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A MATRIX IN THE SCHWARZ BLOCK FORM AND THE STABILITY OF MATRIX POLYNOMIALS

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ABSTRACT

The Schwarz matrix was established by H. R. Schwarz in 1956. He used several elementary transformations to transform a given system matrix to the Schwarz matrix. Since then numerous authors have investigated the properties and applications of the Schwarz matrix. Also, various transformation matrices which relate a given system matrix to the Schwarz matrix have been established. However, most of the earlier developed transformation matrices were too complicated to implement, and also they were restricted to single variable systems only. In this research a matrix which consists of block elements is established in the Schwarz block form via a linear transformation. A new block-transformation matrix is established for transforming the companion block form to the Schwarz block form. A sufficient condition has been derived for determining the stability of a multivariable system whose characteristics are expressed by a polynomial matrix. At the same time, to determine the stability of multivariable systems, the direct extension of the well-known scalar Routh theorem to the matrix Routh theorem has also been studied in this research.

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CHAPTER I

INTRODUCTION

In 1956, H. R. Schwarz[1] published a paper in which he showed how to transform a system matrix to a specific matrix form, which is now called the Schwarz matrix form. Later the Schwarz matrix was extensively used to construct Liapunov functions[3], to prove the Hurwitz criterion[2] and to evaluate the system performance[4]. For instance, Kalman and Bertram[3] used the Schwarz matrix to find the Liapunov functions. Parks[2] used it to prove the Hurwitz criterion via the Second method of Liapunov. On the other hand, Diamesis[4] used the Schwarz matrix to evaluate the system performance. The transformation matrices, which relate various matrix forms and the Schwarz form, have been established by numerous authors[7-15]. Chen and Chu[9] constructed a transformation matrix which links the Schwarz matrix and the companion matrix by using the Routh array elements. However, all existing methods[7-15] deal only with the system matrix which has scalar elements but not block elements. In this research, a matrix which consists of block elements is constructed in the Schwarz block form and a linear transformation matrix which consists of block elements is established to transform the matrix in the companion block form to the matrix in the Schwarz block form. A sufficient condition is then derived for determining the stability of a multivariable system whose characteristics are expressed by a polynomial matrix.

CHAPTER II

THE SCHWARZ MATRIX FOR SINGLE VARIABLE SYSTEMS

The Schwarz matrix is an important matrix for constructing Liapunov functions, proving the Hurwitz criterion and evaluating performance measures in system analysis. However, most of the methods suggested for obtaining the Schwarz matrix are too complicated to implement. In 1966 Chen and Chu[9] established an effective method by which a matrix in the phase variable form can be easily converted into a matrix in the Schwarz form. Their procedure is briefly reviewed in this Chapter.

Consider the following characteristic equation

$$s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n = 0 \qquad (1)$$

The corresponding phase variable form is

$$\dot{X} = AX \qquad (2)$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \cdot & \cdot & 0 \\ 0 & 0 & 1 & 0 & 0 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & \cdot & \cdot & 1 \\ -a_n & -a_{n-1} & \cdot & \cdot & \cdot & \cdot & -a_2 & -a_1 \end{bmatrix}$$

The system matrix A can be transformed to the matrix in the Schwarz form using the transformation

$$Y = PX \tag{3}$$

Substituting Eq. (3) into Eq. (2) yields

$$\dot{Y} = PAP^{-1}Y$$

or
$$\dot{Y} = BY$$

where
$$B = PAP^{-1}$$

The matrix B is the required matrix in the Schwarz form. Chen and Chu[7] constructed the matrices B and P in terms of the elements of the Routh array[16] constructed from Eq. (1). The Routh array can be written as

$$\begin{array}{ccccc} 1 & a_2 & a_4 & a_6 & \cdots \\ a_1 & a_3 & a_5 & a_7 & \cdots \\ \dfrac{a_1a_3-a_1a_2}{a_1} & \cdot & \cdot & \cdot & \cdots \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \end{array} \tag{4}$$

and can be represented in terms of double subscripted notations as follows:

$$\begin{matrix} c_{11} & c_{12} & c_{13} & c_{14} & \cdot & \cdot \\ c_{21} & c_{22} & c_{23} & \cdot & \cdot & \cdot \\ c_{31} & c_{32} & c_{33} & \cdot & \cdot & \cdot \\ c_{41} & c_{42} & \cdot & \cdot & \cdot & \cdot \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \end{matrix} \tag{5}$$

The Schwarz matrix B and the transformation matrix P are expressed as follows:

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \frac{-c_{n+1,1}}{c_{n-1,1}} & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{-c_{51}}{c_{31}} & 0 & 1 & 0 \\ 0 & 0 & \frac{-c_{41}}{c_{21}} & 0 & 1 \\ 0 & 0 & 0 & \frac{-c_{31}}{c_{11}} & \frac{-c_{21}}{c_{11}} \end{bmatrix} \tag{6}$$

and

$$
P = \begin{bmatrix}
1 & . & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & . & . & . & . & . & . & . & . \\
\frac{c_{n-1,2}}{c_{n-1,1}} & . & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & . & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
\frac{c_{n-3,3}}{c_{n-3,1}} & . & \frac{c_{62}}{c_{61}} & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & . & 0 & \frac{c_{52}}{c_{51}} & 0 & 1 & 0 & 0 & 0 \\
\frac{c_{n-5,4}}{c_{n-5,1}} & . & \frac{c_{43}}{c_{41}} & 0 & \frac{c_{42}}{c_{41}} & 0 & 1 & 0 & 0 \\
0 & . & 0 & \frac{c_{33}}{c_{31}} & 0 & \frac{c_{32}}{c_{31}} & 0 & 1 & 0 \\
. & . & \frac{c_{24}}{c_{21}} & 0 & \frac{c_{23}}{c_{21}} & 0 & \frac{c_{22}}{c_{21}} & 0 & 1
\end{bmatrix} \tag{7}
$$

Both the matrices B and P were constructed by using the Routh array elements[9] in Eq. (5). The structures of the matrices B and P are quite simple. If the transformation matrix P for a second order system is required, then we can take the principal minor from the lower right corner of P, as shown by the dotted lines. Similarly, the P matrix for 3rd, 4th,...etc. order systems can be found by taking the respective principal minors from the right lower corner of P.

An illustrative example is shown as follows:

Example:

For a given fourth order system

$$\ddddot{x} + 10\dddot{x} + 35\ddot{x} + 50\dot{x} + 24x = 0$$

the matrix in the Schwarz form is required. The corresponding matrix in the phase variable form is expressed as follows:

$$\dot{x} = AX$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -24 & -50 & -35 & -10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

The Routh array for the system can be written as the following:

$$\begin{array}{lll|lll} 1 & 35 & 24 & c_{11} & c_{12} & c_{13} \\ 10 & 50 & & c_{21} & c_{22} & \\ 30 & 24 & & c_{31} & c_{32} & \\ 42 & & & c_{41} & & \\ 24 & & & c_{51} & & \end{array}$$

Substituting the value of these Routh array elements into Eq. (7), we obtain

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0.8 & 0 & 1 & 0 \\ 0 & 5 & 0 & 1 \end{bmatrix}$$

The required Schwarz matrix is

$$B = PAP^{-1}$$

$$B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -0.8 & 0 & 1 & 0 \\ 0 & -4.2 & 0 & 1 \\ 0 & 0 & -30 & -10 \end{bmatrix}$$

The procedure given in this chapter will be extended to a multivariable system which has a system matrix with block elements.

CHAPTER III

A MATRIX IN THE SCHWARZ BLOCK FORM FOR MULTIVARIABLE SYSTEMS

Section 1

Following the simple procedure of the previous chapter, we can very easily obtain a matrix in the Schwarz block form for a given system matrix which is expressed in the companion block form.

Consider a set of linear time invariant ordinary differential equations in the differential polynomial matrix form

$$\sum_{i=1}^{n+1} B_i D^{i-1} X(t) = [0], \qquad B_{n+1} = I \tag{8}$$

$$D^{i-1} X(0) = [\mathcal{L}_{i-1}] \qquad i = 1,2,3\ldots n$$

where $X(t)$ is the m dimensional state of the system, B_i is an mxm real constant matrix and the differential operator D is $D = \frac{d}{dt}$, matrix I is an identity matrix and [0] is a null matrix. The corresponding state equation of Eq. (8) in the companion block form is

$$[\dot{X}] = [B]\ [X] \tag{9}$$

$$[X(0)] = [\alpha]$$

where

$$[B] = \begin{bmatrix} 0 & I & 0 & 0 & 0 & . & . & 0 \\ 0 & 0 & I & 0 & 0 & . & . & 0 \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & . & . & I \\ -B_1 & -B_2 & -B_3 & -B_4 & -B_5 & . & . & -B_n \end{bmatrix}$$

The dimensions of matrix [B], each block element in [B], and the state vector [X] are (nxm)x(nxm), mxm and (nxm)x1 respectively. Matrix [B] is the matrix in the companion block form[17]. Eq. (9) can be transformed to the Schwarz block form using a linear transformation $[K_1]$ as follows:

Let $[y] = [K_1][X]$

Then

$$[\dot{y}] = [K_1][B][K_1^{-1}][y]$$
$$= [A][y] \qquad (10)$$

Where

$$[A] = \begin{bmatrix} 0 & I & 0 & . & 0 & 0 \\ -A_1 & 0 & I & . & 0 & 0 \\ 0 & -A_2 & 0 & . & 0 & 0 \\ . & . & . & . & . & . \\ 0 & 0 & 0 & . & 0 & I \\ 0 & 0 & 0 & . & -A_{n-1} & -A_n \end{bmatrix}$$

The dimension of each block element in matrix [A] is mxm. Matrix [A] is the matrix in the Schwarz block form. By following the approach proposed by Chen & Chu[9], the linear transformation matrix $[K_1]$ which relates the coordinates [X] and [y] in Eqs. (9) & (10) is constructed and extended as follows:

$$[y] = [K_1][X] \tag{11}$$

Where

$$[K_1] = \begin{bmatrix} I & . & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & . & . & . & . & . & . & . & . \\ C^{-1}_{n-1,1}C_{n-1,2} & . & I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & . & 0 & I & 0 & 0 & 0 & 0 & 0 \\ C^{-1}_{n-3,1}C_{n-3,3} & . & C^{-1}_{61}C_{62} & 0 & I & 0 & 0 & 0 & 0 \\ 0 & . & 0 & C^{-1}_{51}C_{52} & 0 & I & 0 & 0 & 0 \\ C^{-1}_{n-5}C_{n-5,4} & . & C^{-1}_{41}C_{43} & 0 & C^{-1}_{41}C_{42} & 0 & I & 0 & 0 \\ 0 & . & 0 & C^{-1}_{31}C_{33} & 0 & C^{-1}_{31}C_{32} & 0 & I & 0 \\ . & . & C^{-1}_{21}C_{24} & 0 & C^{-1}_{21}C_{23} & 0 & C^{-1}_{21}C_{22} & 0 & I \end{bmatrix} \tag{12}$$

Matrix $[K_1]$ in Eq. (12) is quite similar to the $[K_1]$ matrix of Chen & Chu.[9] The block elements $C_{i,j}$ having dimension mxm in equation (12) can be obtained from the matrix Routh array[17].

Before constructing the matrix Routh array define $\ell = \frac{n}{2} + 1$ if n is an even number, otherwise $\ell = \frac{n+1}{2}$. Also define the double subscript notations $c_{1,j}$ and $c_{2,j}$ as follows:

$$
\begin{aligned}
C_{1,j} &= B_{n+3-2j}\,, \qquad j = 1,2,3,\ldots,\ell \\
C_{2,j} &= B_{n+2-2j}\,, \qquad j = 1,2,\ldots\ldots\ell \qquad (13) \\
C_{11} &= I
\end{aligned}
$$

The B_i are the positive real constant matrices shown in Eq. (8). The matrix Routh array and the matrix Routh algorithm are expressed as follows:

$$
\begin{array}{ll}
 & C_{11} \quad C_{12} \quad C_{13} \quad C_{14} \quad \cdots \\
H_1 = C_{11}C_{21}^{-1} & \\
 & C_{21} \quad C_{22} \quad C_{23} \quad C_{24} \quad \cdots \\
H_2 = C_{21}C_{31}^{-1} & \\
 & C_{31} \quad C_{32} \quad C_{33} \quad \cdots \\
H_3 = C_{31}C_{41}^{-1} & \\
 & C_{41} \quad C_{42} \quad C_{43} \quad \cdots \\
 & C_{51} \quad C_{52} \quad \cdots \\
 & C_{61} \quad C_{62} \quad \cdots \\
 & C_{71} \quad \cdots
\end{array}
\qquad (14)
$$

$$
H_n = C_{n,1}C_{n+1,1}^{-1} \quad \begin{array}{l} C_{n,1} \\ C_{n+1,1} \end{array}
$$

where

$$C_{i,j} = C_{i-2,j+1} - H_{i-2}C_{i-1}C_{j+1} \qquad j=1,2,\ldots$$

$$i=3,4,\ldots$$

$$H_i = C_{i,1}(C_{i+1,1})^{-1} \qquad i=1,2,\ldots,n$$

$$\det\,(C_{i+1,1}) \neq 0$$

The matrices H_i in Eq. (14) are called the matrix quotient. The procedure to construct the transformation matrix $[K_1]$ and the Schwarz matrix [A] is shown in the following section.

Section 2

To obtain a matrix in the Schwarz block form we perform the following linear transformation:

$$[z] = [K_2][y]$$

From Eq. (10) we have an alternate matrix [F] in the Schwarz block form as follows:

$$\begin{aligned}[\dot{z}] &= [K_2][A][K_2^{-1}][z] \\ &= [F][z]\end{aligned}$$

A detailed expression is

$$[\dot{z}] = [K_2][K_1][B][K_1]^{-1}[K_2]^{-1}[z]$$

$$= [K][B][K]^{-1}[z]$$

$$= [F][z] \tag{15}$$

where

$$[F] = \begin{bmatrix} 0 & H_{n-1}^{-1} & 0 & . & 0 & 0 & 0 \\ -H_n^{-1} & 0 & H_{n-2}^{-1} & . & 0 & 0 & 0 \\ 0 & -H_{n-1}^{-1} & 0 & . & 0 & 0 & 0 \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & 0 & H_2^{-1} & 0 \\ 0 & 0 & 0 & . & -H_3^{-1} & 0 & H_1^{-1} \\ 0 & 0 & 0 & . & 0 & -H_2^{-1} & -H_1^{-1} \end{bmatrix} \tag{16}$$

Matrix [F] is the required matrix in the Schwarz block form, which can be constructed by the matrix quotients H_i, $i=1,2,...,n$ obtained from the matrix Routh array, although, it is observed that the approach proposed here to find the Schwarz block form is applicable only when $\det(C_{i+1}) \neq 0$.

The linear transformation matrix [K] which links the coordinates [X] in the companion block form and the coordinates [z] in the Schwarz block form can be written as follows:

$$
[K] = \begin{bmatrix}
H_n C_{n+1,1} & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \cdots & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
H_{n-2} C_{n-1,2} & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \cdots & H_6 C_{71} & 0 & 0 & 0 & 0 & 0 \\
H_{n-4} C_{n-3,3} & \cdots & 0 & H_5 C_{61} & 0 & 0 & 0 & 0 \\
0 & \cdots & H_4 C_{52} & 0 & H_4 C_{51} & 0 & 0 & 0 \\
H_{n-6} C_{n-5,4} & \cdots & 0 & H_3 C_{42} & 0 & H_3 C_{41} & 0 & 0 \\
0 & \cdots & H_2 C_{33} & 0 & H_2 C_{32} & 0 & H_2 C_{31} & 0 \\
\cdot & \cdots & 0 & H_1 C_{23} & 0 & H_1 C_{22} & 0 & H_1 C_{21}
\end{bmatrix} \tag{17}
$$

The matrix [K] is a triangular block matrix. All the block elements in Eq. (17) can be directly obtained from the matrix Routh array in Eq. (14). For example, the block elements in the main diagonal which are shown by the dotted diagonal line are obtained by premultiplying the matrix quotients H_i to the block elements $C_{i,1}$, which are located in the first column of the matrix Routh array. The block elements of the first lower diagonal in [K], shown by the second dotted diagonal line, are determined by premultiplying the matrix quotients H_i to the block elements $C_{i,2}$. The $C_{i,2}$ are located in the second column of the matrix Routh array. The sizes of matrices [F] and [K] are determined by the degree of the polynomial matrix and the order of the matrix coefficients in Eq. (8). In other words, when the degree of a polynomial matrix is n=4 and the dimension of each matrix coefficient is m then the corresponding 4mx4m matrices [F] and [K] are taken from the right hand side lower corner of the matrices [F] and [K] as shown in Eqs. (16) and (17).

In the above paragraphs we have described the Schwarz block matrix [F] and the transformation matrix [K]. However, the question is how to obtain these matrices.

In section 3 we will show the steps to construct these matrices [F] and [K] using the induction approach.

Section 3

Case 1: Second Order System (m=2)

The differential matrix polynomial for a second order multivariable system is

$$\ddot{X}(t)+[B_2]\dot{X}(t)+[B_1]X(t)=[0] \tag{18}$$

where B_1 and B_2 are real constant mxm matrix coefficients. $X(t)$ is the 2-dimensional state of the system.

