COMPARISONS OF PIECEWISE RITZ AND WEIGHTED-RESIDUAL METHODS

A Thesis

Presented to

the Faculty of the Department of Mechanical Engineering

College of Engineering

University of Houston

In Partial Fulfillment

of the Requirements for the Degree

Master of Science in Mechanical Engineering

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by

Oscar E. Alvarez

May 1971

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ABSTRACT

To date, finite element techniques using the Ritz Method have found wide use in solving differential equations in the applied sciences, particularly in structural analysis. Finite Element Techniques using weighted-residual methods, however, have not been tried. This thesis develops Finite Element Techniques using the Collocation, Subdomain, Galerkin, and Least-Squares weighted-residual methods, and compares them to the Ritz Method.

Numerical results are presented for a one-dimensional problem using all weighted-residual methods and the Ritz Method. From a standpoint of accuracy and ease of formulation and programming, the Subdomain Method compared favorably with the Ritz Method.

The solution of a two-dimensional problem was attempted. The weighted-residual techniques led to a singular matrix. This was due, probably, to the poor choice of an approximation family.

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Chapter 1

INTRODUCTION

The analysis of many problems in engineering and the applied sciences depends upon obtaining solutions to differential equations. In all but the simplest of cases, an analytic solution is difficult or impossible to obtain because of complicating factors such as the number of dimensions involved, nonlinearity of the equations (coefficients of the derivatives being themselves functions of the dependent variable), complex geometry of the boundary, etc. Because of this, techniques have been developed to approximate the solution of the problem.

As early as 1795, Carl Fredrich Gaus developed an approximating technique [1]*, the Least-Squares Method. This technique depended upon finding the best fit solution among an assumed family of solutions. Other methods, Collocation [2], Subdomain [3], and Galerkin's [4], used the same approach but changed the criterion for determining the best fit. In 1908, W. Ritz [5] proposed a method whereby, when possible, the equation was first reformulated into its variational statement. A best fit was then obtained for the solution of the problem

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*Numbers in brackets refer to identically numbered entries in the References at the conclusion of this thesis.

in this variational form. These approximating techniques were applied to the entire domain of the problem.

Another approximating technique, the Finite-Difference Method, sought numerical solutions to the differential equations by approximating the equation by the corresponding difference equation. This technique requires many calculations in order to obtain a good approximation, a drawback when calculations had to be performed manually.

The rapid development of computer hardware since the late forties has given rise to a corresponding development of finite techniques for solving differential equations. These techniques generally took advantage of the computer's ability to perform many repetitive calculations with lightning rapidity.

Because the computer removed the obvious computational drawback of finite difference methods, these methods were first used in computer application. In the 1950's, the Finite-Element Technique was developed. This technique breaks the domain of the problem into smaller elements. The Ritz Method is then applied in an implicit manner in which the approximate solution is expressed in terms of the initially unknown nodal values of the dependent variables.

In this thesis, the other approximating techniques (Collocation, Subdomain, Galerkin, Least-Squares) are applied in a manner analogous to that employed for (Ritz) finite elements. That is, the entire domain is subdivided into finite elements. The various methods are then applied in a

piecewise manner. This piecewise application of the weighted-residual method has not been attempted previously.

The various Finite-Element Techniques are then compared from the standpoint of programming efficiency, machine running time, accuracy of solution, and rate of convergence of solution with respect to the number of elements.

Chapter 2

SURVEY OF PREVIOUS RESEARCH

The major contribution to the development of the Finite-Element Method has been in the field of structural analysis. This is a natural development when one views the network formed by the elements as a structure of beams and when one recognizes that variational techniques (Ritz Method) have long been standard techniques in structural analysis. Research in the field developed as follows.

In the 1940's, Henrikoff [8], McHenry [9], and Newmark [10] suggested that the analysis of elasticity in plates and continuous bodies could be approximated by the analysis of elasticity in a network or lattice. At this time, Henrikoff noted that such a method has the advantage of overcoming the complex geometry of the boundary, but that a major disadvantage was "the great amount of labor of computation. . . ." In 1955 and 1956, Argyris [11] and Turner [12], et al., actually approximated continuous problems with networks and introduced the term "finite element." This work used the Ritz Method for problems in structural analysis. These researchers justified their method by means of physical analogy arguments rather than theoretical mathematical considerations. Although they were unaware of the fact, the theoretical justification for their method had already been

developed by Courant [12] in 1943. In his work, Courant actually solved a plane torsion problem by subdividing the domain into elements and applying the Ritz Method. Following this, many workers in structural analysis considered such questions as new element shapes, varying element size, numbers of nodes per element, etc., and applied them to specific structural problems. Over fifty such references are cited by Zienkiewicz [14] in his review of the development of the Finite-Element Method.

The earliest reference to the use of Finite-Element Methods in areas other than structural analysis is Zienkiewicz [15] in 1965 and Wilson and Nickell [16] in 1966. These researchers both dealt with the application of the method to heat-conduction problems. Taylor [17] (1967), Martin [18] (1968), and Atkinson [19] (1969) applied the method to fluid mechanics.

Although much work has been done applying the Ritz Method to finite elements, there has been very little work using the concept of subdividing the domain into elements and applying weighted residual methods.

Chapter 3

DEVELOPMENT OF THE TECHNIQUES

In this section, the classical approximating techniques for the solution of differential equations are briefly developed and are then formulated in the Finite-Element Method.

As the development of the classical "whole domain" techniques follows Crandall's [6] explanation, the reader may refer to this source for more details. All techniques of approximation have in common the choice of a trial solution which, because of undetermined parameters or undetermined functions, actually represents a whole family of possible approximations. A "best fit" approximation is then chosen from within the family. The various techniques differ in the criteria for choosing this "best fit." Since the techniques do choose the best solution from a prescribed family, their success depends, to some extent, on the family selected. That is, the methods will not overcome a very poor choice of initial family.

3.1 General Formulation

The discussion will be limited to the system,

$$L_{2m}[Y] = F; \quad B_i[Y] = g_i, \quad i = 1,\ldots,m \ , \qquad (3.1)$$

where $L_{2m}[Y]$ is an expression containing Y and its derivatives up to order 2m, F is a function of the dependent variable(s) only, $B_i[Y]$ is an expression in Y and its derivatives normal to the boundary which applies to the entire closed boundary of the domain of the problem, g_i are the values known at every boundary point, and the system of equations,

$$B_i[Y] = g_i \quad i = 1,\ldots,m \ ,$$

is linearly independent. We consider the trial solution to have the linear form,

$$Y = \sum_{i=1}^{r} \alpha_i \phi_i \ , \tag{3.2}$$

where the ϕ_i (i=1,...,r) are known functions in the domain D, $\phi_1 = 1$, the α_i (i=1,...,r) are undetermined constant parameters, and the set $\phi_1,\ldots,\phi_r$ is linearly independent.

This trial solution could be written in matrix notation as

$$Y = \langle \phi_1, \phi_2, \ldots, \phi_r \rangle \begin{Bmatrix} \alpha_1 \\ \vdots \\ \alpha_r \end{Bmatrix} \ , \tag{3.3}$$

where $\langle\ \rangle$ is a row vector and $\{\}$ is a column vector. In the Finite-Element Method, the domain is considered to be divided into K elements. The approximating techniques are then applied to obtain a different "best fit" for each element of the domain. The boundary conditions of each element are

provided by geometric consideration which makes them implicitly defined in terms of the real boundary conditions of the entire domain. Each element is considered to have n nodes. Then for the J^{th} element, a trial solution is

$$ {}_{J}Y = \langle {}_{J}\phi_1, {}_{J}\phi_2, \ldots, {}_{J}\phi_r \rangle \begin{Bmatrix} {}_{J}\alpha_1 \\ \vdots \\ {}_{J}\alpha_r \end{Bmatrix} \tag{3.4} $$

If r is set equal the number of nodes, n, and if the equation is evaluated at each node, the system becomes:

$$ \begin{aligned} {}_{J}Y_1 &= \langle {}_{J,1}\phi_1, {}_{J,1}\phi_2, \ldots, {}_{J,1}\phi_n \rangle \begin{Bmatrix} {}_{J}\alpha_1 \\ \vdots \\ {}_{J}\alpha_n \end{Bmatrix} \\ &\vdots \qquad\quad \vdots \qquad\quad \vdots \qquad\quad \vdots \\ {}_{J}Y_n &= \langle {}_{J,n}\phi_1, {}_{J,n}\phi_2, \ldots, {}_{J,n}\phi_n \rangle \begin{Bmatrix} {}_{J}\alpha_1 \\ \vdots \\ {}_{J}\alpha_n \end{Bmatrix} \end{aligned} \tag{3.5} $$

where ${}_{J}Y_1, \ldots, {}_{J}Y_n$ are the nodal values of the dependent variable in the J^{th} element and ${}_{J,1}\phi_1, \ldots, {}_{J,i}\phi_n$ $(i=1,\ldots,n)$ are the linearly independent known functions evaluated at the n nodes of the J^{th} element. This can be written as

$$ \begin{Bmatrix} {}_{J}Y_1 \\ \vdots \\ {}_{J}Y_n \end{Bmatrix} = \begin{bmatrix} {}_{J,1}\phi_1 & \cdots & {}_{J,1}\phi_n \\ \vdots & & \vdots \\ {}_{J,n}\phi_1 & \cdots & {}_{J,n}\phi_n \end{bmatrix} \begin{Bmatrix} {}_{J}\alpha_1 \\ \vdots \\ {}_{J}\alpha_n \end{Bmatrix} \tag{3.6} $$

or, in short notation, as

$$\{ {}_J Y_N \} = [{}_J \Phi_N] \{ {}_J \alpha_N \} \quad . \tag{3.7}$$

Since the ${}_J Y_N$ are implicitly known and the ${}_J \Phi_N$ functions are linearly independent, the unknown coefficients can be expressed as a function of the nodal values of the element J,

$$\{ {}_J \alpha_N \} = [{}_J \Phi_N]^{-1} \{ {}_J Y_N \} \quad , \tag{3.8}$$

where $[]^{-1}$ denotes the inverse of the matrix. Then from Equations (3.4) and (3.8) with r = n, it follows that

$${}_J Y = \langle {}_J \Phi_1, {}_J \Phi_2, \ldots, {}_J \Phi_n \rangle [{}_J \Phi_N]^{-1} \{ {}_J Y_N \} \tag{3.9}$$

which represents the trial solution as a function of the n nodal values of the element J.

Thus, just as in the "whole domain" technique, the values of the r parameters $\alpha_1, \ldots, \alpha_r$ need to be evaluated. In the Finite-Element Method, the values of the parameters

$$\begin{bmatrix} {}_1\alpha_1, {}_1\alpha_2, \ldots {}_1\alpha_n \\ \vdots \\ {}_K\alpha_1, {}_K\alpha_2, \ldots {}_K\alpha_n \end{bmatrix} \tag{3.10}$$

need to be evaluated by using a best fit technique. Because the topology of each element is uniform, this presents no major problem for computer application. To derive the criteria for one element is sufficient.

There are two basic types of techniques for getting the best fit among the family of possible approximations, once the family is fixed: the Weighted-Residual Methods and the Stationary Functional Method. In either case, the result of applying the technique is a set of n simultaneous equations for each element.

3.2 Boundary Condition

In the derivations to follow, each element is treated as an independent domain. The boundary conditions of the element are represented through the initially unknown nodal values on the element boundary. Then in an implicit manner, the nodes of this element are expressed in terms of the known boundary condition of the whole domain D.

3.3 Weighted Residual Techniques

The residual of the system (3.1) is defined as:

$$R = F - L_{2m}[Y] \quad . \tag{3.11}$$

For the J^{th} element using the trial solution (3.4), the residual is of the form

$$\begin{aligned} {}_JR &= {}_JF - L_{2m}[{}_JY] \\ &= {}_JF - L_{2m}\left[\langle 1, {}_J\Phi_2, \ldots, {}_J\Phi_n \rangle \, [{}_J\Phi_N]^{-1} \, \{{}_JY_N\} \right] \end{aligned} \tag{3.12}$$

which is equal to zero for the exact solution.

Within a fixed family, a "good" approximation may be described as one which maintains ${}_JR$ relatively small. The

criterion of the Weighted Residual-Methods is that the various weighted averages of R should vanish.

3.3.1 Collocation

The residual ${}_JR$ (3.12) is set to zero at the n nodes of the element J.

$$\begin{aligned} {}_JR(1) &= 0 \\ &\vdots \\ {}_JR(n) &= 0 \ . \end{aligned} \tag{3.13}$$

3.3.2 Subdomain

The J^{th} element is subdivided into n subdomains, usually according to a simple pattern. The integral of the residual ${}_JR$ (3.12) over each subdomain is then set equal to zero to provide n simultaneous equations for the nodal value.

$$\begin{aligned} \int_{SD_1} {}_JR\,dSD_1 &= 0 \\ &\vdots \\ \int_{SD_m} {}_JR\,dSD_m &= 0 \ , \end{aligned} \tag{3.14}$$

where SD_i is the area of subdomain i.

3.3.3 Galerkin

This method takes the n integrals over the entire element weighted by the n trial solution functions.

$$\int_{D_J} {}_J\Phi_K \ {}_JR\,dD_J = 0 \qquad (K=1,\ldots,n) \tag{3.15}$$

where D_J is the domain of element J (where ${}_J\Phi_1 = 1$) within each element.

3.3.4 Least Squares

The integral of the square of the residual is minimized with respect to the undetermined parameters, which in this case are the nodal values, to provide n simultaneous equations for the J^{th} element,

$$\begin{aligned} \frac{\partial}{\partial {}_JY_1} \int_{D_J} {}_JR^2 dD_J &= 0, \\ \vdots \qquad\qquad & \\ \frac{\partial}{\partial {}_JY_N} \int_{D_J} {}_JR^2 dD_J &= 0. \end{aligned} \tag{3.16}$$

3.4 Stationary Functional Method

The Ritz Method: Solving a given boundary-value problem is equivalent to solving the corresponding variational statement of the problem, i.e., extremizing the functional ${}_JI$ associated with the initial differential equation. The Ritz Method is an approximation technique for solving variational problems. The trial family (3.9) is substituted into the functional

$${}_JI = \int_{D_J} {}_JG({}_JY) dD_J \ , \tag{3.17}$$

where the function ${}_JG$ is derived from the differential equation and D_J is the domain of the J^{th} element. By the

method, the functional is then minimized with respect to the unknown parameters (nodal values) of the trial family,

$$\begin{aligned} \frac{\partial J^I}{\partial J^Y_1} &= 0 \\ &\vdots \\ \frac{\partial J^I}{\partial J^Y_n} &= 0 \quad . \end{aligned} \tag{3.18}$$

Chapter 4

APPLICATION TO SPECIFIC EXAMPLES

In this section, the Finite-Element Method is applied to specific problems. The different techniques developed in Chapter 3 are then used to seek the best solution within the family of possible approximations.

4.1 One-Dimensional Problem with Quadratic Approximation

The general form of differential equation to be considered is as follows:

$$L(Y) - F = \frac{d}{dX}(PY') - QY - F = 0, \qquad (4.1)$$

in the interval (0,1) where Y' is $\frac{dY}{dX}$, and P = P(X), Q = Q(X), F = F(X) are functions of the independent variable, with the boundary condition

$$Y(0) = Y(1) = 0.$$

4.1.1 Formulation

In the formulation of the problem, it was assumed that the functions (PX), Q(X) have the constant value, one, and the function F(X) is equal to the independent variable X. Hence the actual equation is

$$\frac{d}{dX}(Y') - Y - X = 0.$$

Note that even when P,Q are not constants in the whole domain, a good approximation could still be sought by evaluating P and Q within each element.

A "best fit" solution will be sought from the family,

$$\Phi_1 = 1, \quad \Phi_2 = X, \quad \Phi_3 = X^2 \qquad (r=3).$$

In finite-element terms, Equation (3.4) yields a trial solution for the J^{th} element of the form.

$$ {}_JY = \langle 1, X, X^2 \rangle \begin{Bmatrix} {}_J\alpha_1 \\ {}_J\alpha_2 \\ {}_J\alpha_3 \end{Bmatrix} . \qquad (4.2)$$

The domain D, which is a line, is divided into a finite number of elements. Because there are three undetermined parameters, each element has three nodes. The trial solution (4.2) is expressed as a function of the three nodal values of the J^{th} element by using (3.9):

$$ {}_JY = \frac{1}{{}_J\Delta} \langle 1, X, X^2 \rangle \begin{bmatrix} {}_Ja_1 & {}_Ja_2 & {}_Ja_3 \\ {}_Jb_1 & {}_Jb_2 & {}_Jb_3 \\ {}_Jc_1 & {}_Jc_2 & {}_Jc_3 \end{bmatrix} \begin{Bmatrix} {}_JY_1 \\ {}_JY_2 \\ {}_JY_3 \end{Bmatrix} \qquad (4.3)$$

where $X = {}_JX_1 + S_J$. S_J is the local coordinate of the J^{th} element. ${}_JX_1$ is the X coordinate of the first node of the J^{th} element (see Figure 4.1).

$$ [{}_J\Phi_N]^{-1} = \frac{1}{{}_J\Delta} \begin{bmatrix} {}_Ja_1 & {}_Ja_2 & {}_Ja_3 \\ {}_Jb_1 & {}_Jb_2 & {}_Jb_3 \\ {}_Jc_1 & {}_Jc_2 & {}_Jc_3 \end{bmatrix} . \qquad (4.4)$$

$_J\Delta = \left|[_J\Phi_N]\right|$, the determinant of $[_J\Phi_N]$.

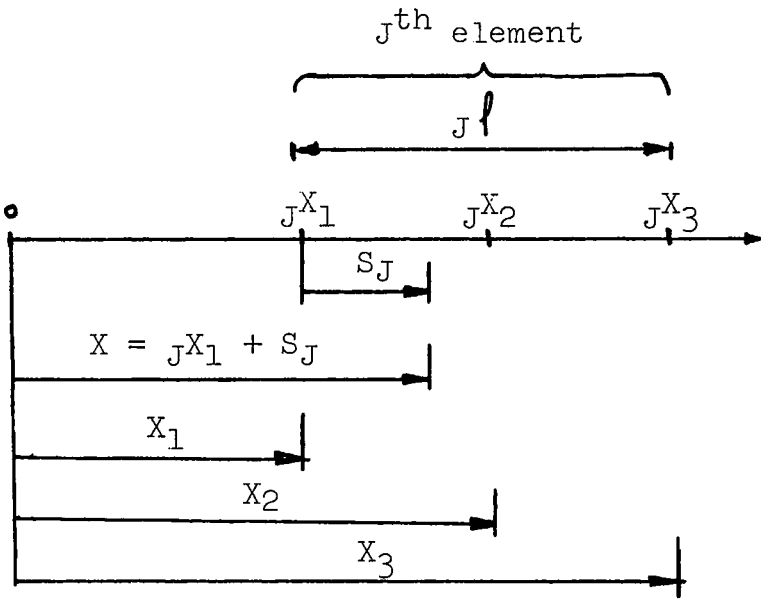


Figure 4.1. Relationship of Local and X Coordinate

When the trial solution (4.3) is substituted into (3.12), the expression of the residual becomes

$$_JR = \left(\frac{2\,{}_JP}{{}_J\Delta} \langle {}_Ja_3, {}_Jb_3, {}_Jc_3 \rangle - \frac{{}_JQ}{{}_J\Delta} \langle 1, X, X^2 \rangle \right.$$

$$\left. \begin{bmatrix} {}_Ja_1 & {}_Ja_2 & {}_Ja_3 \\ {}_Jb_1 & {}_Jb_2 & {}_Jb_3 \\ {}_Jc_1 & {}_Jc_2 & {}_Jc_3 \end{bmatrix} \right) \begin{Bmatrix} {}_JY_1 \\ {}_JY_2 \\ {}_JY_3 \end{Bmatrix} - F = 0 \qquad (4.5)$$

$$
\begin{aligned}
{}_JX_1 \leq X \leq {}_JX_1 + {}_J\ell/3 \\
{}_JX_1 + {}_J\ell/3 \leq X \leq {}_JX_1 + 2_J\ell/3 \\
{}_JX_1 + 2_J\ell/3 \leq X \leq {}_JX_1 + {}_J\ell
\end{aligned}
$$

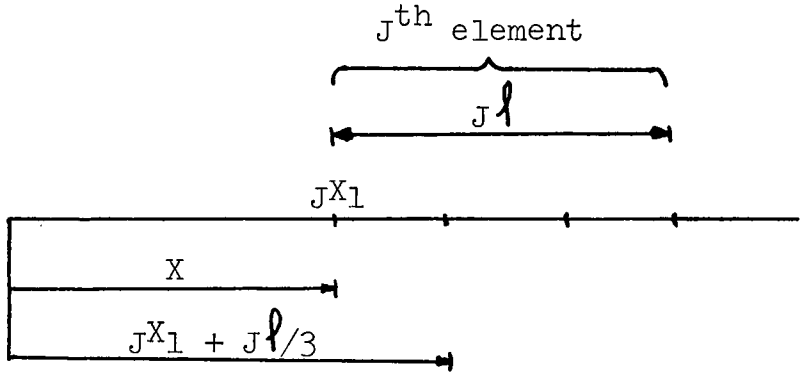


Figure 4.2. Division of the Element for Subdomain Method

Equations (3.14) and (4.5) yield:

$$
\left(\frac{2_J P_J G_{1,i}}{J\Delta} \langle A_{K,3} \rangle - \frac{JQ}{J\Delta} \langle {}_JG_{1,i}, {}_JG_{2,i}, {}_JG_{3,i} \rangle [A_{K,K}] \right) \left\{ {}_JY_K \right\} + {}_JG_{2,i} = 0 \tag{4.10}
$$

where

$$
{}_JG_{k,m} = \frac{({}_JX_{m+1})^k - ({}_JX_m)^k}{k} \tag{4.11}
$$

and

$$
{}_JX_t = {}_JX_1 + (t - 1)\ {}_J\ell/3 \qquad (t=1,2,3,4) \tag{4.12}
$$

and i specifies the subdomain within the J^{th} element.

or, in short notation, as

$$ {}_JR = \left(\frac{2\,{}_JP}{{}_J\Delta} \langle A_{K,3} \rangle - \frac{{}_JQ}{{}_J\Delta} \langle X \rangle [A_{K,K}]\right) \left\{ {}_JY_K \right\} - F = 0 \, . $$

The various weighted-residual methods use this residual in various expressions to determine a "best fit" solution.

4.1.1.1 Collocation

The residual is set to zero at the three nodes of the J^{th} element (see Figure 4.1),

$$ \begin{aligned} X_1 &= {}_JX_1 \\ X_2 &= {}_JX_1 + {}_J\ell/2 \\ X_3 &= {}_JX_1 + {}_J\ell \end{aligned} \tag{4.6} $$

where ${}_J\ell$ is the length of the J^{th} element.

From Equations (4.5) and (4.6), it follows that

$$ {}_JR({}_JX_1) = \left(\frac{2\,{}_JP}{{}_J\Delta} \langle A_{K,3} \rangle - \frac{{}_JQ}{{}_J\Delta} \langle 1, {}_JX_1, {}_JX_1^2 \rangle [A_{K,K}]\right) \left\{ {}_JY_K \right\} - {}_JX_1 = 0. \tag{4.7} $$

Similarly,

$$ {}_JR({}_JX_1 + {}_J\ell/2) = 0, \tag{4.8} $$

$$ {}_JR({}_JX_1 + {}_J\ell) = 0. \tag{4.9} $$

For all the elements, Equations (4.7) through (4.9) are combined to evaluate the nodal values of Y_J for the element.

