A GENERAL DEFECT RELATION AND HEIGHT INEQUALITY FOR DIVISORS IN SUBGENERAL POSITION

A Dissertation Presented to
the Faculty of the Department of Mathematics
University of Houston

In Partial Fulfillment
of the Requirements for the Degree
Doctor of Philosophy

By
Saud Hussein
August 2016

A GENERAL DEFECT RELATION AND HEIGHT INEQUALITY FOR DIVISORS IN SUBGENERAL POSITION

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DEDICATION

My mom did not get a chance for an education, but made sure her six kids did. I remember once trying to explain to her why I liked math so much, by describing prime numbers. It wasn't working, my mom was smart but did not know how to multiply. Then I had an idea. I handed my mom some dollar bills and asked,

"See if you can give the same number of bills to each person in the room, if you can have as many people as you would like in the room." She immediately started counting the stack of bills and imagined passing out the money.

"If you can only do this by giving each person a dollar, then we call that special number of bills you have a prime number." She got it, explain any problem in terms of money, and my mom was like John Nash.

For mom - (1944-2013)

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ABSTRACT

In this dissertation, we describe a paper [HR16] that improves on the conditions that imply holomorphic curves and integral points are degenerate or not Zariski-dense. Specifically, we show that for a holomorphic curve into a projective variety of dimension n intersecting q divisors in subgeneral position whose sum is equidegreelizable, if q is greater than or equal to n^2, then the curve is degenerate. This is an improvement from $2n^2$ under the same conditions in paper [Ru15a]. To achieve this result, we borrow methods from [SR15] that combine divisors in pairs and uses a joint filtration result from linear algebra. Lastly, a pointwise filtration approach, first considered by Corvaja, Levin, and Zannier [PCZ09], is used to give further improvements such that if q is greater than or equal to $n^2 - n$, then the curve is degenerate. This pointwise filtration may be constructed by using linear algebra on the power series locally representing the sections.

Contents

Chapter 1

Introduction and Background

This chapter describes some of the key results in Diophantine Approximation and Nevenlinna theory that have led to the paper [HR16] discussed in chapters two and three. We begin with a summary of the main results of [HR16] and how they relate to similar work in [Ru15a], [Ru15b], and [MR16].

§1.1 Introduction

Let X be a complex projective variety and let D be an effective Cartier divisor on X. A continuous metric $\|\cdot\|$ on the line bundle associated to D, $\mathcal{O}_X(D)$, determines the *Weil function for D*

$$\lambda_D(x) = -\log \|s_D(x)\|,$$

where s_D is the canonical section of $\mathcal{O}_X(D)$, that is $D = (s_D)$.

Let $f : \mathbb{C} \to X$ be a holomorphic map whose image is not contained in the support of divisor D on X. The *proximity function of f with respect to D* is defined by

$$m_f(r, D) = \int_0^{2\pi} \lambda_D(f(re^{i\theta}))\frac{d\theta}{2\pi},$$

where λ_D is the Weil function associated to D. The *counting function of f with respect to D* is defined by

$$N_f(r, D) = \int_1^r \frac{n_f(t, D)}{t}dt$$

where $n_f(t, D)$ is the number of zeros of $\rho \circ f$ inside $\{|z| < t\}$, counting multiplicities, where ρ is a local defining function of D. The *height* or *characteristic function of f with respect to D* is given by

$$T_{f,D}(r) := m_f(r, D) + N_f(r, D).$$

Let $D_1, \ldots, D_q$ be effective divisors on a projective variety X. We say $D_1, \ldots, D_q$ are in *m-subgeneral position* on X if for any subset $I \subseteq \{1, \ldots, q\}$ with $|I| \leq m + 1$,

$$\dim \bigcap_{i \in I} \text{Supp } D_i \leq m - |I|,$$

where $\dim \emptyset = -1$. In particular, the supports of any $m+1$ divisors in m-subgeneral position have empty intersection. If $m = \dim X$, then we say the divisors are in *general position* on X.

For a divisor D on a projective variety X of dimension n, we denote $h^0(D) := \dim H^0(X, \mathcal{O}_X(D))$. We will use the standard notion of intersection theory (see [Ful98] for a thorough modern account or [Laz04]). The notation D^n denotes the intersection number of the n-fold intersection of D with itself. A divisor D on a projective variety X is said to be *numerically effective*, or *nef*, if $D.C \geq 0$ for any closed integral curve C on X, and D is said to be *big* if there are positive integers c_1, c_2 such that $c_1 N^n \leq \dim H^0(X, \mathcal{O}_X(ND)) \leq c_2 N^n$ for N big enough. For an equivalent definition of big for a nef divisor, we have the following lemma.

Lemma 1.1.1 ([Laz04], Corollary 1.4.41)**.** *Suppose D is a nef Cartier divisor on a projective variety X with $\dim X = n$. Then for any positive integer N,*

$$h^0(ND) = \frac{D^n}{n!} N^n + O(N^{n-1}).$$

In particular, D is big if and only if $D^n > 0$. See Theorem 2.2.16 in [Laz04].

The following definition is from Levin [Lev09].

Definition 1.1.2 ([Lev09], Definition 9.6)**.** Let X be a complex projective variety of dimension n and let $D = D_1 + \cdots + D_q$ be a sum of effective divisors on X. Then D is said to have *equidegree with respect to* $D_1, \ldots, D_q$ if

$$D_i . D^{n-1} = \frac{1}{q} D^n$$

for $1 \leq i \leq q$. We say that D is *equidegreelizable* (with respect to $D_1, \ldots, D_q$) if there exist real numbers $r_i > 0$ such that if $D' = r_1 D_1 + \cdots + r_q D_q$, then D' has equidegree with respect to $r_1 D_1, \ldots, r_q D_q$ (where we extend intersections to $\mathbb{R}$-divisors in the canonical way).

Note that, in general, divisors rD and D have the same support for any divisor D and real number r. Our main use of equidegree depends on the following lemma.

Lemma 1.1.3 ([Lev09], Lemma 9.7)**.** *Let X be a projective variety of dimension n.*

If $D_j, 1 \leq j \leq q$, are big and nef, then $\sum_{j=1}^{q} D_j$ is equidegreelizable with respect to $D_1, \ldots, D_q$.

The following two theorems for divisors in subgeneral position on a complex projective variety make use of the concept of equidegree.

Theorem A ([Ru15a], Theorem 5.6)**.** *Let X be a complex projective variety of dimension $n \geq 2$ and $D = D_1 + \cdots + D_q$ a sum of big and nef Cartier divisors, in m-subgeneral position on X. Let $r_i > 0$ be real numbers such that $D' := \sum_{i=1}^{q} r_i D_i$ has equidegree (such numbers exist due to Lemma 1.1.3). We further assume there exists a positive integer N_0 such that the linear system $|ND_i|$ ($i = 1, \ldots, q$) is base-point free for $N \geq N_0$. Let $f : \mathbb{C} \to X$ be a Zariski dense holomorphic map. Then, for $\epsilon > 0$ small enough,*

$$\sum_{j=1}^{q} r_j m_f(r, D_j) \leq \left(\frac{2mn}{q} - \epsilon\right) \left(\sum_{j=1}^{q} r_j T_{f,D_j}(r)\right) \|_E,$$

where $\|_E$ means the inequality holds for all $r > 0$ except for a possible set E with finite Lebesgue measure.

Under the additional assumption that divisors are without irreducible common components, Charles Mills and Min Ru [MR16] gave the following result.

Theorem B ([MR16], Analytic Main Theorem)**.** *Let X be a complex projective variety of dimension $n \geq 2$ and $D_1, \ldots, D_q$ big and nef Cartier divisors in m-subgeneral position on X. Let $r_i > 0$ be real numbers such that $D := r_1 D_1 + \cdots + r_q D_q$ has equidegree (such real numbers exist due to Lemma 1.1.3). Assume* *there exists a positive integer N_0 such that the linear system $|ND_i|$ $(i = 1, \dots, q)$ is base-point free for $N \geq N_0$. We further assume $D_1, \dots, D_q$ have no irreducible components in common. Let $f : \mathbb{C} \to X$ be a Zariski dense holomorphic map. Then*

$$\sum_{j=1}^{q} r_j m_f(r, D_j) \leq \left(\frac{[(m+1)/2]}{(1+\alpha)} \frac{2n}{q} \right) \left(\sum_{j=1}^{q} r_j T_{f,D_j}(r) \right) \|_E,$$

where $[x]$ denotes the greatest integer $\leq x$ and $\alpha > 0$ is a constant.

In chapter two, using methods borrowed from [SR15], we prove the following theorem.

Main Theorem. (Analytic Part) *Let X be a complex projective variety of dimension $n \geq 2$ and $D_1, \dots, D_q$ big and nef Cartier divisors in m-subgeneral position on X. Let $r_i > 0$ be real numbers such that $D := r_1 D_1 + \cdots + r_q D_q$ has equidegree (such real numbers exist due to Lemma 1.1.3). Assume there exists a positive integer N_0 such that the linear system $|ND_i|$ $(i = 1, \dots, q)$ is base-point free for $N \geq N_0$. Let $f : \mathbb{C} \to X$ be a Zariski dense holomorphic map. Then*

$$m_f(r, D) < \frac{m(m-1)}{(m+n-2)} \frac{2n}{q} \frac{1}{(1+C)} T_{f,D}(r)$$

holds for all $r > 0$ outside a set of finite Lebesgue measure, where

$$C = \frac{\min_{1 \leq j \leq q} \{D^{n-2}.(r_j D_j)^2\}}{6qn^2 4^n D^n} > 0.$$

Let $f : \mathbb{C} \to X$ be a holomorphic map and D a divisor on X. The *Nevanlinna defect of f with respect to D* is

$$\delta_f(D) := \liminf_{r \to \infty} \frac{m_f(r, D)}{T_{f,D}(r)}.$$

The defect measures the extent to which the counting function of f $N_f(r, D)$ is smaller than the maximum indicated by the First Main Theorem $T_{f,D}(r) = m_f(r, D) + N_f(r, D) + O(1)$.

Corollary 1.1.4 (Defect Relation)**.** *Assume the conditions of the Main Theorem. The defect relation is then*

$$\delta_f(D) \leq \frac{m(m-1)}{(m+n-2)} \frac{2n}{q} \frac{1}{(1+C)}.$$

If $\delta_f(D) < 1$, then every holomorphic map $f : \mathbb{C} \to X \backslash D$ is not Zariski-dense.

Corollary 1.1.5. *Let X be a complex projective variety of dimension $n \geq 2$ and $D = D_1 + \cdots + D_q$ a sum of big and nef Cartier divisors in general position on X. If $q \geq n^2$, then every holomorphic mapping $f : \mathbb{C} \to X \backslash D$ must be constant.*

Proof. By the First Main theorem, $\delta_f(D) = 1$. Now assume f is also Zariski-dense and $q \geq n^2$. Then by the Main Theorem defect relation, since $m = n$ and $C > 0$, $\delta_f(D) < \dfrac{n^2}{q} \leq 1$, a contradiction, and so f is not Zariski-dense. Therefore, $\overline{f(\mathbb{C})} = Y \subseteq X$ for some proper subvariety Y of X with $\dim Y = k < n$. Since $D_1, \ldots, D_q$ are in general position on X and $D_i \cap Y = \emptyset$, $1 \leq i \leq q$, $D_1, \ldots, D_q$ are in n-subgeneral position on Y. So repeating the steps on the holomorphic map $f : \mathbb{C} \to Y$, $\dim \overline{f(\mathbb{C})} < k$, and by induction $\dim \overline{f(\mathbb{C})} = 0$. Thus $\overline{f(\mathbb{C})} = \{x\}$, i.e., f is constant. □

Theorem A, Theorem B, and the Main Theorem above have counterparts in Diophantine Approximation, so we recall some standard notations in Diophantine

Approximation. Let k be a number field and let $\mathcal{O}_k$ be the ring of integers of k. We have a set M_k of absolute values, or places, of k consisting of one place for each nonzero prime ideal $\mathfrak{p}$ of $\mathcal{O}_k$, one place for each real embedding $\sigma : k \to \mathbb{R}$, and one place for each pair of conjugate embeddings $\sigma, \overline{\sigma} : k \to \mathbb{C}$. The completion of k with respect to v is denoted by k_v. We normalize our absolute values so that

$$\|p\|_v = p^{-[k_v:\mathbb{Q}_p]/[k:\mathbb{Q}]}$$

if v corresponds to $\mathfrak{p}$ and $\mathfrak{p}|p$,

$$\|x\|_v = |\sigma(x)|^{1/[k:\mathbb{Q}]}$$

if v corresponds to the real embedding σ, and

$$\|x\|_v = |\sigma(x)|^{2/[k:\mathbb{Q}]}$$

if v corresponds to the pair of conjugate embeddings $\sigma, \overline{\sigma}$. Let D be a Cartier divisor on a projective variety X, both defined over a number field k. Let $v \in M_k$. We denote by $\lambda_{D,v}$ the Weil function for D relative to v.

Theorem C ([Ru15b], Theorem 4.1)**.** *Let k be a number field and $S \subseteq M_k$ a finite set containing all archimedean places. Let X be a projective variety of dimension $n \geq 2$ and $D = D_1, \ldots, D_q$ a sum of big and nef Cartier divisors in m-subgeneral position on X, both defined over k. Let $r_i > 0$ be real numbers such that $D' := r_1 D_1 + \cdots + r_q D_q$ is equidegree (such numbers exist due to Lemma 1.1.3). We further assume there exists a positive integer N_0 such that the linear system $|ND|$ is base-point free for $N \geq N_0$. Then, for $\epsilon > 0$ small enough,*

$$\sum_{j=1}^{q} r_j m_S(P, D_j) \leq \left(\frac{2mn}{q} - \epsilon\right)\left(\sum_{j=1}^{q} r_j h_{D_j}(P)\right)$$

holds for all $P \in X(k)$ outside a proper Zariski closed subset of X.

Theorem D ([MR16], Arithmetic Main Theorem)**.** *Let k be a number field and $S \subseteq M_k$ a finite set containing all archimedean places. Let X be a projective variety, defined over k, of dimension $n \geq 2$, and let $D_1, \ldots, D_q$ be big and nef Cartier divisors on X, defined over k. Let $r_i > 0$ be real numbers such that $D := r_1 D_1 + \cdots + r_q D_q$ has equidegree (such real numbers exist due to Lemma 1.1.3). Assume there exists a positive integer N_0 such that the linear system $|ND_i|$ $(i = 1, \ldots, q)$ is base-point free for $N \geq N_0$ and that $D_1, \ldots, D_q$ are in m-subgeneral position on X. We further assume $D_1, \ldots, D_q$ have no irreducible components in common. Then*

$$\sum_{j=1}^{q} r_j m_S(P, D_j) \leq \left(\frac{[(m+1)/2]}{(1+\alpha)} \frac{2n}{q} \right) \left(\sum_{j=1}^{q} r_j h_{D_j}(P) \right),$$

holds for all $P \in X(k)$ outside a proper Zariski closed subset of X, where $\alpha > 0$ is a constant.

Main Theorem. (Arithmetic Part) *Let k be a number field and $S \subseteq M_k$ a finite set containing all archimedean places. Let X be a projective variety, defined over k, of dimension $n \geq 2$, and let $D_1, \ldots, D_q$ be big and nef Cartier divisors on X, defined over k. Let $r_i > 0$ be real numbers such that $D := r_1 D_1 + \cdots + r_q D_q$ has equidegree (such real numbers exist due to Lemma 1.1.3). Assume there exists a positive integer N_0 such that the linear system $|ND_i|$ $(i = 1, \ldots, q)$ is base-point free for $N \geq N_0$ and that $D_1, \ldots, D_q$ are in m-subgeneral position on X. Then*

$$m_S(P, D) < \frac{m(m-1)}{(m+n-2)} \frac{2n}{q} \frac{1}{(1+C)} h_D(P)$$

holds for all $P \in X(k)$ outside a proper Zariski closed subset of X, where

$$C = \frac{\min_{1 \le j \le q} \{D^{n-2}.(r_j D_j)^2\}}{6qn^2 4^n D^n} > 0.$$

To compare Theorem A, Theorem B, and the Analytic Main Theorem,

$$\delta_f(D) < \begin{cases} \frac{2mn}{q}, & \text{Theorem A [Ru15a]} \\ \frac{m(m-1)}{(m+n-2)} \frac{2n}{q}, & \text{Main Theorem [HR16]} \\ \left[\frac{m+1}{2}\right] \frac{2n}{q} = \begin{cases} \frac{(m+1)n}{q}, & \text{odd } m \\ \frac{mn}{q}, & \text{even } m \end{cases}, & \text{Theorem B [MR16].} \end{cases}$$

Since $\frac{(m-1)}{(m+n-2)} < 1$ for $2 \le n \le m$, [HR16] gives a sharper result than [Ru15a]. The even case for [MR16] gives a sharper result than [HR16] and [Ru15a], as expected, since Theorem B has an extra condition on the divisors. The odd case depends on the specific values.

