MULTIFUNCTION FIXED POINT THEORY AND ITS APPLICATION TO AN INITIAL-VALUE PROBLEM

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ABSTRACT

In this dissertation the primary concern is with showing the existence of a solution to the initial-value problem

$$\dot{x}(t) \in F(t,x(t))$$

$$x(0) = x_0.$$

The function x is once-differentiable on a closed interval of real numbers having left endpoint zero into a Banach space. The multifunction F maps the cross product of the interval with the Banach space into the Banach space and x_0 is in the Banach space.

The initial-value problem is transposed, using the Bochner integral, into a multifunction fixed point problem in the space of continuous functions on the interval into the Banach space. Several multifunction fixed point theorems are obtained in solving the transposed problem. Each of these results is dependent, either directly or indirectly, on the multifunction being condensing with respect to a measure of non-compactness. As a result, both the concept of a measure of non-compactness and the concept of a condensing multifunction are treated.

In addition, the idea of a monotone multifunction is developed and a role found for it in the fixed point theory. Finally, the topological structure of the solution set to the initial-value problem is investigated.

TABLE OF CONTENTS

INTRODUCTION

The primary concern of this dissertation is to investigate the existence of a solution to the initial-value problem

$$\dot{x}(t) \in F(t, x(t))$$
$$x(0) = x_0.$$

Here x is a function on an interval of real numbers into a Banach space, $\dot{x}$ denotes dx/dt, x_0 is in the Banach space and F is a point-to-set function, henceforth referred to as a multifunction, on the cross-product of the interval with the Banach space into the Banach space.

The above problem is transposed into a corresponding integral problem which in fact turns out to be a multifunction fixed point problem. Thus, the first chapter is basically devoted to providing some background as regards integration in Banach spaces and existing multifunction fixed point theory. The integral used throughout is the Bochner integral essentially as developed in Hille and Phillips [24].* In the endeavor to show the existence of a solution to the initial-value problem several new multifunction fixed point results are obtained.

The pioneer work on the problem was done in the mid thirties by Marchaud [29] and Zaremba [42]. More recently, Castaing [5], Filippov [12], [13], [14] and Hermes [23] have extended this work operating in Euclidean n-space. Through private communication it was also learned that F. De Blasi and V. Lakshmikantham have investigated the problem.

Recent advances in multifunction fixed point theory allow its application to the above initial-value problem not unlike the manner

*Throughout this dissertation a bracketed number refers to the corresponding reference in the bibliography.

in which the Brouwer, Kakutani and Schauder fixed point theories have been applied to show the existence of a solution when the kernel F is restricted to be single-valued. Among the conditions necessary to ensure the existence of a fixed point for the multifunction arising in the transposition of the initial-value problem into an integral problem is that it satisfy a certain Lipschitz-like condition.

Single-valued mappings satisfying this condition are referred to in the literature as condensing, densifying or concentrative. This class of mappings was studied quite extensively during the early seventies by Danes [7], [8], [9], Darbo [10], Furi and Vignoli [19], [20], [21], Nussbaum [34] and Sadovskii [38]. Akin to the class of condensing mappings is the class of k-set-contraction mappings as studied by Gatica and Kirk [22], Nussbaum [34], Petryshyn [36] and Potter [37]. These mappings are also referred to in Martin [31] as α-Lipschitz mappings.

More recently, condensing and k-set-contraction multifunctions have been investigated by Fitzpatrick and Petryshyn [16], [17], [18], Himmelberg, Porter and Van Vleck [25] and Martelli [30]. The k-set-contraction multifunctions, specifically for the case $k<1$, have also been studied as a subclass of the ultimately compact multifunctions by Fitzpatrick and Petryshyn [15].

Central to the idea of a condensing or k-set-contraction multifunction is the notion of a measure of non-compactness. The best known measure of non-compactness was introduced by Kuratowski [28] and is presented in Example 1.4.7. The concept of a measure of non-compactness has since been generalized by Darbo [10], Sadovskii [38] and, under the name "compactness guage", by Jones [26]. Following the guidelines set by Sadovskii in [38] Section 1.4 is given to developing a general concept of measure of non-compactness for locally convex topological vector spaces.

The idea of a g-contractive multifunction as introduced in Section 2.2 is a take-off on one for single-valued functions due to Furi and Vignoli [20]. The class of g-contractive multifunctions is an extension of the class of Banach contractive multifunctions as studied by Filippov [12], Fitzpatrick and Petryshyn [15], Nadler [33] and Smithson [39].

The set relations as defined in Section 1.3 and subsequently applied in Chapter III follow from the work of Smithson [40]. Fixed Point Theorem 3.1.3 is an outgrowth from the standard fixed point theorems for monotone single-valued functions which map an order interval into itself. For example, see the work of Amann [1].

The development of Section 4.1 follows along the same lines as that given in [15] except that the results here apply to a more general class of measures on non-compactness. Utilizing a fixed point theorem due to Kakutani and Tychonoff, later extended to multifunctions by Bohnenblust and Karlin [3], Castaing [4] and Hermes [23] have shown the existence of a solution to the initial-value problem in Euclidean n-space under conditions somewhat less general than those given in Theorem 4.2.6.

Definitions and statements of theorems from the literature have been interspersed throughout the dissertation. However, unless otherwise specified, all given proofs are the author's.

CHAPTER I

1.1 Notation

The following notation will be used throughout the remainder of the dissertation.

The set of all subsets of a point set X will be denoted by 2^X.

If A and B are point sets then

$$A|B = \{a | a\varepsilon A \text{ and } a\notin B\}.$$

If X is a topological space and $A \subset X$ then the closure of A in X will be denoted by $Cl_X(A)$ or more simply $\overline{A}$ when X is understood.

Suppose that (X,d) is a metric space and A is a nonempty subset of X. The diameter of A is defined by

$$di(A) = \sup \{d(x,y) | x,y \; \varepsilon \; A\}.$$

If $x\varepsilon X$ then

$$d(x,A) = \inf \{d(x,y) | y\varepsilon A\}.$$

If $\varepsilon > 0$ then

$$B(A,\varepsilon) = \{x\varepsilon X | d(x,A) < \varepsilon\}.$$

In the event that A is a singleton {a} write $B(a,\varepsilon)$ for $B(A,\varepsilon)$.

Suppose that A and B are nonempty compact subsets of the metric space X. The Hausdorff distance between A and B is defined by

$$d_H(A,B) = \max \left\{ \{\sup \{d(a,B) | a\varepsilon A\}\}, \{\sup \{d(b,A) | b\varepsilon B\}\} \right\}.$$

The distance function d_H is a metric on the set of compact subsets of X. If the metric space X is complete so is the metric space of compact subsets of X with metric d_H.

A subset A of X is precompact provided given $\varepsilon > 0$ there exists a finite set $\{a_i\}_{i=1}^n$ in A such that

$$A \subset \bigcup_{i=1}^n B(a_i,\varepsilon).$$

A subset A of X is relatively compact provided $\bar{A}$ is compact in X.

If X is a complete metric space then a subset A of X is precompact if and only if it is relatively compact.

Unless otherwise specified all vector spaces, including normed linear spaces and Banach spaces, will be assumed to have scalar field the complex numbers.

Suppose that X is a vector space. If A and B are nonempty subsets of X and η is a scalar then

i) $A+B = \{a+b \mid a \in A \text{ and } b \in B\}$;

ii) $\eta A = \{\eta a \mid a \in A\}$;

iii) $A-B = A+(-1)B$.

If A is a singleton $\{a\}$ write $a+B$ for $A+B$. If $A \subset X$ and η is a scalar then $\phi + A = \phi$ and $\eta\phi = \phi$.

A subset A of X is convex provided that whenever $x,y \in A$ and $\eta \in [0,1]$ the vector $(1-\eta)x + \eta y$ is in A.

If X is a topological vector space and $A \subset X$ then the convex hull of A, denoted co(A), is the intersection of all convex sets containing A. The closed convex hull of A, denoted $\overline{co}(A)$, is the intersection of all closed convex sets containing A.

The following two results appear in [ ].

<u>Lemma 1.1.1</u>

If X is a vector space and $A \subset X$ then

$$co(A) = \{\sum_{i=1}^{n} \eta_i a_i \mid \eta_i \geq 0, \ \sum_{i=1}^{n} \eta_i = 1 \text{ and } a_i \in A\}.$$

<u>Lemma 1.1.2</u>

If X is a topological vector space and $A \subset X$ then $\overline{co}(A) = \overline{co(A)}$.

The set of all closed subsets of a topological space X will be denoted by Kl(X) and the set of all compact subsets by Kp(X). If X is a vector space the set of all convex subsets of X will be denoted by K(X). The set of all closed convex subsets of a topological vector space X will be denoted by KlK(X) and the set of all compact convex subsets by KpK(X). Finally, if X is the appropriate space and P is one of K, Kl, Kp, KlK or KpK then the subset of nonempty elements of P(X) will be denoted by $\underline{P}(X)$.

If X is a Banach space and M is a compact metric space the Banach space of continuous functions on M into X will be denoted by C(M,X) where the norm of $f \in C(M,X)$ is given by

$$||f||_C = \sup \{||f(t)||: t \in M\}.$$

The following two results are proved in [31].

Lemma 1.1.3

If X is a Banach space and A is a precompact subset of X then $\overline{co}(A)$ is compact in X.

Theorem 1.1.4

If X is a Banach space, M a compact metric space and $A \subset C(M,X)$ then A is precompact in C(M,X) if and only if A is bounded, equicontinuous and $\{f(m) | f \in A\}$ is precompact in X for every $m \in M$.

1.2 Calculus

Properties concerning the measurability, integrability and differentiability of functions having their range in a Banach space are discussed here.

Until otherwise specified X will denote a Banach space and T a complete measure space with a positive σ-finite measure μ acting on Ω a σ-algebra of subsets of T.

The proofs for results not proved or referenced in this section can be found in [24].

Definition 1.2.1

If f and f_n, n = 1,2,···, are function on T into X then the sequence $\{f_n\}_{n=1}^{\infty}$ converges to f on T

i) almost everywhere (a.e.) provided there exists a μ-null set E in T such that $\lim_{n\to\infty} ||f(t) - f_n(t)|| = 0$ for every $t \in T|E$;

ii) almost uniformly provided that for every ε>0 there exists E_ε in Ω with $\mu(E_\varepsilon) < \varepsilon$ such that $\{f_n\}_{n=1}^{\infty}$ converges uniformly to f on $T|E_\varepsilon$.

Definition 1.2.2

A function f:T →X is

i) countably-valued if it assumes at most a countable number of values in X and assumes each value on an element of Ω;

ii) separably-valued if f(T) is separable in X;

iii) almost separably-valued if there exists a μ-null set E such that f(T|E) is separable in X;

iv) μ-measurable if there exists a sequence of countably-valued functions converging almost everywhere in T to f.

Lemma 1.2.3

A function f:T →X is μ-measurable if and only if f is almost separably-valued on T and $f^{-1}(G) \in \Omega$ for every open (closed) set G in X.

Proof Suppose that f is μ-measurable on T. By a result in [24] f is the almost uniform limit of a sequence $\{f_k\}_{k=1}^{\infty}$ of countably-valued functions. That is, there exists $A_j \in \Omega$, j = 1,2,···, such that $\mu(A_j) < 1/j$ and $\{f_k\}_{k=1}^{\infty}$ converges uniformly to f on $T|A_j$.

Let $A = \bigcap_{j=1}^{\infty} A_j$. Then $A \in \Omega$, μ(A) = 0 and $\{f_k\}_{k=1}^{\infty}$ converges pointwise to f on T|A. The set $\bigcup_{k=1}^{\infty} f_k(T|A)$ is countable and hence its closure is a separable subset of X containing f(T|A). Thus, f is almost separably-valued.

Let G be an open subset of X and

$$G_n = \{x\varepsilon X | B(x,1/n) \subset G\} \text{ for } n = 1,2,\cdots.$$

Consider $t\varepsilon T|A$. Then $f(t)\varepsilon G$ if and only if there exists positive integers n and K such that $f_k(t)\varepsilon G_n$ for all $k \geq K$. Now,

$$f^{-1}(G)|A = \left\{ \bigcup_{m,n=1}^{\infty} \left(\bigcap_{k=m}^{\infty} f_k^{-1}(G_n) \right) \right\} |A.$$

Since $\{f_k\}_{k=1}^{\infty}$ is a sequence of countably-valued functions, for each k and n the set $f_k^{-1}(G_n)$ is the union of a countable collection of measurable sets and hence is itself measurable. Thus, $f^{-1}(G)|A\varepsilon\Omega$ and hence $f^{-1}(G)\varepsilon\Omega$ since T is a complete measure space.

Conversely, suppose f is separably-valued and $f^{-1}(G)\varepsilon\Omega$ for every open set G in X. Let E be a μ-null set such that $f(T|E)$ is separable in X and $\{x_n\}_{n=1}^{\infty}$ a sequence dense in $f(T|E)$. Given $\varepsilon>0$ let

$$C_n = \left\{ f^{-1}(B(x_n,\varepsilon)) \right\} | \left\{ \bigcup_{k \neq n}^{\infty} f^{-1}(B(x_k,\varepsilon)) \right\} \text{ for } n = 1,2,\cdots,$$

and $C = \bigcup_{n=1}^{\infty} C_n$.

Then the C_n are pairwise disjoint, $C_n\varepsilon\Omega$ for each n and $E \cup C = T$.

Define $f_\varepsilon : T \to X$ by

$$f_\varepsilon(t) = \begin{cases} x_n, & \text{if } t\varepsilon C_n \text{ for some } n \\ \theta, & \text{if } t\varepsilon T|C \text{ (Here } \theta \text{ is the zero of X).} \end{cases}$$

Then f_ε is countably-valued and $\lim_{\varepsilon \to 0_+} f_\varepsilon(t) = f(t)$ for each $t\varepsilon C$. Thus, f is measurable on T.

Since for every function $f:T \to X$ and every subset H of X, $f^{-1}(X|H) = T|f^{-1}(H)$ the result for the case when G is closed follows immediately from the preceding argument.

<u>Definition 1.2.4</u>

A countably-valued function $f:T \to X$ is integrable (Bochner) provided the function $||f||$ on T into the real numbers is Lebesgue integrable and then the integral on $E \in \Omega$ is defined by

$$\int_E f(s)d\mu = \sum_{k=1}^{\infty} x_k \mu(E_k \cap E)$$

where $f(t) = x_k$ on $E_k \in \Omega$ for $k = 1,2,\cdots$.

Definition 1.2.5

A function $f:T \to X$ is integrable (Bochner) provided there exists a sequence of countably-valued integrable functions $\{f_n\}_{n=1}^{\infty}$ converging almost everywhere to f such that

$$\lim_{n\to\infty} \int_T ||f(s) - f_n(s)||d\mu = 0.$$

The integral on $E \in \Omega$ is then defined by

$$\int_E f(s)d\mu = \lim_{n\to\infty} \int_E f_n(s)d\mu.$$

Remark The above integral is shown to be well-defined in [24].

Theorem 1.2.6

A function $f:T \to X$ is integrable (Bochner) if and only if f is μ-measurable and $||f||$ is Lebesgue integrable on T with $\int_T ||f(s)||d\mu < \infty$.

Definition 1.2.7

Let $L(T,X,\mu)$ or $L(T,X)$, when μ is understood, denote the set of integrable functions on T into X where functions which disagree only on a μ-null set are identified via the usual equivalence relation.

Lemma 1.2.8

The set $L(T,X)$ is a Banach space when endowed with the norm defined by

$$||f||_L = \int_T ||f(s)||d\mu \text{ for } f \in L(T,X).$$

Lemma 1.2.9

If $f,g \in L(T,X)$ and η is in the scalar field of X then $f+g$ and ηf are in $L(T,X)$ and

i) $\int_E (f+g)(s)d\mu = \int_E f(s)d\mu + \int_E g(s)d\mu;$

ii) $\int_E \eta f(s)d\mu = \eta\int_E f(s)d\mu$;

for $E \in \Omega$.

Lemma 1.2.10

If $\{E_n\}_{n=1}^{\infty}$ is a sequence of pairwise disjoint elements in Ω, $E = \bigcup_{n=1}^{\infty} E_n$ and $f \in L(T,X)$ then

$$\int_E f(s)d\mu = \sum_{n=1}^{\infty} \int_{E_n} f(s)d\mu.$$

Lemma 1.2.11

If a sequence $\{f_n\}_{n=1}^{\infty}$ in $L(T,X)$ converges almost everywhere to a function f and if there exists a function $g \in L(T,[0,\infty))$ such that for all n, $||f_n(t)|| \leq g(t)$ for all $t\in T$ then $f \in L(T,X)$ and

$$\lim_{n\to\infty} \int_E f_n(s)d\mu = \int_E f(s)d\mu$$

for $E \in \Omega$.

Lemma 1.2.12

If $f \in L(T,X)$ and $E \in \Omega$ then

$$||\int_E f(s)d\mu|| \leq \int_E ||f(s)||d\mu$$

where the latter integral is the Lebesgue integral on E.

Lemma 1.2.13

If $f \in L(T,X)$ then the set function m defined on Ω by

$$m(E) = \int_E f(s)d\mu$$

is absolutely continuous.

Lemma 1.2.14

If a sequence $\{f_n\}_{n=1}^{\infty}$ converges weakly to f in $L(T,X)$ then

$$\lim_{n\to\infty} \int_E f_n(s)d\mu = \int_E f(s)d\mu$$

for $E \in \Omega$.

Proof Suppose that $E \in \Omega$ and consider $x^* \in X^*$, the conjugate space of X. Then, let F^* be the element in the conjugate space of $L(T,X)$ defined by

$$F^*(f) = x^*\int_E f(s)d\mu,\ f \in L(T,X).$$

Since

$$x^*(\lim_{n\to\infty} \int_E f_n(s)d\mu) = \lim_{n\to\infty} F^*(f_n) = F^*(f) = x^*(\int_E f(s)d\mu),$$

$$\lim_{n\to\infty} \int_E f_n(s)d\mu = \int_E f(s)d\mu.$$

<u>Lemma 1.2.15</u>

If $0<\mu(T)<\infty$ and $f:T\to X$ is a countably-valued integrable function then

$$\int_T f(s)d\mu \in \mu(T)\overline{co}(\{f(s):s\in T\}).$$

<u>Proof</u> Fix x_o an element in the range of f. Then by Definition 1.2.4

$$\int_T f(s)d\mu = \lim_{k\to\infty}\left(\left(\sum_{n=1}^{k} \mu(E_n)x_n\right) + \mu(T|\bigcup_{i=1}^{k} E_i)x_o\right)$$

where $f(t) = x_n$ on $E_n \in \Omega$ for each n. Indeed,

$$\lim_{k\to\infty} \mu(T|\bigcup_{i=1}^{k} E_i) = \mu(\phi) = 0.$$

Since for each k

$$1/\mu(T)\left(\left(\sum_{n=1}^{k} \mu(E_n)\right) + \mu(T|\bigcup_{i=1}^{k} E_i)\right) = 1$$

and μ is a positive measure

$$\int_T f(s)d\mu \in \mu(T)\overline{co}(\{f(s):s\in T\})$$

by Lemma 1.1.1.

