THE EFFECT OF TENSION ON THE DYNAMIC BEHAVIOR OF ECCENTRIC SHAFTS ROTATING IN FLUID MEDIUM

A Dissertation

Presented to

the Faculty of the Department of Mechanical Engineering

University of Houston

In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

---

by

Victor Prodonoff

May 1972

ACKNOWLEDGMENTS

The author wishes to express his sincere appreciation to the following:

Dr. C. D. Michalopoulos, his advisor, for suggesting the subject of this dissertation, for providing initial literature and for guidance and assistance throughout the entire research;

Conselho Nacional de Pesquisas, a research institute of the Brazilian Government, for financial support for his doctoral study program;

Mrs. Shirley Brunson who kindly read the manuscript;

and above all to Vivina, his wife, who always encouraged him in pursuing graduate study despite the sacrifice and homesickness (Rio de Janeiro!) that it would require, which she bore in silence, and to Victor Jr. and Viviane, his children, whose smiles impel the author to accept any challenge.

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ABSTRACT

Using Euler-Bernoulli beam theory an investigation of the dynamic behavior of an eccentric rotating shaft, subject to linearly varying or constant tension, was made. The shaft has distributed mass and elasticity and is suspended in a fluid. Initial lack of straightness was also included in the analysis. The local mass eccentricity is assumed to be a deterministic function of the axial coordinate.

For the variable-tension case the response was determined for a vertical shaft simply supported at the top and vertically guided at the bottom. The constant-tension case was analyzed for a shaft simply supported at its ends. The solution was obtained using modal analysis. It is in series form and is expressed in terms of characteristic functions of the free vibration shaft.

External damping was linearized by equating the energy dissipated per revolution by quadratic and equivalent viscous damping.

Displacements and stresses were computed along the shaft at a specific speed of rotation. Also maximum stress

and displacement were computed for speeds in the neighborhood of a natural frequency. Results are given in graphical form for several values of the tension and different eccentricity functions.

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LIST OF SYMBOLS

a, b	components of shaft eccentricity in the U and V directions, respectively
a_n, b_n, c_n, d_n	coefficients of the four independent solutions in the power series expansion of ϕ_r
a_r, b_r, c_r, d_r	Fourier coefficients in the eigenvalue series expansion of a, b, u_o and v_o, respectively
$a_{ro}, b_{ro}, c_{ro}, d_{ro}$	Eqs. (5.43) - (5.46)
$a^*_{ro}, b^*_{ro}, c^*_{ro}, d^*_{ro}$	Eqs. (5.57) - (5.60)
ς	viscous damping coefficient, slug/ft. sec.
d	inside diameter of shaft
e	arbitrary unit of eccentricity
e_r	$a_r + i\, b_r$
f_r	$c_r + i\, d_r$
g	$\gamma L^3/EI$, dimensionless pseudo-weight parameter
h	$T_o L^2/EI$, dimensionless tension
h_r	$\int_0^1 \phi_r^2(\xi)\, d\xi$
i	$\sqrt{-1}$

k_r	$\left[m\omega_r^2 L^4/EI\right]^{1/4}$, non-dimensional rth natural frequency
m	mass per unit length participating in motion
m_o	mass per unit length of the surrounding fluid, with the same shape as a solid shaft
p	number of disks
$p_r(t)$, $q_r(t)$	normal coordinates, unknown in the eigenvalue series expansion of u and v, respectively
t	time
u, v	displacements of the shaft central axis in the U and V directions, respectively
u_o, v_o	components of the initial lack of straightness in the U and V directions, respectively
x, y	displacements of the shaft central axis in the X and Y directions, respectively
z	distance along the Z-direction
A_r	rth arbitrary constant
C_R	$\sum_{n=0}^{\infty} a_n^{(r)} \Big/ \sum_{n=0}^{\infty} b_n^{(r)}$
D	outside diameter of shaft
EI	bending stiffness of the shaft
H_r	$\int_0^L \phi_r^2(z)\, dz$, normalizing constant
L	length of shaft

M_o	EIe/L^2
$M_u(z)$, $M_v(z)$	components of bending moment measured along the U and V directions
R_m^2	$1 + m_o/m$
$R_{o,r}$	ratio of the rth natural frequency to the speed of the shaft, for zero tension along the shaft
R_r	ratio of the rth natural frequency to the speed of the shaft
$S_u(z)$, $S_v(z)$	components of shear force in the U and V directions
$T(z)$	tension at a section with distance z from the bottom of the shaft
T_o	tension at the bottom end of the shaft; or constant tension along the shaft
$T_u(z), T_v(z), T_z(z)$	components of shaft tension in the U, V and Z directions
UVZ	rotating reference
XYZ	fixed (inertial) reference
Z	axis of rotation; centerline of bearings
α	c/m, viscous damping coefficient, sec^{-1}
α_o	$c/m\Omega$, non-dimensional viscous damping coefficient
γ	weight per unit length of the shaft, in the fluid
ϵ_i	one half of the thickness of the ith disk
ϵ_{oi}	one half of the non-dimensional thickness of the ith disk

ζ_i, η_i	non-dimensional components of eccentricity of the ith disk in the U and V directions, respectively
η_r	$p_r + i\, q_r$, complex normal coordinate
λ	Eqs. (4.44); or index of the solution for the power series expansion of ϕ_r, Eq. (5.12)
λ_i	ratio of the mass per unit length of the ith disk plus shaft to the mass per unit length of the shaft
$\mu(\xi), \nu(\xi)$	non-dimensional components of the deflection in the U and V directions, respectively
$\mu_0(\xi), \nu_0(\xi)$	non-dimensional components of the initial lack of straightness in the U and V directions, respectively
ξ	non-dimensional distance along Z-axis
ξ_i	non-dimensional distance of the ith disk
σ	maximum bending stress at a section
σ_0	EIe/L^2D^3
ϕ_r	rth modal shape
ψ_r	Eq. (5.26)
$\omega_{0,r}$	rth natural frequency for zero tension along the shaft
ω_r	rth natural frequency
$M_\mu(\xi), M_\nu(\xi)$	non-dimensional components of the bending moment measured along the U and V directions, respectively
$M_{\mu\nu}(\xi)$	total non-dimensional bending moment at a section

Ω	speed of rotation of the shaft

Subscripts

i	refers to ith disk
n	nth coefficient in the power series expansion of ϕ_r
r	refers to rth normal mode
uvz	refers to UVZ reference
u, v, z	refers to components along the U , V and Z axes
xyz	refers to XYZ reference

Superscripts

(r)	refers to rth normal mode
*	refers to a shaft with constant tension

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Chapter 1

INTRODUCTION

The dynamic behavior of rotating shafts has been the object of a great deal of attention in the past. Early studies consider only heavy discs mounted on a massless elastic shaft. Due to the inadequacy of that theory to many of the modern rotors, intensive investigations of shafts with distributed mass and elasticity have been made over the past decade or so.

Jeffcott [1][1] was the first to establish in a rational basis the theory of the whirling of a shaft. He considered a single (heavy) mass attached to a thin elastic shaft. Linear damping was included. The behavior of the shaft was studied close to the natural frequency of the system. This theory was experimentally proved by Taylor [2], who also simplified Jeffcott's analysis, using non-dimensional parameters. An important step forward was made by Robertson [3] who introduced the rotating system, providing

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[1] Numbers in brackets designate References at the end of the dissertation.

a better understanding of the problem. Johnson [4], [5] first introduced the equations for distributed mass and elasticity, considering a rotating body in general and analyzing the response with normal coordinates. As special cases of rotating bodies, he considered: (1) thin circular shaft, simply supported at the ends, (2) the previous case with a heavy rigid wheel attached to the shaft, and (3) thin uniform ring. Following Johnson's approach, Bishop [6] particularized and extended the theory of rotating bodies to the case of rotating flexible shafts. The response of unbalance was developed by modal analysis using normal coordinates. Based on this theory a modal balancing was first suggested for flexible rotors. Parkinson [7] summarized work done by Bishop and co-workers between 1959 and 1967, explaining in simple terms the fundamental behavior of rotating shafts. Ariaratnam [8] extended Bishop's theory to the case of a shaft with unequal stiffness, including also the effect of gravity force.

Closely related to the problem of a rotating shaft subject to non-uniform tension is that of a vibrating beam with the same kind of tension. Graham et al. [9] derived equations for a drill string considering elastic, dynamic and drag forces. The actual string was then assumed to be

a short beam under constant tension at its ends and a long perfectly flexible cable under variable tension in its central part. Exact solutions were found for both cases and the solution for the drill string was obtained joining the beam and cable solutions subject to boundary conditions at each joint. The analysis was for a plane motion and forcing functions were considered as boundary conditions. Huang et al. [10] analyzed a similar problem as Graham, but obtained an improved solution, with the tension varying throughout the beam. However, their analysis was restricted to a two-dimensional case.

At present no one has considered the response of a eccentric rotating shaft, with distributed mass and elasticity and variable tension over its full length and in a spatial rather than a plane configuration.

Many shafts have to run through several natural frequencies to reach the running speed. Thus the study around the fundamental frequency does not give the true picture of the problem. The objective of this dissertation is to show the behavior of the shaft under tension around the natural frequency closer to the operational speed. The instability and the influence of tension on the displacement and stress are discussed. Where applicable, a comparison

is made between the response of a shaft under linearly varying tension and one with constant tension (average value).

The differential equations of motion of an eccentric rotating shaft subject to non-uniform tension are derived in Chapter 3. They are two partial differential equations of fourth order with constant and variable coefficients. The system is then solved by modal analysis, using characteristic functions of a rotating shaft. The basic theory is developed in Chapter 4 for any kind of non-uniform tension. Two particular cases are studied in Chapter 5, namely: (1) linearly varying tension and (2) constant tension. In Appendix 1, the problem of constant tension is also solved by means of Fourier Sine Transforms. Numerical examples are given in Chapter 6, where the eccentricity functions chosen are defined there.

Chapter 2

STATEMENT OF THE PROBLEM

The object of the present study is to determine analytically the effect of linearly varying or constant tension on the dynamic behavior of an eccentric shaft, rotating in a fluid medium. The analysis considers a shaft with distributed mass and elasticity. The eccentricity is assumed to be a deterministic function of the axial coordinate. The effect of small initial lack of straightness is also considered.

In the derivation of the differential equations of motion for a shaft with non-uniform axial tension in Chapter 3, the following assumptions are made:

1. The shaft material is linearly elastic.

2. Only lateral deflection is considered which is assumed small enough such that linear theory can be applied.

3. Transverse shear and rotatory inertia are negligible.

4. The diameter (or characteristic dimension of cross section) of the shaft is small compared to its length, so that Euler-Bernoulli theory for beams is valid.

5. Internal damping is negligible by comparison to external damping.

The shaft under consideration is shown in Fig. 1, loaded axially by its own weight and the force T_o . Fig. 2 is a cross section of the shaft, showing the center of mass (CM) and the components of shaft eccentricity (a and b) of the section.

As for the constant-tension case everything above holds but the shaft being simply supported at both ends.

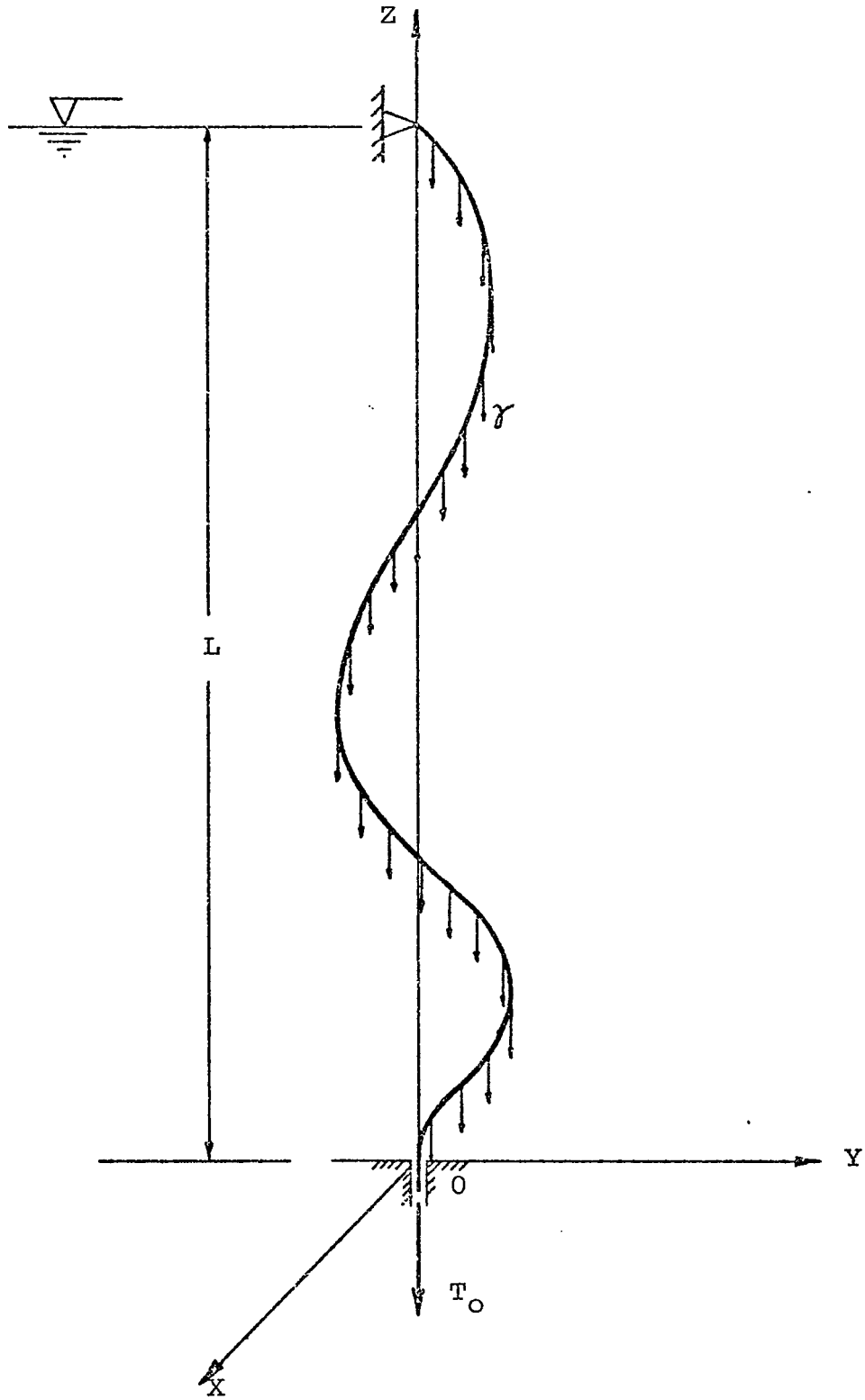

Fig.1 Shaft with Linearly-Varying Tension in its Deflected Position

Chapter 3

DERIVATION OF THE GOVERNING DIFFERENTIAL EQUATIONS

The differential equations of motion for the shaft described in the previous chapter are derived here using Newton's Second Law. For this aim the equilibrium is set up for an element of the shaft of length ΔL and mass Δm (shown in Fig. 3) acted upon the following external forces: gravity, tension, shear, lift and drag. According to the assumptions made in Chapter 2, the mass of the element can be thought of as concentrated in its center of mass.

The equations in a rotating system show advantages over those of the fixed one since in the latter, time-variable coefficients govern the equations. The rotating system is then used to describe the motion.

In this chapter there is no restriction on the variation of axial tension with distance along the shaft.

