Defect Analysis of 1D Spring-Mass Systems via Laplace Transform and Asymptotics

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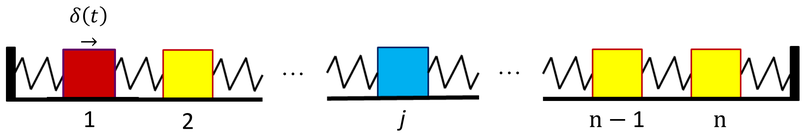
INTRODUCTION

Spring-mass systems have utility in modeling multiple physical phenomena involving elastic deformation and wave propagation. In the scope of this project: we focus on linear spring-mass systems and use experimental data to analytically locate and enumerate defects along the chain. A particular application of this project concerns limited access pipelines, where we can determine the number of blockages present and their severity.

Consider a chain of n blocks linked in series with springs within two rigid, immobile walls on a flat surface. All springs are massless, identical, and uniformly stiff (k = 1 N/m). Frictional effects dampen each block in the chain uniformly (d = 0.1). All blocks are assigned an integer indicative of their location along the chain (1, 2, …, n) and are identical in size and shape. Most blocks have a fixed mass of 1 kg, but some of them have unknown mass of non-unity (the " defects"). The locations of each defect are also unknown.

If we apply an impulse to the first block and observe ONLY the vibrations of the first block, is it possible to:

- count the number of defects in the chain?
- recover each defects' masses and locations?



(In the case of one defect: the defective block has mass m_j and is situated at location j.)

THEORY

- Establish the system of differential equations describing the motions of each block.

$$\begin{cases} \delta(t) = x_1'' + dx_1' + 2x_1 - x_2 \\ 0 = x_2'' + dx_2' + 2x_2 - x_1 - x_3 \\ \vdots \\ 0 = m_j x_j'' + dx_j' + 2x_j - x_{j-1} - x_{j+1} \\ \vdots \\ 0 = x_{n-1}'' + dx_{n-1}' + 2x_{n-1} - x_{n-2} - x_{n+1} \\ 0 = x_n'' + dx_n' + 2x_n - x_{n-1} \end{cases}$$

- Transform the system into the Laplace domain and rewrite as a matrix equation.

$$(A + E)\tilde{x} = \mathbf{b}$$

$$A = \begin{pmatrix} h & 1 & 0 & \cdots & 0 \\ 1 & h & 1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & 1 & h & 1 \\ 0 & \cdots & 0 & 1 & h \end{pmatrix} \quad E = \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & (1-m_j)s^2 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$h = -(s^2 + ds + 2)$$

- Explicitly invert A with a method from "Analytical Inverstion of Symmetric Tridiagonal Matrices" by Hu and O'Connell. ($A^{-1} = R$)

$$R_{i,j} = \frac{\cosh[\lambda(n + 1 - i - j)] - \cosh[\lambda(n + 1 - |j - i|)]}{2\sinh(\lambda)\sinh[\lambda(n + 1)]}$$

$$\lambda = \operatorname{arccosh}(-h/2)$$

- Compute an alternate form of $\widetilde{x_1}(s)$ with the matrix equation.

$$\widetilde{x_1}(s) = -R_{1,1} + (1 - m_j)s^2 \frac{|R_{1,j}|^2}{1 + (1 - m_j)s^2 R_{j,j}}$$

  - This equation is for one defect. When an arbitrary number of defects are present, there are additional additive terms of similar structure representing each defect (one term per defect).

METHOD

- Algebraically manipulate the alternate form of $\widetilde{x_1}(s)$ under full expansion such that all terms containing a^j remain on one side of the equation.

$$\frac{D}{s^2}(\widetilde{x_1}(s) + R_{1,1}) = a^{-2j}\frac{(1 - m_j)|r_{1,j}|^2}{D + (1 - m_j)s^2 r_{j,j}}$$

where: $a = \frac{1}{2}(-h + \sqrt{h^2 - 4})$

$$D = 1 + a^{-2n-4} - a^{-2} - a^{-2n-2}$$

$$r_{1,j} = a^{-2} + a^{2j-2n-2} - 1 - a^{2j-2n-4}$$

$$r_{j,j} = a^{-2j} + a^{2j-2n-2} - 1 - a^{-2n-2}$$

- Insert a new parameter j_{test} and define the left side as a function.

$$g(s, j_{\text{test}}) = \frac{D}{s^2}(\widetilde{x_1}(s) + R_{1,1})a^{2j_{\text{test}}}$$

$$g(s, j_{\text{test}}) \approx \frac{1 - m_j}{m_j}a^{2j_{\text{test}} - 2j} \quad \text{(for large values of } s\text{)}$$

- Systematically determine j by plotting $g(s, j_{\text{test}})$ for j_{test} from 2 to n. If…
  - $j_{\text{test}} < j$: $\lim_{s\to\infty}[g(s, j_{\text{test}})] = 0$; Block is not a defect.
  - $j_{\text{test}} > j$: $\lim_{s\to\infty}[g(s, j_{\text{test}})] = \pm\infty$; Block is not a defect.
  - $j_{\text{test}} = j$: $\lim_{s\to\infty}[g(s, j_{\text{test}})] = \frac{1 - m_j}{m_j}$; Block is a defect.
- m_j can be recovered from the finite limit.

$$m_j \approx \lim_{s\to\infty}\left[\frac{1}{1 + g(s, j_{\text{test}})}\right]$$