<u>Lemma 1.2.16</u>

If $0<\mu(T)<\infty$, E a μ-null subset of T and $f:T\to X$ an integrable function then

$$\int_T f(s)d\mu \in \mu(T)\overline{co}(\{f(s):s\in T|E\}).$$

<u>Proof</u> Let n be a positive integer. By a result of Hille and Phillips [24] there exists a sequence $\{A_j\}_{j=1}^{\infty}$ of nonempty disjoint elements of Ω so that $T = \bigcup_{j=1}^{\infty} A_j$ and for any choice of $s_j \in A_j$ the function f_n defined on

T by

1.2.17 $$f_n(t) = f(s_j) \text{ for } t \varepsilon A_j$$

is countably-valued, μ-integrable and

$$\int_T ||f_n(s) - f(s)||d\mu < 1/n.$$

Select a sequence $\{s_j\}_{j=0}^{\infty}$ such that $s_o \varepsilon T|E$ and for $j \geq 1$

$$s_j \varepsilon \begin{cases} A_j|E, \text{ if } A_j \not\subset E \\ A_j, \text{ if } A_j \subset E. \end{cases}$$

Let f_n be the function defined as in 1.2.17 for this choice of $\{s_j\}_{j=1}^{\infty}$ and $E_o = \{\bigcup A_k | A_k \subset E\}$.

Now, define a function g_n on T by

$$g_n(t) = \begin{cases} f_n(t), \text{ if } t \notin E_o \\ f(s_o), \text{ if } t \varepsilon E_o. \end{cases}$$

Since E_o is a μ-null subset of T, g_n is countably-valued and μ-integrable since f_n is. Also,

$$\begin{aligned} ||\int_T g_n(s)d\mu - \int_T f(s)d\mu|| &\leq \int_T ||g_n(s) - f(s)||d\mu \\ &= \int_T ||f_n(s) - f(s)||d\mu \\ &< 1/n. \end{aligned}$$

Thus, $\lim_{n\to\infty} \int_T g_n(s)d\mu = \int_T f(s)d\mu$.

By Lemma 1.2.15

$$\int_T f(s)d\mu = \lim_{n\to\infty} \mu(T) \sum_{i=1}^{m(n)} a_i^n g_n(t_i^n)$$

where $t_i^n \varepsilon T$, $a_i^n \geq 0$ for $1 \leq i \leq m(n)$ and $\sum_{i=1}^{m(n)} a_i^n = 1$ for all n. However, for each i and n, $t_i^n \varepsilon A_j$ for some j and hence there exists $s_i^n \varepsilon T|E$ such that $g_n(t_i^n) = f(s_i^n)$. Thus,

$$\int_T f(s)d\mu = \lim_{n\to\infty} \mu(T) \sum_{i=1}^{m(n)} a_i^n f(s_i^n) \varepsilon \mu(T)\overline{co}(\{f(s)|s\varepsilon T|E\})$$

by Lemma 1.1.1.

Lemma 1.2.18

If $\{f_n\}_{n=1}^{\infty}$ is a sequence in L(T,X) converging to f then there exists a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ converging pointwise almost everywhere on T to f.

Proof Select a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ such that

$$\int_T ||f_{n_{k+1}}(s) - f_{n_k}(s)||d\mu < 2^{-k}.$$

Let $h_m = \sum_{k=1}^{m} ||f_{n_{k+1}} - f_{n_k}||$ and $h = \sum_{k=1}^{\infty} ||f_{n_{k+1}} - f_{n_k}||$.

Then, $\int_T h_m(s)d\mu \leq 1$ for each m and hence, by an application of Fatou's Lemma, $\int_T h(s)d\mu \leq 1$. Thus, $h(t) < \infty$ a.e. on T so that

$$\left\{ f_{n_1}(t) + \sum_{k=1}^{m}\left(f_{n_{k+1}} - f_{n_k}\right)(t)\right\}_{m=1}^{\infty}$$

converges a.e. on T. But, $f_{n_1} + \sum_{k=1}^{m}\left(f_{n_{k+1}} - f_{n_k}\right) = f_{n_m}$ so that

$$\lim_{k\to\infty} f_{n_k}(t) = f(t) \text{ a.e. on T.}$$

In the following λ will denote Lebesgue measure on the real line. Also, J will denote an interval of real numbers having endpoints c and d with c<d and $c = -\infty$ and/or $d = \infty$ will be permissible. If $J_o = [a,b]$, $J_o \subset J$, and $f \in L(J_o,X,\lambda)$ then $\int_a^b f(s)d\lambda$ may be written for $\int_{J_o} f(s)d\lambda$.

Lemma 1.2.19

If $J_o = [a,b] \subset J$ and $f \in L(J_o,X,\lambda)$ then

$$||\int_a^b f(s)d\lambda|| \leq (b-a)\sup\{||f(s)||:s\in J_o\}.$$

Proof This result is an obvious extension of Lemma 1.2.12.

Definition 1.2.20

A function $f:J \to X$ is differentiable at $t\in J$ provided there exists $\dot{f}(t)$ in X such that

$$\lim_{\substack{h\to 0\\ t+h\in J}} \frac{f(t+h) - f(t)}{h} = \dot{f}(t)$$

A function $f:J \to X$ is differentiable on J if it is differentiable at every point of J.

Lemma 1.2.21

If $f \in L(J,X,\lambda)$ and $c \in J$ then the function defined by $F(t) = \int_c^t f(s)d\lambda$ is differentiable a.e. on J and $\dot{F}(t) = f(t)$ a.e. on J.

Lemma 1.2.22

If $J_0 = [a,b] \subset J$ and $f:J \to X$ is a function differentiable on J_0 such that $\dot{f}(t) = \theta$ on J_0 then f is a constant function on J_0.

Proof By a mean-value theorem due to Martin [31] for every $t \in J_0$

$$f(t) - f(a) \in (t-a)\overline{co}(\dot{f}(s) | s \in [a,t]) = \{\theta\}.$$

Thus, $f(t) = f(a)$ on J_0.

Lemma 1.2.23

If $J_0 = [a,b] \subset J$ and $f,g:J \to X$ are functions differentiable on J_0 with $\dot{f}(t) = \dot{g}(t)$ on J_0 then for some $C \in X$, $f(t) = g(t) + C$ on J_0.

Proof Since $(f\dot{-}g)(t) = \dot{f}(t) - \dot{g}(t) = \theta$ on J_0, by Lemma 1.2.22, f-g is a constant function on J_0. Hence, there exists $C \in X$ such that

$$f(t) = g(t) + C \text{ for every } t \in J_0.$$

Lemma 1.2.24

If $J_0 = [a,b] \subset J$ and $f,g:J \to X$ are continuous functions such that g is differentiable on J_0 with $\dot{g}(t) = f(t)$ for every $t \in J_0$ then

$$\int_a^b f(s)d\lambda = g(b) - g(a).$$

Proof Let $G(t) = \int_a^t f(s)d\lambda$ on J_0. Then, as in Martin [31], for every $t \in J_0$, $\dot{G}(t) = f(t)$. Thus, $\dot{G}(t) = \dot{g}(t)$ on J_0 and by Lemma 1.2.23 there exists $C \in X$ such that $G(t) = g(t) + C$ on J_0.

Now, $G(b) - G(a) = g(b) - g(a)$ and $G(a) = \theta$. Thus,

$$\int_a^b f(s)d\lambda = G(b) = g(b) - g(a).$$

1.3 Partial Orderings on a Banach Space

The concept of a partial ordering on a Banach space defined here generally follows one developed by Krasnoselskii in [27].

Definition 1.3.1

If X is a Banach space then $K \subset X$ is a cone provided

i) K is closed in X;

ii) if $u, v \in K$ then $\alpha u + \beta v \in K$ for all $\alpha, \beta \in [0,\infty)$;

iii) $K \cap (X|K) = \theta$.

Definition 1.3.2

A cone K is normal in a Banach space X provided there exists $\varepsilon > 0$ such that for all $x, y \in K$ with $||x|| = ||y|| = 1$,

$$||x+y|| \geq \varepsilon.$$

Definition 1.3.3

A relation $\leq'$ on a Banach space X is a partial ordering on X provided

i) $x \leq' y$ implies $tx \leq' ty$ if $t \in [0,\infty)$ and $ty \leq' tx$ if $t \in (-\infty, 0)$;

ii) $x \leq' y$ and $y \leq' x$ imply $x = y$;

iii) $x \leq' y$ and $z \leq' w$ imply $x+z \leq' y+w$;

iv) $x \leq' y$ and $y \leq' z$ imply $x \leq' z$.

Remark If K is a cone in a Banach space X then the relation $\leq'$ defined on X by $x \leq' y$ provided $y-x \in K$ is readily seen to be a partial ordering.

Definition 1.3.4

If $(X, \leq')$ is a partially ordered Banach space and $u, v \in X$ with $u \leq' v$ then the set

$$[u,v] = \{x \mid u \leq' x \leq' v\}$$

is called an order interval with endpoints u and v.

Definition 1.3.5

If $(X,\leq')$ is a partially ordered Banach space then a sequence $\{x_n\}_{n=1}^{\infty}$ in X is increasing if $x_n \leq' x_{n+1}$ for all n, decreasing if $x_{n+1} \leq' x_n$ for all n and monotone if it is either increasing or decreasing.

Definition 1.3.6

If $(X,\leq_1)$ and $(Y,\leq_2)$ are partially ordered Banach spaces and f is a function on X into Y then f is increasing if $x \leq_1 z$ implies $f(x) \leq_2 f(z)$, decreasing if $x \leq_1 z$ implies $f(z) \leq_2 f(x)$ and monotone if it is either increasing or decreasing.

Definition 1.3.7

A norm in a Banach space partially ordered by a cone K is semi-monotonic provided there exists a real number κ such that if $x \leq' y$ in K then

$$||x|| \leq \kappa ||y||.$$

The following result is due to Krasnoselskii [27].

Lemma 1.3.8

A cone K in a Banach space X is normal if and only if the norm on X is semi-monotonic.

Definition 1.3.9

If X is a Banach space partially ordered by a cone K and $A,B \in 2^X$ then A is left-set related to B, written $A \underset{l}{<} B$, provided that

$$(B-a) \cap K \neq \phi \text{ for every } a \in A.$$

Definition 1.3.10

If X is a Banach space partially ordered by a cone K and $A,B \in 2^X$ then A is right-set related to B, written $A \underset{r}{<} B$, provided that

$$(b-A) \cap K \neq \phi \text{ for every } b \in B.$$

Remark Definitions 1.3.9 and 1.3.10 are indeed different for let X be the real numbers with their usual ordering, $A = \{1,2\}$ and $B = \{0,3\}$.

Then $A \underset{l}{<} B$ but it is not true that $A \underset{r}{<} B$.

1.4 Measures of Non-compactness

The idea of a measure of non-compactness as defined in this section is generally attributed to Sadovskii [38]. Unless otherwise specified X will denote a locally convex topological vector space in the following.

Definition 1.4.1

A closed convexity in X is a set $\Lambda \subset 2^X$ with the property that if $A \in \Lambda$ then $\overline{co}(A) \in \Lambda$.

Definition 1.4.2

If Λ is a closed convexity in X and $(T, \leq, 0)$ is a totally ordered set with minimal element 0 then a function $\gamma : \Lambda \to T$ is a measure of non-compactness (mnc) provided that for all $A \in \Lambda$

$$\gamma(\overline{co}(A)) = \gamma(A).$$

The quadruple (X, Λ, T, γ) is called a non-compactness measurable (ncm) space.

Definition 1.4.3

If (X, Λ, T, γ) is a ncm space then γ is

i) monotonic if $A, B \in \Lambda$ and $A \subset B$ imply $\gamma(A) \leq \gamma(B)$;

ii) semi-additive if whenever A, B and $A \cup B$ are in Λ then

$$\gamma(A \cup B) = \text{lub}\,\{\gamma(A), \gamma(B)\};$$

iii) non-singular if for all $x \in X$, $\{x\} \in \Lambda$ and $\gamma(\{x\}) = 0$;

iv) 1-regular if $A \in \Lambda$ and $\gamma(A) = 0$ imply that A is precompact;

v) 2-regular if $A \in \Lambda$ and A precompact imply that $\gamma(A) = 0$;

vi) algebraically semi-additive if T is a monoid and if $A, B \in \Lambda$ implies $A+B \in \Lambda$ and $\gamma(A+B) \leq \gamma(A) + \gamma(B)$;

vii) invariant under shifts if $A \in \Lambda$ implies $x+A \in \Lambda$ and

$$\gamma(x+A) = \gamma(A) \text{ for all } x\varepsilon X;$$

viii) semi-homogeneous if $T = [0,\infty)$ and if $A\varepsilon\Lambda$ and η a complex number imply $\eta A \,\varepsilon\, \Lambda$ and $\gamma(\eta A) = |\eta|\gamma(A)$.

<u>Example 1.4.4</u>

Let X be a locally convex topological vector space, $T = \{0,1\}$ with its usual ordering and $\Lambda = 2^X$. Define $\gamma:\Lambda \to T$ by

$$\gamma(A) = \begin{cases} 0, & \text{if } A \text{ is precompact} \\ 1, & \text{if } A \text{ is not precompact.} \end{cases}$$

Then it is easy to verify that (X,Λ,T,γ) is a ncm space satisfying i) through viii) of Definition 1.4.3.

<u>Example 1.4.5</u>

Let X be a normed linear space, Λ the set of all bounded subsets of X and $T = [0,\infty)$. Then, Martin [31] has shown that (X,Λ,T,di) is a ncm space satisfying i), iii), iv), vi), vii) and viii) of Definition 1.4.3.

<u>Example 1.4.6</u>

Let X be a Banach space, $C \,\varepsilon\, \underline{K}(X)$, Λ the set of all bounded subsets of X and $T = [0,\infty)$. Define $\delta_C:\Lambda \to T$ by

$$\delta_C(A) = \inf\{d>0 \mid A \subset \bigcup_{i=1}^{n} B(c_i,d),\ c_i\varepsilon C,\ n \text{ a positive integer}\}.$$

Then (X,Λ,T,δ_C) is a ncm space.

Indeed, consider $A\varepsilon\Lambda$. It is immediate that $\delta_C(\bar{A}) = \delta_C(A)$. Thus, it suffices to show that $\delta_C(co(A)) = \delta_C(A)$.

Let $d>\delta_C(A)$. Then there exists a positive integer n and $c_i \,\varepsilon\, C$, $1\le i\le n$, such that $A \subset \bigcup_{i=1}^{n} B(c_i,d)$. Let $V = co(\{c_i\}_{i=1}^{n})$ and $x \,\varepsilon\, co(A)$. Then,

$$x = \sum_{k=1}^{p} \eta_k a_k \text{ where } \eta_k \ge 0,\ 1\le k\le p,\ \sum_{k=1}^{p} \eta_k = 1 \text{ and } a_k \,\varepsilon\, A.$$

Now, define a function h on A into the set $\{1,\cdots,n\}$ such that

$$||a - c_{h(a)}|| < d \text{ for all } a\varepsilon A.$$

Then

$$||x - \sum_{k=1}^{p} \eta_k c_{h(a_k)}|| = ||\sum_{k=1}^{p} \eta_k (a_k - c_{h(a_k)})|| < d.$$

Let

$$B = \underbrace{[0,1]\times\cdots\times[0,1]}_{n \text{ times}}\times\{c_1\}\times\cdots\times\{c_n\}$$

and

$$D = \{(\psi_1,\cdots,\psi_n,c_1,\cdots,c_n) | \sum_{i=1}^{n} \psi_i = 1,\ \psi_i\varepsilon[0,1]\}.$$

Then D is a subset of the compact set B and hence is precompact. Let $f:B \to X$ be defined by

$$f(\psi_1,\cdots,\psi_n,c_1,\cdots,c_n) = \sum_{i=1}^{n} \psi_i c_i,\ \psi_i\varepsilon[0,1] \text{ for } 1\le i\le n.$$

Then f is continuous on B and so is g, the restriction of f to D. Since g maps D onto V the set V is precompact in X. Thus, given $\varepsilon>0$ there exists a finite set $\{v_i\}_{i=1}^{m}$ in V such that

$$V \subset \bigcup_{i=1}^{m} B(v_i,\varepsilon).$$

In particular, there exists $j \varepsilon \{1,\cdots,m\}$ such that

$$||\sum_{k=1}^{p} \eta_k c_{h(a_k)} - v_j|| < \varepsilon.$$

Hence,

$$||x-v_j|| \le ||x - \sum_{k=1}^{p} \eta_k c_{h(a_k)}|| + ||\sum_{k=1}^{p} \eta_k c_{h(a_k)} - v_j|| < d+\varepsilon.$$

That is,

$$x \varepsilon B(v_j,d+\varepsilon).$$

Since d was arbitrarily larger than $\delta_C(A)$ and $\delta_C(A) \le \delta_C(co(A))$,

$$\delta_C(co(A)) = \delta_C(A).$$

Remark The mnc δ_C is referred to in the literature as the ball measure of non-compactness relative to C. In [38] Sadovskii shows that the ball measure of non-compactness relative to C satisfies i) through viii) of definition 1.4.3.

Example 1.4.7

Let X be a Banach space, Λ the set of all bounded subsets of X and $T = [0,\infty)$. Define $\alpha:\Lambda \to T$ by

$$\alpha(A) = \inf \{d>0 \mid A \text{ can be covered by a finite number of sets of diameter less than } d\}.$$

Then Martin [31] has shown that (X,Λ,T,α) is a ncm space.

Remark In [38] Sadovskii shows that α satisfies i) through viii) of Definition 1.4.3. In addition, he shows that if $C = X$ in Example 1.4.6 then α and δ_X are equivalent in the sense that there exists $m,M \in (0,\infty)$ such that for every $A\in\Lambda$, the bounded subsets of X,

$$m(\delta_X(A)) \leq \alpha(A) \leq M(\delta_X(A)).$$

In fact, he shows that $m = 1$ and $M = 2$ always works.

The measure of non-compactness α was defined by Kuratowski and is referred to in the literature as the set measure of non-compactness. The following theorem is also due to Kuratowski [28].

Theorem 1.4.8

Let X be a Banach space and $\{A_n\}_{n=1}^{\infty}$ a sequence of nonempty closed subsets of X such that $A_1 \supset A_2 \supset \cdots$. If $\lim_{n\to\infty} \alpha(A_n) = 0$ then $A = \bigcap_{n=1}^{\infty} A_n$ is a nonempty compact set and $\{A_n\}_{n=1}^{\infty}$ converges to A in the Hausdorff metric.

Example 1.4.9

Let X be a Banach space, M a compact metric space, Λ the set of all bounded subsets of X and $T = [0,\infty)$. In addition, suppose that (X,Λ,T,γ) is a ncm space with γ a monotonic mnc. Furthermore, let $X_C = C(M,X)$ and Λ_C the set of all bounded subsets of X_C.

