FORCED VIBRATIONS OF DAMPED SYSTEMS WITH BILINEAR NONSYMMETRIC ELASTICITY

A Thesis

Presented to

the Faculty of the Department of Mechanical Engineering

University of Houston

In Partial Fulfillment

of the Requirements for the Degree

Master of Science in Mechanical Engineering

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by

Jayant Mandke

August 1969

ACKNOWLEDGMENT

The author is extremely grateful to Dr. C. D. Michalopoulos for his guidance, encouragement, helpful suggestions and comments in the course of this investigation, and for his thorough revision of the manuscript.

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ABSTRACT

A single degree-of-freedom viscously damped system with bilinear, nonsymmetric restoring force is analyzed. The differential equations governing the motion of the system are solved in closed form by matching the solutions for the positive and negative parts of a cycle. The fourth order Runge-Kutta method is also employed to solve the system numerically. Steady-state response curves are plotted for several ratios of the spring constants and damping. Both force and motion inputs are considered.

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LIST OF SYMBOLS

M	Absolute mass or Dynamic mass
K_1	Stiffness of the spring on positive side
K_2	Stiffness of the spring on negative side
c	Viscous damping coefficient
ω	Frequency of the exciting force
..	Frequency of excitation of the top support
m	Absolute mass of sphere
L	Length of cable
R	Radius of sphere
C_D	Drag coefficient
	Density of water
A	Area of cross section of cable
E	Young's Modulus of Elasticity for cable

LIST OF FIGURES

CHAPTER I

INTRODUCTION

Since Duffing's [1][1] classical work in 1918, problems in vibrations of mechanical systems with nonlinear restoring forces have received considerable attention. Other pioneers in the field include Martienssen, Ludeke, Den Hartog, Raucher, and Jacobsen.

Duffing and Ludeke [2] studied a nonlinear restoring force as shown in Fig. 1(e,f). Den Hartog along with Heiles [3] and Mikina [4] studied various combinations of linear springs as shown in Fig. 1(a,b,c,d). But their work was chiefly limited to symmetrical restoring forces. Jacobsen and Jesperson [5] extended an analytical method due to Den Hartog [6] and a graphical method due to Martienssen for solving symmetric as well as unsymmetric nonlinear restoring force problems. Raucher [7] used a different analytical method for problems having nonsymmetric restoring forces. Most of the nonlinear symmetric spring forces treated by earlier investigators are similar to the types shown in Fig. 1.

Various analytical methods have been developed over the years for solving nonlinear problems.

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[1]Numbers in brackets refer to the Bibliography at the end of the thesis.

The purpose of this thesis is to study analytically the response of a single-degree-of-freedom system possessing the spring characteristics shown in Fig. 2.

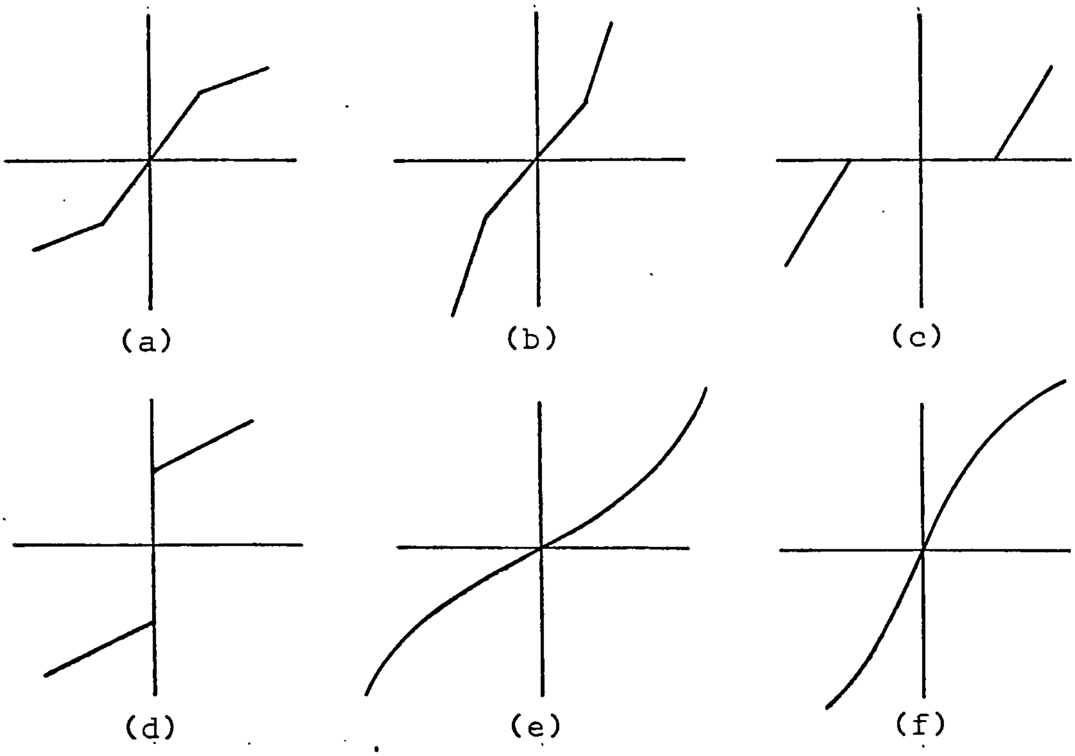

Fig. 1. Symmetric Restoring Forces

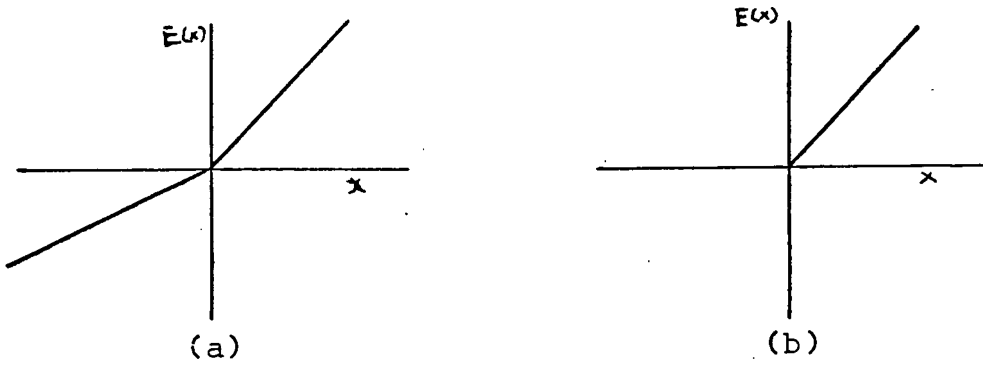

Fig. 2. Unsymmetric Restoring Forces

Jacobsen and Jesperson [5] gave an analytical expression for the steady-state response of system with the restoring force shown in Fig. 2(a), using a two term approximation. But they have neither considered damping nor given response curves. They studied such a system with base excitation in connection with the use of nonlinear springs in safeguarding buildings against seismic disturbances.

The motivation for the present problem lies in a practical situation. When a certain heavy mass is suspended by a cable fixed on a heaving ship or floating platform, the cable has an equivalent spring characteristic of the type shown in Fig. 2(b). In order to simplify the problem, the restoring force of the type shown in Fig. 2(a) is considered. The mass is acted upon by a sinusoidal force and it is constrained to oscillate in a horizontal direction. It should be noted that the spring characteristic shown in Fig. 2(b) is a special case of that shown in Fig. 2(a). Later the case of a vertically oscillating mass with motion input is studied. Although in practice the mass which may be suspended in the sea experiences quadratic damping, the problem has been studied for viscous damping. The reason for this is to focus attention on the effects of the spring nonlinearities on the response of the system. The method for dealing with quadratic damping is explained later.

In Chapter II, the differential equations governing the response and their solutions are given. In Chapter III, displacement vs. time and amplitude vs. frequency curves are presented. Chapter IV contains discussion of results and conclusions. Appendix I contains the solution of equations for a vertically oscillating mass. Appendix II gives a listing of the Fortran IV program statements used in obtaining the response.

CHAPTER II

DIFFERENTIAL EQUATIONS OF MOTION

GOVERNING THE RESPONSE

CASE I: Mass Oscillating Horizontally

With Force Input

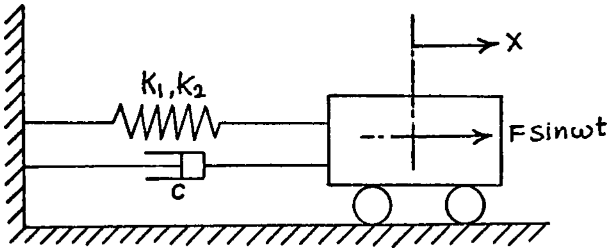

Fig. 3. Horizontally Oscillating System

The spring characteristic for the above system is as follows:

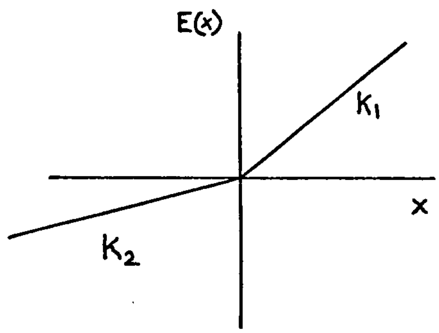

Fig. 4. Spring Characteristic for the System Shown in Fig. 3.

The differential equations of motion are

$$M\ddot{x} + c\dot{x} + K_1 x = F \sin \omega t, \text{ for } x > 0 \qquad (1)$$

$$M\ddot{x} + c\dot{x} + K_2 x = F \sin \omega t, \text{ for } x < 0 \qquad (2)$$

These equations being linear over a range can be solved analytically. Consider first Eq. (1), i.e., the differential equation governing the motion in the region $x > 0$. This equation has the familiar complementary solution

$$x_c = e^{-\frac{c}{2M}t}\left[A_1 \sin \omega_{d_1} t + B_1 \cos \omega_{d_1} t\right] \qquad (3)$$

and the particular solution

$$x_p = P_1 \sin \omega t + Q_1 \cos \omega t \qquad (4)$$

where A_1 and B_1 are arbitrary constants which depend on the initial conditions, and

$$\omega_{d_1} = \sqrt{\frac{K_1}{M} - \left(\frac{c}{2M}\right)^2}$$

$$P_1 = \frac{F\,(K_1 - M\omega^2)}{[K_1 - M\omega^2]^2 + (c\omega)^2}$$

$$Q_1 = \frac{-Fc\omega}{[K_1 - M\omega^2]^2 + (c\omega)^2}$$

The complete solution is, therefore:

$$x = e^{-\frac{c}{2M}t}\left[A_1 \sin \omega_{d_1} t + B_1 \cos \omega_{d_1} t\right]$$

$$+ P_1 \sin \omega t + Q_1 \cos \omega t \qquad (5)$$

It has been assumed in deriving Eq. (5) that $\left(\frac{c}{2M}\right)^2 < \frac{K_1}{M}$, i.e., the system is under-damped.

