Linear stability of a growing or collapsing bubble in a slightly viscous liquid

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A simplified form of the stability equation appropriate for liquids of small viscosity undergoing nearly spherically symmetric flow is derived on the basis of earlier results. This equation is then applied to the analysis of the stability characteristics of the spherical shape for growing and collapsing cavitation bubbles. It is found that viscosity does not remove the well-known instability associated with the collapse process, although it does delay the growth of higher order modes. This feature explains the relatively small number of microbubbles to which a cavitation bubble gives rise upon fragmentation.

I. INTRODUCTION

The stability of the spherical shape for an expanding or collapsing cavitation bubble in an unbounded liquid of negligible viscosity has been the object of a number of studies.[1-4] On the basis of the linear approximation[1-3] it is possible to conclude that the growth process is substantially stable (in the sense that the perturbation amplitude remains bounded) whereas the collapse process is highly unstable. In this case the amplitude of the distortions of the spherical shape is found to oscillate with increasing frequency and amplitude as the bubble radius decreases. This behavior leads to the expectation of the eventual fragmentation of the bubble. This conjecture, which is in agreement with experiment, has also been confirmed by the fully nonlinear study of Chapman and Plesset.[4]

The corresponding problem for viscous fluids has received little attention[5] because of the lack of an appropriate perturbation equation. This equation has recently been made available,[6] but because of its complex mathematical structure it does not easily lend itself to investigation in the general case. Therefore, in the present study we limit our considerations to a slightly viscous liquid for which a substantial simplification becomes possible. It is found that the addition of viscosity leads to only a minor modification of the collapse instability essentially consisting of delaying the growth of higher-order distortions. More strongly affected is the growth process: The initial behavior may cease to be oscillatory, and the perturbation amplitude tends to zero instead of to a finite quantity as the bubble radius increases.

II. THE STABILITY EQUATION

The equation to be obtained in this section has already been presented without proof in Ref. 7; additional details are available in Refs. 6 and 8.

We take a spherical polar coordinate system (r, θ, ϕ) centered at the centroid of the bubble. The shape of the free surface is specified as

$$r_s(\theta, \phi; t) = R(t) + a_n(t) Y_n^m(\theta, \phi), \qquad (1)$$

where $R(t)$ is the average radius, $Y_n^m(\theta, \phi)$ is a spherical harmonic, and $a_n(t)$ is the amplitude of the nth order distortion. In the linearized approximation the equation for $a_n(t)$ does not depend on the degree m of the spherical harmonic,[1,3,6] and is found to be[6]

$$\frac{d^2a_n}{dt^2} + \left[\frac{3}{R}\frac{dR}{dt} - 2(n-1)(n+1)(n+2)\frac{\nu}{R^2}\right]\frac{da_n}{dt} + (n-1)\left[-\frac{1}{R}\frac{d^2R}{dt^2} + 2(n+1)(n+2)\frac{\nu}{R^3}\frac{dR}{dt} + (n+1)(n+2)\frac{\sigma}{\rho R^3}\right]a_n$$
$$+ n(n+1)(n+2)\frac{\nu}{R^2}T[R(t), t] + n(n+1)\frac{1}{R^2}\frac{dR}{dt}\int_R^\infty \left[\left(\frac{R}{s}\right)^3 - 1\right]\left(\frac{R}{s}\right)^n T(s,t)\,ds = 0\,. \qquad (2)$$

Here, ν is the kinematic viscosity, ρ is the liquid density, and σ is the surface tension. The quantity $T(r, t)$ is the toroidal component of the vorticity field $\boldsymbol{\omega}$ which is expressed as

$$\boldsymbol{\omega} = \nabla \times (T Y_n^m \hat{\mathbf{r}}) + \nabla \times \nabla \times (S Y_n^m \hat{\mathbf{r}}),$$

where $\hat{\mathbf{r}}$ is the unit vector in the radial direction and $S(r, t)$ is the poloidal component of $\boldsymbol{\omega}$. The function $T(r, t)$ is found to satisfy the following equation, which is a direct consequence of the linearized vorticity equation for the flow[6]

$$\nu\frac{\partial^2 T}{\partial r^2} - \frac{\partial T}{\partial t} - \frac{\partial}{\partial r}\left[\left(\frac{R}{r}\right)^2\frac{dR}{dt}T\right] - n(n+1)\frac{\nu}{r^2}T = 0\,. \qquad (3)$$