If $A\in\Lambda_C$ define $A_M = \{f(t) \mid f\in A,\ t\in M\}$ and $\gamma_C:\Lambda_C \to T$ by

$$\gamma_C(A) = \gamma(A_M).$$

Then $(X_C, \Lambda_C, T, \gamma_C)$ is a ncm space.

To see this consider $A \varepsilon \Lambda_C$. Since $(\overline{co}(A))_M \subset \overline{co}(A_M)$ and γ is monotonic

$$\gamma_C(\overline{co}(A)) = \gamma((\overline{co}(A))_M) \le \gamma(\overline{co}(A_M)) = \gamma(A_M) = \gamma_C(A)$$

Also, $A_M \subset (\overline{co}(A))_M$ so that

$$\gamma_C(A) = \gamma(A_M) \le \gamma((\overline{co}(A))_M) = \gamma_C(\overline{co}(A)).$$

Thus,

$$\gamma_C(\overline{co}(A)) = \gamma_C(A).$$

The idea of defining γ_C in this manner is essentially due to Sadovskii [38].

In the following sequence of lemma's the ncm space involved will be as in Example 1.4.9.

<u>Lemma 1.4.10</u>

If γ is monotonic then so is γ_C.

<u>Proof</u> Consider $A,B \varepsilon \Lambda_C$ with $A \subset B$. Since $A_M \subset B_M$ and γ is monotonic

$$\gamma_C(A) = \gamma(A_M) \le \gamma(B_M) = \gamma_C(B).$$

<u>Lemma 1.4.11</u>

If γ is semi-additive then so is γ_C.

<u>Proof</u> Suppose A, B and $A \cup B$ are in Λ_C. Since γ is semi-additive and $(A \cup B)_M = A_M \cup B_M$

$$\begin{aligned} \gamma_C(A \cup B) &= \gamma((A \cup B)_M) \\ &= \gamma(A_M \cup B_M) \\ &= \text{lub }\{\gamma(A_M), \gamma(B_M)\} \\ &= \text{lub }\{\gamma_C(A), \gamma_C(B)\}. \end{aligned}$$

<u>Lemma 1.4.12</u>

If γ is 2-regular then γ_C is non-singular.

Proof Consider $f\varepsilon X_C$. Then $f(M) \varepsilon Kp(X)$ so that $\{f\}\varepsilon\Lambda_C$. Since γ is 2-regular

$$\gamma_C(\{f\}) = \gamma(f(M)) = 0.$$

Lemma 1.4.13

If γ is monotonic and 1-regular then γ_C is 1-regular when restricted to equicontinuous sets in Λ_C.

Proof Suppose A is an equicontinuous set in Λ_C with $\gamma_C(A) = 0$. Since γ is monotonic and $\{f(m)|f\varepsilon A\}\subset A_M$ for every $m\varepsilon M$

$$0 \le \gamma(\{f(m)|f\varepsilon A\}) \le \gamma(A_M) = \gamma_C(A) = 0.$$

Thus, since γ is 1-regular A is precompact by Theorem 1.1.4.

Lemma 1.4.14

If γ is monotonic and algebraically semi-additive then γ_C is algebraically semi-additive.

Proof Suppose $A,B\varepsilon\Lambda_C$. Then $A+B \varepsilon \Lambda_C$, $A_M+B_M\varepsilon \Lambda$ and $(A+B)_M\subset A_M+B_M$. Thus, since γ is monotonic and algebraically semi-additive

$$\gamma_C(A+B) = \gamma((A+B)_M) \le \gamma(A_M+B_M) \le \gamma(A_M) + \gamma(B_M) = \gamma_C(A) + \gamma_C(B).$$

Lemma 1.4.15

If γ is monotonic, 2-regular and algebraically semi-additive then γ_C is invariant under shifts.

Proof Consider $f\varepsilon X_C$ and $A\varepsilon\Lambda_C$. Then $f+A \varepsilon \Lambda_C$ and since γ is monotonic, 2-regular and algebraically semi-additive

$$\begin{aligned}\gamma_C(f+A) &= \gamma((f+A)_M)\\ &\le \gamma(f(M)+A_M)\\ &\le \gamma(f(M)) + \gamma(A_M)\\ &= \gamma(A_M)\\ &= \gamma_C(A)\end{aligned}$$

Lemma 1.4.16

If γ is semi-homogeneous then so is γ_C.

Proof Suppose that $A\varepsilon\Lambda_C$ and η is a complex number. Then

$$\gamma_C(\eta A) = \gamma((\eta A)_M) = \gamma(\eta(A_M)) = |\eta|\gamma(A_M) = |\eta|\gamma_C(A).$$

Example 1.4.17

Let X be a Banach space, T = $[0,\infty)$, Λ the bounded subsets of X and (M,d) a compact metric space. Also, let X_C and Λ_C be as in Example 1.4.9.

For $\eta>0$ and $A\varepsilon\Lambda_C$ define

$$\omega(\eta,A) = \begin{cases} \sup\ \{||f(m_1) - f(m_2)||:d(m_1,m_2)<\eta \text{ and } f\varepsilon A\}, \text{ if } A \neq \phi \\ 0, \text{ if } A = \phi. \end{cases}$$

The modulus of continuity of A is then defined by

$$\omega(A) = \lim_{\eta\to 0_+} \omega(\eta,A).$$

It is not difficult to show that (X_C,Λ_C,T,ω) is a ncm space. Indeed, consider $A\varepsilon\Lambda_C$ and $\eta>0$.

If $A = \phi$ then evidently $\omega(A) = \omega(\overline{co}(A))$, so suppose that $A \neq \phi$. Then, given $\varepsilon>0$,

$$\begin{aligned}\omega(\eta,\bar{A}) &= \sup\ \{||g(m_1) - g(m_2)||:d(m_1,m_2)<\eta,\ g\varepsilon\bar{A}\} \\ &\le \{\sup\ \{||g(m_1) - f_g(m_1)||\} + \sup\ \{||f_g(m_1) - f_g(m_2)||\} \\ &\quad + \sup\ \{||f_g(m_2) - g(m_2)||\}:d(m_1,m_2)<\eta,\ g\varepsilon\bar{A},\ f_g\varepsilon A \\ &\quad \text{and } ||f_g-g||_C < \varepsilon/2\} \\ &\le \varepsilon/2 + \omega(\eta,A) + \varepsilon/2 \\ &= \omega(\eta,A) + \varepsilon.\end{aligned}$$

Thus, $\omega(\eta,\bar{A}) \le \omega(\eta,A)$ for every $\eta>0$ and so $\omega(\bar{A}) \le \omega(A)$.

Evidently, $\omega(\eta,A) \le \omega(\eta,\bar{A})$ for every $\eta>0$. Hence, $\omega(A) \le \omega(\bar{A})$ and

$$\omega(\bar{A}) = \omega(A).$$

Therefore, it suffices to show that $\omega(co(A)) = \omega(A)$. Now,

$$\omega(\eta,\mathrm{co}(A)) = \sup\left\{\left\|\sum_{k=1}^{p}\beta_k f_k(m_1) - \sum_{k=1}^{p}\beta_k f_k(m_2)\right\| : d(m_1,m_2)<\eta,\right.$$

$$\left.\beta_k \geq 0,\ \sum_{k=1}^{p}\beta_k = 1,\ f_k \varepsilon A\right\}$$

$$\leq \sup\left\{\left\|\sum_{k=1}^{p}\beta_k(f_k(m_1) - f_k(m_2))\right\| : m_1,\ m_2,\ \beta_k \text{ and } f_k \right.$$

$$\left.\text{are as above}\right\}$$

$$\leq \sup\left\{\sum_{k=1}^{p}\beta_k\|f_k(m_1) - f_k(m_2)\| : m_1,\ m_2,\ \beta_k \text{ and } f_k\right.$$

$$\left.\text{are as above}\right\}$$

$$\leq \sup\{\sum_{k=1}^{p}\beta_k\omega(\eta,A) : \beta_k \text{ are as above}\}$$

$$= \omega(\eta,A)$$

Thus, $\omega(\mathrm{co}(A)) \leq \omega(A)$.

Certainly, $\omega(\eta,A) \leq \omega(\eta,\mathrm{co}(A))$ for all $\eta>0$. Therefore,

$$\omega(A) \leq \omega(\mathrm{co}(A)) \text{ and } \omega(A) = \omega(\mathrm{co}(A)).$$

Remarks It is readily verified that the mnc ω satisfies conditions i) and ii) of Definition 1.4.3. It also satisfies condition iii) since a continuous function on a compact set into a Banach space is uniformly continuous. Conditions iv) and v) are satisfied if either ω is restricted to $A \varepsilon \Lambda_C$ such that the set $\{f(m) | f \varepsilon A\}$ is precompact for every $m \varepsilon M$ or if X is the real or complex numbers. The former follows from Theorem 1.1.4 and the latter from the Arzela-Ascoli Theorem.

Condition vi) of Definition 1.4.3 is satisfied by ω, for suppose $A,B \varepsilon \Lambda_C$. If one of A or B is empty then it is obvious that

$$\omega(A+B) = \omega(A) + \omega(B),$$

so suppose that A and B are both nonempty. Then $A+B\varepsilon\Lambda_C$ and for all $\eta>0$

$$\omega(\eta,A+B) = \sup\{||(f+g)(m_1) - (f+g)(m_2)||:f\varepsilon A,\ g\varepsilon B,\ d(m_1,m_2)<\eta\}$$

$$\leq \sup\{||f(m_1) - f(m_2)||:f\varepsilon A,\ d(m_1,m_2)<\eta\}$$
$$+ \sup\{||g(m_1) - g(m_2)||:g\varepsilon B,\ d(m_1,m_2)<\eta\}$$

$$\leq \omega(\eta,A) + \omega(\eta,B).$$

Thus, $\omega(A+B) \leq \omega(A) + \omega(B)$.

It is easy to see that ω satisfies condition vii). Indeed, consider $f\varepsilon X_C$ and $A\varepsilon\Lambda_C$. If A is empty then it is immediate that

$$\omega(f+A) = \omega(A),$$

so suppose that A is not empty. Then given $\varepsilon>0$ there exists $\beta>0$ such that

$$||f(m_1) - f(m_2)||<\varepsilon \text{ whenever } d(m_1,m_2)<\beta$$

Therefore, if $\eta\leq\beta$ then

$$\omega(\eta,A) = \sup\{||g(m_1) - g(m_2)||:g\varepsilon A,\ d(m_1,m_2)<\eta\}$$

$$\leq \sup\{||g(m_1) - f(m_1) + f(m_1) - f(m_2) + f(m_2) - g(m_2)||:$$
$$g\varepsilon A,\ d(m_1,m_2)<\eta\}$$

$$\leq \sup\{||(f+g)(m_1) - (f+g)(m_2)||:g\varepsilon A,\ d(m_1,m_2)<\eta\}$$
$$+ \sup\{||f(m_1) - f(m_2)||:d(m_1,m_2)<\eta\}$$

$$\leq \omega(\eta,f+A) + \varepsilon.$$

Thus, $\omega(A) \leq \omega(f+A)$. By conditions iii) and vii)

$$\omega(f+A) \leq \omega(\{f\}) + \omega(A) = \omega(A)$$

and hence $\omega(f+A) = \omega(A)$.

Finally, ω satisfies condition viii), for suppose $A\varepsilon\Lambda_C$, β is a complex number and $\eta>0$. If A is empty then trivially

$$\omega(\beta A) = |\beta|\omega(A),$$

so suppose that A is not empty. Then

$$\begin{aligned}\omega(\eta,\beta A) &= \sup\{||\beta f(m_1) - \beta f(m_2)||: f\varepsilon A,\ d(m_1,m_2)<\eta\}\\ &= |\beta|\sup\{||f(m_1) - f(m_2)||: f\varepsilon A,\ d(m_1,m_2)<\eta\}\\ &= |\beta|\omega(\eta,A).\end{aligned}$$

Thus, $\omega(\beta A) = |\beta|\omega(A)$.

In the event that ω is restricted to $A\varepsilon\Lambda_C$ such that $\{f(m)|\,f\varepsilon A\}$ is precompact for each $m\varepsilon M$, Nussbaum [34], without referring to ω as a mnc, has shown that α and ω are equivalent on X_C in the sense of the Remark following Example 1.4.7. In fact, $m = 1/2$ and $M = 1$ always works.

1.5 Multifunctions

The text by Berge [2] and the paper by Smithson [41] were used as references in the development of this section.

Definition 1.5.1

Suppose X and Y are point sets. A function on X into 2^Y is called a multifunction.

Definition 1.5.2

If X and Y are point sets, F and G multifunctions on X into 2^Y, $B\subset X$ and $A\subset Y$ then

i) $F(B) = \bigcup_{b\varepsilon B} F(b)$;

ii) $F^-(A) = \{x\varepsilon X|\,F(x)\cap A \neq \phi\}$;

iii) the multifunction $F\cap G$ on X into 2^Y is defined by

$$F\cap G(x) = F(x)\cap G(x), \quad x\varepsilon X.$$

Definition 1.5.3

If X is a point set, Y a vector space, F and G multifunctions on X into 2^Y and η a complex number then

i) F+G is a multifunction on X into 2^Y defined by

$$F+G(x) = F(x) + G(x), \quad x \in X;$$

ii) ηF is a multifunction on X into 2^Y defined by

$$\eta F(x) = \eta(F(x)), \quad x \in X;$$

iii) F-G is a multifunction on X into 2^Y defined by

$$F-G = F+(-1)G.$$

Definition 1.5.4

Suppose X and Y are point sets. If P is a property of sets then a multifunction $F:X \to 2^Y$ is point P provided F(x) has property P for all $x \in X$.

Remark If X and Y are point sets then a single-valued function f on X into Y defined by f(x) = y and the multifunction F on X into 2^Y defined by $F(x) = \{y\}$ will be identified by the map $y \to \{y\}$.

In the following X and Y will denote topological spaces unless otherwise specified.

Definition 1.5.5

A multifunction $F:X \to 2^Y$ is upper semi-continuous (usc) at $x \in X$ provided that for every V an open set in Y containing F(x) there exists an open set U in X containing x such that $F(z) \subset V$ for all $z \in U$. A multifunction $F:X \to 2^Y$ is upper semi-continuous on X if it is usc at every point of F.

The following result can be found in Smithson [41].

Lemma 1.5.6

A multifunction $F:X \to 2^Y$ is usc if and only if $F^-(A)$ is closed in X whenever A is closed in Y.

Definition 1.5.7

A multifunction $F:X \to 2^Y$ is closed provided that whenever $x \in X$ and $y \in Y$ with $y \notin F(x)$ there exists open sets U in X containing x and V in Y containing y such that for all $z \in U$, $F(z) \cap V = \phi$.

Lemma 1.5.8

If $F:X \to 2^Y$ is a closed multifunction then it is point closed.

Proof Suppose, by way of contradiction, that $F(x)$ is not closed in Y for some $x \in X$. Then there exists a point $y \in Y|F(x)$ such that every open set containing y meets $F(x)$. However, this contradicts the definition of F being a closed multifunction.

Lemma 1.5.9

Suppose that $F:X \to 2^Y$ is a closed multifunction. If $\{x_\alpha\}_{\alpha \in A}$ is a net in X that converges to $\hat{x}$ and $\{y_\alpha\}_{\alpha \in A}$ is a net in Y that converges to $\hat{y}$ such that $y_\alpha \in F(x_\alpha)$ for every $\alpha \in A$ then $\hat{y} \in F(\hat{x})$.

Proof The graph $\Psi = \{(x,y) | y \in F(x)\}$ of F is closed in $X \times Y$. Indeed, since F is a closed multifunction, for every $(x,y) \in (X \times Y)|\Psi$ there exists open sets U in X containing x and V in Y containing y such that

$$U \times V \subset (X \times Y)|\Psi.$$

Since $(x_\alpha, y_\alpha) \in \Psi$ for each $\alpha \in A$ and $\{(x_\alpha, y_\alpha)\}_{\alpha \in A}$ converges to $(\hat{x}, \hat{y})$ in the product topology $(\hat{x}, \hat{y}) \in \Psi$, that is, $\hat{y} \in F(\hat{x})$.

Definition 1.5.10

A multifunction $F:X \to 2^Y$ is σ-closed provided that if $\{x_n\}_{n=1}^{\infty}$ is a sequence in X converging to x and $\{y_n\}_{n=1}^{\infty}$ is a sequence in Y converging to y such that $y_n \in F(x_n)$ for all n then $y \in F(x)$.

Lemma 1.5.11

If X and Y are metric spaces then a multifunction $F:X \to 2^Y$ is closed if and only if it is σ-closed.

Proof Suppose that F is σ-closed. By way of contradiction, suppose that F is not closed. Then there exist $x\varepsilon X$ and $y\varepsilon Y$ with $y\notin F(x)$ such that for every open set U in X containing x and every open set V in Y containing y there exists z, depending on U and V, in U such that $V\cap F(z) \neq \phi$.

Thus, there exists a sequence $\{x_n\}_{n=1}^{\infty}$ in X such that $x_n\varepsilon B(x,1/n)$ for all n and a sequence $\{y_n\}_{n=1}^{\infty}$ in Y such that $y_n\varepsilon B(y,1/n)\cap F(x_n)$ for all n. Since F is σ-closed $y\varepsilon F(x)$, a contradiction.

Conversely, suppose that F is closed. Then, by Lemma 1.5.9, F is σ-closed.

Lemma 1.5.12

If Y is a regular topological space then every point closed usc multifunction $F:X \to 2^Y$ is a closed multifunction.

Proof Suppose $x\varepsilon X$ and $y\varepsilon Y$ with $y\notin F(x)$. Since F(x) is a closed subset of Y there are open sets V containing y and W containing F(x) in Y such that $W\cap V = \phi$.

Since F is usc there exists an open set U in X containing x such that $F(U)\subset W$. That is, if $z\varepsilon U$ then $F(z)\cap V = \phi$ and hence F is a closed multifunction.

The subsequent result can be found in Smithson [41].

Lemma 1.5.13

If $F:X \to 2^Y$ is a point compact usc multifunction and $A\varepsilon Kp(X)$ then

$$F(A) \varepsilon Kp(Y).$$

Lemma 1.5.14

If $F:X \to 2^Y$ is a closed multifunction and $A\varepsilon Kp(X)$ then $F(A)\varepsilon Kl(Y)$.

Proof Suppose $\{y_\alpha\}_{\alpha\varepsilon B}$ is a net in F(A) converging to y. Then there exists a net $\{x_\alpha\}_{\alpha\varepsilon B}$ in A such that $y_\alpha \varepsilon F(x_\alpha)$ for every $\alpha\varepsilon B$.

Since A is compact there exists $x\varepsilon A$ and a subnet of $\{x_\alpha\}_{\alpha\varepsilon B}$ that converges to x. Thus, since F is a closed multifunction,

$$y\varepsilon F(x) \subset F(A)$$

by Lemma 1.5.9, hence $F(A)\varepsilon Kl(Y)$.

Definition 1.5.15

Suppose X and Y are point sets and $F:X \to 2^Y$ is a point nonempty multifunction. A single-valued function $f:X \to Y$ with the property that

$$f(x)\varepsilon F(x) \text{ for all } x\varepsilon X$$

is called a selector for F on X.