3.1 Definitions

Consider a vertical shaft with uniform section as shown in Fig. 1. To obtain the equilibrium equations, the following coordinate systems are used:

XYZ - fixed (inertial) reference with Z-axis vertical, positive upward,

UVZ - rotating reference with Z-axis the same as in XYZ ; the U and V axes are perpendicular to Z and rotate with a constant angular velocity Ω .

In connection with these references, there are two sets of unit vectors:

$\bar{i}, \bar{j}, \bar{k}$ - unit vectors for XYZ reference,

$\bar{e}_u, \bar{e}_v, \bar{e}_z$ - unit vectors for UVZ reference.

A (circular) cross section of the shaft is shown in Fig. 2, where the following are defined:

$u(z), v(z)$ - displacements of the shaft axis,

$a(z), b(z)$ - components of shaft eccentricity, constant at each section in UVZ reference,

$\bar{R}$ - position vector of origin of UVZ with respect to XYZ reference,

$\bar{\rho}$ - position vector of the center of mass (CM) of the element,

$\bar{\omega}$ - angular velocity vector of UVZ system,

$\bar{r}$ - position vector of the centroid of the element.

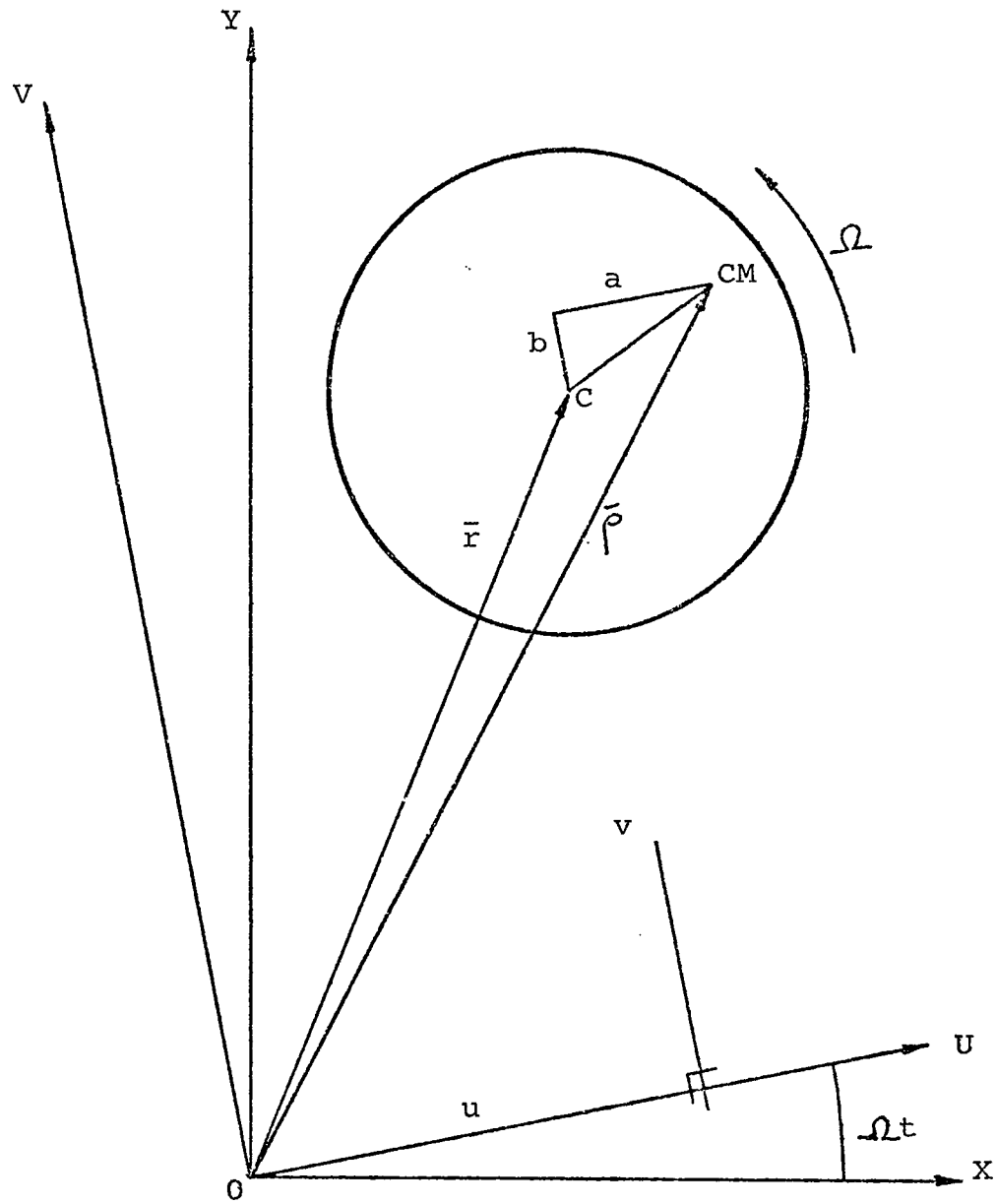

Fig. 2 (Circular) Cross Section of the Shaft. XYZ - Fixed Reference; UVZ - Rotating Reference; u, v - Components of Shaft Displacement; a, b - Components of Shaft Eccentricity; C - Centroid; CM - Center of Mass

Also, it is implied by Fig. 2 that the whirling frequency of the shaft is assumed to be synchronous with shaft rotational speed and takes place about the vertical Z -axis.

3.2 Differential Equations of Motion

3.2.1 Acceleration

Let $\bar{a}_{xyz}$ be the absolute acceleration of center of mass of the slice (as observed from the XYZ reference). Then

$$\bar{a}_{xyz} = \ddot{\bar{R}} + \ddot{\bar{\rho}}\big)_{uvz} + 2\,\bar{\omega} \times \dot{\bar{\rho}}\big)_{uvz} + \dot{\bar{\omega}} \times \bar{\rho} + \bar{\omega} \times (\bar{\omega} \times \bar{\rho}) \tag{3.1}$$

where the dots have their familiar meaning. The vectors $\bar{R}$, $\bar{\rho}$ and $\bar{\omega}$ are given by (see Fig. 2)

$$\bar{R} = 0 \quad \text{(origins always coincident)},$$

$$\bar{\rho} = (u+a)\bar{e}_u + (v+b)\bar{e}_v + z\,\bar{e}_z \,,$$

$$\bar{\omega} = \Omega\,\bar{k} \,.$$

Taking the time derivatives of these vectors and substituting in Eq. (3.1), $\bar{a}_{xyz}$ becomes

$$\bar{a}_{xyz} = \left[\ddot{u} - 2\Omega\dot{v} - \Omega^2(u+a)\right]\bar{e}_u + \left[\ddot{v} + 2\Omega\dot{u} - \Omega^2(v+b)\right]\bar{e}_v \,. \tag{3.2}$$

3.2.2 External Forces

The components of tension in the UZ plane are shown in Fig. 3. The net tension and gravity force acting on the element are:

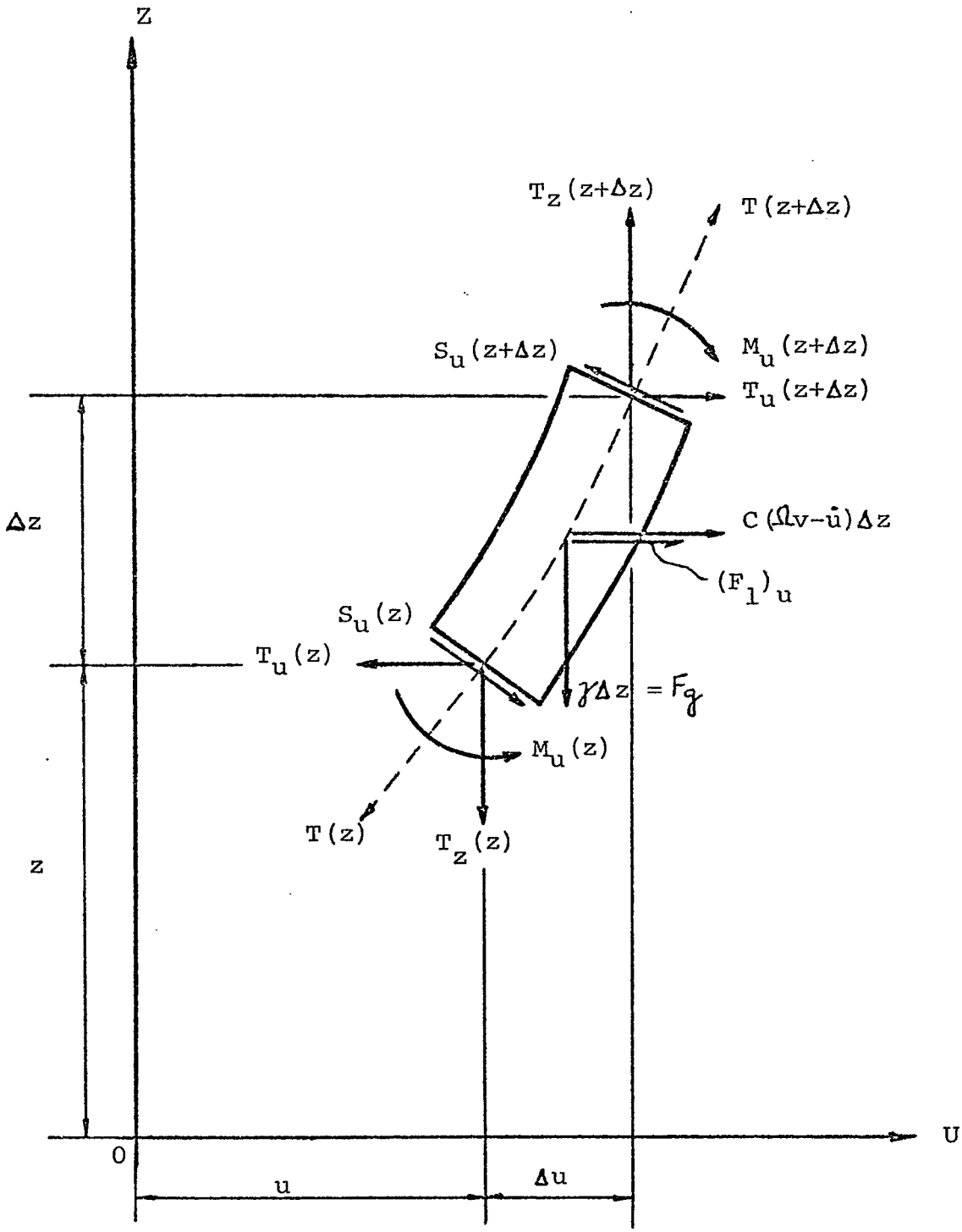

Fig.3 Forces and Bending Moment on a Shaft Element of ΔL Length - Plane UZ

$$F_t = \left[T_u(z+\Delta z) - T_u(z)\right]\bar{e}_u + \left[T_v(z+\Delta z) - T_v(z)\right]\bar{e}_v +$$

$$\left[T_z(z+\Delta z) - T_z(z) - F_g\right]\bar{e}_z$$

where $T_u(z+\Delta z)$ is the component of tension in the u direction, at a distance $z+\Delta z$ from the bottom. The assumption of small displacements leads to

$$F_t = \left[\left(T\frac{\Delta u}{\Delta z}\right)_{z+\Delta z} - \left(T\frac{\Delta u}{\Delta z}\right)_z\right]e_u + \left[\left(T\frac{\Delta v}{\Delta z}\right)_{z+\Delta z} - \left(T\frac{\Delta v}{\Delta z}\right)_z\right]\bar{e}_v +$$

$$+ \left[T(z+\Delta z) - T(z) - F_g\right]\bar{e}_z \tag{3.3}$$

where T is the total tension at a section.

The drag forces are linearized by equating the energies dissipated per cycle by quadratic and equivalent viscous damping as shown in Appendix 2. The linearized force is given by

$$F_d = -(\Delta z)\, c\, \bar{v}_{xyz} \tag{3.4}$$

where c is the equivalent viscous damping coefficient (constant) and $\bar{v}_{xyz}$ is the absolute centroidal velocity of the element, given by

$$\bar{v}_{xyz} = \dot{\bar{R}} + \dot{\bar{r}}\big)_{uvz} + \bar{\omega} \times \bar{r} .$$

From Fig. 2, the vector $\bar{r}$ is

$$\bar{r} = u\,\bar{e}_u + v\,\bar{e}_v + z\,\bar{e}_z\,,$$

which yields for the velocity

$$\bar{v}_{xyz} = (\dot{u} - \Omega v)\bar{e}_u + (\dot{v} + \Omega u)\bar{e}_v\,.$$

Substitution of this expression into (3.4) yields for the viscous force

$$\bar{F}_d = -\Delta z\, C\,(\dot{u} - \Omega v)\bar{e}_u - \Delta z\, C\,(\dot{v} + \Omega u)\bar{e}_v\,. \qquad (3.4a)$$

The resultant shear force, denoted by $\bar{F}_s$, is

$$\bar{F}_s = \left[-S_u(z+\Delta z) + S_u(z)\right]\bar{e}_u + \left[-S_v(z+\Delta z) + S_v(z)\right]\bar{e}_v \qquad (3.5)$$

where the components along the Z-direction have been neglected. Note that in Fig. 3 only the U-component of F_s , S_u , is shown.

The lift force can be written as[1]

$$\bar{F}_\ell = \Delta z\left[(F_\ell)_u\,\bar{e}_u + (F_\ell)_v\,\bar{e}_v\right]. \qquad (3.6)$$

3.2.3 Scalar Equations in the U and V Directions

Using Eqs. (3.2), (3.3), (3.4a), (3.5) and (3.6) the equation of motion in the U-direction is

$$m\,\Delta z\left[\ddot{u} - 2\Omega v + \Omega^2(u+a)\right] = -S_u(z+\Delta z) + S_u(z) +$$

$$+\left(T\frac{\Delta u}{\Delta z}\right)_{z+\Delta z} - \left(T\frac{\Delta u}{\Delta z}\right)_z - \Delta z\, C\,(\dot{u} - \Omega v) + \Delta z\,(F_\ell)_u$$

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[1] See explanation at the end of the chapter.

where m is the total mass per unit length. Dividing both sides of this equation by $m\Delta z$ and letting Δz approach zero, the following differential equation is obtained:

$$\ddot{u} - 2\Omega\dot{v} + \Omega^2(u+a) = -\frac{1}{m}\frac{\partial S_u}{\partial z} + \frac{1}{m}\frac{\partial}{\partial z}\left(T\frac{\partial u}{\partial z}\right) - \frac{c}{m}(\dot{u} - \Omega v) + (F_\ell)_u \quad (3.7)$$

where $(F_\ell)_u$ is the U-component of the lift force per unit mass.