4.1.1.2 Subdomain

The J^{th} element is subdivided into three subdomains (see Figure 4.2):

4.1.1.3 Galerkin

When Equation (3.15) with ${}_J\phi_1 = 1$, ${}_J\phi_2 = X$, ${}_J\phi_3 = X^2$, is combined with the expression of the residual (4.5), the following is obtained:

$$\left(\frac{2\,{}_JP\,{}_JH_{i+1}}{{}_J\Delta} \langle A_{K,3} \rangle - \frac{{}_JQ}{{}_J\Delta} \langle {}_JH_{i+1}, {}_JH_{i+2}, {}_JH_{i+3} \rangle \right.$$

$$\left. [A_{K,K}] \right) \left\{ {}_JY_K \right\} + {}_JH_{i+2} = 0 \qquad (4.13)$$

$$(i=1,2,3)$$

where

$${}_JH_k = \frac{{}_JX_3^k - {}_JX_1^k}{k} \qquad (4.14)$$

and

$${}_JX_3 = {}_JX_1 + {}_J\ell \quad . \qquad (4.15)$$

See Figure 4.1.

4.1.1.4 Least Squares

Equations (3.16) can be reformulated into

$$\int_{D_J} {}_JR \frac{\partial\, {}_JR}{\partial\, {}_J\alpha_1} d{}_JD = 0; \quad \int_{D_J} {}_JR \frac{\partial\, {}_JR}{\partial\, {}_J\alpha_2} d{}_JD = 0; \quad \int_{D_J} {}_JR \frac{\partial\, {}_JR}{\partial\, {}_J\alpha_3} d{}_JD = 0. \qquad (4.16)$$

With Equations (4.2) and (3.12), it follows that

$${}_JR = 2\,{}_J\alpha_3\,{}_JP - {}_JQ({}_J\alpha_1 + {}_J\alpha_2 X + {}_J\alpha_3 X^2) - X = 0. \qquad (4.17)$$

The partial derivatives with respect to the unknown coefficients are (from Equation 4.17):

$$\frac{\partial\, {}_JR}{\partial\, {}_J\alpha_1} = -{}_JQ; \quad \frac{\partial\, {}_JR}{\partial\, {}_J\alpha_2} = -{}_JQX; \quad \frac{\partial\, {}_JR}{\partial\, {}_J\alpha_3} = 2\,{}_JP - {}_JQX^2. \qquad (4.18)$$

Equation (4.18) is substituted into Equations (4.16) to yield:

$$ {}_JI_1 = -{}_JQ\left(\int_{D_J} {}_JR\,dD_J \right) = 0 \qquad (4.19) $$

$$ {}_JI_2 = -{}_JQ\left(\int_{D_J} {}_JRX\,dD_J \right) = 0 \qquad (4.20) $$

$$ {}_JI_3 = 2_JP\left(\int_{D_J} {}_JR\,dD_J \right) - Q\left(\int_{D_J} {}_JRX^2\,dD_J \right) = 0. \qquad (4.21) $$

It can be seen that Equations (4.19) and (4.20) are the same as Equations (3.15) for K = 1,2 (i.e., ${}_J\Phi_1 = 1$, ${}_J\Phi_2 = X$) respectively, and Equation (4.21) is a combination of Equations (3.15) for i = 1,3 (i.e., ${}_J\Phi_1 = 1$, ${}_J\Phi_3 = X^2$). Therefore, application of the Least-Square Method is seen to result in a series of equations (4.19, 4.20, 4.21) which are, in fact, linear combinations of the equations resulting from the Galerkin Method.

4.1.1.5 Ritz

The variational statement of Equation (4.1) is [7]:

$$ I = \int_0^{J\ell} [{}_JP({}_JY')^2 + {}_JQ_JY^2 + 2_JF_JY]\,dX \qquad (4.22) $$

The expressions for $({}_JY')^2$ and Y^2 are found from Equation (4.3). These are substituted into Equation (4.22). When the partial derivatives with respect to the nodal values are taken, the final equation is of the form:

$$ {}_{J}T_{i}\,{}_{J}L_{1} + {}_{J}G_{i}\,{}_{J}L_{2} + {}_{J}W_{i}\,{}_{J}L_{3} - {}_{J}Z_{i} = 0 \qquad (4.23) $$

where

$$ {}_{J}L_{i} = ({}_{J}a_{i}\,{}_{J}Y_{1} + {}_{J}b_{i}\,{}_{J}Y_{2} + {}_{J}c_{i}\,{}_{J}Y_{3}) \quad (i=1,2,3) \qquad (4.24) $$

$$ {}_{J}T_{1} = {}_{J}Q \left\langle {}_{J}V_{1}, {}_{J}V_{2}, \frac{2}{3}{}_{J}V_{3} \right\rangle \begin{Bmatrix} {}_{J}a_{1} \\ {}_{J}a_{2} \\ {}_{J}a_{3} \end{Bmatrix} \qquad (4.25) $$

$$ {}_{J}G_{1} = {}_{J}Q \left\langle {}_{J}V_{2}, \frac{2}{3}{}_{J}V_{3}, \frac{1}{2}{}_{J}V_{4} \right\rangle \begin{Bmatrix} {}_{J}a_{1} \\ {}_{J}a_{2} \\ {}_{J}a_{3} \end{Bmatrix} + 2\,{}_{J}P \left\langle {}_{J}V_{1}, {}_{J}V_{2} \right\rangle \begin{Bmatrix} {}_{J}a_{2} \\ {}_{J}a_{3} \end{Bmatrix} \qquad (4.26) $$

$$ {}_{J}W_{1} = {}_{J}Q \left\langle \frac{2}{3}{}_{J}V_{3}, \frac{1}{2}{}_{J}V_{4}, \frac{2}{5}{}_{J}V_{5} \right\rangle \begin{Bmatrix} {}_{J}a_{1} \\ {}_{J}a_{2} \\ {}_{J}a_{3} \end{Bmatrix} + {}_{J}P \left\langle 2\,{}_{J}V_{2}, \frac{8}{3}{}_{J}V_{3} \right\rangle \begin{Bmatrix} {}_{J}a_{2} \\ {}_{J}a_{3} \end{Bmatrix} \qquad (4.27) $$

$$ {}_{J}Z_{1} = {}_{J}\Delta \left\langle {}_{J}V_{2}, \frac{2}{3}{}_{J}V_{3}, \frac{1}{2}{}_{J}V_{4} \right\rangle \begin{Bmatrix} {}_{J}a_{1} \\ {}_{J}a_{2} \\ {}_{J}a_{3} \end{Bmatrix} \qquad (4.28) $$

and

$$ {}_{J}V_{i} = ({}_{J}X_{1} + {}_{J}\text{f})^{i} - ({}_{J}X_{1})^{i} \quad (i=1,\ldots,5). \qquad (4.29) $$

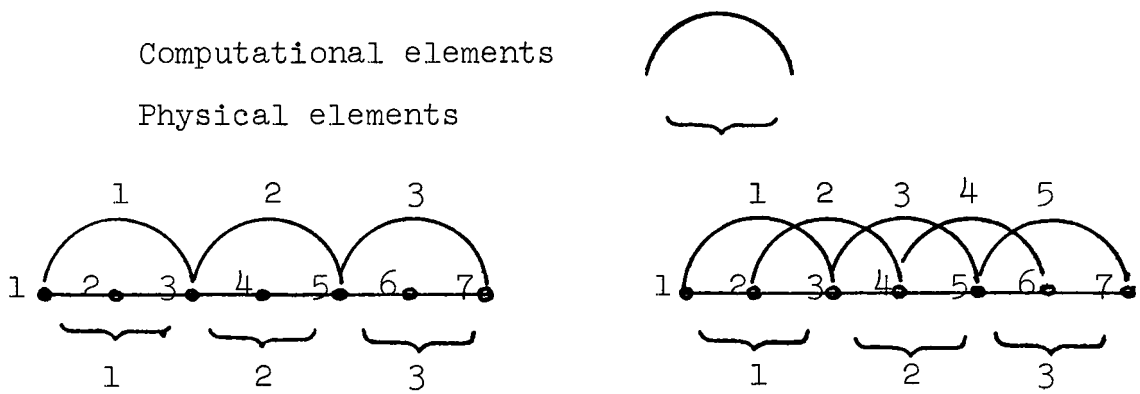
$({}_{J}T_{2}, {}_{J}G_{2}, {}_{J}W_{2}, {}_{J}Z_{2})$ and $({}_{J}T_{3}, {}_{J}G_{3}, {}_{J}W_{3}, {}_{J}Z_{3})$ can be obtained by substituting the b's and the c's (see Equation 4.3), instead of the a's respectively in Equations (4.25), (4.26), (4.27), and (4.28). By substituting Equation (4.24) into Equation (4.23), Equation (4.23) can be expressed as a function of the nodal values as follows:

$$ \left\langle {}_{J}T_{i}, {}_{J}G_{i}, {}_{J}W_{i} \right\rangle \left[{}_{J}A_{K,K} \right] \left\{ {}_{J}Y_{K} \right\} - {}_{J}Z_{i} = 0. \qquad (4.30) $$

These equations are then used to determine ${}_JY_1,\ldots{}_JY_3$.

4.1.1.6 Element Combination Method

To obtain the general equation, it is necessary to add the contribution of all the elements [15] (see Figure 4.3a). This technique is used in the (Ritz) Finite-Element Method. However, it did not work for the one-dimensional piecewise weighted-residual methods. It is possible that this failure is due to the isolation of middle nodes of the element from the boundary. They have no implicit linkage with the external boundaries of the whole domain. An overlapping element technique that put each node as the boundary of at least one element was used (see Figure 4.3b).



(a) Ritz Element Combination

(b) Element Combination Used

Figure 4.3. Element Combination

After the nodal values are determined for the J^{th} element, a continuous solution within the element is then

obtained by substituting $\{{}_{J}Y_{N}\}$ into Equation (3.9). In this way, ${}_{J}Y$ can be computed at any point within the element. In particular, the solution obtained using a large element can be compared, point by point, with a solution obtained using a smaller element at the nodes of the smaller elements.

4.1.2 Results

The equations derived in Chapter 4.1 were then programmed to obtain solutions for 1, 2, 4, 8, and 16 elements. The solution was compared to the analytic solution, $Y = [\sin(X)/\sin(1)] - X$. Charts 1 through 5 display results obtained for each method and compare them to the analytic solution.

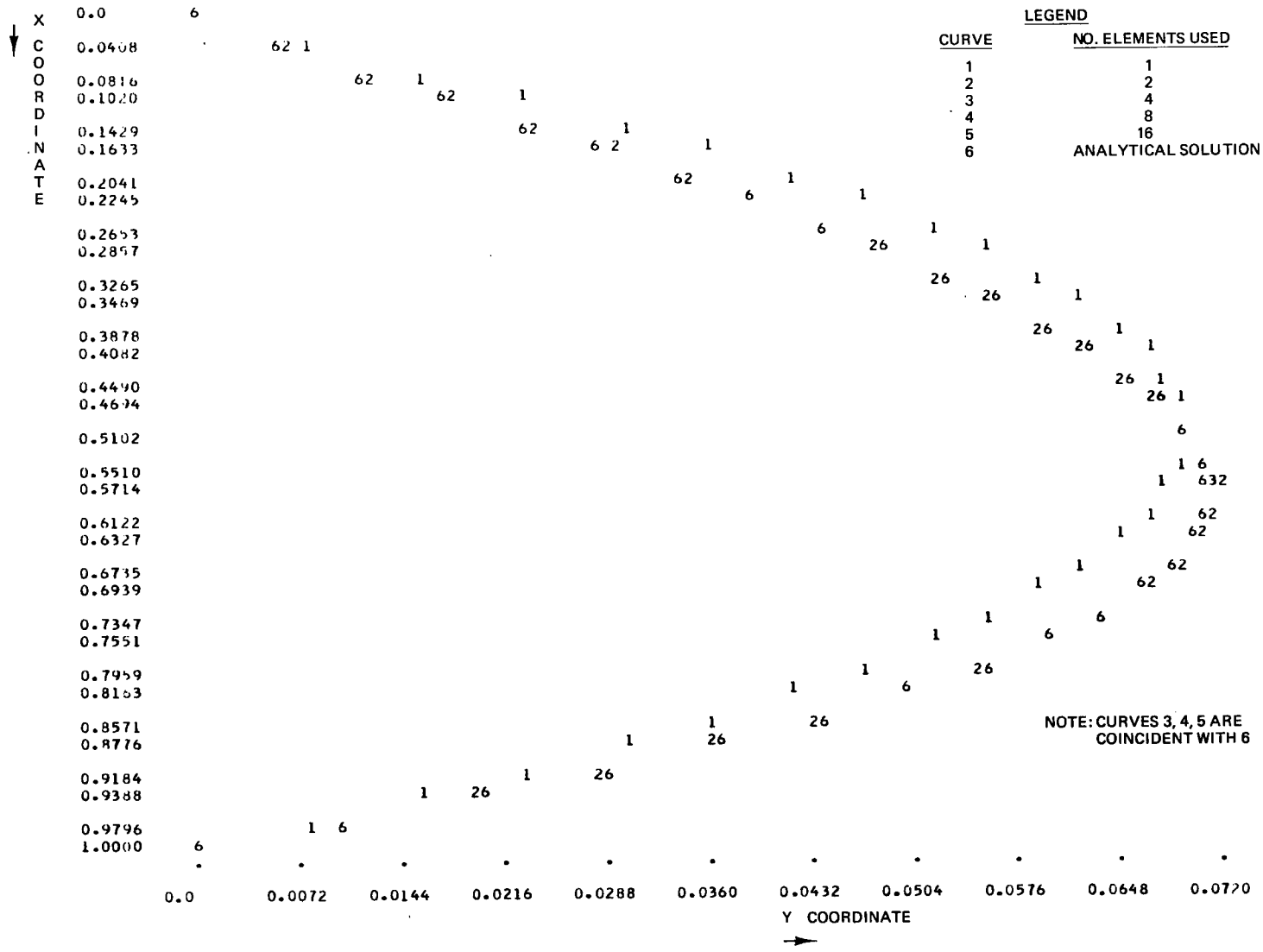
The curve formed by the symbol "1" is that obtained by the classical "whole-domain" method. As the curve identification symbol increases through 5, the number of elements increases to 16 (i.e., 33 nodes). The "6" curve corresponds to the exact solution.

Whenever a symbol does not appear, it is because the point has essentially the same value as a finer (more elements) approximation. For example, in Chart 1, approximations 4 and 5 are essentially the same as the analytic solution, 6.

Chart 6 represents the convergence of the various methods to the analytic solution. The absolute error, E, is measured by the sum of the absolute value of the difference between the approximation and the analytic solution at 33 points in the domain

CHART 1

RITZ METHOD



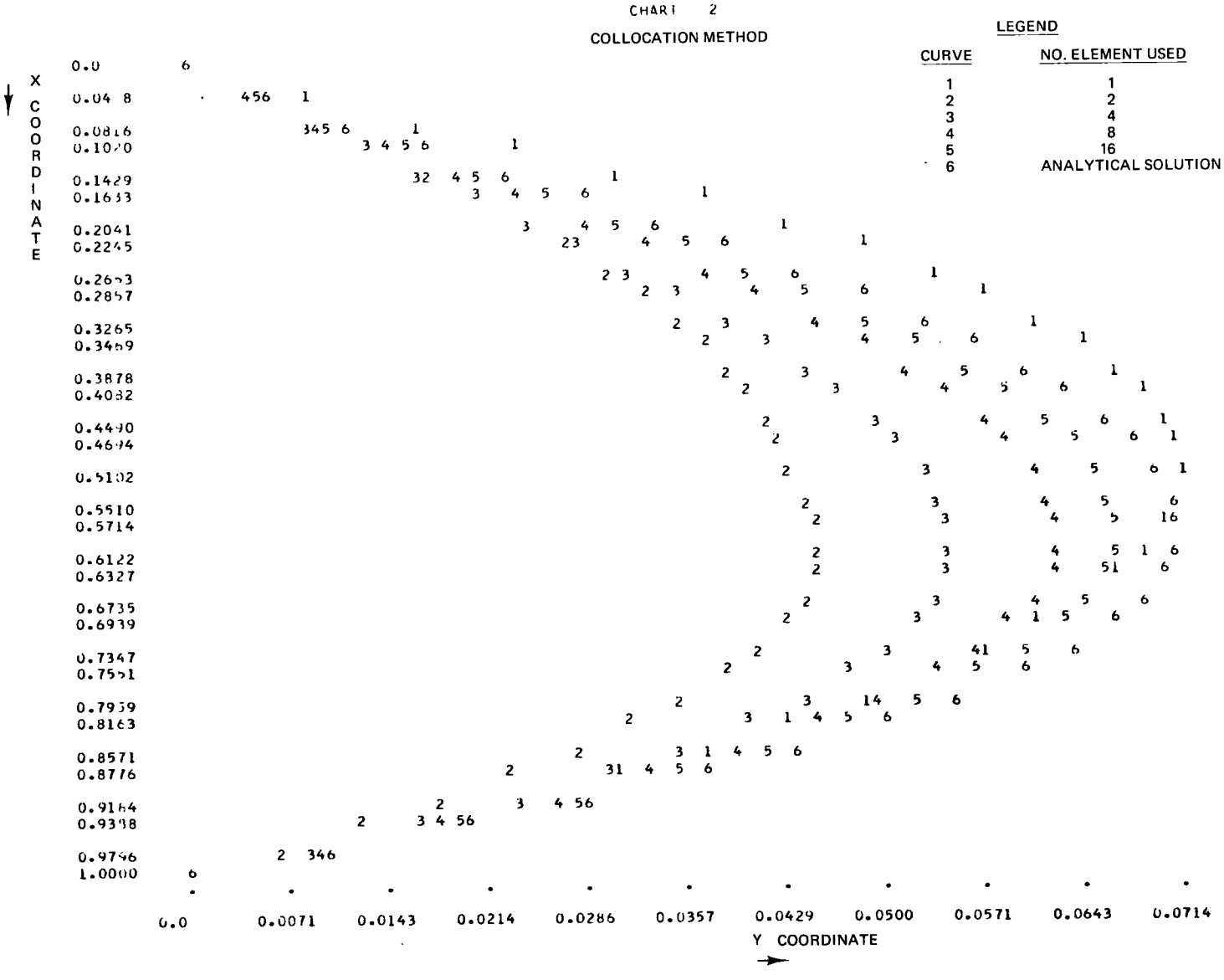


CHART 2
COLLOCATION METHOD
LEGEND
CURVE
NO. ELEMENT USED
1
1
2
2
3
4
4
8
5
16
6
ANALYTICAL SOLUTION
X COORDINATE
0.0
0.04 8
0.0816
0.1020
0.1429
0.1633
0.2041
0.2245
0.2653
0.2857
0.3265
0.3469
0.3878
0.4082
0.4490
0.4694
0.5102
0.5510
0.5714
0.6122
0.6327
0.6735
0.6939
0.7347
0.7551
0.7959
0.8163
0.8571
0.8776
0.9184
0.9388
0.9796
1.0000
0.0
0.0071
0.0143
0.0214
0.0286
0.0357
0.0429
0.0500
0.0571
0.0643
0.0714
Y COORDINATE