When $\underline{x}$ is less than zero, the governing equation is

$$M\ddot{x} + c\dot{x} + K_2 x = F \sin \omega t \tag{2}$$

which has the solution

$$x = e^{-\frac{c}{2M}t}\left[A_2 \sin \omega_{d_2} t + B_2 \cos \omega_{d_2} t\right] + P_2 \sin \omega t + Q_2 \cos \omega t \tag{6}$$

where A_2 and B_2 are arbitrary constants, and

$$\omega_{d_2} = \sqrt{\frac{K_2}{M} - \left(\frac{c}{2M}\right)^2}$$

$$P_2 = \frac{F\left[K_2 - M\omega^2\right]}{\left[K_2 - M\omega^2\right]^2 + (c\omega)^2}$$

$$Q_2 = \frac{-Fc\omega}{\left[K_2 - M\omega^2\right]^2 + (c\omega)^2}$$

The solution is different from (6) when $K_2=0$.

The response to sinusoidal exciting force is obtained as follows. At $t = 0$, let $x = x_0$ and $\dot{x} = \dot{x}_0$. Using these initial conditions, the arbitrary constants A_1 and B_1 in Eq. (5) are evaluated. This equation is used to evaluate x and $\dot{x}$ at time $0 + \Delta t$. Then if $x > 0$, a new value for x and $\dot{x}$ is computed from Eq. (5) for $t = 2\Delta t$. Assume now that $\underline{x}$ becomes zero when $t = t_0$. At this instant let the velocity $\dot{x}(t_0)$ be V_0. This velocity and the condition $x=0$ constitute the initial conditions for evaluating the constants A_2 and B_2 in Eq. (6). Then new values of x and $\dot{x}$ are computed at $t = t_0 + \Delta t$

using Eq. (6). Similarly, more time steps are taken until $x \approx 0$. Again, the new velocity $\dot{x}$ corresponding to the new t_0 and $x = x_0 \approx 0$ are used as initial conditions to compute the A_1 and B_1 in Eq. (5). The values of x and $\dot{x}$ are subsequently computed for time $t + \Delta t$ from Eq. (5) and the process is continued. It should be noted that the instant $t = 0$ is a special case of $t = t_0$. It is therefore necessary to determine the constants A_1, B_1, A_2 and B_2 for general initial conditions, i.e., $x = x_0$ and $\dot{x} = V_0$ at $t = t_0$, including the case $t = 0$. The evaluation of the constants for the three different cases is given below:

(a) <u>When $t = 0$, $x = x_0$ and $\dot{x} = V_0$</u>

Differentiating Eq. (5), one obtains

$$\dot{x} = e^{-\frac{c}{2M}t}\left[\left\{-\frac{c}{2M}\sin\omega_{d_1}t + \omega_{d_1}\cos\omega_{d_1}t\right\}A_1 - \left\{\frac{c}{2M}\cos\omega_{d_1}t + \omega_{d_1}\sin\omega_{d_1}t\right\}B_1\right] + P_1\omega\cos\omega t - Q_1\omega\sin\omega t \tag{7}$$

At $t = 0$, let $x = x_0$ and $\dot{x} = V_0$. Substituting in Eqs. (5) and (7) and solving for A_1 and B_1, one has

$$B_1 = x_0 - Q_1 \tag{8}$$

$$A_1 = \left(V_0 + \frac{c}{2M}B_1 - P_1\omega\right)\Big/\omega_{d_1} \tag{9}$$

(b) When $t = t_0$, $x = x_0$, $\dot{x} = V_0$ and $x > 0$

Let

$$\begin{aligned} T_1 &= P_1 \sin \omega t_0 + Q_1 \cos \omega t_0 \\ T_2 &= (x_0 - T_1)\, e^{\frac{c}{2M} t_0} \\ T_3 &= \omega (P_1 \cos \omega t_0 - Q_1 \sin \omega t_0) \\ T_4 &= \omega_{d_1} \frac{\cos \omega_{d_1} t_0}{\sin \omega_{d_1} t_0} - \frac{c}{2M} \end{aligned} \tag{10}$$

Substituting (b) into Eqs. (5) and (7) and using Eq. (10), one finds for A_1 and B_1,

$$B_1 = \frac{\sin \omega_{d_1}}{\omega_{d_1}} \left[T_2 . T_4 - (V_0 - T_3)\, e^{\frac{c}{2M} t_0} \right] \tag{11}$$

$$A_1 = (T_2 - B_1 \cos \omega_{d_1} t_0) \Big/ \sin \omega_{d_1} t_0 \tag{12}$$

(c) When $t = t_0$, $x = x_0$, $\dot{x} = V_0$ and $x < 0$

When K_2 is sufficiently large such that $\frac{K_2}{M} > \left(\frac{c}{2M}\right)^2$, the constants A_2 and B_2 in Eq. (6) are evaluated in the same way as in the previous case. It is sufficient to replace in Eqs. (11) and (12) P_1 by P_2, Q_1 by Q_2 and ω_{d_1} by ω_{d_2} to obtain

$$B_2 = \frac{\sin \omega_{d_2} t_0}{\omega_{d_2}} \left[T_2 . T_4 - (V_0 - T_3)\, e^{\frac{c}{2M} t_0} \right] \tag{13}$$

$$A_2 = (T_2 - B_2 \cos \omega_{d_2} t_0) \Big/ \sin \omega_{d_2} t_0 \tag{14}$$

where $\omega_{d_2} = \sqrt{\frac{K_2}{M} - \left(\frac{c}{2M}\right)^2}$

CASE II: Vertically Oscillating Mass With Motion Input

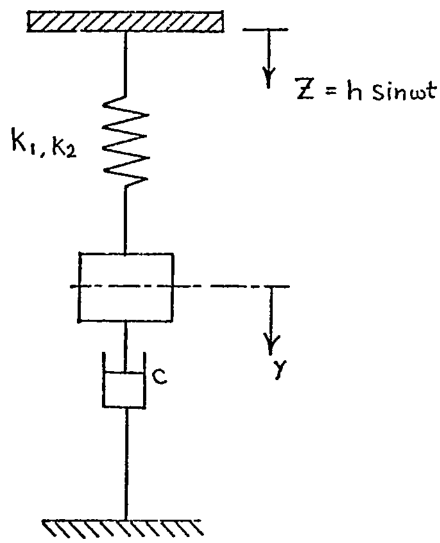

Fig. 5 Vertically Oscillating System

The spring characteristic is shown in Fig. 2. The differential equations of motion are

$$M\ddot{y} + c\dot{y} + k_1(y-z) = F \quad , \text{for } (y-z) > 0 \qquad (15)$$

$$M\ddot{y} + c\dot{y} + k_2(y-z) = F \,, \quad \text{for } (y-z) < 0 \qquad (16)$$

Eqs. (15) and (16) are solved analytically and the response is calculated in the same way as done in Case I. The solutions of Eqs. (15) and (16) are given in Appendix I. While applying Eqs. (15) and (16) to a mass suspended from a floating platform by a cable and oscillating in sea water, M is the dynamic mass, (mass of the body plus virtual mass of water) and F is the weight of the mass in water.

CASE III: A Sphere Suspended by Cable Supported to a Floating Platform Oscillating Sinusoidally

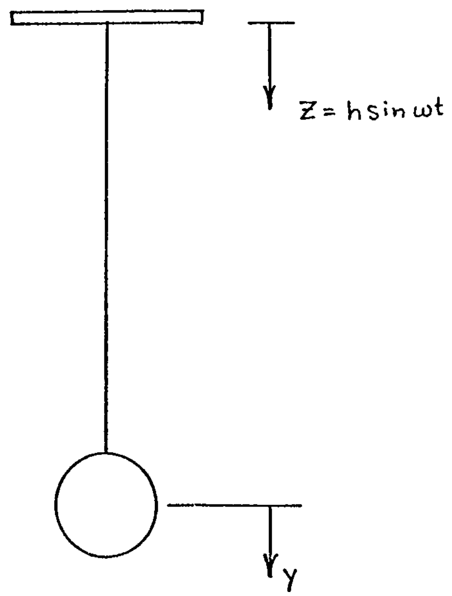

Fig. 6 Quadratically Damped System

Let L be the free length of the cable in feet,

m the mass of sphere in slugs,

R the radius of sphere in feet,

h the amplitude of excitation at the top in feet,

C_D the drag coefficient

ρ the density of water in lbs/ft^3

D the damping coefficient

A the cross sectional area of the cable in square inches.

E Young's Modulus of Elasticity for the cable in lb/in^2

M the dynamic mass,

= Mass of sphere + Virtual Mass of water accelerating with it.

F the weight of sphere in water

K the stiffness of cable

Then

$$D = C_D \times \pi R^2 \cdot \frac{\rho}{32.2 \times 2}$$

$$K = \frac{AE}{L} \quad lb/ft.$$

$$M = m + \frac{1}{2} \cdot \frac{4}{3} \pi R^3 \times \frac{\rho}{32.2}$$

The equations of Motion are

$$M\ddot{y} + D|\dot{y}|\dot{y} + K(y-z) = F, \quad \text{for } (y-z) > 0 \tag{17}$$

$$M\ddot{y} + D|\dot{y}|\dot{y} = F, \quad \text{for } (y-z) < 0 \tag{18}$$

Eqs. (17) and (18) can be solved numerically using Runge-Kutta method. They can also be solved analytically if the damping is linearized and an equivalent viscous damping coefficient is used.

As shown by Thompson [8], the equivalent viscous damping coefficient in the present case is

$$C_{eq} = \frac{8}{3\pi} \times \omega \times D \times X$$

where X is the amplitude of a linear system with the same parameters. Using C_{eq}, Eqs. (17) and (18) become:

$$M\ddot{y} + C_{eq}\dot{y} + K(y-z) = F, \quad \text{for } (y-z) > 0$$

and $$M\ddot{y} + c_{eq}\dot{y} \mp F, \quad \text{for } (y-z) < 0$$

However, in the present study, greater emphasis is placed on the effects of the bilinear, nonsymmetric spring characteristic (Fig.2.a,b) than on the effects of quadratic damping. Using viscous damping Eqs. (17) and (18) become identical to those of Case II with $K_2=0$, and they are subsequently solved in the manner shown there.

CHAPTER III

RESULTS

CASE I:

The system shown in Fig. 3 was studied for various values of the ratio $\frac{K_2}{K_1} < 1$, keeping K_1 constant and reducing K_2. In all the results shown in Figs. 7 to 19, the parameters used are: M=1 slug, K_1= lb/ft, F=215.

The steady-state amplitude response curves for various values of the ratio $\frac{K_2}{K_1}$ were plotted in nondimensional form. The amplitudes were nondimensionalized by dividing by $x_{o_1} = F/K_1$ and plotted vs. the frequency ratio $\frac{\omega}{p}$ where ω is the frequency of the forcing function and $p = \sqrt{\frac{K_1}{M}}$

In order to note the effect of damping, for a given ratio of $\frac{K_2}{K_1}$, the damping parameter $\zeta_1 = c/2Mp$ was varied from 0.127 to 0.25. These curves were derived directly from the displacement vs. time curves which were obtained by means of the computer program explained in Chapter II. The displacement was computed as a function of time up to 100 sec. for higher frequencies and up to 200 sec. for lower frequencies. This is equivalent to at least ten cycles. The actual computations showed that this is sufficient for steady-state conditions to be reached.

Two methods can be used in plotting the amplitude response curves. In order to bring out the effect of asymmetry of the spring characteristic, Raucher [7] recommended that the amplitudes on the positive side and the corresponding amplitudes on the negative side vs. frequency be plotted above and below the frequency axis respectively. But other investigators, like Jacobsen [5] adopted the method of using the average of the maximum positive and negative displacements as amplitude. In the present study the two amplitudes, the positive and the negative, have been plotted separately in Figs. 7 to 10, and the average amplitudes are plotted in Figs. 11 to 14.