The shear S_u can be eliminated, using a moment equation in the U-direction. From Fig. 3

$$\left[S_u(z+\Delta z) + S_u(z) - \left(T\frac{\Delta u}{\Delta z}\right)_{z+\Delta z} - \left(T\frac{\Delta u}{\Delta z}\right)_z\right]\frac{\Delta z}{2} +$$
$$\left[T(z+\Delta z) + T(z)\right]\frac{\Delta u}{2} - M_u(z+\Delta z) + M_u(z) = 0.$$

Dividing now by Δz and letting $\Delta z \to 0$ the tension terms cancel out, leaving

$$S_u = \frac{\partial M_u}{\partial z}. \quad (3.8)$$

The bending moment M_u is related to the net curvature of a beam by the well-known relation

$$M_u = EI\frac{\partial^2(u-u_0)}{\partial z^2} \quad (3.9)$$

where EI is the bending stiffness of the beam and $u_0(z)$ is the coordinate representing the initial lack of straightness of the shaft, constant with time in the UVZ

system. Using Eqs. (3.8) and (3.9) a substitution can be made for $\frac{\partial S_u}{\partial z}$ in Eq. (3.7), which with the notation

$$\alpha = \frac{G}{m} \tag{3.10}$$

yields

$$\ddot{u} - 2\Omega v - \Omega^2(u+a) = -\frac{EI}{m}\frac{\partial^4(u-u_0)}{\partial z^4} + \frac{1}{m}\frac{\partial}{\partial z}\left(T\frac{\partial u}{\partial z}\right) - \alpha(\dot{u} - \Omega v) + (F_\ell)_u \tag{3.11a}$$

The equilibrium equation in the V-direction is easily obtained if one recalls that in going from the U-axis to the V-axis, a positive rotation of 90° is necessary. Calling j the operator that performs this transformation, such that $ju = v$, $jv = -u$, $j\frac{\partial}{\partial t}(u) = \frac{\partial}{\partial t}(ju)$, etc., the equation in the V-direction is

$$\ddot{v} + 2\Omega\dot{u} - \Omega^2(v+b) = -\frac{EI}{m}\frac{\partial^4(v-v_0)}{\partial z^4} + \frac{1}{m}\frac{\partial}{\partial z}\left(T\frac{\partial v}{\partial z}\right) - \alpha(\dot{v} + \Omega u) + (F_\ell)_v \,. \tag{3.11b}$$

No expression in the literature is available at present for the transient lift force of a body moving in a real fluid. The steady lift force for an ideal fluid is

$$\bar{F}_{\ell s} = \frac{1}{2}\rho_0 \, \Delta z \left(\bar{V}_{0,xyz} \times \bar{\Gamma}\right)$$

where $\bar{V}_{0,xyz}$ is the steady velocity, $\bar{\Gamma}$ is the circulation

vector and ρ_o is the mass density of the surrounding fluid. If the same procedure used for the other forces is applied here (dividing by $m\Delta z$ and letting $\Delta z \rightarrow 0$), it can be shown that the steady lift force per unit mass is

$$\bar{F}_{\ell p} = \frac{m_o}{m}\left(\Omega^2 u\, \bar{e}_u + \Omega^2 v\, \bar{e}_v\right) \tag{3.12}$$

where m_o is the mass per unit length of the fluid with the same shape as the (solid) shaft.

The steady inertia "force" per unit mass is $\Omega^2 u\, \bar{e}_u + \Omega^2 v\, \bar{e}_v$, so by Eq. (3.12) the steady lift force is a fraction of the inertia "force".

Chapter 4

SOLUTION OF GENERAL EQUATIONS BY MODAL ANALYSIS

The differential equations of motion, Eqs. (3.11a,b), will be solved in this chapter by eigenfunction expansions. It is assumed that the displacements u and v of the shaft central axis can be represented by the following series of orthogonal functions $\phi_r(z)$:

$$u(z,t) = \sum_{r=1}^{\infty} p_r(t)\ \phi_r(z) \tag{4.1a}$$

$$v(z,t) = \sum_{r=1}^{\infty} q_r(t)\ \phi_r(z) \tag{4.1b}$$

where

p_r, q_r are unknown functions of time (known as normal coordinates),

ϕ_r is the rth modal shape, or rth normal function.

It should be noted that such an expansion into orthogonal functions is always possible.

The functions $\phi_r(z)$ depend on the particular kind of tension and boundary conditions of the freely

vibrating shaft. Section 4.1 shows how to obtain them. In order to find $p_r(t)$ and $q_r(t)$, the orthogonality of the functions $\phi_r(z)$ is necessary and is shown in Section 4.2. The question of stability (critical speeds) arises while solving for $p_r(t)$ and $q_r(t)$ and this is discussed at the end of the chapter.

4.1 The Characteristic Functions of a Shaft Running in Ideal Bearings

Dropping the terms involving u_o, v_o, a, b, α and also $(F_\ell)_u$ and $(F_\ell)_v$ from Eqs. (3.11a,b), i.e. disregarding the effects of lack of straightness, eccentricity, damping and lift force, the following differential equations governing free vibration of the shaft are obtained:

$$\ddot{u} - 2\Omega\dot{v} - \Omega^2 u = -\frac{EI}{m}\frac{\partial^4 u}{\partial z^4} + \frac{1}{m}\frac{\partial}{\partial z}\left(T\frac{\partial u}{\partial z}\right) \tag{4.2a}$$

$$\ddot{v} + 2\Omega\dot{u} - \Omega^2 v = -\frac{EI}{m}\frac{\partial^4 v}{\partial z^4} + \frac{1}{m}\frac{\partial}{\partial z}\left(T\frac{\partial v}{\partial z}\right). \tag{4.2b}$$

Let a complex variable ζ be defined by

$$\zeta = u + iv = r e^{i\theta}. \tag{4.3}$$

If Eq. (4.2b) is multiplied by the imaginary i and added to

Eq. (4.2a) the following equation results:

$$\ddot{\xi} + 2i\Omega\dot{\xi} - \Omega^2\xi = -\frac{EI}{m}\frac{\partial^4\xi}{\partial z^4} + \frac{1}{m}\frac{\partial}{\partial z}\left(T\frac{\partial\xi}{\partial z}\right). \tag{4.4}$$

Another complex quantity is now introduced through the relation

$$\psi = \xi e^{i\Omega t} = r\, e^{i(\theta+\Omega t)} \tag{4.5}$$

which can be visualized by reference to Fig. 4 as the complex variable which defines the position of point C with respect to the XYZ system. This variable can also be written as

$$\psi = x + iy. \tag{4.6}$$

If Eq. (4.4) is multiplied by $e^{i\Omega t}$, the left-hand side can be shown to be $\ddot{\psi}$. Thus, using Eq. (4.5),

$$\frac{EI}{m}\frac{\partial^4\psi}{\partial z^4} - \frac{1}{m}\frac{\partial}{\partial z}\left(T\frac{\partial\psi}{\partial z}\right) + \ddot{\psi} = 0. \tag{4.7}$$

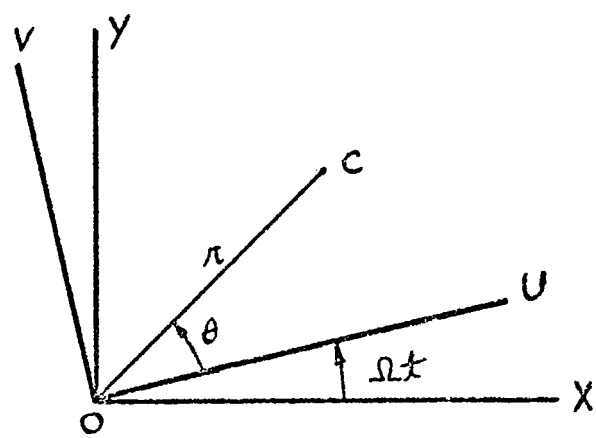


Fig. 4 Position of pt. C in the complex plane:

$$\xi = r e^{i\theta}; \quad \psi = r\, e^{i(\theta+\Omega t)}.$$

The solution of Eq. (4.7) is assumed to be of the form

$$\psi(z,t) = K\ \phi(z)\ \mathcal{T}(t) \tag{4.8}$$

where K is a complex constant, $\phi(z)$ a function of z only and $\mathcal{T}(t)$ a function of time t. Substituting $\psi(z,t)$ in Eq. (4.7) and dividing by $\phi\,\mathcal{T}$, one obtains (by the method of separation of variables) the following two ordinary differential equations:

$$\frac{EI}{m}\frac{d^4\phi}{dz^4} - \frac{1}{m}\frac{d}{dz}\left(T\frac{d\phi}{dz}\right) - \omega^2\phi = 0, \tag{4.9}$$

$$\ddot{\mathcal{T}} + \omega^2\mathcal{T} = 0. \tag{4.10}$$

Eq. (4.10) has the well-known solution $\mathcal{T} = A\cos\omega t + B\sin\omega t$. If K is expressed in its real and imaginary parts, ψ can be written as

$$\psi = (k_1 + i\,k_2)(A\cos\omega t + B\sin\omega t)\ \phi(z).$$

Separation of complex and imaginary parts gives the following two equations:

$$x(z,t) = \left[A_1\cos\omega t + B_1\sin\omega t\right]\phi(z) \tag{4.11a}$$

$$y(z,t) = \left[A_2\cos\omega t + B_2\sin\omega t\right]\phi(z) \tag{4.11b}$$

where $A_1 = k_1 A$, $B_1 = k_1 B$, $A_2 = k_2 A$, $B_2 = k_2 B$ are arbitrary constants. Thus, the most general motion of the point $C\ (x, y)$, position of shaft central axis at a height z , is an ellipse on a plane parallel to the XY plane, with angular velocity ω .

Equation (4.9) has to be solved for a specific tension $T\ (z)$, with the appropriate boundary conditions. It constitutes an eigenvalue problem with ω_r as the eigenvalue and $\phi_r\ (z)$ as the eigenfunction. The latter is the modal shape used in the expansion of $u\ (z, t)$ and $v\ (z, t)$, Eqs. (4.1a,b).

For certain types of tension $T\ (z)$ it is possible to evaluate the natural frequencies and mode shapes of the shaft in a closed form. However, in general this is not possible; approximate methods must be used, such as truncated Power Series or Fourier Cosine and Sine Series. Fortunately, only a few functions $\phi_r\ (z)$ and ω_r in practice are necessary for the solution of the problem of rotating shafts with acceptable accuracy.

In Chapter 5, the determination of $\phi_r(z)$ is presented for two different cases: (1) linearly varying tension, simply supported at the top and vertically guided at the bottom and (2) constant tension, simply supported at both ends.

4.2 Orthogonality of the Mode Shapes

Consider two distinct modal functions $\phi_r(z)$ and $\phi_s(z)$ associated with the natural frequencies ω_r and ω_s, respectively. Since each function is a solution of Eq. (4.9), one can write

$$\frac{d^4\phi_r}{dz^4} - \frac{1}{EI}\frac{d}{dz}\left(T\frac{d\phi_r}{dz}\right) - k_r^4\,\phi_r = 0 \tag{4.12}$$

$$\frac{d^4\phi_s}{dz^4} - \frac{1}{EI}\frac{d}{dz}\left(T\frac{d\phi_s}{dz}\right) - k_s^4\,\phi_s = 0 \tag{4.13}$$

where $k_r^4 = \dfrac{m\omega_r^2}{EI}$ and $k_s^4 = \dfrac{m\omega_s^2}{EI}$. (4.14), (4.15)

Multiplying Eq. (4.12) by ϕ_s and (4.13) by ϕ_r, subtracting the second from the first and integrating between 0 and L, yields

$$\int_0^L \phi_s \frac{d^4\phi_r}{dz^4}\,dz - \int_0^L \phi_r \frac{d^4\phi_s}{dz^4}\,dz + \frac{1}{EI}\int_0^L \phi_r \frac{d}{dz}\left(T\frac{d\phi_s}{dz}\right)dz -$$

$$- \frac{1}{EI}\int_0^L \phi_s \frac{d}{dz}\left(T\frac{d\phi_r}{dz}\right)dz - \left(k_r^4 - k_s^4\right)\int_0^L \phi_r\,\phi_s\,dz = 0. \tag{4.16}$$

Integrating by parts the first, third and fourth integrals and simplifying, the following is obtained:

$$\left[\phi_s \frac{d^3\phi_r}{dz^3} - \frac{d\phi_s}{dz}\frac{d^2\phi_r}{dz^2} + \frac{d^2\phi_s}{dz^2}\frac{d\phi_r}{dz} - \frac{d^3\phi_s}{dz^3}\phi_r\right]_0^L +$$

$$+ \frac{1}{EI}\left\{\left[\phi_r T \frac{d\phi_s}{dz}\right]_0^L - \left[\phi_s T \frac{d\phi_r}{dz}\right]_0^L\right\} - (k_r^4 - k_s^4)\int_0^L \phi_s \phi_r \, dz = 0. \tag{4.17}$$

For any combination of the 3 common boundary conditions (pinned, clamped or sliding end) the expressions inside the brackets are zero. Also they are zero if the tension vanishes at the free end of the shaft.

If it is considered that the shaft meets the above requirements, Eq. (4.17) reduces to

$$(k_r^4 - k_s^4)\int_0^L \phi_s \phi_r \, dz = 0. \tag{4.18}$$

Since $r \neq s$, or $k_r \neq k_s$, the integral in Eq. (4.18) must vanish. For the special case $r = s$, the integral is a non-zero quantity H_r, called the normalizing constant. Summarizing,

$$\int_0^L \phi_r \phi_s \, dz = \begin{cases} 0, & \text{for } r \neq s \quad (4.19) \\ H_r, & \text{for } r = s. \quad (4.20) \end{cases}$$

Thus, it has been shown that the modal shapes

$\phi_r(z)$ form a complete set of orthogonal functions in the interval $(0, L)$ for shafts which have pinned, clamped or sliding ends in any combination, or with free ends (zero shear force and bending moment) in which the tension vanishes.

4.3 Normal Coordinates of the Shaft

It remains now to obtain the expressions for the normal coordinates $p_r(t)$ and $q_r(t)$ in order to evaluate u and v from Eqs. (4.1a,b). The ϕ_r's are supposed known, although in general it is not an easy task to find them unless the tension is of a very special function of the axial coordinate. Eqs. (4.1a,b), repeated here for convenience, are

$$u(z,t) = \sum_{r=1}^{\infty} p_r(t)\, \phi_r(z) \tag{4.1a}$$

$$v(z,t) = \sum_{r=1}^{\infty} q_r(t)\, \phi_r(z). \tag{4.1b}$$

It is assumed also that the following expansions are valid:

$$a(z) = \sum_{r=1}^{\infty} a_r\, \phi_r(z) \tag{4.21}$$

$$b(z) = \sum_{r=1}^{\infty} b_r\, \phi_r(z) \tag{4.22}$$

$$\frac{EI}{m}\frac{d^4 u_0(z)}{dz^4} = \sum_{n=1}^{\infty} c_n \omega_n^2 \phi_n(z) \tag{4.23}$$

$$\frac{EI}{m}\frac{d^4 v_0(z)}{dz^4} = \sum_{n=1}^{\infty} d_n \omega_n^2 \phi_n(z) \tag{4.24}$$

where ω_n is the natural frequency associated to $\phi_n(z)$ in Eq. (4.9), and a_n, b_n, c_n, d_n are constants obtained from the expressions

$$a_n = \frac{1}{H_n}\int_0^L a(z)\,\phi_n(z)\,dz \tag{4.25}$$

$$b_n = \frac{1}{H_n}\int_0^L b(z)\,\phi_n(z)\,dz \tag{4.26}$$

$$c_n = \frac{EI}{m\,\omega_n^2 H_n}\int_0^L \frac{d^4 u_0(z)}{dz^4}\,\phi_n(z)\,dz \tag{4.27}$$

$$d_n = \frac{EI}{m\,\omega_n^2 H_n}\int_0^L \frac{d^4 v_0(z)}{dz^4}\,\phi_n(z)\,dz\,. \tag{4.28}$$

Eq. (4.25), for example, is obtained by multiplying Eq. (4.21) by $\phi_s(z)$, integrating in the interval $(0, L)$ and making use of Eqs. (4.19) and (4.20).