CHART 3

SUBDOMAIN METHOD

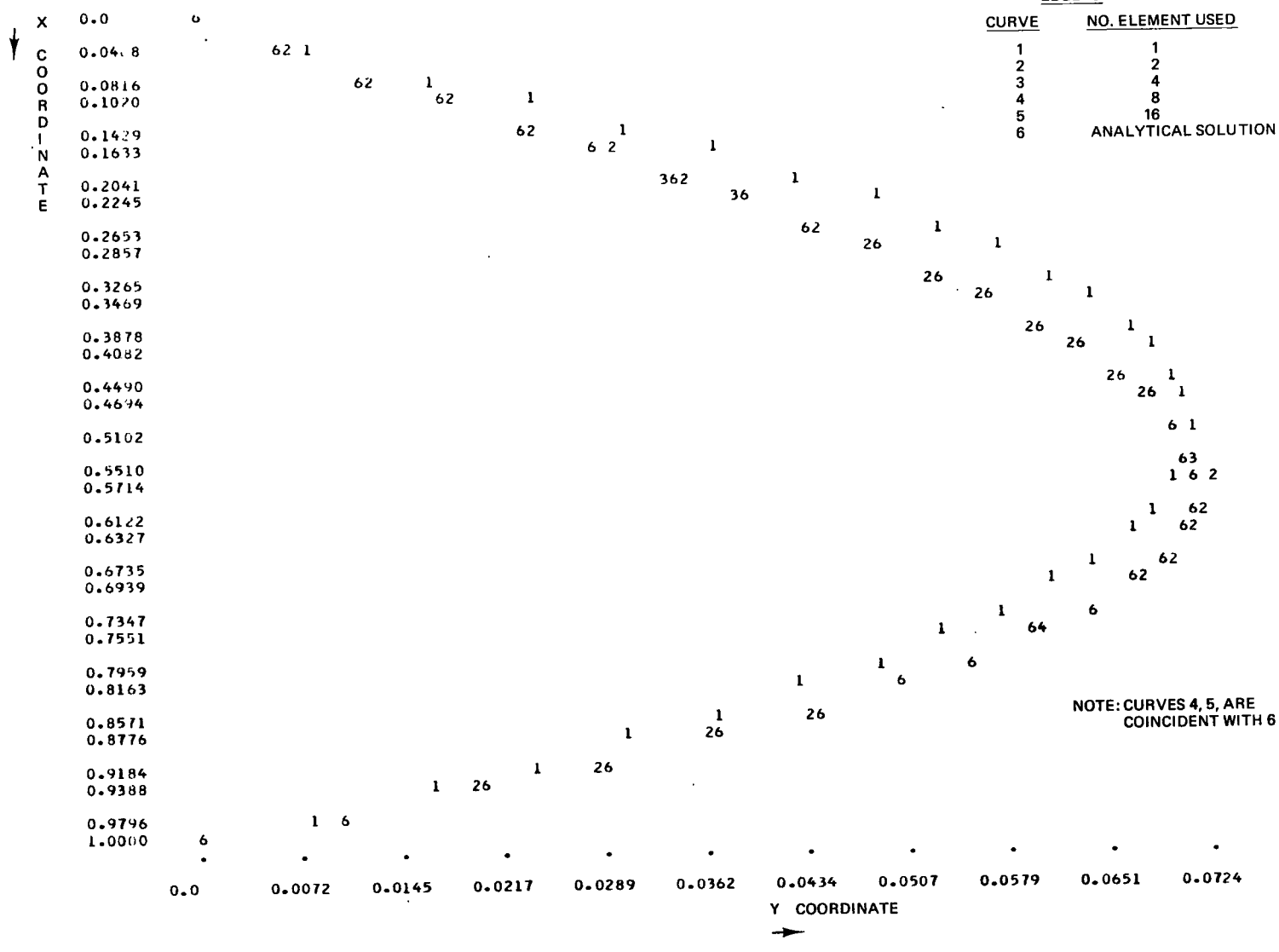


CHART 4

GALERKIN'S METHOD

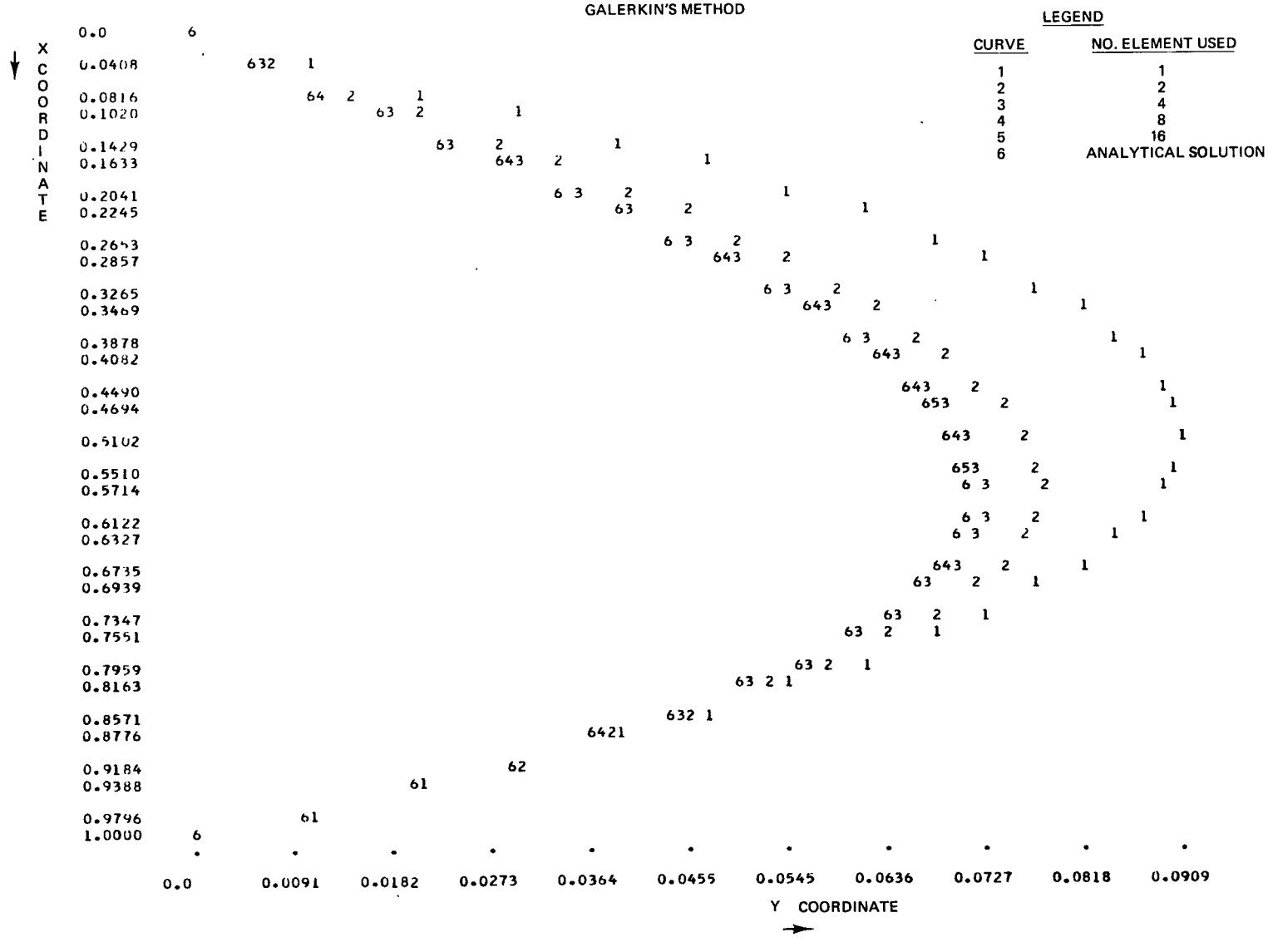


CHART 5

LEAST SQUARE METHOD

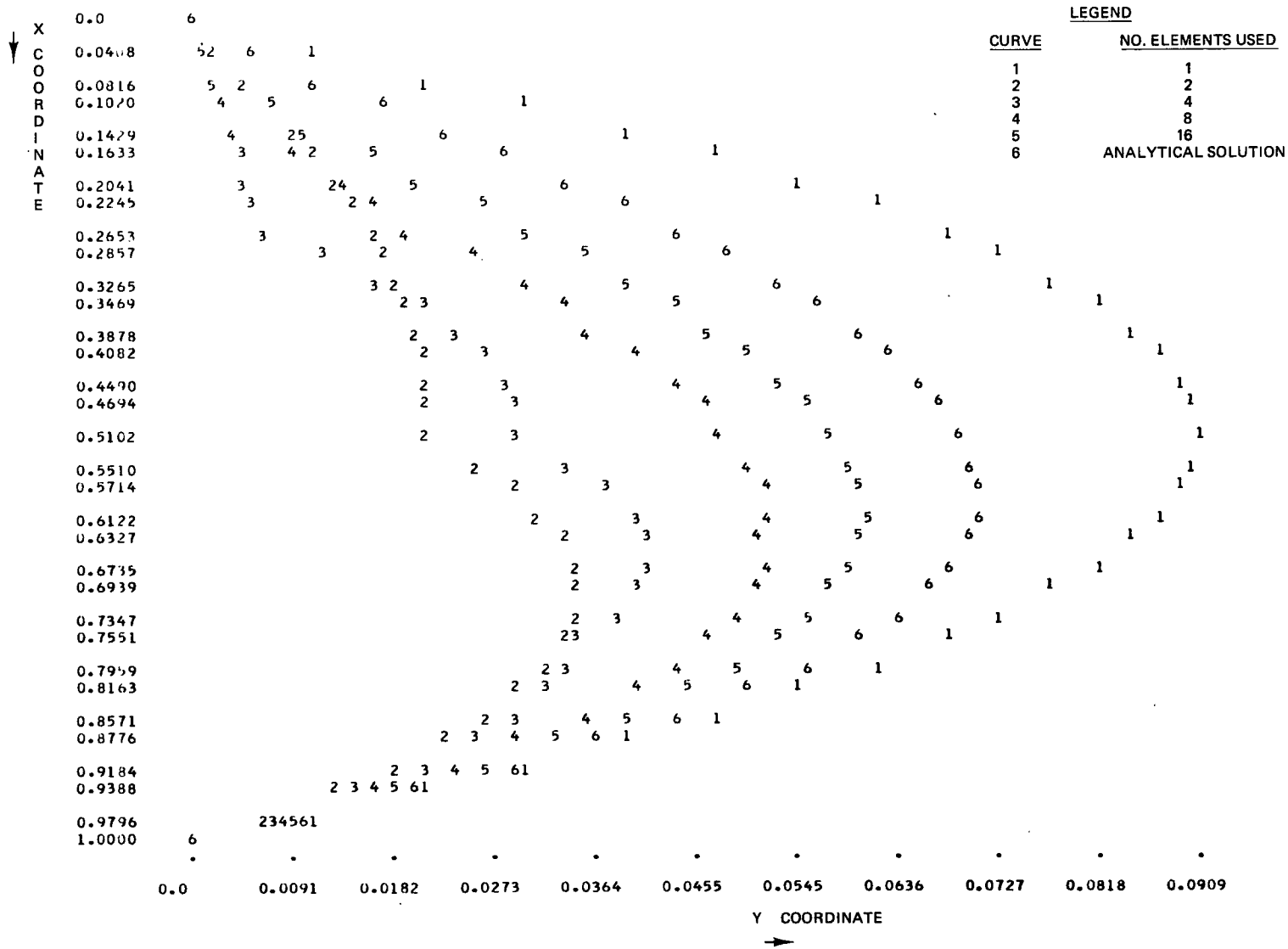


CHART 6

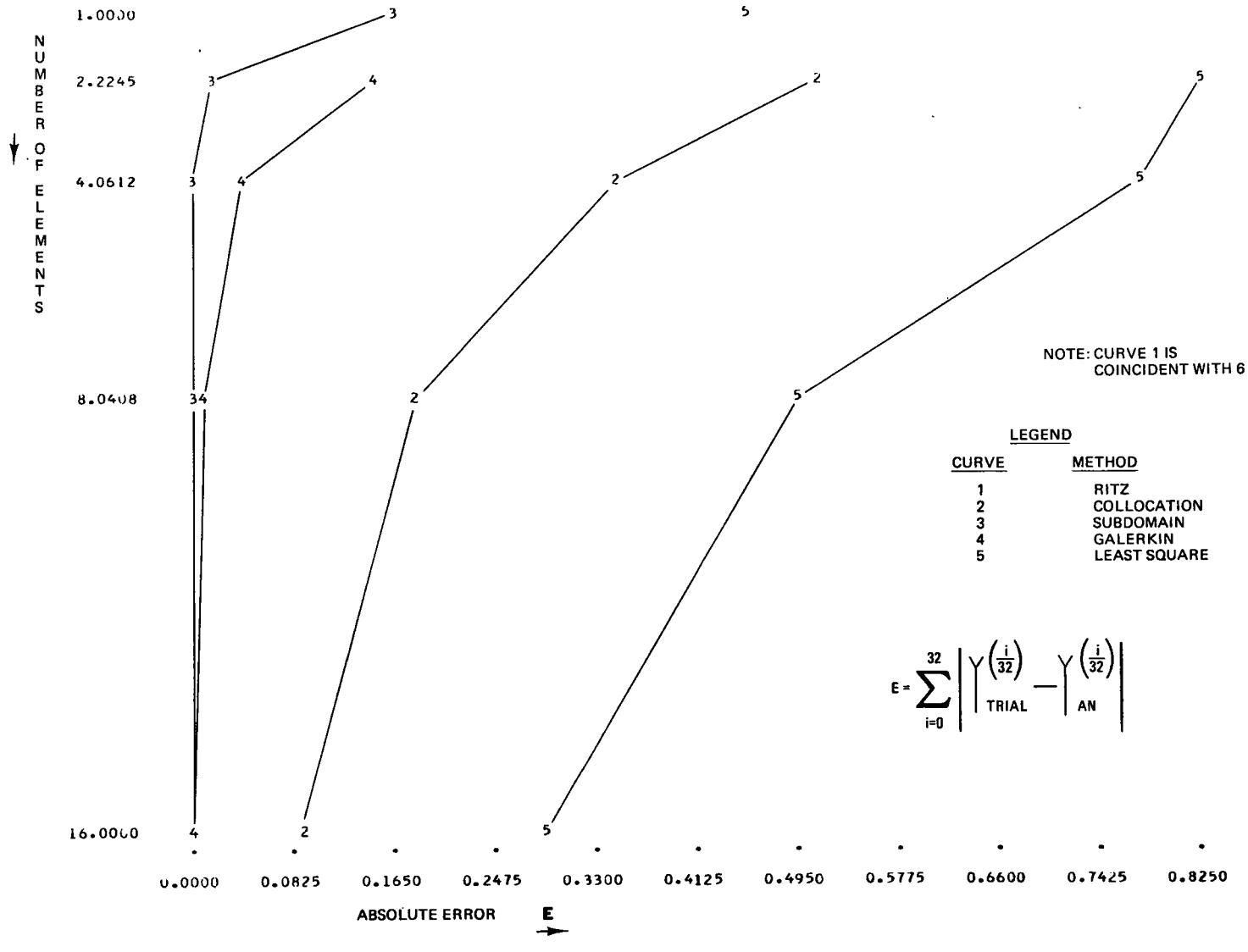
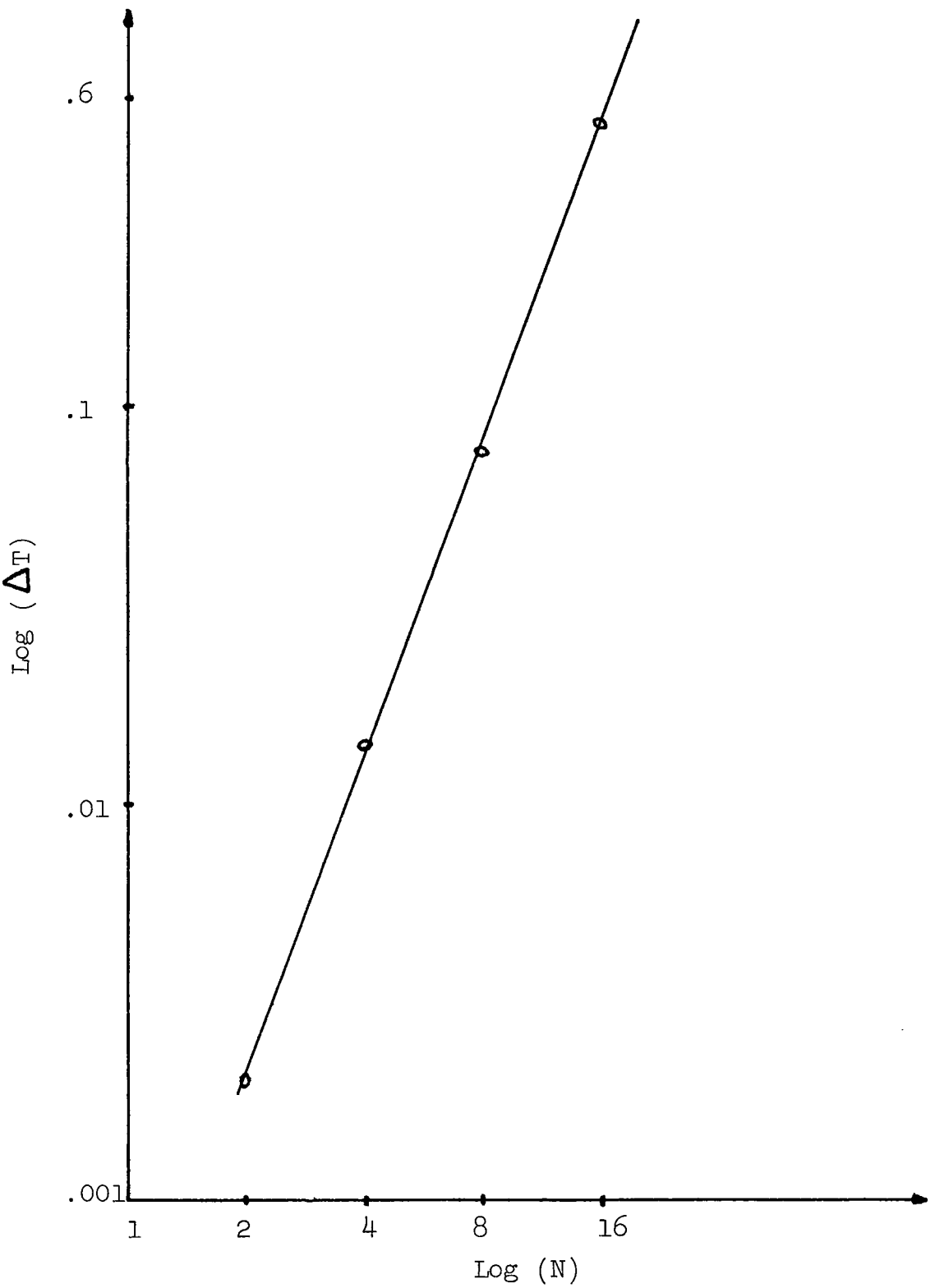

ABSOLUTE ERROR
NUMBER OF ELEMENTS
1.0000
2.2245
4.0612
8.0408
16.0000
0.0000
0.0825
0.1650
0.2475
0.3300
0.4125
0.4950
0.5775
0.6600
0.7425
0.8250
ABSOLUTE ERROR E
NOTE: CURVE 1 IS COINCIDENT WITH 6
LEGEND
CURVE METHOD
1 RITZ
2 COLLOCATION
3 SUBDOMAIN
4 GALERKIN
5 LEAST SQUARE
$E = \sum_{i=0}^{32} \left| Y\left(\frac{i}{32}\right)_{TRIAL} - Y\left(\frac{i}{32}\right)_{AN} \right|$

Chart 7

Computational Time (ΔT) vs. No. of Elements (N)



$$E = \sum_{i=0}^{32} \left| Y_{trial}\left(\frac{i}{32}\right) - Y_{analy.}\left(\frac{i}{32}\right) \right| , \qquad (4.31)$$

where the Y_{trial} is obtained as discussed in Section 4.1.1.6. This error represents the total area between the exact (analytic) solution curve and the approximate solution curve. It is a convenient measure of the relative error for the entire domain.

The computer time required for a solution by all methods for a given number of elements was approximately the same. Chart 7 presents computer time required to obtain the solutions for various numbers of elements plotted on a log-log graph. ΔT is the time differential between one and n elements.

4.2 Two-Dimensional Problem with Linear Approximation

4.2.1 Formulation

Consider the differential equation

$$\frac{\partial G}{\partial X} - Y = 0 \qquad (4.32)$$

in the domain

$$0 \leq X \leq 1, \; 0 \leq Y \leq 1$$

where

$$G = G(X,Y)$$

under the boundary condition

$$X = 1, \; G(1,Y) = 0. \qquad (4.33)$$

A "best fit" solution will be sought from the family

$$\phi_1 = 1; \; \phi_2 = X; \; \phi_3 = Y \; . \qquad (4.34)$$

Stating the problem in finite-element terms, Equation (3.4) yields a trial solution for the J^{th} element of the form

$$ {}_JG = \langle 1, X, Y \rangle \begin{Bmatrix} {}_J\alpha_1 \\ {}_J\alpha_2 \\ {}_J\alpha_3 \end{Bmatrix} . \tag{4.35} $$

Because of the three undetermined parameters, the triangle with a node in each vertex has been used to divide the domain D.

The trial solution (4.35) is expressed as a function of the three nodal values of the J^{th} element by using (3.9). Then,

$$ {}_JG = \frac{1}{2_J\Delta} \langle 1, X, Y \rangle \begin{bmatrix} {}_Ja_1 & {}_Ja_2 & {}_Ja_3 \\ {}_Jb_1 & {}_Jb_2 & {}_Jb_3 \\ {}_Jc_1 & {}_Jc_2 & {}_Jc_3 \end{bmatrix} \begin{Bmatrix} {}_JG_1 \\ {}_JG_2 \\ {}_JG_3 \end{Bmatrix} , \tag{4.36} $$

where

$$ [{}_J\Phi_N]^{-1} = \frac{1}{2_J\Delta} \begin{bmatrix} {}_Ja_1 & {}_Ja_2 & {}_Ja_3 \\ {}_Jb_1 & {}_Jb_2 & {}_Jb_3 \\ {}_Jc_1 & {}_Jc_2 & {}_Jc_3 \end{bmatrix} \tag{4.37} $$

and

$$ {}_J\Delta = \tfrac{1}{2} \left| [{}_J\Phi_N] \right| = \text{Area of the triangle.} \tag{4.38} $$

With the trial solution (4.36), the expression of the residual (3.12) becomes

$$ {}_JR = \frac{1}{2_J\Delta} \langle {}_Jb_1, {}_Jb_2, {}_Jb_3 \rangle \begin{Bmatrix} {}_JG_1 \\ {}_JG_2 \\ {}_JG_3 \end{Bmatrix} - Y = 0 . \tag{4.39} $$

The residual is used in the various methods in order to determine a "best fit" solution.

4.2.1.1 Collocation

The residual is set to zero at the three nodes of the J^{th} element (see Figure 4.4).

Node	X	Y
1	${}_JX_1$	${}_JY_1$
2	${}_JX_2$	${}_JY_2$
3	${}_JX_3$	${}_JY_3$

(4.40)

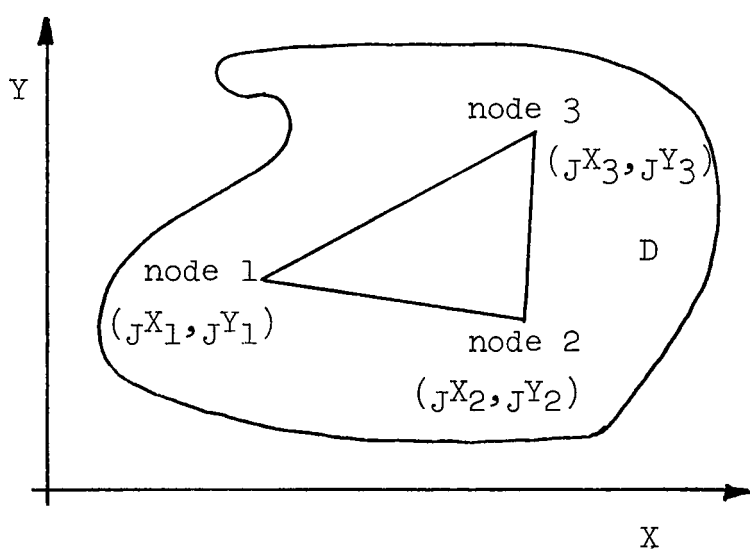


Figure 4.4. Two-Dimensional Triangular Element

Using Equations (4.39) and (4.40) results in

$$ {}_JR({}_JX_1, {}_JY_1) = \frac{1}{2_J\Delta} \langle {}_Jb_1, {}_Jb_2, {}_Jb_3 \rangle \begin{Bmatrix} {}_JG_1 \\ {}_JG_2 \\ {}_JG_3 \end{Bmatrix} - Y_1 = 0. \quad (4.41) $$

Similarly,

$$ {}_JR({}_JX_2, {}_JY_2) = 0 \tag{4.42} $$

and

$$ {}_JR({}_JX_3, {}_JY_3) = 0. \tag{4.43} $$

Equations (4.41) through (4.43) are then used to evaluate the nodal values of the element.

4.2.1.2 Subdomain

The J^{th} element was subdivided into three subdomains as shown in Figure 4.5, where the $({}_JX_4, {}_JY_4)$ are the coordinates of the median of the triangle,

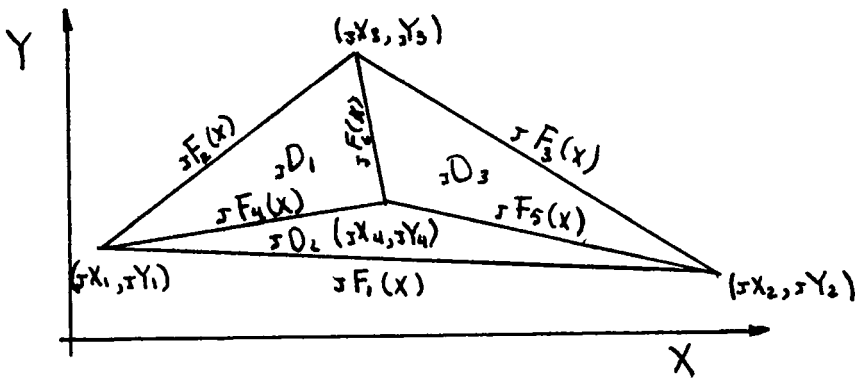


Figure 4.5. Triangle Partition for the Subdomain Method

and ${}_JF_i(X) = {}_JN_iX + {}_JM_i$ $(i=1,\ldots,6)$ where ${}_JN_i$ is the slope of the line ${}_JF_i(X)$ and ${}_JM_i$ is the point of intersection with Y coordinate. Then, Equations (3.14) and (4.39) yield

$$ \frac{1}{6} \langle {}_Jb_1, {}_Jb_2, {}_Jb_3 \rangle \begin{Bmatrix} {}_JG_1 \\ {}_JG_2 \\ {}_JG_3 \end{Bmatrix} - {}_JI_i = 0 \ , \tag{4.44} $$

$$ (i=1,2,3) $$

where

$$ {}_{J}I_{1} = \int_{{}_{J}X_{1}}^{{}_{J}X_{4}} \int_{{}_{J}F_{4}(X)}^{{}_{J}F_{2}(X)} YdYdX + \int_{{}_{J}X_{4}}^{{}_{J}X_{3}} \int_{{}_{J}F_{6}(X)}^{{}_{J}F_{2}(X)} YdYdX \qquad (4.45) $$

$$ {}_{J}I_{2} = \int_{{}_{J}X_{1}}^{{}_{J}X_{4}} \int_{{}_{J}F_{1}(X)}^{{}_{J}F_{4}(X)} YdYdX + \int_{{}_{J}X_{4}}^{{}_{J}X_{2}} \int_{{}_{J}F_{1}(X)}^{{}_{J}F_{5}(X)} YdYdX \qquad (4.46) $$

$$ {}_{J}I_{3} = \int_{{}_{J}X_{4}}^{{}_{J}X_{3}} \int_{{}_{J}F_{5}(X)}^{{}_{J}F_{6}(X)} YdYdX + \int_{{}_{J}X_{3}}^{{}_{J}X_{2}} \int_{{}_{J}F_{5}(X)}^{{}_{J}F_{3}(X)} YdYdX \qquad (4.47) $$

4.2.1.3 Galerkin

By using Equation (3.15) with ${}_{J}\Phi_{1} = 1$, ${}_{J}\Phi_{2} = X$, ${}_{J}\Phi_{3} = Y$, and the expression of the residual (4.39), the following is obtained:

$$ \tfrac{1}{2} \left\langle {}_{J}b_{1}, {}_{J}b_{2}, {}_{J}b_{3} \right\rangle \begin{Bmatrix} {}_{J}G_{1} \\ {}_{J}G_{2} \\ {}_{J}G_{3} \end{Bmatrix} - I_{y} = 0 \qquad (4.48) $$

$$ \frac{I_{x}}{2_{J}\Delta} \left\langle {}_{J}b_{1}, {}_{J}b_{2}, {}_{J}b_{3} \right\rangle \begin{Bmatrix} {}_{J}G_{1} \\ {}_{J}G_{2} \\ {}_{J}G_{3} \end{Bmatrix} - I_{xy} = 0 \qquad (4.49) $$

$$\frac{I_y}{2_J\Delta} \langle {}_Jb_1, {}_Jb_2, {}_Jb_3 \rangle \begin{Bmatrix} {}_JG_1 \\ {}_JG_2 \\ {}_JG_3 \end{Bmatrix} - I_{y2} = 0 \qquad (4.50)$$

where

$$I_y = \int_{D_J} Y dD_J \qquad (4.51)$$

$$I_y = \int_{D_J} XY dD_J \qquad (4.52)$$

$$I_{y2} = \int_{D_J} Y^2 dD_J \, . \qquad (4.53)$$

4.2.2 Results

The system used has no known variation statement. Hence, it is impossible to formulate a Ritz Method. Due to the poor results of the one-dimensional problem using the Least-Squares Technique, this method was not developed for the two-dimensional problem.

