\frac{(\omega+\omega_1)T}{4}\right]^2} - \frac{\sin^4\left[\frac{(\omega-\omega_1)T}{4}\right]}{\left[\frac{(\omega-\omega_1)T}{4}\right]^2}\right\}$$

$$(5.80)\quad \int_{-\infty}^{\infty} \sin \omega_1\tau\, e^{-j\omega\tau}\, d\tau = \pi\left[\delta(\omega+\omega_1) - \delta(\omega-\omega_1)\right]$$

$$(5.81)\quad \int_{-\infty}^{\infty} \sin \omega_2\tau\, e^{-j\omega\tau}\, d\tau = \pi\left[\delta(\omega+\omega_2) - \delta(\omega-\omega_2)\right]$$

Substitution into Equation 5.61 yields the time varying power spectral density for coherent FSK.

The imaginary components appear because $R_{FSKC}(t_1,\tau)$ is not an even function of τ.

$$\begin{aligned}
(5.82)\quad S_{FSKC}(t_1,\omega) = {} & \frac{A^2V^2\pi}{8}\left[\delta(\omega+\omega_1) + \delta(\omega-\omega_1)\right]\left[1 + \cos 2(\omega_1 t_1+\Theta_{1c})\right] \\
& - j\,\frac{A^2V^2\pi}{8}\sin 2(\omega_1 t_1+\Theta_{1c})\left[\delta(\omega+\omega_1) - \delta(\omega-\omega_1)\right] \\
& + \frac{A^2V^2\pi T}{8}\left\{1 + \cos 2(\omega_1 t_1+\Theta_{1c})\right\}\left\{\frac{\sin^4\left[\frac{(\omega+\omega_1)T}{4}\right]}{\left[\frac{(\omega+\omega_1)T}{4}\right]^2} + \frac{\sin^4\left[\frac{(\omega-\omega_1)T}{4}\right]}{\left[\frac{(\omega-\omega_1)T}{4}\right]^2}\right\} \\
& - j\,\frac{A^2V^2\pi T}{8}\sin 2(\omega_1 t_1+\Theta_{1c})\left\{\frac{\sin^4\left[\frac{(\omega+\omega_1)T}{4}\right]}{\left[\frac{(\omega+\omega_1)T}{4}\right]^2} - \frac{\sin^4\left[\frac{(\omega-\omega_1)T}{4}\right]}{\left[\frac{(\omega-\omega_1)T}{4}\right]^2}\right\} \\
& + \frac{A^2V^2\pi}{8}\left[\delta(\omega+\omega_2) + \delta(\omega-\omega_2)\right]\left[1 + \cos 2(\omega_2 t_1+\Theta_{2c})\right] \\
& - j\,\frac{A^2V^2\pi}{8}\sin 2(\omega_2 t_1+\Theta_{2c})\left[\delta(\omega+\omega_2) - \delta(\omega-\omega_2)\right]
\end{aligned}$$

$$+ \frac{A^2V^2\pi T}{8}\left\{1+\cos 2(\omega_2 t_1+\Theta_{2c})\right\}\left\{\frac{\sin^4\left[\frac{(\omega+\omega_2)T}{4}\right]}{\left[\frac{(\omega+\omega_2)T}{4}\right]^2}+\frac{\sin^4\left[\frac{(\omega-\omega_2)T}{4}\right]}{\left[\frac{(\omega-\omega_2)T}{4}\right]^2}\right\}$$

$$- j\frac{A^2V^2\pi T}{8}\sin 2(\omega_2 t_1+\Theta_{2c})\left\{\frac{\sin^4\left[\frac{(\omega+\omega_2)T}{4}\right]}{\left[\frac{(\omega+\omega_2)T}{4}\right]^2}-\frac{\sin^4\left[\frac{(\omega-\omega_2)T}{4}\right]}{\left[\frac{(\omega-\omega_2)T}{4}\right]^2}\right\}$$