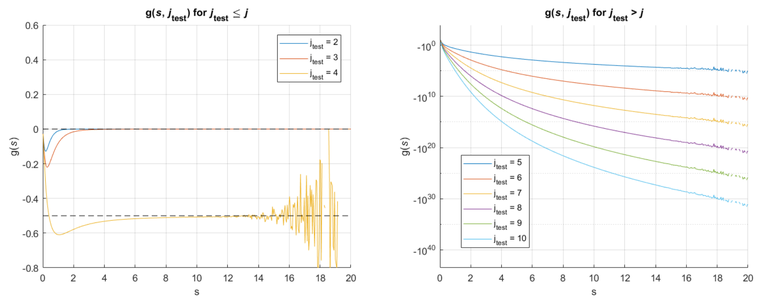
- Repeat iteratively. Loop back to the algebraic isolation of the next successive defect's factor each time a defect is discovered, and terminate at the end of the chain. The iteration count of this algorithm gives the number of defective blocks.
- The lack of access to a laboratory setting compelled numerical synthesis of the vibrational data for $\widetilde{x_1}(s)$ from the matrix equation $(A + E)\tilde{x} = \mathbf{b}$.
  - Such a system is impossible to solve exactly, so $\widetilde{x_1}(s)$ is numerically approximate (though, to a very high degree of precision).
- The alternate form of $\widetilde{x_1}(s)$, being analytically exact, acted as smoothed noise-free data and was considered "perfect."

RESULTS

One Defect

- Data for $\widetilde{x_1}(s)$ was synthesized with the following parameters:
  - $n = 10$; $m_j = 2$; $j = 4$

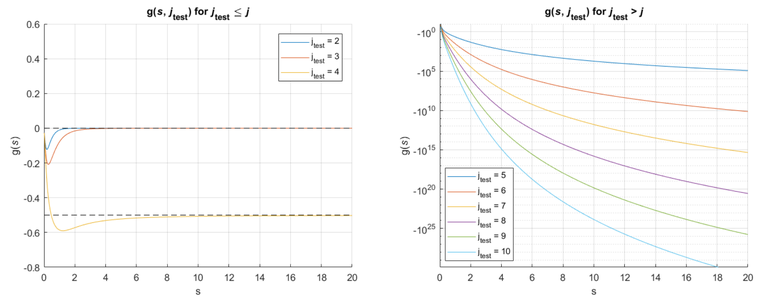
Plots of g using the synthetic data of $\widetilde{x_1}(s)$:



- Algorithm detected one defect of correct mass at the right location.
  - Convergence to zero for $j_{\text{test}} < j$ and divergence for $j_{\text{test}} > j$.
- As s grows large, numerical instabilities overtake smoothness of g.
- Extremely small numerical imprecisions in the simulation of $\widetilde{x_1}(s)$ (~10^{-150}) due to software limitations are blown up as s grows.
  - Analogous to measurement errors and cannot be remedied.

RESULTS (cont.)

Plots of g using the "perfect" data of $\widetilde{x_1}(s)$:



- Expected behavior is observed.
- g is smooth everywhere.
- Noise-free data is impossible to obtain in reality.

Two Defects

$$\frac{\widetilde{x_1}(s) + R_{1,1}}{s^2}a^{2j_{\text{test}}} \approx \left(\frac{1 - m_{j_1}}{m_{j_1}}a^{-2j_1} + \frac{1 - m_{j_2}}{m_{j_2}}a^{-2j_2}\right)a^{2j_{\text{test}}}$$

- Data for $\widetilde{x_1}(s)$ was synthesized with the following parameters:
  - $n = 20$; $m_{j_1} = 3$; $j_1 = 5$; $m_{j_2} = 8$; $j_2 = 16$
- Algorithm found the first defect with correct mass at the right location.
- Defect #2 is irrecoverable for the following reasons:
  - Multiplicative coupling of m_{j_1} and m_{j_2}. The alternate form of $\widetilde{x_1}(s)$ requires knowledge of m_{j_2} when trialing for j_2.
  - Exacerbated numerical imprecision. Finite convergence is never observed if m_{j_2} is approximated out during the second iteration.
  - Exponentiation rules. When trialing for j_2, no such value of j_{test} can force convergence to a finite value.

CONCLUSION & FUTURE WORK

- Though theoretically sound, this algorithm managed to detect only the first defect in the chain. Furthermore, the numerical instabilities in plotting $g(s, j_{\text{test}})$ irreconcilably conflicted with the expectation of smooth, gradual convergence to a finite value due to the imprecision of $\widetilde{x_1}(s)$ – fundamentally similar to measurement noise. Using exact data removed the instability, but that is impossible to procure in reality.
- In the future: we'll revisit the problem in the time domain, where the first block vibration curve can be found by rewriting the system of differential equations into a first order system and solving with matrix exponentials.

$$\mathbf{u}' = \mathbf{f} + A\mathbf{u}$$

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & \cdots & \cdots \\ -2k & -d & k & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & 1 & \ddots & \ddots \\ k & 0 & -2k & -d & k & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \quad \mathbf{f} = \begin{pmatrix} 0 \\ \delta(t) \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{2n} \end{pmatrix}$$

REFERENCES

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- Hu, G. Y., & Robert F. O'Connell. (1996). Analytical inversion of symmetric tridiagonal matrices. *Journal of Physics A: Mathematical and General, 29*(7), 1511-1512.