The corresponding state equation of Eq. (18) in the companion block form is

$$[\dot{X}] = [B][X]$$

or

$$[\dot{X}] = \begin{bmatrix} 0 & I \\ -B_1 & -B_2 \end{bmatrix} [X] \tag{19}$$

and the matrix Routh array for the given matrix polynomial is

$$\begin{array}{ll} I & B_1 \\ B_2 & \\ B_1 & \end{array}$$

The positions of the matrix Routh array can be indicated by matrix double subscript notations as follows:

$$\begin{array}{lll} & C_{11} & C_{12} \\ H_1=C_{11}C_{21}^{-1} & & \\ & C_{21} & \\ H_2=C_{21}C_{31}^{-1} & & \\ & C_{31} & \end{array}$$

Eq. (19) can be written by using these new Routh array elements as :

$$[\dot{X}] = \begin{bmatrix} 0 & I \\ -C_{12} & -C_{21} \end{bmatrix} [X]$$

The corresponding Schwarz form is

$$[\dot{Z}] = [K][B][K]^{-1}[z]$$
$$= [F][z]$$

where

$$[F] = [K][B][K]^{-1}$$

or

$$[F][K] = [K][B] \tag{20}$$

Now, consider the transformation matrix [K]

$$[K] = \begin{bmatrix} H_2C_{31} & 0 \\ 0 & H_1C_{21} \end{bmatrix} \tag{21}$$

The matrix [F] in the new proposed Schwarz block form is

$$[F] = \begin{bmatrix} 0 & F_3 \\ -F_2 & -F_1 \end{bmatrix}$$

To determine the matrix [F] impose the following condition:

$$[F][K] = [K][B]$$

or

$$[F][K] = \begin{bmatrix} 0 & F_3 \\ -F_2 & -F_1 \end{bmatrix} \begin{bmatrix} H_2C_{31} & 0 \\ 0 & H_1C_{21} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & F_3H_1C_{21} \\ -F_2H_2C_{31} & -F_1H_1C_{21} \end{bmatrix} \qquad (22)$$

and

$$[K][B] = \begin{bmatrix} H_2C_{31} & 0 \\ 0 & H_1C_{21} \end{bmatrix} \begin{bmatrix} 0 & I \\ -C_{12} & -C_{21} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & H_2C_{31} \\ -H_1C_{21}C_{12} & -H_1C_{21}C_{21} \end{bmatrix} \qquad (23)$$

A comparison of Eqs. (22) and (23) yields

$$F_1 = H_1^{-1}$$
$$F_2 = H_2^{-1}$$
$$F_3 = H_1^{-1}$$

Thus

$$[F] = \begin{bmatrix} 0 & H_1^{-1} \\ -H_2^{-1} & -H_1^{-1} \end{bmatrix} \qquad (24)$$

Matrix [F] in Eq. (24) is the required matrix in the Schwarz block form. Now we go one step ahead and consider the third order system.

Case 2: Third Order System

For a third order system the differential matrix equation is as follows:

$$\dddot{X} + [B_3]\ddot{X} + [B_2]\dot{X} + [B_1]X = [0] \tag{25}$$

The state equation in the companion block form for the above equation can be written as

$$[\dot{X}] = \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & I \\ -B_1 & -B_2 & -B_3 \end{bmatrix} [X]$$

and the matrix Routh array for (25) is;

$$\begin{array}{ll} I & B_2 \\ B_3 & B_1 \\ (B_2 - B_3^{-1}B_1) & \\ B_1 & \end{array}$$

The matrix Routh array can be expressed in the matrix double subscript notation as shown below.

$$
\begin{array}{lcl}
H_1=C_{11}C_{21}^{-1} & < & \begin{array}{ll} C_{11} & C_{12} \\ C_{21} & C_{22} \end{array} \\
H_2=C_{21}C_{31}^{-1} & < & \begin{array}{l} C_{21} \\ C_{31} \end{array} \\
H_3=C_{31}C_{41}^{-1} & < & \begin{array}{l} C_{31} \\ C_{41} \end{array}
\end{array}
\tag{26}
$$

Eq. (20) can be re-written using these new matrix Routh array elements as

$$
[\dot{X}] = \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & I \\ -C_{22} & -C_{12} & -C_{21} \end{bmatrix} [X]
$$

The corresponding state equation in the Schwarz block form is

$$
\begin{aligned}
[\dot{z}] &= [K][B][K]^{-1}[z] \\
&= [F][z]
\end{aligned}
$$

where the matrix in the Schwarz block form can be expressed in the following form:

$$
[F] = \begin{bmatrix} 0 & F_5 & 0 \\ -F_4 & 0 & F_3 \\ 0 & -F_2 & -F_1 \end{bmatrix}
$$

The suggested transformation matrix [K] for this third order system is

$$[K] = \begin{bmatrix} H_3C_{41} & 0 & 0 \\ 0 & H_2C_{31} & 0 \\ H_1C_{22} & 0 & H_1C_{21} \end{bmatrix} \tag{27}$$

The block elements of [K] are found from the matrix Routh array of Eq. (26). Here again the matrix [F] can be determined by using the equality of Eq. (20), or by imposing the condition

$$[F][K] = [K][B]$$

The detailed block structure is

$$[F][K] = \begin{bmatrix} 0 & F_5 & 0 \\ -F_4 & 0 & F_3 \\ 0 & -F_2 & -F_1 \end{bmatrix} \begin{bmatrix} H_3C_{41} & 0 & 0 \\ 0 & H_2C_{31} & 0 \\ H_1C_{22} & 0 & H_1C_{21} \end{bmatrix}$$

$$\begin{bmatrix} 0 & F_5H_2C_{31} & 0 \\ -F_4H_3C_{41}+F_3H_1C_{22} & 0 & F_3H_1C_{21} \\ -F_1H_1C_{22} & -F_2H_2C_{31} & -F_1H_1C_{21} \end{bmatrix} \tag{28}$$

and

$$[K][B] = \begin{bmatrix} H_3C_{41} & 0 & 0 \\ 0 & H_2C_{31} & 0 \\ H_1C_{22} & 0 & H_1C_{21} \end{bmatrix} \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & I \\ -C_{22} & -C_{12} & -C_{21} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & H_3C_{41} & 0 \\ 0 & 0 & H_2C_{31} \\ -H_1C_{21}C_{22} & H_1C_{22}-H_1C_{21}C_{12} & -H_1C_{21}C_{21} \end{bmatrix} \tag{29}$$

From Eqs. (28) and (29)

$$\begin{aligned} F_1 &= H_1^{-1} \\ F_2 &= H_2^{-1} \\ F_3 &= H_1^{-1} \\ F_4 &= H_3^{-1} \\ F_5 &= H_2^{-1} \end{aligned}$$

Therefore the matrix in the Schwarz block form is

$$F = \begin{bmatrix} 0 & H_2^{-1} & 0 \\ -H_3^{-1} & 0 & H_1^{-1} \\ 0 & -H_2^{-1} & -H_1^{-1} \end{bmatrix} \tag{30}$$

The structure of matrix [F] is the same as the matrix proposed in Eq. (16). Next we continue to search for the pattern of these matrices for a fourth order system.

Case 3: Fourth Order System

The differential matrix equation for a fourth order equation is

$$\ddddot{X} + [B_4]\dddot{X} + [B_3]\ddot{X} + [B_2]\dot{X} + [B_1]X = [0] \qquad (31)$$

The state equation in the companion block form of the above equation is

$$[\dot{X}] = \begin{bmatrix} 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ -B_1 & -B_2 & -B_3 & -B_4 \end{bmatrix} [X] \qquad (32)$$

We establish the matrix Routh array for Eq. (31) as follows:

$$\begin{array}{lll} I & B_3 & B_1 \\ B_4 & B_2 & \\ (B_3-B_4^{-1}B_2) & B_1 & \\ \left(B_2 - B_4(B_3-B_4^{-1}B_2)^{-1}B_1\right) & & \\ B_1 & & \end{array}$$

The matrix Routh array in terms of a matrix double subscript notation can be written as follows:

$$\begin{array}{llll}
 & C_{11} & C_{12} & C_{13} \\
H_1=C_{11}C_{21}^{-1} & & & \\
 & C_{21} & C_{22} & \\
H_2=C_{21}C_{31}^{-1} & & & \\
 & C_{31} & C_{32} & \\
H_3=C_{31}C_{41}^{-1} & & & \\
 & C_{41} & & \\
H_4=C_{41}C_{51}^{-1} & & & \\
 & C_{51} & &
\end{array} \tag{33}$$

Thus (32) becomes

$$[\dot{X}] = \begin{bmatrix} 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ -C_{13} & -C_{22} & -C_{12} & -C_{21} \end{bmatrix} [X]$$

The corresponding state equation in the Schwarz block form is

$$\begin{aligned} [\dot{z}] &= [K][B][K]^{-1}[z] \\ &= [F][z] \end{aligned}$$

where the Schwarz matrix [F] is expressed in the following form:

$$[F] = \begin{bmatrix} 0 & F_7 & 0 & 0 \\ -F_6 & 0 & F_5 & 0 \\ 0 & -F_4 & 0 & F_3 \\ 0 & 0 & -F_2 & -F_1 \end{bmatrix}$$

The suggested transformation matrix [K] is

$$[K] = \begin{bmatrix} H_4C_{51} & 0 & 0 & 0 \\ 0 & H_3C_{41} & 0 & 0 \\ H_2C_{32} & 0 & H_2C_{31} & 0 \\ 0 & H_1C_{22} & 0 & H_1C_{21} \end{bmatrix} \quad (34)$$

The block elements of [K] are found from the matrix Routh array in (33). Once again we check the equality of the following matrix equation:

$$[F][K] = [K][B]$$

where

$$[F][K] = \begin{bmatrix} 0 & F_7 & 0 & 0 \\ -F_6 & 0 & F_5 & 0 \\ 0 & -F_4 & 0 & F_3 \\ 0 & 0 & -F_2 & -F_1 \end{bmatrix} \begin{bmatrix} H_4C_{51} & 0 & 0 & 0 \\ 0 & H_3C_{41} & 0 & 0 \\ H_2C_{32} & 0 & H_2C_{31} & 0 \\ 0 & H_1C_{22} & 0 & H_1C_{21} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & F_7H_3C_{41} & 0 & 0 \\ -F_6H_4C_{51}+F_5H_2C_{32} & 0 & F_5H_2C_{31} & 0 \\ 0 & -F_4H_3C_{41}+F_3H_1C_{22} & 0 & F_3H_1C_{21} \\ -F_2H_2C_{32} & -F_1H_1C_{22} & -F_2H_2C_{31} & -F_1H_1C_{21} \end{bmatrix}$$