If we assume $D_1, \ldots, D_q$ are in general position on X, then $m = n$, and so

$$\delta_f(D) < \begin{cases} \frac{2n^2}{q}, & \text{Theorem A [Ru15a]} \\ \frac{n^2}{q}, & \text{Main Theorem [HR16]} \\ \left[\frac{n+1}{2}\right]\frac{2n}{q} = \begin{cases} \frac{(n+1)n}{q}, & \text{odd } n \\ \frac{n^2}{q}, & \text{even } n \end{cases}, & \text{Theorem B [MR16].} \end{cases}$$

Thus [HR16] gives a sharper result or as good as [Ru15a] and [MR16] in the general position case.

§1.2 Background Material

As first noticed by Charles Freeman Osgood in 1981 and then further developed by Paul Vojta in 1987, there is a formal analogy between Nevanlinna theory in complex analysis and certain results in Diophantine Approximation in number theory. The correspondence can be described in both a qualitative and quantitative way. As a simple example of a qualitative correspondence, Siegel's Theorem and the Little Picard theorem will first be described in their respective section.

§1.2.1 Diophantine Approximation

In number theory, the field of Diophantine Approximation, named after Diophantus of Alexandria, deals with the approximation of real numbers by rational numbers.

Let α be an algebraic number of degree d over $\mathbb{Q}$ and let $\epsilon > 0$. In 1921, Carl Ludwig Siegel proved there are only finitely many pairs of relatively prime integers a and b ($b > 0$) satisfying the inequality

$$\left|\alpha - \frac{a}{b}\right| < \frac{1}{b^{2\sqrt{d}+1+\epsilon}}.$$

Using this result, Siegel in 1929 proved the following.

Theorem 1.2.1 (Siegel's Theorem)**.** *Let C be a smooth algebraic curve of genus one defined over a number field k. Then all sets of integral points on C are finite.*

In 1955, Klaus Roth [Rot55] improved on Siegel's inequality, resulting in the best possible approximation by rationals of this form. That is, Roth's theorem fails when

$\epsilon = 0$ by Gustav Dirichlet's approximation theorem (1840).

Theorem 1.2.2 (Roth's Theorem)**.** *Let α be an algebraic number over $\mathbb{Q}$ and let $\epsilon > 0$. Then there are only finitely many pairs of relatively prime integers a and b $(b > 0)$ satisfying the inequality*

$$\left|\alpha - \frac{a}{b}\right| < \frac{1}{b^{2+\epsilon}}.$$

By taking the log of both sides, the statement may be restated as

$$\log \frac{1}{\alpha - \frac{a}{b}} < (2+\epsilon)\log b$$

holds for all but finitely many pairs of relatively prime integers a and b $(b > 0)$. Roth's theorem can be generalized to any number field k.

Theorem 1.2.3 ([Ru01] Theorem B1.2.7, Roth)**.** *Let k be a number field with extension degree $d = [k : \mathbb{Q}]$ and let $S \subseteq M_k$ be a finite set containing all archimedean places. Let $a_1, \ldots, a_q \in k$ be distinct. Then, for any $\epsilon > 0$,*

$$\sum_{j=1}^{q} m_S(x, a_j) \leq (2+\epsilon)h(x)$$

holds for all but finitely many $x \in k$, where

$$h(x) = \frac{1}{d}\sum_{v \in M_k} \log^+ \|x\|_v$$

and

$$m_S(x, a_j) = \frac{1}{d}\sum_{v \in S} \log^+ \frac{1}{\|x - a_j\|_v}.$$

In 1972, Wolfgang Schmidt gave a higher-dimensional generalization of Roth's theorem. Let k be a number field with extension degree $d = [k : \mathbb{Q}]$. The *logarithmic height* of $x = [x_0 : \cdots : x_n] \in \mathbb{P}^n(k)$ is defined by

$$h(x) = \frac{1}{d} \sum_{v \in M_k} \log^+ \|x\|_v = \frac{1}{d} \sum_{v \in M_k} \log^+ \left(\max_{0 \le i \le n} \{\|x_i\|_v\} \right).$$

Let $H = \{[x_0 : \cdots : x_n] \in \mathbb{P}^n(k) \mid a_0x_0 + \cdots + a_nx_n = 0\}$ be the projective hyperplane defined by the coefficient vector $a = (a_0, \ldots, a_n) \in k^{n+1}$. On $x \in \mathbb{P}^n(k) \backslash H$, the *Weil function* for H relative to $v \in S$ is defined by

$$\lambda_{H,v}(x) = \frac{1}{d} \log \frac{(n+1)\|a\|_v \|x\|_v}{\|a_0x_0 + \cdots + a_nx_n\|_v}.$$

Using these definitions, the following is Schmidt's subspace theorem, as generalized by Paul Vojta in 1997.

Theorem 1.2.4 ([Voj97] Schmidt's Subspace Theorem)**.** *Let k be a number field and let $S \subseteq M_k$ be a finite set containing all archimedean places. Let $H_1, \ldots, H_q$ be hyperplanes in $\mathbb{P}^n(k)$ with corresponding Weil functions $\lambda_{H_1,v}, \ldots, \lambda_{H_q,v}$ for each $v \in S$. Then there exists a finite union of hyperplanes $Z \subseteq \mathbb{P}^n(k)$, depending only on $H_1, \ldots, H_q$ (and not k or S), such that for any $\epsilon > 0$,*

$$\sum_{v \in S} \max_J \sum_{j \in J} \lambda_{H_j,v}(x) \le (n+1+\epsilon)h(x)$$

holds for all $x \in \mathbb{P}^n(k) \backslash Z$, where the max is taken over all subsets $J \subseteq \{1, \ldots, q\}$ such that the hyperplanes H_j, $j \in J$, are in general position on $\mathbb{P}^n(k)$.

In 2002, Pietro Corvaja and Umberto Zannier gave a new proof [CZ02] of Siegel's theorem using Schmidt's Subspace Theorem. Corvaja and Zannier further developed their method to higher dimensions in 2004 ([CZ04a], [CZ04b]). To describe one application of their method, it is convenient to borrow a definition from Aaron Levin [Lev09]. Let X be a smooth projective variety over a number field k and let D be a divisor on X. Also, let $\overline{k}(X)$ denote the function field of X over $\overline{k}$ and let $L(D)$ be the $\overline{k}$-vector space $L(D) = \{f \in \overline{k}(X) \mid (f) \geq -D\}$.

Definition 1.2.5 ([Lev09] Definition 8.1)**.** Let D be an effective divisor on a smooth projective variety X defined over a number field k. Then D is called a *very large divisor* on X if for every $P \in D$, there exists a basis B of $L(D)$ such that

$$\sum_{f \in B} \mathrm{ord}_E(f) > 0$$

for every irreducible component E of D passing through P. An effective divisor D is called *large* if it has the same support as some very large divisor on X.

Theorem 1.2.6 ([Lev09] Theorem 8.3A, Corvaja-Zannier)**.** *Let X be a smooth projective variety over a number field k and let $S \subseteq M_k$ be a finite set containing all archimedean places. Let D be a large divisor on X. Then any set of (D, S)-integral points on X is not Zariski-dense.*

Corvaja and Zannier also proved in 2004 an extension of Schmidt's Subspace Theorem with polynomials of arbitrary degree instead of linear forms. Their result states that the set of solutions in $\mathbb{P}^n(k)$ (k a number field) of the inequality being considered is not Zariski-dense ([CZ04a] Theorem 3).

Jan-Hendrik Evertse and Roberto Ferretti in 2008 [EF08] generalized the results of Corvaja and Zannier in which the solutions are taken from an arbitrary projective variety instead of $\mathbb{P}^n$. By a projective variety of $\mathbb{P}^n$, we mean a geometrically irreducible Zariski-closed subset of $\mathbb{P}^n$. The following theorem is a slight reformulation of their main result by Levin in 2014.

Theorem 1.2.7 ([Lev14] Theorem 3.1, Evertse-Ferretti)**.** *Let X be a projective variety of dimension n and let $D_1, \ldots, D_q$ be Cartier divisors in general position on X, all defined over a number field k. Let $S \subseteq M_k$ be a finite set containing all archimedean places. Assume there exist an ample divisor A on X, defined over k, and positive integers d_i such that $D_i \sim d_i A$ for all i. Then, for every $\epsilon > 0$,*

$$\sum_{j=1}^{q} \frac{m_S(P, D_j)}{d_j} \leq (n + 1 + \epsilon) h_A(P)$$

holds for all k-rational points $P \in X$ outside a proper Zariski closed subset of X.

Levin in his 2014 paper also proves Theorem 1.2.7 remains true if we replace linear equivalence by numerical equivalence ([Lev14] Theorem 3.2).

§1.2.2 Nevanlinna Theory

In the field of complex analysis, Nevanlinna theory is part of the theory of meromorphic functions. Developed by brothers Rolf and Frithiof Nevanlinna in the 1920s, it deals with the distribution of values of holomorphic and meromorphic functions. We can think of the original Nevanlinna theory as a generalization of Emile Picard's classic Little Picard Theorem.

Theorem 1.2.8 (Little Picard Theorem)**.** *Let $f : \mathbb{C} \to \mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$ be a meromorphic function. If the image of f omits three distinct points in $\mathbb{P}^1(\mathbb{C})$, then f must be constant.*

Remark 1.2.9. Siegel's theorem 1.2.1 may be used to state a result similar to the Little Picard's theorem but in the context of a number field.

To state Nevanlinna's results for a meromorphic function $f : \mathbb{C} \to \mathbb{C}$, we first need to define three functions.

Definition 1.2.10. Let $f : \mathbb{C} \to \mathbb{C}$ be a meromorphic function. The *proximity function* of f is defined by

$$m_f(r) = \int_0^{2\pi} \log^+ |f(re^{i\theta})| \frac{d\theta}{2\pi}$$

for all $r > 0$. Also, define

$$m_f(r, \infty) = m_f(r) \quad \text{and} \quad m_f(r, a) = m_{\frac{1}{f-a}}(r)$$

when $a \in \mathbb{C}$.

For $r > 0$, let $n_f(r)$ be the number of poles of f in the open disc $|z| < r$ of radius r, counted with multiplicity, and let $n_f(0)$ be the order of the pole at $z = 0$.

Definition 1.2.11. Let $f : \mathbb{C} \to \mathbb{C}$ be a meromorphic function. The *counting function* of f is defined by

$$N_f(r) = \int_0^r (n_f(t) - n_f(0)) \frac{dt}{t} + n_f(0) \log r.$$

Also, define

$$N_f(r, \infty) = N_f(r) \quad \text{and} \quad N_f(r, a) = N_{\frac{1}{f-a}}(r)$$

when $a \in \mathbb{C}$.

Now, using the previous two definitions, we have the last of the definitions needed.

Definition 1.2.12. Let $f : \mathbb{C} \to \mathbb{C}$ be a meromorphic function. The *characteristic (height) function* of f is the function $T_f : (0, \infty) \to \mathbb{R}$ defined by

$$T_f(r) = m_f(r) + N_f(r).$$

Theorem 1.2.13 (Nevanlinna's First Main Theorem)**.** *Let $f : \mathbb{C} \to \mathbb{C}$ be a non-constant meromorphic function and let $a \in \mathbb{C}$. Then*

$$T_f(r) = m_f(r, a) + N_f(r, a) + O(1),$$

where the constant $O(1)$ depends only on f and a.

Remark 1.2.14. The First Main Theorem gives an upper bound on the counting function N_f and can be thought of as a generalization of the fundamental theorem of algebra.

Theorem 1.2.15 (Nevanlinna's Second Main Theorem)**.** *Let $f : \mathbb{C} \to \mathbb{P}^1(\mathbb{C})$ be a non-constant meromorphic function and let $a_1, \ldots, a_q \in \mathbb{P}^1(\mathbb{C})$ be distinct. Then, for every $\epsilon > 0$,*

$$\sum_{j=1}^{q} m_f(r, a_j) \leq (2 + \epsilon) T_f(r)$$

holds for all $r > 0$ outside a set of finite Lebesgue measure.

Remark 1.2.16. Nevanlinna's Second Main Theorem corresponds to Roth's theorem 1.2.3 and may be used to prove the Little Picard theorem.

Nevanlinna's First and Second Main theorems give us quantitative descriptions of the theory while the Little Picard theorem is an example of a qualitative view of the same basic theory. Since most of the time in applications, qualitative results suffice, the following definition is convenient.

Definition 1.2.17. Let $f : \mathbb{C} \to \mathbb{P}^1(\mathbb{C})$ be a meromorphic function and let $a \in \mathbb{C} \cup \{\infty\}$. The *defect* of a is defined by

$$\delta_f(a) = \liminf_{r \to \infty} \frac{m_f(r, a)}{T_f(r)}.$$

By the First Main Theorem, $0 \leq \delta_f(a) \leq 1$ for every $a \in \mathbb{C} \cup \{\infty\}$, so by the Second Main theorem,

$$\sum_{a \in \mathbb{C}} \delta_f(a) \leq 2.$$

In 1933, Henri Cartan [Car33] gave a higher dimensional generalization of Nevanlinna's Second Main theorem. To state this theorem, we need to define the Nevanlinna functions with respect to a holomorphic curve $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ and a hyperplane $H \subseteq \mathbb{P}^n(\mathbb{C})$. So let $H = \{[z_0 : \cdots : z_n] \in \mathbb{P}^n(\mathbb{C}) \mid a_0 z_0 + \cdots + a_n z_n = 0\}$ be the projective hyperplane defined by the coefficient vector $a = (a_0, \ldots, a_n) \in \mathbb{C}^{n+1}$ and let $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic curve with $f(\mathbb{C}) \not\subseteq H$. For $z \in \mathbb{C}$, the *Weil function of f with respect to H* is defined by

$$\lambda_H(f(z)) = \log \frac{\|f(z)\| \, \|a\|}{|\langle f(z), a \rangle|}.$$

The *proximity function of f with respect to H* is defined by

$$m_f(r, H) = \int_0^{2\pi} \lambda_H(f(re^{i\theta})) \frac{d\theta}{2\pi}.$$

For $r > 0$, let $n_f(r, H)$ be the number of zeroes of $\langle f(z), a \rangle$ in the open disc $|z| < r$ of radius r, counted with multiplicity, and let $n_f(0, H) = \lim_{t \to \infty} n_f(t, H)$. Then the *counting function of f with respect to H* is defined by

$$N_f(r, H) = \int_0^r (n_f(t, H) - n_f(0, H)) \frac{dt}{t} + n_f(0, H) \log r.$$

So, the *height of f with respect to H* is defined as $T_{f,H}(r) = m_f(r, H) + N_f(r, H)$. The First Main Theorem may be shown to hold for hyperplanes in projective space, so the height of f, denoted by $T_f(r)$, depends on a hyperplane only up to a constant $O(1)$. Finally, with these definitions, the following is Cartan's Second Main Theorem, as generalized by Paul Vojta in 1997.

Theorem 1.2.18 ([Voj97] Theorem 1, Cartan's Second Main Theorem). *Let $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be a linearly non-degenerate holomorphic curve (i.e., the image of f is not contained in any proper subspace of $\mathbb{P}^n(\mathbb{C})$) and let $H_1, \ldots, H_q$ be hyperplanes in $\mathbb{P}^n(\mathbb{C})$. Then, for every $\epsilon > 0$,*

$$\int_0^{2\pi} \max_J \sum_{j \in J} \lambda_{H_j}(f(re^{i\theta})) \frac{d\theta}{2\pi} \leq (n + 1 + \epsilon) T_f(r)$$

holds for all $r > 0$ outside a set of finite Lebesgue measure, where the max is taken over all subsets $J \subseteq \{1, \dots, q\}$ such that the hyperplanes H_j, $j \in J$, are in general position on $\mathbb{P}^n(\mathbb{C})$.