Substitution of Eqs. (4.1a,b) and (4.21)-(4.24) into Eqs. (3.11a,b) yields (disregarding for now $(F_\ell)_u$ and $(F_\ell)_v$)

$$\sum_{n=1}^{\infty}\ddot{p}_n\phi_n - 2\Omega\sum_{n=1}^{\infty}\dot{q}_n\phi_n - \Omega^2\left[\sum_{n=1}^{\infty}(p_n+a_n)\phi_n\right] = -\frac{EI}{m}\sum_{n=1}^{\infty}p_n\frac{d^4\phi_n}{dz^4} +$$

$$+\sum_{n=1}^{\infty}c_n\omega_n^2\phi_n + \frac{1}{m}\frac{\partial}{\partial z}\left[T\sum_{n=1}^{\infty}p_n\frac{d\phi_n}{dz}\right] - \alpha\left[\sum_{n=1}^{\infty}\dot{p}_n\phi_n - \Omega\sum_{n=1}^{\infty}q_n\phi_n\right], \tag{4.29a}$$

$$\sum_{n=1}^{\infty}\ddot{q}_n\phi_n + 2\Omega\sum_{n=1}^{\infty}\dot{p}_n\phi_n - \Omega^2\left[\sum_{n=1}^{\infty}(q_n+b_n)\phi_n\right] = -\frac{EI}{m}\sum_{n=1}^{\infty}q_n\frac{d^4\phi_n}{dz^4} +$$

$$+\sum_{n=1}^{\infty}d_n\omega_n^2\phi_n + \frac{1}{m}\frac{\partial}{\partial z}\left[T\sum_{n=1}^{\infty}q_n\frac{d\phi_n}{dz}\right] - \alpha\left[\sum_{n=1}^{\infty}\dot{q}_n\phi_n + \Omega\sum_{n=1}^{\infty}p_n\phi_n\right]. \tag{4.29b}$$

The first plus the third term in the right-hand side of these equations can be simplified. For Eq. (4.29a),

$$-\frac{EI}{m}\sum_{n=1}^{\infty}p_n\frac{d^4\phi_n}{dz^4} + \frac{1}{m}\frac{\partial}{\partial z}\left[T\sum_{n=1}^{\infty}p_n\frac{d\phi_n}{dz}\right]$$

$$= \sum_{n=1}^{\infty}p_n\left[-\frac{EI}{m}\frac{d^4\phi_n}{dz^4} + \frac{1}{m}\frac{d}{dz}\left(T\frac{d\phi_n}{dz}\right)\right]$$

$$= -\sum_{n=1}^{\infty}p_n\omega_n^2\phi_n$$

since the expression in the brackets is equal to $-\omega_n^2\phi_n$, according to Eq. (4.9). Similarly, the first and third

terms of the right-hand side of Eq. (4.29b) are equivalent to

$$-\sum_{r=1}^{\infty} q_r \, \omega_r^2 \, \phi_r \, .$$

If these two last expressions are introduced in Eqs. (4.29), the following equations are obtained:

$$\sum_{r=1}^{\infty} \left[\ddot{p}_r - 2\Omega \dot{q}_r - \Omega^2 p_r - \Omega^2 a_r + \omega_r^2 p_r - \omega_r^2 c_r + \alpha \dot{p}_r - \alpha \Omega q_r \right] \phi_r = 0$$

$$\sum_{r=1}^{\infty} \left[\ddot{q}_r + 2\Omega \dot{p}_r - \Omega^2 q_r - \Omega^2 b_r + \omega_r^2 q_r - \omega_r^2 d_r + \alpha \dot{q}_r + \alpha \Omega p_r \right] \phi_r = 0 \, .$$

This pair of equations are satisfied if all the coefficients of the functions ϕ_r vanish simultaneously. That is to say

$$\ddot{p}_r + \alpha \dot{p}_r + (\omega_r^2 - \Omega^2) p_r - 2\Omega \dot{q}_r - \alpha \Omega q_r = \omega_r^2 c_r + \Omega^2 a_r \qquad (4.30a)$$

$$\ddot{q}_r + \alpha \dot{q}_r + (\omega_r^2 - \Omega^2) q_r + 2\Omega \dot{p}_r + \alpha \Omega p_r = \omega_r^2 d_r + \Omega^2 b_r \, . \qquad (4.30b)$$

In order not to digress (from the main purpose of finding solutions for p_r and q_r), the above equations will be solved now. However, in Section 4.5 the stability of the system will be discussed, starting with Eqs. (4.30).

Define now three new complex quantities

$$\eta_r = p_r + i \, q_r \qquad (4.31a)$$

$$e_r = a_r + i \, b_r \qquad (4.31b)$$

$$f_r = c_r + i\, d_r \,. \tag{4.31c}$$

Multiplying Eq. (4.30b) by the imaginary unit $\underline{i}$ and adding it to Eq. (4.30a) yields

$$\ddot{\eta}_r + (\alpha + i2\Omega)\dot{\eta}_r + \left[(\omega_r^2 - \Omega^2) + i\alpha\Omega\right]\eta_r = \omega_r^2 f_r + \Omega^2 e_r \,. \tag{4.32}$$

This is a second-order differential equation in the complex variable η_r, with constant complex coefficients and constant forcing functions. The solution is then easily obtained by familiar methods [11] . Introducing the notation

$$\alpha = 2\mu_r \omega_r \,, \tag{4.33}$$

$$\nu_r = \sqrt{1-\mu_r^2} \,, \tag{4.34}$$

Equation (4.32) admits the solution

$$\eta_r = A\, e^{-\mu_r\omega_r t - i(\nu_r\omega_r + \Omega)t} + B\, e^{-\mu_r\omega_r t + i(\nu_r\omega_r - \Omega)t} +$$

$$+ \frac{f_r\, e^{-i\theta}}{\left[(1-\Omega^2/\omega_r^2)^2 + 4\mu_r^2\Omega^2/\omega_r^2\right]^{1/2}} + \frac{e_r\, e^{-i\theta}}{\left[(1-\omega_r^2/\Omega^2)^2 + 4\mu_r^2\omega_r^2/\Omega^2\right]^{1/2}} \,, \tag{4.35}$$

where $$\theta = \tan^{-1}\frac{\alpha\Omega}{\omega_r^2 - \Omega^2} = \tan^{-1}\frac{2\mu_r\omega_r\Omega}{\omega_r^2 - \Omega^2} \,. \tag{4.36}$$

It should be noted that $\mu_r = \frac{\alpha}{2\omega_r} = \frac{\zeta}{2m\omega_r}$ is the damping ratio of the rth mode.

The first two terms of Eq. (4.35) constitute the homogeneous solution of Eq. (4.32), where A and B are arbitrary complex constants. They represent two inward spiral motions in the complex plane. The first term gives a motion counter to that of the rotation of the shaft and the second term in a direction which depends on the damping coefficient α, since $\nu_r = f(\alpha)$. Both motions are damped by the term $e^{-\mu_r \omega_r t}$. This agrees with the result of Section 4.5 which says that the damped system is always stable. If damping is not present, i.e., $\alpha = 0$, it follows that $\mu_r = 0$ and $\nu_r = 1$. In such a case, the first two terms represent circular motions, the first being counter to that of the rotating axis, and the second being dependent upon the values of ω_r and Ω.

The last two terms of Eq. (4.35), the particular solution of Eq. (4.32), represent a steady configuration in the complex plane. Relation (4.36) shows an important well-known characteristic of these particular solutions. If $\Omega = \omega_r$, the imposed speed equal to one of the natural frequencies of the shaft, it follows that $\theta = \frac{\pi}{2}$. Then, from Eq. (4.35)

$$\eta_{r(p)} = \frac{-(f_r + e_r)\, i}{2\mu_r}$$

where $\eta_{r(p)}$ is the particular solution only. This equation shows that the rth component of the deflection of the shaft lags 90° behind the total rth component of the exciting forces. If $\alpha = 0$ ($\mu_r = 0$) this rth component of deflection is unstable.

The remaining part of this dissertation deals with the steady-state solution of the rotating shaft. For this purpose, the particular solution of Eq. (4.32) follows in detail.

The particular solution $\eta_{r(p)}$ is assumed to be constant, say $\eta_{r(p)} = K$. Substituting in Eq. (4.32), it follows that

$$\eta_{r(p)} = \frac{\omega_r^2 f_r + \Omega^2 e_r}{(\omega_r^2 - \Omega^2) + i\alpha\Omega} .$$

Rationalizing,

$$\eta_{r(p)} = \frac{(\omega_r^4 - \omega_r^2\Omega^2) f_r + (\omega_r^2\Omega^2 - \Omega^4) e_r - i\alpha\Omega\omega_r^2 f_r - i\alpha\Omega^3 e_r}{(\omega_r^2 - \Omega^2)^2 + \alpha^2\Omega^2} \quad (4.37)$$

Substituting Eqs. (4.31) in the above expression and separating the real and imaginary parts, the following expressions for $p_{r(p)}$ and $q_{r(p)}$ are obtained:

$$p_{r(p)} \equiv U_r = \frac{(\omega_r^2 - \Omega^2)(\omega_r^2 c_r + \Omega^2 a_r) + \alpha\Omega(\omega_r^2 d_r + \Omega^2 b_r)}{(\omega_r^2 - \Omega^2)^2 + \alpha^2\Omega^2} , \quad (4.38a)$$

$$q_{r(p)} \equiv V_r = \frac{(\omega_r^2 - \Omega^2)(\omega_r^2 d_r + \Omega^2 b_r) - \alpha\Omega(\omega_r^2 c_r + \Omega^2 a_r)}{(\omega_r^2 - \Omega^2)^2 + \alpha^2\Omega^2}, \tag{4.38b}$$

where the particular solutions of $p_r(t)$ and $q_r(t)$ have been renamed by U_r and V_r, respectively. The constants a_r, b_r, c_r and d_r are given by Eqs. (4.25)-(4.28).

If the steady lift force, mentioned at the end of Chapter 3, is included in the steady-state analysis, it can be shown that the particular solutions are

$$U_r = \frac{[\omega_r^2 - (1 + m_o/m)\Omega^2](\omega_r^2 c_r + \Omega^2 a_r) + \alpha\Omega(\omega_r^2 d_r + \Omega^2 b_r)}{[\omega_r^2 - (1 + m_o/m)\Omega^2]^2 + \alpha^2\Omega^2} \tag{4.38c}$$

$$V_r = \frac{[\omega_r^2 - (1 + m_o/m)\Omega^2](\omega_r^2 d_r + \Omega^2 b_r) - \alpha\Omega(\omega_r^2 c_r + \Omega^2 a_r)}{[\omega_r^2 - (1 + m_o/m)\Omega^2]^2 + \alpha^2\Omega^2}. \tag{4.38d}$$

Eqs. (4.38c,d) indicate that the resonance frequencies (for $\alpha = 0$) are not the same as the natural frequencies of the shaft. The new values are smaller and given by

$$\Omega_{res} = (1 + m_o/m)^{-1/2}\,\omega_r,$$

depending on the ratio m_o/m.

4.4 Displacements, Bending Moments and Stresses

From Eqs. (4.1), the series solutions for the displacements in the rotating reference UVZ are

$$u(z) = \sum_{n=1}^{\infty} U_n \, \phi_n(z) \tag{4.39a}$$

$$v(z) = \sum_{n=1}^{\infty} V_n \, \phi_n(z) \tag{4.39b}$$

in which the time t has been dropped from the notation $u(z,t)$ and $v(z,t)$, meaning that $u(z)$ and $v(z)$ are the steady-state solutions of the displacements.

The bending moment in the UZ plane has the following expression

$$M_u(z,t) = EI \frac{\partial^2 (u - u_0)}{\partial z^2} .$$

With the same notation used above, the steady-state bending moment $M_u(z)$ can be obtained with the expression of $u(z)$ from Eq. (4.39a).

$$M_u(z) = EI \sum_{n=1}^{\infty} U_n \frac{d^2 \phi_n(z)}{dz^2} - EI \frac{d^2 u_0(z)}{dz^2} \tag{4.40a}$$

A similar expression holds for $M_v(z)$.

$$M_v(z) = EI \sum_{n=1}^{\infty} V_n \frac{d^2 \phi_n(z)}{dz^2} - EI \frac{d^2 v_0(z)}{dz^2} . \tag{4.40b}$$

The maximum positive bending stress of a circular shaft is

$$\sigma(z) = \frac{32\, M(z)}{\pi D^3 \left[1 - \left(\frac{d}{D}\right)^4 \right]} \tag{4.41}$$

where D and d are the outside and inside diameters of the shaft, respectively, and

$$M(z) = \left[M_u^2(z) + M_v^2(z) \right]^{1/2} . \tag{4.42}$$

4.5 Stability of the System

Although some conclusions have been drawn in connection with Eq. (4.35) it is intended in this section to study in greater detail the problem of stability.

The free motion of the shaft is governed by the homogeneous part of the differential equations (4.30). These are

$$\ddot{p}_n + \alpha \dot{p}_n + (\omega_n^2 - \Omega^2) p_n - 2\Omega \dot{q}_n - \alpha \Omega q_n = 0 \tag{4.43a}$$

$$\ddot{q}_n + \alpha \dot{q}_n + (\omega_n^2 - \Omega^2) q_n + 2\Omega \dot{p}_n + \alpha \Omega p_n = 0 . \tag{4.43b}$$

Solutions are sought in the form

$$p_n(t) = F e^{\lambda t} \tag{4.44a}$$

$$q_n(t) = G e^{\lambda t} \tag{4.44b}$$

where F, G and λ are real constants. Substitution in Eqs. (4.43) gives, after simplification

$$\left[\lambda^2 + \alpha\lambda + (\omega_n^2 - \Omega^2)\right] F - \left[2\Omega\lambda + \alpha\Omega\right] G = 0$$

$$\left[2\Omega\lambda + \alpha\Omega\right] F - \left[\lambda^2 + \alpha\lambda + (\omega_n^2 - \Omega^2)\right] G = 0.$$

For non-trivial solutions of F and G, the determinant of their coefficients must vanish, which gives

$$\lambda^4 + 2\alpha\lambda^3 + \left[2(\omega_n^2 + \Omega^2) + \alpha^2\right]\lambda^2 + 2\alpha(\omega_n^2 + \Omega^2)\lambda + \left[(\omega_n^2 - \Omega^2)^2 + \alpha^2\Omega^2\right] = 0 \tag{4.45}$$

or
$$\lambda^4 + A_3\lambda^3 + A_2\lambda^2 + A_1\lambda + A_0 = 0 \tag{4.46a}$$

where
$$A_3 = 2\alpha \tag{4.46b}$$

$$A_2 = 2(\omega_n^2 + \Omega^2) + \alpha^2 \tag{4.46c}$$

$$A_1 = 2\alpha(\omega_n^2 + \Omega^2) \tag{4.46d}$$

$$A_0 = (\omega_n^2 - \Omega^2)^2 + \alpha^2\Omega^2. \tag{4.46e}$$

In practice it is not necessary to determine the roots of Eq. (4.46a) explicitly; it is sufficient to know the sign of the real part of the roots. If all roots λ have negative real values, the solutions for $p_n(t)$ and $q_n(t)$ will always be bounded. If one or more λ has

positive real value, the amplitude will increase exponentially, making the motion unstable. For the fourth-order algebraic equation (4.46a) in λ, the condition of stability (non-positive real parts) is given by Routh's criteria [12] :

(1) all coefficients A_i ($i = 0, 1, 2, 3$) must be of the same sign (positive, the sign of λ^4) and

(2) the following inequality must be true

$$A_1 A_2 A_3 > A_1^2 + A_3^2 A_0 .$$

The first condition is automatically satisfied, since all coefficients are positive ($\alpha > 0$).

Using Eqs. (4.46b,c,d,e), the second condition is equivalent to

$$4\alpha^2 \omega_n^2 (4\Omega^2 + \alpha^2) > 0,$$

which is always satisfied.