Figs. 15 to 19 give displacement vs. time curves, at various frequencies for the last two cycles before the runs were discontinued, to show the variation of displacement with time and that steady-state is reached. These curves correspond to $\frac{K_2}{K_1} = 0.3$ and $\zeta_1 = 0.127$.

CASE II:

Figs. 20 to 23 give the response curves for the vertically oscillating mass under sinusoidal motion of the support. Here, the average of the maximum positive and negative displacements is plotted vs. frequency. The parameters used for this system are: M=1 slug, F=2 lbs.,

K_1= 3 lb/ft, ζ_1=0.2. These parameters were picked to correspond to the physical system described previously in Case III.

The results obtained using the semi-analytical method given in Chapter II were in good agreement with the results obtained using the fourth order Runge-Kutta method.

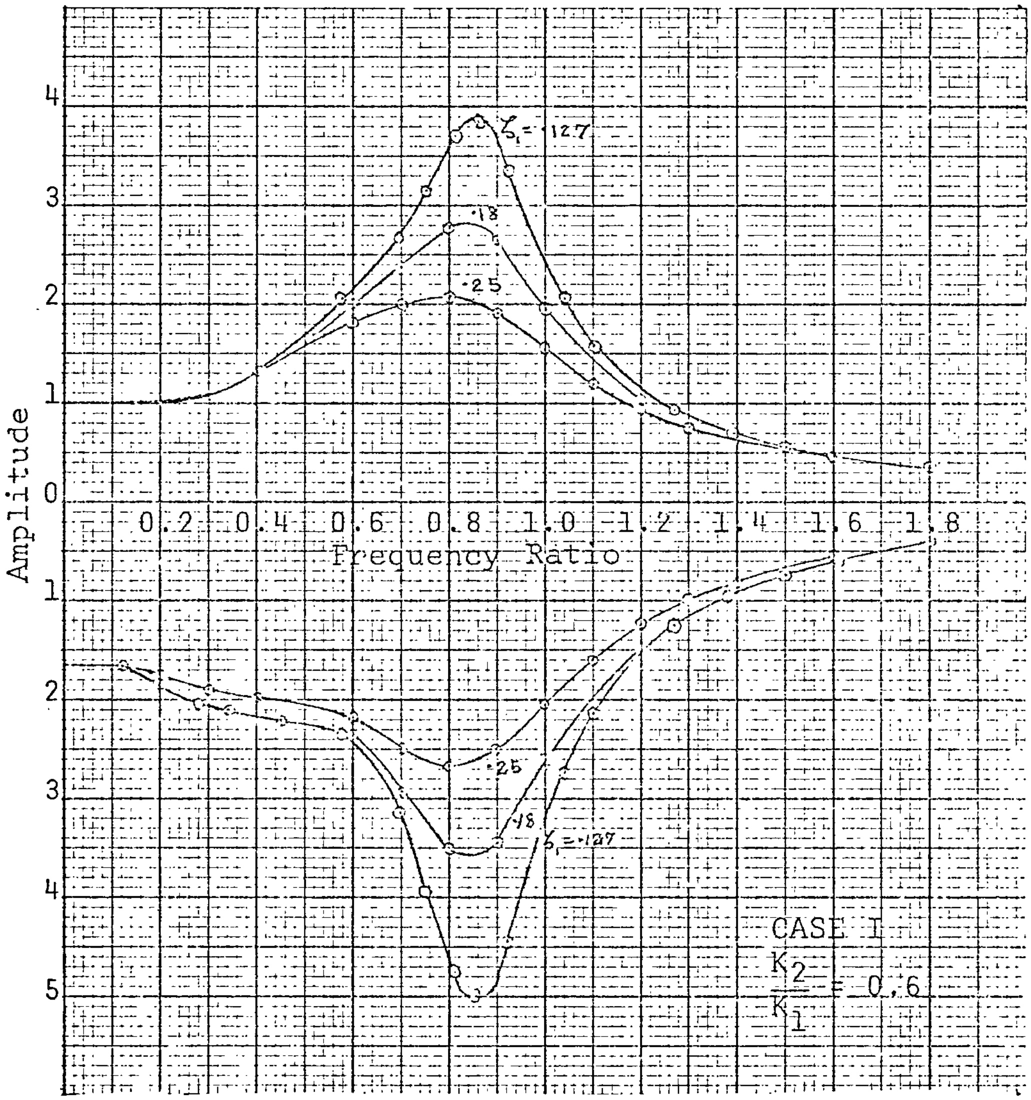

Fig. 7. Amplitude-Frequency Spectra

for $\frac{K_2}{K_1} = 0{\cdot}6$.