Remark 1.2.19. Cartan's Second Main Theorem corresponds to Schmidt's Subspace Theorem 1.2.4.

In 2004, Min Ru generalized Cartan's Second Main Theorem to non-linear hypersurfaces. To state this result, we need the Nevanlinna functions with respect to a holomorphic curve $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ and a hypersurface $D \subseteq \mathbb{P}^n(\mathbb{C})$. So let $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic curve and without loss of generality, assume its set of entire component functions $f_0, \dots, f_n$ have no common zeros. For $z = re^{i\theta} \in \mathbb{C}$, the *height function of f* is defined by

$$T_f(r) = \frac{1}{2\pi} \int_0^{2\pi} \log \left\| f(re^{i\theta}) \right\| d\theta,$$

where

$$\|f(z)\| = \max\{|f_0(z)|, \dots, |f_n(z)|\}.$$

Let $D \subseteq \mathbb{P}^n(\mathbb{C})$ be a hypersurface of degree d defined by a homogeneous polynomial $Q : \mathbb{P}^n(\mathbb{C}) \to \mathbb{C}$. The *proximity function of f with respect to D* is defined by

$$m_f(r, D) = \int_0^{2\pi} \log \frac{\left\| f(re^{i\theta}) \right\|^d}{|Q(f)(re^{i\theta})|} \frac{d\theta}{2\pi}.$$

Theorem 1.2.20 ([Ru04] Main Theorem). *Let $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be an algebraically non-degenerate holomorphic curve and let $D_1, \dots, D_q$ be hypersurfaces in $\mathbb{P}^n(\mathbb{C})$ of* *degree $d_1, \ldots, d_q$ in general position. Then, for every $\epsilon > 0$,*

$$\sum_{j=1}^{q} \frac{m_f(r, D_j)}{d_j} \leq (n+1+\epsilon)T_f(r)$$

holds for all $r > 0$ outside a set of finite Lebesgue measure.

Let f and D be as specified in the theorem. Define the defect

$$\delta_f(D) = \liminf_{r\to\infty} \frac{m_f(r, D)}{dT_f(r)}.$$

Then the theorem gives us the *defect relation*

$$\sum_{j=1}^{q} \delta_f(D_j) \leq n+1$$

where we assume all hypersurfaces have the same degree d.

Remark 1.2.21. The above defect relation proves a conjecture made by Bernard Shiffman in 1979 and corresponds to the work of Corvaja and Zannier ([CZ04a] Theorem 3). Phillip Griffiths conjectures the following sharp defect relation holds in this setting,

$$\sum_{j=1}^{q} \delta_f(D_j) \leq \frac{n+1}{d},$$

where we assume all hypersurfaces have the same degree d.

In 2009, Ru further generalized Theorem 1.2.20, giving a defect relation for algebraically non-degenerate holomorphic mappings into an arbitrary smooth complex projective variety, rather than just the projective space, intersecting possible nonlinear hypersurfaces.

Theorem 1.2.22 ([Ru09] Main Result Theorem)**.** *Let $V \subseteq \mathbb{P}^N(\mathbb{C})$ be a smooth complex projective variety of dimension n and let $D_1, \ldots, D_q$ be hypersurfaces in $\mathbb{P}^N(\mathbb{C})$ of degree $d_1, \ldots, d_q$ in general position in V. Let $f : \mathbb{C} \to V$ be an algebraically non-degenerate holomorphic map. Then, for every $\epsilon > 0$,*

$$\sum_{j=1}^{q} \frac{m_f(r, D_j)}{d_j} \leq (n + 1 + \epsilon) T_f(r)$$

holds for all $r > 0$ outside a set of finite Lebesgue measure.

Remark 1.2.23*.* The above result corresponds to Evertse-Ferretti Theorem 1.2.7.

§1.2.3 Complex Geometry

In this section, we gather some definitions and examples in complex analysis and geometry useful to Diophantine Approximation and Nevanlinna Theory.

Let X be a complex manifold.

Definition 1.2.24. A *holomorphic line bundle over* X is a complex manifold L together with a surjective holomorphic map $\pi : L \to X$ satisfying the following conditions:

(i) For each $x \in X$, the fiber $L_x = \pi^{-1}(x)$ over x is endowed with the structure of a one-dimensional complex vector space.

(ii) There exists an open covering $\{U_i\}$ of X and biholomorphic maps

$$\phi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C}$$

called *local trivializations of* L *over* U_i satisfying the following conditions:

(a) $\pi_{U_i} \circ \phi_i = \pi$ where $\pi_{U_i} : U_i \times \mathbb{C} \to U_i$ is the projection map;

(b) for each $q \in U_i$, the restriction of ϕ_i to L_q is a vector space isomorphism from L_q to $\{q\} \times \mathbb{C} \cong \mathbb{C}$.

Definition 1.2.25. Let $\pi : L \to X$ be a holomorphic line bundle over X and let $\{U_i\}$ be an open covering of X. Suppose $\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}$ and $\phi_\beta : \pi^{-1}(U_\beta) \to U_\beta \times \mathbb{C}$ are local trivializations of L with $U_\alpha \cap U_\beta \neq \emptyset$. Then the composition map

$$\phi_\alpha \circ \phi_\beta^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{C} \to (U_\alpha \cap U_\beta) \times \mathbb{C}$$

given by $\phi_\alpha \circ \phi_\beta^{-1}(x,z) = (x, g_{\alpha\beta}(x)z)$ induces a non-vanishing holomorphic function $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{C}^*$ called a *transition function* of L.

The transition function system $\{g_{\alpha\beta}\}$ clearly satisfies the following cocycle conditions:

(i) $g_{\alpha\alpha}(x) = 1$ for every $x \in U_\alpha$

(ii) $g_{\alpha\beta} g_{\beta\gamma} = g_{\alpha\gamma}$ on $U_\alpha \cap U_\beta \cap U_\gamma$

Conversely, given a system of non-vanishing holomorphic functions $\{g_{\alpha\beta}\}$ satisfying the cocycle conditions, $\{U_i, g_{\alpha\beta}\}$ represents a holomorphic line bundle over X, where $\{U_i\}$ is an open covering of X. Explicitly, let $L = (\bigcup_i U_i \times \mathbb{C})/\sim$, where $\sim$ is the equivalence relation defined by $(x_\alpha, z_\alpha) \sim (x_\beta, z_\beta) \iff x_\alpha = x_\beta,\ z_\alpha = g_{\alpha\beta}(x) z_\beta$ on $U_\alpha \cap U_\beta$. Then the holomorphic map $\pi : L \to X$ defined by $L([x,z]) = x$ is clearly surjective and so is a holomorphic line bundle over X.

Example 1.2.26. The projection map $p_1 : X \times \mathbb{C} \to X$ is clearly a holomorphic line bundle. This is called the *trivial line bundle*, denoted by $\mathcal{O}_X$.

The set of all holomorphic line bundles over X forms a group.

Definition 1.2.27. Let $\pi_1 : L_1 \to X$ and $\pi_2 : L_2 \to X$ be holomorphic line bundles over X. A biholomorphic map $f : L_1 \to L_2$ is called a *line bundle isomorphism* and L_1 is said to be isomorphic to L_2 if $\pi_1 = \pi_2 \circ f$ and the restriction of f to each fiber is linear. The set of all such isomorphism classes of holomorphic line bundles over X forms an abelian group with the group operation the tensor product $\otimes$ and is called the *Picard group of* X, denoted by $\mathrm{Pic}(X)$. Line bundles isomorphic to the trivial line bundle $\mathcal{O}_X$ is the zero element of this group.

Definition 1.2.28. Let $\pi : L \to X$ be a holomorphic line bundle over X. A *holomorphic section of* L is a holomorphic map $s : X \to L$ such that $\pi \circ s = Id_X$. This means $s(x)$ is an element of the fiber $L_x = \pi^{-1}(x)$ for each $x \in X$. If a section is defined only on an open subset $U \subseteq X$, then it is called a *holomorphic local section of* L *over* U. The *zero section of* L is the holomorphic section of L defined by $s(x) = 0 \in L_x$ for each $x \in X$.

The set of all holomorphic sections of L forms a complex vector space denoted by $H^0(X, L)$. Let $s \in H^0(X, L)$ and let $\{U_i, g_{\alpha\beta}\}$ represent the holomorphic line bundle L. Then there exists a set of holomorphic functions $\{s_i\}$ such that $s_\alpha = g_{\alpha\beta}s_\beta$ on $U_\alpha \cap U_\beta$. To demonstrate this, the concept of local frames are needed.

Definition 1.2.29. Let $\pi : L \to X$ be a holomorphic line bundle over X and let $U \subseteq X$ be open. Then a *local frame for* L *over* U is a nowhere zero holomorphic local section of L over U. The value of the local frame at each $x \in U$ serves as a basis for each fiber L_x and so must be nonzero. If there exists a nowhere zero holomorphic section of L over all of X, then the section is called a global frame.

Example 1.2.30. Let $p_1 : X \times \mathbb{C} \to X$ be the trivial line bundle. Then the holomorphic section $e : X \to X \times \mathbb{C}$ defined by $e(x) = (x, 1)$ is clearly a global frame for this line bundle. Now let $\pi : L \to X$ be a holomorphic line bundle and let $U \subseteq X$ be open. If $\phi : \pi^{-1}(U) \to U \times \mathbb{C}$ is a local trivialization of L over U, then the holomorphic local section $e_U : U \to L$ defined by $e_U(x) = \phi^{-1}(x, 1)$ is a local frame for L over U.

Let $s \in H^0(X, L)$ and let $\{U_i, g_{\alpha\beta}\}$ represent the holomorphic line bundle L. Also, let $\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}$ be a local trivialization of L over U_α and let $e_\alpha : U_\alpha \to L$ be the local frame for L over U_α described in the last example. For each $x \in U_\alpha$, $e_\alpha(x)$ is a basis for fiber L_x, so we can locally write $s = s_\alpha e_\alpha$ where $s_\alpha : U_\alpha \to \mathbb{C}$ is some holomorphic function. Since

$$\begin{aligned} e_\beta(x) = \phi_\beta^{-1}(x, 1) = \phi_\alpha^{-1} \circ (\phi_\alpha \circ \phi_\beta^{-1}(x, 1)) &= \phi_\alpha^{-1}(x, g_{\alpha\beta}(x)) \\ &= g_{\alpha\beta}(x)\phi_\alpha^{-1}(x, 1) \\ &= g_{\alpha\beta}(x)e_\alpha(x), \end{aligned}$$

then for $x \in U_\alpha \cap U_\beta$,

$$s(x) = s_\alpha(x)e_\alpha(x) = s_\beta(x)e_\beta(x) = s_\beta(x)g_{\alpha\beta}(x)e_\alpha(x).$$

But the local frame e_α is nowhere zero, so

$$s_\alpha(x) = g_{\alpha\beta}(x)s_\beta(x).$$

So each $s \in H^0(X, L)$ induces a set of holomorphic functions $\{s_i\}$ on $\{U_i\}$.

Definition 1.2.31. Let $\pi : L \to X$ be a holomorphic line bundle over X represented by $\{U_i, g_{\alpha\beta}\}$. A holomorphic section $s \in H^0(X, L)$ inducing a set of meromorphic functions $\{s_i\}$ on $\{U_i\}$ such that $s_\alpha = g_{\alpha\beta}s_\beta$ on $U_\alpha \cap U_\beta$ is called a *meromorphic section of* L.

Definition 1.2.32. A *base-point* of a holomorphic line bundle $\pi : L \to X$ over X is a point $x \in X$ where for every $s \in H^0(X, L)$, $s(x) = 0$. A holomorphic line bundle without any such points is called *base-point-free*.

Let $\pi : L \to X$ be a holomorphic line bundle over X represented by $\{U_i, g_{\alpha\beta}\}$ and let $\{s_0, \ldots, s_N\}$ be a basis for the vector space of sections $H^0(X, L)$. If L is base-point-free and $\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}$ is a local trivialization of L over U_α, then

$$\Phi_\alpha : U_\alpha \to \mathbb{P}^n(\mathbb{C}) \quad \text{defined by} \quad \Phi_\alpha(x) = [\phi_\alpha(s_0(x)) : \cdots : \phi_\alpha(s_N(x))]$$

is a holomorphic map since for any section $s \in H^0(X, L)$,

$$\phi_\alpha \circ s : U_\alpha \to U_\alpha \times \mathbb{C} \quad \text{is given by} \quad \phi_\alpha \circ s(x) = (x, z),$$

so each $x \in U_\alpha$ passes through the map Φ_α unchanged. The line bundle L is base-point-free, so the set of local trivializations of L gives a well-defined holomorphic map from all of X to $\mathbb{P}^n(\mathbb{C})$.

Definition 1.2.33. The vector space $H^0(X, L)$ associated to a base-point-free holomorphic line bundle $\pi : L \to X$ is called a *complete linear system*. The line bundle is called *very ample* if the map $\Phi_\alpha : U_\alpha \to \mathbb{P}^n(\mathbb{C})$ described above is a holomorphic embedding and *ample* if the n^{th} tensor product of L, denoted $L^{\otimes n}$, is very ample for some $n \in \mathbb{N}$.

Definition 1.2.34. Let $\{U_i\}$ be an open covering of X and let $\psi_\alpha : U_\alpha \to \mathbb{C}$ be a non-vanishing meromorphic function on each U_α. If the ratio

$$\psi_\alpha / \psi_\beta : U_\alpha \cap U_\beta \to \mathbb{C}^*$$

is a holomorphic function for every α and β, then $\{(U_i, \psi_i)\}$ is called a *Cartier divisor* D on X. If each function ψ_α is holomorphic, then D is called *effective*.

Assume all the conditions in the above definition and let $g_{\alpha\beta} = \psi_\alpha/\psi_\beta$. Since each function $g_{\alpha\beta}$ satisfies the cocycle conditions, $\{U_i, g_{\alpha\beta}\}$ defines a holomorphic line bundle over X, denoted by $\mathcal{O}_X(D)$.

A different notion of a divisor is defined (locally) by the set of zeros of a holomorphic function $f : U \to \mathbb{C}$, i.e., $D_U = \{x \in U \mid f(x) = 0\}$, where $U \subseteq X$ is open. To fully define this divisor, we first need a couple of definitions.

Definition 1.2.35. A *hypersurface* of X is a subset of X, locally given as the zero set of a holomorphic function (called a *local defining function*), and is of codimension one. A hypersurface that can not be written as the union of two proper hypersurfaces is called an *irreducible hypersurface.* So every hypersurface is a union of its irreducible hypersurfaces.

Definition 1.2.36. A *Weil divisor* on X is a formal linear combination

$$D = \sum n_i Y_i$$

of irreducible hypersurfaces Y_i (called *prime divisors*) of X. We assume the sum is *locally finite*, i.e., for any $x \in X$, there exists an open neighborhood U of x such that only finitely many $n_i \neq 0$ with $Y_i \cap U \neq \emptyset$. A prime divisor Y_i with $n_i \neq 0$ is called a *(irreducible) component* of D and the *support of* D, denoted by Supp D, is the union of these components. The set of all Weil divisors on X, denoted by Div(X), is a group under addition. If every integer $n_i \geq 0$, then D is called *effective* and is written $D \geq 0$.

Let $D = \sum n_i Y_i$ be a Weil divisor on smooth X. Then there exists an open covering $\{U_i\}$ of X such that each irreducible hypersurface Y_i is locally defined by some holomorphic function $f_{i\alpha} : U_\alpha \to \mathbb{C}$, i.e., $Y_i \cap U_\alpha = \{x \in U_\alpha \mid f_{i\alpha}(x) = 0\}$. Set $f_\alpha = \prod_i f_{i\alpha}^{n_i} : U_\alpha \to \mathbb{C}^*$. Then f_α is a non-vanishing meromorphic function with $f_\alpha/f_\beta : U_\alpha \cap U_\beta \to \mathbb{C}^*$ a holomorphic function for every α and β. Thus $\{(U_i, f_i)\}$ defines a Cartier divisor on X.