The conclusion is that, with damping present, the free motion of the shaft is always stable.

The undamped system can be analysed from Eq. (4.45), in which α is taken equal to 0. Then

$$\lambda^4 + 2(\omega_n^2 + \Omega^2)\lambda^2 + (\omega_n^2 - \Omega^2)^2 = 0. \tag{4.47}$$

This equation has the following roots:

$$\lambda^2 = -(\omega_n^2 + \Omega^2) \pm 2\omega_n \Omega . \tag{4.48}$$

Supposing now that the speed of rotation Ω is equal to one of the natural frequencies of shaft, i.e., $\Omega = \omega_n$, the roots are

$$\lambda^2 = -2\omega_n^2 \pm 2\omega_n^2 \quad , \text{ or}$$

$$\lambda_1 = 0$$

$$\lambda_2 = 0$$

$$\lambda_3 = +i\,2\omega_n$$

$$\lambda_4 = -i\,2\omega_n$$

Then with $\Omega = \omega_n$ ($n = 1, 2, 3, \cdots$), the value $\lambda = 0$ is a double root of Eq. (4.47), which corresponds to solutions of the form

$$p_n(t) = F_1 + F_2 t \qquad\qquad q_n(t) = G_1 + G_2 t \ ,$$

making the system unstable, since $p_n(t)$ and $q_n(t)$ increase with time without bound.

The stability has been discussed based on the assumption of tension varying only with the distance z, since Eq. (4.9) was obtained by the method of separation of variables, in which is implied that the tension is $T(z)$. A tension varying also with time, $T(z, t)$, would require a slightly different approach. However, the case with time-varying tension is of less practical significance and beyond the scope of this study.

Chapter 5

SPECIAL CASES OF TENSION

The solution of the general equations by modal analysis was derived in the last chapter as a series expansion in the normal coordinates and modal shapes of the associated free-vibrations problem. The former, for a steady-state case, were given by Eqs. (4.38 c,d), while the latter were shown to satisfy the differential equation (4.9) which is repeated here for convenience

$$\frac{EI}{m}\frac{d^4\phi}{dz^4} - \frac{1}{m}\frac{d}{dz}\left(T\frac{d\phi}{dz}\right) - \omega^2\phi = 0. \tag{5.1}$$

In this chapter, Eq. (5.1) will be solved for particular tension functions, $T(z)$, and boundary conditions. Two cases of tension are considered: (1) linearly varying tension and (2) constant tension. The first case involves a shaft with one end simply supported and the other guided (sliding), while in the second case a simply-supported shaft (at both ends) is considered. Following the solution of Eq. (5.1), final expressions for displacements, bending moments and stresses are given.

5.1 Shaft Under Linearly Varying Tension

5.1.1 Free-Vibration Analysis

Let the tension at a section of the shaft be given by

$$T(z) = T_o + \gamma z \tag{5.2}$$

in which γ is the weight per unit length of the shaft (in the fluid). The value of γ is

$$\gamma = (m - m_o)\, g_o \tag{5.3}$$

where g_o is the acceleration of gravity.

Substituting Eq. (5.2) into Eq. (5.1) and multiplying by m/EI, yields

$$\frac{d^4\phi}{dz^4} - \frac{1}{EI}\frac{d}{dz}\left[(T_o + \gamma z)\frac{d\phi}{dz}\right] - \frac{\omega^2 m}{EI}\phi = 0. \tag{5.4}$$

The boundary conditions are

$$\phi(0) = 0, \tag{5.5a}$$

$$\frac{d\phi}{dz}(0) = 0, \tag{5.5b}$$

$$\phi(L) = 0, \tag{5.5c}$$

$$\frac{d^2\phi}{dz^2}(L) = 0. \tag{5.5d}$$

The homogeneous Eq. (5.4) with the homogeneous boundary conditions (5.5) constitute an eigenvalue problem.

Let a dimensionless variable ξ be defined as

$$\xi = \frac{z}{L}. \tag{5.6}$$

Eq. (5.4) can now be written as

$$\frac{d^4\phi}{d\xi^4} - g\xi\frac{d^2\phi}{d\xi^2} - h\frac{d^2\phi}{d\xi^2} - g\frac{d\phi}{d\xi} - k^4\phi = 0 \tag{5.7}$$

where g, h and k are three dimensionless parameters defined by

$$g = \frac{\gamma L^3}{E I}, \tag{5.8}$$

$$h = \frac{T_0 L^2}{E I}, \tag{5.9}$$

$$k^4 = \frac{m\omega^2 L^4}{E I}. \tag{5.10}$$

With the introduction of the independent variable ξ, the boundary conditions (5.5) become

$$\phi(0) = 0, \tag{5.11a}$$

$$\frac{d\phi}{d\xi}(0) = 0, \tag{5.11b}$$

$$\phi(1) = 0, \tag{5.11c}$$

$$\frac{d^2\phi}{d\xi^2}(1) = 0. \tag{5.11d}$$

Equation (5.7) will be solved by assuming for the dependent variable ϕ a power series in ξ near the point $\xi = 0$. The reason for this expansion is due to the presence of the variable coefficient, arising from the variable axial force. As a consequence of this variable

coefficient, the function ϕ may not be expressible in terms of known elementary functions. Thus, it is assumed that

$$\phi(\xi) = \xi^{\lambda} \sum_{n=0}^{\infty} c_n \xi^n \tag{5.12}$$

where the c_n's and λ are constants to be determined. Because the coefficients involved in Eq. (5.7) are analytic for any finite value of the variable ξ, the function $\phi(\xi)$ can be represented by the assumed series, being absolutely and uniformly convergent everywhere in the finite domain [13].

The differential equation is of the fourth order, thus admitting four linearly independent solutions, which can be obtained by the method of Frobenius.

Substituting Eq. (5.12) into (5.7) an equation involving a power series in ξ is obtained, whose sum is equal to zero. In order to satisfy the equation, each coefficient in the series must be zero. The first of these coefficients gives the indicial equation

$$c_0 \left[\lambda(\lambda-1)(\lambda-2)(\lambda-3) \right] = 0$$

which is satisfied by

$$\lambda_1 = 0, \quad \lambda_2 = 1, \quad \lambda_3 = 2, \quad \lambda_4 = 3 \tag{5.13}$$

with c_0 arbitrary. The differences between the values of

λ's are integer numbers. In such cases the method of Frobenius assures solutions of the type of (5.12) only for the largest value of λ. But since the equation to be solved is ordinary at $\xi = 0$, the method still gives the four independent solutions for the other remaining λ's, with a suitable choice of the arbitrary constants [14]. The other coefficients in the power series equation have to be set equal to zero for each value of λ, thus giving four independent solutions. Letting a_n, b_n, c_n and d_n be the coefficients of Eq. (5.12) for the four solutions, one obtains

$$a_0 = 1, \qquad a_1 = 0, \qquad a_2 = \frac{h}{20}, \qquad a_3 = \frac{g}{40},$$

$$a_n = \frac{h}{(n+3)(n+2)} a_{n-2} + \frac{ng}{(n+3)(n+2)(n+1)} a_{n-3} + \frac{k^4}{(n+3)(n+2)(n+1)n} a_{n-4}, \qquad (n \geq 4)$$

$$b_0 = 1, \qquad b_1 = 0, \qquad b_2 = \frac{h}{12}, \qquad b_3 = \frac{g}{30},$$

$$b_n = \frac{h}{(n+2)(n+1)} b_{n-2} + \frac{(n-1)g}{(n+2)(n+1)n} b_{n-3} + \frac{k^4}{(n+2)(n+1)(n)(n-1)} b_{n-4}, \qquad (n \geq 4)$$

$$c_0 = 1, \qquad c_1 = 0, \qquad c_2 = 0, \qquad c_3 = \frac{g}{24},$$

$$c_n = \frac{h}{(n+1)n} c_{n-2} + \frac{(n-2)\,g}{(n+1)n(n-1)} c_{n-3} + \frac{k^4}{(n+1)n(n-1)(n-2)} c_{n-4}\,,$$

$$(n \geq 4)$$

$$d_0 = 1, \qquad d_1 = 0, \qquad d_2 = 0, \qquad d_3 = 0,$$

$$d_n = \frac{h}{n(n-1)} d_{n-2} + \frac{(n-3)\,g}{n(n-1)(n-2)} d_{n-3} + \frac{k^4}{n(n-1)(n-2)(n-3)} d_{n-4}\,.$$

$$(n \geq 4)$$

(5.14)

Using the values of λ given by (5.13), the series solution can be written as

$$\phi(\xi) = A\sum_{n=0}^{\infty} a_n \xi^{n+3} + B\sum_{n=0}^{\infty} b_n \xi^{n+2} + C\sum_{n=0}^{\infty} c_n \xi^{n+1} + D\sum_{n=0}^{\infty} d_n \xi^{n}$$

(5.15)

where A, B, C and D are arbitrary constants and a_n, b_n, c_n, d_n given by Eqs. (5.14). The arbitrary constants A, B, C and D may be obtained using the boundary conditions (5.11). Application of condition (5.11a) gives

$$D = 0. \tag{5.16}$$

From (5.15), with (5.16), the first derivative of $\phi(\xi)$ is equal to

$$\frac{d\phi}{d\xi} = A\sum_{n=0}^{\infty}(n+3)a_n\xi^{n+2} + B\sum_{n=0}^{\infty}(n+2)b_n\xi^{n+1} +$$

$$+ C\sum_{n=0}^{\infty}(n+1)c_n\xi^{n}.$$

Applying condition (5.11b) yields

$$C = 0. \tag{5.17}$$

Substituting (5.16) and (5.17) in (5.15), condition (5.11c) gives

$$A\sum_{n=0}^{\infty}a_n + B\sum_{n=0}^{\infty}b_n = 0. \tag{5.18}$$

With $C = D = 0$, the second derivative of $\phi(\xi)$ is

$$\frac{d^2\phi}{d\xi^2} = A\sum_{n=0}^{\infty}(n+3)(n+2)a_n\xi^{n+1} + B\sum_{n=0}^{\infty}(n+2)(n+1)b_n\xi^{n}. \tag{5.19}$$

Applying condition (5.11d) to (5.19), the following equation is obtained:

$$A\sum_{n=0}^{\infty}(n+3)(n+2)a_n + B\sum_{n=0}^{\infty}(n+2)(n+1)b_n = 0. \tag{5.20}$$

Equations (5.18) and (5.20) form a set of two homogeneous equations in the unknowns A and B. For non-trivial solution the determinant of their coefficients must vanish, i.e.

$$\begin{vmatrix} \sum_{n=0}^{\infty} a_n & \sum_{n=0}^{\infty} b_n \\ \sum_{n=0}^{\infty} (n+3)(n+2)\, a_n & \sum_{n=0}^{\infty} (n+2)(n+1)\, b_n \end{vmatrix} = 0 .$$

Expanding and dividing by $-\sum_{n=0}^{\infty} b_n$, one finally obtains

$$\sum_{n=0}^{\infty} (n+3)(n+2)\, a_n - \left(\sum_{n=0}^{\infty} a_n \Big/ \sum_{n=0}^{\infty} b_n \right) \sum_{n=0}^{\infty} (n+2)(n+1)\, b_n = 0 . \tag{5.21}$$

This is the frequency equation, which gives the value of the natural frequencies. The ω_n's are related to the coefficients a_n and b_n through Eqs. (5.14) and (5.10). Once the frequency equation has been solved, the power series expansion of $\phi(\xi)$ can be obtained.

From Eq. (5.18), the constant B can be written as

$$B = -A \frac{\sum_{n=0}^{\infty} a_n}{\sum_{n=0}^{\infty} b_n} , \tag{5.22}$$

Values of B, C and D, given by (5.22), (5.17)

and (5.16), can now be substituted in (5.15) and written as functions of the original variable z ($\xi = z/L$). The function $\phi(z)$ then becomes

$$\phi_r(z) = A_r \left[\sum_{n=0}^{\infty} a_n^{(r)} \left(\frac{z}{L}\right)^{n+3} - \left(\sum_{n=0}^{\infty} a_n^{(r)} \Big/ \sum_{n=0}^{\infty} b_n^{(r)} \right) \sum_{n=0}^{\infty} b_n^{(r)} \left(\frac{z}{L}\right)^{n+2} \right] \tag{5.23}$$

where the subscript r has been introduced for $\phi(z)$, since for each value of ω_r there is a correspondent modal function $\phi_r(z)$.

5.1.2 Steady-State Response of the Shaft.

The displacements for steady-state conditions are given by

$$u(z) = \sum_{r=1}^{\infty} U_r \, \phi_r(z) \qquad \text{and} \qquad v(z) = \sum_{r=1}^{\infty} V_r \, \phi_r(z),$$

Eqs. (4.39). Substitution of U_r and V_r from Eqs. (4.38 c,d) yields the final expressions for the displacements. These are

$$u(z) = \sum_{r=1}^{\infty} \frac{(\omega_r^2 - R_m^2 \Omega^2)(\omega_r^2 c_r + \Omega^2 a_r) + \alpha \Omega (\omega_r^2 d_r + \Omega^2 b_r)}{(\omega_r^2 - R_m^2 \Omega^2)^2 + \alpha^2 \Omega^2} \phi_r(z) \tag{5.24a}$$

$$v(z) = \sum_{r=1}^{\infty} \frac{(\omega_r^2 - R_m^2 \Omega^2)(\omega_r^2 d_r + \Omega^2 b_r) - \alpha \Omega (\omega_r^2 c_r + \Omega^2 a_r)}{(\omega_r^2 - R_m^2 \Omega^2)^2 + \alpha^2 \Omega^2} \phi_r(z) \tag{5.24b}$$

where $$R_m^2 = 1 + m_o/m \tag{5.24c}$$

and $\phi_n(z)$ is given by Eq. (5.23). Similarly, substituting Eq. (5.23) into (4.40) with (4.38 c,d) the following expressions for the bending moments are obtained:

$$M_u(z) = \frac{EI}{L^2}\sum_{n=1}^{\infty}\left[\frac{(\omega_n^2 - R_m^2\Omega^2)(\omega_n^2 c_n + \Omega^2 a_n) + \alpha\Omega(\omega_n^2 d_n + \Omega^2 b_n)}{(\omega_n^2 - R_m^2\Omega^2)^2 + \alpha^2\Omega^2}\,\psi_n(z)\right] -$$

$$- EI\frac{d^2 u_o(z)}{dz^2}, \tag{5.25a}$$

$$M_v(z) = \frac{EI}{L^2}\sum\left[\frac{(\omega_n^2 - R_m^2\Omega^2)(\omega_n^2 d_n + \Omega^2 b_n) - \alpha\Omega(\omega_n^2 c_n + \Omega^2 a_n)}{(\omega_n^2 - R_m^2\Omega^2)^2 + \alpha^2\Omega^2}\,\psi_n(z)\right] -$$

$$- EI\frac{d^2 v_o(z)}{dz^2}. \tag{5.25b}$$

where

$$\psi_n(z) = A_n\left[\sum_{n=0}^{\infty}(n+3)(n+2)a_n^{(n)}\left(\frac{z}{L}\right)^{n+1} - C_R\sum_{n=0}^{\infty}(n+2)(n+1)b_n^{(n)}\left(\frac{z}{L}\right)^{n}\right] \tag{5.26}$$

in which $$C_R = \sum_{n=0}^{\infty} a_n^{(n)} \Big/ \sum_{n=0}^{\infty} b_n^{(n)}. \tag{5.27}$$