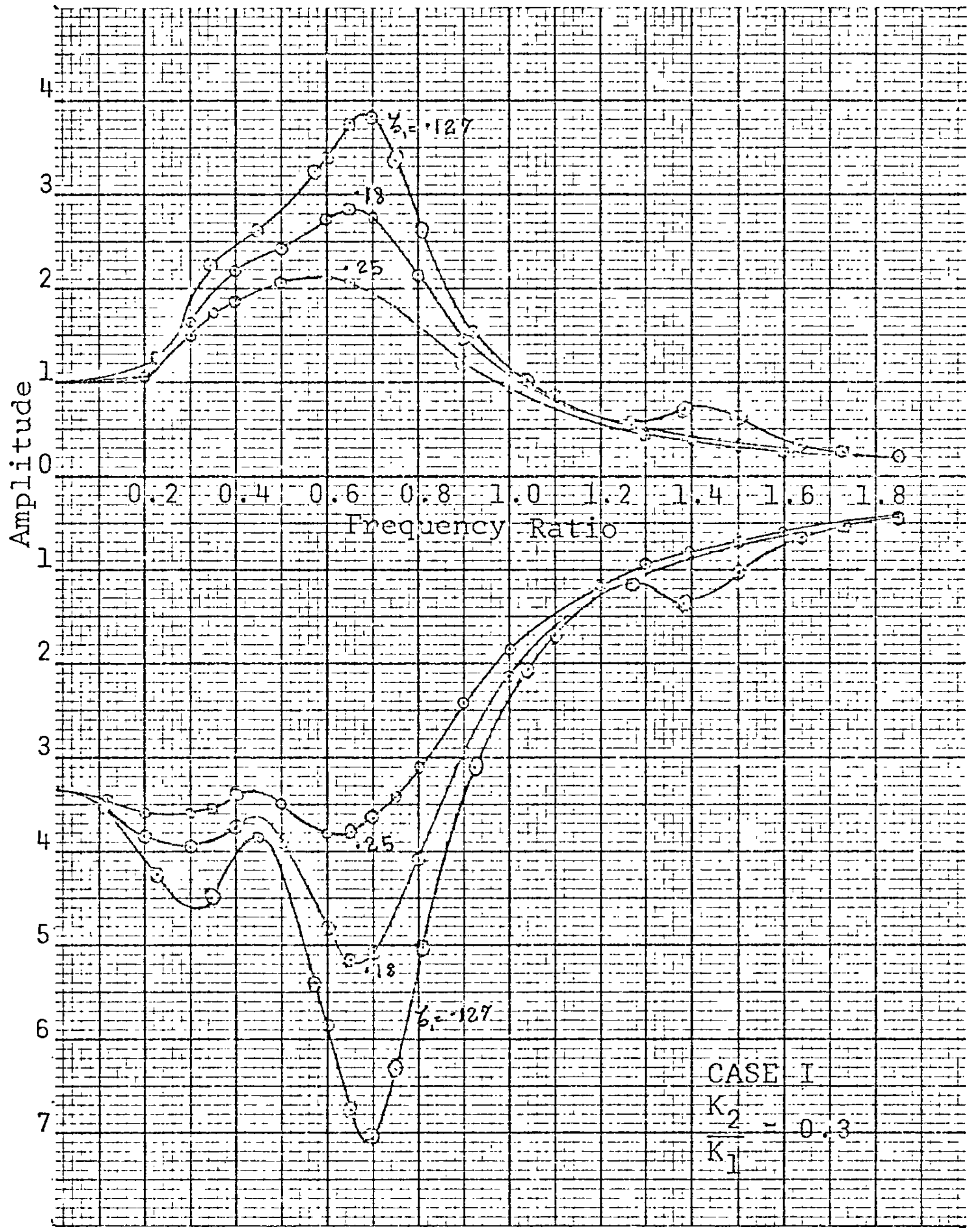

Fig. 8. Amplitude-Frequency Spectra

for $\frac{K_2}{K_1} = 0.3$

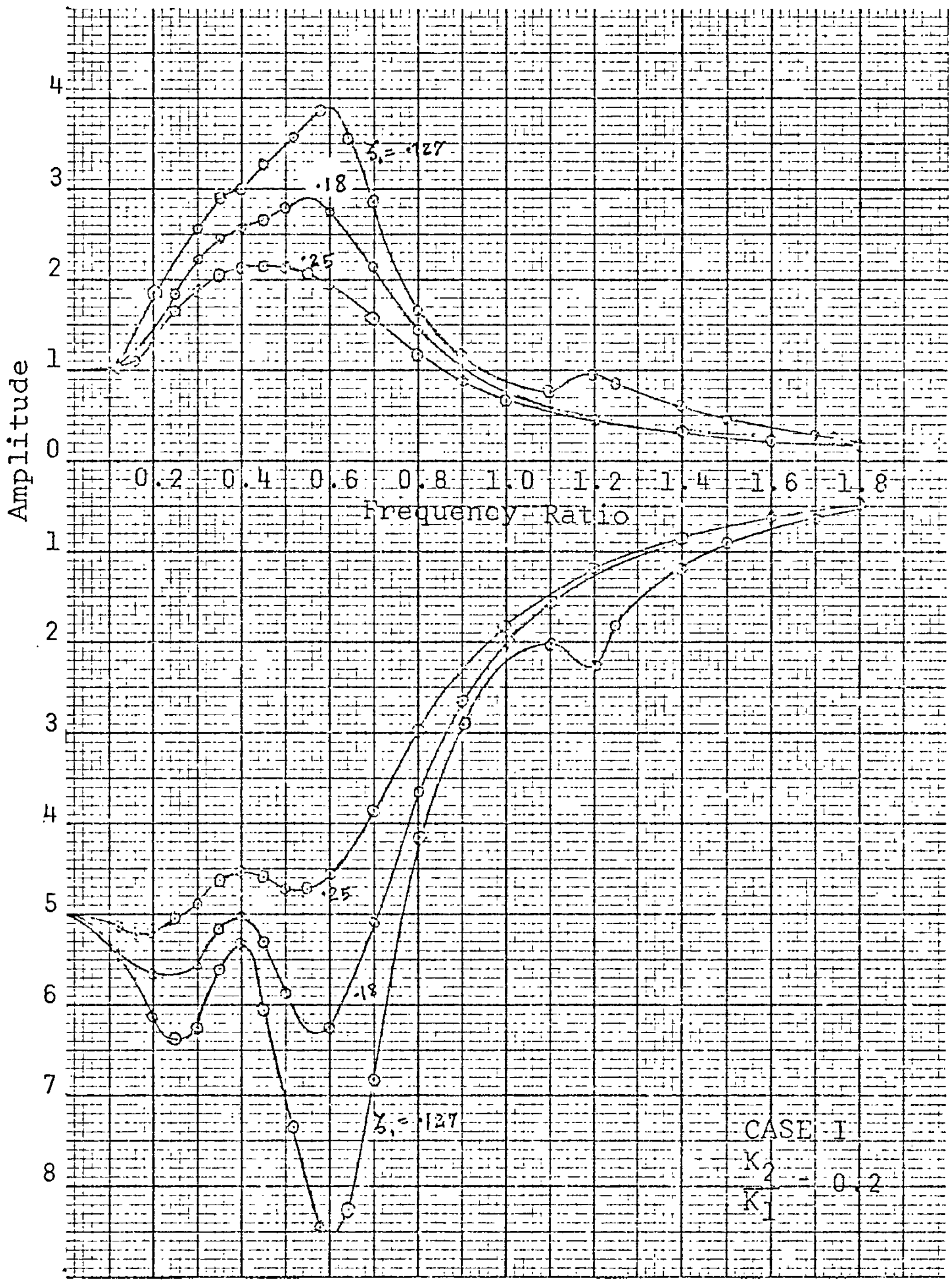

Fig. 9. Amplitude-Frequency Spectra

for $\frac{K_2}{K_1} = 0.2$

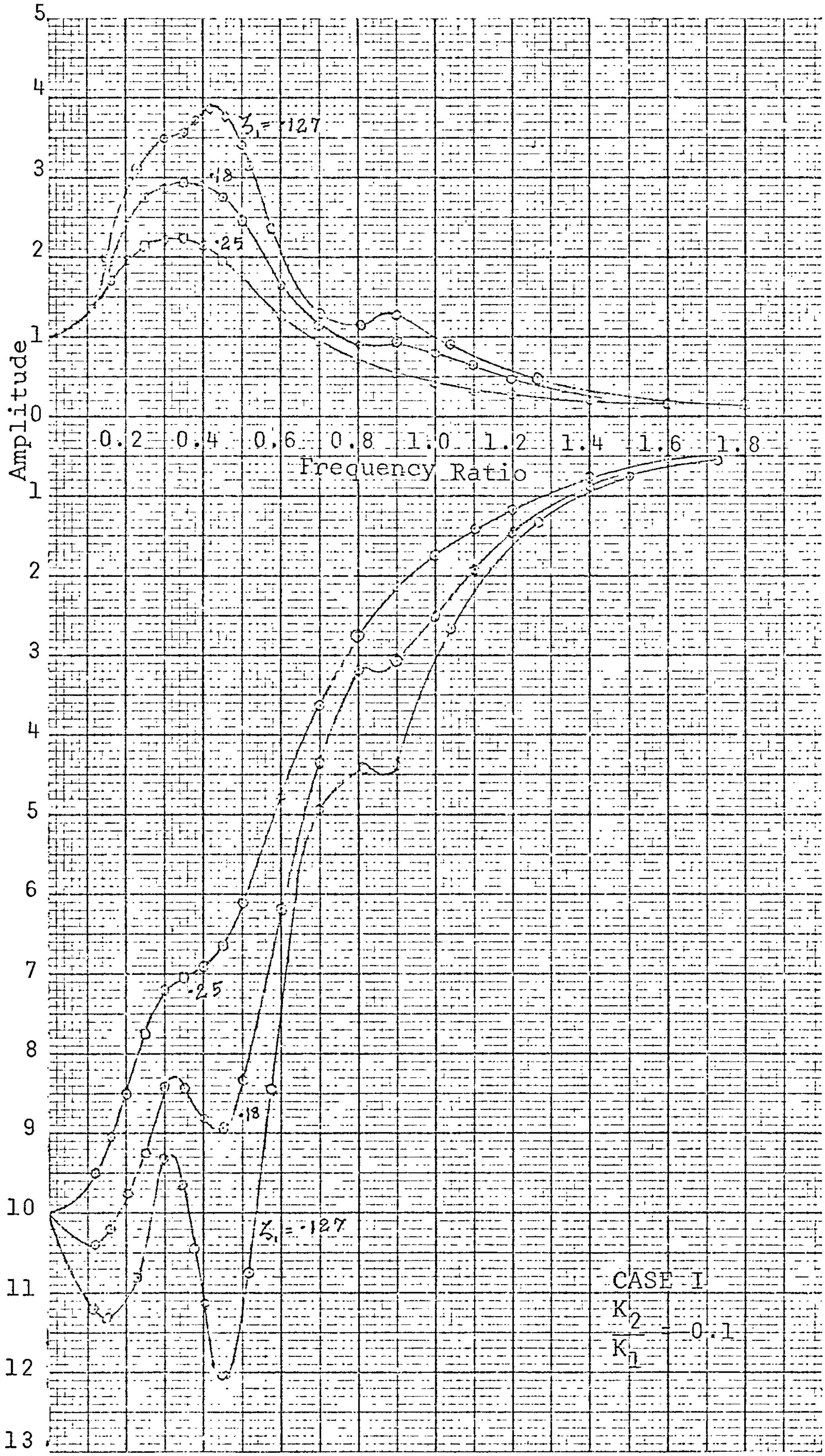

Fig. 10. Amplitude-Frequency Spectra
for $K_2/K_1 = 0.1$

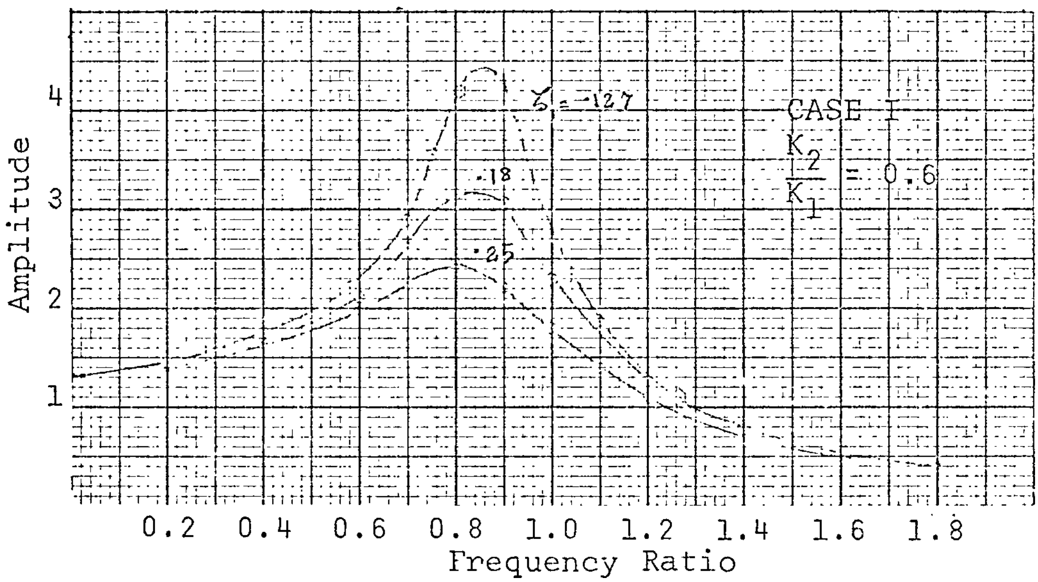

Fig. 11. (Average) Amplitude-Frequency Spectra for $\frac{K_2}{K_1} = .6$

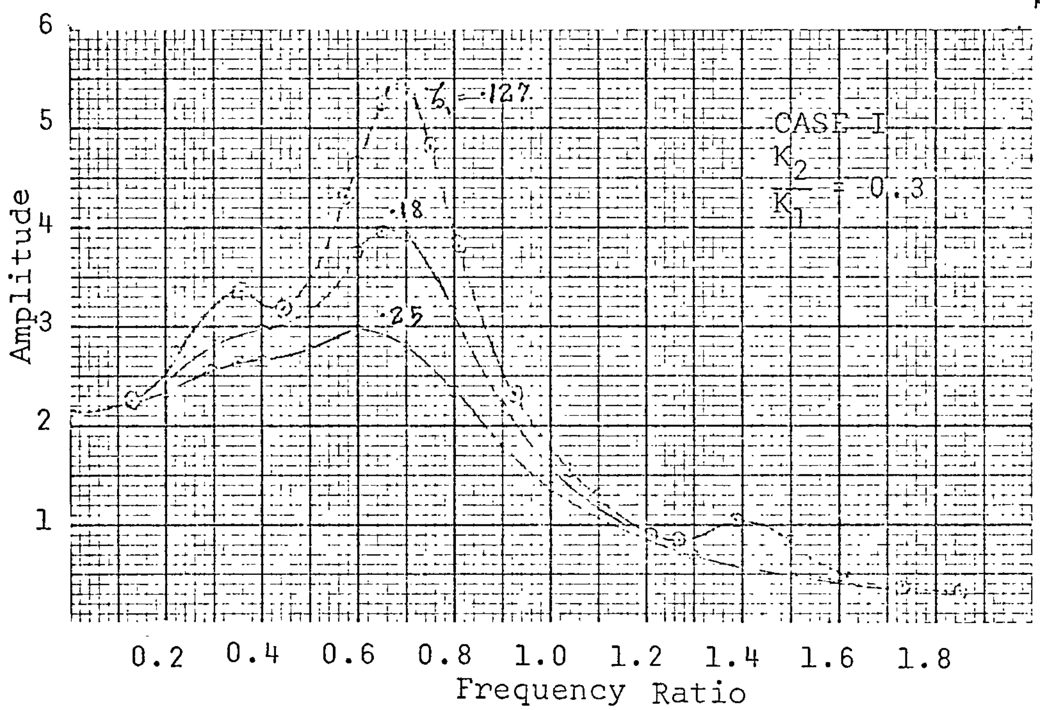

Fig. 12. (Average) Amplitude-Frequency Spectra for $\frac{K_2}{K_1} = .3$

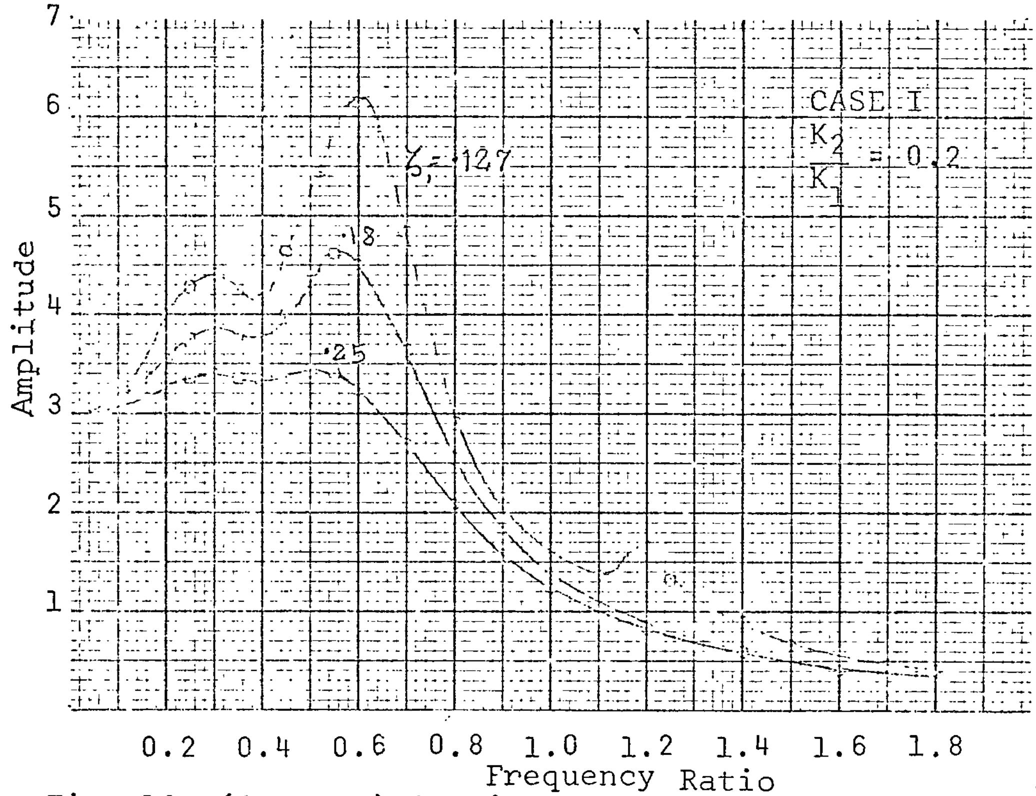

Fig. 13. (Average) Amplitude-Frequency Spectra for $\frac{K_2}{K_1} = 0.2$

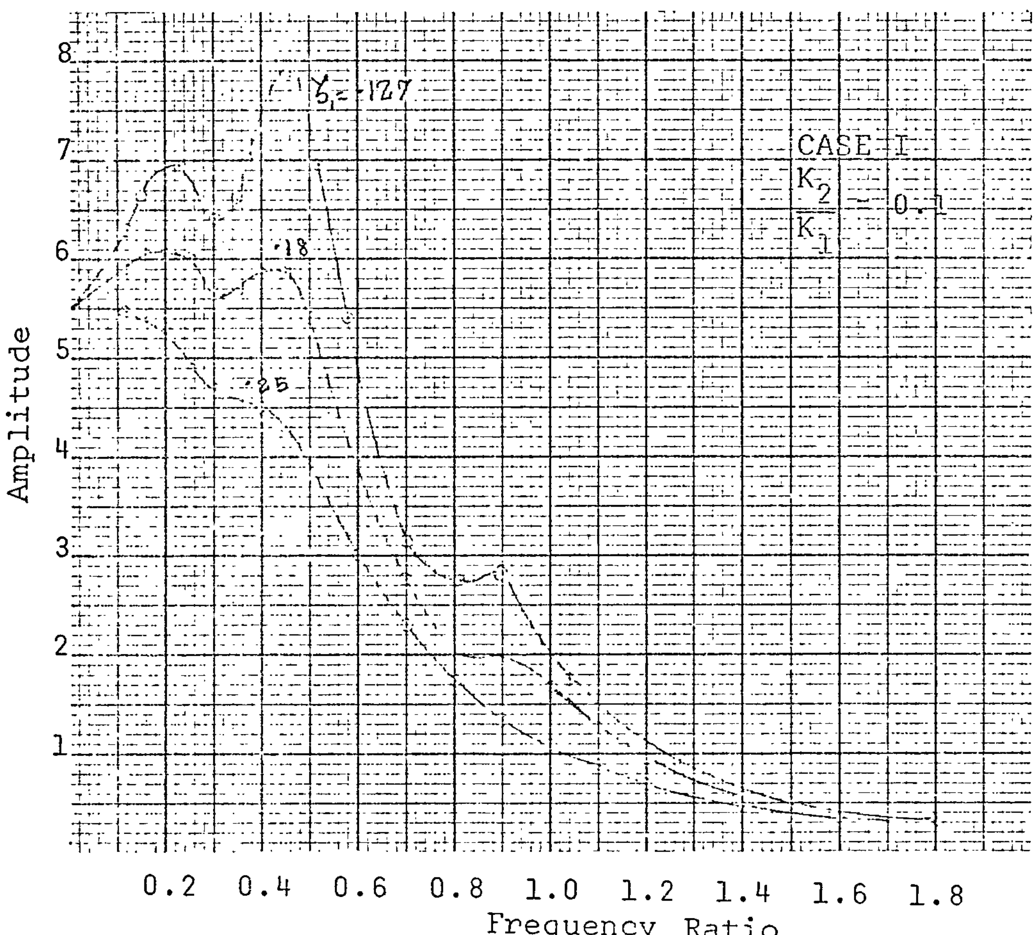