The condition of vanishing tangential stress at the bubble surface $r = R(t)$ requires that $T(r, t)$ satisfy[6]

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$$2\nu R^{n-1}\int_R^\infty s^{-n}T(s,t)\,ds+\nu T[R(t),t]$$
$$=\frac{2}{n+1}\nu\left[(n+2)\frac{da_n}{dt}-(n-1)\frac{1}{R}\frac{dR}{dt}a_n\right]. \quad (4)$$

Although the kinematic viscosity ν can be canceled from both sides of this equation, it has been left in to emphasize the fact that this condition is to be applied only for a viscous fluid.

This equation shows that, when $\nu>0$, vorticity is continuously generated at the free surface and propagates into the body of the liquid by diffusion and convection according to Eq. (3). If the initial vorticity vanishes, the free surface is the only source of vorticity and from the form of Eq. (3) we may draw the following qualitative picture of the behavior of the function $T(r,t)$. Suppose first that the bubble has a constant radius so that $dR/dt=0$. Then, it is evident from the parabolic nature of the equation that $T(r,t)$ will be appreciable only in a layer of thickness of the order of $(\nu t)^{1/2}$ adjacent to the bubble surface. If we now allow for a nonzero radial velocity, we expect that this layer is stretched and becomes thinner when $dR/dt>0$, whereas it becomes thicker when $dR/dt<0$. In conclusion, we may expect that in the case of small viscosity and moderate decrease of the bubble radius, the integrals containing $T(r,t)$ in Eqs. (2) and (4) play a minor role in the behavior of the amplitude. We shall examine this matter in greater detail in Sec. V. Here, let us proceed to eliminate $T[R(t),t]$ between (2) and (4) disregarding the integral terms to find the following approximate perturbation equation for spherically symmetric, slightly viscous flows

$$\frac{d^2a_n}{dt^2}+\left[\frac{3}{R}\frac{dR}{dt}+2(n+2)(2n+1)\frac{\nu}{R^2}\right]\frac{da_n}{dt}$$
$$+(n-1)\left[(n+1)(n+2)\frac{\sigma}{R^3\rho}-\frac{1}{R}\frac{d^2R}{dt^2}\right.$$
$$\left.+2(n+2)\frac{\nu}{R^3}\frac{dR}{dt}\right]a_n=0. \quad (5)$$

This equation embodies several known results which will be briefly mentioned. For instance, for $n=1$ and $R=$ const one obtains Levich's well-known expression for the drag force on a translating bubble of fixed radius.[9] Similarly, for the case of translation in an inviscid liquid, it implies $R^3(t)\,da_1/dt=$ const, which is an expression of the conservation of liquid momentum (Ref. 10, Sec. 92; Ref. 11, Sec. 11). Both of these results are independent of the assumed smallness of the coefficient a_1 once the hypothesis of spherical shape has been made. Finally, for constant radius, one can read from (5) the expression for the natural frequency and damping constant of small oscillations given by Lamb (Ref. 10, p. 475 and p. 641). The stability equation given by Plesset[1] can be obtained by setting $\nu=0$.

The evolution in time of the mean radius $R(t)$, to the present approximation, can be obtained from the well-known Rayleigh–Plesset equation[7,12]

$$R\frac{d^2R}{dt^2}+\frac{3}{2}\left(\frac{dR}{dt}\right)^2=\frac{1}{\rho}\left(p_i-p_\infty-\frac{2\sigma}{R}\right)-4\frac{\nu}{R}\frac{dR}{dt}, \quad (6)$$

in which p_i is the internal cavity pressure and p_∞ is the ambient pressure, which will be taken to be a constant. Since in this study we are dealing with the dynamics of cavitation bubbles, namely, of vapor bubbles in a relatively cold liquid, we shall neglect vaporization and condensation effects and we shall take $p_i=$ const in (6).[7]

