MONOTONE SIMPLICIAL FUNCTIONS
ON COMBINATORIAL SPHERES

A Thesis

Presented to

the Faculty of the Department of Mathematics

University of Houston

In Partial Fulfillment

of the Requirements for the Degree

Master of Science

---

by

James Daniel Borkowski

June 1968

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ABSTRACT

Let f be a monotone simplicial function from a triangulated combinatorial n-sphere S^n onto a triangulated combinatorial n-manifold M^n. It is shown that f is point-like if and only if $f^{-1}(v)$ is algebraically $[n/2]$-connected for each vertex v of M^n, provided that $n=3$. The proof is accomplished along lines which it is hoped will lead to a proof for other dimensions, perhaps for all n, in at least a modified version of the statement.

Several related conjectures are investigated by showing that some of the lemmas used to prove the main theorem are true in all dimensions, or at least in all but two. The particular difficulties encountered by the author in trying to prove these conjectures are explained.

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CHAPTER I

INTRODUCTION AND DEFINITIONS

I. INTRODUCTION

In June, 1962, Ross Finney [5]* proved that if there exists a pointlike, simplicial mapping of M onto T, where M is a triangulated 3-sphere and T is a triangulated topological space, then T is a 3-sphere. Following this result, C. L. Wiginton [20] established that if f is a monotone simplicial mapping from a triangulated 3-sphere onto a 3-manifold M^3, then if the inverse image of each vertex is simply connected, M^3 is a 3-sphere.

In hopes that a similar result might be developed for higher dimensions, the author has attempted to extend Wiginton's main theorem to higher dimensions. The result of this effort is a new proof of this theorem in chapter IV and the discussion of the related conjectures in chapter V.

II. DEFINITIONS

Definition 1.1. An n-cell is a topological space homeomorphic to either $I^n = \{(x_1,\ldots,x_n) \in E^n \ni 0 \le x_i \le 1\}$, the standard n-cube, or $B^n = \{(x_1,\ldots,x_n) \in E^n \ni \sum_{i=1}^{n} x_i^2 \le 1\}$, the standard n-ball. An n-cell is also referred to as an n-ball.

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*Throughout this paper a bracketed number refers to the corresponding reference in the bibliography.

Definition 1.2. An n-sphere is a topological space homeomorphic to $S^n = \{(x_1, \ldots, x_{n+1}) \in E^{n+1} \ni \sum_{i=1}^{n+1} x_i^2 = 1\}$, the standard n-sphere.

Definition 1.3. An n-manifold M^n is a separable metric space such that all its points have a neighborhood homeomorphic to either E^n or B^n. An interior point of M^n is a point having a neighborhood homeomorphic to E^n and a boundary point is a point having no neighborhood homeomorphic to E^n. The interior of M^n, int M^n, is the union of all the interior points of M^n. The boundary of M^n, Bd M^n or $\dot{M}^n$, is the complement in M^n of int M^n.

Definition 1.4. A manifold M is called closed if it is compact and $\dot{M} = \emptyset$, bounded if it is compact and $\dot{M} \neq \emptyset$, open if it is not compact and $\dot{M} = \emptyset$.

Definition 1.5. If A is a closed subset of an n-manifold M^n, then a component of $M^n - A$ is called a complementary domain of A.

Definition 1.6. A subset A of an n-sphere S^n is pointlike if $S^n - A$ is homeomorphic to E^n. A mapping g of an n-sphere onto a space Y is pointlike if the set $g^{-1}(y)$ is pointlike for each $y \in Y$.

Definition 1.7. A continuous function $f: M \to G$ is said to be monotone if the inverse image of each $g \in G$ is connected.

Definition 1.8. A subset A of N, where N is an n-manifold, is cellular if there exists a sequence of n-cells

$\langle C_i \rangle$ such that $A=\cap C_i$ and for each i, $C_{i+1} \subset \text{int } C_i$.

Definition 1.9. An *n-simplex* $A(n\geq 0)$, where $A\subset E^p$, $p\geq n$, is the convex hull of $(n+1)$ linearly independent points, called *vertices* of A. The convex hull of any $(k+1)$ sub-collection of these vertices is a simplex called a *k-face* of A, where $k=-1,0,1,\ldots,n$. Thus the empty set is a (-1) face of A, and $A<A$. If B is a face of A, this fact is denoted by $B<A$.

Definition 1.10. The *join* of two simplexes A and B, written *AB*, is the simplex spanned by the vertices of A and B taken together. If this collection of vertices is not linearly independent, the join is not defined, and A and B are said to be *not joinable*. Otherwise, they are *joinable*.

Definition 1.11. A *finite simplicial complex*, or *complex*, K in E^p is a finite collection of simplexes such that:

(1) $A\in K$ and $B<A \Longrightarrow B\in K$

(2) $A\in K$ and $B\in K \Longrightarrow (A\cap B)<A$ and $(A\cap B)<B$

Definition 1.12. A *polyhedron* $|K|$ is the point-set determined from a complex K by taking the union of the simplexes in K. A topology on $|K|$ is the relative topology inherited from E^p.

Definition 1.13. A *triangulation of a topological space* is a finite simplicial complex K such that the polyhedron $|K|$ is homeomorphic to the topological space. A triangulation of a topological space is often called simply a <u>triangulated topological space</u>.