Fig. 14. (Average) Amplitude-Frequency Spectra for $\frac{K_2}{K_1} = 0.1$

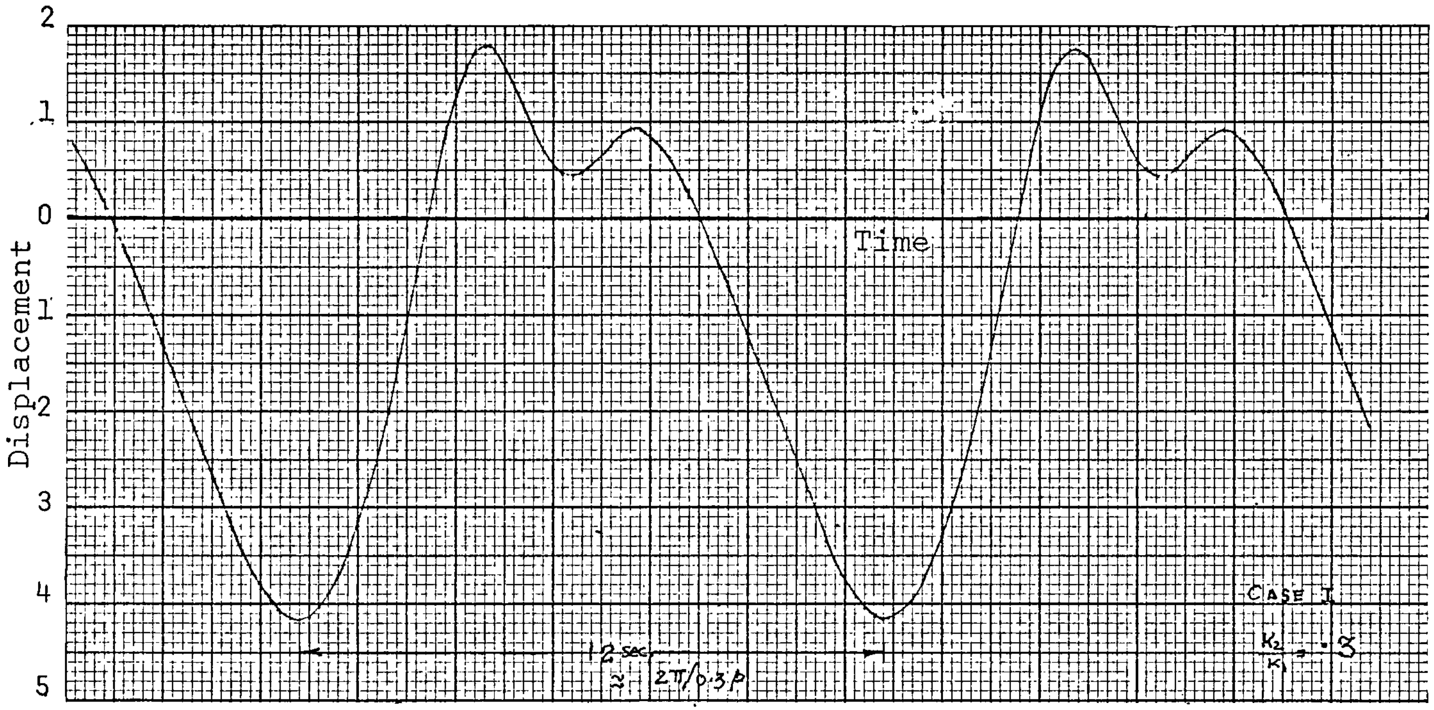

Fig. 15 Displacement vs. Time for $\frac{\omega}{p} = 0.3$

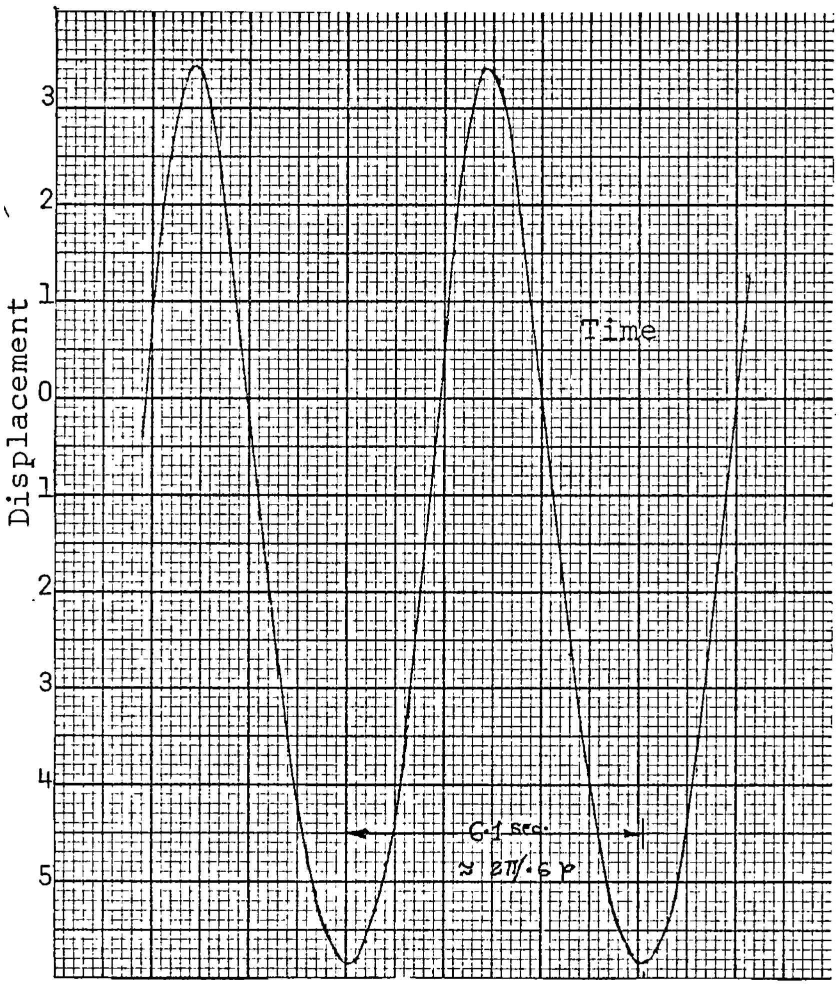

Fig. 16 Displacement vs. Time for $\frac{\omega}{p} = .6$

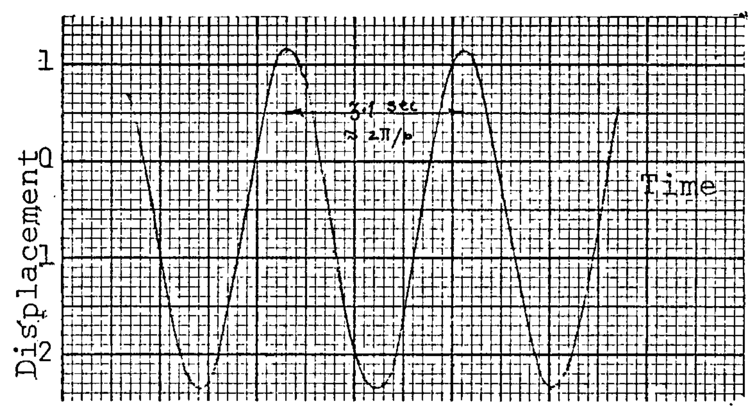

Fig. 17 Displacement vs. Time for $\frac{\omega}{p} = 1.0$

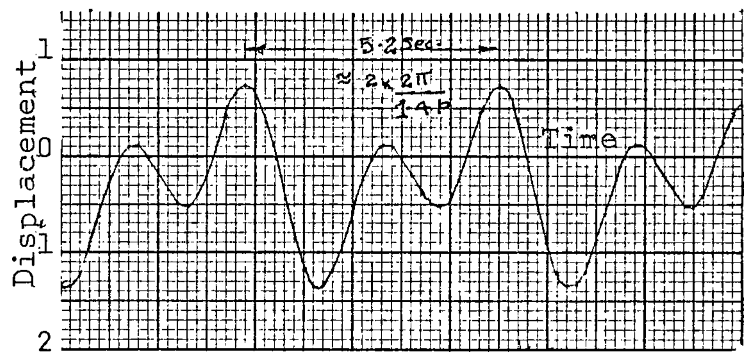

Fig. 18 Displacement vs. Time for $\frac{\omega}{p} = 1.4$

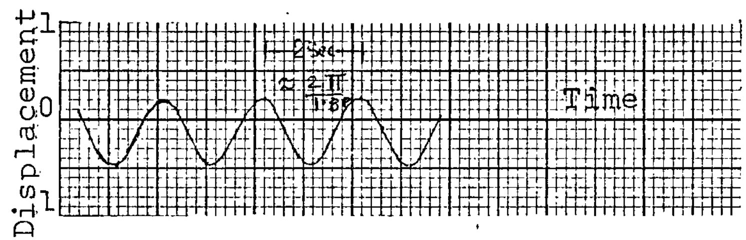

Fig. 19 Displacement vs. Time for $\frac{\omega}{p} = 1.8$

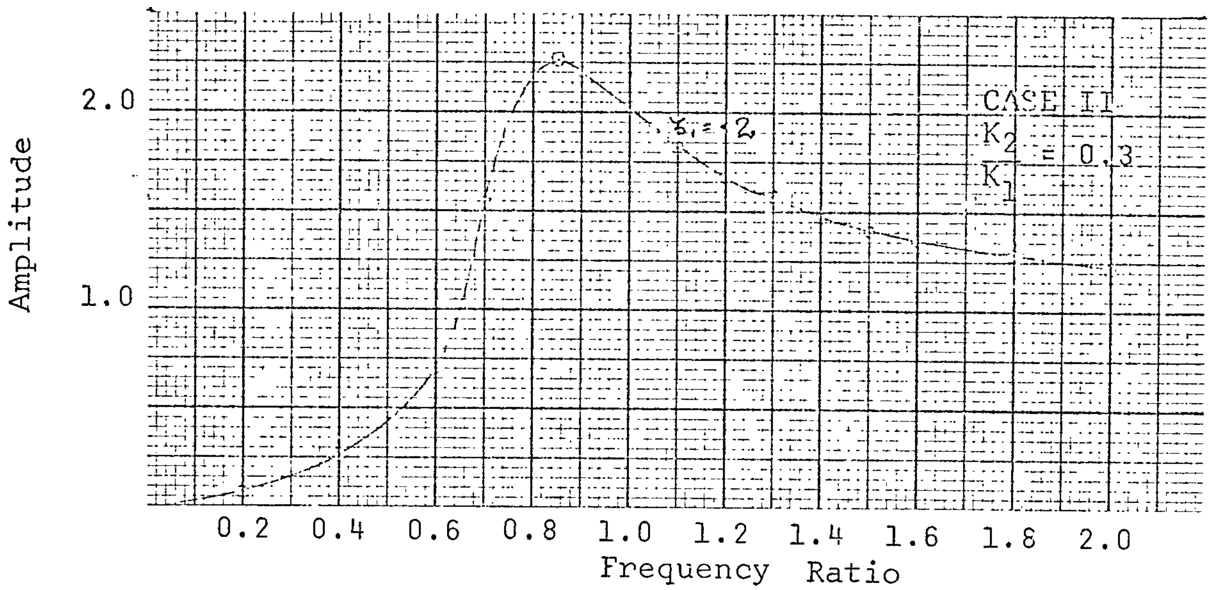

Fig. 20. (Average) Amplitude-Frequency Spectra for $\frac{K_2}{K_1} = .3$

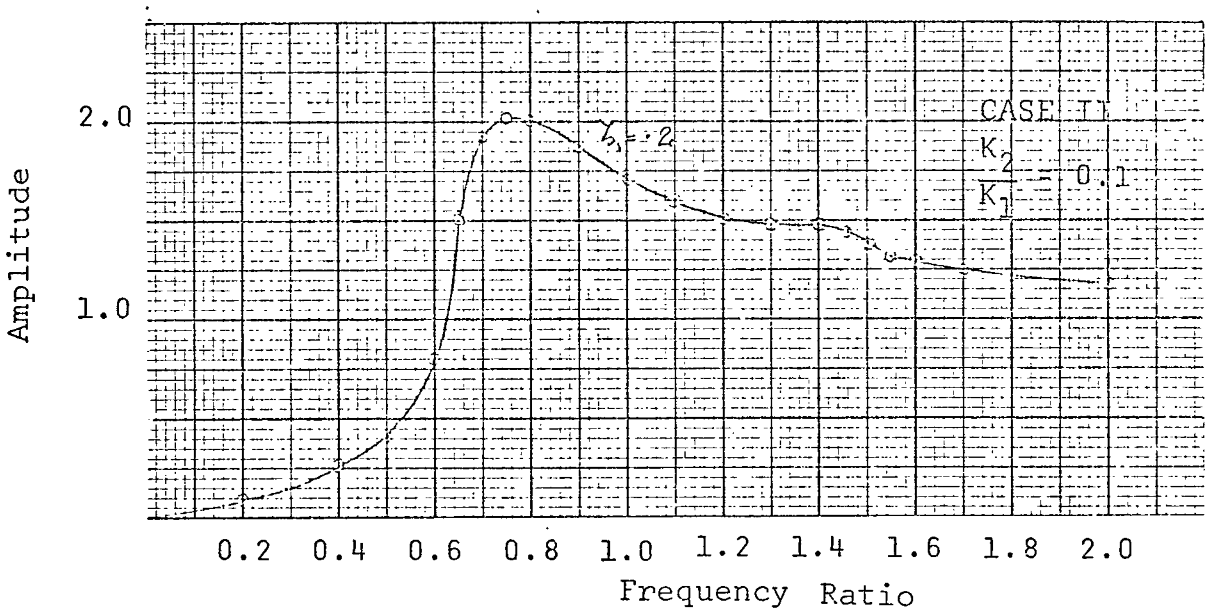

Fig. 21. (Average) Amplitude-Frequency Spectra for $\frac{K_2}{K_1} = .1$