It will be convenient in the following to deal with nondimensional quantities and write

$$R^*=\frac{R}{R_0},\quad a_n^*=\frac{a_n}{a_0},\quad t^*=|(p_i-p_\infty)/\rho|^{1/2}t/R_0,$$
$$N=\frac{\nu}{R_0|(p_i-p_\infty)/\rho|^{1/2}},\quad S=\frac{2\sigma}{|p_i-p_\infty|R_0}. \quad (7)$$

The reference quantities R_0 and a_0 will be taken as the initial values of R and a_n, respectively. From here on we shall use only these variables, although the asterisks will be dropped for convenience. Expressed in these terms Eqs. (6) and (5) take the form

$$R\frac{d^2R}{dt^2}+\frac{3}{2}\left(\frac{dR}{dt}\right)^2=\frac{p_i-p_\infty}{|p_i-p_\infty|}-\frac{1}{R}\left(S+4N\frac{dR}{dt}\right), \quad (8)$$

$$\frac{d^2a_n}{dt^2}+\left[\frac{3}{R}\frac{dR}{dt}+2(n+2)(2n+1)\frac{N}{R^2}\right]\frac{da_n}{dt}+(n-1)$$
$$\times\left[(n+1)(n+2)\frac{S}{2R^3}-\frac{1}{R}\frac{d^2R}{dt^2}+2(n+2)\frac{N}{R^3}\frac{dR}{dt}\right]a_n=0. \quad (9)$$

With the following change in the dependent variable

$$a_n=R^{-3/2}\exp\left[-(n+2)(2n+1)N\int_0^t R^{-2}dt\right]b_n, \quad (10)$$

the last equation becomes

$$d^2b_n/dt^2+G(t)b_n=0, \quad (11)$$

where

$$G(t)=\frac{3}{4}\frac{1}{R^2}\left(\frac{dR}{dt}\right)^2-\left(n+\frac{1}{2}\right)\frac{1}{R}\frac{d^2R}{dt^2}+\frac{1}{2}(n-1)(n+1)$$
$$\times(n+2)\frac{S}{R^3}-3(n+2)\frac{N}{R^3}\frac{dR}{dt}-(n+2)^2(2n+1)^2\frac{N^2}{R^4}. \quad (12)$$

III. COLLAPSING CAVITIES

In Fig. 1 the numerical solution of the Rayleigh–Plesset equation (8) is shown for a collapsing cavity with initial conditions $R(0)=1$, $dR(0)/dt=0$. The parameters S and N have the value $S=10^{-3}$ (corresponding, for instance, to a dimensional initial radius of 0.14 cm in water) and $N=0$. The solution is found to be independent of the actual values of S and N, if chosen within a realistic range for collapse in slightly viscous liquids.

Before illustrating the results obtained by the numerical integration of the perturbation equation (9), it is interesting to analytically investigate the asymptotic behavior of a_n as $R\to 0$. To this end we shall neglect the viscous term in Eq. (8) to integrate once with the result[7]

$$(dR/dt)^2=\tfrac{2}{3}(R^{-3}-1)+SR^{-1}(R^{-2}-1); \quad (13)$$

a vanishing initial velocity has been assumed for sim-

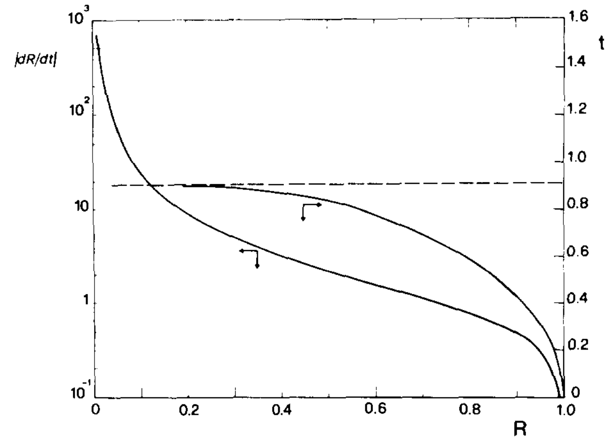

FIG. 1. The dimensionless time and velocity are shown as functions of the dimensionless radius for a collapsing cavity with $R(0)=1$, $dR(0)/dt=0$, $S=10^{-3}$ and $N=0$. For $R\to 0$ the total collapse time approaches the value $t=0.915$ given by Rayleigh.[12,7] The curves for $N=10^{-3}$ are indistinguishable from those shown.