Definition 1.2.37. Let $f : X \to \mathbb{C}$ be a meromorphic function. The Weil divisor associated to f is

$$(f) = \sum_{Y \subseteq X} \mathrm{ord}_Y(f) Y$$

where the sum is over all irreducible hypersurfaces $Y \subseteq X$. The *order* of f along Y, denoted by $\mathrm{ord}_Y(f)$, is the largest integer n such that $f = g^n h$ where g is a local defining function for Y and h is a holomorphic function not zero on Y. A divisor of this form is called *principal*. Two divisors D_1 and D_2 are said to be *linearly equivalent*, denoted by $D_1 \sim D_2$, if $D_1 - D_2$ is a principal divisor.

If D_1 and D_2 are linearly equivalent divisors on X, then $\mathcal{O}_X(D_1) \cong \mathcal{O}_X(D_2)$. Also, a divisor D on X is principal if and only if $\mathcal{O}_X(D) \cong \mathcal{O}_X$. Thus $\mathrm{Pic}(X) \cong \mathrm{Div(X)}/\sim$.

Let $\{(U_i, \psi_i)\}$ be a Cartier divisor on X. Then $\psi_\alpha : U_\alpha \to \mathbb{C}$ is a non-vanishing meromorphic function and $\psi_\alpha/\psi_\beta : U_\alpha \cap \mathbb{C}^*$ is a holomorphic function for every α and β. So for any irreducible hypersurface $Y \subseteq X$ with $Y \cap U_\alpha \cap U_\beta \neq \emptyset$, $\mathrm{ord}_Y(\psi_\alpha) = \mathrm{ord}_Y(\psi_\beta)$. Thus the order is well-defined for each $Y \subseteq X$, so $D = \sum_{Y \subseteq X} \mathrm{ord}_Y(\psi_Y) Y$ is a Weil divisor on X.

Let $\pi : L \to X$ be a holomorphic line bundle over X represented by $\{U_i, g_{\alpha\beta}\}$ and let $s \in H^0(X, L)$ be a non-zero section. The induced set of holomorphic functions $\{s_i\}$ satisfy $s_\alpha = g_{\alpha\beta} s_\beta$ on $U_\alpha \cap U_\beta$ for every α and β. So $(s) = \{(U_i, s_i)\}$ defines a Cartier divisor on X with $L \cong \mathcal{O}_X((s))$. Conversely, let $D = \{(U_i, f_i)\}$ be an effective Cartier divisor on X. Since D is effective, each f_i is a holomorphic function and $\mathcal{O}_X(D)$ is represented by $\{U_i, f_\alpha / f_\beta\}$. The holomorphic functions $\{f_i\}$ are induced by a non-zero section $s \in H^0(X, \mathcal{O}_X(D))$ with $D = (s)$.

Definition 1.2.38. The section $s \in H^0(X, \mathcal{O}_X(D))$ with $D = (s)$ described above is called the *canonical section* and is denoted by s_D.

Definition 1.2.39. Let D be a divisor on X. The *complete linear system* of D, denoted $|D|$, is the set of effective divisors linearly equivalent to D. A *base-point* of $|D|$ is a point $x \in X$ such that $x \in \text{Supp } D'$ for every $D' \in |D|$.

Example 1.2.40. Let $s \in H^0(X, \mathcal{O}_X)$, i.e., a section of the trivial line bundle $p_1 : X \times \mathbb{C} \to X$. Then there exists some holomorphic function $f : X \to \mathbb{C}$ such that $s : X \to X \times \mathbb{C}$ is given by $s(x) = (x, f(x))$.

Example 1.2.41. Let $X = \mathbb{P}^n(\mathbb{C})$ and let

$$H = \{[z_0 : \cdots : z_n] \in \mathbb{P}^n(\mathbb{C}) \mid a_0 z_0 + \cdots + a_n z_n = 0\}$$

be the projective hyperplane defined by the coefficient vector $a = (a_0, \dots, a_n) \in \mathbb{C}^{n+1}$. The standard open covering of $\mathbb{P}^n(\mathbb{C})$ is $\{U_i\}_{i=0}^n$ with

$$U_i = \{[z_0 : \cdots : z_n] \in \mathbb{P}^n(\mathbb{C}) \mid z_i \neq 0\}, \quad i = 0, \dots, n.$$

For each $i = 1, \ldots, n$, define the holomorphic function $f_i : U_i \to \mathbb{C}$ by

$$f_i([z_0 : \cdots : z_n]) = \frac{a_0 z_+ \cdots a_n z_n}{z_i}.$$

The hyperplane divisor H is locally defined by the zero set of each f_i. The holomorphic line bundle over $\mathbb{P}^n(\mathbb{C})$, $\mathcal{O}_X(H)$, has transition functions $g_{ij} = \frac{f_i}{f_j} = \frac{z_j}{z_i}$ on $U_i \cap U_j$.

Definition 1.2.42. The holomorphic line bundle over $\mathbb{P}^n(\mathbb{C})$ described in the last example for any hyperplane H represents the isomorphism class of line bundles called the *hyperplane line bundle*, denoted by $\mathcal{O}_{\mathbb{P}^n}(1)$.

Example 1.2.43. Let

$$L = \{([x_0 : \cdots : x_n], (z_0, \ldots, z_n)) \in \mathbb{P}^n(\mathbb{C}) \times \mathbb{C}^{n+1} \mid (z_0, \ldots, z_n) \in [x_0 : \cdots : x_n]\}$$

and let $\pi : L \to \mathbb{P}^n(\mathbb{C})$ be defined by $\pi(x, z) = x$. Also, let $\{U_i\}$ be the standard open covering of $\mathbb{P}^n(\mathbb{C})$. For each $i = 0, \ldots, n$, define the map $\phi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C}$ by $\phi_i([x_0 : \cdots : x_n], (z_0, \ldots, z_n)) = ([x_0 : \cdots : x_n], z_i)$. Notice

$$\phi_j^{-1}([x_0 : \cdots : x_n], 1) = ([x_0 : \cdots : x_n], (z_0, \ldots, z_n)/z_j),$$

so

$$\phi_i \circ \phi_j^{-1}([x_0 : \cdots : x_n], 1) = \phi_i([x_0 : \cdots : x_n], (z_0, \ldots, z_n)/z_j)) = ([x_0 : \cdots : x_n], z_i/z_j).$$

So the functions $g_{ij} = \frac{z_i}{z_j}$ on $U_i \cap U_j$ define a holomorphic line bundle over $\mathbb{P}^n(\mathbb{C})$, denoted $\mathcal{O}_{\mathbb{P}^n}(-1)$, and is called the *tautological line bundle*. It is the dual of $\mathcal{O}_{\mathbb{P}^n}(1)$.

Chapter 2

An Improved Defect Relation in Nevanlinna Theory

In this chapter, we prove the Analytic Main Theorem in [HR16]. First, we collect lemmas used in this chapter and the next.

§2.1 Common Lemmas

Lemma 2.1.1 ([Aut09], Lemma 4.2)**.** *Suppose E is a big and base-point free Cartier divisor on a projective variety X of dimension n, and let F be a nef Cartier divisor on X such that $F - E$ is also nef. Let $\beta > 0$ be a positive real number. Then for any positive integers N, m with $1 \leq m \leq \beta N$, we have*

$$h^0(NF - mE) \geq \frac{F^n}{n!}N^n - \frac{F^{n-1}.E}{(n-1)!}N^{n-1}m + \frac{(n-1)F^{n-2}.E^2}{n!}N^{n-2}\min\{m^2, N^2\} + O(N^{n-1})$$

where O depends on β.

Let D_1 and D_2 be two effective divisors on X. We define

$$\mathrm{lcm}\,(D_1, D_2) = \sum_E \max\{\mathrm{ord}_E\, D_1, \mathrm{ord}_E\, D_2\}E,$$

where the sum runs over all prime divisors E on X.

Lemma 2.1.2. *Let $D_1, \ldots, D_q$ be effective divisors in m-subgeneral position on a projective variety X of dimension $n \geq 2$. Then for any subset of m divisors $\{D_{i_1}, \ldots, D_{i_m}\} \subseteq \{D_1, \ldots, D_q\}$,*

$$\sum_{\substack{\mu,\nu=1 \\ \mu\neq\nu}}^{m} \mathrm{lcm}\,(D_{i_\mu}, D_{i_\nu}) \geq (m+n-2)\sum_{\alpha=1}^{m} D_{i_\alpha}.$$

Proof. Fix $\mu \in \{1, \ldots, m\}$. We will first show that every irreducible component E of D_{i_μ} can belong to at most $m-n$ divisors $D_{i_\nu}, \nu \neq \mu$. For sake of contradiction, assume there exists an irreducible element E of D_{i_μ} belonging to at least $m-n+2$ divisors D_{i_α}. Then

$$E \subseteq \bigcap_\alpha \mathrm{Supp}\ D_{i_\alpha},$$

with α indexing the divisors E belongs to, so

$$\dim \bigcap_\alpha \mathrm{Supp}\ D_{i_\alpha} \geq \dim E = n-1 > m-(m-n+2) = n-2.$$

This contradicts $D_1, \ldots, D_q$ are in m-subgeneral position. So any irreducible component E of D_{i_μ} can belong to at most $m-n$ divisors $D_{i_\nu}, \nu \neq \mu$, and so

$$\sum_{\substack{\nu=1 \\ \nu\neq\mu}}^{m} \mathrm{lcm}\,(D_{i_\mu}, D_{i_\nu}) \geq (m-1-(m-n))D_{i_\mu} + \sum_{\substack{\nu=1 \\ \nu\neq\mu}}^{m} D_{i_\nu}$$

$$= (n-1)D_{i_\mu} + \sum_{\substack{\nu=1\\ \nu\neq\mu}}^{m} D_{i_\nu}.$$

Summing over all μ proves the claim. $\square$

Let D be a divisor on a projective variety X. Let σ_0 be the set of all prime divisors occurring in D. Write

$$D = \sum_{E\in\sigma_0} (\operatorname{ord}_E D)E.$$

We call $\operatorname{ord}_E D$ the *coefficient of E in D.*

Lemma 2.1.3. *Let $D_1, \ldots, D_q$ be effective divisors in m-subgeneral position on a projective variety X of dimension $n \geq 2$ and let σ_0 be the set of all prime divisors occurring in $D_1, \ldots, D_q$. Then for each*

$$\sigma \in \Sigma = \left\{ \sigma \subseteq \sigma_0 \mid \bigcap_{E\in\sigma} E \neq \emptyset \right\},$$

there are m divisors

$$D_{i_1}, \ldots, D_{i_m} \in \{D_1, \ldots, D_q\}$$

such that the prime divisors $E \in \sigma$ only occur in $\{D_{i_1}, \ldots, D_{i_m}\}$.

Proof. Let σ be a subset of all prime divisors occurring in $D_1, \ldots, D_q$ with $\bigcap_{E\in\sigma} E \neq \emptyset$. To the contrary, assume there are not m divisors

$$D_{i_1}, \ldots, D_{i_m} \in \{D_1, \ldots, D_q\}$$

such that the prime divisors $E \in \sigma$ only occur in $\{D_{i_1}, \ldots, D_{i_m}\}$. Since $\bigcap_{E\in\sigma} E \neq \emptyset$ with the prime divisors $E \in \sigma$ occurring in at least $m+1$ of the divisors $D_1, \ldots, D_q$, this contradicts $D_1, \ldots, D_q$ being in m-subgeneral position on X. $\square$

Lemma 2.1.4 ([CZ04b], Lemma 3.2)**.** *Let V be a vector space of finite dimension d. Let $V = W_1 \supset W_2 \supset \cdots \supset W_h$ and $V = W_1^* \supset W_2^* \supset \cdots \supset W_{h^*}^*$ be two filtrations of V. Then there exists a basis $v_1, \ldots, v_d$ of V which contains a basis of each W_j and W_j^*.*

§2.2 Main Theorem

We make use of the following generalized Cartan's Second Main Theorem for holomorphic curves.

Theorem 2.2.1 ([Ru97] Theorem 2.1, [Voj97] Theorem 1)**.** *Let $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be a linearly non-degenerate holomorphic curve and $H_1, \ldots, H_q$ hyperplanes in $\mathbb{P}^n(\mathbb{C})$ with corresponding Weil functions $\lambda_{H_1}, \ldots, \lambda_{H_q}$. Then for any $\epsilon > 0$,*

$$\int_0^{2\pi} \max_J \sum_{j\in J} \lambda_{H_j}(f(re^{i\theta}))\frac{d\theta}{2\pi} \leq (n+1+\epsilon)T_f(r)$$

holds for all $r > 0$ outside a set of finite Lebesgue measure, where the max is taken over all subsets $J \subseteq \{1, \ldots, q\}$ such that the hyperplanes H_j, $j \in J$, are in general position on $\mathbb{P}^n(\mathbb{C})$.

We also need a lemma from Vojta.

Lemma 2.2.2 ([Voj07], Lemma 20.7)**.** *Let X be a complex projective variety and D an effective divisor on X. Write*

$$D = \sum_{E\in\sigma_0} (\mathrm{ord}_E\, D)E$$

and let

$$\Sigma := \left\{ \sigma \subseteq \sigma_0 \mid \bigcap_{E \in \sigma} E \neq \emptyset \right\}.$$

For each $\sigma \in \Sigma$, *let*

$$D_\sigma := \sum_{E \notin \sigma} (\operatorname{ord}_E D) E.$$

Choose a Weil function for each such D_σ. *Then there exists a constant* C_0, *depending only on* X *and* D, *such that*

$$\min_{\sigma \in \Sigma} \lambda_{D_\sigma}(x) \leq C_0$$

for all $x \in X$.

We are now ready for the proof. For the convenience of the reader, we restate the Main Theorem.

Main Theorem. *Let* X *be a complex projective variety of dimension* $n \geq 2$ *and* $D_1, \ldots, D_q$ *big and nef Cartier divisors in* m*-subgeneral position on* X. *Let* $r_i > 0$ *be real numbers such that* $D := r_1 D_1 + \cdots + r_q D_q$ *has equidegree (such real numbers exist due to Lemma 1.1.3). Assume there exists a positive integer* N_0 *such that the linear system* $|ND_i|$ $(i = 1, \ldots, q)$ *is base-point free for* $N \geq N_0$. *Let* $f : \mathbb{C} \to X$ *be a Zariski dense holomorphic map. Then*

$$m_f(r, D) < \frac{m(m-1)}{(m+n-2)} \frac{2n}{q} \frac{1}{(1+C)} T_{f,D}(r)$$

holds for all $r > 0$ *outside a set of finite Lebesgue measure, where*

$$C = \frac{\min_{1 \leq j \leq q} \{D^{n-2}.(r_j D_j)^2\}}{6qn^2 4^n D^n} > 0.$$

Proof. Let

$$\alpha = \frac{\min_{1\le j\le q}\{D^{n-2}.(r_jD_j)^2\}}{3n^3 4^n D^n}.$$

Notice $\alpha > 0$ by Lemma 1.1.1, since D_j, $1 \le j \le q$, are big and nef, and so

$$\min_{1\le j\le q}\{D^{n-2}.(r_jD_j)^2\} \ge \min_{1\le j\le q}\{(r_jD_j)^n\} > 0.$$

Since D has equidegree with respect to $r_1D_1, \ldots, r_qD_q$,

$$r_iD_i.D^{n-1} = \frac{1}{q}D^n, \quad 1 \le i \le q.$$

So by the density of $\mathbb{Q}$ in $\mathbb{R}$, choose (positive) rational numbers $a_1, \ldots, a_q$ such that both

$$|a_j - r_j| \le \frac{\delta_1}{2}\left(\min_{1\le i\le q} r_i\right)\min\left\{1, \frac{1}{\frac{m(m-1)}{(m+n-2)}\frac{2n}{q}\frac{1}{\left(1+\frac{n\alpha}{q}\right)}}\right\}, \quad 1 \le j \le q, \tag{2.2.1}$$

and

$$\left|\frac{D'^n}{a_iD_i.D'^{n-1}} - q\right| < \delta_2, \quad 1 \le i \le q, \quad D' = a_1D_1 + \cdots + a_qD_q, \tag{2.2.2}$$

where δ_1 and δ_2 will be chosen later such that $\delta_1, \delta_2 \le 1$. Note that

$$|a_j - r_j| \le \frac{\delta_1}{2}\left(\min_{1\le i\le q} r_i\right)\min\left\{1, \frac{1}{\frac{m(m-1)}{(m+n-2)}\frac{2n}{q}\frac{1}{\left(1+\frac{n\alpha}{q}\right)}}\right\} \le \frac{1}{2}r_j, \quad 1 \le j \le q,$$

so

$$D'^n \ge \frac{1}{2^n}D^n \quad \text{and} \quad D'^n \le 2^nD^n. \tag{2.2.3}$$