(35)

and

$$[K][B]=\begin{bmatrix} H_4C_{51} & 0 & 0 & 0 \\ 0 & H_3C_{41} & 0 & 0 \\ H_2C_{32} & 0 & H_2C_{31} & 0 \\ 0 & H_1C_{22} & 0 & H_1C_{21} \end{bmatrix} \begin{bmatrix} 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ -C_{13} & -C_{22} & -C_{12} & -C_{21} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & H_4C_{51} & 0 & 0 \\ 0 & 0 & H_3C_{41} & 0 \\ 0 & H_2C_{32} & 0 & H_2C_{31} \\ -H_1C_{21}C_{13} & -H_1C_{21}C_{22} & (H_1C_{22}-H_1C_{21}C_{12}) & -H_1C_{21}C_{21} \end{bmatrix} \quad (36)$$

From Eqs. (35) and (36) we have

$$F_1 = H_1^{-1}$$
$$F_2 = H_2^{-1}$$
$$F_3 = H_1^{-1}$$
$$F_4 = H_3^{-1}$$
$$F_5 = H_2^{-1}$$
$$F_6 = H_4^{-1}$$
$$F_7 = H_3^{-1}$$

The required matrix in the Schwarz block form is

$$[F] = \begin{bmatrix} 0 & H_3^{-1} & 0 & 0 \\ -H_4^{-1} & 0 & H_2^{-1} & 0 \\ 0 & -H_3^{-1} & 0 & H_1^{-1} \\ 0 & 0 & -H_2^{-1} & -H_1^{-1} \end{bmatrix} \qquad (37)$$

Here once more we have the expected structure of the Schwarz matrix [F] and the transformation matrix [K]. Following the development of the pattern of the matrices [F] and [K] in Eqs. (24), (30) and (37), we can proceed to construct the corresponding matrices [F] and [K] for a fifth order system.

Fifth Order System

The differential matrix equation for a fifth order system is

$$\overset{\cdot\cdot\cdot\cdot}{X} + [B_5]\overset{\cdot\cdot\cdot\cdot}{X} + [B_4]\dddot{X} + [B_3]\ddot{X} + [B_2]\dot{X} + [B_1]X = [0]$$

The state equation in the companion block form becomes

$$[\dot{X}] = \begin{bmatrix} 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \\ -B_1 & -B_2 & -B_3 & -B_4 & -B_5 \end{bmatrix} [X] \qquad (38)$$

The matrix Routh array can be written as follows:

$$
\begin{array}{llll}
 & C_{11} & C_{12} & C_{13} \\
H_1 = C_{11}C_{21}^{-1} & & & \\
 & C_{21} & C_{22} & C_{23} \\
H_2 = C_{21}C_{31}^{-1} & & & \\
 & C_{31} & C_{32} & \\
H_3 = C_{31}C_{41}^{-1} & & & \\
 & C_{41} & C_{42} & \\
H_4 = C_{41}C_{51}^{-1} & & & \\
 & C_{51} & & \\
H_5 = C_{51}C_{61}^{-1} & & & \\
 & C_{61} & &
\end{array}
$$

The alternate representation of Eq. (38) is

$$
[\dot{X}] = \begin{bmatrix} 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \\ -C_{23} & -C_{13} & -C_{22} & -C_{12} & -C_{21} \end{bmatrix} [X] \tag{39}
$$

The corresponding state equation in the Schwarz block form is

$$
[\dot{z}] = [K][B][K]^{-1}[z] \tag{40}
$$

$$
[\dot{z}] = [F][z] \tag{41}
$$

Following the previous procedure we find the matrix

[F] and the transformation matrix [K]:

$$[F] = \begin{bmatrix} 0 & H_4^{-1} & 0 & 0 & 0 \\ -H_5^{-1} & 0 & H_3^{-1} & 0 & 0 \\ 0 & -H_4^{-1} & 0 & H_2^{-1} & 0 \\ 0 & 0 & -H_3^{-1} & 0 & H_1^{-1} \\ 0 & 0 & 0 & -H_2^{-1} & -H_1^{-1} \end{bmatrix} \qquad (42)$$

and

$$[K] = \begin{bmatrix} H_5C_{61} & 0 & 0 & 0 & 0 \\ 0 & H_4C_{51} & 0 & 0 & 0 \\ H_3C_{42} & 0 & H_3C_{41} & 0 & 0 \\ 0 & H_2C_{32} & 0 & H_2C_{31} & 0 \\ H_1C_{23} & 0 & H_1C_{22} & 0 & H_1C_{21} \end{bmatrix} \qquad (43)$$

This supports the following procedure for the development of the Schwarz block matrix [F] and the transformation matrix [K]. Hence, we can establish a general structure for the matrices [F] and [K].

General Schwarz Matrix In Block Form

From Eq. (24), (30), (37), and (42), we can describe the pattern of the matrix [F] as follows:

(1) The principal minors of the matrix [F] located at the lower right corner represent the Schwarz block matrices for the system. For example, the second lower principal minor is the Schwarz matrix for a second order system, while the third lower principal minor is the Schwarz matrix for a third order system.

(2) All the block elements on the main diagonal are null matrices except the lowest right corner block element which is the inverse of the matrix quotient H_1.

(3) The block elements of the first lower diagonal are negative of the inverses of the matrix quotients H_i, $i=1,2,\ldots,n+1$, which are found from the matrix Routh array.

(4) Each block elements in the 2nd, 3rd,...etc. lower diagonals are null matrices.

(5) The block elements of the first upper diagonal are the inverse of the matrix quotients H_i, $i=1,2,\ldots,n_1$ which are found from the matrix Routh array.

(6) Each block element in the 2nd, 3rd,...etc. upper diagonal is a null matrix.

The general [F] matrix is shown as follows:

$$[F] = \begin{bmatrix} 0 & H_{n-1}^{-1} & 0 & \cdot & 0 & 0 & 0 & 0 \\ -H_n^{-1} & 0 & H_{n-2}^{-1} & \cdot & 0 & 0 & 0 & 0 \\ 0 & -H_{n-1}^{-1} & 0 & \cdot & 0 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & 0 & H_3^{-1} & 0 & 0 \\ 0 & 0 & 0 & \cdot & -H_4^{-1} & 0 & H_2^{-1} & 0 \\ 0 & 0 & 0 & \cdot & 0 & -H_3^{-1} & 0 & H_1^{-1} \\ 0 & 0 & 0 & \cdot & 0 & 0 & -H_2^{-1} & -H_1^{-1} \end{bmatrix}$$

General Transformation Matrix

From Eqs. (21), (27), (34) and (43), the structure of the general transformation matrix [K] can be recognized as follows:

1. The principal minors of the matrix [K], from the lower right corner to the upper right corner, represent the transformation matrices for the systems of increasing order. For example, the second lower principal minor is the transformation matrix for a second order system, and the third lower principal minor is the transformation matrix for a third order system.
2. The matrix [K] is a triangular matrix with block elements.
3. The block elements of the main diagonal in the matrix [K] are ob-

tained by the respective product of the matrix quotients H_i and the block elements $C_{i,1}$ in the first column of the matrix Routh array. The main diagonal is shown by the first dotted line.

4. The block elements of the first lower diagonal in the matrix [K] are null matrices.

5. The block elements of the second lower diagonal in the matrix [K] are determined by the respective products of the matrix quotients H_i and the block elements $C_{1,2}$ in the second column of the matrix Routh array. The second lower diagonal is shown by the second dotted line.

6. The remaining block elements in the lower triangle are found by following the guidelines of steps (4) and (5).

The general structure of the transformation matrix [K] is shown as below,

$$[K]=\begin{bmatrix} H_nC_{n+1,1} & \cdot & \cdot & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ H_{n-2}C_{n-1,2} & \cdot & \cdot & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cdot & \cdot & H_6C_{71} & 0 & 0 & 0 & 0 & 0 \\ H_{n-4}C_{n-3,3} & \cdot & \cdot & 0 & H_5C_{61} & 0 & 0 & 0 & 0 \\ 0 & \cdot & \cdot & H_4C_{52} & 0 & H_4C_{51} & 0 & 0 & 0 \\ H_{n-6}C_{n-5,4} & \cdot & \cdot & 0 & H_3C_{42} & 0 & H_3C_{41} & 0 & 0 \\ 0 & \cdot & \cdot & H_2C_{33} & 0 & H_2C_{32} & 0 & H_2C_{31} & 0 \\ \cdot & \cdot & \cdot & 0 & H_1C_{23} & 0 & H_1C_{22} & 0 & H_1C_{21} \end{bmatrix}$$

ILLUSTRATIVE EXAMPLES

Example 1

To construct the proposed linear transformation matrix and the matrix in the Schwarz block form, consider a differential matrix equation

$$\sum_{i=1}^{n+1=5} B_i D^{i-1} X(t) = [0]$$

or

$$B_5 \ddddot{X}(t) + B_4 \dddot{X}(t) + B_3 \ddot{X}(t) + B_2 \dot{X}(t) + B_1 X(t) = [0] \tag{44}$$