Definition 1.5.16

Suppose (T,μ) is a measure space, Y a topological space and $F:T \to 2^Y$ a multifunction such that F(t) is nonempty μ-a.e. on T. A single-valued function $f:X \to Y$ with the property that $f(t)\varepsilon F(t)$ μ-a.e. on T is called a μ-selector for F on T.

Definition 1.5.17

Suppose (T,μ) is a measure space and Y a topological space. A multifunction $F:T \to 2^Y$ is μ-measurable provided $F^-(A)$ is μ-measurable in T for every closed subset A of Y.

Definition 1.5.18

Suppose (T,μ) is a measure space and Y a Banach space. A multifunction $F:T \to 2^Y$ is integrably bounded on T provided there exists a μ-integrable function p on T into the real numbers such that

$$\sup \{||y||:y\varepsilon F(t)\} \leq p(t) \text{ for every } t\varepsilon T.$$

The function p is called an integral bound for F.

Definition 1.5.19

Suppose (T,μ) is a measure space and Y a Banach space. A multifunction

$F:T \to 2^Y$ is integrable on T provided the set

$$B = \{f | f \text{ is an integrable } \mu\text{-selector for } F \text{ on } T\}$$

is nonempty. The integral of F on T is then defined by

$$\int_T F(s)d\mu = \{\int_T f(s)d\mu | f\varepsilon B\}.$$

<u>Definition 1.5.20</u>

Suppose that X is a point set and $F:X \to 2^X$ is a multifunction. A point $x\varepsilon X$ such that $x\varepsilon F(x)$ is called a fixed point of F.

<u>Definition 1.5.21</u>

Suppose that $(X,\leq_1)$ and $(Y,\leq_2)$ are Banach spaces partially ordered by cones. A multifunction $F:X \to 2^Y$ is left-set (respectively, right-set) monotone on X provided that if $x\leq_1 z$ in X then

$$F(x) \underset{l}{<} (\text{respectively, } \underset{r}{<}) \; F(z).$$

1.6 <u>Contractive and Condensing Multifunctions</u>

The paper by Sadovskii [38] was used as a reference in the discussion here of condensing and k-set-contraction multifunctions. It should be noted, however, that the work done there is restricted to single-valued functions.

Throughout this section Q will denote a point set.

<u>Definition 1.6.1</u>

Suppose that X and Y are metric spaces and $k\geq 0$. A multifunction $F:X \to Kp(Y)$ is a k-contraction provided that for all $x,z \;\varepsilon\; X$

$$d_H(F(x),F(z)) \leq kd(x,z).$$

<u>Definition 1.6.2</u>

If X is a metric space and Y a topological space then a multifunction $F:Q\times X \to 2^Y$ is precompact (respectively, compact) provided that $F(Q\times A)$ is precompact (respectively, compact) in Y for every bounded subset A of X.

<u>Definition 1.6.3</u>

Suppose that (X,Λ_1,T,γ_1) and (Y,Λ_2,T,γ_2) are ncm spaces. A multifunction $F:Q\times X \to 2^Y$ is $\gamma_1\gamma_2$-condensing if $F(Q\times A)\varepsilon\Lambda_2$ for every $A\varepsilon\Lambda_1$ and if

$$\gamma_2(F(Q\times A)) \geq \gamma_1(A)$$

implies that A is precompact.

<u>Remark</u> If $X = Y$ and $\gamma_1 = \gamma_2 = \gamma$ in Definition 1.6.3 then F is said to be γ-condensing or simply condensing. Since T is a totally ordered set the last condition in Definition 1.6.3 is equivalent to demanding that if A is not precompact then $\gamma_2(F(Q\times A)) < \gamma_1(A)$.

<u>Definition 1.6.4</u>

Suppose that (X,Λ_1,T,γ_1) and (Y,Λ_2,T,γ_2) are ncm spaces, $k\geq 0$ and T a subset of the real numbers. A multifunction $F:Q\times X \to 2^Y$ is a k-$\gamma_1\gamma_2$-set-contraction provided that for every $A\varepsilon\Lambda_1$, $F(Q\times A)\varepsilon\Lambda_2$ and

$$\gamma_2(F(Q\times A)) \leq k(\gamma_1(A)).$$

<u>Remarks</u> In order to apply the preceding definitions to multifunctions defined on an appropriate space X into the set of all subsets of an appropriate space Y identify X with $Q\times X$, where $Q = \{q\}$ is a singleton, by the map $x \to (q,x)$.

Suppose that (X,Λ_1,T,γ_1) and (Y,Λ_2,T,γ_2) are ncm spaces where Λ_1 and Λ_2 are the bounded subsets of X and Y respectively. If γ_2 is a 2-regular mnc then every precompact, hence every compact, multifunction on X into 2^Y is obviously a 0-$\gamma_1\gamma_2$-set-contraction.

The subsequent result follows immediately from the two preceding definitions and Definition 1.4.3.

<u>Lemma 1.6.5</u>

Suppose that (X,Λ_1,T,γ_1) and (Y,Λ_2,T,γ_2) are ncm spaces where γ_1 is a 1-regular mnc. If $0\leq k<1$ and $F:X \to 2^Y$ is a k-$\gamma_1\gamma_2$-set-contraction then

F is $\gamma_1\gamma_2$-condensing.

The following result is proved by Fitzpatrick and Petryshyn in [15].

Lemma 1.6.6

Suppose that (X,Λ_X,T,δ_X) and (Y,Λ_Y,T,γ_Y) are ncm spaces as in Example 1.4.6. If $F:X \to Kp(Y)$ is a k-contraction then F is a k-$\delta_X\delta_Y$-set contraction.

Lemma 1.6.7

Suppose that (X,Λ_1,T,γ_1) and (Y,Λ_2,T,γ_2) are ncm spaces where Λ_1 and Λ_2 are the bounded subsets of X and Y respectively. If γ_2 is monotonic, algebraically semi-additive and 2-regular, $F:X \to 2^Y$ is a precompact multifunction and $G:X \to 2^Y$ is a k-$\gamma_1\gamma_2$-set-contraction then F+G is a k-$\gamma_1\gamma_2$-set-contraction.

Proof Consider $A\varepsilon\Lambda_1$. Then

$$\begin{aligned}\gamma_2(F+G(A)) &\le \gamma_2(F(A) + G(A))\\ &\le \gamma_2(F(A)) + \gamma_2(G(A))\\ &\le 0 + k(\gamma_1(A))\\ &= k(\gamma_1(A)).\end{aligned}$$

CHAPTER II

2.1 A Definition of Solution to the Initial-value Problem

A definition of a solution to the initial-value problem as posed in the Introduction is provided in this section. The definition differs from that of the classical solution as given by Filippov [12] in that only a local solution defined almost everywhere in a neighborhood of zero is sought here.

Definition 2.1.1

Suppose that T, X and Y are point sets. If $x:T \to X$ is a single-valued function, $s \in T$ and $F:T \times X \to 2^Y$ is a multifunction then

i) the multifunction $F_x:T \to 2^Y$ is defined by

$$F_x(t) = F(t,x(t)), \quad t \in T;$$

ii) the multifunction $F_s:X \to 2^Y$ is defined by

$$F_s(z) = F(s,z), \quad z \in X.$$

Unless otherwise specified X will denote a Banach space, J a closed interval of real numbers $[0,T]$ with $0<T<\infty$, and $F:J \times X \to 2^X$ a multifunction. Also, unless otherwise stated, any reference to measure on the real line will be to Lebesgue measure and it will be denoted by λ.

Definition 2.1.2

If $G:J \to 2^X$ is a multifunction and J_o is a subinterval of J then

i) $IS(G,J_o)$ will denote the set of all integrable λ-selectors on J_o for G restricted to J_o;

ii) $MS(G,J_o)$ will denote the set of all measurable λ-selectors on J_o for G restricted to J_o.

Definition 2.1.3

A solution to the initial-value problem

$$\dot{x}(t) \in F(t,x(t)) \qquad (2.1.4)$$

$$x(0) = x_o,\ x_o \in X$$

is an ordered pair (x,J_o) where $J_o = [0,T_o]$, $0<T_o\leq T$ and $x:J_o \to X$ is a once differentiable function a.e. on J_o such that $\dot{x}$ is a λ-selector on J_o for F_x restricted to J_o and $x(0) = x_o$.

Lemma 2.1.5

Suppose that $J_o = [0,T_o]$, $0<T_o\leq T$ and $x_o \in X$. If Γ is the multifunction on $C(J_o,X)$ into the subsets of itself defined by

$$\Gamma(x) = \{y \mid y(t) = x_o + \int_0^t f_x(s)d\lambda \text{ where } f_x \in IS(F_x,J_o)\} \qquad (2.1.6)$$

then every fixed point of Γ is a solution to 2.1.4.

Proof Indeed, Γ is a multifunction on $C(J_o,X)$ into the subsets of itself by Lemma 1.2.13.

Suppose that x is a fixed point of Γ. Then

$$x(t) = x_o + \int_0^t f_x(s)d\lambda \text{ where } f_x \in IS(F_x,J_o).$$

Certainly, $x(0) = x_o$ and by Lemma 1.2.21, $\dot{x}(t) = f_x(t)$ a.e. on J_o. Thus, $\dot{x}(t) \in F(t,x(t))$ a.e. on J_o and hence x is a solution to 2.1.4.

Remark The multifunction 2.1.6 corresponding to the initial-value x_o will be denoted by (Γ,x_o) or simply Γ when x_o is understood.

Lemma 2.1.7

Suppose that X is a separable reflexive Banach space and $x_o \in X$. If $J_o = [0,T_o]$ with $0<T_o\leq T$ and $x \in C(J_o,X)$ such that

i) F_x restricted to J_o is integrably bounded;

ii) $F_x(t) \in K1K(X)$ a.e. on J_o;

then the multifunction (Γ,x_o) maps x into $KpK(C(J_o,X))$.

Proof If $\Gamma(x)$ is empty then the result is trivial, so suppose that $\Gamma(x)$ is not empty.

Consider $\{y_n\}_{n=1}^{\infty}$ a sequence in $\Gamma(x)$. Then for every n

$$y_n(t) = x_o + \int_0^t f_n(s)d\lambda \text{ where } f_n \varepsilon IS(F_x, J_o).$$

By i) and a result of Castaing [6] the set $\{f_n | n = 1,2,\cdots\}$ is precompact in the weak topology of $L(J_o, X)$. Thus, by a result due to Eberlein and Smulian found in [11] there exists a subsequence $\{f_k\}_{k=1}^{\infty}$ of $\{f_n\}_{n=1}^{\infty}$ converging weakly to some $f \varepsilon L(J_o, X)$.

Let y be the function defined on J_o by

$$y(t) = x_o + \int_0^t f(s)d\lambda.$$

According to Lemma 1.2.14 for every $t\varepsilon J_o$

$$\lim_{k\to\infty} \int_0^t f_k(s)d\lambda = \int_0^t f(s)d\lambda.$$

By a result found in [24], for each k there exists a finite set of non-negative numbers $\{a_i^k\}_{i=1}^{m(k)}$ such that $\sum_{i=1}^{m(k)} a_i^k = 1$ and

$$\lim_{k\to\infty} \sum_{i=1}^{m(k)} a_i^k f_{k+i} = f \text{ in } L(J\ ,X).$$

Let $h_k = \sum_{i=1}^{m(k)} a_i^k f_{k+i}$. By Lemma 1.2.18 there exists a subsequence $\{h_j\}_{j=1}^{\infty}$ of $\{h_k\}_{k=1}^{\infty}$ converging a.e. to f. Let

$$E = \{t\varepsilon J_o | f_n(t) \notin F_x(t) \text{ for some integer } n\} \cup \{t\varepsilon J_o | F_x(t) \notin KlK(X)\}$$
$$\cup \{t\varepsilon J_o | \{h_j(t)\}_{j=1}^{\infty} \text{ does not converge to } f(t)\}.$$

Then, for $t \varepsilon J_o | E$

$$f(t) \varepsilon \bigcap_{k=1}^{\infty} \overline{co}\Big(\bigcup_{i=k}^{\infty} f_i(t)\Big) \subset \overline{co}(F_x(t)) = F_x(t).$$

Since E is a λ-null set $f(t) \varepsilon F_x(t)$ a.e. on J_o and hence $y \varepsilon \Gamma(x)$.

To verify that $\Gamma(x)$ is compact in $C(J_o,X)$ it suffices to show that $\Gamma(x)$ is sequentially compact, that is, it suffices to show that $\{y_k\}_{k=1}^{\infty}$ converges to y in $C(J_o,X)$.

Now, $\Gamma(x)$ is uniformly equicontinuous on J_o. Indeed, consider $z \in \Gamma(x)$. Then for every $t \in J_o$

$$z(t) = x_o + \int_0^t f_x(s)d\lambda \text{ where } f_x \in IS(F_x,J_o).$$

Let the function p_x be an integral bound for F_x restricted to J_o. Then, by Lemma 1.2.13, the function p on J_o into the real numbers defined by

$$p(t) = \int_0^t p_x(s)d\lambda$$

is continuous, hence uniformly continuous, on J_o. Thus, by Lemma 1.2.10 and Lemma 1.2.12, given $\varepsilon>0$ there exists $\delta>0$ such that if $s,t \in J_o$ with $s \le t$ and $|t-s|<\delta$ then

$$||z(t) - z(s)|| \le \int_s^t ||f_x(s)||d\lambda \le \int_s^t p_x(s)d\lambda \le |p(t) - p(s)| < \varepsilon.$$

Therefore, $\Gamma(x)$ is uniformly equicontinuous on J_o.

The set $S = \{y\} \bigcup \{y_k | k = 1,2,\cdots\}$ is also uniformly equicontinuous since $y \in C(J_o,X)$. Thus, given $\varepsilon>0$ there exists $\delta>0$ such that if $s,t \in J_o$ with $s \le t$ and $|t-s|<\delta$ then

$$||z(t) - z(s)|| < \varepsilon/3 \text{ for every } z \in S.$$

Let the set $\{t_i\}_{i=1}^{n}$ be such that $0 = t_1 < \cdots < t_n = T_o$ and

$$\max\{t_i - t_{i-1} | 2 \le i \le n\} < \delta.$$

Then for each $i \in \{1,\cdots,n\}$ there exists a positive integer N_i such that for $j \ge N_i$

$$||y_j(t_i) - y(t_i)|| < \varepsilon/3.$$

Let $N = \max \{N_i | 1 \leq i \leq n\}$ and consider $t \varepsilon J_o$. Then there exists $m \varepsilon \{1, \cdots, n\}$ such that $t_m \leq t$, $|t - t_m| < \delta$ and thus for $j \geq N$

$$\begin{aligned} ||y_j(t) - y(t)|| &\leq ||y_j(t) - y_j(t_m)|| + ||y_j(t_m) - y(t_m)|| \\ &\quad + ||y(t_m) - y(t)|| \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 \\ &= \varepsilon \end{aligned}$$

Hence, $\{y_k\}_{k=1}^{\infty}$ converges to y and $\Gamma(x)$ is compact in $C(J_o, X)$.

In order to see that $\Gamma(x)$ is convex consider $y_1, y_2 \varepsilon \Gamma(x)$ and $\eta \varepsilon [0,1]$. Then

$$y_i(t) = x_o + \int_0^t f_i(s) d\lambda \text{ where } f_i \varepsilon IS(F_x, J_o) \text{ for } i = 1,2.$$

Thus,

$$\eta y_1(t) + (1-\eta) y_2(t) = x_o + \int_0^t [\eta f_1(s) - (1-\eta) f_2(s)] d\lambda, \quad t \varepsilon J_o.$$

Let

$$E_o = \{t \varepsilon J_o | f_i(t) \notin F_x(t) \text{ for some } 1 \leq i \leq 2\} \cup \{t \varepsilon J_o | F_x(t) \notin K(X)\}.$$

Since E_o is a λ-null set and for $t \varepsilon J_o | E_o$

$$\eta f_1(t) + (1-\eta) f_2(t) \varepsilon F_x(t),$$

$\eta f_1 + (1-\eta) f_2 \varepsilon IS(F_x, J_o)$. Therefore $\eta y_1 + (1-\eta) y_2 \varepsilon \Gamma(x)$ and $\Gamma(x)$ is convex.

2.2 A Fixed Point Theorem for g-contractive Multifunctions

In this section the definition of a g-contractive multifunction is given and a fixed point theorem for g-contractive multifunctions is proved. The fixed point theorem will be applicable in obtaining a solution to the initial-value problem 2.1.4 in section 4.

Definition 2.2.1

Suppose that X and Y are metric spaces and $g:[0,\infty)\to[0,\infty)$ is an increasing function continuous on the right with the property that $g(r)<r$ for $r>0$. A multifunction $G:X\to Kp(Y)$ is g-contractive provided

$$d_H(G(x),G(z)) \le g(d(x,z)) \text{ for } x,z\in X.$$

Remark Every k-contraction for k<1 is a g-contraction.

Lemma 2.2.2

Suppose that (X,Λ_X,T,δ_X) and (Y,Λ_Y,T,δ_Y) are ncm spaces as in Example 1.4.6. If $G:X\to \underline{Kp}(Y)$ is a g-contractive multifunction then G is a $\delta_X\delta_Y$-condensing multifunction.

Proof Consider $A\in\Lambda_X$ with $\delta_X(A) = d>0$. Then, given $\varepsilon>0$ there exists $x_i\in X$, $1\le i\le n$, such that

$$A\subset\bigcup_{i=1}^{n} B(x_i,d+\varepsilon).$$

Since G is point compact for each $i\in\{1,\cdots,n\}$ there exists $y_j\in Y$, $1\le j\le m(i)$, such that

$$G(x_i)\subset\bigcup_{j=1}^{m(i)} B(y_j^i,\varepsilon).$$

It is not difficult to verify that

$$G(A)\subset\bigcup_{i=1}^{n}\bigcup_{j=1}^{m(i)} B(y_j^i,g(d+\varepsilon)+\varepsilon).$$

Indeed, if $y\in G(A)$ then $y\in G(x)$ for some $x\in A$ and there exists $p\in\{1,\cdots,n\}$ such that

$$||x_p-x|| \le d+\varepsilon.$$

Since $d_H(G(x_p),G(x)) \le g(d+\varepsilon)$ there exists $w\in G(x_p)$ such that

$$||w-y|| \le g(d+\varepsilon).$$

In addition, there exists $q\in\{1,\cdots,m(p)\}$ such that $y_q^p \in G(x_p)$ and

$$||w-y_q^p|| < \varepsilon.$$

Hence,

$$||y-y_q^p|| \leq ||y-w|| + ||w-y_q^p|| < g(d+\varepsilon) + \varepsilon.$$

Since g is continuous on the right $\lim_{\varepsilon\to 0_+} g(d+\varepsilon) = g(d) < d$ and so there exists $\varepsilon>0$ such that

$$g(d+\varepsilon) + \varepsilon < d.$$

Thus, $\delta_Y(G(A)) < \delta_X(A)$ and G is $\delta_X\delta_Y$-condensing.