To clear out the denominators, define the divisor $\widetilde{D} = dD'$, where d is the product of the denominators of $a_1, \ldots, a_q$. Notice that

$$\frac{\widetilde{D}^n}{da_iD_i.\widetilde{D}^{n-1}} = \frac{(dD')^n}{da_iD_i.(dD')^{n-1}} = \frac{d^nD'^n}{d^n(a_iD_i).D'^{n-1}} = \frac{D'^n}{a_iD_i.D'^{n-1}},$$

so by (2.2.2),

$$\left|\frac{\widetilde{D}^n}{da_i D_i.\widetilde{D}^{n-1}} - q\right| < \delta_2 \quad 1 \le i \le q. \tag{2.2.4}$$

Let $x = f(z) \in X$. By Lemma 2.2.2, there exists a divisor $\widetilde{D}_\sigma$ on X and a Weil function $\lambda_{\widetilde{D}_\sigma}$ such that

$$\lambda_{\widetilde{D}_\sigma}(x) \le C_0, \tag{2.2.5}$$

where σ is some subset of prime divisors occurring in $\widetilde{D}$ with $\bigcap_{E\in\sigma} E \neq \emptyset$ and C_0 is a positive constant depending only on X and $\widetilde{D}$. Write

$$\widetilde{D} = \widetilde{D}_0 + \widetilde{D}_\sigma = \sum_{E\in\sigma}(\operatorname{ord}_E \widetilde{D})E + \widetilde{D}_\sigma$$

and select Weil functions for $\widetilde{D}$ and $\widetilde{D}_0$. Then by (2.2.5) and the additivity of Weil functions,

$$\lambda_{\widetilde{D}}(x) = \lambda_{\widetilde{D}_0}(x) + \lambda_{\widetilde{D}_\sigma}(x) = \lambda_{\widetilde{D}_0}(x) + O(1). \tag{2.2.6}$$

Select Weil functions for each D_i, $i = 1, \ldots, q$, and for each prime divisor $E \in \sigma$. Since $D_1, \ldots, D_q$ are in m-subgeneral position, then by Lemma 2.1.3, there are

$$D_{1,z}, \ldots, D_{m,z} \in \{D_1, \ldots, D_q\}$$

such that the prime divisors $E \in \sigma$ only occur in $D_{1,z}, \ldots, D_{m,z}$. So by (2.2.6) and the additivity of Weil functions,

$$\begin{aligned}\sum_{j=1}^{q} da_j \lambda_{D_j}(x) &= \lambda_{\widetilde{D}}(x) = \lambda_{\widetilde{D}_0}(x) + O(1) \\ &= \sum_{E\in\sigma}(\operatorname{ord}_E \widetilde{D})\lambda_E(x) + O(1)\end{aligned} \tag{2.2.7}$$

$$\leq \sum_{\alpha=1}^{m} da_{\alpha,z}\lambda_{D_{\alpha,z}}(f(z)) + O(1).$$

Also, by Lemma 2.1.2, for each $z \in \mathbb{C}$, $\{D_{1,z}, \ldots, D_{m,z}\}$,

$$\sum_{\substack{\mu,\nu=1 \\ \mu\neq\nu}}^{m} \operatorname{lcm}(da_{\mu,z}D_{\mu,z}, da_{\nu,z}D_{\nu,z}) \geq (m+n-2)\sum_{\alpha=1}^{m} da_{\alpha,z}D_{\alpha,z}. \tag{2.2.8}$$

Select Weil functions for each divisor $\operatorname{lcm}(da_{\mu,z}D_{\mu,z}, da_{\nu,z}D_{\nu,z})$, $\mu, \nu = 1, \ldots, m$, $\mu \neq \nu$. Then by (2.2.7), (2.2.8), and the additivity of Weil functions,

$$\begin{aligned}\sum_{j=1}^{q} da_j\lambda_{D_j}(x) &\leq \sum_{\alpha=1}^{m} da_{\alpha,z}\lambda_{D_{\alpha,z}}(f(z)) + O(1) \\ &\leq \frac{1}{m+n-2}\sum_{\substack{\mu,\nu=1 \\ \mu\neq\nu}}^{m} \lambda_{\operatorname{lcm}(da_{\mu,z}D_{\mu,z}, da_{\nu,z}D_{\nu,z})}(f(z)).\end{aligned} \tag{2.2.9}$$

Fix $D_{\mu,z} \in \{D_{1,z}, \ldots, D_{m,z}\}$. Then for $N \geq N_0$, consider the following filtration for the vector space $H^0(X, \mathcal{O}_X(N\widetilde{D}))$,

$$H^0(X, \mathcal{O}_X(N\widetilde{D})) = W_0 \supset W_1 \supset \cdots \supset W_i \supset \cdots \supset W_N \supset W_{N+1} \supset \cdots \supset \{0\},$$

where $W_k = H^0(X, \mathcal{O}_X(N\widetilde{D} - kda_{\mu,z}D_{\mu,z}))$. Let B be a basis of $H^0(X, \mathcal{O}_X(N\widetilde{D}))$ associated to this filtration. Also, let $\varphi_{N\widetilde{D}} : X \to \mathbb{P}^M(\mathbb{C})$ be the canonical morphism associated to $N\widetilde{D}$ and let $M = h^0(N\widetilde{D}) - 1$. Since $\phi^*_{N\widetilde{D}}\mathcal{O}_{\mathbb{P}^M}(1) = \mathcal{O}_X(N\widetilde{D})$, every section $s \in H^0(X, \mathcal{O}_X(N\widetilde{D}))$ corresponds to a hyperplane $H \subseteq \mathbb{P}^M(\mathbb{C})$ such that $\varphi^*_{N\widetilde{D}}H = (s)$. Note when we write $s \in H^0(X, \mathcal{O}_X(N\widetilde{D} - kda_{\mu,z}D_{\mu,z})) \subseteq H^0(X, \mathcal{O}_X(N\widetilde{D}))$, we mean $s \otimes s_{D_{\mu,z}}^{kda_{\mu,z}} \in H^0(X, \mathcal{O}_X(N\widetilde{D}))$ where $s_{D_{\mu,z}}$ is the canonical section of $\mathcal{O}_X(D_{\mu,z})$, so

$$\varphi^*_{N\widetilde{D}}H = (s) \geq kda_{\mu,z}D_{\mu,z}. \tag{2.2.10}$$

Let $\mathcal{H}_\mu$ be the set of hyperplanes (also depending on z) corresponding to the basis B. Since B is a basis of $H^0(X, \mathcal{O}_X(N\widetilde{D}))$, the hyperplanes in $\mathcal{H}_\mu$ are in general position. Then by (2.2.10),

$$\sum_{H\in\mathcal{H}_\mu} \varphi^*_{N\widetilde{D}} H \geq \left(\sum_{k=0}^{\infty} k \dim\left(W_k/W_{k+1}\right)\right) da_{\mu,z}D_{\mu,z} = \left(\sum_{k=1}^{\infty} \dim W_k\right) da_{\mu,z}D_{\mu,z}$$
$$= \left(\sum_{k=1}^{\infty} h^0(N\widetilde{D} - kda_{\mu,z}D_{\mu,z})\right) da_{\mu,z}D_{\mu,z}. \tag{2.2.11}$$

Now applying Lemma 2.1.1, with $F = \widetilde{D}, E = da_{\mu,z}D_{\mu,z}$, and $\beta := \frac{\widetilde{D}^n}{n\widetilde{D}^{n-1}.(da_{\mu,z}D_{\mu,z})}$, gives us

$$\sum_{k=1}^{\infty} h^0(N\widetilde{D} - kda_{\mu,z}D_{\mu,z}) \tag{2.2.12}$$
$$\geq \sum_{k=1}^{[\beta N]} \left(\frac{\widetilde{D}^n}{n!}N^n - \frac{\widetilde{D}^{n-1}.(da_{\mu,z}D_{\mu,z})}{(n-1)!}N^{n-1}k + \frac{A}{n!}N^{n-2}\min\{k^2, N^2\}\right) + O(N^n)$$
$$\geq \left(\frac{\widetilde{D}^n}{n!}\beta - \frac{\widetilde{D}^{n-1}.(da_{\mu,z}D_{\mu,z})}{(n-1)!}\frac{\beta^2}{2} + \frac{A}{n!}g(\beta)\right) N^{n+1} + O(N^n)$$
$$= \left(\frac{\beta}{2} + \frac{A}{\widetilde{D}^n}g(\beta)\right)\widetilde{D}^n \frac{N^{n+1}}{n!} + O(N^n)$$
$$\geq \left(\frac{\beta}{2} + \widetilde{\alpha}\right)\widetilde{D}^n \frac{N^{n+1}}{n!} + O(N^n),$$

where $A := (n-1)\widetilde{D}^{n-2}.(da_{\mu,z}D_{\mu,z})^2$, $\widetilde{\alpha} = \dfrac{\min_{1\leq j\leq q}\{\widetilde{D}^{n-2}.(da_jD_j)^2\}}{\widetilde{D}^n}g(\beta)$, and $g : \mathbb{R}^+ \to \mathbb{R}^+$ is the function defined by

$$g(x) = \begin{cases} \frac{x^3}{3}, & 0 < x \leq 1 \\ x - \frac{2}{3}, & x \geq 1. \end{cases}$$

Returning to (2.2.4),

$$\frac{\widetilde{D}^n}{\widetilde{D}^{n-1}.(da_{\mu,z}D_{\mu,z})} = \frac{\widetilde{D}^n}{(da_{\mu,z}D_{\mu,z}).\widetilde{D}^{n-1}} > q - \delta_2$$

implies

$$\beta = \frac{\widetilde{D}^n}{n\widetilde{D}^{n-1}.(da_{\mu,z}D_{\mu,z})} > \frac{q-\delta_2}{n}. \tag{2.2.13}$$

Since $\widetilde{D}^n \geq \widetilde{D}^{n-1}.(da_{\mu,z}D_{\mu,z})$, then $\beta \geq \frac{1}{n}$, so $g(\beta) \geq \frac{1}{3n^3}$. Also, since $\widetilde{D} = dD'$, (2.2.3) implies

$$\begin{aligned}\widetilde{\alpha} = \frac{\min_{1\leq j\leq q}\{\widetilde{D}^{n-2}.(da_jD_j)^2\}}{\widetilde{D}^n}g(\beta) &\geq \frac{\min_{1\leq j\leq q}\{D'^{n-2}.(a_jD_j)^2\}}{3n^3D'^n}\\ &\geq \frac{\min_{1\leq j\leq q}\{D^{n-2}.(r_jD_j)^2\}}{3n^34^nD^n} = \alpha.\end{aligned} \tag{2.2.14}$$

Using (2.2.13) and (2.2.14), we can write (2.2.12) as

$$\begin{aligned}\sum_{k=1}^{\infty} h^0(N\widetilde{D} - kda_{\mu,z}D_{\mu,z}) &\geq \left(\frac{\beta}{2} + \widetilde{\alpha}\right)\widetilde{D}^n\frac{N^{n+1}}{n!} + O(N^n)\\ &> \left(\frac{q-\delta_2}{2n} + \alpha\right)\widetilde{D}^n\frac{N^{n+1}}{n!} + O(N^n),\end{aligned}$$

and using this inequality in (2.2.11),

$$\begin{aligned}\sum_{H\in\mathcal{H}_\mu}\varphi^*_{N\widetilde{D}}H &\geq \left(\sum_{k=1}^{\infty} h^0(N\widetilde{D} - kda_{\mu,z}D_{\mu,z})\right)da_{\mu,z}D_{\mu,z}\\ &> \left(\left(\frac{q-\delta_2}{2n} + \alpha\right)\widetilde{D}^n\frac{N^{n+1}}{n!} + O(N^n)\right)da_{\mu,z}D_{\mu,z}.\end{aligned}$$

Fix another $D_{\nu,z} \in \{D_{1,z},\ldots,D_{m,z}\}$. Then similar steps give us

$$\sum_{H\in\mathcal{H}_\nu}\varphi^*_{N\widetilde{D}}H > \left(\left(\frac{q-\delta_2}{2n} + \alpha\right)\widetilde{D}^n\frac{N^{n+1}}{n!} + O(N^n)\right)da_{\nu,z}D_{\nu,z}.$$

Consider the two filtrations of $H^0(X,\mathcal{O}_X(N\widetilde{D}))$ coming from looking at the order of vanishing along $D_{\mu,z}$ and $D_{\nu,z}$, as described previously. Let B be the basis of $H^0(X,\mathcal{O}_X(N\widetilde{D}))$ that Lemma 2.1.4 gives with respect to these two filtrations. Let

$\mathcal{H}_{\mu,\nu}$ be the corresponding set of hyperplanes in $\mathbb{P}^M(\mathbb{C})$. Then by the definition of B and similar steps as before,

$$\sum_{H\in\mathcal{H}_{\mu,\nu}} \varphi^*_{N\widetilde{D}}H > \left(\left(\frac{q-\delta_2}{2n}+\alpha\right)\widetilde{D}^n\frac{N^{n+1}}{n!}+O(N^n)\right) da_{\mu,z}D_{\mu,z}$$

and

$$\sum_{H\in\mathcal{H}_{\mu,\nu}} \varphi^*_{N\widetilde{D}}H > \left(\left(\frac{q-\delta_2}{2n}+\alpha\right)\widetilde{D}^n\frac{N^{n+1}}{n!}+O(N^n)\right) da_{\nu,z}D_{\nu,z}.$$

It follows that

$$\sum_{H\in\mathcal{H}_{\mu,\nu}} \varphi^*_{N\widetilde{D}}H > \left(\left(\frac{q-\delta_2}{2n}+\alpha\right)\widetilde{D}^n\frac{N^{n+1}}{n!}+O(N^n)\right) \operatorname{lcm}(da_{\mu,z}D_{\mu,z}, da_{\nu,z}D_{\nu,z}).$$

By the additivity of Weil functions,

$$\begin{aligned}&\sum_{H\in\mathcal{H}_{\mu,\nu}} \lambda_{\varphi^*_{N\widetilde{D}}H}(f(z))\\ &> \left(\left(\frac{q-\delta_2}{2n}+\alpha\right)\widetilde{D}^n\frac{N^{n+1}}{n!}+O(N^n)\right)\lambda_{\operatorname{lcm}(da_{\mu,z}D_{\mu,z},da_{\nu,z}D_{\nu,z})}(f(z)),\end{aligned}$$

and summing over all $m(m-1)$ distinct $\mu,\nu\in\{1,\dots,m\}$,

$$\begin{aligned}&\left(\left(\frac{q-\delta_2}{2n}+\alpha\right)\widetilde{D}^n\frac{N^{n+1}}{n!}+O(N^n)\right)\sum_{\substack{\mu,\nu=1\\ \mu\neq\nu}}^{m}\lambda_{\operatorname{lcm}(da_{\mu,z}D_{\mu,z},da_{\nu,z}D_{\nu,z})}(f(z))\\ &< \sum_{\substack{\mu,\nu=1\\ \mu\neq\nu}}^{m}\sum_{H\in\mathcal{H}_{\mu,\nu}}\lambda_{\varphi^*_{N\widetilde{D}}H}(f(z)) \le m(m-1)\left(\max_{\mathcal{H}_{\mu,\nu}}\sum_{H\in\mathcal{H}_{\mu,\nu}}\lambda_{\varphi^*_{N\widetilde{D}}H}(f(z))\right)\end{aligned}$$

or

$$\begin{aligned}&\sum_{\substack{\mu,\nu=1\\ \mu\neq\nu}}^{m}\lambda_{\operatorname{lcm}(da_{\mu,z}D_{\mu,z},da_{\nu,z}D_{\nu,z})}(f(z))\\ &< \frac{m(m-1)}{\left(\frac{q-\delta_2}{2n}+\alpha\right)\widetilde{D}^n\frac{N^{n+1}}{n!}+O(N^n)}\left(\max_{\mathcal{H}_{\mu,\nu}}\sum_{H\in\mathcal{H}_{\mu,\nu}}\lambda_{\varphi^*_{N\widetilde{D}}H}(f(z))\right).\end{aligned}$$