Using Eq. (13), the above equation (44) can be rewritten as

$$C_{11} \overset{(4)}{X}(t) + C_{21} \overset{(3)}{X}(t) + C_{12} \ddot{X}(t) + C_{22} \dot{X}(t) + C_{13} X(t) = [0]$$

where

$$C_{11} = B_5 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad C_{12} = B_3 = \begin{pmatrix} 11.4 & -1.6 \\ -2.2 & 5.8 \end{pmatrix}$$

$$C_{13} = B_1 = \begin{pmatrix} 5 & 3 \\ 3 & 4 \end{pmatrix} \qquad C_{21} = B_4 = \begin{pmatrix} 2 & 2 \\ 2 & 4 \end{pmatrix}$$

$$C_{22} = B_2 = \begin{pmatrix} 10.4 & 2.4 \\ 4 & 10 \end{pmatrix}$$

A matrix in the Schwarz block form and the linear transformation matrix which transforms the companion block form to the Schwarz block form can be constructed as follows:

The state equation in the companion block form is

$$[\dot{X}] = [B][X]$$

where

$$[B]=\begin{bmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \begin{pmatrix} -5 & -3 \\ -3 & -4 \end{pmatrix} & \begin{pmatrix} -10.4 & -2.4 \\ -4 & -10 \end{pmatrix} & \begin{pmatrix} -11.4 & 1.6 \\ 2.2 & -5.8 \end{pmatrix} & \begin{pmatrix} -2 & -2 \\ -2 & -4 \end{pmatrix} \end{bmatrix}$$

As shown in Eq. (15) and (16), the state equation in the Schwarz block form is

$$[\dot{Z}] = [F][Z]$$

where

$$[F] = \begin{bmatrix} 0 & H_3^{-1} & 0 & 0 \\ -H_4^{-1} & 0 & H_2^{-1} & 0 \\ 0 & -H_3^{-1} & 0 & H_1^{-1} \\ 0 & 0 & -H_2^{-1} & -H_1^{-1} \end{bmatrix}$$

Substitute above the numerical value of matrix quotients H_i, which are obtained from the matrix Routh array for Eq. (44).

$$[F] = \begin{bmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} \frac{14}{5} & \frac{-12}{5} \\ \frac{-6}{5} & \frac{-8}{5} \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} \frac{-8}{4} & \frac{-14}{4} \\ \frac{-7}{4} & \frac{-15}{4} \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} \frac{10}{4} & -1 \\ 0 & \frac{1}{2} \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} \frac{-14}{5} & \frac{12}{5} \\ \frac{6}{5} & \frac{-8}{5} \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} \frac{5}{7.5} & \frac{-5}{7.5} \\ \frac{-5}{7.5} & \frac{-10}{7.5} \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} \frac{-10}{4} & 1 \\ 0 & \frac{-1}{2} \end{pmatrix} & \begin{pmatrix} \frac{-5}{7.5} & \frac{5}{7.5} \\ \frac{5}{7.5} & \frac{10}{7.5} \end{pmatrix} \end{bmatrix}$$

The linear transformation between [X] and [Z] coordinates is

$$[Z] = [K][X]$$

The linear transformation matrix [K] can be found from the lower right corner of Eq. (14) as in the following:

$$[K] = \begin{bmatrix} H_4C_{51} & 0 & 0 & 0 \\ 0 & H_3C_{41} & 0 & 0 \\ H_2C_{32} & 0 & H_2C_{31} & 0 \\ 0 & H_1C_{22} & 0 & H_1C_{21} \end{bmatrix}$$

$$\text{or} \quad [K] = \begin{bmatrix} \begin{pmatrix} 6 & -2 \\ -2 & 2 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} \frac{22}{5} & \frac{22}{5} \\ 6 & 8 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 2 & 2 \\ 2 & 4 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 8.4 & -2.6 \\ -7.2 & -6.2 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{bmatrix}$$

Example 2

Consider another differential matrix equation

$$B_5\ddddot{X}(t)+B_4\dddot{X}(t)+B_3\ddot{X}(t)+B_2\dot{X}(t)+B_1X(t) =$$

$$C_{11}\ddddot{X}(t)+C_{21}\dddot{X}(t)+C_{12}\ddot{X}(t)+C_{22}\dot{X}(t)+C_{13}X(t) = [0] \qquad (45)$$

where

$$C_{11} = B_5 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$C_{12} = B_3 = \begin{pmatrix} -37.05 & -78.8 \\ 33 & 65 \end{pmatrix},$$

$$C_{13} = B_1 = \begin{pmatrix} -10.5 & -23 \\ -0.1 & -0.6 \end{pmatrix},$$

$$C_{21} = B_4 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix},$$

and $$C_{22} = B_2 = \begin{pmatrix} -43.1 & -94.6 \\ -6.05 & -16.3 \end{pmatrix}$$

The matrix in the Schwarz block form and the linear transformation matrix [K] are desired to be constructed:

According to Eq. (14) the matrix Routh array is written as follows:

$$C_{11}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\quad C_{12}=\begin{pmatrix}-37.5 & -78.8\\ 33 & 65\end{pmatrix}\quad C_{13}=\begin{pmatrix}-10.5 & -23\\ -0.1 & -0.6\end{pmatrix}$$

$$H_1=\begin{pmatrix}1 & -1\\ -1 & 2\end{pmatrix}$$

$$C_{21}=\begin{pmatrix}2 & 1\\ 1 & 1\end{pmatrix}\quad C_{22}=\begin{pmatrix}-43.1 & -94.6\\ -6.05 & -16.3\end{pmatrix}$$

$$H_2=\begin{pmatrix}4 & 1\\ 1 & 0.5\end{pmatrix}$$

$$C_{31}=\begin{pmatrix}0 & -0.5\\ 2 & 3\end{pmatrix}\quad C_{32}=\begin{pmatrix}-10.5 & -23\\ -0.1 & -0.6\end{pmatrix}$$

$$H_3=\begin{pmatrix}1.125 & 0.25\\ 0.25 & 0.5\end{pmatrix}$$

$$C_{41}=\begin{pmatrix}-1 & -2\\ 4.5 & 7\end{pmatrix}$$

$$H_4=\begin{pmatrix}0.1 & -0.5\\ -0.5 & 7.5\end{pmatrix}$$

$$C_{51}=\begin{pmatrix}-10.5 & -23\\ -0.1 & -0.6\end{pmatrix}$$

The state equation in the companion block form is

$$[\dot{X}] = [B][X]$$

where

$$[B]=\begin{bmatrix}
\begin{pmatrix}0&0\\0&0\end{pmatrix} & \begin{pmatrix}1&0\\0&1\end{pmatrix} & \begin{pmatrix}0&0\\0&0\end{pmatrix} & \begin{pmatrix}0&0\\0&0\end{pmatrix}\\
\begin{pmatrix}0&0\\0&0\end{pmatrix} & \begin{pmatrix}0&0\\0&0\end{pmatrix} & \begin{pmatrix}1&0\\0&1\end{pmatrix} & \begin{pmatrix}0&0\\0&0\end{pmatrix}\\
\begin{pmatrix}0&0\\0&0\end{pmatrix} & \begin{pmatrix}0&0\\0&0\end{pmatrix} & \begin{pmatrix}0&0\\0&0\end{pmatrix} & \begin{pmatrix}1&0\\0&1\end{pmatrix}\\
\begin{pmatrix}10.5&23\\0.1&0.6\end{pmatrix} & \begin{pmatrix}43.1&94.6\\6.05&16.3\end{pmatrix} & \begin{pmatrix}37.05&78.8\\-33&-65\end{pmatrix} & \begin{pmatrix}-2&-1\\-1&-1\end{pmatrix}
\end{bmatrix}$$

and the state equation in the Schwarz block form is

$$[\dot{Z}] = [F][Z]$$

where

$$[F]=\begin{bmatrix}
\begin{pmatrix}0&0\\0&0\end{pmatrix} & \begin{pmatrix}1&-0.5\\-0.5&2.25\end{pmatrix} & \begin{pmatrix}0&0\\0&0\end{pmatrix} & \begin{pmatrix}0&0\\0&0\end{pmatrix}\\
\begin{pmatrix}-15&-1\\-1&-0.2\end{pmatrix} & \begin{pmatrix}0&0\\0&0\end{pmatrix} & \begin{pmatrix}0.5&-1\\-1&4\end{pmatrix} & \begin{pmatrix}0&0\\0&0\end{pmatrix}\\
\begin{pmatrix}0&0\\0&0\end{pmatrix} & \begin{pmatrix}-1&0.5\\0.5&-2.25\end{pmatrix} & \begin{pmatrix}0&0\\0&0\end{pmatrix} & \begin{pmatrix}2&1\\1&1\end{pmatrix}\\
\begin{pmatrix}0&0\\0&0\end{pmatrix} & \begin{pmatrix}0&0\\0&0\end{pmatrix} & \begin{pmatrix}-0.5&1\\1&-4\end{pmatrix} & \begin{pmatrix}-2&-1\\-1&-1\end{pmatrix}
\end{bmatrix}$$

The linear transformation matrix [F] which relates the [X] and the [Z] coordinates is

$$[Z] = [K][X]$$

where

$$[K] = \begin{bmatrix} \begin{pmatrix} -1 & -2 \\ 4.5 & 7 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & -0.5 \\ 2 & 3 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} -42.1 & -92.6 \\ -10.55 & -23.3 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} -37.05 & -78.3 \\ 31 & 62 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{bmatrix}$$

CHAPTER IV

ROUTH ALGORITHM FOR SCALAR POLYNOMIALS

The well-known Routh Theorem[16] is commonly used to determine the absolute stability of single variable systems. The advantage of using the Routh criterion is that the stability of a system can be determined without actually solving for the roots of the characteristic equation. In this research the scalar Routh theorem for single variable systems has been extended to the matrix Routh theorem. Therefore, it is quite appropriate to start by reviewing the scalar Routh array.