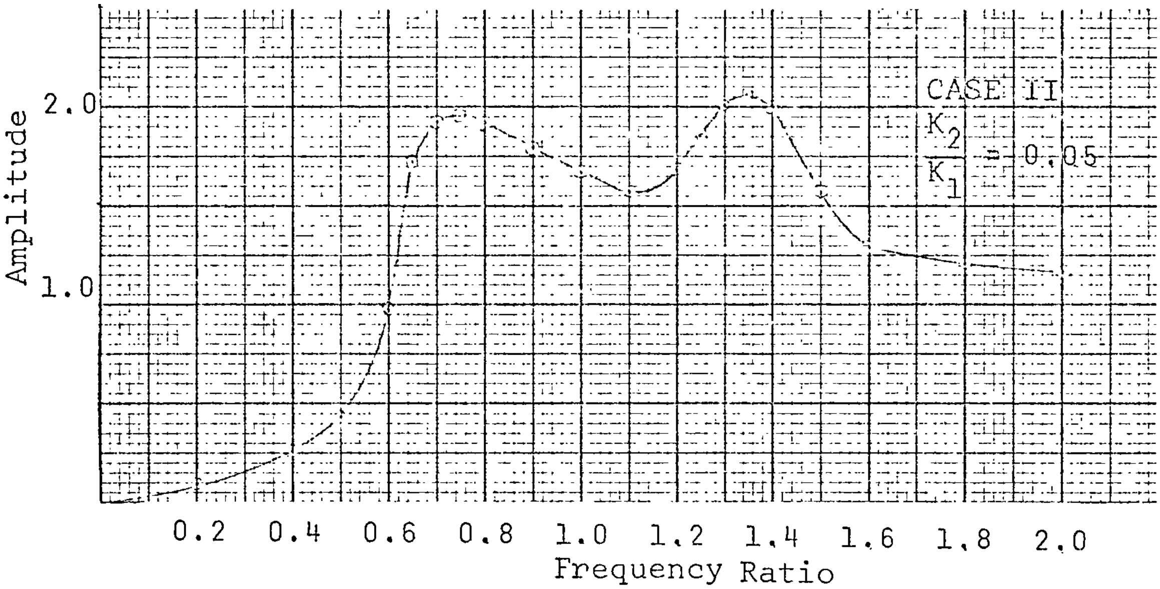

Fig. 22. (Average) Amplitude-Frequency Spectra for $\frac{K_2}{K_1} = \cdot 05$

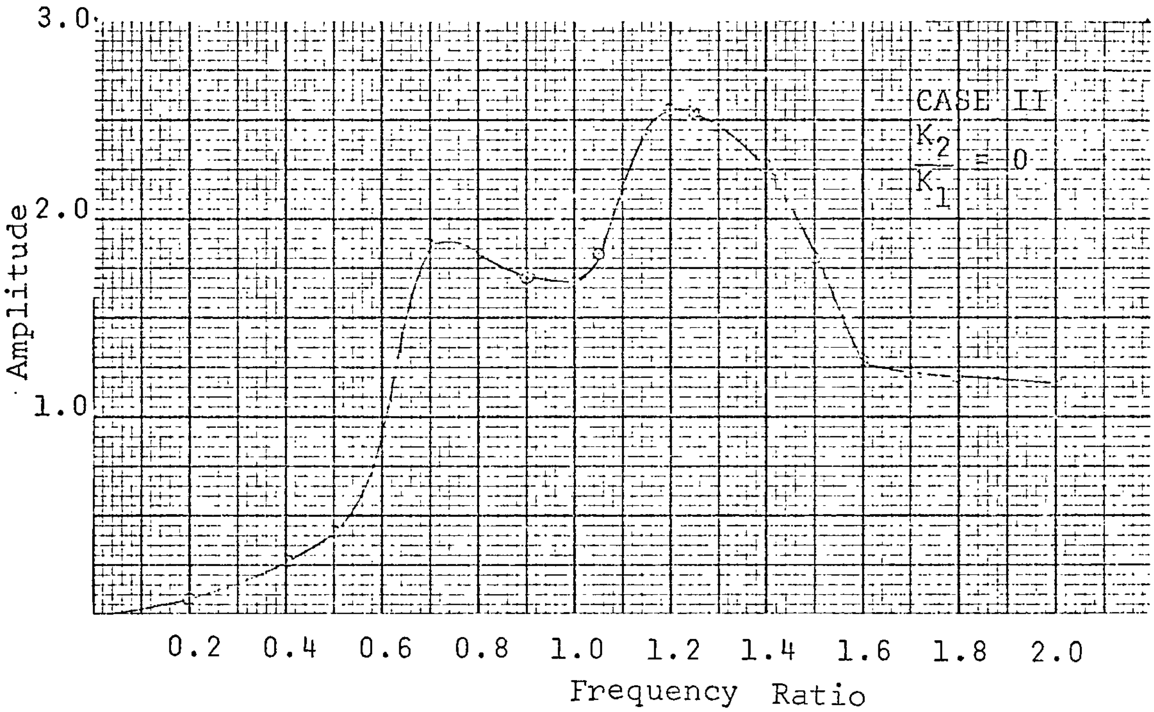

Fig. 23. (Average) Amplitude-Frequency Spectra for $\frac{K_2}{K_1} = 0$

CHAPTER IV

DISCUSSION OF RESULTS AND CONCLUSIONS

Since the spring is softer on one side and harder on the other, it is obvious that the mass should spend more time in the part of the cycle which corresponds to the softer side than that corresponding to the harder side. Also, the maximum displacement on the softer side should be greater than that on the harder side. This, in fact, is the case and it is demonstrated by the displacement vs. time curves in Figs. 15 to 19.