<u>Definition 1.14</u>. If K and L are complexes, then a <u>simplicial map</u> or <u>simplicial function</u> $f:K\to L$ is a function $f:|K|\to|L|$ such that:

(1) f is continuous

(2) f maps vertices to vertices, and whenever $\{v_o,\ldots,v_p\}$ span a simplex of K, then $\{f(v_o),\ldots,f(v_p)\}$ span a simplex of L

(3) f is <u>barycentric</u>; that is, if $\alpha=\sum_{i=1}^{n} t_i v_i$, then $f(\alpha)=\sum_{i=1}^{n} t_i f(v_i)$

<u>Definition 1.15</u>. If A is a simplex of the complex K, then the <u>star of A in K</u> denoted <u>st(A,K)</u>, is the subcomplex of K made up of all the simplexes of K having A as a face together with all their faces. The <u>link of A in K</u>, denoted <u>lk(A,K)</u> is the subcomplex of K made up of all the simplexes of K in st(A,K) which do not intersect A.

<u>Definition 1.16</u>. If A^k is a k-dimensional simplex and $\alpha\subset|A|$, where $\{v_i\}_{i=0,k}$ are the vertices of A^k and $\alpha=\sum_{i=0}^{k} t_i v_i$ so that $t_i\subset[0,1]$ and $\sum_{i=0}^{k} t_i = 1$, then the $\{t_i\}$ are called <u>barycentric coordinates</u> of α. In particular, the point $\hat{A}^k$, which has barycentric coordinates $\{t_j\}_{j=0,k}$ such that $t_j=1/k+1$ for all j is called the <u>barycenter</u> of A^k.

<u>Definition 1.17</u>. The <u>dimension</u>, n, of a complex

K^n is the dimension of the highest dimensional simplex of K^n.

<u>Definition 1.18</u>. If K and L are complexes, L is a <u>subdivision</u> of K if $|K|=|L|$ and every simplex of L lies in a simplex of K.

<u>Definition 1.19</u>. If A is a simplex, the <u>boundary</u> $\overset{\circ}{A}$ of A is the complex consisting of all the proper faces of A. If K^n is a complex which is <u>homogeneous</u>; that is, made up of n-simplexes and their faces, the <u>boundary</u> $\overset{\circ}{K}{}^n$ of K^n is the (n-1)-complex made up of the (n-1)-simplexes of the boundary $\overset{\circ}{A}$ of each of the n-dimensional simplexes $A \in K^n$ (except for those (n-1)-simplexes which are faces of an even number of n-simplexes of K^n) plus all their faces.

<u>Definition 1.20</u>. If K and L are complexes, then the <u>join</u> of K and L, written <u>KL</u>, is the complex $\{AB \ni A \in K, B \in L\}$, where AB is defined for all such A and B.

<u>Definition 1.21</u>. Let A be a simplex of the complex K and α be a point in int A. The construction of L from K such that $L=\{K-st(A,K)\} \cup \alpha \overset{\circ}{A} lk(A,K)$ is called an <u>elementary starring</u> of K at α, written $K \to L$. A <u>stellar subdivision</u> of K, denoted σK, is any subdivision of K obtained by a finite sequence of elementary starrings.

<u>Definition 1.22</u>. A <u>derived subdivision</u> of a complex K is a subdivision obtained by starring all the simplexes of K in some order such that if $B<A$, then A is starred before B.

If the starring is done at the barycenter of each simplex, the subdivision is said to be *barycentric derived*, or simply *barycentric*. The n-th such construction is called the *n-th derived subdivision* of K.

Definition 1.23. A function f from a complex K onto a complex L is called *piece-wise linear* (pwl) if there exists subdivisions αK and βL relative to which f is simplicial.

Definition 1.24. A simplicial mapping f from a complex K onto a complex L such that f is a homeomorphism is called an *isomorphism*, and K is said to be *isomorphic* to L, denoted $K \cong L$. If f is only a piece-wise linear homeomorphism, then we write $K \simeq L$, and say that K and L are *combinatorially equivalent*.

Definition 1.25. A *combinatorial n-ball* is a complex pwl homeomorphic to the triangulated standard n-ball B^n. A *combinatorial n-sphere* is a complex pwl homeomorphic to the triangulated standard n-sphere S^n.

Definition 1.26. A *combinatorial n-manifold* is a homogeneous n-dimensional complex K such that for any vertex v of K, $lk(v,K)$ is a combinatorial (n-1)-ball if $v \subset \dot{K}$ and a combinatorial (n-1)-sphere if $v \not\subset \dot{K}$.

Definition 1.27. Suppose K is a complex such that $K = L \cup A$, where A is a simplex, $A = aB$, and $L \cap A = a\dot{B}$ (i.e., the face B opposite the vertex a in A is a "free" face of the simplex A in K), then the operation of going from the complex

K to the complex $L=K-A-B$ (recall that removing a simplex from a complex does not remove all its faces), is called an elementary (simplicial) collapsing, denoted $K\searrow L$. The inverse operation, denoted $L\nearrow K$, is called an elementary (simplicial) expansion. Finite sequences of either are called, respectively, (simplicial) collapsings and (simplicial) expansions. If K collapses to a vertex $v\in K$, we say K is collapsible, and denote this by $K\searrow 0$.

Definition 1.28. A subcomplex L of the complex K is complete or full in K if, for every simplex $A\in K$ with all vertices in L, then $A\in L$.

Definition 1.29. If K_1 is a complex such that $K_1=K_o\cup B^n$, where B^n is a combinatorial ball and $B^n\cap K_o=B^{n-1}\subset \dot{B}^n$, then the operation of going from K_1 to K_o, denoted by $K_1\rightarrow K_o$, is called an elementary geometrical collapsing. The inverse operation $K_o\rightarrow K_1$ is termed an elementary geometrical expansion. Finite sequences of either are called, respectively, geometrical collapsings and geometrical expansions. If K has dimension n, and the only B^k used are such that $k=n$, then the collapsings (or expansions) are called regular.

Definition 1.30. If K is a complex and $X\subset|K|$, then $N(X,K)=\{A\in K\ni|st(A,K)|\cap X\neq\emptyset\}$ is called the (closed) simplicial neighborhood of X in K. Notice that N(X,K) is the complex formed by taking all simplexes of K which intersect X, plus all their faces.