Suppose that the characteristic equation of a linear system is written in the general form:

$$F(S) = A_o S^n + A_1 S^{n-1} + A_2 S^{n-2} + \ldots + A_{n-1} S + A_n = 0 \qquad (45a)$$

In order that there are no roots of Eq. (45a) with positive real parts, it is necessary (but not sufficient) that[23]

(1) All the coefficients of the polynomial have the same sign and

(2) None of the coefficients vanish.

The two necessary conditions given above can be checked by inspection. However they are not sufficient, it is quite likely that a polynomial with all positive coefficients has roots in the right half S-plane. However, this difficulty can be overcome by using the Routh theorem. In 1877, Routh developed a necessary and sufficient condition for the absolute stability of a characteristic equation. The Routh theorem is reviewed as follows:

1. Write the chracteristic polynomial in S in the following form:

$$a_o s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n = 0$$

where the coefficients are real numbers. We assume that $a_n \neq 0$, that is, any zero root has been removed.

2. Check the signs of the coefficients of the characteristic polynomial. If there exists any sign change, or any coefficients vanish, then the characteristic equation is not asymptotically stable. If all coefficients have a positive sign, then proceed with the following processes.

3. Arrange the coefficients of the polynomial in rows and columns according to the following pattern:

$$
\begin{array}{llllll}
s^n & a_1 & a_2 & a_4 & a_6 & \cdots \\
s^{n-1} & a_1 & a_3 & a_5 & a_7 & \cdots \\
s^{n-2} & b_1 & b_2 & b_3 & b_5 & \cdots \\
s^{n-3} & c_1 & c_2 & c_3 & c_4 & \cdots \\
s^{n-4} & d_1 & d_2 & d_3 & d_4 & \cdots \\
\cdot & & & & & \\
\cdot & & & & & \\
\cdot & & & & & \\
s^2 & e_1 & e_2 & & & \\
s^1 & f_1 & & & & \\
s^o & g_1 & & & &
\end{array}
$$

The coefficients b_1, b_2, b_3, etc. are evaluated as follows:

$$b_1 = \frac{a_1 a_2 - a_o a_3}{a_1}$$

$$b_2 = \frac{a_1a_4-a_oa_5}{a_1}$$

$$b_3 = \frac{a_1a_6-a_oa_7}{a_1}$$

$$\cdot \quad \cdot \quad \cdot$$

Evaluation of the b's continued until the remaining ones are all zero. The same pattern of cross multiplying the coefficients of the two previous rows is followed in evaluating the c's, d's, etc. e.g.

$$c_1 = \frac{b_1a_3-a_1b_2}{b_1}$$

$$\cdot \quad \cdot \quad \cdot$$

$$d_1 = \frac{c_1c_2-c_1c_2}{c_1}$$

$$\cdot \quad \cdot \quad \cdot$$

This process is continued until the n^{th} row has been completed. The Routh array can be much conveniently written in the double subscript notation as follows

$$\begin{array}{lllll}
 & C_{11} & C_{12} & C_{13} & C_{14} \cdot \cdot \\
H_1 = \dfrac{C_{11}}{C_{27}} \Big\langle & & & & \\
 & C_{21} & C_{22} & C_{23} & C_{24} \cdot \cdot \\
H_2 = \dfrac{C_{21}}{C_{31}} \Big\langle & C_{31} & C_{32} & C_{33} & \\
 & \cdot & \cdot & \cdot & \cdot \quad \cdot \cdot
\end{array}$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$H_n = \frac{C_{n,1}}{C_{n+1,1}} \left< \begin{array}{l} C_{n,1} \\ \\ C_{n+1,1} \end{array} \right.$$

where

$$C_{i,j} = C_{i-2,j+1} - H_{i-2}C_{i-1,j+1} \qquad j=1,2,\ldots$$

$$i=3,4,\ldots$$

$$H_i = C_{i,1}(C_{i+1,1})^{-1} \qquad i=1,2,\ldots n$$

The Routh's[16] stability criterion states that the number of the roots of the characteristic equation with a positive real part is equal to the number of changes in sign of the coefficients of the first column of the array. It should be noted that exact values of the term in the first column need not be known, instead only the signs are needed. The necessary and sufficient condition that all the roots of the chracteristic equation lie in the left half S-plane is that all the coefficients of the characteristic polynomial are positive and all terms in the first column of the Routh array have positive signs.

The Routh theorem can be illustrated by the following examples.

Example 1

Consider the following characteristic equation

$$S^4+2S^3+3S^2+4S+5 = 0$$

Follow the procedure just presented and construct the Routh array as in the following:

$$\begin{array}{lllll}
 & & 1 & 3 & 5 \\
 & h_1 = \frac{1}{2} & & & \\
 & & 2 & 4 & \\
 & h_2 = 2 & & & \\
 & & 1 & 5 & \\
\text{Sign change} & h_3 = -\frac{1}{6} & & & \\
 & & -6 & & \\
 & h_4 = -\frac{6}{5} & & & \\
 & & 5 & &
\end{array}$$

Since there are two sign changes in the first column of the Routh array, the equation has two roots located in the right half S-plane. Consequently all h's are also not positive.

Example 2

Consider the following characteristic equation:

$$(S+1)(S+2)(S+3)(S+4) = S^4+10S^3+55S^2+50S+24 = 0$$

The Routh array is constructed as follows:

$$\begin{array}{llll}
 & 1 & 55 & 24 \\
h_1 = \frac{1}{10} & & & \\
 & 10 & 50 & \\
h_2 = \frac{1}{5} & & & \\
 & 50 & 24 & \\
h_3 = \frac{250}{226} & & & \\
 & \frac{226}{5} & & \\
h_4 = \frac{226}{120} & & & \\
 & 24 & &
\end{array}$$

There is no sign change in the first column of the Routh array, therefore the characteristic equation is stable. In this case, we observe that all the h's are also positive.

In this chapter, we have gone through the procedures of the Routh algorithm and the Routh array, and we have reviewed the Routh theorem.

In the next chapter the scalar Routh theorem will be extended to the matrix Routh theorem.

CHAPTER V

A SUFFICIENT CONDITION FOR THE STABILITY OF A POLYNOMIAL MATRIX

In a single variable system the Routh criterion[16] is applied to the characteristic polynomial of a linear system to determine the absolute stability. In other words, the scalar polynomial in the form of (8) is arranged in the Routh array in (14), then the Routh criterion is applied. If the scalars $C_{i,1}$ in the first column of the Routh array have no sign changes or all elements $C_{i,1}$, $i=1,2,\ldots$ are positive real, then the system is asymptotically stable.

In the same fashion it is interesting to investigate whether a multivariable system whose characteristic polynomial matrix is shown in (8) is asymptotically stable if the block elements $C_{i,1}$, $i=1,2,\ldots$ in the first column of the matrix Routh array in Eq. (4) are positive definite matrices. In other words can the Routh criterion be extended to the matrix Routh criterion? In this research we will partially answer this question.

Consider the differential matrix equation

$$\sum_{i=1}^{n+1} B_i D^{i-1} X(t) = [0], \qquad B_{n+1} = I \tag{46}$$

$$D^{i-1} X(0) = [\alpha_{i-1}] \qquad i=1,2,\ldots,n$$

where X(t) is the m-dimensional state of the system and the B_i are mxm real constant matrices. As discussed in the previous chapters, the corresponding state equation in the companion block form is

$$[\dot{X}] = [B][X]$$

The corresponding state equation in the Schwarz block form is

$$\begin{aligned}[\dot{Z}] &= [K][B][K^{-1}][Z] \\ &= [F][Z]\end{aligned} \tag{47}$$

Since the stability of a system is invariant under the transpose operation of the system matrix, we consider the following system:

$$[\dot{q}] = [F^T][q] \qquad (48)$$
$$= [G][q]$$

The matrix [F] in Eq. (48) is defined in Eq. (47) and the transpose of the matrix [F] is defined as [G]. If the matrix quotients H_i in Eq. (13) are positive definite and symmetric real matrices, then we can consider the following quadratic equation[3,22]:

$$V = [q^T][P][q] \qquad (49)$$

where

$$[P] = \begin{bmatrix} H_n & 0 & . & 0 & 0 \\ 0 & H_{n-1} & . & 0 & 0 \\ . & . & . & . & . \\ 0 & 0 & . & H_2 & 0 \\ 0 & 0 & . & 0 & H_1 \end{bmatrix}$$