Using this result, (2.2.9) becomes

$$\begin{aligned}
&\sum_{j=1}^{q} da_j \lambda_{D_j}(x) \\
&\leq \frac{1}{m+n-2} \sum_{\substack{\mu,\nu=1 \\ \mu\neq\nu}}^{m} \lambda_{\mathrm{lcm}\,(da_{\mu,z}D_{\mu,z}, da_{\nu,z}D_{\nu,z})}(f(z)) \qquad (2.2.15)\\
&< \frac{m(m-1)}{(m+n-2)\left(\left(\frac{q-\delta_2}{2n}+\alpha\right)\widetilde{D}^n\frac{N^{n+1}}{n!}+O(N^n)\right)} \left(\max_{\mathcal{H}_{\mu,\nu}} \sum_{H\in\mathcal{H}_{\mu,\nu}} \lambda_{\varphi^*_{N\widetilde{D}}H}(f(z))\right).
\end{aligned}$$

Let $\mathcal{H}_z = \bigcup_{\substack{\mu,\nu=1 \\ \mu\neq\nu}}^{m} \mathcal{H}_{\mu,\nu}$ for each $z \in \mathbb{C}$ and denote $\mathcal{H}_T = \bigcup_{z\in\mathbb{C}} \mathcal{H}_z$. Then by the functoriality of Weil functions, for any $z \in \mathbb{C}$,

$$\max_{\mathcal{H}_{\mu,\nu}} \sum_{H\in\mathcal{H}_{\mu,\nu}} \lambda_{\varphi^*_{N\widetilde{D}}H}(f(z)) \leq \max_{J} \sum_{H\in J} \lambda_{\varphi^*_{N\widetilde{D}}H}(f(z)) = \max_{J} \sum_{H\in J} \lambda_H((\varphi_{N\widetilde{D}} \circ f)(z)),$$

where the max is taken over all subsets $J \subseteq \mathcal{H}_T$ consisting of hyperplanes in general position on $\mathbb{P}^M(\mathbb{C})$. Hence (2.2.15) can be written as

$$\begin{aligned}
&\sum_{j=1}^{q} da_j \lambda_{D_j}(x) \\
&< \frac{m(m-1)}{(m+n-2)\left(\left(\frac{q-\delta_2}{2n}+\alpha\right)\widetilde{D}^n\frac{N^{n+1}}{n!}+O(N^n)\right)} \left(\max_{J} \sum_{H\in J} \lambda_H((\varphi_{N\widetilde{D}} \circ f)(z))\right).
\end{aligned}$$

We can finally integrate both sides and apply Cartan's Theorem 2.2.1 (with $\epsilon = 1$) to the curve $\varphi_{N\widetilde{D}} \circ f : \mathbb{C} \to \mathbb{P}^M(\mathbb{C})$ and to the set of hyperplanes $\mathcal{H}_T$, so

$$\sum_{j=1}^{q} da_j m_f(r, D_j) < \frac{m(m-1)}{(m+n-2)\left(\left(\frac{q-\delta_2}{2n}+\alpha\right)\widetilde{D}^n\frac{N^{n+1}}{n!}+O(N^n)\right)} (M+2) T_{\varphi_{N\widetilde{D}}\circ f}(r) \qquad (2.2.16)$$

holds for all $r > 0$ outside a set of finite Lebesgue measure.

Using Lemma 1.1.1,

$$M+1=h^0(N\widetilde{D})=\frac{\widetilde{D}^n}{n!}N^n+O(N^{n-1}),$$

so by the functoriality of height functions,

$$\begin{aligned}(M+2)T_{\varphi_{N\widetilde{D}}\circ f}(r)&=(M+2)T_{f,N\widetilde{D}}(r)\\&=N(M+2)T_{f,\widetilde{D}}(r)\\&=\left(\frac{\widetilde{D}^n}{n!}N^{n+1}+O(N^n)+N\right)T_{f,\widetilde{D}}(r).\end{aligned}$$

Thus, by (2.2.16),

$$\begin{aligned}\sum_{j=1}^q da_j m_f(r,D_j)&<\frac{m(m-1)}{(m+n-2)\left(\left(\frac{q-\delta_2}{2n}+\alpha\right)\widetilde{D}^n\frac{N^{n+1}}{n!}+O(N^n)\right)}(M+2)T_{\varphi_{N\widetilde{D}}\circ f}(r)\\&=\left(\frac{m(m-1)}{(m+n-2)}\right)\frac{\frac{\widetilde{D}^n}{n!}N^{n+1}+O(N^n)+N}{\left(\frac{q-\delta_2}{2n}+\alpha\right)\widetilde{D}^n\frac{N^{n+1}}{n!}+O(N^n)}T_{f,\widetilde{D}}(r)\end{aligned}$$

holds for all $r>0$ outside a set of finite Lebesgue measure.

Now, choose $N\geq N_0$ such that

$$\begin{aligned}\sum_{j=1}^q da_j m_f(r,D_j)&<\left(\frac{m(m-1)}{(m+n-2)}\right)\frac{\frac{\widetilde{D}^n}{n!}N^{n+1}+O(N^n)+N}{\left(\frac{q-\delta_2}{2n}+\alpha\right)\widetilde{D}^n\frac{N^{n+1}}{n!}+O(N^n)}T_{f,\widetilde{D}}(r)\\&=\left(\frac{m(m-1)}{(m+n-2)}\right)\frac{1+O(\frac{1}{N})+O(\frac{1}{N^n})}{\left(\frac{q-\delta_2}{2n}+\alpha\right)+O(\frac{1}{N})}\sum_{j=1}^q da_jT_{f,D_j}(r)\\&\leq\left(\frac{m(m-1)}{(m+n-2)}\right)\frac{1}{\left(\frac{q-\delta_2}{2n}+\frac{2}{3}\alpha\right)}\sum_{j=1}^q da_jT_{f,D_j}(r).\end{aligned}$$

Let $\delta_2=\min\{1,\frac{n\alpha}{3}\}$. Notice $\delta_2>0$ since we saw earlier that $\alpha>0$. Now,

$$\sum_{j=1}^q da_j m_f(r,D_j)<\left(\frac{m(m-1)}{(m+n-2)}\right)\frac{1}{\left(\frac{q-\delta_2}{2n}+\frac{2}{3}\alpha\right)}\sum_{j=1}^q da_jT_{f,D_j}(r)$$

$$\leq \frac{m(m-1)}{(m+n-2)} \frac{2n}{q} \frac{1}{\left(1+\frac{n\alpha}{q}\right)} \sum_{j=1}^{q} da_j T_{f,D_j}(r),$$

and after canceling d on both sides,

$$\sum_{j=1}^{q} a_j m_f(r, D_j) < \frac{m(m-1)}{(m+n-2)} \frac{2n}{q} \frac{1}{\left(1+\frac{n\alpha}{q}\right)} \sum_{j=1}^{q} a_j T_{f,D_j}(r). \tag{2.2.17}$$

To write this result in terms of divisor $D = r_1 D_1 + \cdots + r_q D_q$, note (2.2.1) gives us

$$\sum_{j=1}^{q} r_j m_f(r, D_j) \leq \sum_{j=1}^{q} a_j m_f(r, D_j) + \frac{\delta_1}{2} \left(\min_{1\leq i\leq q} r_i\right) \sum_{j=1}^{q} m_f(r, D_j)$$

and

$$\sum_{j=1}^{q} a_j T_{f,D_j}(r) \leq \sum_{j=1}^{q} r_j T_{f,D_j}(r) + \frac{\delta_1}{2} \left(\min_{1\leq i\leq q} r_i\right) \frac{1}{\frac{m(m-1)}{(m+n-2)} \frac{2n}{q} \frac{1}{\left(1+\frac{n\alpha}{q}\right)}} \sum_{j=1}^{q} T_{f,D_j}(r).$$

Using these two inequalities, along with (2.2.17) and the First Main Theorem,

$$\begin{aligned}
&\sum_{j=1}^{q} r_j m_f(r, D_j) \leq \sum_{j=1}^{q} a_j m_f(r, D_j) + \frac{\delta_1}{2} \left(\min_{1\leq i\leq q} r_i\right) \sum_{j=1}^{q} m_f(r, D_j) \\
&< \frac{m(m-1)}{(m+n-2)} \frac{2n}{q} \frac{1}{\left(1+\frac{n\alpha}{q}\right)} \sum_{j=1}^{q} a_j T_{f,D_j}(r) + \frac{\delta_1}{2} \left(\min_{1\leq i\leq q} r_i\right) \sum_{j=1}^{q} T_{f,D_j}(r) \\
&\leq \frac{m(m-1)}{(m+n-2)} \frac{2n}{q} \frac{1}{\left(1+\frac{n\alpha}{q}\right)} \sum_{j=1}^{q} a_j T_{f,D_j}(r) + \frac{\delta_1}{2} \sum_{j=1}^{q} r_j T_{f,D_j}(r) \\
&\leq \frac{m(m-1)}{(m+n-2)} \frac{2n}{q} \frac{1}{\left(1+\frac{n\alpha}{q}\right)} \left(\sum_{j=1}^{q} r_j T_{f,D_j}(r) + \frac{\frac{\delta_1}{2}\left(\min_{1\leq i\leq q} r_i\right)}{\frac{m(m-1)}{(m+n-2)} \frac{2n}{q} \frac{1}{\left(1+\frac{n\alpha}{q}\right)}} \sum_{j=1}^{q} T_{f,D_j}(r) \right) \\
&+ \frac{\delta_1}{2} \sum_{j=1}^{q} r_j T_{f,D_j}(r) \\
&\leq \frac{m(m-1)}{(m+n-2)} \frac{2n}{q} \frac{1}{\left(1+\frac{n\alpha}{q}\right)} \sum_{j=1}^{q} r_j T_{f,D_j}(r) + \frac{\delta_1}{2} \sum_{j=1}^{q} r_j T_{f,D_j}(r) + \frac{\delta_1}{2} \sum_{j=1}^{q} r_j T_{f,D_j}(r)
\end{aligned}$$

$$= \left(\frac{m(m-1)}{(m+n-2)} \frac{2n}{q} \frac{1}{\left(1+\frac{n\alpha}{q}\right)} + \delta_1 \right) \sum_{j=1}^{q} r_j T_{f,D_j}(r).$$

To write this inequality in a form useful for the defect relation, let

$$\begin{aligned}\sum_{j=1}^{q} r_j m_f(r, D_j) &< \left(\frac{m(m-1)}{(m+n-2)} \frac{2n}{q} \frac{1}{\left(1+\frac{n\alpha}{q}\right)} + \delta_1 \right) \sum_{j=1}^{q} r_j T_{f,D_j}(r) \\ &\leq \frac{m(m-1)}{(m+n-2)} \frac{2n}{q} \frac{1}{(1+\frac{n\alpha}{2q})} \sum_{j=1}^{q} r_j T_{f,D_j}(r).\end{aligned}$$

That is,

$$\frac{m(m-1)}{(m+n-2)} \frac{2n}{q} \frac{1}{\left(1+\frac{n\alpha}{q}\right)} + \delta_1 \leq \frac{m(m-1)}{(m+n-2)} \frac{2n}{q} \frac{1}{(1+\frac{n\alpha}{2q})},$$

so let

$$\delta_1 = \min\left\{1, \frac{m(m-1)}{(m+n-2)} \frac{2n}{q} \left(\frac{1}{1+\frac{n\alpha}{2q}} - \frac{1}{1+\frac{n\alpha}{q}} \right)\right\}.$$

Again, we saw that $\alpha > 0$, so $\delta_1 > 0$. Thus

$$\sum_{j=1}^{q} r_j m_f(r, D_j) < \frac{m(m-1)}{(m+n-2)} \frac{2n}{q} \frac{1}{(1+C)} \sum_{j=1}^{q} r_j T_{f,D_j}(r)$$

or

$$m_f(r, D) < \frac{m(m-1)}{(m+n-2)} \frac{2n}{q} \frac{1}{(1+C)} T_{f,D}(r)$$

holds for all $r > 0$ outside a set of finite Lebesgue measure, where

$$C = \frac{n\alpha}{2q} = \frac{\min_{1\leq j\leq q}\{D^{n-2}.(r_j D_j)^2\}}{6qn^2 4^n D^n} > 0. \qquad \square$$

Chapter 3

An Improved Height Inequality in Diophantine Approximation

In this chapter, we prove the Arithmetic Main Theorem in [HR16]. The proof is similar to the analytic case in the previous chapter. In general, using Vojta's dictionary [Voj07], statements in Nevanlinna Theory may be translated into Diophantine Approximation and vice versa, including the proofs in some cases.

§3.1 Main Theorem

We make use of the following generalized Schmidt's Subspace Theorem (see [Voj97]).

Theorem 3.1.1. *Let k be a number field and $S \subseteq M_k$ a finite set containing all archimedean places. Let $H_1, \ldots, H_q$ be hyperplanes in $\mathbb{P}^n(k)$ with corresponding Weil functions $\lambda_{H_1}, \ldots, \lambda_{H_q}$. Then there exists a finite union of hyperplanes $Z \subseteq \mathbb{P}^n(k)$,* *depending only on $H_1, \ldots, H_q$ (and not k or S), such that for any $\epsilon > 0$,*

$$\sum_{v \in S} \max_J \sum_{j \in J} \lambda_{H_j,v}(P) \leq (n+1+\epsilon)h(P)$$

holds for all $P \in \mathbb{P}^n(k) \backslash Z$, where the max is taken over all subsets $J \subseteq \{1, \ldots, q\}$ such that the hyperplanes H_j, $j \in J$, are in general position on $\mathbb{P}^n(k)$.

We also need a lemma from Vojta.

Lemma 3.1.2 ([Voj07], Lemma 20.7)**.** *Let X be a projective variety over a number field k and let D be an effective divisor on X. Write*

$$D = \sum_{E \in \sigma_0} (\operatorname{ord}_E D)E$$

and let

$$\Sigma := \left\{ \sigma \subseteq \sigma_0 \mid \bigcap_{E \in \sigma} E \neq \emptyset \right\}.$$

For each $\sigma \in \Sigma$, let

$$D_\sigma := \sum_{E \notin \sigma} (\operatorname{ord}_E D)E.$$

For each place $v \in M_k$, choose a Weil function for each such D_σ. Then there exists a M_k-constant $(C_v)_{v \in M_k}$, depending only on X and D, such that

$$\min_{\sigma \in \Sigma} \lambda_{D_\sigma,v}(P) \leq C_v$$

for all $P \in X(\mathbb{C}_v)$ and all $v \in M_k$.

We are now ready for the proof. For the convenience of the reader, we restate the Main Theorem.