plicity. From (13) and (12) the following approximate expression for $G(t)$ is readily obtained

$$G(t)=-\tfrac{3}{2}(S+\tfrac{2}{3})nR^{-5}+3(S+\tfrac{2}{3})^{1/2}(n+2)NR^{-9/2}-(n+2)^2(2n+1)^2N^2R^{-4}+O(R^{-3})\,. \tag{14}$$

Notice that the viscous correction which tends to oppose the unstable behavior brought about by the first term is $O(R^{1/2})$ smaller. Therefore we do not expect the instability to be suppressed by viscosity. Indeed, with the change of variables

$$x=4(n+2)(2n+1)N(S+\tfrac{2}{3})^{-1/2}R^{1/2}(t)\,, \tag{15a}$$

$$b_n=Rc_n\,, \tag{15b}$$

Eq. (11) becomes

$$\frac{d^2c_n}{dt^2}+\left(-\frac{1}{4}+\frac{k}{x}+\frac{\frac{1}{4}-\mu^2}{x^2}\right)c_n=0\,, \tag{16}$$

where the approximate form (14) of $G(t)$ has been used and

$$k=3(2n+1)^{-1}\,,\quad \mu^2=-6(n-1)+\tfrac{1}{4}\,. \tag{17}$$

Equation (16) is the standard form of Whittaker's equation[13] and its solution may be written as

$$c_n=e^{-1/2x}[Ax^{1/2+\mu}F(\tfrac{1}{2}+\mu-k,\,1+2\mu,\,x)+Bx^{1/2-\mu}F(\tfrac{1}{2}-\mu-k,\,1-2\mu,\,x)]\,, \tag{18}$$

where A, and B are integration constants and $F(a,c,x)$ is the confluent hypergeometric function. A noteworthy feature of this result is its independence of the viscous constant N which has been eliminated by means of the transformations (15a) and (10). Recalling that $F(a,c,0)=1$, we readily obtain the leading order behavior of the solutions as $x\to 0$ from (18). In terms of the original variables one has

$$a_n\sim R^{-1/4}\exp\left\{\pm i[6(n-1)-\tfrac{1}{4}]^{1/2}\log R-(n+2)(2n+1)N\int_0^t R^{-2}dt\right\}, \tag{19}$$

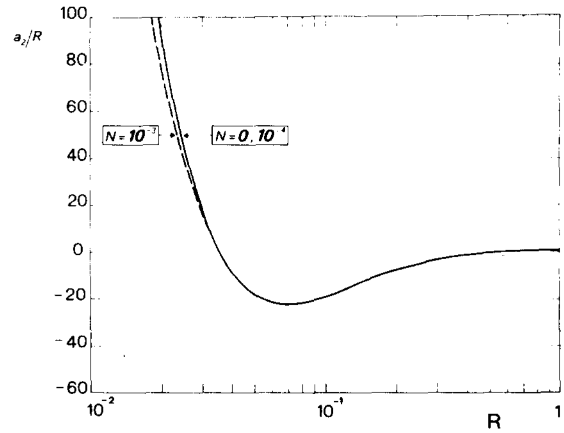

FIG. 2. The dimensionless quantity a_2/R versus the dimensionless radius for a collapsing cavity with $S=10^{-3}$, $N=0$, 10^{-4}, 10^{-3}. The curves corresponding to the first two values of N are indistinguishable.

which, for $N=0$, coincides with the result for the inviscid case.[2,3] Since the integral in (19) is a finite quantity, it is seen that viscosity does not remove the $R^{-1/4}$ singularity. However, the instant at which the singular behavior starts to dominate is determined by a competition between this integral and the $R^{-1/4}$ term. It is clear therefore that the beginning of the growth of the nth order distortion of the spherical shape is delayed by an amount which increases exponentially with $2n^2N$. In practice, this feature leads to a suppression of the higher-order modes. Indeed, by the time they start to grow, the bubble will already have fragmented owing to the instability of the lower order modes. This observation may explain why collapsing bubbles are found to give rise to only a small number of microbubbles upon fragmentation (see, e.g., Ref. 14, Fig. 11, and Ref. 15), a feature which cannot be explained by the inviscid theory. Indeed, the increase in the oscillation frequency with n and the independence of n of the growth rate predicted by that theory makes the breakup into a very large number of fragments much more probable.