$$+ \frac{A^2V^2\pi}{4}\cos(\omega_2 t_1+\Theta_{2c})\cos(\omega_1 t_1+\Theta_{1c})\left\{\left[\delta(\omega-\omega_1)+\delta(\omega+\omega_1)\right]\right.$$

$$\left.- T\left[\frac{\sin^4\left\{\frac{(\omega-\omega_1)T}{4}\right\}}{\left\{\frac{(\omega-\omega_1)T}{4}\right\}^2}+\frac{\sin^4\left\{\frac{(\omega+\omega_1)T}{4}\right\}}{\left\{\frac{(\omega+\omega_1)T}{4}\right\}^2}\right]\right\}$$

$$- \frac{A^2V^2\pi}{4}\cos(\omega_2 t_1+\Theta_{2c})\sin(\omega_1 t_1+\Theta_{1c})\left\{\left[\delta(\omega+\omega_1)-\delta(\omega-\omega_1)\right]\right.$$

$$\left.+ jT\left[\frac{\sin^4\left\{\frac{(\omega+\omega_1)T}{4}\right\}}{\left\{\frac{(\omega+\omega_1)T}{4}\right\}^2}-\frac{\sin^4\left\{\frac{(\omega-\omega_1)T}{4}\right\}}{\left\{\frac{(\omega-\omega_1)T}{4}\right\}^2}\right]\right\}$$

$$+ \frac{A^2V^2\pi}{4}\cos(\omega_1 t_1+\Theta_{1c})\cos(\omega_2 t_1+\Theta_{2c})\left\{\left[\delta(\omega-\omega_2)+\delta(\omega+\omega_2)\right]\right.$$

$$\left.- T\left[\frac{\sin^4\left\{\frac{(\omega-\omega_2)T}{4}\right\}}{\left\{\frac{(\omega-\omega_2)T}{4}\right\}^2}+\frac{\sin^4\left\{\frac{(\omega+\omega_2)T}{4}\right\}}{\left\{\frac{(\omega+\omega_2)T}{4}\right\}^2}\right]\right\}$$

$$- \frac{A^2V^2\pi}{4}\cos(\omega_1 t_1+\Theta_{1c})\sin(\omega_2 t_1+\Theta_{2c})\left\{\left[\delta(\omega+\omega_2)-\delta(\omega-\omega_2)\right]\right.$$

$$\left.+ jT\left[\frac{\sin^4\left\{\frac{(\omega+\omega_2)T}{4}\right\}}{\left\{\frac{(\omega+\omega_2)T}{4}\right\}^2}-\frac{\sin^4\left\{\frac{(\omega-\omega_2)T}{4}\right\}}{\left\{\frac{(\omega-\omega_2)T}{4}\right\}^2}\right]\right\}$$

Time Averaged Power Spectral Density

It should be observed that the expression for the time varying power spectral density of coherent FSK (Equation 5.82) when time averaged with respect to t_1 reduces (in a manner analogous to that of Equation 3.69 through Equation 3.74, the terms which are a function of t_1 average to zero) to the expression for the power spectral density of non-coherent FSK.

$$(5.83) \quad \overline{S_{FSKC}(t_1,\omega)} = \frac{A^2V^2\pi}{8}\left\{\delta(\omega+\omega_1) + \delta(\omega-\omega_1)\right\}$$

$$+ \frac{A^2V^2\pi T}{8}\left\{\frac{\sin^4\left[\frac{(\omega+\omega_1)T}{4}\right]}{\left[\frac{(\omega+\omega_1)T}{4}\right]^2} + \frac{\sin^4\left[\frac{(\omega-\omega_1)T}{4}\right]}{\left[\frac{(\omega-\omega_1)T}{4}\right]^2}\right\}$$

$$+ \frac{A^2V^2\pi}{8}\left\{\delta(\omega+\omega_2) + \delta(\omega-\omega_2)\right\}$$

$$+ \frac{A^2V^2\pi T}{8}\left\{\frac{\sin^4\left[\frac{(\omega+\omega_2)T}{4}\right]}{\left[\frac{(\omega+\omega_2)T}{4}\right]^2} + \frac{\sin^4\left[\frac{(\omega-\omega_2)T}{4}\right]}{\left[\frac{(\omega-\omega_2)T}{4}\right]^2}\right\}$$