Definition 1.31. If L is a subcomplex of K, then $S_L K$ will denote the barycentric subdivision obtained by starring the simplexes of K-L in order of decreasing dimension.

Definition 1.32. If K is a subcomplex of a combinatorial manifold M^n, then by a regular neighborhood of K in M is meant a subcomplex U(K,M) such that:

(1) U(K,M) is a combinatorial n-manifold.

(2) U(K,M) collapses geometrically to K.

Definition 1.33. Suppose $A \subset U \subset B$, where U is open in B. Then U is called a cartesian product neighborhood of A if there exists a homeomorphism $h: A \times (-1,1) \to U$ such that $h(x \times \{0\}) = x$, for all $x \in A$. If there exists such a U, A is said to be bicollared by U.

Definition 1.34. If M^n is a combinatorial n-manifold with $X \subset |M^n|$ such that under some subdivision α of M^n there exists a $K \subset \alpha M^n$ where $X = |K|$, then X is called a combinatorial subspace of M^n.

Definition 1.35. A combinatorial n-manifold M^n is said to be algebraically g-connected if the inclusion mapping of every combinatorial subspace $X^k \subset |M^n|$, $k \leq g$, into M^n is homotopic to a constant.

Definition 1.36. A combinatorial n-manifold M^n is said to be geometrically g-connected if every g-dimensional combinatorial subspace of M^n is contained in a combinatorial n-ball of M^n.

Definition 1.37. A combinatorial closed n-manifold is called a homotopy sphere if it is connected and algebraically $[n/2]$-connected, where $[x]$ is the greatest integer less than or equal to x.

Definition 1.38. A simplex A^p is said to be oriented if some arbitrary fixed ordering of its vertices has been determined. The equivalence class of even permutations of this fixed ordering is the positively oriented simplex, denoted $+A^p$ and the equivalence class of odd permutations of the ordering is called the negatively oriented simplex. An oriented complex is a complex which has all oriented simplexes.

Definition 1.39. Let K denote an oriented complex and G denote an arbitrary abelian group. Then an m-dimensional chain on the complex K with coefficients in the group G is a function c_m on the oriented m-simplexes of K with values in G such that if $c_m(+A^m)=g$, $g \in G$, then $c_m(-A^m)=-g$. The collection of all such m-dimensional chains on K will be denoted by $C_m(K,G)$. Note: Addition can be defined by $({}_1c_m + {}_2c_m)\ (A^m) \equiv {}_1c_m(A^m)+{}_2c_m(A^m)$, where the addition on the right is the group addition in G, and that if G is the integers mod2, there is no necessity to have oriented simplexes. $C_m(K,G)$ forms a commutative group under the described addition.

Definition 1.40. Let K be an oriented complex. There is associated with every pair of simplexes A^m and B^{m-1}, where A and B differ in dimension by 1, an <u>incidence number</u> defined as follows:

(1) $[A^m, B^{m-1}] = 0$, if $B^{m-1} \nless A^m$

(2) $[A^m, B^{m-1}] = \pm 1$, if $B^{m-1} < A^m$

In order to decide in (2) which sign to use, consider the vertices of B^{m-1}. They must be the same as the vertices of A^m, with say v_i left out. If that particular ordering is $+B^{m-1}$ and $(v_i, v_o, \ldots, \cancel{v}_i, \ldots, v_m)$ is $+A^m$, then $[A^m, B^{m-1}] = +1$; if it is $-A^m$, then $[A^m, B^{m-1}] = -1$. If that particular ordering is $-B^{m-1}$ and $(v_i, v_o, \ldots, \cancel{v}_i, \ldots, v_m)$ is $+A^m$, then $[A^m, B^{m-1}] = -1$; if it is $-A^m$, then $[A^m, B^{m-1}] = +1$.

<u>Definition 1.41</u>. An <u>elementary m-chain</u> on K is an m-chain c_m such that $c_m\ (\pm A^m) = \pm g_o$ for some particular simplex $A^m \in K$, and $c_m\ (B^m) = 0$, for any other m-simplex of K. Thus any m-chain may be written as the formal linear combination of elementary m-chains; $c_m = \sum g_i \cdot A_i^m$, where $g_i = c_m\ (+A_i^m)$

<u>Definition 1.42</u>. The <u>boundary operator</u> is defined by:

$$\partial(g_o \cdot A_o^m) = \sum_{B_i^{m-1} < A_o} [A_o^m, B_i^{m-1}] \cdot g_o \cdot B_i^{m-1},$$

where $[A_o^m, B_i^{m-1}]$ is the incidence number. We further define:

$$\partial(\sum_i g_i \cdot A_i^m) = \sum_i \partial(g_i \cdot A_i^m)$$

Definition 1.43. If $m>0$, then an *m-dimensional cycle on K with coefficients in G* is a chain z_m in $C_m(K,G)$ such that $\partial(z_m) = 0$. The *m-dimensional cycle group of K with coefficients in G* is the collection of such m-cycles together with the commutative addition previously noted and is written *$Z_m(K,G)$*.

Definition 1.44. An *m-boundary* is an m-chain b_m if there exists an (m+1)-chain C_{m+1} in $C_{m+1}(K,G)$ such that $\partial(C_{m+1}) = b_m$. The collection of all m-boundaries of K together with the previously noted addition forms a commutative group written *$B_m(K,G)$*. Notice that $B_m(K,G)$ is a subgroup of $Z_m(K,G)$ since $\partial[\partial(C_{m+1})] = 0$.

Definition 1.45. The factor group $Z_m(K,G) - B_m(K,G)$ is called the *m-th homology group of K over G*, and is denoted by $H_m(K,G)$.