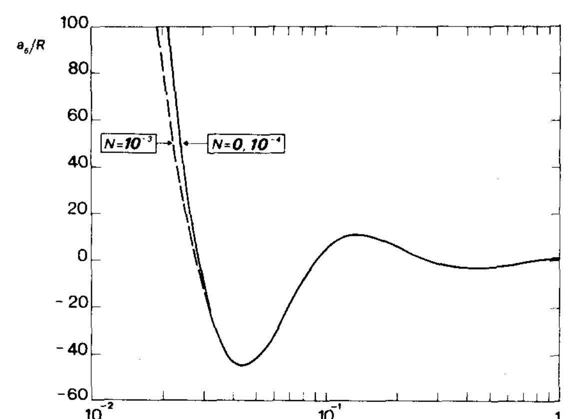

FIG. 3. The dimensionless quantity a_6/R versus the dimensionless radius for a collapsing cavity with $S=10^{-3}$, $N=0$, 10^{-4}, 10^{-3}. The curves corresponding to the first two values of N are indistinguishable.

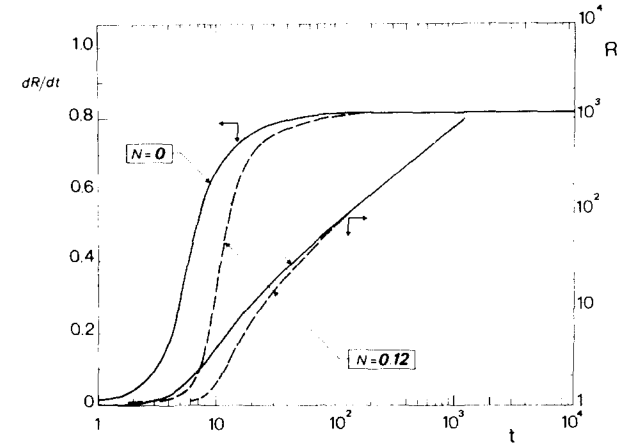

FIG. 4. The dimensionless radius and radial velocity are shown as functions of the dimensionless time for a growing bubble with $R(0)=1$, $dR(0)/dt=0.01$, $S=1$, $N=0$, and $N=0.12$. For large times the velocity approaches the asymptotic value[7] $(2/3)^{1/2}$.

The results of the numerical integration of the stability equation (9) for $n=2$ and $n=6$ are shown in Figs. 2 and 3. The initial conditions for R are those of Fig. 1, and for the perturbation amplitude $a_n(0)=1$, $da_n(0)/dt=0$. In these figures a_n/R, which is the significant quantity governing bubble breakup, is plotted versus the radius for the inviscid case, $N=0$, and for $N=10^{-4}$ and $N=10^{-3}$. In water, these values would correspond to initial dimensional radii of 0.1 and 0.01 cm, respectively if $p_\infty - p_i \simeq 1$ atm. For the lowest order mode, $n=2$, it is seen that the results are affected very little by viscosity: for $N=10^{-4}$ they are indistinguishable from the inviscid ones, while for $N=10^{-3}$ small differences appear. The effect is somewhat greater for $n=6$, but again viscosity introduces appreciable differences only for the largest value of N. The shape of the curve shows that this difference consists mainly in a delay of the growth, as had been anticipated.

IV. GROWING CAVITIES

Figure 4 shows numerical solutions of the Rayleigh–Plesset equation for a growing cavity. Again, the initial radius, assumed equal to the equilibrium value,[7] has been taken as the reference length so that $R(0)=1$, $S=1$. An initial velocity $dR(0)/dt=0.01$ has also been taken. Curves are shown for the inviscid case ($N=0$) and for $N=0.12$, which would correspond to an initial dimensional radius of 0.83×10^{-4} cm for $p_i - p_\infty = 1$ atm in water. Notice that viscosity merely increases the duration of the so-called latency stage, with negligible consequences on the later phases of the growth.