<u>Lemma 2.2.3</u>

Suppose that (X,Λ,T,δ_X) is a ncm space as in Example 1.4.6 and $G:X\to 2^X$ is a closed condensing multifunction. Then any bounded sequence $\{x_n\}_{n=1}^{\infty}$ with the property that

$$\lim_{n\to\infty} d(x_n,G(x_n)) = 0$$

has a subsequence that converges to a fixed point of G.

<u>Proof</u> Suppose that $\{x_n\}_{n=1}^{\infty}$ is a bounded sequence in X such that

$$\lim_{n\to\infty} d(x_n,G(x_n)) = 0.$$

Let $S = \{x_n|n=1,2,\cdots\}$. Then, given $\varepsilon>0$, there exists subsets T and U of S such that T is finite, $U\subset B(G(S),\varepsilon)$ and $S = T\cup U$.

Since δ_X is monotonic, semi-additive and non-singular

$$\delta_X(S) \leq \max\{\delta_X(T),\delta_X(U)\} = \max\{0,\delta_X(U)\} \leq \delta_X(G(S)) + \varepsilon.$$

Thus, $\delta_X(G(S)) \geq \delta_X(S)$ and S is precompact in accordance with Definition 1.6.3. Therefore there exists a subsequence $\{x_k\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ converging to $x\varepsilon X$. Also, there exists a sequence $\{z_k\}_{k=1}^{\infty}$ in X such that $z_k\varepsilon G(x_k)$ for each k and

$$\lim_{k\to\infty} ||x_k-z_k|| = 0.$$

Thus, the sequence $\{z_k\}_{k=1}^{\infty}$ also converges to x and since G is closed

$$x \varepsilon G(x)$$

by Lemma 1.5.11.

Theorem 2.2.4

Suppose that X is a Banach space and $G:X \to \underline{Kp}(X)$ is a closed g-contractive multifunction. If G(X) is bounded then G has a fixed point.

Proof By Lemma 2.2.2 G is δ_X-condensing.

Making use of the point compactness of G construct the following sequence in X. Choose $x_o \varepsilon X$ and let $x_1 \varepsilon G(x_o)$ be such that

$$||x_o - x_1|| = d(x_o, G(x_o)).$$

For n>1 let $x_n \varepsilon G(x_{n-1})$ be such that

$$||x_{n-1} - x_n|| = d(x_{n-1}, G(x_{n-1})).$$

Then by Lemma 2.2.3 it suffices to show that $\lim_{n\to\infty} d(x_n, G(x_n)) = 0$. Now, for n>1

$$\begin{aligned} d(x_n, G(x_n)) &\le d_H(G(x_{n-1}), G(x_n)) \\ &\le g(||x_{n-1} - x_n||) \\ &= g\Big(d(x_{n-1}, G(x_{n-1}))\Big) \\ &\le d(x_{n-1}, G(x_{n-1})). \end{aligned}$$

Let $s_n = d(x_n, G(x_n))$ for $n = 1,2,\cdots$. Then $\{s_n\}_{n=1}^{\infty}$ is a non-negative decreasing sequence of real numbers and hence converges to some $s\varepsilon[0,\infty)$.

Since $s_n \le g(s_{n-1})$ and g is continuous on the right $s \le g(s)$. However, since $g(s) < s$ for $s>0$ it must be that $s = 0$. That is,

$$\lim_{n\to\infty} d(x_n, G(x_n)) = 0.$$

2.3 The Measurability of Multifunctions

This section is devoted to a discussion of some of the measurability properties of multifunctions that will be necessary in the application of the fixed point theorem of the preceding section to the multifunction 2.1.6.

In the following T will denote a locally compact space with a positive measure μ acting on a σ-algebra Ω of subsets of T and Y will denote a separable Banach space.

Lemma 2.3.1

If G and H are μ-measurable multifunctions on T into $\underline{Kp}(Y)$ then G+H is a μ-measurable multifunction on T into $\underline{Kp}(Y)$.

Proof Certainly, G+H is point nonempty. To verify that G+H is point compact it suffices to show that if $A,B \in \underline{Kp}(Y)$ then so is A+B.

Therefore, consider $A,B \in \underline{Kp}(Y)$. Then $A \times B$ is a compact subset of $Y \times Y$ with respect to the product topology. Since addition is continuous from $Y \times Y$ into Y the set A+B is the continuous image of a compact set and hence is itself compact.

Thus, $G+H(t) = G(t) + H(t) \in \underline{Kp}(Y)$ for every $t \in T$. The multifunction G+H is μ-measurable on T by a result of Castaing [5].

Lemma 2.3.2

Suppose μ is a complete measure and G and H are μ-measurable multifunctions on T into 2^Y such that G(t) and H(t) are both in $\underline{Kp}(Y)$ μ-a.e. on T. Then G+H is a μ-measurable multifunction on T with

$$G+H(t) \in \underline{Kp}(Y) \ \mu\text{-a.e. on } T.$$

Proof Let

$$E = \{t \in T \mid \text{one of } G(t) \text{ or } H(t) \text{ is not in } \underline{Kp}(Y)\}.$$

Then, define multifunctions G_E and H_E on T into $\underline{Kp}(Y)$ by

$$G_E(t) = \begin{cases} G(t), & \text{if } t \in T | E \\ \{\theta\}, & \text{if } t \in E \end{cases}$$

and

$$H_E(t) = \begin{cases} H(t), & \text{if } t \in T | E \\ \{\theta\}, & \text{if } t \in E. \end{cases}$$

Each of G_E and H_E are μ-measurable for consider A a closed subset of Y. Then

$$G_E^-(A) = \{t\varepsilon T : G_E(t) \cap A \neq \phi\}$$
$$= \begin{cases} G^-(A)|E, \text{ if } \theta \notin A \\ G^-(A) \cup E, \text{ if } \theta \varepsilon A. \end{cases}$$

Since μ is complete G_E is a μ-measurable multifunction on T since G is in accordance with Definition 1.5.17. The argument for the μ-measurability of H_E is analogous.

By Lemma 2.3.1 the multifunction G_E+H_E is μ-measurable on T into $\underline{Kp}(Y)$. Given A a closed subset of Y let

$$E_A = [(G+H)^-(A)] \cap E.$$

Again, since μ is complete E_A is a μ-null set. Also,

$$(G+H)^-(A) = \{t\varepsilon T | [G+H(t)] \cap A \neq \phi\}$$
$$= \begin{cases} [(G_E+H_E)\ (A)] \cup E_A, \text{ if } \theta \notin A \\ \{[(G_E+H_E)\ (A)] | E\} \cup E_A, \text{ if } \theta \varepsilon A. \end{cases}$$

Thus, G+H is μ-measurable on T since G_E+H_E is.

Finally, if $t\varepsilon T|E$ then $G+H(t)\ \varepsilon\ \underline{Kp}(Y)$ and hence $G+H(t)\ \varepsilon\ \underline{Kp}(Y)$ μ-a.e. on T.

<u>Lemma 2.3.3</u>

If $H:T \to \underline{Kp}(Y)$ is a μ-measurable multifunction then there exists a μ-measurable selector, h, for H on T such that for every $t\varepsilon T$

2.3.4 $$h(t)\ \varepsilon\ \{v\varepsilon H(t): ||v|| = d(\theta,H(t))\}.$$

<u>Proof</u> Let $\{u_n\}_{n=1}^{\infty}$ be a sequence dense in Y with $u_1 = \theta$. Using the point compactness of H construct the following sequence of multifunctions on T. Let

$$H_1(t) = \{v\varepsilon H(t): ||v-u_1|| = d(u_1,H(t))\}$$

and for n>1 let

$$H_n(t) = \{v\varepsilon H_{n-1}(t):||v-u_n|| = d(u_n,H_{n-1}(t))\}.$$

Certainly, H_n is point closed for each n. Thus, since H is point compact and for all n, $H_n(t) \subset H(t)$ for every $t\varepsilon T$, the multifunction H_n is point compact for every n. Also, by a result of Castaing [5], H_n is μ-measurable on T for every n.

Since for every $t\varepsilon T$

$$H_1(t) \supset H_2(t) \supset \cdots,$$

the multifunction $h = \bigcap_{n=1}^{\infty} H_n$ is point nonempty by Lemma 1.4.8 and is μ-measurable on T by another result of Castaing [5].

It is not difficult to show that h is single-valued on T, for consider $t\varepsilon T$ and suppose $y,z \varepsilon h(t)$ with $y \neq z$. Then there exists $\varepsilon>0$ such that $B(y,\varepsilon) \cap B(z,\varepsilon) = \phi$. There also exists a positive integer j such that $u_j \varepsilon B(y,\varepsilon)$. However, since $y,z \varepsilon H_j(t)$,

$$||y-u_j|| = ||z-u_j||,$$

a contradiction. Thus, $y = z$ and h is single-valued.

Therefore, by Definitions 1.5.2 ii) and 1.5.17 and Lemma 1.2.3, h is a μ-measurable selector for H on T safisfying 2.3.4.

Lemma 2.3.5

If $H:T \to 2^Y$ is a μ-measurable multifunction such that $H(t) \varepsilon \underline{Kp}(Y)$ μ-a.e. on T then there exists a μ-measurable μ-selector, h, for H on T such that

$$h(t) \varepsilon \{v\varepsilon H(t):||v|| = d(\theta,H(t))\} \ \mu\text{-a.e. on T}.$$

Proof Let

$$E = \{t\varepsilon T:H(t) \notin \underline{Kp}(Y)\}.$$

Then, define a multifunction $H_E:T \to \underline{Kp}(Y)$ by

$$H_E(t) = \begin{cases} H(t), & \text{if } t\varepsilon T|E \\ \{\theta\}, & \text{if } t\varepsilon E. \end{cases}$$

As argued in the proof of Lemma 2.3.2, H_E is μ-measurable on T. Thus, by Lemma 2.3.3 there exists a μ-measurable selector, h, for H_E on T satisfying 2.3.4. If $t\varepsilon T|E$ then $h(t) \varepsilon H_E(t) = H(t)$ and

$$h(t) \varepsilon \{v\varepsilon H(t):||v|| = d(\theta,H(t))\}.$$

Since E is a μ-null set the result follows.

Lemma 2.3.6

If $G:T \to 2^Y$ is a μ-measurable, μ-integrably bounded multifunction such that $G(t) \varepsilon \underline{Kp}(Y)$ μ-a.e. on T then G is integrable on T.

Proof This is an immediate consequence of the preceding lemma and Theorem 1.2.6.

2.4 A Solution to the Initial-value Problem

Certain contractive conditions on the kernel F of the initial-value problem 2.1.4 which guarantee a fixed point for the corresponding multifunction Γ and hence a solution to the initial-value problem are ascertained here.

In the following X will denote a separable reflexive Banach space and F, J and λ will be as in Section 2.1. Any reference to measurability on the real line will be to Lebesgue measurability.

Theorem 2.4.1

Suppose g is a function as in Definition 2.2.1, $x_o \varepsilon X$ and $J_o = [0,T_o]$ with $0<T_o \le \min\{T,1\}$. If for all $x,y \varepsilon C(J_o,X)$

i) F_x is measurable and integrably bounded when restricted to J_o;

ii) $F_x(t) \varepsilon \underline{KpK}(X)$ a.e. on J_o;

iii) $\sup\{d_H(F_x(t),F_y(t)):t\varepsilon J_o\} \le g(||x-y||_C)$;

then the multifunction (Γ,x_o) is g-contractive on $C(J_o,X)$.

<u>Proof</u> If $x \in C(J_o,X)$ then by i) and Lemma 2.3.6, $\Gamma(x)$ is nonempty so that Γ maps $C(J_o,X)$ into $\underline{KpK}(C(J_o,X))$ by Lemma 2.1.7.

Consider $x,y \in C(J_o,X)$. If $z \in \Gamma(x)$ then

$$z(t) = x_o + \int_0^t f_x(s)d\lambda \text{ where } f_x \in IS(F_x,J_o),\ t \in J_o.$$

Now, define a multifunction H on J_o into 2^X by

$$H = F_y - f_x.$$

Then H is measurable on J_o and $H(t) \in \underline{Kp}(X)$ a.e. on J_o by Lemma 2.3.2. Therefore, by Lemma 1.2.3 and Lemma 2.3.5 there exists $h \in MS(H,J_o)$ such that

$$h(t) \in \{v \in H(t) : ||v|| = d(\theta,H(t))\} \text{ a.e. on } J_o.$$

Since both h and f_x are measurable on J_o

$$h = f_y - f_x$$

where $f_y \in MS(F_y,J_o)$. Recalling how H was defined and making use of iii) it is clear that

$$\begin{aligned} ||f_y(t) - f_x(t)|| &= d(f_x(t),F_y(t)) \\ &\leq d_H(F_y(t),F_x(t)) \\ &\leq g(||x-y||_C) \text{ a.e. on } J_o. \end{aligned}$$

As a result of i) and Lemma 2.3.6 the function w defined on J_o by

$$w(t) = x_o + \int_0^t f_y(s)d\lambda$$

is in $\Gamma(y)$. Thus,

$$\begin{aligned} ||z-w||_C &= \sup \{||\int_0^t [f_y(s) - f_x(s)]d\lambda|| : t \in J_o\} \\ &\leq \int_0^{T_o} ||f_y(s) - f_x(s)||d\lambda \end{aligned}$$

$$\le \int_0^{T_0} g(||x-y||_C)d\lambda$$

$$\le g(||x-y||_C)$$

by Lemma 1.2.12 and Lemma 1.2.19. That is,

$$\sup\{d(z,\Gamma(y)) : z\varepsilon\Gamma(x)\} \le g(||x-y||_C).$$

Reversing the roles of x and y gives the inequality

$$\sup\{d(w,\Gamma(x)) : w\varepsilon\Gamma(y)\} \le g(||x-y||_C).$$

Hence,

$$d_H(\Gamma(x),\Gamma(y)) \le g(||x-y||_C)$$

and Γ is g-contractive on $C(J_o,X)$.

Theorem 2.4.2

Suppose that $x_o \varepsilon X$, $J_o = [0,T_o]$ with $0<T_o\le\min\{T,1\}$ and for all x,y in $C(J_o,X)$

i) F_x is measurable and integrably bounded when restricted to J_o;

ii) $F_x(t) \;\varepsilon\; \underline{KpK}(X)$ a.e. on J_o;

iii) $\sup\{d_H(F_x(t),F_y(t)) : t\varepsilon J_o\} \le g(||x-y||_C)$.

If $F(J_o xX)$ is bounded then the multifunction (Γ,x_o) has a fixed point in $C(J_o,X)$ which is a solution to the initial-value problem 2.1.4.

Proof According to Theorem 2.4.1 and its proof Γ is a g-contractive multifunction on $C(J_o,X)$ into $\underline{KpK}(C(J_o,X))$.

It is not difficult to show that Γ is a closed multifunction. Indeed, suppose that $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are sequences in $C(J_o,X)$ converging to x and y respectively with $y_n \;\varepsilon\; \Gamma(x_n)$ for all n. Then, given $\varepsilon>0$ there exists an integer N such that for n>N

$$||y-y_n||_C < \varepsilon/2 \text{ and } ||x-x_n||_C < \varepsilon/2.$$

Thus, for $n\ge N$

$$d(y,\Gamma(x)) \le ||y-y_n||_C + d(y_n,\Gamma(x))$$

$$\leq ||y-y_n||_C + d_H(\Gamma(x_n),\Gamma(x))$$
$$\leq ||y-y_n||_C + g(||x-x_n||_C)$$
$$< \varepsilon.$$

Since according to Lemma 2.1.7, $\Gamma(x)$ is a closed subset of $C(J_o,X)$, $y \in \Gamma(x)$ and by Lemma 1.5.11, Γ is a closed multifunction.

As a result of the hypothesis that $F(J_o xX)$ is bounded and Lemma 1.2.12, $\Gamma(C(J_o,X))$ is bounded in $C(J_o,X)$. Therefore, by Theorem 2.2.4 the multifunction (Γ,x_o) has a fixed point and by Lemma 2.1.5 it is a solution to the initial-value problem 2.1.4.

CHAPTER III

3.1 A Fixed Point Theorem in an Order Interval

In this section a multifunction fixed point theorem is obtained by requiring that a condensing multifunction satisfy a monotone condition on an order interval of a ncm space.

For the duration of this section $(X,\leq')$ will denote a Banach space partially ordered by a cone K and $\underset{l}{<}$ and $\underset{r}{<}$ will respectively denote the induced left and right set relations of Definitions 1.3.9 and 1.3.10.

Lemma 3.1.1

If $\{x_n\}_{n=1}^{\infty}$ is a monotone sequence in a compact subset of X then it converges.

Proof Suppose that $\{x_n\}_{n=1}^{\infty}$ is an increasing sequence in a compact subset of X. Then there exists a subsequence $\{x_{k(n)}\}_{n=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ converging to $x \varepsilon X$.

Further, suppose that $\{x_{i(n)}\}_{n=1}^{\infty}$ is a subsequence of $\{x_n\}_{n=1}^{\infty}$ converging to $y \varepsilon X$. Then there exists a subsequence $\{x_{j(n)}\}_{n=1}^{\infty}$ of $\{x_{i(n)}\}_{n=1}^{\infty}$ such that $k(n) \leq j(n)$ for all n. Since K is a closed set

$$y-x = \lim_{n\to\infty} (x_{j(n)}-x_{k(n)}) \; \varepsilon \; K.$$

That is, $x\leq' y$. There also exists a subsequence $\{x_{p(n)}\}_{n=1}^{\infty}$ of $\{x_{k(n)}\}_{n=1}^{\infty}$ such that $i(n) \leq p(n)$ for all n. Again, since K is closed

$$x-y = \lim_{n\to\infty} (x_{p(n)}-x_{i(n)}) \; \varepsilon \; K.$$

Thus, $y\leq' x$ and $x = y$.

It is immediate that $\{x_n\}_{n=1}^{\infty}$ converges to x, for suppose not. Then there exists $\varepsilon>0$ and a subsequence $\{x_k\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ such that $x_k \notin B(x,\varepsilon)$ for all k. However, $\{x_k\}_{k=1}^{\infty}$ being in a compact subset of X has a convergent subsequence which by the above must converge to x, a contradiction.

The proof for the case when $\{x_n\}_{n=1}^{\infty}$ is decreasing is analogous.

Lemma 3.1.2

An order interval $[u,v]$ in $(X,\leq')$ is closed and convex.

Proof Suppose that $[u,v]$ is an order interval in X and $x \in \overline{[u,v]}$. Then there exists a sequence $\{x_n\}_{n=1}^{\infty}$ in $[u,v]$ converging to x. Since $v-x_n \in K$ for every n and K is a closed set

$$v-x = \lim_{n\to\infty} v-x_n \in K$$

so that $x\leq' v$. Similarly, since $x_n-u \in K$ for every n,

$$x-u = \lim_{n\to\infty} x_n-u \in K.$$

That is, $u\leq' x$, hence $x \in [u,v]$ and $[u,v]$ is closed.