In order to note the effect of the spring nonlinearity Figs. 7 to 13 can be compared with the classical response curves for a linear system. With the decrease of the ratio $\frac{K_2}{K_1}$, the point of resonance, or maximum amplitude shifts to the left.

At smaller value of ζ_1, for certain values of the ratio $\frac{\omega}{p} > 1$, the period of the response becomes twice that of the exciting force (see Fig. 18). At such points, a hump is observed on the amplitude vs. frequency curves. Such a phenomenon is due to what is called subharmonic response of order two, and the jump in amplitude is due to subharmonic resonance. This phenomenon has been observed in systems governed by Duffing's equation [9],[10] and also in self-excited systems. At larger values of ζ_1 (=.25) such subharmonics are not observed. For the same ratio $\frac{K_2}{K_1}$, an increase in ζ_1 also results in shifting the point of resonance to the left. It is observed that as $\frac{K_2}{K_1}$ decreases, the region of subharmonic response shifts to the left. The effect becomes more significant with a decrease of $\frac{K_2}{K_1}$. For $\frac{K_2}{K_1} = 0.6$, the curves are almost similar to the classical ones, except for the fact that the amplitude on the negative side is greater than that on the positive side.

At lower frequencies, again there is a jump in the amplitude. Such a jump has been observed by Jacobsen and Jesperson [5] and Wylie [11]. They called such a jump "pseudo-resonance." In both references no proper physical interpretation for such a phenomenon was given. Perhaps it might be due to higher order harmonics attaining resonance. This jump becomes more significant at smaller values of the ratio $\frac{K_2}{K_1}$.

The case $\frac{K_2}{K_1} = 0$ is not studied in the case of a horizontally oscillating system, since here, in general, the mass will oscillate harmonically either only in the positive region or only in the negative region, depending on the conditions imposed. When the mass enters the negative region from the positive, the energy of the system, instead of being converted, at least partly, to potential energy, like in ordinary systems, is totally dissipated in damping. However, this is not true in the case of a vertically oscillating mass.

The curves plotted for the vertically oscillating system show certain similar features. Here again, the point of resonance shifts to the left as the ratio K_2/K_1 decreases. Here a subharmonic of order two is generated for $\frac{K_2}{K_1} = 0.1, 0.05,$ and zero. The effect of subharmonic resonance occurring at $\frac{\omega}{p} \approx 1.15$ becomes much more significant than the natural resonance occurring at $\frac{\omega}{p} \approx 0.7$, for $K_2 = 0$ (see Fig. 23). At low frequencies it is observed that the mass oscillates harmonically completely within the positive region. In the physical system of Case III, it can be predicted that the cable may not buckle under certain conditions. These are: (a) low frequency, (b) low amplitude of excitation and sufficiently large initial extension, (c) very low damping. It should be noted that no pseudo-resonance was observed in Case II, which indicates that the oscillations are linear in that range, i.e., no buckling occurred for the conditions used in obtaining the curves shown in Figs. 20 to 23.

It was found in the present study that the amplitude-frequency curve for both types of systems is a single-valued curve. The jump phenomena associated with many nonlinear systems do not occur in the present case. Such type of single-valued relationship between amplitude and frequency has been observed in the case of bilinear hysteresis by Caughey [12] and Iwan [13].

At frequencies higher than the resonant, subharmonics might be generated at low damping and large nonlinearity. Most of the analysis found in textbooks [14], [15], [16] on this subject is pertaining to Duffing's equation. A generalized analytical explanation for the reasons of generation of such subharmonics pertaining to this case has to be carried out. There seems to be a lack of physical interpretation of what has been called pseudo-resonance [5], [11] and this phenomenon, perhaps, should be investigated further.

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APPENDIX I

The differential equations of motion are:

$$M\ddot{y} + c\dot{y} + K_1(y-z) = F, \quad \text{for } (y-z) > 0 \tag{15}$$

$$M\ddot{y} + c\dot{y} + K_2(y-z) = F, \quad \text{for } (y-z) < 0 \tag{16}$$

Let $y - z = x$ where $z = h \sin \omega t$

Then $\dot{y} = \dot{x} + \dot{z}$ and $\ddot{y} = \ddot{x} + \ddot{z}$

Substituting into (15) and (16)

$$M\ddot{x} + c\dot{x} + K_1 x = F + M\omega^2 h \sin \omega t - c\omega h \cos \omega t \tag{19}$$

$$M\ddot{x} + c\dot{x} + K_2 x = F + M\omega^2 h \sin \omega t - c\omega h \cos \omega t \tag{20}$$

Solution of Eq. (19) is:

$$x = e^{-\frac{c}{2M}t}\left[A_1 \sin \omega_{d_1} t + B_1 \cos \omega_{d_1} t\right]$$

$$+ P_1 \sin \omega t + Q_1 \cos \omega t + \frac{F}{K_1} \tag{21}$$

and

$$\dot{x} = e^{-\frac{c}{2M}t}\left[A_1\left\{-\frac{c}{2M} \sin \omega_{d_1} t + \omega_{d_1} \cos \omega_{d_1} t\right\} - B_1\left\{\frac{c}{2M} \cos \omega_{d_1} t + \omega_{d_1} \sin \omega_{d_1} t\right\}\right]$$

$$+ P_1 \omega \cos \omega t - Q_1 \omega \sin \omega t \tag{22}$$

where A_1 and B_1 are arbitrary constants and,

$$\omega_{d_1} = \sqrt{\frac{K_1}{M} - \left(\frac{c}{2M}\right)^2}$$

$$P_1 = \frac{h\left[(K_1 - M\omega^2) M\omega^2 - (c\omega)^2\right]}{\left[(K_1 - M\omega^2)^2 + (c\omega)^2\right]}$$

$$Q_1 = -hc\omega K_1 / \left[(K_1 - M\omega^2)^2 + (c\omega)^2\right]$$

Solution of Eq. (20) is similar to that of Eq. (19) when $K_2 \neq 0$. In such a case:

$$x = e^{-\frac{c}{2M}t}\left[A_2 \sin \omega_{d_2} t + B_2 \cos \omega_{d_2} t\right]$$

$$+ P_2 \sin \omega t + Q_2 \cos \omega t + F/K_2 \qquad (23)$$

and

$$\dot{x} = e^{-\frac{c}{2M}t}\left[A_2\left\{-\frac{c}{2M} \sin \omega_{d_2} t + \omega_{d_2} \cos \omega_{d_2} t\right\}\right.$$

$$\left. - B_2\left\{\frac{c}{2M} \cos \omega_{d_2} t + \omega_{d_2} \sin \omega_{d_2} t\right\}\right]$$

$$+ P_2 \omega \cos \omega t - Q_2 \omega \sin \omega t \qquad (24)$$

where A_2 and B_2 are arbitrary constants and

$$\omega_{d_2} = \sqrt{\frac{K_2}{M} - \left(\frac{c}{2M}\right)^2}$$

$$P_2 = \frac{h\left[(K_2 - M\omega^2) M\omega^2 - (c\omega)^2\right]}{(K_2 - M\omega^2)^2 +}$$

$$Q_2 = -hc\omega k_2 / \left[(K_2 - M\omega^2)^2 + (c\omega)^2\right]$$

However, the solution of Eq. (20) is entirely different when $K_2 = 0$. In such a case solution is:

$$X = A_3 + B_3 e^{-\frac{c}{M}t} - h \sin \omega t + \frac{F}{C} t \qquad (25)$$

and

$$\dot{X} = -B_3 \frac{c}{M} e^{-\frac{c}{M}t} - h \omega \cos \omega t + \frac{F}{C} \qquad (26)$$

where A_3 and B_3 are arbitrary constants.

Using the above analytical solutions, the response of the system shown in Fig. 5 is obtained in the same way as described in Chapter II, with respect to Eqs. (5) and (6).

The evaluation of constants for the different cases is given below:

(a) <u>When $t = 0$, $x = x_0$ and $\dot{x} = V_0$</u>.

Substituting conditions (a) into Eqs. (21) and (22), one obtains,

$$B_1 = x_0 - Q_1 - F/K_1$$

and

$$A_1 = \left(V_0 + \frac{c}{2M} B_1 - P_1 \omega\right) / \omega_{d_1}$$

(b) <u>When $t = t_0$, $x = x_0$, $\dot{x} = V_0$ and $x > 0$</u>

Let

$$T_1 = P_1 \sin \omega t_0 + Q_1 \cos \omega t_0 + F/K_1$$

$$T_2 = (x_0 - T_1)\, e^{\frac{c}{2M} t_0} \qquad (27)$$

$$T_3 = \omega\left(P_1 \cos \omega t_0 - Q_1 \sin \omega t_0\right)$$

$$T_4 = \omega_{d_1} \frac{\cos \omega_{d_1} t_0}{\sin \omega_{d_1} t_0} - \frac{c}{2M}$$

Substituting conditions (b) into Eqs. (21) and (22) and using Eqs. (27) one finds A_1 and B_1 as:

$$B_1 = \left[T_2 \cdot T_4 + e^{\frac{c}{2M} t_0}(T_3 - V_0)\right] \frac{\sin \omega_{d_1} t_0}{\omega_{d_1}}$$

and $$A_1 = \left(T_2 - B_1 \cos \omega_{d_1} t_0\right) / \sin \omega_{d_1} t_0 \qquad (28)$$

(c) When $t = t_0$, $x = x_0$, $\dot{x} = V_0$, $x < 0$ and $k_2 \neq 0$.

The values of the constants A_2 and B_2 in such a case are identical to those of A_1 and B_1 obtained in case (b) with ω_{d_1}, P_1 and Q_1 being replaced by ω_{d_2}, P_2 and Q_2 in Eqs. (27) and (28).

(d) When $t = t_0$, $x = x_0$, $\dot{x} = V_0$, $x < 0$, and $k_2 = 0$.