Definition 1.46. Let G_i be the direct sum decomposition of $H_m(K,G)$ such that at most one of the G_i is not a cyclic group, and call the non-cyclic group (if it exists) G_o. The number of generators of G_o is called the *m-th Betti number of K*. If there is no non-cyclic group, then the m-th Betti number is *zero*.

CHAPTER II

BACKGROUND PRELIMINARIES

A classical problem in the study of monotone continuous functions from a topological space X onto a topological Y is to discover conditions on such a function such that X is homeomorphic to Y.

As early as 1929, R.L. Moore [1] showed that the monotone continuous image of a 2-sphere is a cactoid (a continuous curve whose every maximal cyclic element is a 2-sphere), and that every cactoid is a monotone continuous image of a 2-sphere. He also showed that if the inverse image of no point of the image space separates the 2-sphere domain, then the image is either a 2-sphere or a point.

Some highly analagous results were obtained by J.H. Roberts and N.E. Steenrod [15] in 1938. They proved that the class of continuous images of compact 2-manifolds without boundary under monotone transformations is composed of just those continuous curves each of which can be obtained from a generalized cactoid (a continuous curve whose every maximal cyclic element is a 2-manifold and all but a finite number of these are 2-spheres) by making a finite number of identifications. By adding various restrictions on the non-degenerate inverse images of points in the image space, they obtained stronger results, including that if f is a monotone continuous function from a compact, connected 2-manifold X onto Y, then X is homeomorphic to Y if and only if Y contains more than one point and the 1-dimensional Betti number (mod 2) of each of the sets $f^{-1}(y)$, $y \in Y$, is zero.

An interesting sufficient condition that a monotone continuous image of the 3-sphere be homeomorphic to a 3-sphere was published by O.G. Harrold, Jr. [7] in 1958, and is stated as follows:

Theorem 2.1. Let $M=f(S^3)$, where f is a monotone continuous map such that if Y is the set of points in M which have non-degenerate inverse images, and given $y \in \overline{Y}$ and $\epsilon > 0$, there exists a topological 2-sphere $K \subset T = \{z \in M \ni \rho(z,y) < \epsilon\}$, where K separates y and M-T such that $K \cap \overline{Y} = \emptyset$, then M is a topological 3-sphere.

Going in a different direction from Harrold's result, Ross Finney [5] proved this powerful theorem:

Theorem 2.2. Let M be a triangulated 3-sphere and let T be a triangulated topological space. If there exists a pointlike, simplicial mapping M onto T, then T is homeomorphic to M.

In 1965, C.L. Wiginton [20] proved this closely related theorem:

Theorem 2.3. Suppose f is a monotone simplicial mapping from a triangulated 3-sphere S^3 onto a 3-manifold M^3. Then f is pointlike if and only if the inverse image of each vertex is simply connected.

The power of this theorem is seen in the following corollary:

Corollary 2.3.1. Suppose f is a monotone simplicial mapping from a triangulated 3-sphere onto a 3-manifold M^3. Then if the inverse image of each vertex is simply connected, M^3 is a 3-sphere.

This corollary is a result of Theorem 2.2 and Theorem 2.3.

Four theorems of E.C. Zeeman [21] will be employed in the sequel.

Theorem 2.4. A derived simplicial neighborhood of a collapsible polyhedron which is contained in an n-manifold in an n-ball.

Theorem 2.5. Any derived simplicial neighborhood of a combinatorial subspace of a combinatorial manifold is a regular neighborhood.

Theorem 2.6. Any two regular neighborhoods of a combinatorial subspace of a combinatorial manifold are homeomorphic.

Theorem 2.7. A manifold is collapsible if and only if it is a ball.

H. Seifert [17] proved the following extremely useful theorem:

Theorem 2.8. The first Betti number of a compact

3-manifold with boundary is greater than or equal to the sum of the genera of its boundary surfaces.

In 1961, Morton Brown [2] obtained this result:

Theorem 2.9. The monotone union of open n-cells is an open n-cell.

This theorem is used to relate the concepts of pointlikeness and cellularity.

CHAPTER III

THE POINCARÉ CONJECTURE

The original version of the Poincaré Conjecture is stated as follows:

Conjecture 3.1. Suppose M^3 is a connected closed n-manifold which is also simply connected. Then M^3 is homeomorphic to S^3.

Paradoxically, this is still an outstanding question, even though a more general form of the Poincaré Conjecture is known to be true for dimensions greater than 4:

Theorem 3.1. Let M^n be a connected closed combinatorial n-manifold. Then M^n is a homeomorphic to S^n if and only if M^n is a homotopy sphere $(n \geq 5)$.

E.C. Zeeman [22] established this significant result by proving the following relationship between algebraic and geometric connectedness:

Theorem 3.2. Let M^n be a connected combinatorial n-manifold. If M^n is algebraically g-connected and $g \leq n-3$, then M^n is geometrically g-connected.

The generalized Poincaré Conjecture $n \geq 5$ was obtained as a corollary in light of Theorem 3.2 and an earlier result of J.R. Stallings [19]:

Theorem 3.3. Let M^n be a connected closed combinatorial n-manifold, $n>0$. If M^n is geometrically $[n/2]$-

connected, then M^n is homeomorphic to a sphere.

Stallings had used his theorem in 1960 to prove the Poincaré Conjecture for $n \geq 7$, but admitted that his proof was not the best possible, since Zeeman's theorem (3.2) had already been published. Zeeman comments that Theorem 3.2 might be shown for $g \leq n-2$ if use could be made of the additional hypothesis that M^n is a closed manifold. If this could be done, the Poincaré Conjecture would follow for dimensions 3 and 4 since $[n-2] = n/2$ for these dimensions.