To verify that $[u,v]$ is convex consider $x,y \in [u,v]$ and $\eta \in [0,1]$. Then by Definition 1.3.3 i)

$$\eta u \leq' \eta x \leq' \eta v$$

and

$$(1-\eta)u \leq' (1-\eta)y \leq' (1-\eta)v.$$

Thus, by Definition 1.3.3 iii)

$$u \leq' \eta x + (1-\eta)y \leq' v$$

and $[u,v]$ is convex.

Theorem 3.1.3

Suppose that (X,Λ,T,γ) is a ncm space with γ a monotonic, semi-additive, non-singular mnc and $[u,v]$ is an order interval in X such that $[u,v]$ and all its subsets are in Λ. If $G:[u,v] \to 2^X$ is a left-set (respectively, right-set) monotone, closed, γ-condensing multifunction such that

$$\{u\} \underset{l}{<} (\text{respectively, } \underset{r}{<})\ G(u)$$

and

$$G(v) \underset{l}{<} (\text{respectively, } \underset{r}{<})\ \{v\}$$

then G has a fixed point in $[u,v]$.

<u>Proof</u> Suppose that G is left-set monotone with

$$\{u\} \underset{l}{<} G(u) \text{ and } G(v) \underset{l}{<} \{v\}.$$

Using the left-set monotonicity of G construct the following increasing sequence in X. Let $u_o = u$ and for $n \geq 1$ select $u_n \in G(u_{n-1})$ such that $u_{n-1} \leq' u_n$. It is clear that $\{u_n\}_{n=1}^{\infty}$ is in $[u,v]$. Indeed, by induction, u_o is in $[u,v]$ by definition. Assume $u_n \in [u,v]$ for $n \geq 0$. Then since $u_n \leq' v$ and G is left-set monotone

$$G(u_n) \underset{l}{<} G(v) \underset{l}{<} \{v\}.$$

Thus, since $u_{n+1} \in G(u_n)$

$$u = u_o \leq' u_{n+1} \leq' v.$$

Let

$$A = \bigcup_{n=0}^{\infty} \{u_n\} \text{ and } B = \bigcup_{n=1}^{\infty} \{u_n\}.$$

Then, $B \subset G(A)$ and $A = B \cup \{u_o\}$. Since γ is monotonic, semi-additive and non-singular

$$\gamma(A) = \gamma(B \cup \{u_o\}) = \gamma(B) \leq \gamma(G(A)).$$

Thus, since G is γ-condensing A is precompact and by Lemma 3.1.1 and Lemma 3.1.2 the sequence $\{u_n\}_{n=1}^{\infty}$ converges to some $y \in [u,v]$. Since $u_n \in G(u_{n-1})$ for $n \geq 1$ and G is a closed multifunction $y \in G(y)$ by Lemma 1.5.11.

Suppose that G is right-set monotone with

$$\{u\} \underset{r}{<} G(u) \text{ and } G(v) \underset{r}{<} \{v\}.$$

Using the right-set monotonicity of G construct the following decreasing sequence in X. Let $v_o = v$ and for $n \geq 1$ select $v_n \in G(v_{n-1})$ such that $v_n \leq' v_{n-1}$. The sequence $\{v_n\}_{n=1}^{\infty}$ is evidently in $[u,v]$. Indeed, by induction, v_o is in $[u,v]$ by definition. Assume that $v_n \in [u,v]$ for $n \geq 0$. Since $u \leq' v_n$ and G is right-set monotone

$$\{u\} \underset{r}{<} G(u) \underset{r}{<} G(v_n)$$

Thus, since $v_{n+1} \in G(v_n)$,

$$u \leq' v_{n+1} \leq' v_o = v.$$

Let

$$C = \bigcup_{n=0}^{\infty} \{v_n\} \text{ and } D = \bigcup_{n=1}^{\infty} \{v_n\}.$$

Then, $D \subset G(C)$ and $C = D \cup \{v_o\}$. Again, since γ is monotonic, semi-additive and non-singular

$$\gamma(C) = \gamma(D \cup \{v_o\}) = \gamma(D) \leq \gamma(G(C)).$$

Thus, since G is γ-condensing, C is precompact and by Lemmas 3.1.1 and 3.1.2 the sequence $\{v_n\}_{n=1}^{\infty}$ converges to some $z \in [u,v]$. Therefore, since $v_n \in G(v_{n-1})$ for $n \geq 1$ and G is a closed multifunction $z \in G(z)$ by Lemma 1.5.11 and the result follows.

3.2 A Solution to the Initial-value Problem in an Order Interval

The Fixed Point Theorem 3.1.3 is shown to be applicable in determining a solution to the initial-value problem 2.1.4.

Unless otherwise specified $(X, \leq')$ will denote a separable Banach space partially ordered by a cone K and $\underset{l}{<}$ and $\underset{r}{<}$ will respectively denote the induced left and right set relations of Definitions 1.3.9 and 1.3.10. Also, J and λ will be as in Section 2.1 and F will be a multifunction JxX into 2^X. Finally, the partial ordering on $C(J,X)$ induced by the cone

$$K_C = \{x \in C(J,X) : x(t) \in K \text{ for every } t \in J\}$$

will be denoted by $\leq_C$. The left and right set relations of Definitions 1.3.9 and 1.3.10 in $(C(J,X), \leq_C)$ will be denoted by $\underset{lC}{<}$ and $\underset{rC}{<}$ respectively.

Lemma 3.2.1

Suppose that $[u,v]$ is an order interval in $(C(J,X), \leq_C)$ and $x_o \in X$. If for all $x,y \in [u,v]$

i) F_x is measurable and integrably bounded on J;

ii) $F_x(t) \in \underline{Kp}(X)$ a.e. on J;

iii) $x \leq_C y$ implies that $F_x(t) \underset{l}{<}$ (respectively, $\underset{r}{<}$) $F_y(t)$ a.e. on J;

then the multifunction (Γ, x_o) is left-set (respectively, right-set) monotone on $[u,v]$.

Proof First, suppose that if $x,y \in [u,v]$ with $x \leq_C y$ then

$$F_x(t) \underset{l}{<} F_y(t) \text{ a.e. on } J.$$

Consider $x,y \in [u,v]$ with $x \leq_C y$ and suppose that $z \in \Gamma(x)$. Then

$$z(t) = x_o + \int_0^t f_x(s)d\lambda \text{ where } f_x \in IS(F_x,J).$$

Let C be the constant multifunction defined on J by

$$C(t) = K$$

Then by Lemma 2.3.2, Definition 1.3.9 and the fact that K is closed

$$[F_y(t) - f_x(t)] \cap C(t) \in \underline{Kp}(X) \text{ a.e. on } J$$

and the multifunction $F_y - f_x$ is measurable on J. In fact, the multifunction $[F_y - f_x] \cap C$ is measurable on J for consider $A \in Kl(X)$. Then, since K is closed, $A \cap K \in Kl(X)$ and

3.2.2 $([F_y - f_x] \cap C)^-(A) = \{t \in J \mid (F_y(t) - f_x(t)) \cap (K \cap A) \neq \phi\}$.

is a measurable set since $F_y - f_x$ is a measurable function on J. Applying Lemma 2.3.5 to the multifunction $[F_y - f_x] \cap C$ and recalling that f_x is integrable, hence measurable on J, there exists $f_y \in MS(F_y,J)$ such that

$$f_y(t) - f_x(t) \in K \text{ a.e. on } J.$$

Since F_y is integrably bounded on J and by Lemma 2.3.6 the function w defined on J by

$$w(t) = x_o + \int_0^t f_y(s)d\lambda$$

is in $\Gamma(y)$. To verify that $\Gamma(x) \underset{1C}{<} \Gamma(y)$ it suffices, by Definition 1.3.9, to show that $z \leq_C w$.

Let

$$E = \{s \varepsilon J | f_y(s) - f_x(s) \notin K\}.$$

Then E is a λ-null set, and by Lemma 1.2.16 and the fact that K is closed and convex, for every $t \varepsilon J$

$$w(t) - z(t) = \int_0^t [f_y(s) - f_x(s)] d\lambda$$

$$\varepsilon \ t\overline{co}(\{f_y(s) - f_x(s) : s \varepsilon [0,t] | E\})$$

$$\subset K$$

Therefore, $z(t) \leq' w(t)$ for every $t \varepsilon J$, hence $z \leq_C w$ and $\Gamma(x) \underset{1C}{<} \Gamma(y)$.

Now, suppose that if $x,y \ \varepsilon \ [u,v]$ with $x \leq_C y$ then

$$F_x(t) \underset{r}{<} F_y(t) \text{ a.e. on } J.$$

Consider $x,y \ \varepsilon \ [u,v]$ with $x \leq_C y$ and suppose that $\hat{w} \ \varepsilon \ \Gamma(y)$. Then

$$\hat{w}(t) = x_o + \int_0^t f_{\hat{y}}(s) d\lambda \text{ where } f_{\hat{y}} \varepsilon IS(F_y, J).$$

Applying the argument given in the preceding case to the multifunction $[f_{\hat{y}} - F_x] \cap C$, there exists $\hat{z} \ \varepsilon \ C(J,X)$ such that $\hat{z} \leq_C \hat{w}$. Thus, by Definition 1.3.10, $\Gamma(x) \underset{rC}{<} \Gamma(y)$ and the result follows.

<u>Lemma 3.2.3</u>

Suppose that $[u,v]$ is an order interval in $(C(J,X), \leq_C)$ such that u and v are continuously differentiable on J and $x_o \varepsilon X$ with $u(0) \leq' x_o \leq' v(0)$. If the multifunction F satisfies

i) F_u and F_v are measurable on J and $F_u(t)$ and $F_v(t)$ are each in <u>Kp</u>(X) a.e. on J;

ii) $\{\dot{u}(t)\} \underset{1}{<} F_u(t) \underset{1}{<} F_v(t) \underset{1}{<} \{\dot{v}(t)\}$ (respectively, $\{\dot{u}(t)\} \underset{r}{<} F_u(t) \underset{r}{<} F_v(t) \underset{r}{<} \{\dot{v}(t)\}$) a.e. on J;

iii) F_u (respectively, F_v) is integrably bounded on J;

then $\{u\} \underset{lC}{<}$ (respectively, $\underset{rC}{<}$) $\Gamma(u)$ and $\Gamma(v) \underset{lC}{<}$ (respectively, $\underset{rC}{<}$) $\{v\}$.

<u>Proof</u> First, suppose that the left set conditions hold. Again, let C be the constant multifunction defined on J by

$$C(t) = K.$$

Then $[F_u(t) - \dot{u}(t)] \cap C(t) \in \underline{Kp}(X)$ a.e. on J and by an argument similar to that given in 3.2.2 the multifunction $[F_u - \dot{u}] \cap C$ is measurable on J. Thus, by Lemma 2.3.5 there exists $f_u \in MS(F_u, J)$ such that

$$f_u(t) - \dot{u}(t) \in K \text{ a.e. on } J.$$

Since F_u is integrably bounded on J and by Lemma 2.3.6 the function z defined on J by

$$z(t) = x_o + \int_0^t f_u(s) d\lambda$$

is in $\Gamma(u)$.

Let

$$E_o = \{s \in J \mid f_u(s) - \dot{u}(s) \notin K\}.$$

Then, by Lemma 1.2.16 and the facts that E_o is a λ-null set and K is a cone, for every $t \in J$

$$\begin{aligned} \int_0^t f_u(s) d\lambda - \int_0^t \dot{u}(s) d\lambda &= \int_0^t [f_u(s) - \dot{u}(s)] d\lambda \\ &\in t\overline{co}(\{f_u(s) - \dot{u}(s) \mid s \in [0,t] \mid E_o\}) \\ &\subset K \end{aligned}$$

so that

$$x_o + \int_0^t f_u(s) d\lambda - \int_0^t \dot{u}(s) d\lambda \in K + x_o$$

and hence by Lemma 1.2.24

$$x_o + \int_0^t f_u(s) d\lambda - [u(t) - u(0)] \in K + x_o$$

Thus, according to Definition 1.3.1

$$x_o + \int_0^t f_u(s)d\lambda - u(t) \;\varepsilon\; K + [x_o - u(0)] \subset K.$$

That is, $u(t) \leq' z(t)$ for all $t\varepsilon J$, hence $u \leq_C z$ and $\{u\} \underset{1C}{<} \Gamma(u)$.

Suppose that $w \;\varepsilon\; \Gamma(v)$. Then

$$w(t) = x_o + \int_0^t f_v(s)d\lambda \text{ where } f_v \varepsilon IS(F_v, J).$$

Since $F_v(t) \underset{1}{<} \{\dot{v}(t)\}$ a.e. on J, $\dot{v}(t) - f_v(t) \;\varepsilon\; K$ a.e. on J. Let

$$E_1 = \{s\varepsilon J | \dot{v}(s) - f_v(s) \notin K\}.$$

Again, by Lemma 1.2.16 and the fact that E_1 is a λ-null set, for every $t\varepsilon J$

$$\int_0^t \dot{v}(s)d\lambda - \int_0^t f_v(s)d\lambda = \int_0^t [\dot{v}(s) - f_v(s)]d\lambda$$

$$\varepsilon\; t\overline{co}(\{\dot{v}(s) - f_v(s) : s\varepsilon[0,t] | E_1\})$$

$$\subset K$$

so that

$$\int_0^t \dot{v}(s)d\lambda - [x_o + \int_0^t f_v(s)d\lambda] \;\varepsilon\; K - x_o$$

and hence by Lemma 1.2.24

$$[v(t) - v(0)] - [x_o + \int_0^t f_v(s)d\lambda] \;\varepsilon\; K - x_o.$$

Thus,

$$v(t) - [x_o + \int_0^t f_v(s)d\lambda] \;\varepsilon\; K + [v(0) - x_o] \subset K.$$

That is, $w(t) \leq' v(t)$ for all $t\varepsilon J$, hence $w \leq_C v$ and $\Gamma(v) \underset{1C}{<} \{v\}$.

The case involving the right set conditions is proved using an analogous argument.

<u>Lemma 3.2.4</u>

Suppose that X is a separable reflexive Banach space, $x_o \varepsilon X$ and D is a closed subset of $C(J,X)$. If for all $x\varepsilon D$ the multifunction $F:J\times X \to 2^X$ satisfies

i) $F_x(t) \varepsilon K1K(X)$ a.e. on J;

ii) for a.e. $t\varepsilon J$ the multifunction $F_t:X \to 2^X$, when restricted to $x(J)$, is usc;

iii) there exists $p \varepsilon L(J,[0,\infty))$, independent of x, such that $||z|| \le p(t)$ a.e. on J for all $z \varepsilon F_x(t)$;

then the multifunction (Γ,x_o) is closed on D.

<u>Proof</u> Suppose that $\{x_n\}_{n=1}^{\infty}$ is a sequence in D converging to x and $\{y_n\}_{n=1}^{\infty}$ is a sequence in $C(J,X)$ converging to y such that

$$y_n \varepsilon \Gamma(x_n).$$

Then, for every n

$$y_n(t) = x_o + \int_0^t f_n(s)d\lambda \text{ where } f_n \varepsilon IS(F_{x_n},J).$$

According to iii), there exists an integral bound for the set

$$\{f_n | n = 1,2,\cdots\}.$$

Thus, as argued in the proof of Lemma 2.1.7, there exists a subsequence $\{f_k\}_{k=1}^{\infty}$ of $\{f_n\}_{n=1}^{\infty}$ that converges weakly to some $f \varepsilon L(J,X)$. Then, by Lemma 1.2.14, for every $t\varepsilon J$

$$\lim_{k\to\infty} \int_0^t f_k(s)d\lambda = \int_0^t f(s)d\lambda,$$

hence

$$y(t) = x_o + \int_0^t f(s)d\lambda, \ t\varepsilon J.$$

Again, as argued in the proof of Lemma 2.1.7, for every k there exists a finite set of non-negative numbers $\{a_i^k\}_{i=1}^{m(k)}$ such that $\sum_{i=1}^{m(k)} a_i^k = 1$ and

$$\lim_{k\to\infty} \sum_{i=1}^{m(k)} a_i^k f_{k+i} = f \text{ in } L(J,X).$$

For every $t\varepsilon J$ let

$$C(t) = \bigcap_{k=1}^{\infty} \overline{co}(\bigcup_{i=k}^{\infty} F_{x_i}(t)).$$

:or a.e. $t\varepsilon J$

$$.t \quad \varepsilon \bigcap_{k=1}^{\infty} \overline{co}(\bigcup_{i=k}^{\infty} f_i(t)) \subset C(t).$$

$$E = \{t\varepsilon J | f(t) \notin C(t)\},$$

:estricted to $x(J)$ is not usc},

:) $\notin F_{x_n}(t)$ for some integer n},

$$E = \bigcup_{n=1}^{3} E_n.$$

is usc on $x(J)$, given $\varepsilon>0$, there exists $\gamma>0$ such that if $y\varepsilon X$ with $||y-x(t)|| < \delta$ then

$$F_t(y) \subset B(F_x(t),\varepsilon/2).$$

Also, there exists an integer N such that for $n \geq N$

$$||x_n(t) - x(t)|| < \delta.$$

Thus, given $k \geq N$ there exist finite sets $\{\eta_i\}_{i=1}^{\infty}$ of non-negative numbers and $\{z_i\}_{i=1}^{\infty}$ of elements in X with $\sum_{i=1}^{m} \eta_i = 1$ and for each i,

$$z_i \ \varepsilon \ F_{x_j}(t) \text{ for some } j \geq k,$$

such that

$$||f(t) - \sum_{i=1}^{m} \eta_i z_i|| < \varepsilon/2.$$

Let $w_i \ \varepsilon \ F_x(t)$ be such that $||z_i - w_i|| < \varepsilon/2$ for $1 \leq i \leq m$. Then

$$||\sum_{i=1}^{m} \eta_i z_i - \sum_{i=1}^{m} \eta_i w_i|| \leq \sum_{i=1}^{m} \eta_i ||z_i - w_i|| < \varepsilon/2.$$

Thus, since $F_x(t)$ is closed and convex, $f(t) \ \varepsilon \ F_x(t)$. That is, since

E is a λ-null set, $f(t) \in F_x(t)$ a.e. on J, hence $y \in \Gamma(x)$ and Γ is a closed multifunction on D as a result of Lemma 1.5.11.

Remark Suppose that X is partially ordered by a normal cone K and F maps $J \times X$ into 2^K in the preceding lemma. Further, suppose that $D = [u,v]$ is an order interval in $(C(J,X),\leq_C)$ such that F_v is integrably bounded by p on J and the left set relation of condition iii) of Lemma 3.2.1 is satisfied. Then, it is immediate that condition iii) of the preceding lemma is also satisfied. Indeed, if $x \in D$ then for a.e. $t \in J$ there exists a $v_z \in F_v(t)$ for every $z \in F_x(t)$ such that $z \leq' v_z$. As a result of Lemma 1.3.8 there exists a real number κ such that for a.e. $t \in J$, $||z|| \leq \kappa||p(t)||$ for every $z \in F_x(t)$. Since κp is integrable on J condition iii) of the preceding lemma follows.