The equations derived in Section 4.2.1 were then programmed. The resulting system of linear algebraic equation was seen to be singular. The singularity is believed to be due to the poor approximating family since it contains no xy terms and the exact solution is known to be $G = xy + y$.

Due to the labor involved in developing and programming the linear family in the two-dimensional program, it was decided that the application of a quadratic family was beyond the scope of this thesis.

The main difficulty encountered in any of the approximations involving integration (Subdomain, Galerkin, Least-Square) is that a simple algorithm for the area integral similar to that available for the Ritz (Finite-Element) Method [20] has not been developed. Hence, the integration over each element must be carried out. This results in multiple integrals with variable limits (e.g., Equations 4.45 through 4.47).

Chapter 5

CONCLUSIONS AND EXTENSIONS

5.1 One-Dimensional Problem

The following conclusions are made, based on the specific one-dimensional problem used in this thesis:

1. All the techniques approached the analytic solution, with the error decreasing as the number of elements increased.

2. For a given number of elements, the Subdomain and the Ritz methods yield the best approximation.

3. Poorest results were obtained using the Least-Squares method. The error in the Ritz and Subdomain method using only two elements was less than one-half of the error in the Least-Squares method using sixteen elements.

4. For a given number of elements, computation time for all methods was essentially the same.

5. Computation time is generally of the exponential form (see Chart 7).

6. In terms of difficulty of programming and derivation, the methods ranked as follows (listed in order of increasing difficulty):

1. Collocation
2. Subdomain
3. Galerkin
4. Ritz
5. Least Squares

7. For this problem, combining the factors of minimizing error, computation time, and programming difficulty, the Subdomain method would be preferred, followed by the Ritz method.

5.2 Two-Dimensional Problem

Although numerical results were not obtained, the following observations can still be made:

1. For trial family $G = \langle 1,X,Y \rangle \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix}$ all Weighted Residual Techniques led to a singular coefficient matrix.

2. This shows that the approximation $G = \langle 1,X,Y \rangle \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix}$ is a poor approximating family for the analytic solution $G = XY - Y$.

3. A higher approximation family, $G = \langle 1,X^2,Y^2, XY,X,Y \rangle \{\alpha_N\}$, will complicate significantly the equation. Better methods of solving

$$\iint_A L_1^n L_2^m L_3^{\ell} X^K Y^J dA,$$

where

$$L_i = (a_i X + b_i Y + c_i) \quad i=1,2,3$$

are the area coordinates of the element, must be sought.

5.3 General

1. It is possible to apply Weighted-Residual methods to finite elements to obtain approximate solutions of differential equations.

2. In order to achieve any kind of approximation, the Weighted-Residual methods require a higher order of approximation than the Ritz Method does. It is due to the lower order degree of the variational statement with respect to the original differential equation.

3. As stated in Chapter 3, no technique will overcome a poor approximating family.

4. The computer time among the techniques for a given number of elements is not significantly different.

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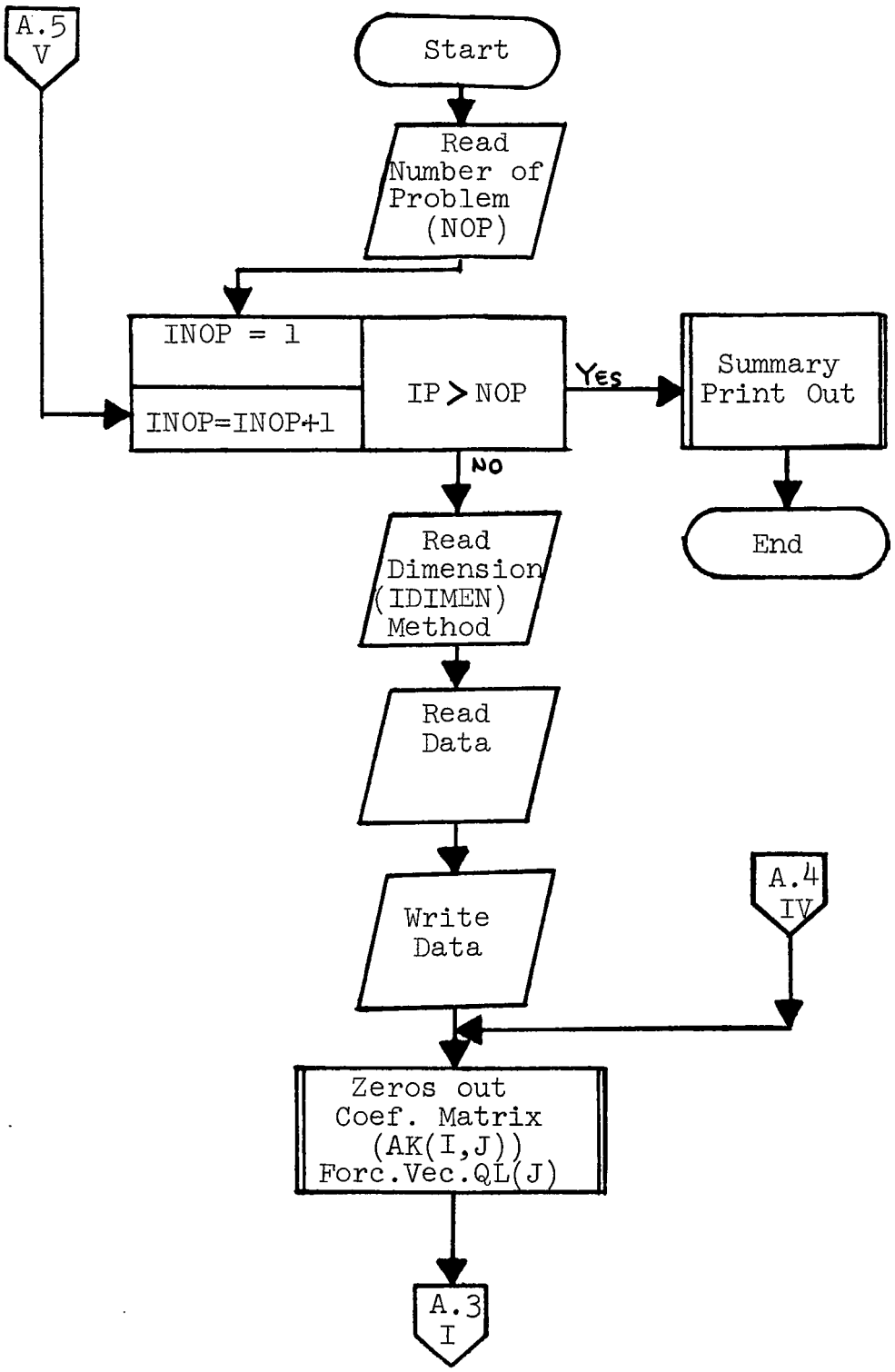
APPENDIX A

Program Flow Charts


A.5
V
Start
Read
Number of
Problem
(NOP)
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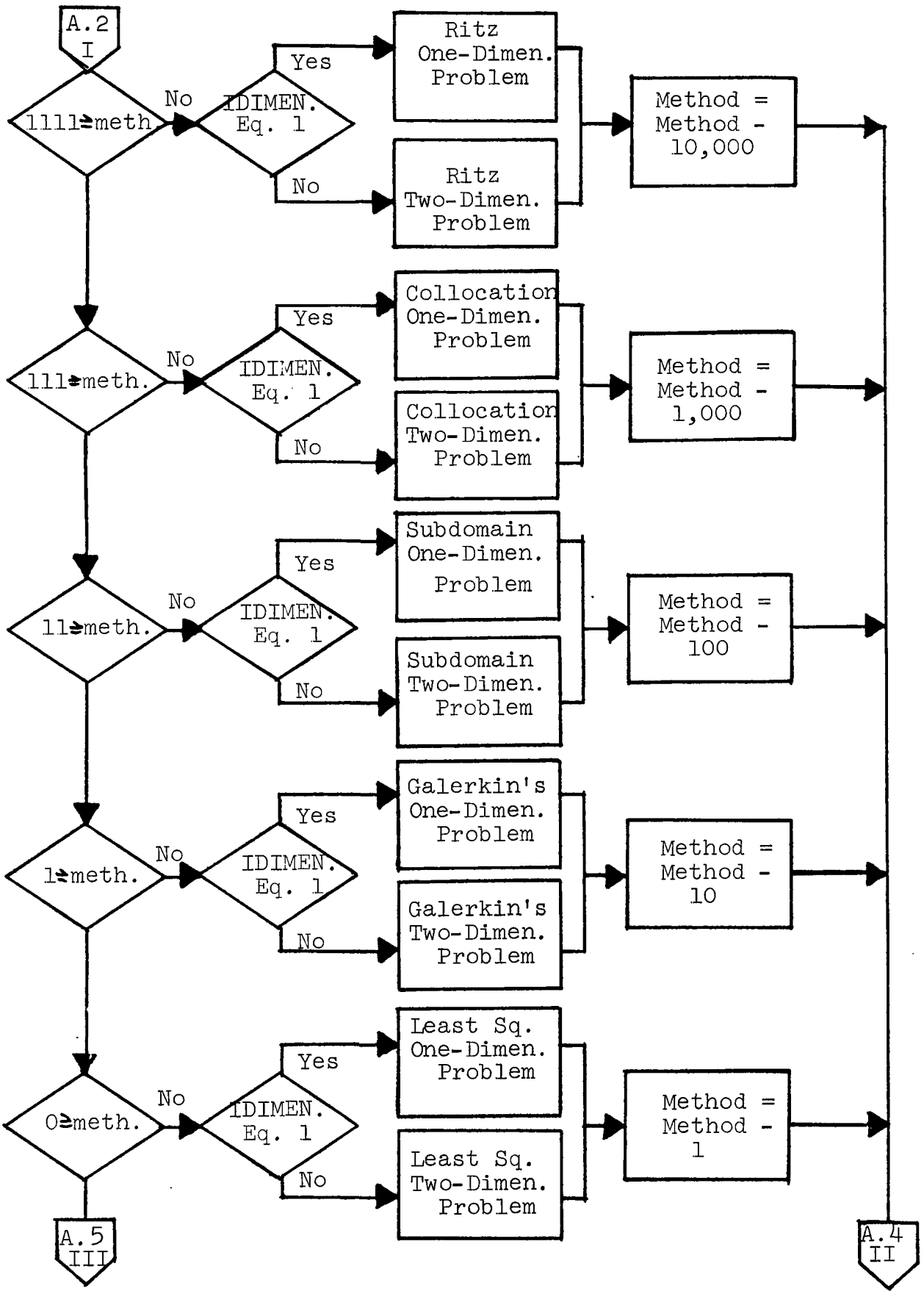
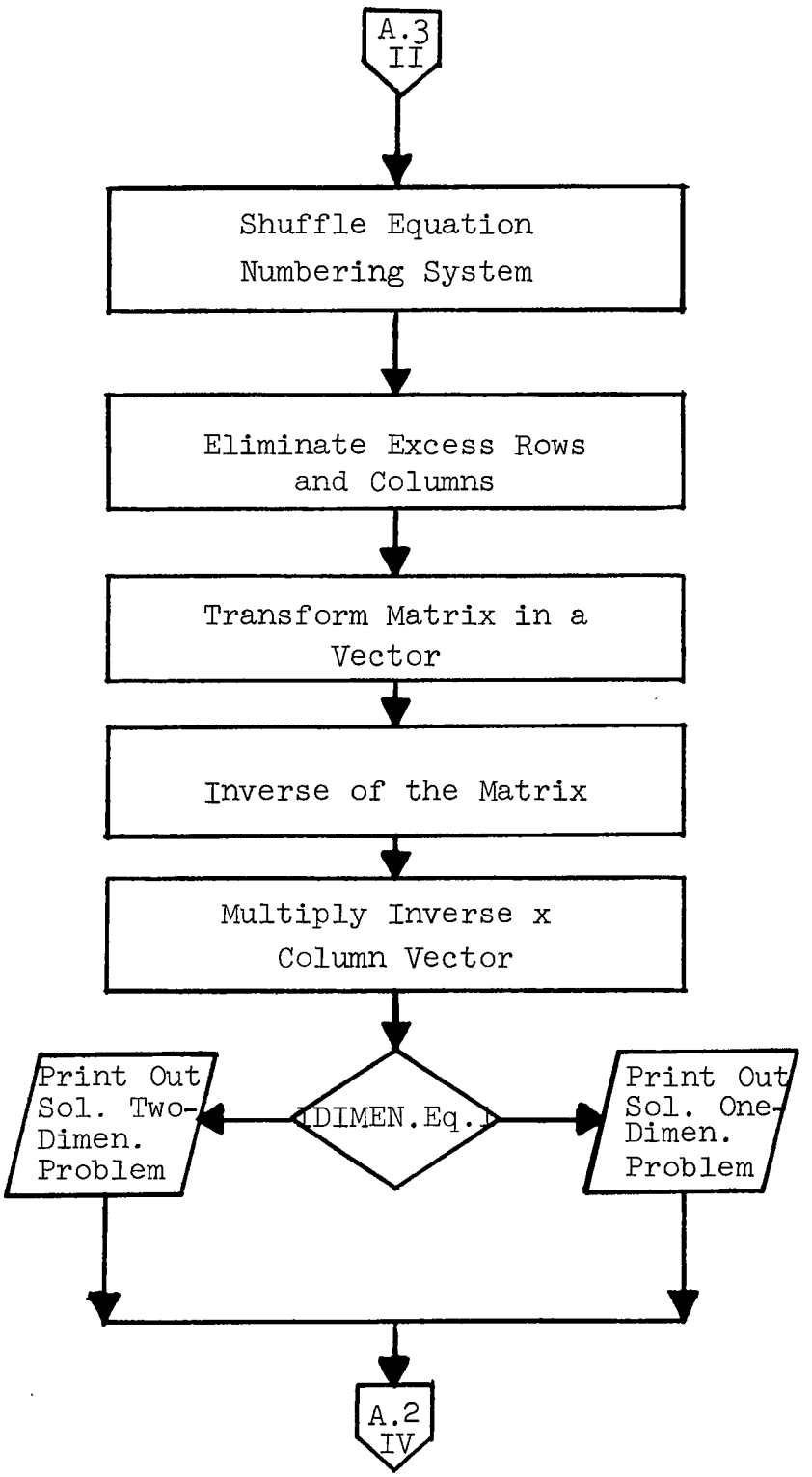
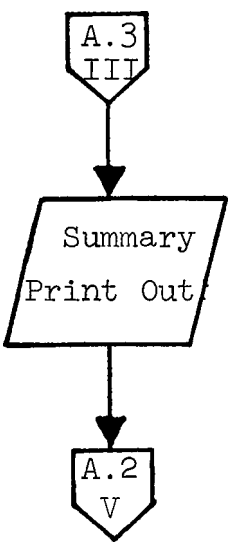
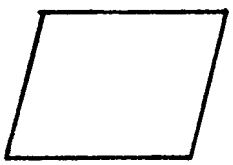
INOP=INOP+1
IP > NOP
Yes
Summary
Print Out
End
NO
Read
Dimension
(IDIMEN)
Method
Read
Data
Write
Data
A.4
IV
Zeros out
Coef. Matrix
(AK(I,J))
Forc.Vec.QL(J)
A.3
I


A.2
I
1111≥meth.
No
IDIMEN.
Eq. 1
Yes
No
Ritz
One-Dimen.
Problem
Ritz
Two-Dimen.
Problem
Method =
Method -
10,000
111≥meth.
No
IDIMEN.
Eq. 1
Yes
No
Collocation
One-Dimen.
Problem
Collocation
Two-Dimen.
Problem
Method =
Method -
1,000
11≥meth.
No
IDIMEN.
Eq. 1
Yes
No
Subdomain
One-Dimen.
Problem
Subdomain
Two-Dimen.
Problem
Method =
Method -
100
1≥meth.
No
IDIMEN.
Eq. 1
Yes
No
Galerkin's
One-Dimen.
Problem
Galerkin's
Two-Dimen.
Problem
Method =
Method -
10
0≥meth.
No
IDIMEN.
Eq. 1
Yes
No
Least Sq.
One-Dimen.
Problem
Least Sq.
Two-Dimen.
Problem
Method =
Method -
1
A.5
III
A.4
II


A.3
II
Shuffle Equation
Numbering System
Eliminate Excess Rows
and Columns
Transform Matrix in a
Vector
Inverse of the Matrix
Multiply Inverse x
Column Vector
Print Out
Sol. Two-
Dimen.
Problem
IDIMEN.Eq.1
Print Out
Sol. One-
Dimen.
Problem
A.2
IV


A.3
III
Summary
Print Out
A.2
V



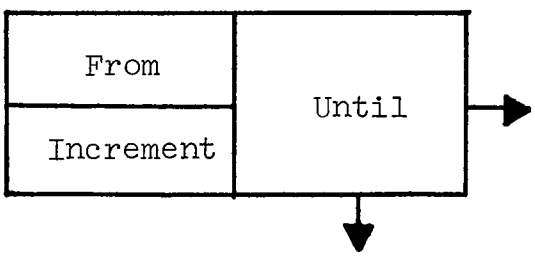
Read, Write

Start, Stop

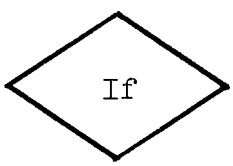
Connector

Offpage Connector

Process



Do Loop



If Statement

Logical Flow

APPENDIX B

Data and Results

Appendix B

DATA AND RESULTS

This section contains the computer output for the one-dimensional problem in order of increasing number of elements. The information for a given number of elements is arranged as follows:

1. Input
2. List of results for Ritz, Collocation, Subdomain, Galerkin, and Least Squares
3. Summary results and time spent
4. Graphic summary

FINITE ELEMENT TO 1 DIMENSIONAL PROBLEM.

METHOD TO USE = 11111

NUMBER OF NODES = 3

NUMBER OF ELEMENTS = 1

NUMBER OF NODAL VALUES GIVEN = 2

NODAL COORDINATES

NODE	X	Y	INITIAL VALUE
1	0.0	0.0	0.0
2	0.50000	0.0	0.0
3	1.00000	0.0	0.0

ELEMENT	NODE NUMBERS	CONSTANTS P(I)	Q(I)	F(I)	PQ(I)
1	1 2 3	1.00000	-1.00000	1.00000	0.0

2 NODAL VALUES GIVEN

NODE	VAL.
1	0.0
3	0.0

RITZ METHOD: ONE DIMENSIONAL PROBLEM USING QUADRATIC APPROXIMATION (METHOD = 10000)

NODE	X-COORDIN	ANALYTICAL	APR METHOD	ERROR
1	0.0	0.0	0.0	0.0
2	0.03125	0.005881	0.008409	-0.002528
3	0.06250	0.011726	0.016276	-0.004550
4	0.09375	0.017499	0.023600	-0.006101
5	0.12500	0.023163	0.030382	-0.007219
6	0.15625	0.028682	0.036621	-0.007939
7	0.18750	0.034021	0.042318	-0.008297
8	0.21875	0.039143	0.047472	-0.008329
9	0.25000	0.044014	0.052083	-0.008070
10	0.28125	0.048597	0.056152	-0.007555
11	0.31250	0.052858	0.059679	-0.006820
12	0.34375	0.056763	0.062663	-0.005900
13	0.37500	0.060276	0.065104	-0.004828
14	0.40625	0.063365	0.067003	-0.003638
15	0.43750	0.065995	0.068359	-0.002365
16	0.46875	0.068133	0.069173	-0.001040
17	0.50000	0.069747	0.069444	0.000302
18	0.53125	0.070805	0.069173	0.001631
19	0.56250	0.071274	0.068359	0.002915
20	0.59375	0.071125	0.067003	0.004122
21	0.62500	0.070327	0.065104	0.005222
22	0.65625	0.068849	0.062663	0.006187
23	0.68750	0.066664	0.059679	0.006985
24	0.71875	0.063742	0.056152	0.007590
25	0.75000	0.060056	0.052083	0.007973
26	0.78125	0.055579	0.047472	0.008107
27	0.81250	0.050285	0.042318	0.007968
28	0.84375	0.044149	0.036621	0.007527
29	0.87500	0.037145	0.030382	0.006763
30	0.90625	0.029251	0.023600	0.005650
31	0.93750	0.020443	0.016276	0.004167
32	0.96875	0.010700	0.008409	0.002290
33	1.00000	0.0	0.0	0.0

COLLOCATION METHOD: ONE DIMENSIONAL PROBLEM USING QUADRATIC APPROXIMATION (METHOD = 01000)

NODE	X-COORDIN	ANALYTICAL	APR METHOD	ERROR
1	0.0	0.0	0.0	0.0
2	0.03125	0.005881	0.008650	-0.002768
3	0.06250	0.011726	0.016741	-0.005015
4	0.09375	0.017499	0.024275	-0.006776
5	0.12500	0.023163	0.031250	-0.008087
6	0.15625	0.028682	0.037667	-0.008985
7	0.18750	0.034021	0.043527	-0.009506
8	0.21875	0.039143	0.048828	-0.009685
9	0.25000	0.044014	0.053571	-0.009558
10	0.28125	0.048597	0.057757	-0.009160
11	0.31250	0.052858	0.061384	-0.008525
12	0.34375	0.056763	0.064453	-0.007690
13	0.37500	0.060276	0.066964	-0.006688
14	0.40625	0.063365	0.068917	-0.005553
15	0.43750	0.065995	0.070312	-0.004318
16	0.46875	0.068133	0.071149	-0.003017
17	0.50000	0.069747	0.071429	-0.001682
18	0.53125	0.070805	0.071149	-0.000345
19	0.56250	0.071274	0.070312	0.000962
20	0.59375	0.071125	0.068917	0.002208
21	0.62500	0.070327	0.066964	0.003362
22	0.65625	0.068849	0.064453	0.004396
23	0.68750	0.066664	0.061384	0.005280
24	0.71875	0.063742	0.057757	0.005985
25	0.75000	0.060056	0.053571	0.006485
26	0.78125	0.055579	0.048828	0.006751
27	0.81250	0.050285	0.043527	0.006758
28	0.84375	0.044149	0.037667	0.006481
29	0.87500	0.037145	0.031250	0.005895
30	0.90625	0.029251	0.024275	0.004976
31	0.93750	0.020443	0.016741	0.003702
32	0.96875	0.010700	0.008650	0.002050
33	1.00000	0.0	0.0	0.0