In his proof of Theorem 3.3, Stallings employed the long unknown Generalized Schoenflies Theorem which was established in early 1960 by Morton Brown [3]:

<u>Theorem 3.4</u>. Suppose h is a homeomorphic embedding of $S^{n-1} \times [0,1]$ in S^n. Then the closure of either complementary domain of $h(S^{n-1} \times 1/2)$ is homeomorphic to an n-cell.

The following unproven conjecture by Zeeman [21] is extremely interesting in that it implies the original Poincaré Conjecture:

<u>Conjecture 3.2</u>. If K^2 is a contractible 2-complex, then $K \times [0,1] \searrow 0$.

Another proof of Theorem 3.1 was given by Stephen Smale [18] using a differential structure which he showed was implied by the combinatorial structure.

M.W. Hirsch [8] pointed out in 1965 that the Poincaré Conjecture is still unknown in $n \geq 3$ if a combinatorial structure is not assumed. He also stated that if a combinatorial structure is assumed, it was easy to show that M^n is combinatorially equivalent to S^n for $n \geq 6$. In the cited paper he showed the following:

Theorem 3.5. Suppose M^5 is a closed combinatorial 5-manifold which is the boundary of an orientable 6-manifold. Then M^5 is combinatorially equivalent to S^5 if and only if M^5 is a homotopy sphere.

In two papers published in 1963, C.D. Papakyriakopoulos [12, 13] proved several other conjectures to be equivalent to the Poincaré Conjecture in hopes that they would lead to a solution of that elusive question.

CHAPTER IV

THE MAIN THEOREM

This chapter will consist in an alternate proof of the previously stated theorem of C.L. Wiginton.

<u>Theorem 4.1</u>. Suppose f is a monotone simplicial function from a combinatorial 3-sphere S^3 onto a combinatorial 3-manifold M^3. Then f is pointlike if and only if the inverse image of each vertex of M^3 is simply connected.

Throughout the following sequence of lemmas, f will be a monotone simplicial function from S^3 onto M^3, where S^3 is a combinatorial 3-sphere and M^3 is a combinatorial 3-manifold.

<u>Lemma 4.1</u>. A subset of an n-sphere S^n is cellular if and only if it is pointlike.

<u>Proof</u>: Suppose A is a pointlike subset of S^n; then there exists a homeomorphism h between S^n-A and E^n-p, where p is a point of E^n. Now there exists a sequence of closed ϵ-neighborhoods $\langle D_i \rangle$ of p such that $\cap D_i = p$. By the Generalized Schoenflies Theorem (3.4), $h^{-1}(\dot{D}_i \times [0,1])$ is a tame bicollared (n-1)-sphere $\hat{B}_i$ so that $h^{-1}(\dot{D}_i \times 1/2)$ is a tame (n-1)-sphere B_i. These $\langle B_i \rangle$ are the boundaries of tame n-cells $\langle C_i \rangle$ of S^n such that $C_{i+1} \subset \text{int } C_i$ and $A \subset \text{int } C_i$ for all i. Since $h[\cap C_i] \subset \cap h(C_i) = p$, it follows that $\cap C_i = A$.

Suppose A is a cellular subset of S_n. Then there exists a sequence of tame n-cells $\langle C_i \rangle$ such that $\cap C_i = A$ and $C_{i+1} \subset \text{int } C_i$ for all i, where $A \subset \text{int } C_i$ for all i. Now $\sim C_i$ is homeomorphic to E^n for all i and $\sim C_i \subset \sim C_{i+1}$, where $\sim C_i$ means the complement of C_i. But $\cup \sim C_i = \sim \cap C_i$, and by Theorem 2.9, $\cup \sim C_i$ is an open n-cell. It follows that A is pointlike.

Lemma 4.2. Let g be a monotone simplicial function from a closed triangulated n-manifold M onto a triangulated n-manifold T. Then T is closed.

Proof: Suppose $\dot{T} \neq \emptyset$ and A is an (n-1)-simplex of $\dot{T}$. Then A must be the face of exactly one n-simplex B of T. Now $g^{-1}|B|$ must be $\cup|\beta_i|$, where the β_i are n-simplexes of M. Consider β_n, where $g(\beta_n)=B$, and let α_n be the (n-1)-face of β_n which maps onto A. Since M has no boundary, α_n must be the face of exactly one other n-simplex γ_n of M. Now γ_n must also map to A, since g is simplicial. Now all (n-1)-faces of γ_n must map into A, and thus every chain of n-simplexes of M which are pairwise connected by (n-1)-simplexes and are connected to γ_n by one of its (n-1)-faces other than α_n must also map to A because of the simplicial nature of g. But these chains cannot fill up all of M since β_n does not map to A. Thus at least one of them has a last n-simplex with an (n-1)-simplex which is the face of that n-simplex only. This implies that M has a non-empty boundary. But M has an empty boundary by hypothesis. The contradiction implies that

$\dot{T} = \emptyset$.

Lemma 4.3. Suppose g is a continuous monotone function from a compact space S onto a space T, S and T are both T_1, and M is a closed subset of S such that $S-M = A\cup B$ where $\overline{A}\cap B = A\cap\overline{B} = \emptyset$ and neither $g(A)$ nor $g(B)$ is contained in $g(M)$. Then $g(M)$ separates T.