A comparison of Equation 5.83 and Equation 5.37 shows that indeed the time average of the time varying power spectral density for coherent FSK is equal to the power spectral density for non-coherent FSK.

$$(5.79) \quad \overline{S_{FSKC}(t_1,\omega)} = S_{FSKN}(\omega)$$

CHAPTER VI

SUMMARY AND CONCLUSIONS

I GENERAL

A general discussion of pulse modulation techniques was developed, leading to the introduction of PCM coding. The ability to time multiplex many channels of information over a single transmission link was given as one of the most important characteristics of pulse modulation. Some general information concerning sampling and sampling rates was introduced. PCM was then defined, with an explanation of "quantizing", "quantization noise", and "encoding". The decided advantage of only having to discern which of two possible voltage levels was transmitted, for a binary PCM code was discussed. Split-phase coding was introduced as one technique used in binary encoding. It was noted that the higher transition density of split-phase codes usually allows more efficient synchronization to be maintained at the receiver.

Ensemble averages and ensemble average autocorrelation functions are introduced. The ensemble average and ensemble average autocorrelation functions for a random PCM split-phase code are presented in Equation 2.3 and Equation 2.10. The definitions of "stationary" and "ergodic" random variables were presented. The power spectral density function is defined and the power spectral density function for a random PCM split-phase code is presented in Equation 2.14.

It was observed that the autocorrelation and power spectral density functions were time varying for coherent ASK, PSK, and FSK where modulated by a random PCM split-phase code. The fact that these functions are time varying implies non-stationary. For coherent ASK, PSK, and FSK where the modulation was a random PCM split-phase code, the time varying autocorrelation and power spectral density functions when time averaged reduced to the corresponding non-coherent autocorrelation and power spectral density functions.

II AMPLITUDE SHIFT KEYING

The ASK case with coherent or non-coherent split-phase code modulation was discussed in Chapter III. The autocorrelation and power spectral density functions were found for both coherent and non-coherent split-phase code modulation. It was observed in Equation 3.22 and Equation 3.66 that maximum sideband power occurs when βV equals one ($\beta V = 1$), corresponding to "on - off" keying of the carrier.

Non-Coherent Modulation

It was observed that the condition for non-coherent modulation of the carrier by the split-phase code was that the carrier frequency not be an integral multiple of the split-phase code bit rate ($f_c \neq K\, f_B$, K = integer). This condition for non-coherent modulation required that the initial phase of the carrier be considered a random variable uniformly distributed between zero and 2π, ($\Theta = \Theta_N$, Θ_N = random variable uniformly distributed between 0 and 2π). Non-coherent ASK with split-phase code modulation is represented pictorially by Figure 4. The autocorrelation function for non-coherent ASK with split-phase code modulation is expressed in Equation 3.8. It should be observed that the autocorrelation function consists of the sum of the carrier autocorrelation function and a term resulting from the product of the split-phase code autocorrelation function and the carrier autocorrelation function. The power spectral density for this case, expressed in Equation 3.22 and pictorially represented by Figure 5 was shown to consist of discrete carrier components and sidebands resulting from the split-phase spectrum being translated to appear about plus and minus the carrier frequency.