Substituting conditions (d) into Eqs. (25) and (26) one obtains A_3 and B_3 as:

$$B_3 = \left(-V_0 - h\omega \cos \omega t_0 + \frac{F}{C}\right)\left(\frac{M}{C}\right) e^{\frac{C}{M} t_0}$$

$$A_3 = x_0 - Q e^{-\frac{C}{M} t} + h \sin \omega t_0 - \frac{F}{C} t_0$$

APPENDIX II

The computer programs are written in Fortran IV and a listing of the statements is included at the end of this Appendix. The first program computes the response using analytical solutions of the differential equations of motion, as shown in Chapter II, and the second program obtains the solution using a fourth order Runge-Kutta method. Although the programs furnished here are for the case of a horizontally oscillating system, very similar programs have been used in the case of a vertically oscillating system.

PROGRAM USING ANALYTICAL SOLUTION FOR EVALUATING

DISPLACEMENT AS A FUNCTION OF TIME

```
      DIMENSION VEC(1500),TIME(1500)
      REAL M,K1,K2
300   READ(5,1,END=400)M,K1,F,ZETA,ALPHA,RA
1     FORMAT(6F10.3)
      C=ZETA*SQRT(K1/M)*2.0*M
      W=RA*SQRT(K1/M)
      WD1=SQRT(K1/M-(C/(2.*M))**2)
      K2=K1*ALPHA
      HH=(C/(2.*M))**2
      GG=K2/M
      P1=F*(K1-M*W**2)/((K1-M*W**2)**2+(C*W)**2)
      Q1=-F*C*W/((K1-M*W**2)**2+(C*W)**2)
      P2=F*(K2-M*W**2)/((K2-M*W**2)**2+(C*W)**2)
      Q2=-F*C*W/((K2-M*W**2)**2+(C*W)**2)
C
C     INITIAL HALF CYCLE STARTING WITH T=0
C
      I=1
      J=1
      T=0.
      X=F/K1
      X1=0.
      VEC(1)=F/K1
      TIME(1)=0.
      B1=X-Q1
      A1=(X1+C/(2.*M)*B1-P1*W)/WD1
      H=.0100000.
      T=T+H
2     J=J+1
      X=EXP(-C/(2.*M)*T)*(A1*SIN(WD1*T)+B1*COS(WD1*T))+P1*SIN(W*T)+
     1 Q1*COS(W*T)
      X1=EXP(-C/(2.*M)*T)*((-C/(2.*M)*SIN(WD1*T)+WD1*COS(WD1*T))*A1-
     1 (C/(2.*M)*COS(WD1*T)+WD1*SIN(WD1*T))*B1)+P1*W*COS(W*T)-Q1*W*SIN
     2 (W*T)
      IF(X)5,4,3
3     T=T+H
      IF(J-10)4,4,5
4     GO TO 2
5     I=I+1
      VEC(I)=X/(F/K1)
      TIME(I)=T-H
      J=J-10
      GO TO 2
C
C     WHEN X<0 AND HH.LT.GG
```

```
C
 5      IF(MM-33)101,100,100
 101    WD2=SQRT(K2/M-(C/(2.*M))**2)
        TT1=P2*SIN(W*T)+G2*COS(W*T)
        TT2=(X-TT1)*EXP(C/(2.*M)*T)
        TT3=W*(P2*COS(W*T)-G2*SIN(W*T))
        TT4=WD2*COS(WD2*T)/SIN(WD2*T)-C/(2.*M)
        B2=SIN(WD2*T)/WD2*(TT2*TT4-(X1-TT3)*EXP(C/(2.*M)*T))
        A2=(TT2-B2*COS(WD2*T))/SIN(WD2*T)
        T=T+H
  7     J=J+1
        X=EXP(-C/(2.*M)*T)*(A2*SIN(WD2*T)+B2*COS(WD2*T))+P2*SIN(W*T)
       1 +G2*COS(W*T)
        X1=EXP(-C/(2.*M)*T)*((-C/(2.*M)*SIN(WD2*T)+WD2*COS(WD2*T))*A2-
       1 (C/(2.*M)*COS(WD2*T)+WD2*SIN(WD2*T))*B2)+P2*W*COS(W*T)-G2*W*
       2 SIN(W*T)
        IF(X)8,15,15
 8      T=T+H
        IF(J-10)9,9,10
  9     GO TO 7
  10    I=I+1
        VEC(I)=X/(F/K1)
        TIME(I)=T-H
        J=J-10
        IF(I-1000)7,200,200
C
C       WHEN K2 IS VERY SMALL AND X<0
C
 100    IF(K2.LT..1)GO TO 103
        T1=EXP(SQRT((C/(2.*M))**2-K2/M)*T)
        T2=EXP(-SQRT((C/(2.*M))**2-K2/M)*T)
        T3=(X-P2*SIN(W*T)-G2*COS(W*T))*EXP(C/(2.*M)*T)
        T4=(X1-P2*W*COS(W*T)+G2*W*SIN(W*T))*EXP(C/(2.*M)*T)
        T5=T1*(SQRT((C/(2.*M))**2-K2/M)-C/(2.*M))
        T6=T2*(SQRT((C/(2.*M))**2-K2/M)+C/(2.*M))
        B2=(T3*T5-T1*T4)/(T2*T5+T1*T6)
        A2=(T3-B2*T2)/T1
        T=T+H
  11    J=J+1
        X=EXP(-C/(2.*M)*T)*(A2*EXP(SQRT((C/(2.*M))**2-K2/M)*T)+B2*EXP
       1 (-SQRT((C/(2.*M))**2-K2/M)*T))+P2*SIN(W*T)+G2*COS(W*T)
        X1=EXP(-C/(2.*M)*T)*(A2*EXP(SQRT((C/(2.*M))**2-K2/M)*T)*(SQRT(
       1 (C/(2.*M))**2-K2/M)+C/(2.*M)))+P2*W*COS(W*T)-G2*W*SIN(W*T)
        IF(X)12,15,15
 12     T=T+H
        IF(J-10)13,13,14
  13    GO TO 11
 14     I=I+1
        VEC(I)=X/(F/K1)
        TIME(I)=T-H
        J=J-10
        IF(I-1000)11,200,200
```

```
C
C      WHEN K2=0 AND X<0
C
 103   B3=(X1+F/(C**2+(M*W)**2)*(M*W*COS(W*T)-C*SIN(W*T)))*(-M/C)*EXP
     1 (C/M*T)
       A3=X-B3*EXP(-C/M*T)+F/((C*W)**2+(M*W**2)**2)*(C*W*COS(W*T)+M*
     1 W**2*SIN(W*T))
       T=T+H
 30    J=J+1
       X=A3+B3*EXP(-C/M*T)-F/((C*W)**2+(M*W**2)**2)*(C*W*COS(W*T)+M*W
     1 **2*SIN(W*T))
       X1=B3*(-C/M)*EXP(-C/M*T)-F/(C**2+(M*W)**2)*(-C*SIN(W*T)+M*W*COS
     1 (W*T))
       IF(X)31,15,15
 31    T=T+H
       IF(J-10)32,32,33
 32    GO TO 30
 33    I=I+1
       VEC(I)=X/(F/K1)
       TIME(I)=T-H
       J=J-10
       IF(I-1000)30,200,200
C
C      WHEN X>0
C
 15    T1=P1*SIN(W*T)+Q1*COS(W*T)
       T2=(X-T1)*EXP(C/(2.*M)*T)
       T3=W*(P1*COS(W*T)-Q1*SIN(W*T))
       T4=WD1*COS(WD1*T)/SIN(WD1*T)-C/(2.*M)
       B1=SIN(WD1*T)/WD1*(T2*T4-(X1-T3)*EXP(C/(2.*M)*T))
       A1=(T2-B1*COS(WD1*T))/SIN(WD1*T)
       T=T+H
 16    J=J+1
       X=EXP(-C/(2.*M)*T)*(A1*SIN(WD1*T)+B1*COS(WD1*T))+P1*SIN(W*T)+
     1 Q1*COS(W*T)
       X1=EXP(-C/(2.*M)*T)*((-C/(2.*M)*SIN(WD1*T)+WD1*COS(WD1*T))*A1-
     1 (C/(2.*M)*COS(WD1*T)+WD1*SIN(WD1*T))*B1)+P1*W*COS(W*T)-Q1*W*SIN
     2 (W*T)
       IF(X)6,6,17
 17    T=T+H
       IF(J-10)17,18,19
 18    GO TO 16
 19    I=I+1
       VEC(I)=X/(F/K1)
       TIME(I)=T-H
       J=J-10
       IF(I-1000)16,200,200
 200   WRITE(6,20)M,K1,F,C,ALPHA,W,RA
 20    FORMAT(5X,7F15.3/)
       WRITE(6,21) WD1,WD2,W
 21    FORMAT(5X,3F20.3)
```

```
      WRITE(6,22)
22    FORMAT(10X,'TIME',10X,'DISPLACEMENT'//)
      DO 23 I=800,1000
23    WRITE(6,24)TIME(I),VEC(I)
24    FORMAT(2F15.4/)
      GO TO 300
400   END
```

PROGRAM USING RUNGE-KUTTA METHOD FOR EVALUATING DISPLACEMENT AS A FUNCTION OF TIME

```
      REAL M,L
      COMMON CE,AM,DM,EK,EK2,W
      DIMENSION VEC(1000),TIME(1000)
20    READ(5,1,END=100)DM,EK,CE,AM,WA,BETA
1     FORMAT(6F10.3)
      W=WA
      EK2=BETA*EK
      T1=0
      Y1=AM/EK
      Y11=0.
      H=.01
      H2=H/2.0
      VEC(1)=Y1
      TIME(1)=T1
21    DO 50 I=1,300
      DO 60 J=1,20
      Y21=FUN(T1,Y1,Y11)
      T2=T1+H2
      Y2=Y1+Y11*H2
      Y12=Y11+Y21*H2
      Y22=FUN(T2,Y2,Y12)
      T3=T1+H2
      Y3=Y1+Y12*H2
      Y13=Y11+Y22*H2
      Y23=FUN(T3,Y3,Y13)
      T4=T1+H
      Y4=Y1+Y13*H
      Y14=Y11+Y23*H
      Y24=FUN(T4,Y4,Y14)
      DELY=H*(Y11+2.*Y12+2.*Y13+Y14)/6.
      DELY1=H*(Y21+2.*Y22+2.*Y23+Y24)/6.
      Y1=Y1+DELY
      Y11=Y11+DELY1
60    T1=T1+H
      VEC(I+1)=Y1
      TIME(I+1)=T1
50    CONTINUE
31    WRITE(6,2)DM,EK,CE,AM,W,BETA
2     FORMAT(/5X,6F15.3/)
      DO 12 I=200,300
12    WRITE(6,3)TIME(I),VEC(I)
3     FORMAT(2F25.4/)
      GO TO 20
100   END
```

```
      FUNCTIØN FUN(T,Y,Y1)
      CØMMØN CE,AM,DM,EK,EK2,W
      IF(Y)10,10,11
   10 FUN=AM*SIN(W*T)/DM-CE/DM*Y1-EK2*Y/DM
      GØ TØ 12
   11 FUN=AM*SIN(W*T)/DM-CE/DM*Y1-EK/DM*Y
   12 RETURN
      END
```