Proof: Since $S = A\cup B\cup M$, then $T = g(A)\cup g(B)\cup g(M)$. Suppose there exists an $x\in[g(A)-g(M)]\cap[g(B)-g(M)]$. Then $g^{-1}(x)\cap A\neq\emptyset$, $g^{-1}(x)\cap B\neq\emptyset$, $g^{-1}(x)\cap M=\emptyset$. This implies that $g^{-1}(x)$ is not connected. Since this contradicts g being monotone it follows that $[g(A)-g(M)]\cap[g(B)-g(M)]=\emptyset$. Suppose that t is a limit point of $g(B)-g(M)$ which is contained in $g(A)-g(M)$. Let $\langle p_i\rangle$ be a sequence of points of $g(B)-g(M)$ which converges to t. Then the set $\{g^{-1}(p_i)\}$ determines a convergent sequence of points $\langle y_i\rangle$ of B. Let z be the sequential limit point of this sequence. Now $z\notin A$ since $A\cap\overline{B}=\emptyset$ by hypothesis. Thus $g(z)\notin g(A)$. But $g(z)=t$, which is an element of $g(A)$. This contradiction implies that $[\overline{g(B)-g(M)}]\cap[g(A)-g(M)]=\emptyset$. A similar argument yields that $[g(B)-g(M)]\cap[\overline{g(A)-g(M)}]=\emptyset$. These two results imply that $g(M)$ separates T.

Lemma 4.4. Suppose g is a monotone simplicial function from a triangulated n-sphere S^n onto a triangulated n-manifold M^n. If C is a set that separates S^n and $g(C)$ is a point $x\in M^n$, then at most one complementary domain of C can fail to map to x.

Proof: If more than one complementary domain of the set C fails to map to x, then x will separate M^n by Lemma 4.3. Every point of M^n has a neighborhood homeomorphic to E^n, since M^n is an n-manifold without boundary by Lemma 4.2. But if x separates M^n, it must separate all of its euclidian neighborhoods, and thus none of them can be homeomorphic to E^n. The contradiction implies the lemma.

Lemma 4.5. Suppose g is a monotone simplicial function from a triangulated n-sphere S^n onto a triangulated n-manifold M^n. If v is a vertex of M^n, then $g^{-1}(v)$ is a connected full proper subcomplex of S^n which does not separate.

Proof: The set $g^{-1}(v)$ is connected since g is monotone and does not separate because of Lemma 4.4. Now $g^{-1}(v)$ must be $\cup|A_i|$ where the A_i are simplexes of S^n, since g is simplicial. The simplexes of $g^{-1}(v)$ are properly joined and finite in number since they belong to the complex S^n. Moreover, if the vertices of any simplex of S^n are all in $g^{-1}(v)$, then the simplex itself must be in $g^{-1}(v)$, since g is barycentric. Thus $g^{-1}(v)$ is a full subcomplex of S^n. It is obviously proper, since v is not all of M^n.

Lemma 4.6. Any regular neighborhood of a connected, simply connected, full subcomplex Z of S^3 which does not separate is a 3-cell.

Proof: Let $U(Z,S^3)$ be a regular neighborhood of Z in some subdivision of S^3. Then $U(Z,S^3)$ can be constructed from

Z by a finite number of elementary geometrical expansions such that the resulting expansion is a 3-dimensional submanifold of S^3. Now $U(Z,S^3)$ is certainly connected, and is also simply connected as can be seen by the following two step inductive argument:

(1) One elementary geometrical expansion of Z must be simply connected or else the resulting expansion cannot collapse to the simply connected Z.

(2) If after m elementary geometrical expansions the result is simply connected, then another elementary geometrical expansion must be simply connected or else collapsing of the (m+1)-th result back to the simply connected m-th result would be impossible.

Since $U(Z,S^3)$ is, in addition to being simply connected, a proper connected closed 3-dimensional submanifold of S^3 which does not separate, it must have zero as its first Betti number. By Theorem 2.8, $U(Z,S^3)$ has a 2-sphere for its boundary and is thus a 3-cell. But Theorem 2.6 states that any two regular neighborhoods of Z must be homeomorphic. Thus all regular neighborhoods of Z are 3-cells.

Lemma 4.7. Let v be a vertex of M^3. Then $f^{-1}(v)$ is pointlike if and only if $f^{-1}(v)$ is simply connected.

Proof: Suppose $f^{-1}(v)$ is pointlike. Then there must exist a homeomorphism h which maps $S^3-f^{-1}(v)$ onto S^3-p such that $h[f^{-1}(v)] = p\subset f^{-1}(v)$. Let $q(x,t) = th(x) + (1-t)I$, where $t \subset [0,1]$ and I is the identity mapping. Then q is a homotopy such that $q(x,1) = h(x)$ and $q(x,0) = I$. Thus $q(f^{-1}(v), 1) = p$ and $q(f^{-1}(v), 0) = f^{-1}(v)$. It follows that $f^{-1}(v)$ is homotopic to a constant, and every simple closed curve in $f^{-1}(v)$ must also be homotopic to a constant. Therefore, $f^{-1}(v)$ is simply connected.

Suppose $f^{-1}(v)$ is simply connected. By Theorem 2.5 any derived simplicial neighborhood of $f^{-1}(v)$ is a regular neighborhood and thus a 3-cell by Lemma 4.6. If $N(f^{-1}(v), S^m{}_{f^{-1}(v)}S^3)$ is an m-th derived simplicial neighborhood of $f^{-1}(v)$, a fine enough subdivision $S^p{}_{f^{-1}(v)}S^3$ of S^3 yields $N(f^{-1}(v), S^p{}_{f^{-1}(v)}S^3) \subset \text{int } N(f^{-1}(v), S^m{}_{f^{-1}(v)}S^3)$, so that there exists a sequence of regular neighborhoods $\langle U_i \rangle$ such that $f^{-1}(v) \subset \text{int } U_i$ and $U_{i+1} \subset \text{int } U_i$ for all i. Since the subdivision may be made as fine as desired, $\cap U_i = f^{-1}(v)$, and $f^{-1}(v)$ is seen to be cellular, and therefore pointlike by Lemma 4.1.

Lemma 4.8. Suppose g is a monotone simplicial function from a triangulated n-sphere S^n onto a triangulated n-manifold M^n, and that p is not a vertex of M^n. Then if $\hat{M}^n$ in a first derived subdivision of M^n with p as a new vertex, there exists a corresponding first derived subdivision $\hat{S}^n$ of S^n such that $g\colon \hat{S}^n \to \hat{M}^n$ is a simplicial monotone function.