Coherent Modulation

It was observed that the condition for coherent modulation of the carrier by the split-phase code was that the carrier frequency be an integral multiple of the split-phase code bit rate ($f_C = K f_B$, K = integer). This condition for coherent modulation required that the initial phase of the carrier be a constant ($\Theta = \Theta_C$ = constant). It was observed from Figure 6 that the minimum system bandwidth requirement occurs when Θ_C equals zero ($\Theta_C = 0$) or some integral multiple of π ($\Theta_C = K\pi$, K = integer), for these conditions no instantaneous amplitude change is required when a transition occurs in the split-phase code. The system bandwidth requirement increases with increased Θ_C from the minimum bandwidth conditions to a maximum when Θ_C equals an odd integral multiple of $\frac{\pi}{2}$ ($\Theta_C = Q \frac{\pi}{2}$, Q = odd integer), where the maximum possible instantaneous amplitude change is required.

The time varying autocorrelation function is expressed in Equation 3.37 and consist of the sum of a time varying carrier autocorrelation function and a term composed of the product of the split-phase autocorrelation function and the time varying carrier autocorrelation function. It was then shown that the time varying autocorrelation function for coherent ASK with split-phase code modulation when time averaged reduced to the autocorrelation function for non-coherent ASK with split-phase code modulation, Equation 3.51.

The time varying power spectral density for coherent ASK with split-phase modulation is expressed in Equation 3.67. As observed from Equation 3.66 the power spectral density consists of discrete time varying carrier components, and time varying sidebands resulting from the split-phase code being translated to appear about plus and minus the carrier frequency. It should be observed that the expression for the time varying power spectral density of coherent ASK modulated by a split-phase code when time averaged reduced to the expression for the power spectral density of non-coherent ASK modulated by a split-phase code, Equation 3.74.

III PHASE SHIFT KEYING

The PSK case with coherent or non-coherent split-phase code modulation was discussed in Chapter IV. The autocorrelation and power spectral density functions were found for both coherent and non-coherent split-phase modulation. It was observed from Equation 4.16 and Equation 4.49 that as the sideband power was maximized the carrier components tend to vanish, and indeed at $\beta V = Q\,\frac{\pi}{2}$, Q = odd integer, the carrier does vanish.

Non-Coherent Modulation

It was observed that the condition for non-coherent modulation of the carrier by the split-phase code was that the carrier frequency not be an integral multiple of the split-phase code bit rate ($f_c \neq K\,f_B$, K = integer). This condition for non-coherent modulation required that the initial phase of the carrier be considered a random variable uniformly distributed between zero and 2π, ($\Theta = \Theta_N$, Θ_N = random variable uniformly distributed between 0 and 2π). Non-coherent PSK with split-phase code modulation is represented pictorially by Figure 7. The autocorrelation function for non-coherent PSK with split-phase code modulation is expressed in Equation 4.9. It should be observed that the autocorrelation function consists of the sum of the carrier autocorrelation function and a term resulting from the product of the split-phase code autocorrelation function and the carrier autocorrelation function. The power-spectral density for this case expressed in Equation 4.16 and pictorially represented by Figure 8 was shown to generally consist of discrete carrier components and sidebands resulting from the split-phase spectrum being translated to appear about plus and minus the carrier frequency.

It was observed that the condition for coherent modulation of the carrier by the split-phase code was that the carrier frequency be an integral multiple of the split-phase code bit rate ($f_c = K\ f_B$, K = integer). This condition for coherent modulation required that the initial phase of the carrier be a constant ($\Theta = \Theta_c$ = constant). It was observed from Figure 9 that the minimum system bandwidth requirement occurs when Θ_c equals zero ($\Theta_c = 0$) or some integral multiple of π ($\Theta_c = K\pi$, K = integer), for these conditions no instantaneous amplitude change is required when a transition occurs in the split-phase code. The system bandwidth requirement increases with increased θ_c from the minimum bandwidth conditions to a maximum when θ_c equals an odd integral multiple of $\frac{\pi}{2}$ ($\theta_c = Q\,\frac{\pi}{2}$, Q = odd integer), where the maximum possible instantaneous amplitude change is required.