Also for this case, it is useful to have some analytical estimates of the behavior of the solutions of (9). The first integral of (8) obtainable for $N=0$ takes the form[7]

$$(dR/dt)^2 = R^{-3}\,U_0 + \tfrac{2}{3}(1-R^{-3}) - SR^{-1}(1-R^{-2}), \tag{20}$$

where $U_0 = dR(0)/dt$. Let us consider the initial, surface-tension dominated, stage of the growth first. In terms of the new variables

$$z = \int_0^t R^{-2}\,dt, \tag{21a}$$

$$d_n = R^{1/2} a_n, \tag{21b}$$

Eq. (9) takes the form

$$\frac{d^2 d_n}{dz^2} + 2(n+2)(2n+1)N\frac{dd_n}{dz} + \left[\tfrac{1}{2}(n-1)(n+1)(n+2)S + F(t)\right]d_n = 0, \tag{22}$$

where

$$F(t) = \tfrac{1}{2}S(n-1)(n+1)(n+2)(R-1) - 3(n+2)NR\frac{dR}{dt} - \tfrac{3}{4}R^2\left(\frac{dR}{dt}\right)^2 - (n-\tfrac{1}{2})R^2\frac{d^2R}{dt^2}. \tag{23}$$

As Fig. 4 shows, during the initial stages the growth is very slow; therefore, the function $F(t)$ differs little from zero, and Eq. (22) describes a slowly modulated damped oscillatory motion. Oscillations will actually be present, however, only if the viscosity parameter satisfies the approximate inequality

$$N \lesssim \frac{1}{2n+1}\left[\frac{1}{2}S\frac{(n-1)(n+1)}{n+2}\right]^{1/2}, \tag{24}$$

and hence, for fixed N, only a finite number of modes may have an oscillatory behavior, the remaining ones being overdamped.

It is also possible to obtain the limiting behavior for large R; a straightforward computation leads to

$$a_n \sim R^{-1}(CR^p + DR^{-p}), \tag{25}$$

where C and D are constants and the quantitiy p is defined as

$$p = [1 - 3(n+2)(n-1)N]^{1/2}. \tag{26}$$

For $N=0$ the well known result $a_n \to$ const is recovered. For finite N, however, Eq. (25) entails that $a_n \to 0$, as was to be expected. Further, it is apparent from (25) and (26) that the decrease in the distortion amplitude is monotonic only for sufficiently small values of n, being oscillatory for larger values.

The numerical solution of (9) for $n=2$, $S=1$, $a_2(0)=1$, $da_2(0)/dt=0$ is shown in Fig. 5. The curve corresponding to the inviscid result is seen to exhibit slightly damped oscillations first, which turn into the aperiodic phase shortly after the beginning of the rapid radial growth. The oscillations become much more rapidly damped for $N=0.05$ (the corresponding dimensional initial radius in water is 2×10^{-4} cm, for $p_i - p_\infty = 1$ atm), and are nearly absent for $N=0.12$, which is just below the critical value given by (24).

A similar behavior is exhibited by the mode $n=6$, which is illustrated in Fig. 6. As is clear from Eq. (22) the initial oscillations are much more rapid now and only the first and the last ones are shown in the figure, along with the envelopes of the maxima and minima in the intermediate stage. Again, the damping of the oscillations in correspondence of the rapid growth is visible. The solution for $N=0.05$ exhibits a much

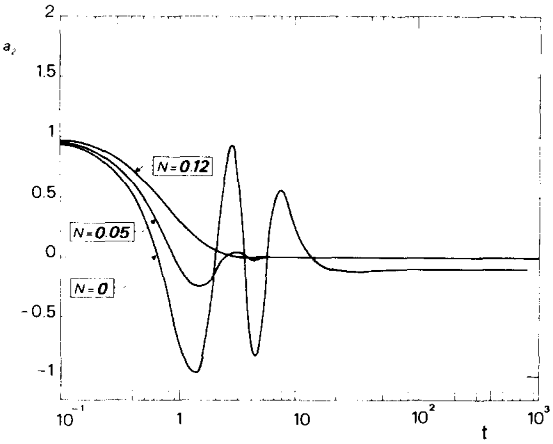

FIG. 5. The dimensionless amplitude a_2 versus dimensionless time for a growing bubble with $a_2(0)=1$, $da_2(0)/dt=0$, $S=1$, $N=0.05$ and 0.12. The initial conditions for R are the same as in Fig. 4.