Proof: The function remains monotone no matter how

S_n and M^n are triangulated. So let $\hat{M}^n$ be a subdivision of M^n where $\{v_i\}$ are the new vertices of $\hat{M}^n$ and $p \subset \{v_i\}$. Each of the v_i was in the interior of some k-simplex $A_i^{\,k}$, where $1 \leq k \leq n$. Now $g^{-1}(v_i) \cap \{{}_iB_j^{\,k}\}$, where $g({}_iB_j^{\,k}) = A_i^{\,k}$, is a collection of single points $\{{}_it_j^{\,k}\}$ each of which is in the interior of one of the ${}_iB_j^{\,k}$, and $g^{-1}(v_i) \cap L$, where $g(L) = A_i^{\,k}$ and L has dimension higher than k, is the convex hull of the elements of $\{{}_it_j^{\,k}\}$ which appear in the k-faces of L that do not collapse. Now let $\{{}_iw_m\}$ be the collection of points each of which is a point interior to one of the convex hulls, where m is the number of hulls generated, and $\{{}_ix_b\}$ be a collection of points each of which is interior to one of the k-dimensional or larger simplexes of $g^{-1}(A_i^{\,k})$ which map only to a vertex of $A_i^{\,k}$. Let the set $[\{{}_it_j^{\,k}\} \cup \{{}_iw_m\} \cup \{{}_ix_b\}]$ be new vertices of $\hat{S}^n$ and complete $\hat{S}^n$ by starring each of the simplexes of S^n beginning with the highest dimensional ones and ending with the lowest. The mapping $g\colon \hat{S}^n \to \hat{M}^n$ is simplicial by construction.

Lemma 4.9. Suppose $p \subset M^3$, $\hat{M}^3$ is a first derived subdivision of M^3 which has p as a vertex and $\hat{S}^3$ is the corresponding derived subdivision of S^3 constructed in Lemma 4.8. Suppose v is a vertex of M^3 such that $[v,p]$ is a 1-simplex of $\hat{M}^3$. Then $f^{-1}[lk(v,\hat{M}^3)]$ is the boundary of $N(f^{-1}(v),\hat{S}^3)$. Furthermore, if $f^{-1}(v)$ is simply connected, then $Bd[N(f^{-1}(v),\hat{S}^3)]$ is a 2-sphere.

Proof: Since f is simplicial and the inverse image of every simplex in $st(v,\hat{M}^3)$ must be simplexes all of which either intersect $f^{-1}(v)$ or faces of such simplexes, it follows that $f^{-1}[st(v,\hat{M}^3)]\subset N(f^{-1}(v),\hat{S}^3)$. Let p be a point of B, where B is some simplex of $\hat{S}^3$ such that $f(B)\in st(v,\hat{M}^3)$. Then B either intersects $f^{-1}(v)$ or is a face of some simplex which intersects $f^{-1}(v)$. If B intersects $f^{-1}(v)$, then f(B) must intersect v since f is simplicial. If B is the face of a simplex which intersects $f^{-1}(v)$, then f(B) must be the face of some simplex which intersects v. In either case f(p) is a point of $st(v,\hat{M}^3)$. So $|N(f^{-1}(v),\hat{S}^3)|\subset f^{-1}(|st(v,\hat{M}^3)|)$ and it follows that they are equal in light of the previously established inclusion. Since $lk(v,\hat{M}^3)$ is the boundary of $st(v,\hat{M}^3)$, then $f^{-1}[lk(v,\hat{M}^3)] = Bd[N(f^{-1}(v),\hat{S}^3)]$.

Suppose $f^{-1}(v)$ is simply connected. Now $N(f^{-1}(v),\hat{S}^3)$ is a 3-cell by Theorem 2.5 and Lemma 4.6. Thus $Bd[N(f^{-1}(v),\hat{S}^3)]$ is a 2-sphere.

Lemma 4.10. If $f^{-1}(v)$ is simply connected for all vertices v of M^3, then $f^{-1}(p)$ is pointlike for every p not a vertex of M^3.

Proof: By Lemma 4.9, for any p, $f^{-1}[lk(v,\hat{M}^3)] = Bd[N(f^{-1}(v),\hat{S}^3)]$, where v is a vertex of M^3 such that [v,p] is a 1-simplex of $\hat{M}^3$. Now $f^{-1}(p)$ must be contained in $Bd[N(f^{-1}(v),\hat{S}^3)]$ since $p\subset lk(v,\hat{M}^3)$, and $Bd[N(f^{-1}(v),\hat{S}^3]$ is a 2-sphere since $f^{-1}(v)$ is simply connected. The fact that S^3

is a combinatorial 3-sphere implies that $lk(v, \hat{M}^3)$ is also a 2-sphere. But a monotone simplicial function from a 2-sphere onto a 2-sphere is pointlike, so that $f^{-1}(p)$ is pointlike on $Bd[N(f^{-1}(v), \hat{S}^3)]$. Thus $f^{-1}(p)$ is cellular on that 2-sphere which implies that there exists a sequence of tame 2-cells $\langle A_i \rangle$ on the 2-sphere such that $f^{-1}(p)$ is contained in the interior of all the A_i and $A_{i+1} \subset \operatorname{int} A_i$ for all i, where $\cap A_i = f^{-1}(p)$. Let $\langle B_i \rangle$ be the sequence of 3-cells in $\hat{S}^3$ such that $B_i = A_i \times [-\frac{1}{i}, \frac{1}{i}]$, and it is readily seen that $f^{-1}(p)$ is cellular in $\hat{S}^3$.

<u>Proof of the Main Theorem (4.1)</u>: Lemmas 4.7 and 4.10 imply the theorem.