Main Theorem. *Let k be a number field and $S \subseteq M_k$ a finite set containing all archimedean places. Let X be a projective variety, defined over k, of dimension $n \geq 2$, and let $D_1, \ldots, D_q$ be big and nef Cartier divisors on X, defined over k. Let $r_i > 0$ be real numbers such that $D := r_1 D_1 + \cdots + r_q D_q$ has equidegree (such real numbers exist due to Lemma 1.1.3). Assume there exists a positive integer N_0 such that the linear system $|ND_i|$ $(i = 1, \ldots, q)$ is base-point free for $N \geq N_0$ and that $D_1, \ldots, D_q$ are in m-subgeneral position on X. Then*

$$m_S(P, D) < \frac{m(m-1)}{(m+n-2)} \frac{2n}{q} \frac{1}{(1+C)} h_D(P)$$

holds for all $P \in X(k)$ outside a proper Zariski closed subset of X, where

$$C = \frac{\min_{1 \leq j \leq q} \{D^{n-2}.(r_j D_j)^2\}}{6qn^2 4^n D^n} > 0.$$

Proof. Let

$$\alpha = \frac{\min_{1 \leq j \leq q} \{D^{n-2}.(r_j D_j)^2\}}{3n^3 4^n D^n}.$$

Notice $\alpha > 0$ by Lemma 1.1.1, since D_j, $1 \leq j \leq q$, are big and nef, and so

$$\min_{1 \leq j \leq q} \{D^{n-2}.(r_j D_j)^2\} \geq \min_{1 \leq j \leq q} \{(r_j D_j)^n\} > 0.$$

Since D has equidegree with respect to $r_1 D_1, \ldots, r_q D_q$,

$$r_i D_i . D^{n-1} = \frac{1}{q} D^n, \quad 1 \leq i \leq q.$$

So by the density of $\mathbb{Q}$ in $\mathbb{R}$, choose (positive) rational numbers $a_1, \ldots, a_q$ such that both

$$|a_j - r_j| \leq \frac{\delta_1}{2} \left(\min_{1 \leq i \leq q} r_i \right) \min \left\{ 1, \frac{1}{\frac{m(m-1)}{(m+n-2)} \frac{2n}{q} \frac{1}{\left(1+\frac{n\alpha}{q}\right)}} \right\}, \quad 1 \leq j \leq q, \tag{3.1.1}$$

and

$$\left|\frac{D'^n}{a_i D_i . D'^{n-1}} - q\right| < \delta_2, \quad 1 \le i \le q, \quad D' = a_1 D_1 + \cdots + a_q D_q, \tag{3.1.2}$$

where δ_1 and δ_2 will be chosen later such that $\delta_1, \delta_2 \le 1$. Note that

$$|a_j - r_j| \le \frac{\delta_1}{2}\left(\min_{1\le i\le q} r_i\right)\min\left\{1, \frac{1}{\frac{m(m-1)}{(m+n-2)}\frac{2n}{q}\frac{1}{\left(1+\frac{n\alpha}{q}\right)}}\right\} \le \frac{1}{2} r_j, \quad 1 \le j \le q,$$

so

$$D'^n \ge \frac{1}{2^n} D^n \quad \text{and} \quad D'^n \le 2^n D^n. \tag{3.1.3}$$

To clear out the denominators, define the divisor $\widetilde{D} = dD'$, where d is the product of the denominators of $a_1, \ldots, a_q$. Notice that

$$\frac{\widetilde{D}^n}{da_i D_i . \widetilde{D}^{n-1}} = \frac{(dD')^n}{da_i D_i . (dD')^{n-1}} = \frac{d^n D'^n}{d^n (a_i D_i) . D'^{n-1}} = \frac{D'^n}{a_i D_i . D'^{n-1}},$$

so by (3.1.2),

$$\left|\frac{\widetilde{D}^n}{da_i D_i . \widetilde{D}^{n-1}} - q\right| < \delta_2, \quad 1 \le i \le q. \tag{3.1.4}$$

Let $P \in X(M_k)$. By Lemma 3.1.2, for each $v \in S$, there exists a divisor $\widetilde{D}_\sigma$ on X and a Weil function $\lambda_{\widetilde{D}_\sigma, v}$ such that

$$\lambda_{\widetilde{D}_\sigma, v}(P) \le C_v, \tag{3.1.5}$$

where σ is some subset of prime divisors occurring in $\widetilde{D}$ with $\bigcap_{E\in\sigma} E \ne \emptyset$ and C_v is a M_k-constant depending only on X and $\widetilde{D}$. Write

$$\widetilde{D} = \widetilde{D}_0 + \widetilde{D}_\sigma = \sum_{E\in\sigma} (\operatorname{ord}_E \widetilde{D}) E + \widetilde{D}_\sigma$$

and select Weil functions for $\widetilde{D}$ and $\widetilde{D}_0$. Then by (3.1.5) and the additivity of Weil functions,

$$\lambda_{\widetilde{D},v}(P) = \lambda_{\widetilde{D}_0,v}(P) + \lambda_{\widetilde{D}_\sigma,v}(P) = \lambda_{\widetilde{D}_0,v}(P) + O_S(1). \tag{3.1.6}$$

Select Weil functions for each D_i, $i = 1, \ldots, q$, and for each prime divisor $E \in \sigma$. Since $D_1, \ldots, D_q$ are in m-subgeneral position, then by Lemma 2.1.3, P is v-adically close to at most m of the divisors D_i, $i = 1, \ldots, q$, so there are

$$D_{1,v}, \ldots, D_{m,v} \in \{D_1, \ldots, D_q\}$$

such that the prime divisors $E \in \sigma$ only occur in $D_{1,v}, \ldots, D_{m,v}$. So by (3.1.6) and the additivity of Weil functions,

$$\begin{aligned}\sum_{j=1}^{q} da_j \lambda_{D_j,v}(P) = \lambda_{\widetilde{D},v}(P) = \lambda_{\widetilde{D}_0,v}(P) + O_S(1) \qquad (3.1.7)\\ = \sum_{E\in\sigma} (\operatorname{ord}_E \widetilde{D}) \lambda_{E,v}(P) + O_S(1)\\ \leq \sum_{\alpha=1}^{m} da_{\alpha,v} \lambda_{D_{\alpha,v},v}(P) + O_S(1).\end{aligned}$$

Also, by Lemma 2.1.2, for each $v \in S$, $\{D_{1,v}, \ldots, D_{m,v}\}$,

$$\sum_{\substack{\mu,\nu=1\\ \mu\neq\nu}}^{m} \operatorname{lcm}(da_{\mu,v} D_{\mu,v}, da_{\nu,v} D_{\nu,v}) \geq (m+n-2) \sum_{\alpha=1}^{m} da_{\alpha,v} D_{\alpha,v}. \tag{3.1.8}$$

Select Weil functions for each divisor $\operatorname{lcm}(da_{\mu,v} D_{\mu,v}, da_{\nu,v} D_{\nu,v})$, $\mu, \nu = 1, \ldots, m$, $\mu \neq \nu$. Then by (3.1.7), (3.1.8), and the additivity of Weil functions,

$$\begin{aligned}\sum_{j=1}^{q} da_j \lambda_{D_j,v}(P) &\leq \sum_{\alpha=1}^{m} da_{\alpha,v} \lambda_{D_{\alpha,v},v}(P) + O_S(1)\\ &\leq \frac{1}{m+n-2} \sum_{\substack{\mu,\nu=1\\ \mu\neq\nu}}^{m} \lambda_{\operatorname{lcm}(da_{\mu,v} D_{\mu,v}, da_{\nu,v} D_{\nu,v}),v}(P). \qquad (3.1.9)\end{aligned}$$

Fix $D_{\mu,v} \in \{D_{1,v}, \ldots, D_{m,v}\}$. Then for $N \geq N_0$, consider the following filtration for the vector space $H^0(X, \mathcal{O}_X(N\widetilde{D}))$,

$$H^0(X, \mathcal{O}_X(N\widetilde{D})) = W_0 \supset W_1 \supset \cdots \supset W_i \supset \cdots \supset W_N \supset W_{N+1} \supset \cdots \supset \{0\},$$

where $W_k = H^0(X, \mathcal{O}_X(N\widetilde{D} - kda_{\mu,v}D_{\mu,v}))$. Let B be a basis of $H^0(X, \mathcal{O}_X(N\widetilde{D}))$ associated to this filtration. Also, let $\varphi_{N\widetilde{D}} : X \to \mathbb{P}^M(k)$ be the canonical morphism associated to $N\widetilde{D}$ and let $M = h^0(N\widetilde{D}) - 1$. Since $\phi^*_{N\widetilde{D}}\mathcal{O}_{\mathbb{P}^M}(1) = \mathcal{O}_X(N\widetilde{D})$, every section $s \in H^0(X, \mathcal{O}_X(N\widetilde{D}))$ corresponds to a hyperplane $H \subseteq \mathbb{P}^M(k)$ such that $\varphi^*_{N\widetilde{D}}H = (s)$. Note when we write $s \in H^0(X, \mathcal{O}_X(N\widetilde{D} - kda_{\mu,v}D_{\mu,v})) \subseteq H^0(X, \mathcal{O}_X(N\widetilde{D}))$, we mean $s \otimes s_{D_{\mu,v}}^{kda_{\mu,v}} \in H^0(X, \mathcal{O}_X(N\widetilde{D}))$ where $s_{D_{\mu,v}}$ is the canonical section of $\mathcal{O}_X(D_{\mu,v})$, so

$$\varphi^*_{N\widetilde{D}}H = (s) \geq kda_{\mu,v}D_{\mu,v}. \tag{3.1.10}$$

Let $\mathcal{H}_\mu$ be the set of hyperplanes (also depending on v) corresponding to the basis B. Since B is a basis of $H^0(X, \mathcal{O}_X(N\widetilde{D}))$, the hyperplanes in $\mathcal{H}_\mu$ are in general position. Then by (3.1.10),

$$\begin{aligned}\sum_{H \in \mathcal{H}_\mu} \varphi^*_{N\widetilde{D}}H &\geq \left(\sum_{k=0}^{\infty} k \dim\left(W_k/W_{k+1}\right)\right) da_{\mu,v}D_{\mu,v} = \left(\sum_{k=1}^{\infty} \dim W_k\right) da_{\mu,v}D_{\mu,v} \\ &= \left(\sum_{k=1}^{\infty} h^0(N\widetilde{D} - kda_{\mu,v}D_{\mu,v})\right) da_{\mu,v}D_{\mu,v}. \end{aligned}\tag{3.1.11}$$

Now applying Lemma 2.1.1, with $F = \widetilde{D}, E = da_{\mu,v}D_{\mu,v}$, and $\beta := \frac{\widetilde{D}^n}{n\widetilde{D}^{n-1}.(da_{\mu,v}D_{\mu,v})}$, gives us

$$\sum_{k=1}^{\infty} h^0(N\widetilde{D} - kda_{\mu,v}D_{\mu,v}) \tag{3.1.12}$$

$$\geq \sum_{k=1}^{[\beta N]} \left(\frac{\widetilde{D}^n}{n!} N^n - \frac{\widetilde{D}^{n-1}.(da_{\mu,v} D_{\mu,v})}{(n-1)!} N^{n-1} k + \frac{A}{n!} N^{n-2} \min\{k^2, N^2\} \right) + O(N^n)$$
$$\geq \left(\frac{\widetilde{D}^n}{n!} \beta - \frac{\widetilde{D}^{n-1}.(da_{\mu,v} D_{\mu,v})}{(n-1)!} \frac{\beta^2}{2} + \frac{A}{n!} g(\beta) \right) N^{n+1} + O(N^n)$$
$$= \left(\frac{\beta}{2} + \frac{A}{\widetilde{D}^n} g(\beta) \right) \widetilde{D}^n \frac{N^{n+1}}{n!} + O(N^n)$$
$$\geq \left(\frac{\beta}{2} + \widetilde{\alpha} \right) \widetilde{D}^n \frac{N^{n+1}}{n!} + O(N^n),$$

where $A := (n-1)\widetilde{D}^{n-2}.(da_{\mu,v} D_{\mu,v})^2$, $\widetilde{\alpha} := \dfrac{\min_{1 \leq j \leq q} \{\widetilde{D}^{n-2}.(da_j D_j)^2\}}{\widetilde{D}^n} g(\beta)$, and $g : \mathbb{R}^+ \to \mathbb{R}^+$ is the function defined by

$$g(x) = \begin{cases} \frac{x^3}{3}, & 0 < x \leq 1 \\ x - \frac{2}{3}, & x \geq 1. \end{cases}$$

Returning to (3.1.4),

$$\frac{\widetilde{D}^n}{\widetilde{D}^{n-1}.(da_{\mu,v} D_{\mu,v})} = \frac{\widetilde{D}^n}{da_{\mu,v} D_{\mu,v}.\widetilde{D}^{n-1}} > q - \delta_2$$

implies

$$\beta = \frac{\widetilde{D}^n}{n\widetilde{D}^{n-1}.(da_{\mu,v} D_{\mu,v})} > \frac{q - \delta_2}{n}. \tag{3.1.13}$$

Since $\widetilde{D}^n \geq \widetilde{D}^{n-1}.(da_{\mu,v} D_{\mu,v})$, then $\beta \geq \frac{1}{n}$, so $g(\beta) \geq \frac{1}{3n^3}$. Also, since $\widetilde{D} = dD'$, (3.1.3) implies

$$\widetilde{\alpha} = \frac{\min_{1 \leq j \leq q} \{\widetilde{D}^{n-2}.(da_j D_j)^2\}}{\widetilde{D}^n} g(\beta) \geq \frac{\min_{1 \leq j \leq q} \{D'^{n-2}.(a_j D_j)^2\}}{3n^3 D'^n}$$
$$\geq \frac{\min_{1 \leq j \leq q} \{D^{n-2}.(r_j D_j)^2\}}{3n^3 4^n D^n} = \alpha. \tag{3.1.14}$$

Using (3.1.13) and (3.1.14), we can write (3.1.12) as

$$\sum_{k=1}^{\infty} h^0(N\widetilde{D} - k da_{\mu,v} D_{\mu,v}) \geq \left(\frac{\beta}{2} + \widetilde{\alpha} \right) \widetilde{D}^n \frac{N^{n+1}}{n!} + O(N^n)$$

$$> \left(\frac{q-\delta_2}{2n}+\alpha\right)\widetilde{D}^n\frac{N^{n+1}}{n!}+O(N^n),$$

and using this inequality in (3.1.11),

$$\sum_{H\in\mathcal{H}_\mu}\varphi^*_{N\widetilde{D}}H \geq \left(\sum_{k=1}^{\infty}h^0(N\widetilde{D}-kda_{\mu,v}D_{\mu,v})\right)da_{\mu,v}D_{\mu,v}$$
$$> \left(\left(\frac{q-\delta_2}{2n}+\alpha\right)\widetilde{D}^n\frac{N^{n+1}}{n!}+O(N^n)\right)da_{\mu,v}D_{\mu,v}.$$

Fix another $D_{\nu,v}\in\{D_{1,v},\ldots,D_{m,v}\}$. Then similar steps give us

$$\sum_{H\in\mathcal{H}_\nu}\varphi^*_{N\widetilde{D}}H > \left(\left(\frac{q-\delta_2}{2n}+\alpha\right)\widetilde{D}^n\frac{N^{n+1}}{n!}+O(N^n)\right)da_{\nu,v}D_{\nu,v}.$$

Consider the two filtrations of $H^0(X,\mathcal{O}_X(N\widetilde{D}))$ coming from looking at the order of vanishing along $D_{\mu,v}$ and $D_{\nu,v}$, as described previously. Let B be the basis of $H^0(X,\mathcal{O}_X(N\widetilde{D}))$ that Lemma 2.1.4 gives with respect to these two filtrations. Let $\mathcal{H}_{\mu,\nu}$ be the corresponding set of hyperplanes in $\mathbb{P}^M(k)$. Then by the definition of B and similar steps as before,

$$\sum_{H\in\mathcal{H}_{\mu,\nu}}\varphi^*_{N\widetilde{D}}H > \left(\left(\frac{q-\delta_2}{2n}+\alpha\right)\widetilde{D}^n\frac{N^{n+1}}{n!}+O(N^n)\right)da_{\mu,v}D_{\mu,v}$$

and

$$\sum_{H\in\mathcal{H}_{\mu,\nu}}\varphi^*_{N\widetilde{D}}H > \left(\left(\frac{q-\delta_2}{2n}+\alpha\right)\widetilde{D}^n\frac{N^{n+1}}{n!}+O(N^n)\right)da_{\nu,v}D_{\nu,v}.$$

It follows that

$$\sum_{H\in\mathcal{H}_{\mu,\nu}}\varphi^*_{N\widetilde{D}}H > \left(\left(\frac{q-\delta_2}{2n}+\alpha\right)\widetilde{D}^n\frac{N^{n+1}}{n!}+O(N^n)\right)\operatorname{lcm}(da_{\mu,v}D_{\mu,v},da_{\nu,v}D_{\nu,v}).$$