The derivative function $\dot{V}$ is

$$\dot{V} = [q^T][PG+G^TP][q] \qquad (50)$$
$$= - [q^T][Q][q]$$

where

$$[P][G] = \begin{bmatrix} 0 & -I & 0 & . & 0 & 0 \\ I & 0 & -I & . & 0 & 0 \\ 0 & I & 0 & . & 0 & 0 \\ . & . & . & . & . & . \\ 0 & 0 & 0 & . & 0 & -I \\ 0 & 0 & 0 & . & I & -I \end{bmatrix}$$

and

$$[Q] = \begin{bmatrix} 0 & 0 & 0 & . & 0 & 0 \\ 0 & 0 & 0 & . & 0 & 0 \\ 0 & 0 & 0 & . & 0 & 0 \\ . & . & . & . & . & . \\ 0 & 0 & 0 & . & 0 & 0 \\ 0 & 0 & 0 & . & 0 & 2I \end{bmatrix}$$

The V function in Eq. (49) is a positive definite quadratic form and the $\dot{V}$ function in Eq. (50) is a negative semi-definite form. Therefore the system in Eq. (47) or Eq. (46) is asymptotically stable and P is a Liapunov function. From the above derivation we obtain a sufficient condition that "a multivariable system, whose characteristic polynomial matrix has the form shown in Eq. (46), is asymptotically stable if the matrix quotients H_i in Eq. (13) are real symmetric positive definite matrices. From Eq. (9) and (16) it can be observed that the $B_n(=H_1^{-1} = C_{21})$ must be symmetric and positive definite for the sufficient condition to apply. It is also noted that, in general, if a system is asymptotically stable, the block elements $C_{i,1}$, i=1,2,..., in the first column of the matrix Routh array and the H_i, i=1,2,..., in the same array are not necessarily symmetric and positive definite matrices. It is seen that the scalar Routh criterion can not, in general, be extended to the matrix case by straightforward replacement of positivity of real numbers by positive definiteness of matrices.

On the other hand, the necessary conditions for an asymptotic stability due to the Routh criterion[16] can be partially extended to the case of matrix polynomials. The necessary conditions are as follows:

1. The determinant of B_1 in Eq. (46) is non zero.
2. The determinants of B_{n+1} and B_1 in Eq. (46) have the same sign if the determinant of $B_{n+1}(=C_{11})$ is non zero.

These conditions can be easily verified from the basic laws of algebra. Thus, in this research we have partially extended the Routh criterion to the matrix Routh criterion for determining the asymptotic stability of a class of matrix polynomials. The following numerical examples are shown to illustrate the procedure for determining the stability of a multivariable system.

Example 1

Consider the differential matrix equation used in Example 2 of Chapter III, and study the stability of the system.

Arrange the matrix Routh array for the given matrix equation.

$$C_{11}\ddddot{X}(t)+C_{21}\dddot{X}(t)+C_{12}\ddot{X}(t)+C_{22}\dot{X}(t)+C_{13}X(t) = [0]$$

$$C_{11}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \quad C_{12}=\begin{pmatrix}-37.05 & -78.8\\ 33 & 65\end{pmatrix} \quad C_{13}=\begin{pmatrix}-10.5 & -2.3\\ -0.1 & -0.6\end{pmatrix}$$

$$H_1=\begin{pmatrix}1 & -1\\ -1 & 2\end{pmatrix} \Big\langle$$

$$C_{21}=\begin{pmatrix}2 & 1\\ 1 & 1\end{pmatrix} \quad C_{22}=\begin{pmatrix}-43.1 & -94.6\\ -6.05 & -16.3\end{pmatrix}$$

$$H_2=\begin{pmatrix}4 & 1\\ 1 & 0.5\end{pmatrix} \Big\langle$$

$$C_{31}=\begin{pmatrix}0 & -0.5\\ 2 & 3\end{pmatrix} \quad C_{32}=\begin{pmatrix}-10.5 & -23\\ -0.1 & -0.6\end{pmatrix}$$

$$H_3=\begin{pmatrix}1.125 & 0.25\\ 0.25 & 0.5\end{pmatrix} \Big\langle$$

$$C_{41}=\begin{pmatrix}-1 & -2\\ 4.5 & 7\end{pmatrix}$$

$$H_4=\begin{pmatrix}0.1 & -0.5\\ -0.5 & 7.5\end{pmatrix} \Big\langle \tag{51}$$

$$C_{51}=\begin{pmatrix}-10.5 & -23\\ -0.1 & -0.6\end{pmatrix}$$

The matrix quotients H_i, $i=1,2,\ldots,4$ in Eq. (51) are positive definite symmetric real matrices. Therefore, the system is asymptotically stable. It is noted that the block elements $C_{i,1}$, $i=1,...,5$ in the first column of the matrix Routh array in Eq. (51) are not all positive definite and/or symmetric real matrices.

It is interesting to note that the characteristic equation $|SI-B|=|SI-F|=0$ and the roots which have negative real parts are:

$$S^8+3S^7+28.95S^6+79.35S^5+206S^4+458.875S^3+221.05S^2$$
$$+48.4S+4 = 0$$

and

$$-0.0239155 \pm j4.27199$$
$$-0.0784809 \pm j2.95637$$
$$-0.189163 \pm j0.165319$$
$$-0.177194$$
$$-2.23969$$

Example 2

Consider the following transfer function matrix $[T(S)]$ which is expressed by the product of two relatively prime polynomial matrices $[N(S)]$ and $[D(S)]^{-1}$ or

$$[T(S)] = [N(S)][D(S)]^{-1}$$

The characteristic polynomial matrix $[D(S)]$ is

$$[D(S)] = B_5S^4+B_4S^3+B_3S^2+B_2S+B_1$$
$$= C_{11}S^4+C_{21}S^3+C_{12}S^2+C_{22}S+C_{13}$$

where

$$C_{11}=B_5=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad C_{12}=B_3=\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}, \quad C_{13}=B_1=\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$C_{21}=B_4=\begin{pmatrix} 2 & 3 \\ 3 & 7 \end{pmatrix}, \quad C_{22}=B_2=\begin{pmatrix} 3 & 4 \\ 4 & 9 \end{pmatrix}$$

It is interesting to determine the stability of this system. Eq. (12b) yields the matrix Routh array as follows:

$$C_{11}=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad C_{12}=\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \quad C_{13}=\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$H_1=\begin{pmatrix} 1.4 & -0.6 \\ -0.6 & 0.4 \end{pmatrix}$$

$$C_{21}=\begin{pmatrix} 2 & 3 \\ 3 & 7 \end{pmatrix} \quad C_{22}=\begin{pmatrix} 3 & 4 \\ 4 & 9 \end{pmatrix}$$

$$H_2=\begin{pmatrix} \frac{3}{2.2} & \frac{4}{2.2} \\ \frac{4}{2.2} & \frac{9}{2.2} \end{pmatrix}$$

$$C_{31}=\begin{pmatrix} 1.2 & -0.2 \\ 0.2 & 1.8 \end{pmatrix} \quad C_{32}=\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$H_3=\begin{pmatrix} 11.6 & -5.4 \\ -5.4 & 4.6 \end{pmatrix}$$

$$C_{41}=\begin{pmatrix} \frac{3}{11} & \frac{4}{11} \\ \frac{4}{11} & \frac{9}{11} \end{pmatrix}$$

$$H_4=\begin{pmatrix} \frac{3}{22} & \frac{4}{22} \\ \frac{4}{22} & \frac{9}{22} \end{pmatrix}$$

$$C_{51}=\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

The matrix quotients H_i, i=1,...,4, in the matrix Routh array are positive definite symmetric real matrices. Therefore the system is stable. It is observed that the block elements $C_{i,1}$, i=1,...,5, in the first column of the matrix Routh array are all positive definite but not symmetric real matrices.

Sometimes, in applying the approach proposed in this research difficulties may be encountered in calculating the matrix quotients H_i as in Eq.(14). This implies that if any $C_{i,1}$ in Eq. (14) is singular, then the H_i cannot be obtained to determine the stability of a matrix polynomial. This limitation can be remedied by multiplying the original matrix polynomial, defined as T(s), by a polynomial matrix defined as E(S), where the roots of the determinant E(S) have negative real parts. Then apply the matrix Routh procedure to the modified matrix polynomial T(S)E(S) or E(S)T(S). It is noted that the stability of the original system is preserved because the stability of a system is invariant under this modification. An alternate way is to perform the transformation $S \to \frac{1}{S}$ to the T(S) and then apply the matrix polynomial defined as $T_1(S)$ $(=T(S)|S \to \frac{1}{S})$. In other words, the $C_{1,j}$ and $C_{2,j}$ in Eq. (12a) are rewritten as follows:

$$C_{1,j} = B_{2j-1} \qquad \text{for} \quad j=1,2,3,...$$

$$C_{2,j} = B_{2j} \qquad \text{for} \quad j=1,2,3,...$$

Again, the stability of the original system is invariant to this modification and the numerically ill-conditioned problem (i.e. $C_{i,1}$ is singular) can be solved. An example is given to illustrate this procedure:

Example 3

Consider that the stability and the structure of the matrix Routh array of the following matrix polynomial T(S) are of interest:

$$T(S) = B_4S^3+B_3S^2+B_2S+B_1$$
$$= C'_{11}S^3+C'_{21}S^2+C'_{12}S+C'_{22} = 0 \qquad (52)$$

where

$$C'_{11} = B_4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad C'_{12} = B_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$C'_{21} = B_3 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \qquad C'_{22} = B_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The determinant of $B_4(=C'_{11})=-1$ and that of $B_1(=C'_{22})=1$. From the derived necessary conditions for asymptotic stability we conclude that the system is unstable because the determinants of B_4 and B_1 have a different sign. Further study of the stability is not necessary. It might be interesting to see the corresponding characteristic equation of this system which can be written as follows:

$$\det T(S) = -S^6-2S^5+3S^2+2S+1 = 0 \qquad (53)$$

Since the first and the last coefficients, which are the determinants of B_4 and B_1 respectively, have different signs, the system is unstable. In order to study the structure of the matrix Routh array of this unstable system and the numerically ill-conditioned problem (i.e. $C_{i,1}$ is singular), we apply the matrix Routh algorithm of Eq. (13) and use the $C'_{i,j}$ in Eq. (52). The matrix Routh array can not be obtained because C'_{21} is singular. This is a numerically ill-conditioned case. Since the stability is invariant between the original system T(S) and the modified system $T_1(S)$ $(=T(S)|\, S \rightarrow \frac{1}{S})$, we can construct the matrix Routh array for $T_1(S)$. $T_1(S)$ can be written as follows:

$$T_1(S) = T(S)\Big|_{S \rightarrow \frac{1}{S}} = C'_{22}S^3 + C'_{12}S^2 + C'_{21}S + C'_{11}$$

$$= C_{11}S^3 + C_{21}S^2 + C_{12}S + C_{22} = 0 \quad (54)$$

where

$$C_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad C_{12} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$C_{21} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad C_{22} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The corresponding matrix Routh array is

$$
\begin{array}{lll}
 & C_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & C_{12} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \\
H_1 = C_{11}C_{21}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & & \\
 & C_{21} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & C_{22} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
H_2 = C_{21}C_{31}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & & \\
 & C_{31} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \\
H_3 = C_{31}C_{41}^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & & \\
 & C_{41} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} &
\end{array}
\tag{43}
$$

Although the C_{12} is singular, we can determine all the H_i's. It is observed that the H_1 and H_2 are symmetric and positive definite matrices, while the H_3 is a symmetric and nonpositive definite matrix.

An alternate method can be described as follows. Construct a new matrix polynomial $T_2(S)$ by multiplying a matrix polynomial $E(S)=(S+1)I$ to the $T(S)$ and then defining the matrix coefficients as $C'_{1,i}$ and $C'_{2,i}$

$$T_2(S) = (S+1)T(S) = C'_{11}S^4 + C'_{21}S^3 + C'_{12}S^2 + C'_{22}S + C'_{13} = 0 \tag{44}$$

where

$$C'_{11} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad C'_{12} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \qquad C'_{13} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$C_{21}' = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \qquad C_{22}' = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

If we like to maintain the consistency of $C_{11}=I$, we may interchange the rows in the $T_2(S)$ and define new matrix coefficients as $C_{1,i}$ and $C_{2,i}$.

$$T_2'(S) = C_{11}S^4+C_{21}S^3+C_{12}S^2+C_{22}S+C_{13}=0$$

where

$$C_{11}= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad C_{12}= \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \qquad C_{13}= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$C_{21} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \qquad C_{22}= \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$$

The corresponding matrix Routh array is

$$\begin{array}{lll}
 & C_{11}= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad C_{12}= \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \quad C_{13}= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
H_1= \frac{1}{3}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} & < & \\
 & C_{21}= \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \quad C_{22}= \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \\
H_2= \frac{1}{7}\begin{pmatrix} 8 & 1 \\ 1 & 8 \end{pmatrix} & < & \\
 & C_{31}= \frac{1}{3}\begin{pmatrix} 5 & 2 \\ 2 & 5 \end{pmatrix} \quad C_{32}= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
H_3= \frac{1}{15}\begin{pmatrix} 17 & 32 \\ 32 & 17 \end{pmatrix} & < & \\
 & C_{41}= \frac{1}{7}\begin{pmatrix} -1 & 6 \\ 6 & -1 \end{pmatrix} \\
H_4= \frac{1}{7}\begin{pmatrix} 6 & -1 \\ -1 & 6 \end{pmatrix} & < & \\
 & C_{51}= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\end{array}$$

No singular matrix appears in the matrix Routh array and all the H_i's can be obtained. It is observed that only the H_3 is a symmetric but non-positive definite matrix.

From the above illustrations we conclude that if any ill conditioned problem occurs in the calculation, then the above methods can be applied to solve the problem.

CHAPTER VI

CONCLUSION

The transformation matrix established by Chen and Chu[9] for transforming the companion form to the Schwarz form has been modified and extended to transform the companion block form to the Schwarz block form. The matrix in the Schwarz block form has been constructed by using the matrix quotients obtained from the matrix Routh array which is constructed from the characteristic polynomial matrix. When the matrix quotients in the matrix Routh array are positive definite and symmetric real matrices, a sufficient condition derived in this research shows that the multivariable system is asymptotically stable. Also, a set of necessary conditions has been derived for the asymptotic stability. Thus, we have partially extended the Routh criterion[16] to the matrix Routh criterion for a class of matrix polynomials. The direct extension of the necessary and sufficient condition of the scalar Routh criterion to the matrix Routh criterion for a general matrix polynomial requires further study.

REFERENCES

[1] H. R. Schwarz, "A Method for Determining Stability of Matrix Differential Equations", Z. Angew. Math. Phys. Vol. 7, pp. 473-500, 1956.

[2] P. C. Parks, "A New Proof of the Hurwitz Stability Criterion by the Second Method of Liapunov With Applications to Optimal Transfer Function", 1963, Joint Automatic Control Conf., ppl. 471-476.

[3] R. E. Kalman and J. E. Bertram, "Control System Analysis and Design via the Second Method of Liapunov", Trans. ASME, Ser. D. pp. 371-393, June 1960.

[4] J. E. Diamessis, "A New Proof for Calculating Systems Performance Measures", Proc. IEEE, Vol. 53, p. 659, June 1964.

[5] M. Wall, Analytic Theory of Continued Fraction, Princeton, N.J., Van Nostrand, 1948.

[6] B.D.O. Anderson, E. I. Jury and M. Mansour, "Schwarz Matrix Properties for Continuous and Discrete Time Systems", Int. J. of Control, Vol. 23, No. 1, pp. 1-16, Jan. 1976.

[7] S. Barnett and C. Storey, Matrix Method in Stability Theory, London, Nelson, 1970.

[8] R. L. Butchart, "An Explicit Solution to the Fokker Planck Equation for an Ordinary Differential Equation", Int. J. of Control, Vol. 1, pp. 201-208, March 1965.

[9] C. F. Chen and H. Chu, "A Matrix for Evaluating Schwarz's Form", IEEE Trans. Automatic Control, Vol. AC-11, pp. 303-305, April 1966.

[10] C. F. Chen and H. Chu, "A General Proof of the P_n matrix and its Inverse," IEEE Trans. Automatic Control, Vol. AC-12, pp. 458-461, Aug. 1967.

[11] S. Barnett and C. Storey, "The Liapunov Matrix Equation and Schwarz's Form", IEEE Trans. Automatic Control, Vol. AC-12, pp. 117-118, Feb. 1967.

[12] S. G. Loo, "A Simplified Proof of a Transformation Matrix Relating the Companion Matrix and the Schwarz Matrix", IEEE Trans. Automatic Control, Vol. 13, pp. 309-310, June 1968.

[13] H. M. Power, "Similarity Transformations Involving the Schwarz Form of a Hessenberg Matrix", IEEE Trans. Automatic Control, Vol. AC-19, pp. 616-617, Oct. 1974.

[14] B. N. Datta, "A Constructive Method for Finding the Schwarz Form of a Hessenberg Matrix", IEEE Trans. Automatic Control, Vol. AC-13, pp. 311-312, June 1968.

[15] I. G. Sarma, M. A. Pai and R. Viswanathan, "On the Transformation to Schwarz Canonical Form", IEEE Trans. Automatic Control, Vol. AC-13, pp. 311-312, June 1968.

[16] E. J. Routh, A Treatise on the Stability of a Given State of Motion, London, 1877.

[17] L. S. Shieh, "On the Inversion of Matrix Routh Array," Int. J. of Control, Vol. 22, No. 6, pp. 861-867, Dec. 1975.

[18] L. S. Shieh and F. F. Gaudiano, "Matrix Continued Fraction Expansion and Inversion by the Generalized Matrix Routh Algorithm," Int. J. of Control, Vol. 20, pp. 727-737, 1974.

[19] L. S. Shieh, H. Z. Chow and R. E. Yates, "Modelling Multivariable Systems with Industrial Specifications," Int. J. of Control 1976 (to be published).

[20] R. J. Schwarz and B. Friedland, Linear Systems, New York, McGraw-Hill Co., 1965.

[21] R. Bellman, Introduction to Matrix Analysis, New York, McGraw-Hill Co., p. 92, 1970.

[22] S. Barnett, Matrices in Control Theory, London, Van Nostrand Reinhold, 1971.

[23] B. C. Kuo, "Automatic Control Systems", Englewood Cliffs, Prentice-Hall, Inc. 1965.