SUBDOMAIN METHOD: ONE DIMENSIONAL PROBLEM USING QUADRATIC APPROXIMATION (METHOD = 00100)

NODE	X-COORDIN	ANALYTICAL	APR METHOD	ERROR
1	0.0	0.0	0.0	0.0
2	0.03125	0.005881	0.008604	-0.002723
3	0.06250	0.011726	0.016653	-0.004927
4	0.09375	0.017499	0.024147	-0.006648
5	0.12500	0.023163	0.031086	-0.007923
6	0.15625	0.028682	0.037469	-0.008787
7	0.18750	0.034021	0.043298	-0.009277
8	0.21875	0.039143	0.048571	-0.009428
9	0.25000	0.044014	0.053289	-0.009276
10	0.28125	0.048597	0.057453	-0.008856
11	0.31250	0.052858	0.061061	-0.008202
12	0.34375	0.056763	0.064114	-0.007351
13	0.37500	0.060276	0.066612	-0.006335
14	0.40625	0.063365	0.068555	-0.005190
15	0.43750	0.065995	0.069942	-0.003948
16	0.46875	0.068133	0.070775	-0.002642
17	0.50000	0.069747	0.071053	-0.001306
18	0.53125	0.070805	0.070775	0.000030
19	0.56250	0.071274	0.069942	0.001332
20	0.59375	0.071125	0.068555	0.002571
21	0.62500	0.070327	0.066612	0.003715
22	0.65625	0.068849	0.064114	0.004735
23	0.68750	0.066664	0.061061	0.005603
24	0.71875	0.063742	0.057453	0.006289
25	0.75000	0.060056	0.053289	0.006767
26	0.78125	0.055579	0.048571	0.007008
27	0.81250	0.050285	0.043298	0.006988
28	0.84375	0.044149	0.037469	0.006679
29	0.87500	0.037145	0.031086	0.006060
30	0.90625	0.029251	0.024147	0.005104
31	0.93750	0.020443	0.016653	0.003790
32	0.96875	0.010700	0.008604	0.002096
33	1.00000	0.0	0.0	0.0

GALERKIN'S METHOD: ONE DIMENSIONAL PROBLEM USING QUADRATIC APPROXIMATION (METHOD = 00010)

NODE	X-COORDIN	ANALYTICAL	APR METHOD	ERROR
1	0.0	0.0	0.0	0.0
2	0.03125	0.005881	0.011009	-0.005127
3	0.06250	0.011726	0.021307	-0.009580
4	0.09375	0.017499	0.030895	-0.013396
5	0.12500	0.023163	0.039773	-0.016610
6	0.15625	0.028682	0.047940	-0.019258
7	0.18750	0.034021	0.055398	-0.021377
8	0.21875	0.039143	0.062145	-0.023002
9	0.25000	0.044014	0.068182	-0.024168
10	0.28125	0.048597	0.073508	-0.024911
11	0.31250	0.052858	0.078125	-0.025266
12	0.34375	0.056763	0.082031	-0.025268
13	0.37500	0.060276	0.085227	-0.024951
14	0.40625	0.063365	0.087713	-0.024348
15	0.43750	0.065995	0.089489	-0.023494
16	0.46875	0.068133	0.090554	-0.022421
17	0.50000	0.069747	0.090909	-0.021162
18	0.53125	0.070805	0.090554	-0.019749
19	0.56250	0.071274	0.089489	-0.018214
20	0.59375	0.071125	0.087713	-0.016588
21	0.62500	0.070327	0.085227	-0.014901
22	0.65625	0.068849	0.082031	-0.013182
23	0.68750	0.066664	0.078125	-0.011461
24	0.71875	0.063742	0.073508	-0.009766
25	0.75000	0.060056	0.068182	-0.008126
26	0.78125	0.055579	0.062145	-0.006566
27	0.81250	0.050285	0.055398	-0.005112
28	0.84375	0.044149	0.047940	-0.003792
29	0.87500	0.037145	0.039773	-0.002628
30	0.90625	0.029251	0.030895	-0.001644
31	0.93750	0.020443	0.021307	-0.000864
32	0.96875	0.010700	0.011009	-0.000309
33	1.00000	0.0	0.0	0.0

LEAST SQUARES METHOD: ONE DIMENSIONAL PROBLEM USING QUADRATIC APPROXIMATION (METHOD = 00001)

NODE	X-COORDIN	ANALYTICAL	APR METHOD	ERROR
1	0.0	0.0	0.0	0.0
2	0.03125	0.005881	0.011009	-0.005127
3	0.06250	0.011726	0.021307	-0.009580
4	0.09375	0.017499	0.030895	-0.013396
5	0.12500	0.023163	0.039773	-0.016610
6	0.15625	0.028682	0.047940	-0.019258
7	0.18750	0.034021	0.055398	-0.021377
8	0.21875	0.039143	0.062145	-0.023002
9	0.25000	0.044014	0.068182	-0.024168
10	0.28125	0.048597	0.073508	-0.024911
11	0.31250	0.052858	0.078125	-0.025266
12	0.34375	0.056763	0.082031	-0.025268
13	0.37500	0.060276	0.085227	-0.024951
14	0.40625	0.063365	0.087713	-0.024348
15	0.43750	0.065995	0.089489	-0.023494
16	0.46875	0.068133	0.090554	-0.022421
17	0.50000	0.069747	0.090909	-0.021162
18	0.53125	0.070805	0.090554	-0.019749
19	0.56250	0.071274	0.089489	-0.018214
20	0.59375	0.071125	0.087713	-0.016588
21	0.62500	0.070327	0.085227	-0.014901
22	0.65625	0.068849	0.082031	-0.013182
23	0.68750	0.066664	0.078125	-0.011461
24	0.71875	0.063742	0.073508	-0.009766
25	0.75000	0.060056	0.068182	-0.008126
26	0.78125	0.055579	0.062145	-0.006566
27	0.81250	0.050285	0.055398	-0.005112
28	0.84375	0.044149	0.047940	-0.003792
29	0.87500	0.037145	0.039773	-0.002628
30	0.90625	0.029251	0.030895	-0.001644
31	0.93750	0.020443	0.021307	-0.000864
32	0.96875	0.010700	0.011009	-0.000309
33	1.00000	0.0	0.0	0.0

X-COORDINATE	ANALYTICAL SOLUTION	RITZ	COLLOCATION	SUBDOMAIN	GALERKIN*S	LEASTSQUARES
0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.50000	0.069746912	0.069444478	0.071428537	0.071052611	0.090909004	0.090909004
1.00000	0.0	0.0	0.0	0.0	0.0	0.0

TIME SPENT IN SEC.

0.033999998122	0.028999999166	0.028999999166	0.028999999166	0.028999999166

CHART A 1

ONE ELEMENT APPROXIMATION

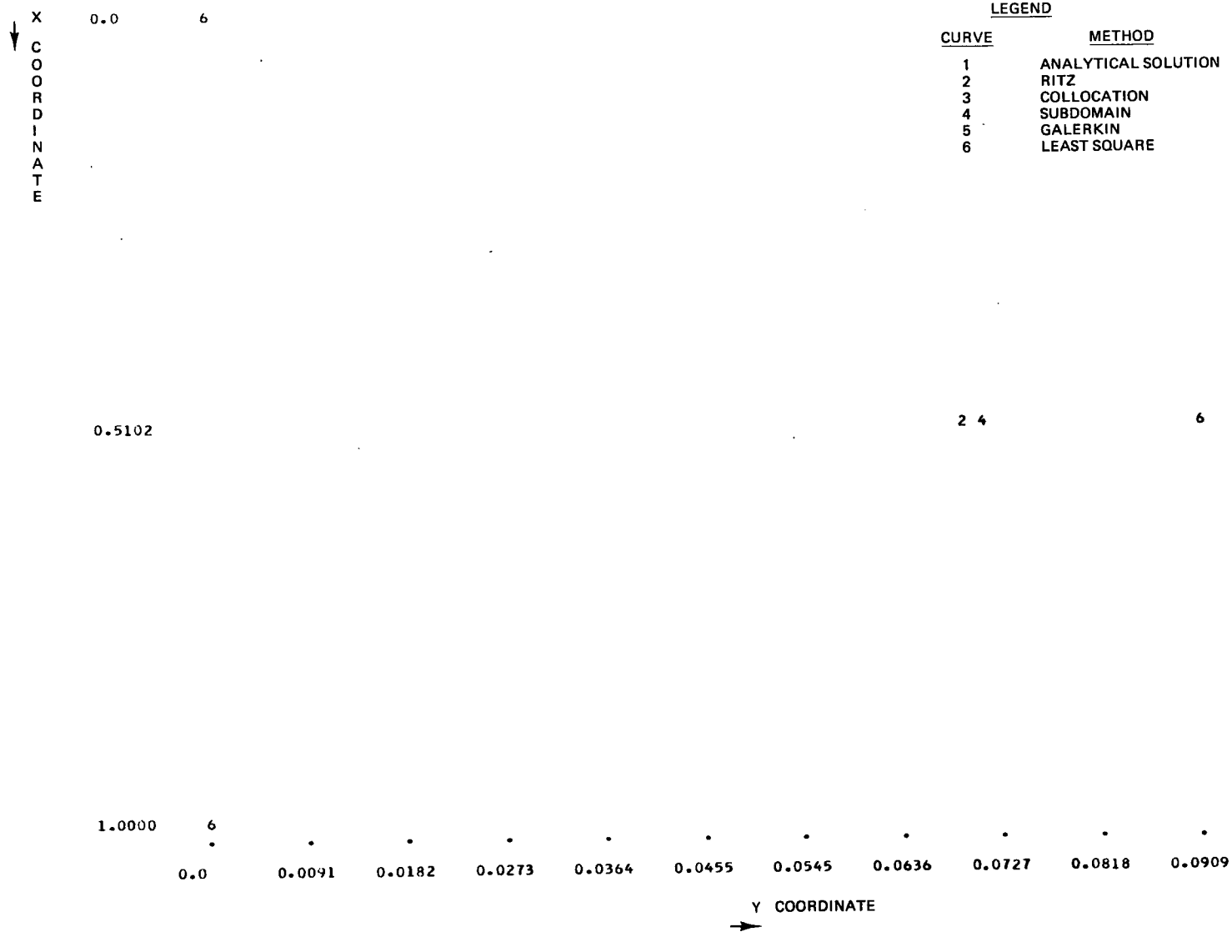
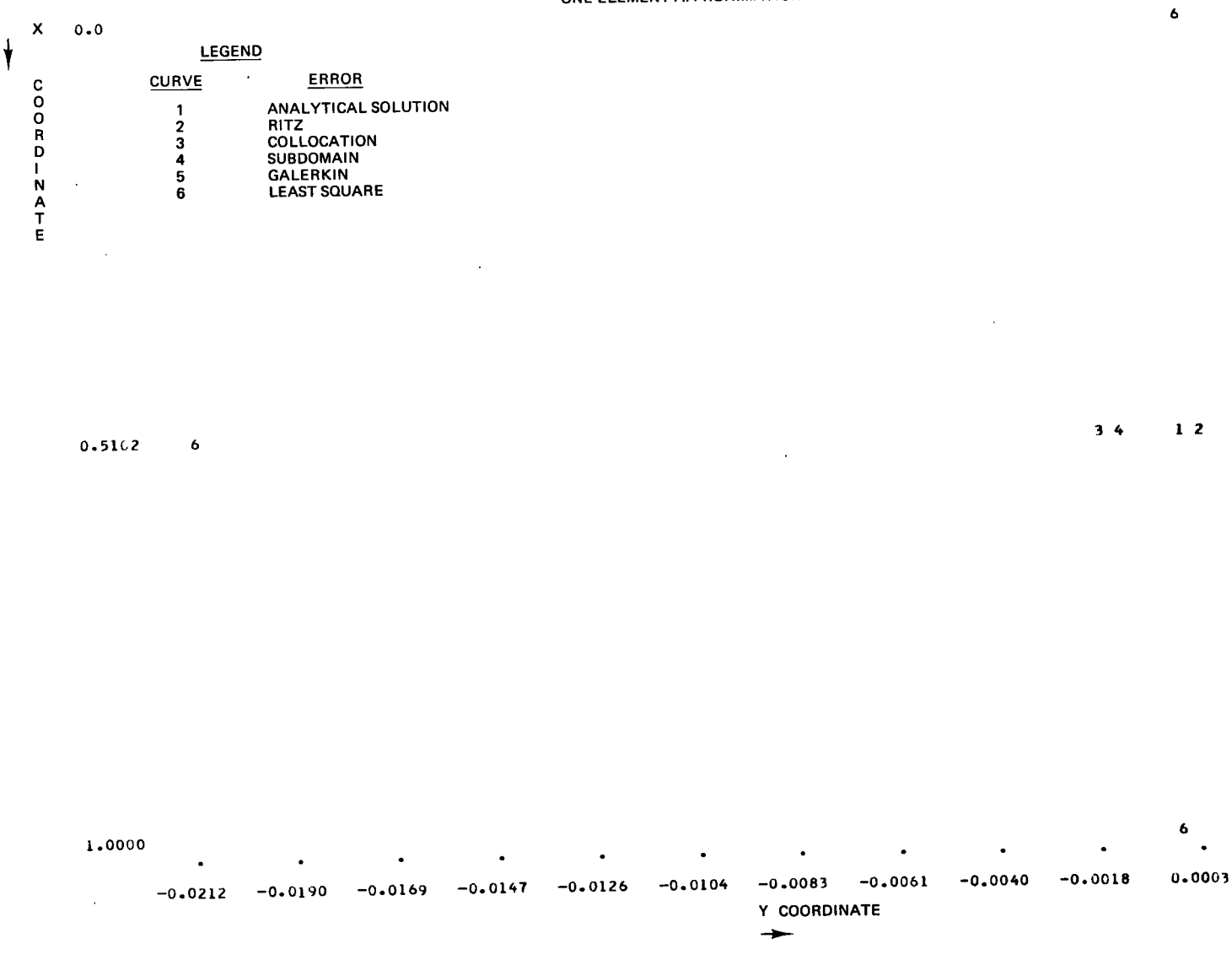


CHART A 2

ONE ELEMENT APPROXIMATION



FINITE ELEMENT TO 1 DIMENSIONAL PROBLEM.

METHOD TO USE = 11111

NUMBER OF NODES = 5

NUMBER OF ELEMENTS = 2

NUMBER OF NODAL VALUES GIVEN = 2

NODAL COORDINATES

NODE	X	Y	INITIAL VALUE
1	0.0	0.0	0.0
2	0.25000	0.0	0.04000
3	0.50000	0.0	0.06000
4	0.75000	0.0	0.50000
5	1.00000	0.0	0.0

ELEMENT	NODE NUMBERS	CONSTANTS P(I)	Q(I)	F(I)	PQ(I)
1	1 2 3	1.00000	-1.00000	1.00000	0.0
2	3 4 5	1.00000	-1.00000	1.00000	0.0

2 NODAL VALUES GIVEN

NODE	VAL.
1	0.0
5	0.0

RITZ METHOD: ONE DIMENSIONAL PROBLEM USING QUADRATIC APPROXIMATION (METHOD = 10000)

NODE	X-COORDIN	ANALYTICAL	APR METHOD	ERROR
1	0.0	0.0	0.0	0.0
2	0.03125	0.005881	0.006499	-0.000618
3	0.06250	0.011726	0.012712	-0.000986
4	0.09375	0.017499	0.018640	-0.001142
5	0.12500	0.023163	0.024283	-0.001121
6	0.15625	0.028682	0.029641	-0.000959
7	0.18750	0.034021	0.034713	-0.000692
8	0.21875	0.039143	0.039500	-0.000357
9	0.25000	0.044014	0.044001	0.000012
10	0.28125	0.048597	0.048218	0.000379
11	0.31250	0.052858	0.052148	0.000710
12	0.34375	0.056763	0.055794	0.000969
13	0.37500	0.060276	0.059154	0.001122
14	0.40625	0.063365	0.062229	0.001136
15	0.43750	0.065995	0.065019	0.000976
16	0.46875	0.068133	0.067523	0.000610
17	0.50000	0.069747	0.069742	0.000005
18	0.53125	0.070805	0.071279	-0.000475
19	0.56250	0.071274	0.072030	-0.000756
20	0.59375	0.071125	0.071995	-0.000870
21	0.62500	0.070327	0.071174	-0.000847
22	0.65625	0.068849	0.069567	-0.000717
23	0.68750	0.066664	0.067173	-0.000509
24	0.71875	0.063742	0.063994	-0.000252
25	0.75000	0.060056	0.060028	0.000028
26	0.78125	0.055579	0.055276	0.000303
27	0.81250	0.050285	0.049738	0.000547
28	0.84375	0.044149	0.043414	0.000735
29	0.87500	0.037145	0.036303	0.000842
30	0.90625	0.029251	0.028407	0.000844
31	0.93750	0.020443	0.019724	0.000719
32	0.96875	0.010700	0.010255	0.000445
33	1.00000	0.0	0.0	0.0

COLLOCATION METHOD: ONE DIMENSIONAL PROBLEM USING QUADRATIC APPROXIMATION (METHOD = 01000)

NODE	X-COORDIN	ANALYTICAL	APR METHOD	ERROR
1	0.0	0.0	0.0	0.0
2	0.03125	0.005881	0.004758	0.001124
3	0.06250	0.011726	0.009242	0.002485
4	0.09375	0.017499	0.013452	0.004047
5	0.12500	0.023163	0.017388	0.005775
6	0.15625	0.028682	0.021050	0.007632
7	0.18750	0.034021	0.024439	0.009582
8	0.21875	0.039143	0.027553	0.011590
9	0.25000	0.044014	0.030394	0.013619
10	0.28125	0.048597	0.032961	0.015636
11	0.31250	0.052858	0.035255	0.017604
12	0.34375	0.056763	0.037274	0.019489
13	0.37500	0.060276	0.039020	0.021257
14	0.40625	0.063365	0.040492	0.022873
15	0.43750	0.065995	0.041689	0.024305
16	0.46875	0.068133	0.042614	0.025519
17	0.50000	0.069747	0.043264	0.026483
18	0.53125	0.070805	0.044539	0.026266
19	0.56250	0.071274	0.045283	0.025991
20	0.59375	0.071125	0.045497	0.025628
21	0.62500	0.070327	0.045181	0.025146
22	0.65625	0.068849	0.044334	0.024516
23	0.68750	0.066664	0.042956	0.023708
24	0.71875	0.063742	0.041048	0.022694
25	0.75000	0.060056	0.038609	0.021447
26	0.78125	0.055579	0.035640	0.019939
27	0.81250	0.050285	0.032140	0.018145
28	0.84375	0.044149	0.028110	0.016039
29	0.87500	0.037145	0.023549	0.013596
30	0.90625	0.029251	0.018457	0.010793
31	0.93750	0.020443	0.012835	0.007607
32	0.96875	0.010700	0.006683	0.004017
33	1.00000	0.0	0.0	0.0

SUBDOMAIN METHOD: ONE DIMENSIONAL PROBLEM USING QUADRATIC APPROXIMATION (METHOD = 00100)

NODE	X-COORDIN	ANALYTICAL	APR METHOD	ERROR
1	0.0	0.0	0.0	0.0
2	0.03125	0.005881	0.006531	-0.000650
3	0.06250	0.011726	0.012776	-0.001050
4	0.09375	0.017499	0.018733	-0.001235
5	0.12500	0.023163	0.024404	-0.001241
6	0.15625	0.028682	0.029787	-0.001105
7	0.18750	0.034021	0.034884	-0.000863
8	0.21875	0.039143	0.039693	-0.000550
9	0.25000	0.044014	0.044216	-0.000202
10	0.28125	0.048597	0.048452	0.000145
11	0.31250	0.052858	0.052400	0.000458
12	0.34375	0.056763	0.056062	0.000701
13	0.37500	0.060276	0.059436	0.000840
14	0.40625	0.063365	0.062524	0.000841
15	0.43750	0.065995	0.065325	0.000670
16	0.46875	0.068133	0.067838	0.000295
17	0.50000	0.069747	0.070065	-0.000318
18	0.53125	0.070805	0.071614	-0.000809
19	0.56250	0.071274	0.072373	-0.001098
20	0.59375	0.071125	0.072341	-0.001216
21	0.62500	0.070327	0.071519	-0.001192
22	0.65625	0.068849	0.069906	-0.001057
23	0.68750	0.066664	0.067503	-0.000839
24	0.71875	0.063742	0.064310	-0.000568
25	0.75000	0.060056	0.060326	-0.000270
26	0.78125	0.055579	0.055552	0.000028
27	0.81250	0.050285	0.049987	0.000298
28	0.84375	0.044149	0.043632	0.000517
29	0.87500	0.037145	0.036486	0.000659
30	0.90625	0.029251	0.028550	0.000700
31	0.93750	0.020443	0.019824	0.000619
32	0.96875	0.010700	0.010307	0.000392
33	1.00000	0.0	0.0	0.0

GALERKIN'S METHOD: ONE DIMENSIONAL PROBLEM USING QUADRATIC APPROXIMATION (METHOD = 00010)

NODE	X-COORDIN	ANALYTICAL	APR METHOD	ERROR
1	0.0	0.0	0.0	0.0
2	0.03125	0.005881	0.007655	-0.001773
3	0.06250	0.011726	0.014926	-0.003200
4	0.09375	0.017499	0.021814	-0.004315
5	0.12500	0.023163	0.028319	-0.005156
6	0.15625	0.028682	0.034440	-0.005758
7	0.18750	0.034021	0.040179	-0.006158
8	0.21875	0.039143	0.045533	-0.006390
9	0.25000	0.044014	0.050505	-0.006491
10	0.28125	0.048597	0.055093	-0.006496
11	0.31250	0.052858	0.059298	-0.006440
12	0.34375	0.056763	0.063120	-0.006357
13	0.37500	0.060276	0.066558	-0.006282
14	0.40625	0.063365	0.069614	-0.006249
15	0.43750	0.065995	0.072285	-0.006290
16	0.46875	0.068133	0.074574	-0.006441
17	0.50000	0.069747	0.076479	-0.006732
18	0.53125	0.070805	0.077771	-0.006967
19	0.56250	0.071274	0.078254	-0.006979
20	0.59375	0.071125	0.077927	-0.006801
21	0.62500	0.070327	0.076790	-0.006463
22	0.65625	0.068849	0.074844	-0.005994
23	0.68750	0.066664	0.072088	-0.005424
24	0.71875	0.063742	0.068522	-0.004780
25	0.75000	0.060056	0.064147	-0.004091
26	0.78125	0.055579	0.058962	-0.003383
27	0.81250	0.050285	0.052968	-0.002683
28	0.84375	0.044149	0.046164	-0.002015
29	0.87500	0.037145	0.038550	-0.001405
30	0.90625	0.029251	0.030127	-0.000877
31	0.93750	0.020443	0.020894	-0.000452
32	0.96875	0.010700	0.010852	-0.000152
33	1.00000	0.0	0.0	0.0

LEAST SQUARES METHOD: ONE DIMENSIONAL PROBLEM USING QUADRATIC APPROXIMATION (METHOD = 00001)

NODE	X-COORDIN	ANALYTICAL	APR METHOD	ERROR
1	0.0	0.0	0.0	0.0
2	0.03125	0.005881	0.002705	0.003176
3	0.06250	0.011726	0.005229	0.006497
4	0.09375	0.017499	0.007573	0.009926
5	0.12500	0.023163	0.009736	0.013427
6	0.15625	0.028682	0.011718	0.016964
7	0.18750	0.034021	0.013520	0.020500
8	0.21875	0.039143	0.015142	0.024001
9	0.25000	0.044014	0.016582	0.027431
10	0.28125	0.048597	0.017843	0.030754
11	0.31250	0.052858	0.018922	0.033936
12	0.34375	0.056763	0.019821	0.036942
13	0.37500	0.060276	0.020539	0.039737
14	0.40625	0.063365	0.021077	0.042288
15	0.43750	0.065995	0.021434	0.044560
16	0.46875	0.068133	0.021611	0.046522
17	0.50000	0.069747	0.021607	0.048140
18	0.53125	0.070805	0.025731	0.045074
19	0.56250	0.071274	0.029124	0.042150
20	0.59375	0.071125	0.031788	0.039337
21	0.62500	0.070327	0.033722	0.036604
22	0.65625	0.068849	0.034926	0.033923
23	0.68750	0.066664	0.035401	0.031263
24	0.71875	0.063742	0.035145	0.028597
25	0.75000	0.060056	0.034160	0.025897
26	0.78125	0.055579	0.032444	0.023135
27	0.81250	0.050285	0.029999	0.020286
28	0.84375	0.044149	0.026824	0.017325
29	0.87500	0.037145	0.022919	0.014226
30	0.90625	0.029251	0.018284	0.010967
31	0.93750	0.020443	0.012919	0.007524
32	0.96875	0.010700	0.006825	0.003875
33	1.00000	0.0	0.0	0.0

X-COORDINATE	ANALYTICAL SOLUTION	RITZ	COLLOCATION	SUBDOMAIN	GALERKIN*S	LEASTSQUARES
0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.25000	0.044013560	0.044001408	0.030394312	0.044216018	0.050505053	0.016582374
0.50000	0.069746912	0.069741845	0.043263964	0.070064843	0.076479018	0.021607026
0.75000	0.060056210	0.060027938	0.038608976	0.060325827	0.064146996	0.034159623
1.00000	0.0	0.0	0.0	0.0	0.0	0.0

TIME SPENT IN SEC.