The relations for displacements, bending moments and stress will be in non-dimensional form by defining the

following quantities and variables:

$R_n = \dfrac{\omega_n}{\Omega}$ ratio of the nth natural frequency to the speed of the shaft, (5.28)

$\alpha_o = \dfrac{\alpha}{\Omega}$ non-dimensional damping coefficient, (5.29)

e arbitrary unit of eccentricity,

$\mu(\xi) = \dfrac{u(\xi)}{e}$ non-dimensional displacement in UOZ plane, (5.30)

$\nu(\xi) = \dfrac{v(\xi)}{e}$ non-dimensional displacement in VOZ plane, (5.31)

$\mu_o(\xi) = \dfrac{u_o(\xi)}{e}$ non-dimensional lack of straightness in UOZ plane, (5.32)

$\nu_o(\xi) = \dfrac{v_o(\xi)}{e}$ non-dimensional lack of straightness in VOZ plane, (5.33)

$M_o = \dfrac{EIe}{L^2}$ a characteristic moment, (5.34)

$M_\mu(\xi) = \dfrac{M_u(\xi)}{M_o}$ non-dimensional bending moment in UOZ plane, (5.35)

$M_\nu(\xi) = \dfrac{M_v(\xi)}{M_o}$ non-dimensional bending moment in VOZ plane, (5.36)

$$M_{\mu\nu}(\xi) = \left[M_\mu^2(\xi) + M_\nu^2(\xi)\right]^{1/2}, \tag{5.37}$$

$$\sigma_0 = \frac{M_0}{D^3} = \frac{EIe}{L^2 D^3} \qquad \text{a characteristic stress,} \tag{5.38}$$

$$\rho(\xi) = \frac{\sigma(\xi)}{\sigma_0} \qquad \text{non-dimensional stress.} \tag{5.39}$$

It is shown in Appendix 3 that, using Eqs. (5.28)-(5.39), the non-dimensional form for the displacements, bending moments and stress are

$$\mu(\xi) = \sum_{r=1}^{\infty} \frac{(R_r^2 - R_m^2)(R_r^2 c_{r0} + a_{r0}) + \alpha_0 (R_r^2 d_{r0} + b_{r0})}{(R_r^2 - R_m^2)^2 + \alpha_0^2} \phi_r(\xi) \tag{5.40a}$$

$$\nu(\xi) = \sum_{r=1}^{\infty} \frac{(R_r^2 - R_m^2)(R_r^2 d_{r0} + b_{r0}) - \alpha_0 (R_r^2 c_{r0} + a_{r0})}{(R_r^2 - R_m^2)^2 + \alpha_0^2} \phi_r(\xi) \tag{5.40b}$$

$$M_\mu(\xi) = \sum_{r=1}^{\infty} \left[\frac{(R_r^2 - R_m^2)(R_r^2 c_{r0} + a_{r0}) + \alpha_0 (R_r^2 d_{r0} + b_{r0})}{(R_r^2 - R_m^2)^2 + \alpha_0^2} \psi_r(\xi) \right] - \frac{d^2}{d\xi^2} \mu_0(\xi) \tag{5.41a}$$

$$M_\nu(\xi) = \sum_{r=1}^{\infty} \left[\frac{(R_r^2 - R_m^2)(R_r^2 d_{r0} + b_{r0}) - \alpha_0 (R_r^2 c_{r0} + a_{r0})}{(R_r^2 - R_m^2)^2 + \alpha_0^2} \psi_r(\xi) \right] - \frac{d^2}{d\xi^2} \nu_0(\xi) \tag{5.41b}$$

$$\rho(\xi) = \frac{32}{\pi \left[1 - \left(\frac{d}{D} \right)^4 \right]} M_{\mu\nu}(\xi) \tag{5.42}$$

where $\phi_r(\xi)$ and $\psi_r(\xi)$ are obtained from Eqs. (5.23) and (5.26) respectively, together with Eq. (5.6) and

$$a_{ro} = \frac{1}{h_r} \int_0^1 \frac{a(\xi)}{e} \phi_r(\xi)\, d\xi, \tag{5.43}$$

$$b_{ro} = \frac{1}{h_r} \int_0^1 \frac{b(\xi)}{e} \phi_r(\xi)\, d\xi, \tag{5.44}$$

$$c_{ro} = \frac{EI}{m\omega_r^2 L^4} \frac{1}{h_r} \int_0^1 \frac{d^4}{d\xi^4} \mu_o(\xi)\, \phi_r(\xi)\, d\xi, \tag{5.45}$$

$$d_{ro} = \frac{EI}{m\omega_r^2 L^4} \frac{1}{h_r} \int_0^1 \frac{d^4}{d\xi^4} \nu_o(\xi)\, \phi_r(\xi)\, d\xi, \tag{5.46}$$

$$h_r = \int_0^1 \phi_r^2(\xi)\, d\xi. \tag{5.47}$$

5.2 Shaft Under Constant Tension

5.2.1 Free Vibration Analysis

With the tension constant along the shaft the differential equation (5.1) can be written as

$$EI \frac{d^4\phi}{dz^4} - T \frac{d^2\phi}{dz^2} - m\omega^2\phi = 0. \tag{5.48}$$

$\phi(z)$ must also satisfy the boundary conditions

for a simply-supported shaft, which are

$$\phi(0) = \frac{d^2\phi}{dz^2}(0) = \phi(L) = \frac{d^2\phi}{dz^2}(L) = 0. \qquad (5.49\ a,b,c,d)$$

The eigenvalue problem stated by Eqs. (5.48) and (5.49) has the solution [15]

$$\phi_r(z) = A_r \sin \frac{r\pi z}{L} \qquad (5.50)$$

$$\omega_r^2 = \frac{EI}{m}\frac{r^4\pi^4}{L^4}\left[1 + \frac{TL^2/EI}{r^2\pi^2}\right]. \qquad (5.51)$$

5.2.2 Steady-State Response of the Shaft

Expressions for the displacements are the same as Eqs. (5.24), with the functions $\phi_r(z)$ given by Eq. (5.50). Thus,

$$u(z) = \sum_{r=1}^{\infty} \frac{(\omega_r^2 - R_m^2\Omega^2)(\omega_r^2 c_r + \Omega^2 a_r) + \alpha\Omega(\omega_r^2 d_r + \Omega^2 b_r)}{(\omega_r^2 - R_m^2\Omega^2)^2 + \alpha^2\Omega^2} A_r \sin\frac{r\pi z}{L} \qquad (5.52a)$$

$$v(z) = \sum_{r=1}^{\infty} \frac{(\omega_r^2 - R_m^2\Omega^2)(\omega_r^2 d_r + \Omega^2 b_r) - \alpha\Omega(\omega_r^2 c_r + \Omega^2 a_r)}{(\omega_r^2 - R_m^2\Omega^2)^2 + \alpha^2\Omega^2} A_r \sin\frac{r\pi z}{L} \qquad (5.52b)$$

where ω_r's are given by Eq. (5.51).

The bending moments in this case are

$$M_u(z) = -\pi^2 \frac{EI}{L^2} \sum_{r=1}^{\infty} \left[\frac{(\omega_r^2 - R_m^2 \Omega^2)(\omega_r^2 c_r + \Omega^2 a_r) + \alpha \Omega (\omega_r^2 d_r + \Omega^2 b_r)}{(\omega_r^2 - R_m^2 \Omega^2)^2 + \alpha^2 \Omega^2} A_r r^2 \sin \frac{r\pi z}{L} \right] - \tag{5.53a}$$

$$- EI \frac{d^2 u_0(z)}{dz^2},$$

$$M_v(z) = -\pi^2 \frac{EI}{L^2} \sum_{r=1}^{\infty} \left[\frac{(\omega_r^2 - R_m^2 \Omega^2)(\omega_r^2 d_r + \Omega^2 b_r) - \alpha \Omega (\omega_r^2 c_r + \Omega^2 a_r)}{(\omega_r^2 - R_m^2 \Omega^2)^2 + \alpha^2 \Omega^2} A_r r^2 \sin \frac{r\pi z}{L} \right] -$$

$$- EI \frac{d^2 v_0(z)}{dz^2}. \tag{5.53b}$$

Non-dimensional forms corresponding to these expressions are obtained using Eqs. (5.28)-(5.36) together with the ratio

$$R_{0,r} = \frac{\omega_{0,r}}{\Omega} \tag{5.54}$$

where $\omega_{0,r}$ is the rth natural frequency of a shaft with zero tension. Substitution in Eqs. (5.52) and (5.53) yields

$$\mu(\xi) = 2 \sum_{r=1}^{\infty} \frac{(R_r^2 - R_m^2)(R_{0,r}^2 c_{r0}^* + a_{r0}^*) + \alpha_0 (R_{0,r}^2 d_{r0}^* + b_{r0}^*)}{(R_r^2 - R_m^2)^2 + \alpha_0^2} \sin r\pi\xi, \tag{5.55a}$$

$$\nu(\xi) = 2 \sum_{r=1}^{\infty} \frac{(R_r^2 - R_m^2)(R_{0,r}^2 d_{r0}^* + b_{r0}^*) - \alpha_0 (R_{0,r}^2 c_{r0}^* + a_{r0}^*)}{(R_r^2 - R_m^2)^2 + \alpha_0^2} \sin r\pi\xi, \tag{5.55b}$$

$$M_\mu(\xi) = -2\pi^2 \sum_{r=1}^{\infty} \left[\frac{(R_r^2 - R_m^2)(R_{o,r}^2 c_{ro}^* + a_{ro}^*) + \alpha_o (R_{o,r}^2 d_{ro}^* + b_{ro}^*)}{(R_r^2 - R_m^2)^2 + \alpha_o^2} r^2 \sin r\pi\xi \right] -$$

$$- \frac{d^2}{d\xi^2} \mu_o(\xi) , \tag{5.56a}$$

$$M_\nu(\xi) = -2\pi^2 \sum_{r=1}^{\infty} \left[\frac{(R_r^2 - R_m^2)(R_{o,r}^2 d_{ro}^* + b_{ro}^*) - \alpha_o (R_{o,r}^2 c_{ro}^* + a_{ro}^*)}{(R_r^2 - R_m^2)^2 + \alpha_o^2} r^2 \sin r\pi\xi \right] -$$

$$- \frac{d^2}{d\xi^2} \nu_o(\xi) , \tag{5.56b}$$

where

$$a_{ro}^* = \int_0^1 \frac{a(\xi)}{e} \sin r\pi\xi \, d\xi , \tag{5.57}$$

$$b_{ro}^* = \int_0^1 \frac{b(\xi)}{e} \sin r\pi\xi \, d\xi , \tag{5.58}$$

$$c_{ro}^* = \int_0^1 \mu_o(\xi) \sin r\pi\xi \, d\xi , \tag{5.59}$$

$$d_{ro}^* = \int_0^1 \nu_o(\xi) \sin r\pi\xi \, d\xi . \tag{5.60}$$

Chapter 6

EXAMPLE PROBLEMS

To provide further insight into the problem, numerical solutions of three examples are presented in this chapter. Also results of the freely-vibrating shaft are shown for the linearly-varying-tension case.

6.1 The Problems and Their Characteristics Parameters

Problem 1 - Linearly Varying Tension

Consider a hollow shaft with the following eccentricity and lack of straightness functions:

$$a(z) = \begin{cases} 0 & \text{for} \quad 0 \le z \le L/4 , \\ 4ez/L - e & \text{for} \quad L/4 \le z \le L/2 , \\ 3e - 4ez/L & \text{for} \quad L/2 \le z \le 3L/4 , \\ 0 & \text{for} \quad 3L/4 \le z \le L , \end{cases} \tag{6.1}$$

$$b(z) = 0 \qquad \text{for} \qquad 0 \le z \le L , \tag{6.2}$$

$$u_o(z) = 0 \qquad \text{for} \qquad 0 \le z \le L , \tag{6.3}$$

$$v_o(z) = 0 \qquad \text{for} \qquad 0 \le z \le L . \tag{6.4}$$

The eccentricity functions are shown in Fig. 5.

In this case, the equations for the displacement, bending moment and stress are (5.40), (5.41) and (5.42), respectively. The parameters a_{ro}, b_{ro}, c_{ro} and d_{ro} involved in these equations are given by Eqs. (5.43)-(5.47). Using Eqs. (6.1)-(6.4) one obtains

$$h_r = \int_0^1 A_r^2 \left[\sum_{n=0}^{\infty} a_n^{(r)} \xi^{n+3} - C_R \sum_{n=0}^{\infty} b_n^{(r)} \xi^{n+2} \right]^2 d\xi \text{, or}$$

$$h_r = A_r^2 \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \left(\frac{a_k^{(r)} a_n^{(r)}}{k+n+7} - 2C_R \frac{a_k^{(r)} b_n^{(r)}}{k+n+6} + C_R^2 \frac{b_k^{(r)} b_n^{(r)}}{k+n+5} \right), \tag{6.5}$$

$$a_{ro} = \frac{1}{h_r} \int_{\frac{1}{4}}^{1/2} (4\xi - 1) A_r \left[\sum_{n=0}^{\infty} a_n^{(r)} \xi^{n+3} - C_R \sum_{n=0}^{\infty} b_n^{(r)} \xi^{n+2} \right] d\xi +$$

$$+ \frac{1}{h_r} \int_{1/2}^{3/4} (3 - 4\xi) A_r \left[\sum_{n=0}^{\infty} a_n^{(r)} \xi^{n+3} - C_R \sum_{n=0}^{\infty} b_n^{(r)} \xi^{n+2} \right] d\xi \text{, or}$$

$$a_{ro} = \frac{A_r}{h_r} \sum_{n=0}^{\infty} \left[\left(\frac{3^{n+5} - 2^{n+6} + 1}{(n+5)(n+4)4^{n+4}} \right) a_n^{(r)} - C_R \left(\frac{3^{n+4} - 2^{n+5} + 1}{(n+4)(n+3)4^{n+3}} \right) b_n^{(r)} \right], \tag{6.6}$$

$$b_{ro} = c_{ro} = d_{ro} = 0 \text{ .} \qquad (6.7), (6.8), (6.9)$$

The surrounding fluid is assumed to be sea water and the geometric parameters of the shaft chosen are:

length = 400 ft ,

$$\left.\begin{array}{l}\text{outside diameter } (D) = 6.625 \text{ in.} \\ \text{inside diameter } (d) = 5.761 \text{ in.}\end{array}\right\} \begin{array}{l}\text{6"ND, Sch.80,} \\ \text{steel}\end{array}$$

unit of eccentricity $e = 0.1\,D$.