By the additivity of Weil functions,

$$\sum_{H\in\mathcal{H}_{\mu,\nu}}\lambda_{\varphi^*_{N\widetilde{D}}H,v}(P)$$

$$> \left(\left(\frac{q-\delta_2}{2n} + \alpha \right) \widetilde{D}^n \frac{N^{n+1}}{n!} + O(N^n) \right) \lambda_{\mathrm{lcm}\,(da_{\mu,v} D_{\mu,v}, da_{\nu,v} D_{\nu,v}),v}(P),$$

and summing over all $m(m-1)$ distinct $\mu, \nu \in \{1, \ldots, m\}$,

$$\left(\left(\frac{q-\delta_2}{2n} + \alpha \right) \widetilde{D}^n \frac{N^{n+1}}{n!} + O(N^n) \right) \sum_{\substack{\mu,\nu=1 \\ \mu \neq \nu}}^{m} \lambda_{\mathrm{lcm}\,(da_{\mu,v} D_{\mu,v}, da_{\nu,v} D_{\nu,v}),v}(P)$$
$$< \sum_{\substack{\mu,\nu=1 \\ \mu \neq \nu}}^{m} \sum_{H \in \mathcal{H}_{\mu,\nu}} \lambda_{\varphi^*_{N\widetilde{D}} H,v}(P) \leq m(m-1) \left(\max_{\mathcal{H}_{\mu,\nu}} \sum_{H \in \mathcal{H}_{\mu,\nu}} \lambda_{\varphi^*_{N\widetilde{D}} H,v}(P) \right)$$

or

$$\sum_{\substack{\mu,\nu=1 \\ \mu \neq \nu}}^{m} \lambda_{\mathrm{lcm}\,(da_{\mu,v} D_{\mu,v}, da_{\nu,v} D_{\nu,v}),v}(P)$$
$$< \frac{m(m-1)}{\left(\frac{q-\delta_2}{2n} + \alpha \right) \widetilde{D}^n \frac{N^{n+1}}{n!} + O(N^n)} \left(\max_{\mathcal{H}_{\mu,\nu}} \sum_{H \in \mathcal{H}_{\mu,\nu}} \lambda_{\varphi^*_{N\widetilde{D}} H,v}(P) \right).$$

Using this result, (3.1.9) becomes

$$\sum_{j=1}^{q} da_j \lambda_{D_j,v}(P)$$
$$\leq \frac{1}{m+n-2} \sum_{\substack{\mu,\nu=1 \\ \mu \neq \nu}}^{m} \lambda_{\mathrm{lcm}\,(da_{\mu,v} D_{\mu,v}, da_{\nu,v} D_{\nu,v}),v}(P) \tag{3.1.15}$$
$$< \frac{m(m-1)}{(m+n-2)\left(\left(\frac{q-\delta_2}{2n} + \alpha \right) \widetilde{D}^n \frac{N^{n+1}}{n!} + O(N^n) \right)} \left(\max_{\mathcal{H}_{\mu,\nu}} \sum_{H \in \mathcal{H}_{\mu,\nu}} \lambda_{\varphi^*_{N\widetilde{D}} H,v}(P) \right). \tag{3.1.16}$$

Let $\mathcal{H}_P = \bigcup_{\substack{\mu,\nu=1 \\ \mu \neq \nu}}^{m} \mathcal{H}_{\mu,\nu}$ for each $P \in X(M_k)$ and denote $\mathcal{H}_T = \bigcup_{P \in X(M_k)} \mathcal{H}_P$. Then by the functoriality of Weil functions, for any $P \in X(M_k)$,

$$\max_{\mathcal{H}_{\mu,\nu}} \sum_{H \in \mathcal{H}_{\mu,\nu}} \lambda_{\varphi^*_{N\widetilde{D}} H,v}(P) \leq \max_{J} \sum_{H \in J} \lambda_{\varphi^*_{N\widetilde{D}} H,v}(P) = \max_{J} \sum_{H \in J} \lambda_{H,v}(\varphi_{N\widetilde{D}}(P)),$$

where the max is taken over all subsets $J \subseteq \mathcal{H}_T$ consisting of hyperplanes in general position on $\mathbb{P}^M(k)$. Hence (3.1.15) can be written as

$$\sum_{j=1}^{q} da_j \lambda_{D_j,v}(P) < \frac{m(m-1)}{(m+n-2)\left(\left(\frac{q-\delta_2}{2n}+\alpha\right)\widetilde{D}^n \frac{N^{n+1}}{n!} + O(N^n)\right)} \left(\max_J \sum_{H \in J} \lambda_{H,v}(\varphi_{N\widetilde{D}}(P))\right).$$

We can finally sum over the places $v \in S$ on both sides and apply Schmidt's Theorem 3.1.1 (with $\epsilon = 1$) to $\mathbb{P}^M(k)$ and to the set of hyperplanes $\mathcal{H}_T$, so

$$\sum_{j=1}^{q} da_j m_S(P, D_j) < \frac{m(m-1)}{(m+n-2)\left(\left(\frac{q-\delta_2}{2n}+\alpha\right)\widetilde{D}^n \frac{N^{n+1}}{n!} + O(N^n)\right)}(M+2)h(\varphi_{N\widetilde{D}}(P)) \tag{3.1.17}$$

holds for all $\varphi_{N\widetilde{D}}(P) \in \mathbb{P}^M(k)\backslash Z$, where Z is a finite union of hyperplanes in $\mathbb{P}^M(k)$ depending only on $\mathcal{H}_T$, not k or S.

Using Lemma 1.1.1,

$$M + 1 = h^0(N\widetilde{D}) = \frac{\widetilde{D}^n}{n!}N^n + O(N^{n-1}),$$

so by the functoriality of height functions,

$$\begin{aligned}(M+2)h(\varphi_{N\widetilde{D}}(P)) &= (M+2)h_{N\widetilde{D}}(P) \\ &= N(M+2)h_{\widetilde{D}}(P) \\ &= \left(\frac{\widetilde{D}^n}{n!}N^{n+1} + O(N^n) + N\right) h_{\widetilde{D}}(P).\end{aligned}$$

Thus, by (3.1.17),

$$\sum_{j=1}^{q} da_j m_S(P, D_j) < \frac{m(m-1)}{(m+n-2)\left(\left(\frac{q-\delta_2}{2n}+\alpha\right)\widetilde{D}^n \frac{N^{n+1}}{n!} + O(N^n)\right)}(M+2)h(\varphi_{N\widetilde{D}}(P))$$

$$= \left(\frac{m(m-1)}{(m+n-2)}\right) \frac{\frac{\widetilde{D}^n}{n!} N^{n+1} + O(N^n) + N}{\left(\frac{q-\delta_2}{2n} + \alpha\right) \widetilde{D}^n \frac{N^{n+1}}{n!} + O(N^n)} h_{\widetilde{D}}(P)$$

holds for all $\varphi_{N\widetilde{D}}(P) \in \mathbb{P}^M(k) \backslash Z$, where Z is a finite union of hyperplanes in $\mathbb{P}^M(k)$ depending on N.

Now, choose $N \geq N_0$ such that

$$\begin{aligned} \sum_{j=1}^{q} da_j m_S(P, D_j) &< \left(\frac{m(m-1)}{(m+n-2)}\right) \frac{\frac{\widetilde{D}^n}{n!} N^{n+1} + O(N^n) + N}{\left(\frac{q-\delta_2}{2n} + \alpha\right) \widetilde{D}^n \frac{N^{n+1}}{n!} + O(N^n)} h_{\widetilde{D}}(P) \\ &= \left(\frac{m(m-1)}{(m+n-2)}\right) \frac{1 + O(\frac{1}{N}) + O(\frac{1}{N^n})}{\left(\frac{q-\delta_2}{2n} + \alpha\right) + O(\frac{1}{N})} \sum_{j=1}^{q} da_j h_{D_j}(P) \\ &\leq \left(\frac{m(m-1)}{(m+n-2)}\right) \frac{1}{\left(\frac{q-\delta_2}{2n} + \frac{2}{3}\alpha\right)} \sum_{j=1}^{q} da_j h_{D_j}(P). \end{aligned}$$

Let $\delta_2 = \min\{1, \frac{n\alpha}{3}\}$. Notice $\delta_2 > 0$ since we saw earlier that $\alpha > 0$. Now,

$$\begin{aligned} \sum_{j=1}^{q} da_j m_S(P, D_j) &< \left(\frac{m(m-1)}{(m+n-2)}\right) \frac{1}{\left(\frac{q-\delta_2}{2n} + \frac{2}{3}\alpha\right)} \sum_{j=1}^{q} da_j h_{D_j}(P) \\ &\leq \frac{m(m-1)}{(m+n-2)} \frac{2n}{q} \frac{1}{\left(1 + \frac{n\alpha}{q}\right)} \sum_{j=1}^{q} da_j h_{D_j}(P), \end{aligned}$$

and after canceling d on both sides,

$$\sum_{j=1}^{q} a_j m_S(P, D_j) < \frac{m(m-1)}{(m+n-2)} \frac{2n}{q} \frac{1}{\left(1 + \frac{n\alpha}{q}\right)} \sum_{j=1}^{q} a_j h_{D_j}(P). \tag{3.1.18}$$

To write this result in terms of divisor $D = r_1 D_1 + \cdots + r_q D_q$, note (3.1.1) gives us

$$\sum_{j=1}^{q} r_j m_S(P, D_j) \leq \sum_{j=1}^{q} a_j m_S(P, D_j) + \frac{\delta_1}{2} \left(\min_{1 \leq i \leq q} r_i\right) \sum_{j=1}^{q} m_S(P, D_j)$$

and

$$\sum_{j=1}^{q} a_j h_{D_j}(P) \leq \sum_{j=1}^{q} r_j h_{D_j}(P) + \frac{\delta_1}{2} \left(\min_{1 \leq i \leq q} r_i\right) \frac{1}{\frac{m(m-1)}{(m+n-2)} \frac{2n}{q} \frac{1}{\left(1 + \frac{n\alpha}{q}\right)}} \sum_{j=1}^{q} h_{D_j}(P).$$

Using these two inequalities, along with (3.1.18) and the First Main Theorem,

$$\begin{aligned}
\sum_{j=1}^{q} r_j m_S(P, D_j) &\leq \sum_{j=1}^{q} a_j m_S(P, D_j) + \frac{\delta_1}{2}\left(\min_{1\leq i\leq q} r_i\right)\sum_{j=1}^{q} m_S(P, D_j) \\
&< \frac{m(m-1)}{(m+n-2)}\frac{2n}{q}\frac{1}{\left(1+\frac{n\alpha}{q}\right)}\sum_{j=1}^{q} a_j h_{D_j}(P) + \frac{\delta_1}{2}\left(\min_{1\leq i\leq q} r_i\right)\sum_{j=1}^{q} h_{D_j}(P) \\
&\leq \frac{m(m-1)}{(m+n-2)}\frac{2n}{q}\frac{1}{\left(1+\frac{n\alpha}{q}\right)}\sum_{j=1}^{q} a_j h_{D_j}(P) + \frac{\delta_1}{2}\sum_{j=1}^{q} r_j h_{D_j}(P) \\
&\leq \frac{m(m-1)}{(m+n-2)}\frac{2n}{q}\frac{1}{\left(1+\frac{n\alpha}{q}\right)}\left(\sum_{j=1}^{q} r_j h_{D_j}(P) + \frac{\frac{\delta_1}{2}\left(\min_{1\leq i\leq q} r_i\right)}{\frac{m(m-1)}{(m+n-2)}\frac{2n}{q}\frac{1}{\left(1+\frac{n\alpha}{q}\right)}}\sum_{j=1}^{q} h_{D_j}(P)\right) \\
&+ \frac{\delta_1}{2}\sum_{j=1}^{q} r_j h_{D_j}(P) \\
&\leq \frac{m(m-1)}{(m+n-2)}\frac{2n}{q}\frac{1}{\left(1+\frac{n\alpha}{q}\right)}\sum_{j=1}^{q} r_j h_{D_j}(P) + \frac{\delta_1}{2}\sum_{j=1}^{q} r_j h_{D_j}(P) + \frac{\delta_1}{2}\sum_{j=1}^{q} r_j h_{D_j}(P) \\
&= \left(\frac{m(m-1)}{(m+n-2)}\frac{2n}{q}\frac{1}{\left(1+\frac{n\alpha}{q}\right)} + \delta_1\right)\sum_{j=1}^{q} r_j h_{D_j}(P).
\end{aligned}$$

To write this inequality in a form useful for the defect relation, let

$$\begin{aligned}
\sum_{j=1}^{q} r_j m_S(P, D_j) &< \left(\frac{m(m-1)}{(m+n-2)}\frac{2n}{q}\frac{1}{\left(1+\frac{n\alpha}{q}\right)} + \delta_1\right)\sum_{j=1}^{q} r_j h_{D_j}(P) \\
&\leq \frac{m(m-1)}{(m+n-2)}\frac{2n}{q}\frac{1}{(1+\frac{n\alpha}{2q})}\sum_{j=1}^{q} r_j h_{D_j}(P).
\end{aligned}$$

That is,

$$\frac{m(m-1)}{(m+n-2)}\frac{2n}{q}\frac{1}{\left(1+\frac{n\alpha}{q}\right)} + \delta_1 \leq \frac{m(m-1)}{(m+n-2)}\frac{2n}{q}\frac{1}{(1+\frac{n\alpha}{2q})},$$

so let

$$\delta_1 = \min\left\{1, \frac{m(m-1)}{(m+n-2)}\frac{2n}{q}\left(\frac{1}{1+\frac{n\alpha}{2q}} - \frac{1}{1+\frac{n\alpha}{q}}\right)\right\}.$$

Again, we saw that $\alpha > 0$, so $\delta_1 > 0$. Thus

$$\sum_{j=1}^{q} r_j m_S(P, D_j) < \frac{m(m-1)}{(m+n-2)} \frac{2n}{q} \frac{1}{(1+C)} \sum_{j=1}^{q} r_j h_{D_j}(P)$$

or

$$m_S(P, D) < \frac{m(m-1)}{(m+n-2)} \frac{2n}{q} \frac{1}{(1+C)} h_D(P)$$

holds for all $P \in X(k)$ outside a Zariski closed subset of X, where

$$C = \frac{n\alpha}{2q} = \frac{\min_{1 \leq j \leq q} \{D^{n-2}.(r_j D_j)^2\}}{6qn^2 4^n D^n} > 0. \qquad \square$$

Chapter 4

Other Approaches

§4.1 Further Improved Qualitative Results

In 2009, Corvaja, Levin, and Zannier [PCZ09] introduced a new approach to the problem of degeneracy of holomorphic curves and integral points.

In Nevanalinna Theory, they proved the following statement.

Theorem 4.1.1. *Let X be a complex projective variety of dimension $n > 2$ and let $D_1, \ldots, D_q$ be ample Cartier divisors such that $D_1 + \cdots + D_q$ is a reduced normal crossings divisor. Assume $q > n^2 - n$. Then there does not exist a holomorphic map $f : \mathbb{C} \to X$ with Zariski dense image. Furthermore, there exists a proper Zariski-closed subset $Y \subsetneq X$ such that the image of any non-constant holomorphic map $f : \mathbb{C} \to X$ is contained in Y.*

The corresponding statement in Diophantine Approximation is the following:

Theorem 4.1.2. *Let k be a number field and $S \subseteq M_k$ a finite set containing all archimedean places. Let X be a projective variety, defined over k, of dimension $n > 2$, and let $D_1, \ldots, D_q$ be ample Cartier divisors on X, defined over k, such that $D_1 + \cdots + D_q$ is a reduced normal crossings divisor. Assume $q > n^2 - n$. Then no set of (D, S)-integral points of X is Zariski dense in X. Furthermore, there exists a proper Zariski-closed subset $Y \subseteq X$, independent of k and S, such that $X \backslash Y$ has only finitely many (D, S)-integral points of X.*

These qualitative results, however, appear to not have a corresponding second main theorem type of statement as of yet.

§4.2 Pointwise Filtration

We saw in chapter two, using the joint filtration lemma 2.1.4, for $q \geq n^2$, every holomorphic mapping $f : \mathbb{C} \to X \backslash D$ must be constant. Theorem 4.1.1 improves this result to $q > n^2 - n$ for varieties of dimension > 2 by using what the authors call a pointwise filtration approach.

In the proof of the Analytic Main Theorem in chapter two, the filtrations involved sections which vanish to a high order along divisors which contain a given point. In [PCZ09], sections are considered which vanish to a high order along the intersection of divisors. Imposing vanishing conditions at a point is significantly less restrictive than imposing vanishing along whole divisors.

This pointwise filtration works by constructing a basis for $H^0(X, \mathcal{O}_X(D))$ which consists of sections that "on average" vanish at a point. This vanishing implies that certain linear forms corresponding to this basis take small values at a point, which allows the successful application of Diophantine Approximation/Nevanlinna Theory. These filtrations may be constructed by using linear algebra on the power series locally representing the sections.

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