0.035999998450	0.032999999821	0.030999999493	0.030999999493	0.030999999493

CHART B 1

TWO ELEMENTS APPROXIMATION

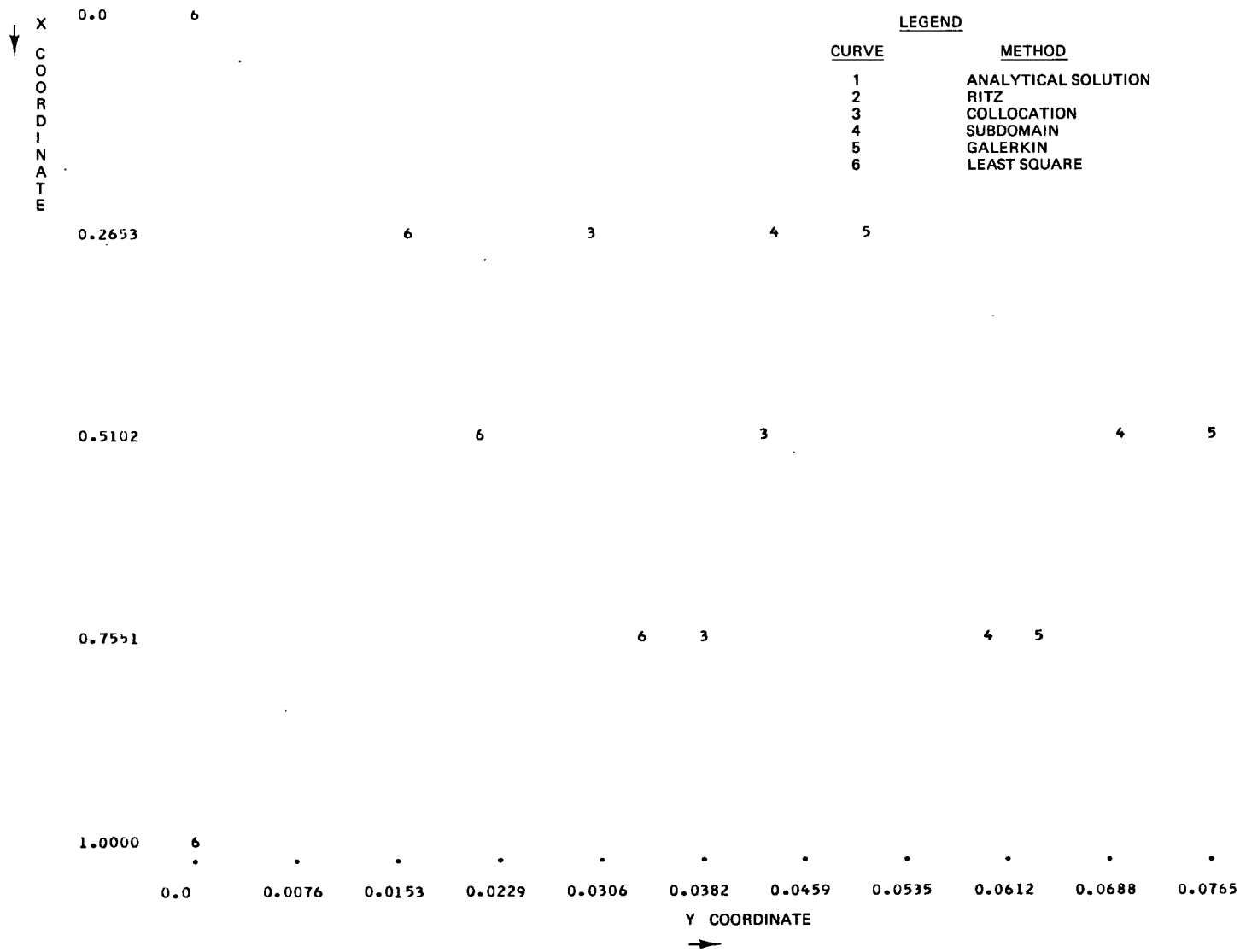
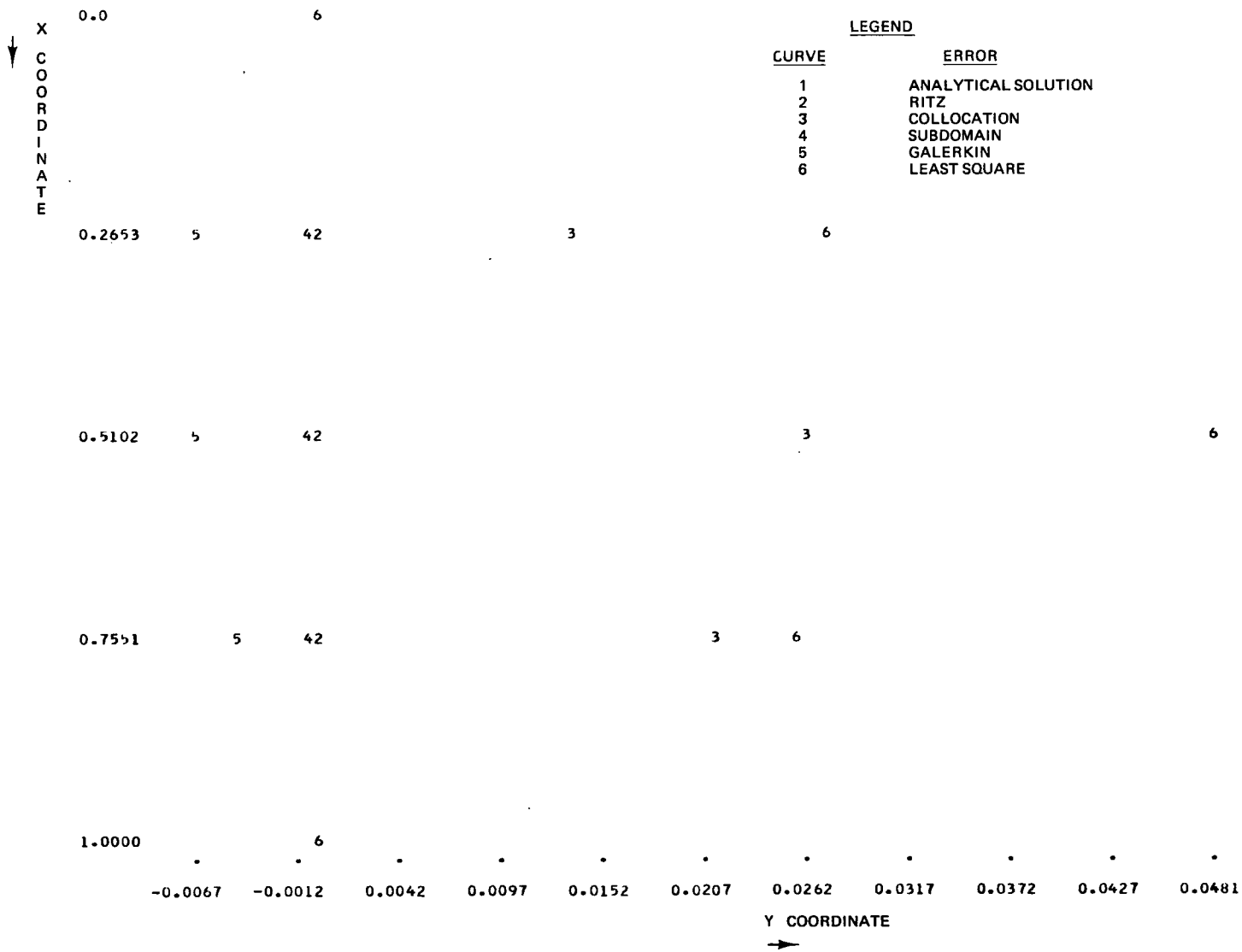


CHART B 2


X COORDINATE
0.0
0.2653
0.5102
0.7551
1.0000
6
5
42
3
LEGEND
CURVE
ERROR
1 ANALYTICAL SOLUTION
2 RITZ
3 COLLOCATION
4 SUBDOMAIN
5 GALERKIN
6 LEAST SQUARE
-0.0067
-0.0012
0.0042
0.0097
0.0152
0.0207
0.0262
0.0317
0.0372
0.0427
0.0481
Y COORDINATE

FINITE ELEMENT TO 1 DIMENSIONAL PROBLEM.

METHOD TO USE = 11111

NUMBER OF NODES = 9

NUMBER OF ELEMENTS = 4

NUMBER OF NODAL VALUES GIVEN = 2

NODAL COORDINATES

NODE	X	Y	INITIAL VALUE
1	0.0	0.0	0.0
2	0.12500	0.0	0.02000
3	0.25000	0.0	0.04000
4	0.37500	0.0	0.05000
5	0.50000	0.0	0.06000
6	0.62500	0.0	0.50000
7	0.75000	0.0	0.55000
8	0.87500	0.0	0.30000
9	1.00000	0.0	0.0

ELEMENT	NODE NUMBERS			CONSTANTS P(I)	Q(I)	F(I)	PQ(I)
1	1	2	3	1.00000	-1.00000	1.00000	0.0
2	3	4	5	1.00000	-1.00000	1.00000	0.0
3	5	6	7	1.00000	-1.00000	1.00000	0.0
4	7	8	9	1.00000	-1.00000	1.00000	0.0

2 NODAL VALUES GIVEN

NODE	VAL.
1	0.0
9	0.0

RITZ METHOD: ONE DIMENSIONAL PROBLEM USING QUADRATIC APPROXIMATION (METHOD = 10000)

NODE	X-COORDIN	ANALYTICAL	APR METHOD	ERROR
1	0.0	0.0	0.0	0.0
2	0.03125	0.005881	0.006007	-0.000126
3	0.06250	0.011726	0.011870	-0.000144
4	0.09375	0.017499	0.017589	-0.000090
5	0.12500	0.023163	0.023163	0.000000
6	0.15625	0.028682	0.028592	0.000090
7	0.18750	0.034021	0.033877	0.000144
8	0.21875	0.039143	0.039018	0.000125
9	0.25000	0.044014	0.044014	-0.000000
10	0.28125	0.048597	0.048716	-0.000119
11	0.31250	0.052858	0.052993	-0.000135
12	0.34375	0.056763	0.056847	-0.000084
13	0.37500	0.060276	0.060276	0.000001
14	0.40625	0.063365	0.063280	0.000085
15	0.43750	0.065995	0.065860	0.000135
16	0.46875	0.068133	0.068016	0.000117
17	0.50000	0.069747	0.069747	0.000000
18	0.53125	0.070805	0.070908	-0.000104
19	0.56250	0.071274	0.071392	-0.000118
20	0.59375	0.071125	0.071198	-0.000073
21	0.62500	0.070327	0.070326	0.000001
22	0.65625	0.068849	0.068775	0.000074
23	0.68750	0.066664	0.066547	0.000117
24	0.71875	0.063742	0.063641	0.000101
25	0.75000	0.060056	0.060057	-0.000000
26	0.78125	0.055579	0.055662	-0.000083
27	0.81250	0.050285	0.050379	-0.000093
28	0.84375	0.044149	0.044205	-0.000057
29	0.87500	0.037145	0.037143	0.000002
30	0.90625	0.029251	0.029191	0.000059
31	0.93750	0.020443	0.020350	0.000093
32	0.96875	0.010700	0.010620	0.000080
33	1.00000	0.0	0.0	0.0

COLLOCATION METHOD: ONE DIMENSIONAL PROBLEM USING QUADRATIC APPROXIMATION (METHOD = 01000)

NODE	X-COORDIN	ANALYTICAL	APR METHOD	ERROR
1	0.0	0.0	0.0	0.0
2	0.03125	0.005881	0.004481	0.001400
3	0.06250	0.011726	0.008823	0.002903
4	0.09375	0.017499	0.013027	0.004472
5	0.12500	0.023163	0.017091	0.006071
6	0.15625	0.028682	0.021017	0.007665
7	0.18750	0.034021	0.024804	0.009216
8	0.21875	0.039143	0.028453	0.010690
9	0.25000	0.044014	0.031962	0.012051
10	0.28125	0.048597	0.035538	0.013059
11	0.31250	0.052858	0.038838	0.014020
12	0.34375	0.056763	0.041864	0.014899
13	0.37500	0.060276	0.044613	0.015663
14	0.40625	0.063365	0.047088	0.016277
15	0.43750	0.065995	0.049287	0.016708
16	0.46875	0.068133	0.051210	0.016923
17	0.50000	0.069747	0.052858	0.016888
18	0.53125	0.070805	0.054090	0.016714
19	0.56250	0.071274	0.054783	0.016492
20	0.59375	0.071125	0.054935	0.016190
21	0.62500	0.070327	0.054547	0.015779
22	0.65625	0.068849	0.053620	0.015230
23	0.68750	0.066664	0.052152	0.014512
24	0.71875	0.063742	0.050145	0.013597
25	0.75000	0.060056	0.047598	0.012459
26	0.78125	0.055579	0.044374	0.011205
27	0.81250	0.050285	0.040372	0.009914
28	0.84375	0.044149	0.035590	0.008558
29	0.87500	0.037145	0.030030	0.007115
30	0.90625	0.029251	0.023691	0.005560
31	0.93750	0.020443	0.016573	0.003870
32	0.96875	0.010700	0.008676	0.002024
33	1.00000	0.0	0.0	0.0

SUBDOMAIN METHOD: ONE DIMENSIONAL PROBLEM USING QUADRATIC APPROXIMATION (METHOD = 00100)

NODE	X-COORDIN	ANALYTICAL	APR METHOD	ERROR
1	0.0	0.0	0.0	0.0
2	0.03125	0.005881	0.006014	-0.000133
3	0.06250	0.011726	0.011884	-0.000158
4	0.09375	0.017499	0.017609	-0.000110
5	0.12500	0.023163	0.023189	-0.000027
6	0.15625	0.028682	0.028625	0.000057
7	0.18750	0.034021	0.033916	0.000105
8	0.21875	0.039143	0.039062	0.000081
9	0.25000	0.044014	0.044064	-0.000050
10	0.28125	0.048597	0.048772	-0.000175
11	0.31250	0.052858	0.053054	-0.000196
12	0.34375	0.056763	0.056912	-0.000149
13	0.37500	0.060276	0.060345	-0.000069
14	0.40625	0.063365	0.063353	0.000012
15	0.43750	0.065995	0.065935	0.000059
16	0.46875	0.068133	0.068093	0.000040
17	0.50000	0.069747	0.069826	-0.000079
18	0.53125	0.070805	0.070989	-0.000185
19	0.56250	0.071274	0.071473	-0.000199
20	0.59375	0.071125	0.071279	-0.000154
21	0.62500	0.070327	0.070406	-0.000079
22	0.65625	0.068849	0.068853	-0.000004
23	0.68750	0.066664	0.066622	0.000042
24	0.71875	0.063742	0.063712	0.000030
25	0.75000	0.060056	0.060123	-0.000067
26	0.78125	0.055579	0.055725	-0.000145
27	0.81250	0.050285	0.050436	-0.000150
28	0.84375	0.044149	0.044256	-0.000107
29	0.87500	0.037145	0.037186	-0.000041
30	0.90625	0.029251	0.029225	0.000025
31	0.93750	0.020443	0.020374	0.000069
32	0.96875	0.010700	0.010632	0.000067
33	1.00000	0.0	0.0	0.0

GALERKIN'S METHOD: ONE DIMENSIONAL PROBLEM USING QUADRATIC APPROXIMATION (METHOD = 00010)

NODE	X-COORDIN	ANALYTICAL	APR METHOD	ERROR
1	0.0	0.0	0.0	0.0
2	0.03125	0.005881	0.006412	-0.000531
3	0.06250	0.011726	0.012631	-0.000904
4	0.09375	0.017499	0.018655	-0.001157
5	0.12500	0.023163	0.024487	-0.001324
6	0.15625	0.028682	0.030124	-0.001442
7	0.18750	0.034021	0.035568	-0.001547
8	0.21875	0.039143	0.040818	-0.001675
9	0.25000	0.044014	0.045875	-0.001861
10	0.28125	0.048597	0.050637	-0.002040
11	0.31250	0.052858	0.054958	-0.002100
12	0.34375	0.056763	0.058839	-0.002076
13	0.37500	0.060276	0.062279	-0.002002
14	0.40625	0.063365	0.065277	-0.001913
15	0.43750	0.065995	0.067835	-0.001841
16	0.46875	0.068133	0.069952	-0.001819
17	0.50000	0.069747	0.071629	-0.001882
18	0.53125	0.070805	0.072727	-0.001922
19	0.56250	0.071274	0.073138	-0.001864
20	0.59375	0.071125	0.072863	-0.001738
21	0.62500	0.070327	0.071902	-0.001575
22	0.65625	0.068849	0.070254	-0.001404
23	0.68750	0.066664	0.067919	-0.001255
24	0.71875	0.063742	0.064898	-0.001155
25	0.75000	0.060056	0.061190	-0.001133
26	0.78125	0.055579	0.056668	-0.001089
27	0.81250	0.050285	0.051253	-0.000968
28	0.84375	0.044149	0.044944	-0.000796
29	0.87500	0.037145	0.037742	-0.000597
30	0.90625	0.029251	0.029647	-0.000396
31	0.93750	0.020443	0.020658	-0.000215
32	0.96875	0.010700	0.010776	-0.000076
33	1.00000	0.0	0.0	0.0

LEAST SQUARES METHOD: ONE DIMENSIONAL PROBLEM USING QUADRATIC APPROXIMATION (METHOD = 00001)

NODE	X-COORDIN	ANALYTICAL	APR METHOD	ERROR
1	0.0	0.0	0.0	0.0
2	0.03125	0.005881	0.001117	0.004764
3	0.06250	0.011726	0.002149	0.009578
4	0.09375	0.017499	0.003094	0.014405
5	0.12500	0.023163	0.003953	0.019210
6	0.15625	0.028682	0.004726	0.023956
7	0.18750	0.034021	0.005413	0.028608
8	0.21875	0.039143	0.006014	0.033130
9	0.25000	0.044014	0.006528	0.037485
10	0.28125	0.048597	0.012104	0.036493
11	0.31250	0.052858	0.016922	0.035936
12	0.34375	0.056763	0.020984	0.035779
13	0.37500	0.060276	0.024288	0.035989
14	0.40625	0.063365	0.026834	0.036531
15	0.43750	0.065995	0.028624	0.037371
16	0.46875	0.068133	0.029656	0.038477
17	0.50000	0.069747	0.029930	0.039816
18	0.53125	0.070805	0.034472	0.036333
19	0.56250	0.071274	0.037899	0.033375
20	0.59375	0.071125	0.040212	0.030913
21	0.62500	0.070327	0.041411	0.028915
22	0.65625	0.068849	0.041496	0.027353
23	0.68750	0.066664	0.040467	0.026197
24	0.71875	0.063742	0.038324	0.025419
25	0.75000	0.060056	0.035066	0.024990
26	0.78125	0.055579	0.034342	0.021237
27	0.81250	0.050285	0.032573	0.017712
28	0.84375	0.044149	0.029758	0.014390
29	0.87500	0.037145	0.025898	0.011247
30	0.90625	0.029251	0.020992	0.008259
31	0.93750	0.020443	0.015040	0.005403
32	0.96875	0.010700	0.008043	0.002657
33	1.00000	0.0	0.0	0.0

X-COORDINATE	ANALYTICAL SOLUTION	RITZ	COLLOCATION	SUBDOMAIN	GALERKIN*S	LEASTSQUARES
0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.12500	0.023162723	0.023162555	0.017091230	0.023189303	0.024486557	0.003952805
0.25000	0.044013560	0.044013705	0.031962305	0.044063833	0.045874942	0.006528359
0.37500	0.060276449	0.060275521	0.044613209	0.060344964	0.062278569	0.024287559
0.50000	0.069746912	0.069746733	0.052858435	0.069825768	0.071628749	0.029930484
0.62500	0.070326686	0.070325613	0.054547209	0.070405722	0.071901858	0.041411240
0.75000	0.060056210	0.060056701	0.047597561	0.060123082	0.061189692	0.035066064
0.87500	0.037145019	0.037142996	0.030030001	0.037185904	0.037742130	0.025897842
1.00000	0.0	0.0	0.0	0.0	0.0	0.0
TIME SPENT IN SEC.						
		0.046999998391	0.042999997735	0.041999999434	0.042999997735	0.041999999434