Problem 2 - Linearly-Varying Tension

Consider a hollow shaft with the following eccentricity and lack of straightness functions:

$$a(z) = \begin{cases} -e & \text{for} \quad 0 \le z \le L/4 \,, \\ 0 & \text{for} \quad L/4 < z < 3L/4 \,, \\ e & \text{for} \quad 3L/4 \le z \le L \,, \end{cases} \tag{6.10}$$

$$b(z) = \begin{cases} 0 & \text{for} \quad 0 \le z < L/2 \,, \\ e & \text{for} \quad L/2 \le z \le 3L/4 \,, \\ 0 & \text{for} \quad 3L/4 < z \le L \,, \end{cases} \tag{6.11}$$

$$u_0(z) = 0 \qquad \text{for} \qquad 0 \le z \le L \,, \tag{6.12}$$

$$v_0(z) = 0 \qquad \text{for} \qquad 0 \le z \le L. \tag{6.13}$$

The eccentricity function for this case is shown in Fig. 6. For this example problem the equations to be used are again (5.40)-(5.47), the parameters a_{ro} , b_{ro} , c_{ro} and d_{ro} being

$$a_{ro} = \frac{1}{h_r} \int_0^{1/4} -A_r \left[\sum_{n=0}^{\infty} a_n^{(r)} \xi^{n+3} - C_R \sum_{n=0}^{\infty} b_n^{(r)} \xi^{n+2} \right] d\xi \; +$$

$$+ \frac{1}{h_r} \int_{3/4}^{1} A_r \left[\sum_{n=0}^{\infty} a_n^{(r)} \xi^{n+3} - C_R \sum_{n=0}^{\infty} b_n^{(r)} \xi^{n+2} \right] d\xi \quad , \text{ or}$$

$$a_{ro} = \frac{A_r}{h_r} \sum_{n=0}^{\infty} \left[\left(\frac{4^{n+4} - 3^{n+4} - 1}{(n+4)4^{n+4}} \right) a_n^{(r)} - C_R \left(\frac{4^{n+3} - 3^{n+3} - 1}{(n+3)4^{n+3}} \right) b_n^{(r)} \right], \quad (6.14)$$

$$b_{ro} = \frac{1}{h_r} \int_{1/2}^{3/4} A_r \left[\sum_{n=0}^{\infty} a_n^{(r)} \xi^{n+3} - C_R \sum_{n=0}^{\infty} b_n^{(r)} \xi^{n+2} \right] d\xi \quad , \text{ or}$$

$$b_{ro} = \frac{A_r}{h_r} \sum_{n=0}^{\infty} \left[\left(\frac{3^{n+4} - 2^{n+4}}{(n+4)4^{n+4}} \right) a_n^{(r)} - C_R \left(\frac{3^{n+3} - 2^{n+3}}{(n+3)4^{n+3}} \right) b_n^{(r)} \right], \quad (6.15)$$

$$c_{ro} = d_{ro} = 0 \quad , \qquad (6.16), \ (6.17)$$

where h_r is given by Eq. (6.5).

The dimensions and surrounding fluid are the same as in the previous problem.

Problem 3 - Constant Tension

Consider a solid shaft with p heavy eccentric discs as shown in Fig. 7a. For the i th disc the following eccentricity functions can be defined:

$$a_i(z) = \begin{cases} 0 & \text{for} \quad 0 \le z < l_i - \epsilon_i, \\ \zeta_i e & \text{for} \quad l_i - \epsilon_i \le z \le l_i + \epsilon_i, \\ 0 & \text{for} \quad l_i + \epsilon_i < z \le L \quad , \end{cases} \qquad (6.18)$$

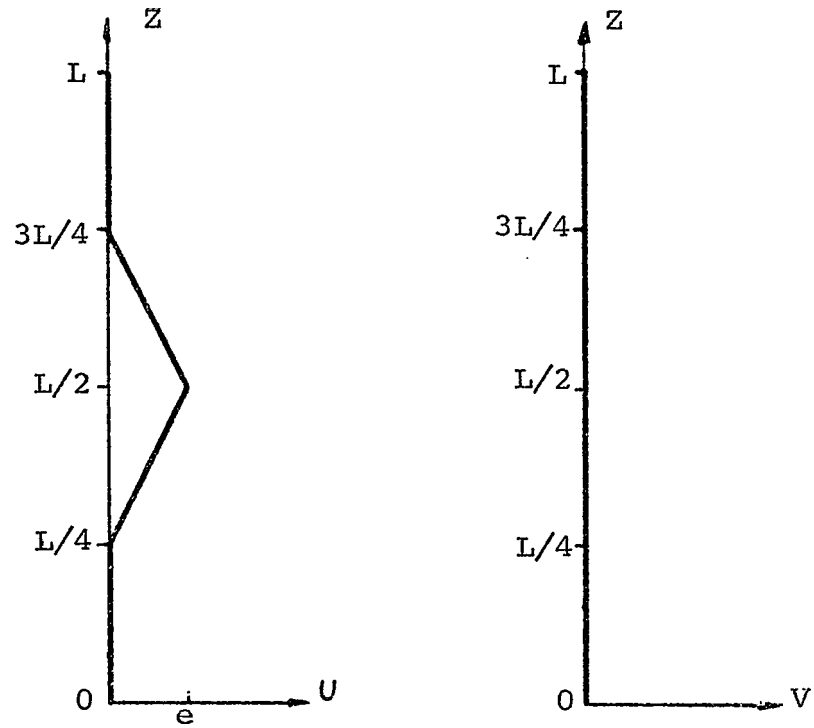


Fig.5 Eccentricity Functions for Example Problem No. 1

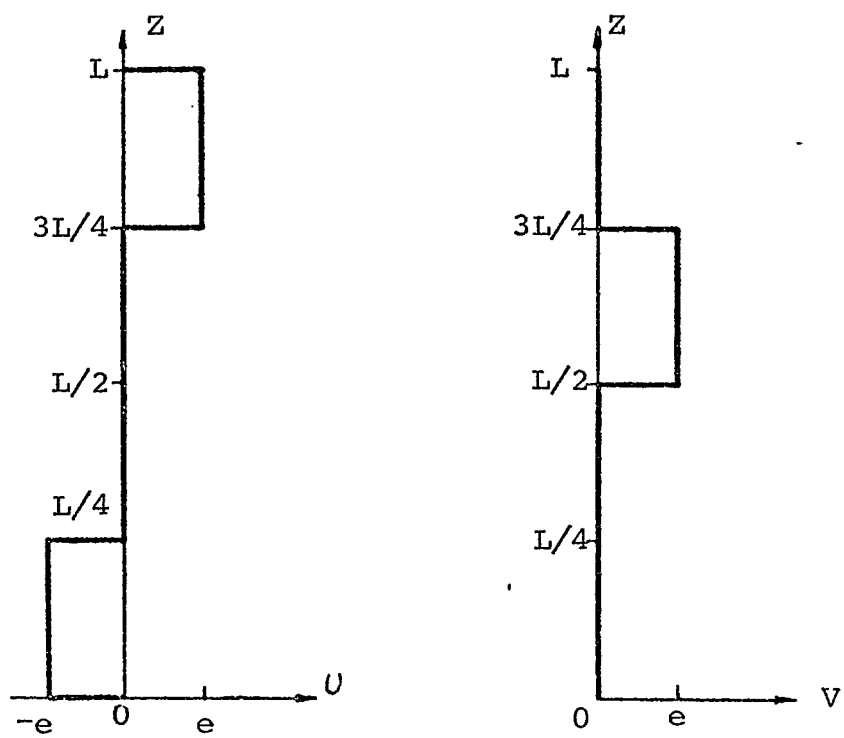


Fig.6 Eccentricity Functions for Example Problem No. 2

$$b_i(z) = \begin{cases} 0 & \text{for} \quad 0 \le z < \ell_i - \epsilon_i, \\ \eta_i e & \text{for} \quad \ell_i - \epsilon_i \le z \le \ell_i + \epsilon_i, \\ 0 & \text{for} \quad \ell_i + \epsilon_i < z \le L \quad , \end{cases} \tag{6.19}$$

where ℓ_i is the distance of the ith disc from the origin, $2\epsilon_i$ is its thickness and $-1 \le \zeta_i \le 1$, $-1 \le \eta_i \le 1$.

In order to consider the influence of the p discs, $a(z)$ and $b(z)$ become now

$$a(z) = \sum_{i=1}^{p} \lambda_i \, a_i(z) \quad , \tag{6.20}$$

$$b(z) = \sum_{i=1}^{p} \lambda_i \, b_i(z) \quad , \tag{6.21}$$

where $\lambda_i = M_i / m$ is the ratio of the mass per unit length of the ith disc plus shaft to the mass per unit length of the shaft.

Let the lack of straightness functions be

$$u_0(z) = 0 \qquad \text{for} \quad 0 \le z \le L \quad , \tag{6.22}$$

$$v_0(z) = 0 \qquad \text{for} \quad 0 \le z \le L \quad . \tag{6.23}$$

For this example problem the equations to be used are (5.55)-(5.60). Introducing the quantities

$$\xi_i = \frac{\ell_i}{L} \qquad \text{non-dimensional distance of the } i\text{th disc,} \tag{6.24}$$

$$2\epsilon_{0i} = \frac{2\epsilon_i}{L} \qquad \text{non-dimensional thickness of the } i\text{ th disc,} \tag{6.25}$$

the parameters a^*_{ro} , b^*_{ro} , c^*_{ro} and d^*_{ro} are

$$a^*_{ro} = \sum_{i=1}^{p} \lambda_i \int_0^1 \frac{a_i(\xi)}{e} \sin r\pi\xi \, d\xi$$

$$= \sum_{i=1}^{p} \lambda_i \int_{\xi_i - \epsilon_{oi}}^{\xi_i + \epsilon_{oi}} \zeta_i \sin r\pi\xi \, d\xi \quad , \quad \text{or}$$

$$a^*_{ro} = \frac{2}{r\pi} \sum_{i=1}^{p} \lambda_i \zeta_i \sin r\pi\xi_i \sin r\pi\epsilon_{oi} \; ; \tag{6.26}$$

similarly

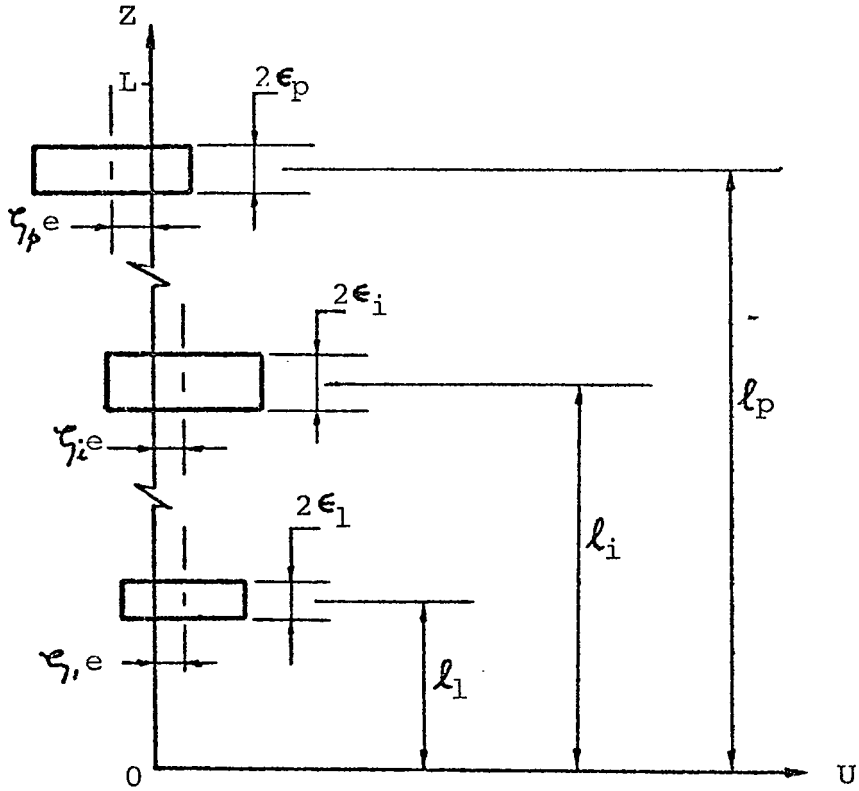
$$b^*_{ro} = \frac{2}{r\pi} \sum_{i=1}^{p} \lambda_i \eta_i \sin r\pi\xi_i \sin r\pi\epsilon_{oi} \; ; \tag{6.27}$$

$$c^*_{ro} = d^*_{ro} = 0 \, . \qquad (6.28), \; (6.29)$$

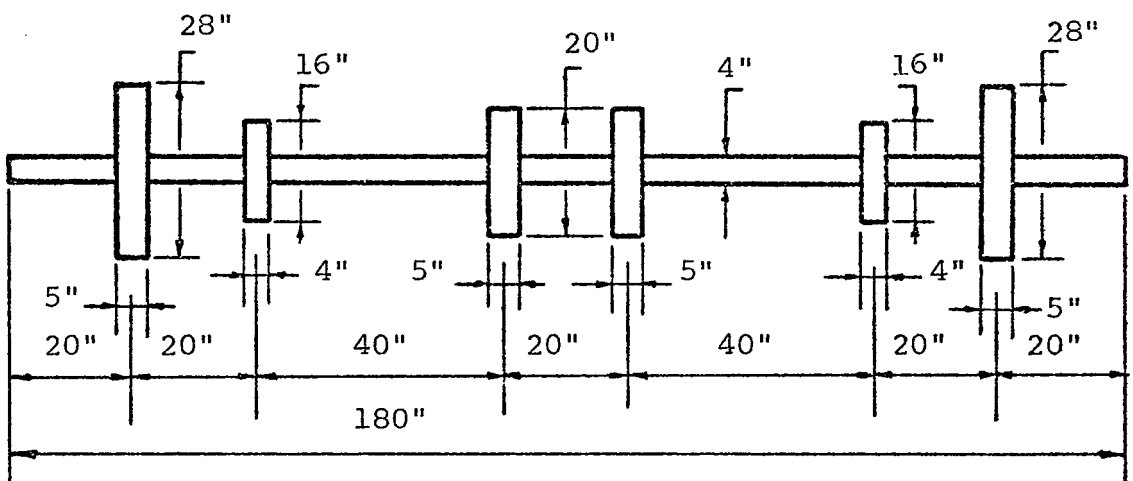
Finally, let the actual values chosen be $p = 6$ and

i	λ_i	ϵ_{oi}	ξ_i	ζ_i	η_i
1	49	.01389	.111	.500	.866
2	16	.01111	.222	.866	.500
3	25	.01389	.445	-.500	.866
4	25	.01389	.555	-.866	.500
5	16	.01111	.778	.500	.866
6	49	.01389	.889	.707	.707

The dimensions of the shaft for this case are as



a)



b)

Fig.7 Example Problem No. 3: a) General Configuration of the Disks on the UZ Plane (replace ζ by η for VZ Plane), b) Actual Dimensions of the Shaft and Disks Considered

follows:

material: steel,

length = 15 ft ,

diameter (D) = 4 in. ,

unit of eccentricity e = 0.001 D .

Surrounding fluid = air.

Shown in Fig. 7b are the shaft and discs considered.

6.2 Discussion of Results

Representative results for the freely vibrating shaft under linearly-varying tension are presented in Figs. 8 through 10 and Table 1. Fig. 8 shows the solution of the frequency equation (5.21) for values of g = 1510 and h = 151. Graphical representation, for those values, of the modal shapes $\phi_r(z)$ and the modal moments are shown in Figs. 9 and 10, respectively. As expected, the amplitudes are greater near the bottom where the tension is smaller but the distances between nodes are greater near the top. Table 1 gives a comparison between the natural frequencies of a shaft under linearly-varying tension and one with constant tension, the constant value being equal to the average value of the first case. A better agreement percentagewise exists for the higher frequencies.

Graphical results for example problem No. 1, for damping coefficients α = 0, 0.16, 0.32 and 1.6, are

presented in Figs. 11 through 15. Fig. 11 is a schematic of the deflection of the shaft central axis, projected on the UZ plane. Fig. 12 is the projection on the perpendicular plane, VZ . While for small damping the influence of the seventh modal shape is clearly evident, for relatively high damping the shaft has a tendency to deflect into a shape resembling the eccentricity distribution. Fig. 13 illustrates the variation along the shaft of the total bending stress in the section. Maximum displacements and stresses for a range of speeds, including the seventh resonant frequency, are shown in Figs. 14 and 15, respectively. In these two figures the value of Ω_o= 75.9 rpm is marked to show the speed of rotation for which Figs. 11 through 13 were computed. The best value of the damping coefficient that fits the physical parameters of example problem No. 1 is α = 0.16.