Lemma 3.2.5

Suppose that $x_0 \in X$ and (X,Λ,T,γ) is a ncm space where X is a Banach space, Λ the set of all bounded subsets of X and γ a mnc satisfying properties i), ii), iii), vii) and viii) of Definition 1.4.3. If $F: J \times X \to 2^X$ is a k-γ-set-contraction multifunction then the multifunction (Γ,x_0) is a kT-γ_C-set-contraction in the ncm space $(X_C,\Lambda_C,T,\gamma_C)$.

Proof For every $x \in C(J,X)$ and $f_x \in IS(F_x,J)$ let

$$E_x^f = \{t \in J \mid f_x(t) \notin F_x(t)\}.$$

Consider $A \in \Lambda_C$. Then

$\gamma_C(\Gamma(A)) = \gamma\big((\Gamma(A))_J\big)$

by the definition in Example 1.4.9,

$$= \gamma(\bigcup\{x_0 + \int_0^t f_x(s)d\lambda : x \in A,\ f_x \in IS(F_x,J),\ t \in J\})$$

by definition,

$$\leq \gamma\Big(x_0 + \bigcup\big\{t\overline{co}(\{f_x(s) : s \in [0,t] \mid E_x^f\}) : x \in A,\ f_x \in IS(F_x,J),\ t \in J\big\}\Big)$$

since γ is monotonic and by Lemma 1.2.16,

$$\leq \gamma\Big(\bigcup\big\{co\big(T\overline{co}(\{f_x(s):s\varepsilon J|E_x^f\}) \cup \{0\}\big):x\varepsilon A,\ f_x\varepsilon IS(F_x,J)\big\}\Big)$$

since γ is monotonic, invariant under shifts and non-singular and by Lemma 1.1.1,

$$\leq \gamma\Big(\bigcup\big\{co\big(T\overline{co}(\bigcup\{F_x(s):s\varepsilon J\}) \cup \{0\}\big):x\varepsilon A\big\}\Big)$$

since γ is monotonic,

$$= \gamma\Big(co\big(T\overline{co}(F(J\times A_J)) \cup \{0\}\big)\Big)$$

by definition,

$$= \gamma\big(T\overline{co}(F(J\times A_J))\big)$$

since γ is semi-additive and non-singular and by Definition 1.4.2,

$$= T\gamma(F(J\times A_J))$$

since γ is semi-homogeneous and by Definition 1.4.2,

$$\leq kT\gamma(A_J)$$

since F is a k-γ-set-contraction,

$$= kT\gamma_C(A)$$

by definition.

Thus, the result follows in accordance with Definition 1.6.4.

In the following theorem (X,Λ,T,γ) will be a ncm space with X a separable reflexive Banach space partially ordered by a cone K, Λ the set of all bounded subsets of X and γ a mnc satisfying properties i), ii), iii), iv), vii) and viii) of Definition 1.4.3.

<u>Theorem 3.2.6</u>

Suppose that $J_o = [0,T_o]$ where $T_o \varepsilon (0,T]$ and $x_o \varepsilon X$. Further, suppose that $[u,v]$ is an order interval in Λ_C, the bounded subsets of $C(J_o,X)$, such that u and v are continuously differentiable on J_o and $u(0) \leq' x_o \leq' v(0)$. If $B = \{x(t):x\varepsilon[u,v],\ t\varepsilon J\}$ and the multifunction $F:J\times X \to 2^X$

satisfies

i) for every $x \in [u,v]$, F_x restricted to J_o is measurable on J_o and $F_x(t) \in \underline{KpK}(X)$ a.e. on J_o;

ii) $x \leq_C y$ in $[u,v]$ implies that $F_x(t) \underset{l}{<}$ (respectively, $\underset{r}{<}$) $F_y(t)$ a.e. on J_o;

iii) $\{\dot{u}(t)\} \underset{l}{<} F_u(t) \underset{l}{<} F_v(t) \underset{l}{<} \{\dot{v}(t)\}$ (respectively, $\{\dot{u}(t)\} \underset{r}{<} F_u(t) \underset{r}{<} F_v(t) \underset{r}{<} \{\dot{v}(t)\}$) a.e. on J_o;

iv) for a.e. $t \in J_o$ the multifunction F_t restricted to $x(J_o)$ is usc for every $x \in [u,v]$;

v) F restricted to $J_o \times B$ is a k-γ-set-contraction with $kT_o<1$;

then the multifunction (Γ,x_o) has a fixed point in $[u,v]$ which is a solution to the initial-value problem 2.1.4.

<u>Proof</u> Suppose that the left set conditions of the lemma are satisfied. Since F is a k-γ-set-contraction on $J_o \times B$ there exists a positive number M that bounds $F(J_o \times B)$, that is,

$$||z|| \leq M \text{ for every } z \in F(J_o \times B).$$

Thus, by Lemma 1.2.19, the constant function defined on J_o by

$$p(t) = M$$

is an integral bound for F_x, independent of the choice of $x \in [u,v]$. Therefore, by i), ii) and Lemma 3.2.1, (Γ,x_o) is left-set monotone on $[u,v]$ and by i), iii) and Lemma 3.2.3

$$\{u\} \underset{lC}{<} \Gamma(u) \text{ and } \Gamma(v) \underset{lC}{<} \{v\}.$$

Also, by i), iv) and Lemmas 3.1.2 and 3.2.4, Γ is closed on $[u,v]$.

Now, the mnc γ_C is monotonic, semi-additive and non-singular on Λ_C by Lemmas 1.4.10, 1.4.11 and 1.4.12 respectively.

Let $\{u_n\}_{n=0}^{\infty}$ be an increasing sequence in $[u,v]$ and $A = \bigcup_{n=0}^{\infty}\{u_n\}$ as in the proof of Theorem 3.1.3. Then, $u_o = u$ and for $n \geq 1$

$$u_n(t) = x_o + \int_0^t f_{n-1}(s)d\lambda \text{ where } f_{n-1}\varepsilon IS(F_{u_{n-1}},J_o),\ t\varepsilon J_o.$$

As a result of Lemmas 1.2.10, 1.2.12 and 1.2.19, given $\varepsilon>0$ and $n\geq 1$, if $r\leq t$ in J_o with $|t-r| < \varepsilon/M$ then

$$||u_n(t) - u_n(r)|| \leq \int_r^t ||f_{n-1}(s)||d\lambda < \varepsilon/M\cdot M = \varepsilon.$$

Thus, the set A is equicontinuous on J_o. Also, consider $x \ \varepsilon \ \Gamma(A)$. Then, for some $n\geq 0$

$$x(t) = x_o + \int_0^t f_n(s)d\lambda \text{ where } f_n\varepsilon IS(F_{u_n},J_o),\ t\varepsilon J_o.$$

By an argument analogous to the preceding one $\Gamma(A)$ is equicontinuous on J_o as well.

As shown in Example 1.4.17 the sets $E \ \varepsilon \ \Lambda_C$ for which both E and $\Gamma(E)$ are equicontinuous on J_o form a closed convexity in $C(J_o,X)$ and by v) and Lemmas 1.4.13 and 1.6.5, Γ is γ_C-condensing when restricted to these sets. According to the proof of Theorem 3.1.3 this is enough to guarantee a fixed point for Γ in $[u,v]$ which is necessarily a solution to the initial-value problem 2.1.4 as a result of Lemma 2.1.5.

Suppose that the right set conditions of the lemma are satisfied now. Then, by the same reasoning as in the preceding case (Γ,x_o) is right-set monotone on $[u,v]$, $\{u\} \leq_{rC} \Gamma(u)$ and $\Gamma(v) \leq_{rC} \{v\}$, and Γ is closed on $[u,v]$. Also, if $\{v_n\}_{n=0}^{\infty}$ is a decreasing sequence and

$$C = \bigcup_{n=0}^{\infty}\{v_n\}$$

as in the proof of Theorem 3.1.3 then, as in the preceding case, C and $\Gamma(C)$ can be shown to be equicontinuous on J_o. This is enough to ensure a solution to the initial-value problem 2.1.4 again, as argued in the preceding case.

CHAPTER IV

4.1 A Fixed Point Theorem in a Compact Set

A multifunction fixed point theorem in the spirit of one given in Fitzpatrick and Petryshyn [15] is proved here.

Throughout this section Y will denote a locally convex topological vector space and D an element of $\underline{\mathrm{Kl}}(Y)$. If $G:D \to 2^Y$ is a multifunction, following Sadovskii [38], define the following transfinite sequence in 2^Y

$$K_o = \overline{co}(G(D))$$

and

$$K_\alpha = \begin{cases} \overline{co}(G(D \cap K_{\alpha-1}), & \text{if } \alpha \text{ is an ordinal of the first kind} \\ \bigcap_{\beta<\alpha} K_\beta, & \text{if } \alpha \text{ is an ordinal of the second kind.} \end{cases}$$

<u>Lemma 4.1.1</u>

If $G:D \to 2^Y$ is a multifunction then

i) $K_\alpha \subset K_\beta$ for $\beta \le \alpha$;

ii) $G(D \cap K_\alpha) \subset K_\alpha$ for all α;

iii) there exists an ordinal δ such that

$$K_\alpha = K_\delta \text{ for } \delta \le \alpha.$$

<u>Proof</u> The proof of i) and ii) is done simultaneously by transfinite induction. Suppose that $\alpha = 0$. Then i) is obvious. Since

$$G(D \cap K_o) \subset G(D) \subset \overline{co}(G(D)) = K_o$$

ii) also follows. Assume that both i) and ii) are true for all $\beta<\alpha$.

If α is an ordinal of the first kind then $\beta<\alpha$ implies that $\beta \le \alpha-1$ and hence $K_{\alpha-1} \subset K_\beta$ by the induction hypothesis for i). Since

$$K_{\alpha-1} \in \mathrm{K1K}(Y)$$

and by the induction hypothesis for ii)

$$K_\alpha = \overline{co}(G(D \cap K_{\alpha-1})) \subset K_{\alpha-1} \subset K_\beta, \ \beta<\alpha,$$

hence, i) follows. As shown in the preceding argument, $K_\alpha \subset K_{\alpha-1}$ so that

$$D \cap K_\alpha \subset D \cap K_{\alpha-1}.$$

Therefore,

$$G(D \cap K_\alpha) \subset \overline{co}(G(D \cap K_{\alpha-1})) = K_\alpha$$

and ii) follows.

If α is an ordinal of the second kind then

$$K_\alpha = \bigcap_{\beta<\alpha} K_\beta$$

so that i) follows immediately. Also,

$$G(D \cap K_\alpha) \subset \overline{co}(G(D \cap K_\beta)) \text{ for } \beta<\alpha.$$

As a result of the induction hypothesis for ii) and the fact that for all $\beta<\alpha$, $K_\beta \in \mathrm{K1K}(Y)$,

$$\overline{co}(G(D \cap K_\beta)) \subset K_\beta, \ \beta<\alpha.$$

Thus,

$$G(D \cap K_\alpha) \subset \bigcap_{\beta<\alpha} K_\beta = K_\alpha$$

and ii) results.

To see the validity of iii) note that when the cardinality of an ordinal δ is greater than the cardinality of 2^Y a repetition must occur in the transfinite sequence $\{K_\alpha\}$. However, since $\{K_\alpha\}$ is decreasing this is equivalent to

$$K_\alpha = K_\delta \text{ for } \delta \leq \alpha.$$

For the remainder of this section the set K_δ corresponding to a a multifunction $G:D \to 2^Y$ will be denoted by $K(G,D)$ or simply K when G is understood.

<u>Lemma 4.1.2</u>

If $G:D \to 2^Y$ is a multifunction then

$$\overline{co}(G(D\cap K)) = K.$$

Proof Since $K = K_\delta = K_{\delta+1}$,

$$\overline{co}(G(D\cap K)) = \overline{co}(G(D\cap K_\delta)) = K_{\delta+1} = K.$$

Definition 4.1.3

An usc multifunction $G:D \to 2^Y$ is ultimately compact provided that

$$K \in Kp(Y).$$

Theorem 4.1.4

Suppose that $G:D \to \underline{Kl}(Y)$ is an ultimately compact multifunction. If A is a nonempty precompact subset of Y such that $G(A) \subset A$ then

$$K \in \underline{Kp}(Y).$$

Proof According to the preceding definition it suffices to show that K is not empty. Following Fitzpatrick and Petryshyn [ ], define a sequence in 2^Y as follows,

$$C_o = G(\bar{A})\cap \bar{A}$$

and for $n\geq 1$

$$C_n = G(C_{n-1})\cap C_{n-1}.$$

Then by Lemma 1.5.14, C_n is compact for each n, hence by Theorem 1.4.8,

$$C = \bigcap_{n=0}^{\infty} C_n \in \underline{Kp}(Y).$$

Now, $C \subset G(C)$. Indeed, if $x \in C$ then $x \in G(C_n)\cap C_n$ for every $n\geq 0$. Therefore, there exists a sequence $\{z_n\}_{n=0}^{\infty}$ such that for all $n\geq 0$,

$$x \in G(z_n) \text{ and } z_n \in C_n \subset \bar{A}.$$

Since $\bar{A}$ is compact there exists a subsequence $\{z_k\}_{k=0}^{\infty}$ of $\{z_n\}_{n=0}^{\infty}$ converging to $z \in \bar{A}$. In accordance with Theorem 1.4.8, given $\varepsilon>0$ there exists an integer N_o such that for $k\geq N_o$

$$||z-z_k|| < \varepsilon/2 \text{ and } d_H(C,C_k) < \varepsilon/2.$$

Thus, $d(z,C) < \varepsilon$ and hence z is in the closed set C. Moreover, since

G is usc and point closed, there exists an integer N such that for $k \geq N$

$$G(z_k) \subset B(G(z),\varepsilon)$$

so that

$$x \in G(z) \subset G(C).$$

It is not difficult to show that $C \subset K_\alpha$ for every α. Indeed, by transfinite induction,

$$C \subset \overline{co}(G(C)) \subset \overline{co}(G(D)) = K_o.$$

Assume that $C \subset K_\beta$ for $\beta<\alpha$. If α is an ordinal of the first kind then

$$C \subset \overline{co}(G(C)) \subset \overline{co}(G(K_{\alpha-1})) = K_\alpha.$$

If α is an ordinal of the second kind then since $C \subset K_\beta$ for $\beta<\alpha$,

$$C \subset \bigcap_{\beta<\alpha} K_\beta = K_\alpha.$$

Thus, $C \subset K$ and K is not empty.

In addition to Y denoting a locally convex topological vector space, (Y,Λ,T,γ) will denote a ncm space for the remainder of this section.

Lemma 4.1.5

If γ is monotonic, D and all its subsets are in Λ and $G:D \to Kp(Y)$ is an usc γ-condensing multifunction then G is ultimately compact.

Proof If K is empty the result is obvious, so suppose that K is not empty.

By Lemma 4.1.2, $\overline{co}(G(D \cap K)) = K$ so that

$$D \cap K \subset \overline{co}(G(D \cap K)).$$

Since γ is monotonic

$$\gamma(D \cap K) \leq \gamma(G(D \cap K)),$$

therefore $D \cap K$ is precompact and hence compact as D and K are in

Kl(Y). As a result of G being usc and point compact $G(D\cap K)$ is compact by Lemma 1.5.13. Thus, by Lemma 1.1.3,

$$\overline{co}(G(D\cap K)) = K$$

is compact.

Theorem 4.1.6

Suppose that $D \in \underline{\text{Kl}}\text{K}(Y)$ with D and all its subsets in Λ and γ is a monotonic, non-singular, semi-additive mnc. If $G:D \to \underline{\text{Kp}}\text{K}(D)$ is an usc γ-condensing multifunction then G has a fixed point.

Proof According to Lemma 4.1.5 and a theorem of Fitzpatrick and Petryshyn [15] it suffices to show that K is nonempty.

Let $x_o \in D$ and define a sequence in 2^Y by

$$G^o(x_o) = \{x_o\}$$

and for $n \geq 1$

$$G^n(x_o) = G(G^{n-1}(x_o))$$

Also, let

$$A = \bigcup_{n=0}^{\infty} G^n(x_o).$$

Then,

$$A = G(A) \cup \{x_o\}$$

and since γ is semi-additive and non-singular

$$\gamma(A) = \gamma(G(A) \cup \{x_o\}) = \gamma(G(A)).$$

Thus, since G is γ-condensing, A is precompact and $K \in \underline{\text{Kp}}(Y)$ by Theorem 4.1.4.

4.2 A Solution to the Initial-value Problem in a Convex Set

In this section conditions on the kernel of the initial-value problem 2.1.4 enabling the application of the Fixed Point Theorem 4.1.6 are investigated.

As previously specified X will denote a Banach space. In addition, (X,Λ,T,γ) will denote a ncm space where Λ is the set of all bounded subsets of X and γ satisfies all the conditions of Definition 1.4.3 except possibly vi). Also, J, F and λ will be as in Section 2.1.

For the purposes of this section suppose that $x_o \varepsilon X$, $T_2 \varepsilon (0,T]$ and $r>0$ such that F restricted to $[0,T_2]\times\overline{B(x_o,r)}$ is a k-γ-set-contraction. Let $B = \overline{B(x_o,r)}$. Then, by Definition 1.6.4, $F([0,T_2]\times B) \varepsilon \Lambda$, hence there exists a positive real number M that bounds $F([0,T_2]\times B)$. Finally, let $T_1 = \min\{r/M,T_2\}$, $T_o \varepsilon (0,T_1]$ be such that $kT_o<1$, $J_o = [0,T_o]$ and

$$C = \{x \varepsilon C(J_o,X):x(J_o) \subset B\}.$$

<u>Lemma 4.2.1</u>

The set C is in <u>Kl</u>K$(C(J_o,X))$.

<u>Proof</u> The set C is nonempty since the constant function defined on J_o by

$$x(t) = x_o$$

is in C.

To verify that C is closed consider a sequence $\{x_n\}_{n=1}^{\infty}$ in C converging to $x \varepsilon X$. Then, given $\varepsilon>0$ there exists an integer N such that for $n \geq N$

$$||x(t) - x_n(t)|| < \varepsilon \text{ for every } t\varepsilon J_o.$$

Thus, for $n \geq N$

$$||x(t) - x_o|| \leq ||x(t) - x_n(t)|| + ||x_n(t) - x_o|| < \varepsilon + r$$

for all $t\varepsilon J_o$. That is, $x(t) \varepsilon B$ for all $t\varepsilon J_o$, hence $x \varepsilon C$ and C is closed.

Consider $x,y \varepsilon C$ and $\eta \varepsilon [0,1]$. Then for all $t\varepsilon J_o$

$$||(\eta x(t) + (1-\eta)y(t)) - x_o|| = ||\eta x(t) - \eta x_o + (1-\eta)y(t) - (1-\eta)x_o||$$

$$\leq \eta||x(t) - x_o|| + (1-\eta)||y(t) - x_o||$$
$$\leq \eta r + (1-\eta)r$$
$$= r.$$

Therefore, C is convex and lies in $\underline{\mathrm{KlK}}(C(J_o,X))$.