CHART C 1

FOUR ELEMENT APPROXIMATION

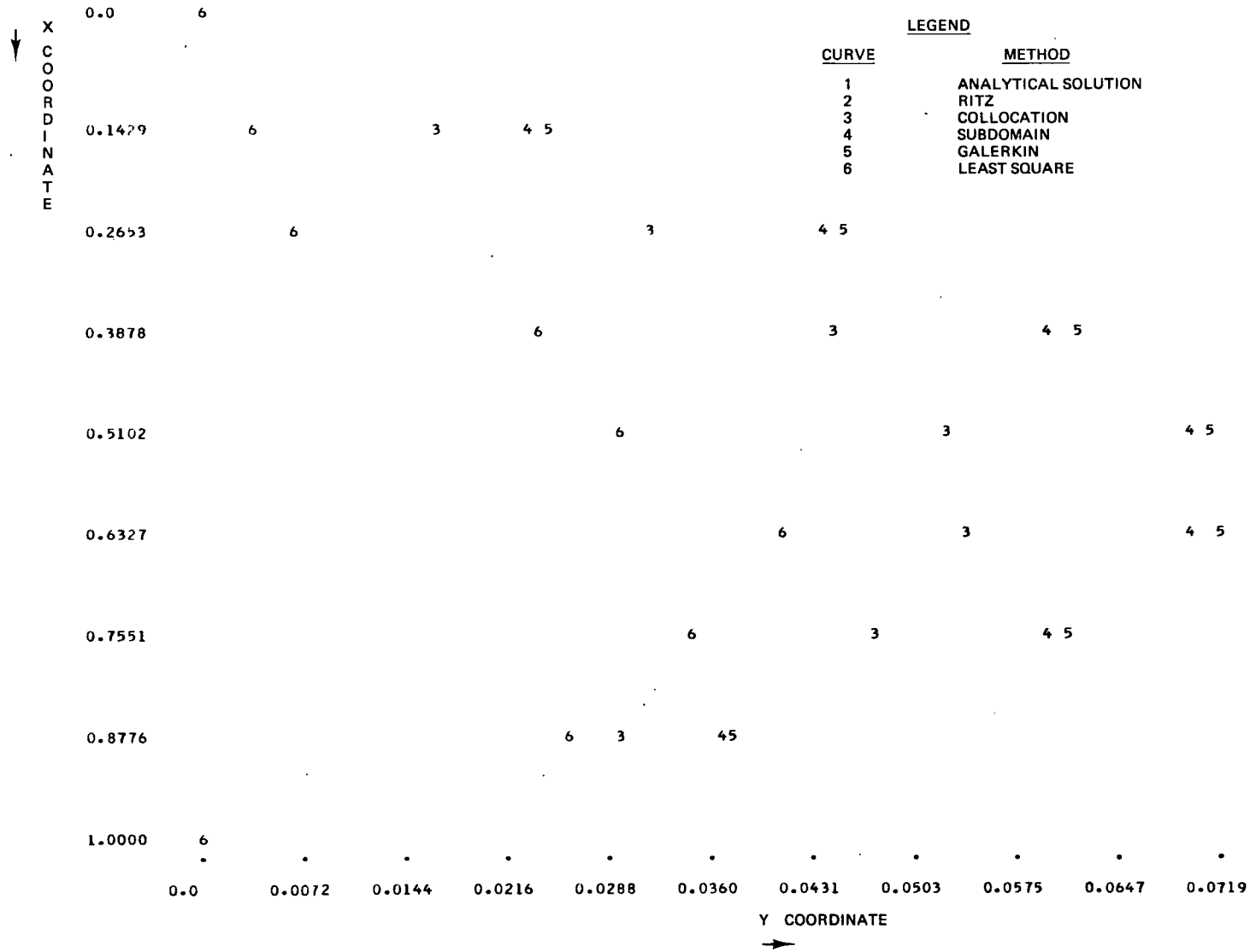
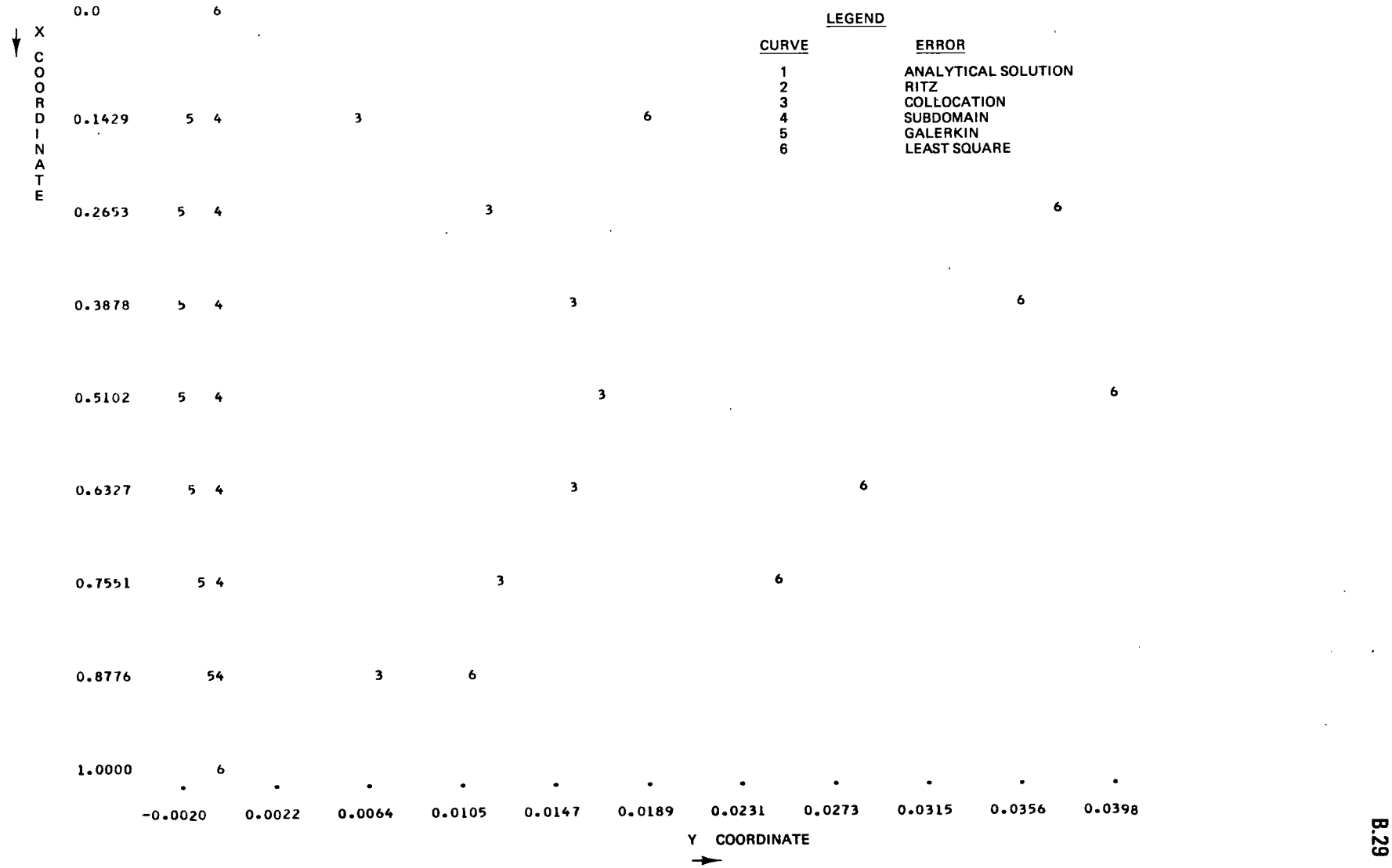


CHART C 2

FOUR ELEMENT APPROXIMATION



FINITE ELEMENT TO 1 DIMENSIONAL PROBLEM.

METHOD TO USE = 11111

NUMBER OF NODES = 17

NUMBER OF ELEMENTS = 8

NUMBER OF NODAL VALUES GIVEN = 2

NODAL COORDINATES

NODE	X	Y	INITIAL VALUE
1	0.0	0.0	0.0
2	0.06250	0.0	0.0
3	0.12500	0.0	0.0
4	0.18750	0.0	0.0
5	0.25000	0.0	0.0
6	0.31250	0.0	0.0
7	0.37500	0.0	0.0
8	0.43750	0.0	0.0
9	0.50000	0.0	0.0
10	0.56250	0.0	0.0
11	0.62500	0.0	0.0
12	0.68750	0.0	0.0
13	0.75000	0.0	0.0
14	0.81250	0.0	0.0
15	0.87500	0.0	0.0
16	0.93750	0.0	0.0
17	1.00000	0.0	0.0

ELEMENT	NODE NUMBERS			CONSTANTS P(I)	Q(I)	F(I)	PQ(I)
1	1	2	3	1.00000	-1.00000	1.00000	0.0
2	3	4	5	1.00000	-1.00000	1.00000	0.0
3	5	6	7	1.00000	-1.00000	1.00000	0.0
4	7	8	9	1.00000	-1.00000	1.00000	0.0
5	9	10	11	1.00000	-1.00000	1.00000	0.0
6	11	12	13	1.00000	-1.00000	1.00000	0.0
7	13	14	15	1.00000	-1.00000	1.00000	0.0
8	15	16	17	1.00000	-1.00000	1.00000	0.0

2 NODAL VALUES GIVEN

NODE	VAL.
1	0.0
17	0.0

RITZ METHOD: ONE DIMENSIONAL PROBLEM USING QUADRATIC APPROXIMATION (METHOD = 10000)

NODE	X-COORDIN	ANALYTICAL	APR METHOD	ERROR
1	0.0	0.0	0.0	0.0
2	0.03125	0.005881	0.005899	-0.000018
3	0.06250	0.011726	0.011726	-0.000000
4	0.09375	0.017499	0.017481	0.000018
5	0.12500	0.023163	0.023163	-0.000000
6	0.15625	0.028682	0.028700	-0.000018
7	0.18750	0.034021	0.034021	-0.000000
8	0.21875	0.039143	0.039126	0.000017
9	0.25000	0.044014	0.044014	-0.000001
10	0.28125	0.048597	0.048615	-0.000018
11	0.31250	0.052858	0.052859	-0.000001
12	0.34375	0.056763	0.056747	0.000016
13	0.37500	0.060276	0.060277	-0.000001
14	0.40625	0.063365	0.063382	-0.000017
15	0.43750	0.065995	0.065996	-0.000001
16	0.46875	0.068133	0.068118	0.000015
17	0.50000	0.069747	0.069748	-0.000002
18	0.53125	0.070805	0.070822	-0.000017
19	0.56250	0.071274	0.071276	-0.000002
20	0.59375	0.071125	0.071112	0.000013
21	0.62500	0.070327	0.070330	-0.000003
22	0.65625	0.068849	0.068866	-0.000017
23	0.68750	0.066664	0.066667	-0.000003
24	0.71875	0.063742	0.063732	0.000010
25	0.75000	0.060056	0.060061	-0.000004
26	0.78125	0.055579	0.055596	-0.000017
27	0.81250	0.050285	0.050289	-0.000004
28	0.84375	0.044149	0.044140	0.000008
29	0.87500	0.037145	0.037149	-0.000004
30	0.90625	0.029251	0.029264	-0.000013
31	0.93750	0.020443	0.020444	-0.000001
32	0.96875	0.010700	0.010690	0.000010
33	1.00000	0.0	0.0	0.0

COLLOCATION METHOD: ONE DIMENSIONAL PROBLEM USING QUADRATIC APPROXIMATION (METHOD = 01000)

NODE	X-COORDIN	ANALYTICAL	APR METHOD	ERROR
1	0.0	0.0	0.0	0.0
2	0.03125	0.005881	0.004947	0.000934
3	0.06250	0.011726	0.009824	0.001902
4	0.09375	0.017499	0.014630	0.002869
5	0.12500	0.023163	0.019366	0.003797
6	0.15625	0.028682	0.024066	0.004616
7	0.18750	0.034021	0.028625	0.005396
8	0.21875	0.039143	0.033043	0.006100
9	0.25000	0.044014	0.037320	0.006694
10	0.28125	0.048597	0.041386	0.007211
11	0.31250	0.052858	0.045171	0.007688
12	0.34375	0.056763	0.048675	0.008087
13	0.37500	0.060276	0.051899	0.008377
14	0.40625	0.063365	0.054774	0.008591
15	0.43750	0.065995	0.057231	0.008764
16	0.46875	0.068133	0.059271	0.008862
17	0.50000	0.069747	0.060895	0.008852
18	0.53125	0.070805	0.062034	0.008770
19	0.56250	0.071274	0.062626	0.008648
20	0.59375	0.071125	0.062670	0.008455
21	0.62500	0.070327	0.062166	0.008160
22	0.65625	0.068849	0.061051	0.007798
23	0.68750	0.066664	0.059265	0.007399
24	0.71875	0.063742	0.056807	0.006935
25	0.75000	0.060056	0.053679	0.006377
26	0.78125	0.055579	0.049820	0.005759
27	0.81250	0.050285	0.045176	0.005109
28	0.84375	0.044149	0.039747	0.004401
29	0.87500	0.037145	0.033534	0.003611
30	0.90625	0.029251	0.026481	0.002769
31	0.93750	0.020443	0.018541	0.001901
32	0.96875	0.010700	0.009714	0.000985
33	1.00000	0.0	0.0	0.0

SUBDOMAIN METHOD: ONE DIMENSIONAL PROBLEM USING QUADRATIC APPROXIMATION (METHOD = 00100)

NODE	X-COORDIN	ANALYTICAL	APR METHOD	ERROR
1	0.0	0.0	0.0	0.0
2	0.03125	0.005881	0.005901	-0.000020
3	0.06250	0.011726	0.011730	-0.000003
4	0.09375	0.017499	0.017486	0.000013
5	0.12500	0.023163	0.023169	-0.000007
6	0.15625	0.028682	0.028708	-0.000026
7	0.18750	0.034021	0.034030	-0.000010
8	0.21875	0.039143	0.039136	0.000007
9	0.25000	0.044014	0.044026	-0.000012
10	0.28125	0.048597	0.048628	-0.000031
11	0.31250	0.052858	0.052873	-0.000015
12	0.34375	0.056763	0.056762	0.000001
13	0.37500	0.060276	0.060293	-0.000017
14	0.40625	0.063365	0.063399	-0.000034
15	0.43750	0.065995	0.066013	-0.000018
16	0.46875	0.068133	0.068135	-0.000002
17	0.50000	0.069747	0.069766	-0.000019
18	0.53125	0.070805	0.070839	-0.000035
19	0.56250	0.071274	0.071294	-0.000019
20	0.59375	0.071125	0.071129	-0.000004
21	0.62500	0.070327	0.070346	-0.000019
22	0.65625	0.068849	0.068882	-0.000033
23	0.68750	0.066664	0.066682	-0.000018
24	0.71875	0.063742	0.063745	-0.000003
25	0.75000	0.060056	0.060072	-0.000016
26	0.78125	0.055579	0.055607	-0.000028
27	0.81250	0.050285	0.050299	-0.000014
28	0.84375	0.044149	0.044148	0.000000
29	0.87500	0.037145	0.037155	-0.000010
30	0.90625	0.029251	0.029269	-0.000019
31	0.93750	0.020443	0.020448	-0.000006
32	0.96875	0.010700	0.010692	0.000008
33	1.00000	0.0	0.0	0.0

GALERKIN'S METHOD: ONE DIMENSIONAL PROBLEM USING QUADRATIC APPROXIMATION (METHOD = 00010)

NODE	X-COORDIN	ANALYTICAL	APR METHOD	ERROR
1	0.0	0.0	0.0	0.0
2	0.03125	0.005881	0.006027	-0.000146
3	0.06250	0.011726	0.011957	-0.000231
4	0.09375	0.017499	0.017790	-0.000291
5	0.12500	0.023163	0.023527	-0.000364
6	0.15625	0.028682	0.029110	-0.000428
7	0.18750	0.034021	0.034469	-0.000448
8	0.21875	0.039143	0.039603	-0.000460
9	0.25000	0.044014	0.044512	-0.000499
10	0.28125	0.048597	0.049128	-0.000531
11	0.31250	0.052858	0.053383	-0.000524
12	0.34375	0.056763	0.057276	-0.000513
13	0.37500	0.060276	0.060807	-0.000530
14	0.40625	0.063365	0.063908	-0.000543
15	0.43750	0.065995	0.066515	-0.000520
16	0.46875	0.068133	0.068626	-0.000493
17	0.50000	0.069747	0.070242	-0.000495
18	0.53125	0.070805	0.071298	-0.000494
19	0.56250	0.071274	0.071733	-0.000459
20	0.59375	0.071125	0.071547	-0.000422
21	0.62500	0.070327	0.070740	-0.000413
22	0.65625	0.068849	0.069250	-0.000401
23	0.68750	0.066664	0.067022	-0.000359
24	0.71875	0.063742	0.064057	-0.000314
25	0.75000	0.060056	0.060353	-0.000297
26	0.78125	0.055579	0.055855	-0.000276
27	0.81250	0.050285	0.050514	-0.000229
28	0.84375	0.044149	0.044329	-0.000181
29	0.87500	0.037145	0.037301	-0.000156
30	0.90625	0.029251	0.029380	-0.000129
31	0.93750	0.020443	0.020522	-0.000079
32	0.96875	0.010700	0.010729	-0.000029
33	1.00000	0.0	0.0	0.0

LEAST SQUARES METHOD: ONE DIMENSIONAL PROBLEM USING QUADRATIC APPROXIMATION (METHOD = 00001)

NODE	X-COORDIN	ANALYTICAL	APR METHOD	ERROR
1	0.0	0.0	0.0	0.0
2	0.03125	0.005881	0.001069	0.004813
3	0.06250	0.011726	0.002094	0.009632
4	0.09375	0.017499	0.003076	0.014423
5	0.12500	0.023163	0.004015	0.019148
6	0.15625	0.028682	0.009574	0.019108
7	0.18750	0.034021	0.013953	0.020068
8	0.21875	0.039143	0.017150	0.021993
9	0.25000	0.044014	0.019168	0.024846
10	0.28125	0.048597	0.025623	0.022973
11	0.31250	0.052858	0.030552	0.022306
12	0.34375	0.056763	0.033953	0.022810
13	0.37500	0.060276	0.035827	0.024450
14	0.40625	0.063365	0.040730	0.022635
15	0.43750	0.065995	0.044311	0.021684
16	0.46875	0.068133	0.046571	0.021562
17	0.50000	0.069747	0.047508	0.022239
18	0.53125	0.070805	0.050474	0.020330
19	0.56250	0.071274	0.052178	0.019096
20	0.59375	0.071125	0.052620	0.018505
21	0.62500	0.070327	0.051800	0.018527
22	0.65625	0.068849	0.052366	0.016483
23	0.68750	0.066664	0.051669	0.014995
24	0.71875	0.063742	0.049708	0.014034
25	0.75000	0.060056	0.046485	0.013571
26	0.78125	0.055579	0.044188	0.011391
27	0.81250	0.050285	0.040600	0.009685
28	0.84375	0.044149	0.035721	0.008427
29	0.87500	0.037145	0.029552	0.007593
30	0.90625	0.029251	0.023860	0.005391
31	0.93750	0.020443	0.017037	0.003406
32	0.96875	0.010700	0.009084	0.001616
33	1.00000	0.0	0.0	0.0

X-COORDINATE	ANALYTICAL SOLUTION	RITZ	COLLOCATION	SUBDOMAIN	GALERKIN*S	LEASTSQUARES
0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.06250	0.011726320	0.011726450	0.009824101	0.011729643	0.011956941	0.002094182
0.12500	0.023162723	0.023163110	0.019365698	0.023169361	0.023526620	0.004014924
0.18750	0.034020722	0.034021154	0.028624788	0.034030311	0.034468506	0.013952516
0.25000	0.044013560	0.044014245	0.037319951	0.044025991	0.044512250	0.019167688
0.31250	0.052858412	0.052859165	0.045170892	0.052873161	0.053382892	0.030551925
0.37500	0.060276449	0.060277425	0.051899474	0.060293235	0.060806908	0.035826746
0.43750	0.065994799	0.065995753	0.057230920	0.066012919	0.066514552	0.044311289
0.50000	0.069746912	0.069748461	0.060894758	0.069766045	0.070242286	0.047508232
0.56250	0.071274221	0.071276248	0.062626064	0.071293712	0.071733415	0.052178465
0.62500	0.070326686	0.070329547	0.062166400	0.070345879	0.070739746	0.051800080
0.68750	0.066663861	0.066667318	0.059264839	0.066682041	0.067022383	0.051668689
0.75000	0.060056210	0.060060546	0.053679019	0.060072426	0.060352813	0.046484932
0.81250	0.050285220	0.050289389	0.045176189	0.050298732	0.050513957	0.040599950
0.87500	0.037145019	0.037148926	0.033533953	0.037154984	0.037301015	0.029551677
0.93750	0.020442843	0.020444330	0.018541459	0.020448353	0.020522308	0.017036989
1.00000	0.0	0.0	0.0	0.0	0.0	0.0
TIME SPENT IN SEC.						
		0.110999941826	0.104999959469	0.104999959469	0.105999946594	0.106999993324

CHART D 1
EIGHT ELEMENT APPROXIMATION

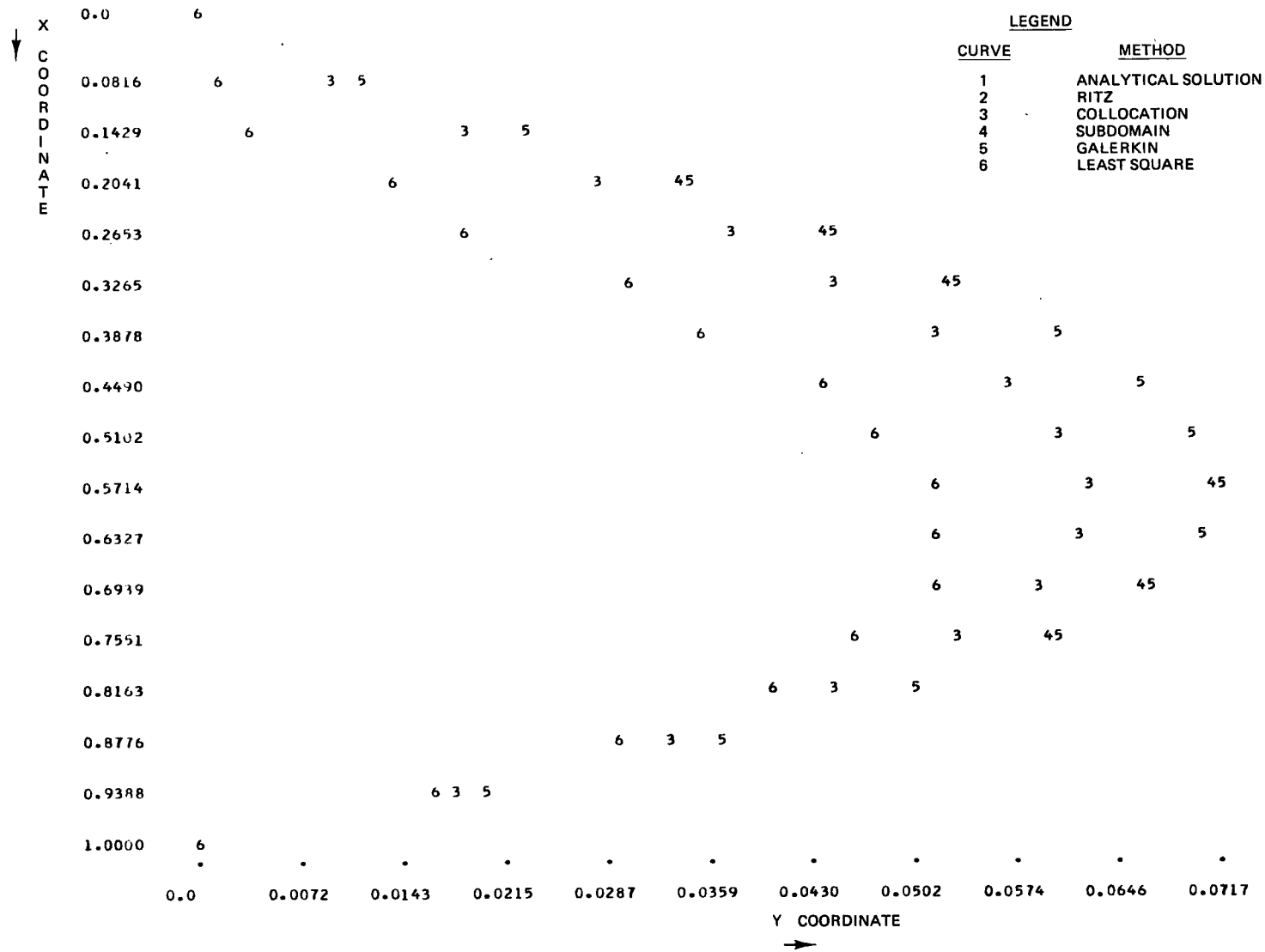
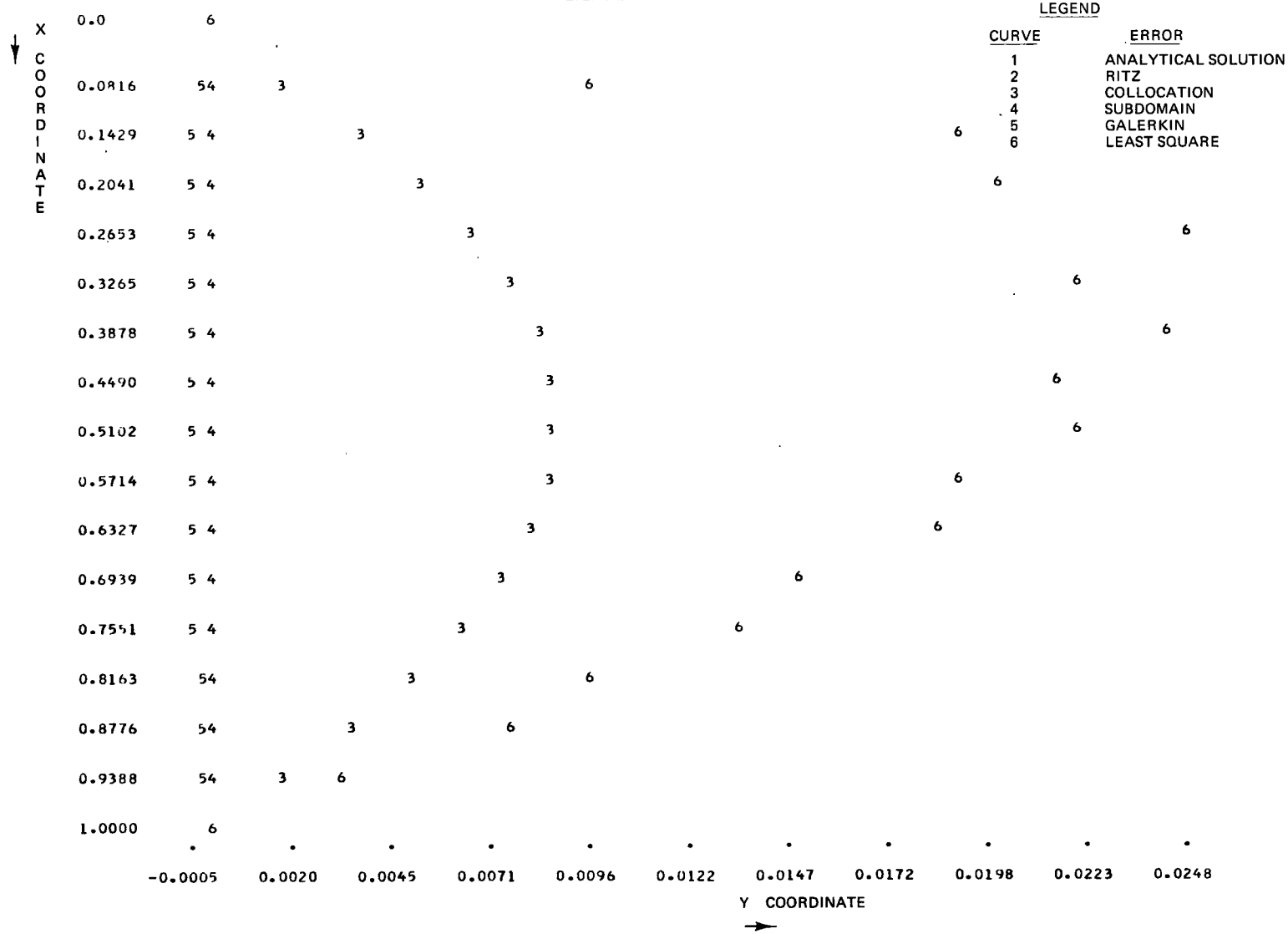