The time varying autocorrelation function is expressed in Equation 4.24 and consist of the sum of a time varying carrier autocorrelation function and a term composed of the product of the split-phase autocorrelation function and the time varying carrier autocorrelation function. It was then shown that the time varying autocorrelation function for coherent PSK with split-phase code modulation when time averaged reduced to the autocorrelation function for non-coherent PSK with split-phase code modulation, Equation 4.31.

The time varying power spectral density for coherent PSK with split-phase modulation is expressed in Equation 4.49. As observed from Equation 4.49 the power spectral density consists of discrete time varying carrier components, and time varying sidebands resulting from the split-phase code being translated to appear about plus and minus the carrier frequency. It should be observed that the expression for the time varying power spectral density of coherent PSK modulated by a split-phase code when time averaged reduced to the expression for the power spectral density of non-coherent PSK modulated by a split-phase code, Equation 4.54.

IV FREQUENCY SHIFT KEYING

The FSK case with coherent or non-coherent split-phase code modulation was discussed in Chapter V. The autocorrelation and power spectral density functions were found for both coherent and non-coherent split-phase code modulation. A technique for generating a Frequency-Shift-Keyed signal was introduced and pictorially represented by Figure 10. This technique transforms the split-phase code modulation signal and its inverse into unipolar signals and uses the resulting orthogonal signals to gate two oscillators of different frequencies.

Non-Coherent Modulation

It was observed that the conditions for non-coherent FSK with split-phase code modulation were that the two carrier frequencies be different and that they not be integral multiples of the bit rate

($f_1 \neq K f_B$, K = integer; $f_2 \neq D f_B$, D = integer; $D \neq K$). These relationships between the two oscillator frequencies and the split-phase code bit rate require that the initial phase angles of the two oscillators (Θ_1 and Θ_2) be considered as random variables, uniformly distributed between the 0 and 2π. Figure 11 pictorially represents non-coherent FSK with split-phase code modulation. The autocorrelation function for non-coherent FSK modulated by a split-phase code is represented by Equation 5.28. The power spectral density for non-coherent FSK with split-phase code modulation is presented in Equation 5.37, and pictorially represented by Figure 12. It should be observed that the spectral density consists of discrete carrier components and sidebands resulting from translation of the split-phase spectrum about plus and minus the two carrier frequencies. The similarity between non-coherent FSK and non-coherent ASK should be observed. Intuitively this similarity results from the fact that non-coherent FSK was shown to be the sum of two "on - off" keyed carriers of different frequency.

Coherent Modulation

It was observed that the conditions for coherent FSK with split-phase modulation were that the carriers be of different frequencies but integral multiples of the split-phase code bit rate ($f_1 = K f_B$, K = integer; $f_2 = D f_B$, D = integer; $D \neq K$). These conditions for coherent FSK require the initial phase of the two oscillators to be constant ($\Theta_1 = \Theta_{1C}$ = constant, $\Theta_2 = \Theta_{2C}$ = constant). It should be observed from Figure 13 that the minimum system bandwidth requirement occurs when the initial phase angles Θ_{1C} and Θ_{2C} are equal to zero or some integral multiple of π. The system bandwidth requirement increases as either Θ_{1c} or Θ_{2c} increases toward an odd integral multiple of $\frac{\pi}{2}$, and reaches a maximum when Θ_{1c} and Θ_{2c} are odd multiples of $\frac{\pi}{2}$ and opposite in sign, for example $\Theta_{1c} = \frac{\pi}{2}$, $\Theta_{2c} = \frac{3\pi}{2} = - \frac{\pi}{2}$.

The time varying autocorrelation function for coherent FSK with split-phase modulation is expressed in Equation 5.51. It was then shown that the time varying autocorrelation function for coherent FSK with split-phase modulation when time averaged reduced to the autocorrelation function for non-coherent FSK modulated with a split-phase code, Equation 5.59. The time varying power spectral density for the coherent FSK case is expressed by Equation 5.82. It should be observed that the expression for the time varying power spectral density for the coherent FSK when time averaged reduced to the expression for the power spectral density for the non-coherent FSK case.

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