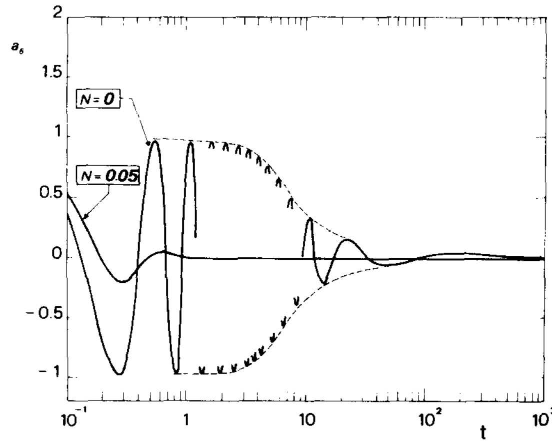

FIG. 6. The dimensionless amplitude a_6 versus dimensionless time for a growing bubble with $a_6(0)=1$, $da_6(0)/dt=0$, $S=1$, $N=0, 0.05$. The initial conditions on R are the same as in Fig. 4. For the curve labeled $N=0$ only the first and the last oscillations are shown, along with the envelopes of maxima and minima (dashed lines).

faster damping, which practically extinguishes the distortion before the beginning of the growth.

V. DISCUSSION

Two obvious sources of error in the preceding results are the use of the linearized approximation, and the neglect of the integrals in Eqs. (2) and (4). Regarding the first point we may rely on the results of the nonlinear calculation performed by Chapman and Plesset[4] to be confident that the linearized approximation has a much wider domain of validity than could be expected. The nonlinear coupling with the other distortion modes introduces an additional factor of damping which, however, does not alter the behavior of the amplitudes too much.

Of greater relevance in the present context is perhaps the second point. It is possible to show that the procedure used to derive the approximate equation (5) is equivalent to using the irrotational, inviscid solution of Plesset[1] to evaluate the dissipation function in the way explained in Ref. 10, p. 639.[16] This procedure tends to over-estimate the velocity gradients and the effect of viscosity on the phenomenon is thus also overestimated.

A very rough estimate of the conditions of validity of our approximation can be obtained if it is assumed that $T(r,t)=T[R(t),t]$ in the layer $R<r<R+\delta$ and $T(r,t)=0$ elsewhere.[8] With this approximation it is possible to evaluate the integrals appearing in (2) and (4). In this way one finds an equation which reduces to (5) provided that

$$n\delta \ll R, \tag{27a}$$

$$\nu \gg \delta^2 R^{-1} |dR/dt|. \tag{27b}$$

The thickness δ can be estimated to be of the order of the diffusion length $(\nu t_0)^{1/2}$, where t_0 is an appropriate time scale for vorticity generation at the bubble surface. For the growth case t_0 can be taken as the inverse of the frequency of surface oscillations, hence $t_0 \sim (\rho R^3/n^3\sigma)^{1/2}$. With this choice for t_0 the first condition (27a) results in $R \gg n\nu^2\rho/\sigma$, which in general will be satisfied for a range of n by most growing bubbles in slightly viscous liquids. The second condition will also be met since, as Figs. 5 and 6 show, there is a significant vorticity generation only as long as the radial velocity is small. Less satisfactory is the situation for the collapse case. Here, both conditions (27) will eventually be violated as $R \to 0$. Nevertheless, for the reasons already mentioned, it is expected that the error introduced by the present procedure results in an over-estimation of the effect of viscosity on the collapse instability.

Finally, we wish to notice that with our estimate $\delta \sim (\nu t_0)^{1/2}$, the second inequality (27b) would seem to impose a condition which is independent of ν. Actually, the physical meaning of (27b) is essentially the requirement that the vortical boundary layer adjacent to the bubble be not excessively distorted by the primary radial flow. Clearly, this requirement is meaningless when $\nu=0$, because then there is no boundary layer. On the other hand, when viscosity is not small, $\delta \sim (\nu t_0)^{1/2}$ is no longer an accurate estimate of the thickness of the boundary layer which will then be influenced in a complicated way by the geometry and the primary flow. In this case, inserting a better estimate of δ into (27b), one would find a condition which becomes independent of ν only in the limit $\nu \to 0$.

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