CHAPTER V

SOME RELATED CONJECTURES

In an effort to extend Theorem 4.1 to higher dimensions in at least some form, Lemmas 4.1, 4.2, 4.3, 4.4, 4.5, and 4.8 were shown to be true for all dimensions, and Lemma 4.9 can easily be proven for all dimensions if the added statement that $Bd[N(f^{-1}(v),\hat{S}^n)]$ is an $(n-1)$-sphere is eliminated. The following are conjectures which were studied together with suggestions for their possible proof.

Conjecture 5.1. Suppose f is a monotone simplicial function from a combinatorial n-sphere S^n onto a combinatorial n-manifold M^n. Then f is pointlike if and only if the inverse image of each vertex of M^n is algebraically $[n/2]$-connected ($n \neq 4$ or 5).

If f is pointlike, it follows that $f^{-1}(v)$ is algebraically $[n/2]$-connected, since $f^{-1}(v)$ is cellular implies that there exists a sequence of tame n-cells $\langle C_i \rangle$ such that $f^{-1}(v) \subset \text{int } C_i$ and $C_{i+1} \subset \text{int } C_i$ for all i, where $\bigcap C_i = f^{-1}(v)$. The difficult part of the proof is the "if" part.

Conjecture 5.1.1. Any regular neighborhood of a connected, algebraically $[n/2]$-connected, full subcomplex Z of S^n which does not separate is an n-cell ($n \neq 4$ or 5).

It is easily seen by induction that a regular neighborhood of Z is algebraically $[n/2]$-connected. If it were true that a compact n-manifold with connected boundary which is algebraically $[n/2]$-connected has a boundary which is algebraically $[(n-1)/2]$-connected, then this conjecture would hold for dimensions $n \neq 4$ or 5 by applying the Poincaré Conjecture for $n \neq 3$ or 4 to establish that the boundary is an $(n-1)$-sphere. This result would readily imply the following version of Lemma 4.7:

Conjecture 5.1.2. Let v be a vertex of M^n. Then $f^{-1}(v)$ is pointlike if and only if $f^{-1}(v)$ is algebraically $[n/2]$-connected ($n \neq 4$ or 5).

The difficult portion of Lemma 4.9 would also follow, namely:

Conjecture 5.1.3. Suppose $f^{-1}(v)$ is algebraically $[n/2]$-connected. Then $Bd[N(f^{-1}(v),\hat{S}^n)]$ is an $(n-1)$ sphere ($n \neq 4$ or 5).

Another difficulty arises, however, in trying to prove Lemma 4.10 for n-dimensions.

Conjecture 5.1.4. If $f^{-1}(v)$ is algebraically $[n/2]$-connected for all vertices v of M^n, then $f^{-1}(p)$ is pointlike for every p not a vertex of M^n ($n \neq 4$ or 5).

The problem is that even if it were known that $Bd[N(f^{-1}(v),\hat{S}^n)]$ is an $(n-1)$-sphere, it is not known whether a monotone, simplicial function from a k-sphere onto a k-sphere is pointlike or not, $k<n$.

Another possibility was suggested by Theorem 3.3:

Conjecture 5.2. Suppose f is a monotone simplicial function from a combinatorial n-sphere S^n onto a combinatorial n-manifold M^n. Then f is pointlike if the inverse image of each vertex of M^n is geometrically [n/2]-connected.

The condition on the inverse images seems to be much stronger, since if the inverse image F^k of a vertex has dimension k, then every [n/2]-combinatorial subspace must lie in a k-ball of F^k. This means that there cannot be any simplexes in F^k of dimension less than [n/2]which are not the faces of [n/2]-simplexes. For this reason, the "only if" part of the theorem was omitted from this conjecture.

Conjecture 5.2.1. Any regular neighborhood of a connected, geometrically [n/2]-connected, full subcomplex Z of S^n which does not separate is an n-cell.

By Theorem 3.2, this conjecture is equivalent to Conjecture 5.1.1 for $n \geq 5$. Again, it can be shown that a regular neighborhood of Z is geometrically [n/2]-connected. Stalling's Theorem (3.3) could be used if it were known that a compact n-manifold with connected boundary which is geometrically [n/2]-connected has a geometrically [(n-1)/2]-connected boundary. The following could easily be shown:

Conjecture 5.2.2. Let v be a vertex of M^n. Then $f^{-1}(v)$ is pointlike if $f^{-1}(v)$ is geometrically [n/2]-connected.

Conjecture 5.2.3. Suppose $f^{-1}(v)$ is geometrically

$[n/2]$-connected. Then $Bd[N(f^{-1}(v), \hat{S}^n)]$ is an $(n-1)$-sphere.

The counterpart of Lemma 4.10 for Conjecture 5.2 has the same difficulties explained in Conjecture 5.1.4.

One other modification was considered in some detail, but no more progress was made:

Conjecture 5.3. Suppose f is a monotone simplicial function from a combinatorial n-sphere S^n onto a combinatorial n-manifold M^n. Then f is pointlike if the inverse image of each vertex of M^n is collapsible.

The reverse implication is not true, since there are examples which show that contractible k-complexes need not be collapsible.

Lemma 5.3.1. A regular neighborhood of a collapsible subcomplex of S^n is an n-cell.

Three of Zeeman's Theorems (2.4, 2.5, and 2.6) taken together imply this lemma. It follows immediately that the next lemma is true:

Lemma 5.3.2. Let v be a vertex of M^n. Then $f^{-1}(v)$ is pointlike if $f^{-1}(v)$ is collapsible.

In trying to establish that $f^{-1}(p)$ is collapsible for p not a vertex of M^n, it could not be ascertained that $f^{-1}(p)$ is not one of those non-collapsible yet contractible k-complexes mentioned above.

These conjectures suggest many areas for further research.

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