<u>Lemma 4.2.2</u>

If (Γ,x_o) is the multifunction 2.1.6 defined on $C(J_o,X)$ then

$$\overline{co}(\Gamma(C)) \subset C.$$

<u>Proof</u> Since $C \in \underline{\mathrm{KlK}}(C(J_o,X))$ it suffices to show that $\Gamma(C) \subset C$. Consider $x \in C$ and $y \in \Gamma(x)$. Then for $t \in J_o$

$$y(t) = x_o + \int_0^t f_x(s)d\lambda \text{ where } f_x \in IS(F_x,J_o).$$

According to Lemma 1.2.13, $y \in C(J_o,X)$ and by Lemmas 1.2.12 and 1.2.19 for every $t \in J_o$

$$||y(t) - x_o|| = ||x_o + \int_0^t f_x(s)d\lambda - x_o||$$
$$\leq \int_0^t ||f_x(s)||d\lambda$$
$$\leq tM$$
$$\leq r.$$

Thus, $y \in C$ and $\Gamma(C) \subset C$.

For the remainder of this section let $D = \overline{co}(\Gamma(C))$.

<u>Lemma 4.2.3</u>

If there exists $x \in C$ such that F_x when restricted to J_o is measurable and $F_x(t) \in \underline{\mathrm{Kp}}(X)$ a.e. on J_o then D is nonempty.

<u>Proof</u> Suppose that x satisfies the hypotheses of the lemma. Since F_x is bounded, hence integrably bounded on J_o by the constant function

$$p(t) = M,$$

the set $IS(F_x, J_o)$ is not empty by Lemma 2.3.6 and therefore D is not empty.

Lemma 4.2.4

The set D is uniformly equicontinuous on J_o.

Proof Consider $y \varepsilon D$. Then, given $\varepsilon>0$ there exist finite sets $\{x_i\}_{i=1}^m$ in C, $\{y_i\}_{i=1}^m$ in D and $\{\eta_i\}_{i=1}^m$ in the non-negative real numbers such that $y_i \varepsilon \Gamma(x_i)$, $1 \leq i \leq m$, $\sum_{i=1}^m \eta_i = 1$ and for all $t \varepsilon J_o$

$$||y(t) - \sum_{i=1}^m \eta_i y_i(t)|| < \varepsilon/3.$$

Now, for every $i \varepsilon \{1,\cdots,m\}$,

$$y_i(t) = x_o + \int_0^t f_i(s)d\lambda \text{ where } f_i \varepsilon IS(F_{x_i}, J_o),\ t \varepsilon J_o.$$

Thus, for $r,t \varepsilon J_o$ with $r \leq t$ and $|t-r| < \varepsilon/3M$,

$$\begin{aligned}||y(t) - y(r)|| &\leq ||y(t) - \sum_{i=1}^m \eta_i y_i(t)|| + ||\sum_{i=1}^m \eta_i y_i(t) - \sum_{i=1}^m \eta_i y_i(r)|| \\ &\quad + ||\sum_{i=1}^m \eta_i y_i(r) - y(r)|| \\ &< \varepsilon/3 + \sum_{i=1}^m \eta_i \int_r^t ||f\ (s)||d\lambda + \varepsilon/3 \\ &\leq \varepsilon/3 + \sum_{i=1}^m \eta_i M(\varepsilon/3M) + \varepsilon/3 \\ &= \varepsilon\end{aligned}$$

as a result of Lemmas 1.2.10, 1.2.12 and 1.2.19.

Lemma 4.2.5

Suppose that X is a separable reflexive Banach space. If for every $x \varepsilon D$

i) $F_x(t) \varepsilon KlK(X)$ a.e. on J_o;

ii) for a.e $t\varepsilon J_o$ the multifunction $F_t:X \to 2^X$ when restricted to $x(J_o)$ is usc;

then (Γ,x_o) is usc on D.

Proof Consider A a closed subset of $C(J_o,X)$. According to Lemma 1.5.6 it suffices to show that

$$\Gamma^-(A) = \{x\varepsilon D:\Gamma(x)\cap A \neq \phi\}$$

is a closed subset of $C(J_o,X)$. To that end, suppose that $\{x_n\}_{n=1}^{\infty}$ is a sequence in $\Gamma^-(A)$ converging to $x \varepsilon C(J_o,X)$. Then, there exists a sequence $\{y_n\}_{n=1}^{\infty}$ in A such that for $t\varepsilon J_o$

$$y_n(t) = x_o + \int_0^t f_n(s)d\lambda \text{ where } f_n\varepsilon IS(F_{x_n},J_o).$$

As noted in the proof of Lemma 4.2.3 the constant function

$$p(t) = M, \ t\varepsilon J_o$$

is an integral bound for each multifunction F_{x_n} on J_o, $n\geq 1$. Thus, as argued in the proof of Lemma 2.1.7, the set $\{f_n|n = 1,2,\cdots\}$ is pre-compact in the weak topology of $L(J_o,X)$ and there exists a subsequence $\{f_k\}_{k=1}^{\infty}$ of $\{f_n\}_{n=1}^{\infty}$ converging weakly to some $f \varepsilon L(J_o,X)$. Then, by Lemma 1.2.14 for every $t\varepsilon J_o$

$$\lim_{n\to\infty} \int_0^t f_n(s)d\lambda = \int_0^t f(s)d\lambda.$$

Also, as shown in the proof of Lemma 3.2.4, $f \varepsilon IS(F_x,J_o)$ and hence the function y defined on J_o by

$$y(t) = x_o + \int_0^t f(s)d\lambda$$

is in $\Gamma(x)$.

To verify that $\Gamma^-(A)$ is closed then, it suffices to show that $y \varepsilon A$. In fact, since the subsequence $\{y_k\}_{k=1}^{\infty}$ of $\{y_n\}_{n=1}^{\infty}$ is in the closed set A it is enough to show that $\{y_k\}_{k=1}^{\infty}$ converges to y in $C(J_o,X)$.

Now, the set $\{y_k | k = 1,2,\cdots\}$ is uniformly equicontinuous on J_o as a result of Lemma 4.2.4. Thus, as in the proof of Lemma 2.1.7, $\{y_k\}_{k=1}^{\infty}$ converges to y in $C(J_o,X)$. Therefore,

$$y \in \Gamma(x) \cap A$$

and $\Gamma^{-}(A)$ is closed.

<u>Theorem 4.2.6</u>

Suppose that X is a separable reflexive Banach space and that there exists $z \in C$ such that F_z when restricted to J_o is measurable and $F_z(t) \in \underline{Kp}(X)$ a.e. on J_o. If for every $x \in D$

i) $F_x(t) \in K1K(X)$ a.e. on J_o;

ii) F_x when restricted to J_o is measurable;

iii) for a.e. $t \in J_o$ the multifunction $F_t : X \to 2^X$ when restricted to $x(J_o)$ is usc;

then (Γ,x_o) has a fixed point in D which is a solution to the initial-value problem 2.1.14.

<u>Proof</u> As a result of Lemma 4.2.3, D is nonempty. The mnc γ_C is monotonic, semi-additive and non-singular on D by Lemmas 1.4.10, 1.4.11 and 1.4.12 respectively. Also, by Lemma 1.4.13, γ_C is 1-regular on D and by Lemmas 1.6.5 and 3.2.5, Γ is γ_C-condensing on D.

Since for every $x \in D$ the multifunction F_x is measurable and integrably bounded on J_o by the constant function

$$p(t) = M, \quad t \in J$$

(Γ,x_o) maps D into $\underline{KpK}(D)$ by Lemmas 4.2.2, 2.1.7 and 2.3.6. Finally, by the preceding lemma Γ is usc on D. Thus, by Theorem 4.1.6, Γ has a fixed point in D and by Lemma 2.1.5 it is a solution to the initial-value problem 2.1.4.

CHAPTER V

5.1 Structure of the Fixed Point Set

In retrospect it is of some interest to note that the hypotheses that the multifunction be closed and condensing with respect to a monotonic mnc are common to each of the Fixed Point Theorems 2.2.4, 3.1.3 and 4.1.6. As the next theorem indicates the presence of these two conditions gives some insight into the topological structure of the fixed point set of the multifunction.

Theorem 5.1.1

Suppose that (Y,Λ,T,γ) is a ncm space with γ a monotonic mnc and $G:Y \to 2^Y$ is a closed γ-condensing multifunction. If S, the set of fixed points of G, is in Λ then S is compact.

Proof Certainly, $S \subset G(S)$. Thus,

$$\gamma(S) \leq \gamma(G(S))$$

and S is precompact according to Definition 1.6.3.

To verify that S is a closed set consider $\{s_\alpha\}$ a net in S converging to s in Y. Then, since $s_\alpha \in G(s_\alpha)$ for every α and G is closed, $s \in G(s)$ by Lemma 1.5.9.

Thus, S is closed and hence compact.

Remark Since in each of the Theorems 2.4.2, 3.2.6 and 4.2.6, Γ is a closed, bounded and γ-condensing multifunction with respect to a monontonic mnc having domain bounded sets, the set of all solutions elicited by any one of these theorems is nonempty and compact.

5.2 Examples of Initial-value Problems

In this section several examples are given of how the initial-value problem 2.1.4 might arise. Of course, many examples arise from the fact that if the kernel of the initial-value problem 2.1.4 is restricted to be single-valued then the problem becomes a standard initial-value problem in ordinary differential equations. This problem can be effectively handled without introducing multifunctions as illustrated by Sadovskii [38]. Thus, the emphasis here will be on examples which have kernels that are possibly not single-valued.

For the remainder of this section J will denote a closed interval of real numbers $[0,T]$ where $0<T<\infty$.

Example 5.2.1

Let X be a Banach space and $f:J\times X \to [0,\infty)$ and $h:J\times X \to X$ single-valued functions. Consider the problem of finding a solution x to the differential inequality

$$||\dot{x}(t) - h(t,x(t))|| \leq f(t,x(t)),\ t\varepsilon J$$
$$x(0) = x_o,\ x_o \varepsilon X.$$

It is easily verified that this problem is equivalent to the initial-value problem

$$\dot{x}(t)\ \varepsilon\ F(t,x(t))$$
$$x(0) = x_o$$

where

$$F(t,x(t)) = \overline{B\Big(h(t,x(t)),f(t,x(t))\Big)}$$

Example 5.2.2

Let $(X,\leq')$ be a partially ordered Banach space and $f:J \to [0,\infty)$ a single-valued function. Consider the problem of finding a solution x to the inequality

$$\theta \leq' \dot{x}(t) \leq' f(t)x(t),\ t\varepsilon J$$
$$x(0) = x_o,\ x_o \varepsilon X.$$

This problem readily translates into the initial-value problem

$$\dot{x}(t) \in F(t,x(t))$$

$$x(0) = x_o$$

where

$$F(t,x(t)) = [\theta,f(t)x(t)].$$

In the event that X is partially ordered by a cone the multifunction F_x is point closed and point convex for every $x \in C(J,X)$ by Lemma 3.1.2.

Example 5.2.3

Let X be Euclidean n-space, Y Euclidean m-space and $f:JxXxY \to X$ a single-valued function. Consider the control theory problem

$$\dot{x}(t) = f(t,x(t),u(t)),\ t \in J$$

$$x(0) = x_o,\ x_o \in X$$

where the control function u may be chosen as any measurable m-vector-valued function with value at time t in a preassigned subset U(t) of Y. This problem is equivalent to the initial-value problem

$$\dot{x}(t) \in F(t,x(t))$$

$$x(0) = x_o$$

where

$$F(t,x(t)) = \{f(t,x(t),u(t)):u \text{ is a measurable selector for } U \text{ on } J\}.$$

Considerable work has been done on this problem by Filippov [14] and Hermes [23] without recourse to multifunction fixed point theory.

5.3 Further Research

There appears to be several avenues related to the initial-value problem 2.1.4 and the fixed point theory developed to solve it open for further research. For example, it might be interesting to investigate the types of conditions that would be necessary to guarantee a solution to the initial-value problem 2.1.4 if the underlying space were other than a Banach space or, in the event that it is a Banach space, if more or less general conditions were placed on it. Along with experimenting with the underlying space an integral, either more or less general than the Bochner integral, might be tried. Also, the effects of generalizing or restricting the measure of non-compactness used in each case could be considered.

It may be of interest as well to attempt the development of a constructive procedure for evaluating certain of the fixed points shown to exist here, at least in the case where the space in which the fixed points are known to exist is Euclidean n-space. Some steps in this direction have already been taken by Merrill [32] who has developed an algorithmic technique for evaluating fixed points of a certain subclass of upper semi-continuous multifunctions having domain Euclidean n-space.

Though the initial-value problem 2.1.4 is known to arise in the area of control theory, further investigation into its role in the areas of linear programming, optimization and econometrics needs to be made. As suggested by Merrill's work in [32] the fixed point theory developed here, aside from its intermediary application to the initial-value problem, may be directly applicable to problems in these areas.

BIBLIOGRAPHY

1. Amann, H., Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces, SIAM Rev., 18 No. 4 (1976), pp. 620-709.

2. Berge, C., Topological Spaces, The Macmillan Company, New York, 1963.

3. Bohnenblust, H.F. and Karlin, S., On a theorem of ville, contributions to the theory of games, Annals of Math. Studies, No. 24, Princeton University Press, Princeton, New Jersey, 1950, pp. 155-160.

4. Castaing, C., Sur les equations differentielles multivoques, C.R. Acad. Sc. Paris, Ser. A 263 (1966), pp. 63-66.

5. Castaing, C., Sur les multi-applications mesurables, Rev. Francaise Informat. Recherche Operationelle, 1 (1967), pp. 91-126.

6. Castaing, C., Un theorme de compacite faible dans L_E^1, Secretariat des Math., No. 44, University of Montpellier, 1969.

7. Danes, J., Fixed point theorems, Nemyckii and Uryson operators, and continuity of nonlinear mappings, Comment. Math. Univ. Carolinae, 11 No. 3 (1970), pp. 481-500.

8. Danes, J., Generalized concentrative mappings and their fixed points, Comment. Math. Univ. Carolinae, 11 No. 1 (1970), pp. 115-135.

9. Danes, J., Some fixed point theorems in metric and Banach spaces, Comment. Math. Univ. Carolinae, 12 No. 1 (1971), pp. 37-50.

10. Darbo, G., Punti uniti in transformazione a condominio non compatto, Rend. Sem. Mat. Univ. Padova, 24 (1955), pp. 84-92.

11. Dunford, N. and Schwartz, J.T., Linear Operators Part I, Interscience Publishers Inc., New York, 1964.

12. Filippov, A.F., Classical solutions of differential equations with multivalued right-hand side, SIAM J. Control, 5 No. 4 (1967), pp. 609-621.

13. Filippov, A.F., Differential equations with many-valued discontinuous right-hand side, Soviet Math. Dokl., 4 (1963), pp. 941-945.

14. Filippov, A.F., On certain questions in the theory of optimal control, SIAM J. Control, Ser. A 1 No. 1 (1962), pp. 76-84.

15. Fitzpatrick, P.M. and Petryshyn, W.V., Degree theory, fixed point theorems and mapping theorems for multivalued noncompact mappings, Trans. Amer. Math. Soc., 194 (1974), pp. 1-25.

16. Fitzpatrick, P.M. and Petryshyn, W.V., Fixed point theorems and the fixed point index for multivalued mappings in cones, J. London Math. Soc., 12 (1975), pp. 75-85.

17. Fitzpatrick, P.M. and Petryshyn, W.V., Fixed point theorems for multivalued noncompact acyclic mappings, Pacific J. Math., 54 No. 2 (1974), pp. 17-23.

18. Fitzpatrick, P.M. and Petryshyn, W.V., Fixed point theorems for multivalued noncompact inward maps, J. Math. Anal. Appl., 46 (1974), pp. 756-767.

19. Furi, M. and Vignoli, A., A fixed point theorem in complete metric spaces, Boll. Un. Mat. Ital., Ser. IV No. 4-5 (1969), pp. 505-509.

20. Furi, M. and Vignoli, A., Fixed points for densifying mappings, Atti. Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur., Ser. 6 47 (1969), pp. 465-467.

21. Furi, M. and Vignoli, A., On α-nonexpansive mappings and fixed points, Atti. Accad. Naz. Lincei Rend. Cl. Sci. Fix. Mat. Natur., Ser. 8 48 (1970), pp. 195-198.

22. Gatica, J. and Kirk, W., A fixed point theorem for k-set-contractions defined in a cone, Pacific J. Math., 53 No. 1 (1974), pp. 131-136.

23. Hermes, H., The generalized differential equation $\dot{x} \in R(t,x)$, Advances in Math., 4 (1970), pp. 149-169.

24. Hille, E. and Phillips, R.S., Functional Analysis and Semi-Groups, American Mathematical Society, Rhode Island, 1957.

25. Himmelberg, C.J., Porter, J.R., Van Vleck, F.S., Fixed point theorems for condensing multifunctions, Proc. Amer. Math. Soc., 23 (1969), pp. 635-641.

26. Jones, G.S., A functional approach to fixed point analysis of noncompact operators, Math. Systems Theory, 6 No. 4 (1971), pp. 375-382.

27. Krasnoselskii, M., Positive Solutions of Operator Equations, P. Noordhoff Ltd., Groningen, The Netherlands, 1964.

28. Kuratowski, C., Topologie, "Monografie Matematiczne", Tom 20, Warszawa, 1958.

29. Marchaud, A., Sur les champs de demi-cones convexes, Bull. Sci. Math., 62 (1938), pp. 229-240.

30. Martelli, M., Some results concerning multivalued mappings defined in Banach spaces, Atti. Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur., 54 (1973), pp. 865-871.

31. Martin, R.H., Nonlinear Operators and Differential Equations in Banach Spaces, John Wiley & Sons, New York, 1976.

32. Merrill, O., Applications and extensions of an algorithm that computes fixed points of certain nonempty convex upper-semi-continuous point-to-set mappings, Technical Report No. 71-7, University of Michigan, 1971.

33. Nadler, S.B., Multivalued contraction mappings, Pacific J. Math., 30 No. 2 (1969), pp. 475-487.

34. Nussbaum, R.D., A generalization of the Axcoli Theorem and an application to functional differential equations, J. Math. Anal. Appl., 35 (1971), pp. 600-610.

35. Nussbaum, R.D., The fixed point index for local condensing maps, Ann. Mat. Pura Appl., 89 (1971), pp. 217-258.

36. Petryshyn, W.V., Structure of the fixed points sets of k-set-contractions, Arch. Rational Mech. Anal., 40 (1971), pp. 312-328.

37. Potter, A.J.B., A fixed point theorem for positive k-set-contractions, Proc. Edinburgh Math. Soc., 19 (1974), pp. 93-102.

38. Sadovskii, B.N., Limit compact and condensing operators, Uspehi Mat. Nauk, 27 (1972), pp. 81-146.

39. Smithson, R.E., Fixed points for contractive multifunctions, Proc. Amer. Math. Soc., 27 No. 1 (1971), pp. 192-194.

40. Smithson, R.E., Fixed points of order preserving multifunctions, Proc. Amer. Math. Soc., 28 No. 1 (1971), pp. 304-310.

41. Smithson, F.E., Multifunctions, Nieuw Arch. Wisk., 20 (1972), pp. 31-53.

42. Zaremba, S.C., Sur les equations au paratingent, Bull. Sci. Math., 60 (1936), pp. 139-160.