CHART D 2

EIGHT ELEMENT APPROXIMATION



FINITE ELEMENT TO 1 DIMENSIONAL PROBLEM.

METHOD TO USE = 11111

NUMBER OF NODES = 33

NUMBER OF ELEMENTS = 16

NUMBER OF NODAL VALUES GIVEN = 2

NODAL COORDINATES

NODE	X	Y	INITIAL VALUE
1	0.0	0.0	0.0
2	0.03125	0.0	0.0
3	0.06250	0.0	0.0
4	0.09375	0.0	0.0
5	0.12500	0.0	0.0
6	0.15625	0.0	0.0
7	0.18750	0.0	0.0
8	0.21875	0.0	0.0
9	0.25000	0.0	0.0
10	0.28125	0.0	0.0
11	0.31250	0.0	0.0
12	0.34375	0.0	0.0
13	0.37500	0.0	0.0
14	0.40625	0.0	0.0
15	0.43750	0.0	0.0
16	0.46875	0.0	0.0
17	0.50000	0.0	0.0
18	0.53125	0.0	0.0
19	0.56250	0.0	0.0
20	0.59375	0.0	0.0
21	0.62500	0.0	0.0
22	0.65625	0.0	0.0
23	0.68750	0.0	0.0
24	0.71875	0.0	0.0
25	0.75000	0.0	0.0
26	0.78125	0.0	0.0
27	0.81250	0.0	0.0
28	0.84375	0.0	0.0
29	0.87500	0.0	0.0
30	0.90625	0.0	0.0
31	0.93750	0.0	0.0
32	0.96875	0.0	0.0
33	1.00000	0.0	0.0

ELEMENT	NODE NUMBERS			CONSTANTS P(I)	Q(I)	F(I)	PQ(I)
1	1	2	3	1.00000	-1.00000	1.00000	1.00000
2	3	4	5	1.00000	-1.00000	1.00000	1.00000
3	5	6	7	1.00000	-1.00000	1.00000	1.00000
4	7	8	9	1.00000	-1.00000	1.00000	1.00000
5	9	10	11	1.00000	-1.00000	1.00000	1.00000
6	11	12	13	1.00000	-1.00000	1.00000	1.00000
7	13	14	15	1.00000	-1.00000	1.00000	1.00000
8	15	16	17	1.00000	-1.00000	1.00000	1.00000
9	17	18	19	1.00000	-1.00000	1.00000	0.0
10	19	20	21	1.00000	-1.00000	1.00000	0.0
11	21	22	23	1.00000	-1.00000	1.00000	0.0
12	23	24	25	1.00000	-1.00000	1.00000	0.0
13	25	26	27	1.00000	-1.00000	1.00000	0.0
14	27	28	29	1.00000	-1.00000	1.00000	0.0
15	29	30	31	1.00000	-1.00000	1.00000	0.0
16	31	32	33	1.00000	-1.00000	1.00000	0.0

2 NODAL VALUES GIVEN

NODE	VAL.
1	0.0
33	0.0

RITZ METHOD: ONE DIMENSIONAL PROBLEM USING QUADRATIC APPROXIMATION (METHOD = 10000)

NODE	X-COORDIN	ANALYTICAL	APR METHOD	ERROR
1	0.0	0.0	0.0	0.0
2	0.03125	0.005881	0.005878	0.000003
3	0.06250	0.011726	0.011720	0.000006
4	0.09375	0.017499	0.017489	0.000009
5	0.12500	0.023163	0.023150	0.000013
6	0.15625	0.028682	0.028666	0.000016
7	0.18750	0.034021	0.034001	0.000019
8	0.21875	0.039143	0.039120	0.000023
9	0.25000	0.044014	0.043988	0.000026
10	0.28125	0.048597	0.048568	0.000029
11	0.31250	0.052858	0.052826	0.000032
12	0.34375	0.056763	0.056728	0.000035
13	0.37500	0.060276	0.060239	0.000037
14	0.40625	0.063365	0.063326	0.000039
15	0.43750	0.065995	0.065953	0.000041
16	0.46875	0.068133	0.068090	0.000043
17	0.50000	0.069747	0.069702	0.000045
18	0.53125	0.070805	0.070758	0.000047
19	0.56250	0.071274	0.071226	0.000048
20	0.59375	0.071125	0.071076	0.000049
21	0.62500	0.070327	0.070276	0.000051
22	0.65625	0.068849	0.068798	0.000051
23	0.68750	0.066664	0.066612	0.000051
24	0.71875	0.063742	0.063691	0.000051
25	0.75000	0.060056	0.060006	0.000051
26	0.78125	0.055579	0.055530	0.000049
27	0.81250	0.050285	0.050238	0.000047
28	0.84375	0.044149	0.044105	0.000043
29	0.87500	0.037145	0.037108	0.000037
30	0.90625	0.029251	0.029221	0.000030
31	0.93750	0.020443	0.020422	0.000021
32	0.96875	0.010700	0.010688	0.000012
33	1.00000	0.0	0.0	0.0

COLLOCATION METHOD: ONE DIMENSIONAL PROBLEM USING QUADRATIC APPROXIMATION (METHOD = 01000)

NODE	X-COORDIN	ANALYTICAL	APR METHOD	ERROR
1	0.0	0.0	0.0	0.0
2	0.03125	0.005881	0.005354	0.000527
3	0.06250	0.011726	0.010673	0.001053
4	0.09375	0.017499	0.015956	0.001543
5	0.12500	0.023163	0.021168	0.001995
6	0.15625	0.028682	0.026272	0.002410
7	0.18750	0.034021	0.031233	0.002787
8	0.21875	0.039143	0.036017	0.003126
9	0.25000	0.044014	0.040587	0.003427
10	0.28125	0.048597	0.044908	0.003689
11	0.31250	0.052858	0.048945	0.003914
12	0.34375	0.056763	0.052664	0.004099
13	0.37500	0.060276	0.056029	0.004247
14	0.40625	0.063365	0.059008	0.004357
15	0.43750	0.065995	0.061566	0.004429
16	0.46875	0.068133	0.063669	0.004464
17	0.50000	0.069747	0.065285	0.004462
18	0.53125	0.070805	0.066381	0.004424
19	0.56250	0.071274	0.066925	0.004349
20	0.59375	0.071125	0.066885	0.004240
21	0.62500	0.070327	0.066231	0.004096
22	0.65625	0.068849	0.064932	0.003918
23	0.68750	0.066664	0.062957	0.003707
24	0.71875	0.063742	0.060279	0.003464
25	0.75000	0.060056	0.056867	0.003189
26	0.78125	0.055579	0.052695	0.002885
27	0.81250	0.050285	0.047734	0.002551
28	0.84375	0.044149	0.041960	0.002189
29	0.87500	0.037145	0.035345	0.001800
30	0.90625	0.029251	0.027865	0.001386
31	0.93750	0.020443	0.019496	0.000946
32	0.96875	0.010700	0.010215	0.000484
33	1.00000	0.0	0.0	0.0

SUBDOMAIN METHOD: ONE DIMENSIONAL PROBLEM USING QUADRATIC APPROXIMATION (METHOD = 00100)

NODE	X-COORDIN	ANALYTICAL	APR METHOD	ERROR
1	0.0	0.0	0.0	0.0
2	0.03125	0.005881	0.005881	-0.000000
3	0.06250	0.011726	0.011726	-0.000000
4	0.09375	0.017499	0.017499	-0.000000
5	0.12500	0.023163	0.023163	-0.000000
6	0.15625	0.028682	0.028682	-0.000000
7	0.18750	0.034021	0.034021	-0.000000
8	0.21875	0.039143	0.039144	-0.000000
9	0.25000	0.044014	0.044014	-0.000001
10	0.28125	0.048597	0.048598	-0.000001
11	0.31250	0.052858	0.052859	-0.000001
12	0.34375	0.056763	0.056764	-0.000001
13	0.37500	0.060276	0.060277	-0.000001
14	0.40625	0.063365	0.063366	-0.000001
15	0.43750	0.065995	0.065996	-0.000001
16	0.46875	0.068133	0.068134	-0.000001
17	0.50000	0.069747	0.069748	-0.000001
18	0.53125	0.070805	0.070806	-0.000001
19	0.56250	0.071274	0.071276	-0.000002
20	0.59375	0.071125	0.071127	-0.000002
21	0.62500	0.070327	0.070328	-0.000002
22	0.65625	0.068849	0.068851	-0.000002
23	0.68750	0.066664	0.066666	-0.000002
24	0.71875	0.063742	0.063744	-0.000002
25	0.75000	0.060056	0.060058	-0.000002
26	0.78125	0.055579	0.055581	-0.000002
27	0.81250	0.050285	0.050287	-0.000001
28	0.84375	0.044149	0.044150	-0.000001
29	0.87500	0.037145	0.037146	-0.000001
30	0.90625	0.029251	0.029252	-0.000001
31	0.93750	0.020443	0.020444	-0.000001
32	0.96875	0.010700	0.010700	-0.000000
33	1.00000	0.0	0.0	0.0

GALERKIN'S METHOD: ONE DIMENSIONAL PROBLEM USING QUADRATIC APPROXIMATION (METHOD = 00010)

NODE	X-COORDIN	ANALYTICAL	APR METHOD	ERROR
1	0.0	0.0	0.0	0.0
2	0.03125	0.005881	0.005918	-0.000037
3	0.06250	0.011726	0.011788	-0.000062
4	0.09375	0.017499	0.017579	-0.000081
5	0.12500	0.023163	0.023258	-0.000095
6	0.15625	0.028682	0.028789	-0.000107
7	0.18750	0.034021	0.034136	-0.000116
8	0.21875	0.039143	0.039266	-0.000123
9	0.25000	0.044014	0.044142	-0.000128
10	0.28125	0.048597	0.048729	-0.000132
11	0.31250	0.052858	0.052992	-0.000134
12	0.34375	0.056763	0.056898	-0.000135
13	0.37500	0.060276	0.060411	-0.000135
14	0.40625	0.063365	0.063499	-0.000134
15	0.43750	0.065995	0.066127	-0.000132
16	0.46875	0.068133	0.068262	-0.000129
17	0.50000	0.069747	0.069873	-0.000126
18	0.53125	0.070805	0.070926	-0.000121
19	0.56250	0.071274	0.071391	-0.000116
20	0.59375	0.071125	0.071236	-0.000111
21	0.62500	0.070327	0.070431	-0.000105
22	0.65625	0.068849	0.068947	-0.000098
23	0.68750	0.066664	0.066755	-0.000091
24	0.71875	0.063742	0.063825	-0.000083
25	0.75000	0.060056	0.060131	-0.000075
26	0.78125	0.055579	0.055646	-0.000066
27	0.81250	0.050285	0.050343	-0.000057
28	0.84375	0.044149	0.044197	-0.000048
29	0.87500	0.037145	0.037184	-0.000039
30	0.90625	0.029251	0.029280	-0.000030
31	0.93750	0.020443	0.020463	-0.000020
32	0.96875	0.010700	0.010710	-0.000010
33	1.00000	0.0	0.0	0.0

LEAST SQUARES METHOD: ONE DIMENSIONAL PROBLEM USING QUADRATIC APPROXIMATION (METHOD = 00001)

NODE	X-COURDIN	ANALYTICAL	APR METHOD	ERROR
1	0.0	0.0	0.0	0.0
2	0.03125	0.005881	0.001199	0.004683
3	0.06250	0.011726	0.002375	0.009351
4	0.09375	0.017499	0.007696	0.009803
5	0.12500	0.023163	0.010553	0.012610
6	0.15625	0.028682	0.017155	0.011527
7	0.18750	0.034021	0.020744	0.013277
8	0.21875	0.039143	0.026715	0.012428
9	0.25000	0.044014	0.030415	0.013599
10	0.28125	0.048597	0.035780	0.012817
11	0.31250	0.052858	0.039255	0.013603
12	0.34375	0.056763	0.043939	0.012824
13	0.37500	0.060276	0.046951	0.013325
14	0.40625	0.063365	0.050844	0.012521
15	0.43750	0.065995	0.053197	0.012798
16	0.46875	0.068133	0.056178	0.011955
17	0.50000	0.069747	0.057700	0.012047
18	0.53125	0.070805	0.059646	0.011159
19	0.56250	0.071274	0.060179	0.011095
20	0.59375	0.071125	0.060966	0.010159
21	0.62500	0.070327	0.060365	0.009962
22	0.65625	0.068849	0.059871	0.008979
23	0.68750	0.066664	0.058000	0.008664
24	0.71875	0.063742	0.056107	0.007636
25	0.75000	0.060056	0.052839	0.007218
26	0.78125	0.055579	0.049433	0.006146
27	0.81250	0.050285	0.044648	0.005638
28	0.84375	0.044149	0.039622	0.004526
29	0.87500	0.037145	0.033208	0.003937
30	0.90625	0.029251	0.026461	0.002790
31	0.93750	0.020443	0.018313	0.002130
32	0.96875	0.010700	0.009749	0.000951
33	1.00000	0.0	0.0	0.0

X-COORDINATE	ANALYTICAL SOLUTION	RITZ	COLLOCATION	SUBDOMAIN	GALERKIN*S	LEASTSQUARES
0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.03125	0.005881298	0.005878091	0.005354419	0.005881347	0.005918216	0.001198611
0.06250	0.011726320	0.011719946	0.010673098	0.011726446	0.011788066	0.002375331
0.09375	0.017498791	0.017489355	0.015956048	0.017499074	0.017579362	0.007696215
0.12500	0.023162723	0.023150165	0.021167528	0.023163076	0.023257948	0.010552566
0.15625	0.028682053	0.028666049	0.026271876	0.028682385	0.028788727	0.017155122
0.18750	0.034020722	0.034001391	0.031233486	0.034021143	0.034136478	0.020743910
0.21875	0.039143085	0.039120350	0.036016878	0.039143570	0.039265815	0.026715070
0.25000	0.044013560	0.043987535	0.040586632	0.044014201	0.044141624	0.030414820
0.28125	0.048596978	0.048567802	0.044907641	0.048597701	0.048728675	0.035780024
0.31250	0.052858412	0.052826218	0.048944827	0.052859180	0.052992322	0.039255463
0.34375	0.056762934	0.056728300	0.052663554	0.056763913	0.056897987	0.043938901
0.37500	0.060276449	0.060239397	0.056029249	0.060277503	0.060411442	0.046951003
0.40625	0.063364804	0.063325644	0.059007909	0.063366055	0.063498795	0.050843854
0.43750	0.065994799	0.065953434	0.061565593	0.065995932	0.066126704	0.053196605
0.46875	0.068132937	0.068089664	0.063668907	0.068134248	0.068262041	0.056178395
0.50000	0.069746912	0.069701850	0.065284848	0.069748282	0.069872618	0.057699673
0.53125	0.070804536	0.070757985	0.066380918	0.070806026	0.070925891	0.059645981
0.56250	0.071274221	0.071226239	0.066924870	0.071275830	0.071390629	0.060178909
0.59375	0.071125150	0.071075857	0.066885352	0.071126699	0.071235895	0.060965765
0.62500	0.070326686	0.070276141	0.066231012	0.070328414	0.070431292	0.060364902
0.65625	0.068849325	0.068798125	0.064931691	0.068851173	0.068947256	0.059870571
0.68750	0.066663861	0.066612422	0.062957227	0.066665709	0.066754580	0.057999942
0.71875	0.063742101	0.063691080	0.060278524	0.063743889	0.063825071	0.056106560
0.75000	0.060056210	0.060005520	0.056866918	0.060057908	0.060131069	0.052838534
0.78125	0.055579185	0.055529743	0.052694615	0.055580862	0.055645660	0.049433041
0.81250	0.050285220	0.050238069	0.047734298	0.050286647	0.050342720	0.044647660
0.84375	0.044148564	0.044105072	0.041959595	0.044149913	0.044197012	0.039622258
0.87500	0.037145019	0.037107538	0.035344835	0.037146129	0.037184097	0.033207793
0.90625	0.029250681	0.029220901	0.027865164	0.029251609	0.029280238	0.026460748
0.93750	0.020442843	0.020421959	0.019496430	0.020443525	0.020462718	0.018312734
0.96875	0.010699630	0.010687985	0.010215491	0.010699987	0.010709625	0.009748768
1.00000	0.0	0.0	0.0	0.0	0.0	0.0

TIME SPENT IN SEC.

0.572999954224	0.509999990463	0.570999979973	0.569999992847	0.578999996185

CHART E 1

SIXTEEN ELEMENT APPROXIMATION

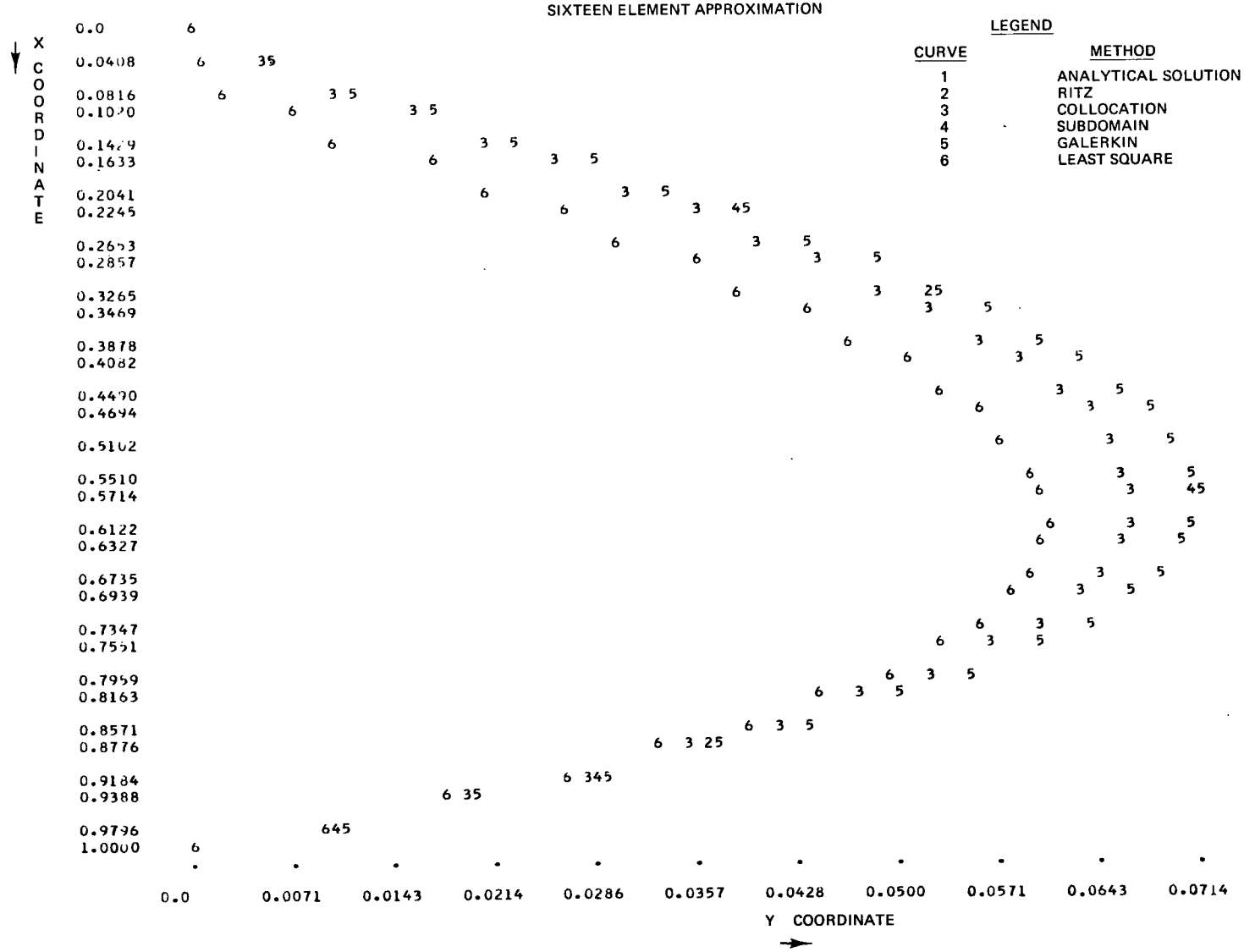


CHART E 2

SIXTEEN ELEMENT APPROXIMATION