Representative results for example problem No. 2, for damping coefficients α = 0, 0.08, 0.32 and 1.6, are presented in Figs. 16 through 20. Fig. 16 is the displacement on the UZ plane. Fig. 17 is the displacement on the perpendicular plane VZ . Fig. 18 shows the variation of the total bending stress along the shaft. Maximum values of deflection and bending stress for a range of speeds, including the seventh resonant frequency, are

shown in Figs. 19 and 20. Fig. 19 illustrates the maximum displacement while Fig. 20 shows the maximum bending stress. For this example problem No.2 the value of the damping coefficient that best fits its physical parameters is $\alpha = 0.32$.

The results for example problem No. 3 are presented in Figs. 21 through 30. The first part, Figs. 21 through 25, depicts the influence of the damping coefficient, for a specific value of the dimensionless tension, $h = 2.16$. The second part, Figs. 26 through 30, demonstrates the influence of the constant tension of the shaft. The latter curves are for the values of nondimensional tension $h = 0$, 1.08, 2.16, 4.32 and 8.64 and for $\alpha = 0.05$ or 3.8. Fig. 21 is the projection of the displacement of the shaft central axis on the UZ plane while Fig. 22 is the projection on the VZ plane. Fig. 23 gives the total bending stress along the shaft axis, as a function of distance. These last three figures were computed for $\Omega_o = 2415$ rpm. Maximum displacements and stresses of the shaft for a range of speeds, including the second resonant frequency, are shown in Figs. 24 and 25, respectively. As already observed in the first two examples, the higher the damping, the smaller the displacements. Figs. 26 through 28 were computed for $\Omega_o = 2415$ rpm, $\alpha = 0.05$ and for the values

of the dimensionless tension mentioned above. Fig. 26 is a schematic of the deflection of the shaft projected on the UZ plane, Fig. 27 is the projection on the VZ plane and Fig. 28 represents the bending stress along the shaft. The influence of the tension on the displacement and stress can be better understood by reference to Figs. 29 and 30. These curves are for a range of speeds which includes the second resonant frequency. As shown, an increase or decrease in the tension which brings the resonant frequency closer to the operational speed Ω_o, will increase the displacement and bending stress. The value of the damping coefficient that fits best the parameters of the third example problem is $\alpha = 0.05$.

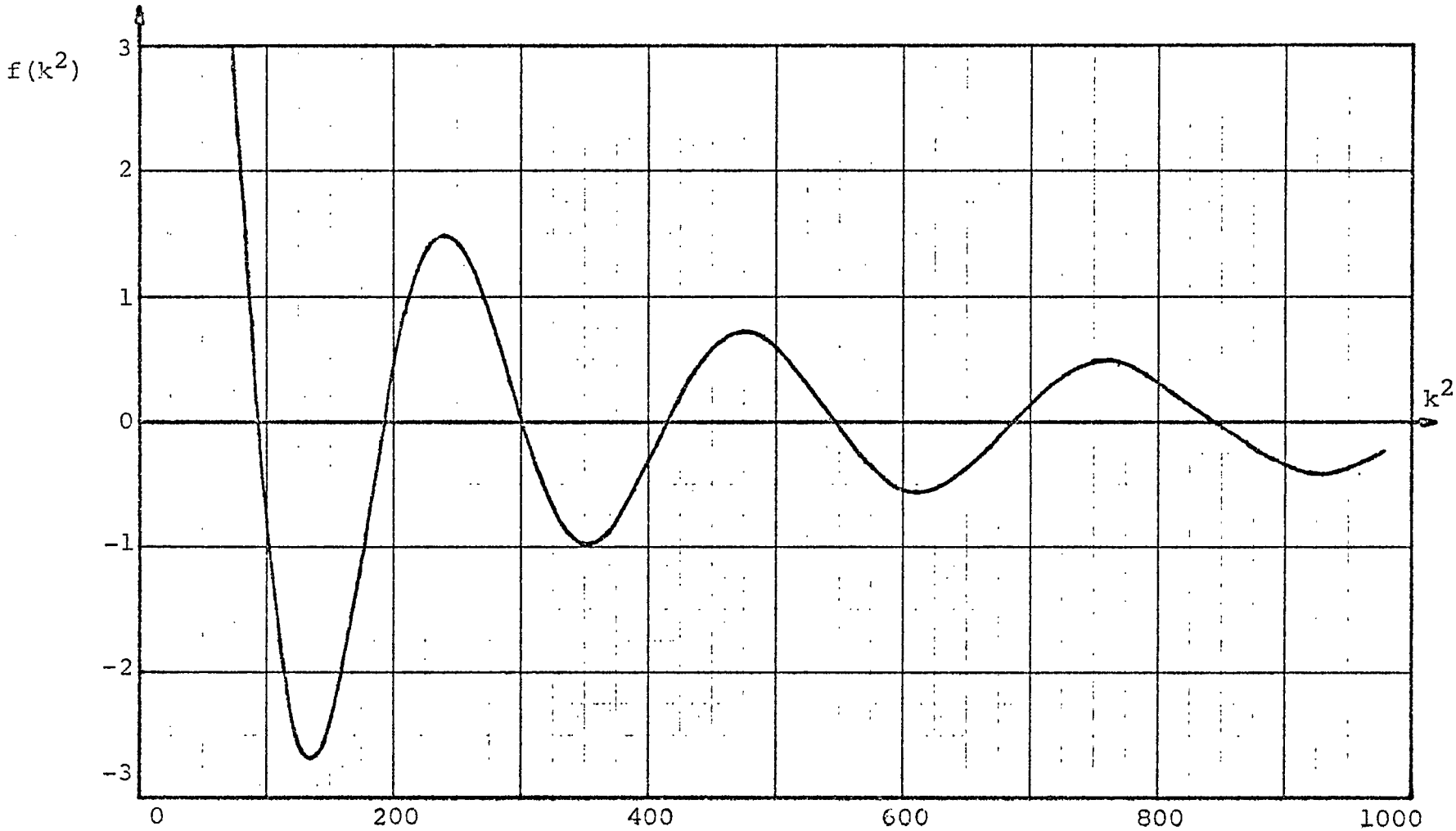


Fig. 8 Solution of the Frequency Equation (5.21), Function $f(k^2)$ vs. k^2, for g=1510 and h=151 (6 in. ND, Sch. 80, L=800 ft)

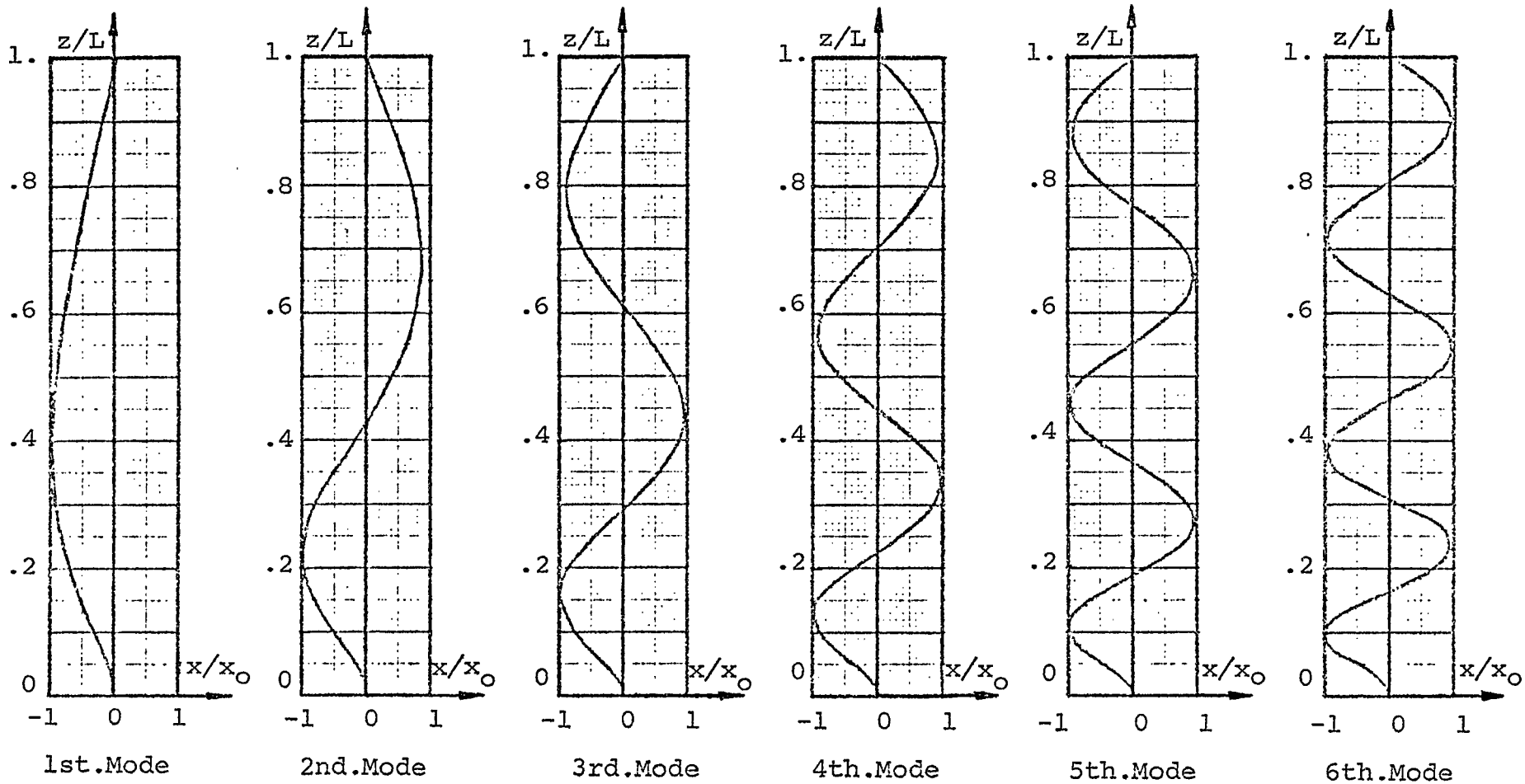


Fig. 9 Modal Shapes

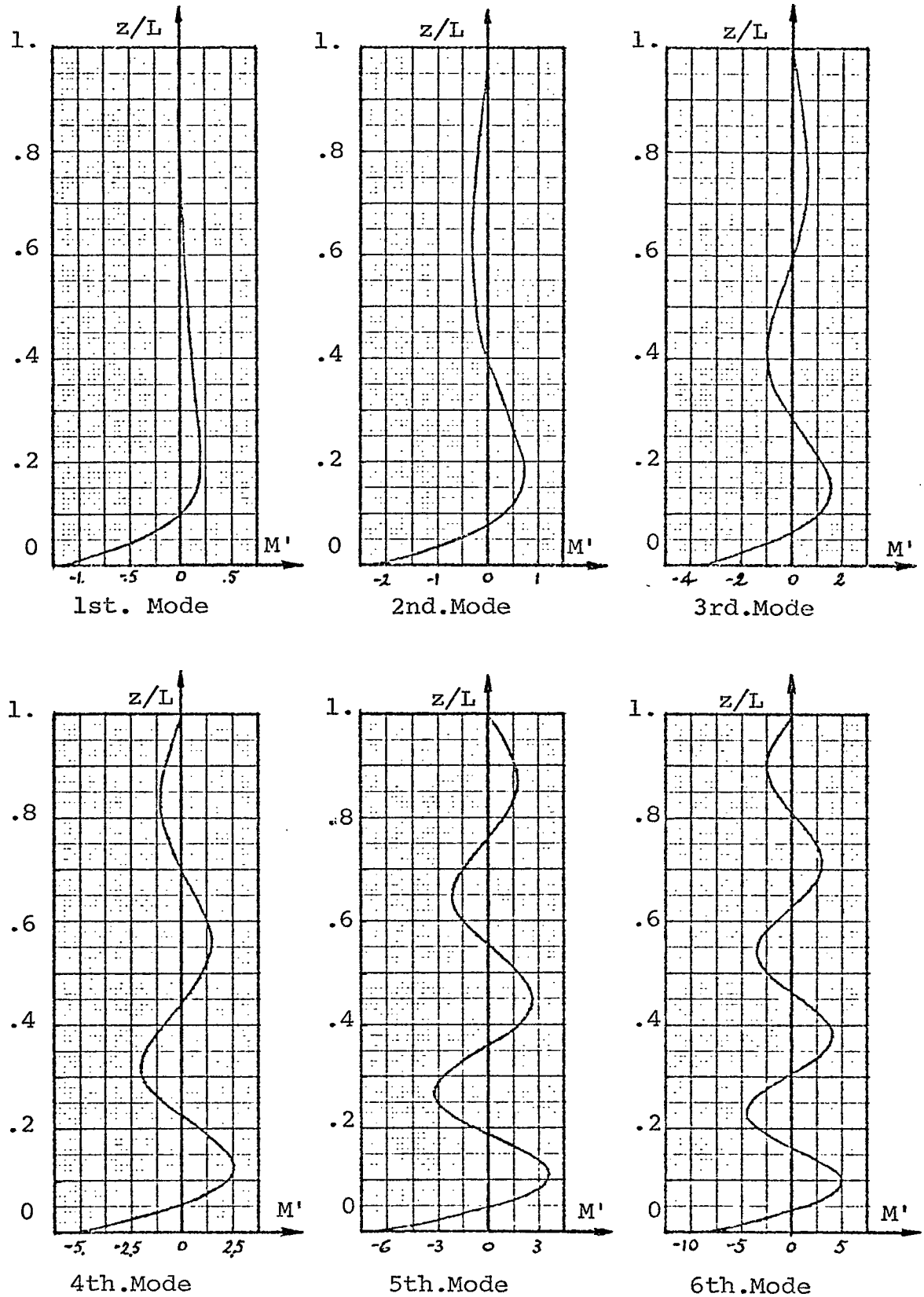


Fig. 10 Modal Moments

$M' = (ML^2/EIx_o)10^{-2}$

Table 1. Comparison of the Natural Frequencies (cpm) of a Shaft under Linearly-Varying Tension and one with Constant Tension (Average Value) for a Guided (Sliding) Shaft at the Bottom End and Simply Supported at the Top.

Frequency number	Linearly-Varying Tension Case	Constant Tension Case
Diameter = 6 in. ND, Sch. 80		L = 200 ft
1	13.92	13.56
2	36.52	36.18
3	70.58	70.30
4	116.65	116.42
5	174.87	174.68
6	245.28	245.11
7	327.90	327.75
Diameter = 6 in. ND, Sch. 80		L = 400 ft
1	6.447	6.357
2	14.142	13.952
3	23.870	23.634
4	36.112	35.861
5	51.115	50.868
6	69.009	68.774
7	89.862	89.641
8	113.71	113.50
9	140.58	140.39
10	170.49	170.30
11	202.92	203.25

Table 1 (continued)

Frequency number	Linearly-Varying Tension Case	Constant Tension Case
Diameter = 6 in. ND, Sch. 80		L = 600 ft
1	4.535	4.631
2	9.496	9.589
3	15.101	15.159
4	21.559	21.564
5	29.012	28.966
6	37.562	37.477
7	47.285	47.171
8	58.233	58.101
9	70.497	70.302
10	84.026	83.797
Diameter = 6 in. ND, Sch. 80		L = 800 ft
1	3.619	3.810
2	7.463	7.743
3	11.593	11.915
4	16.113	16.429
5	21.093	21.372
6	26.582	26.816
7	32.657	32.818
8	39.608	39.424

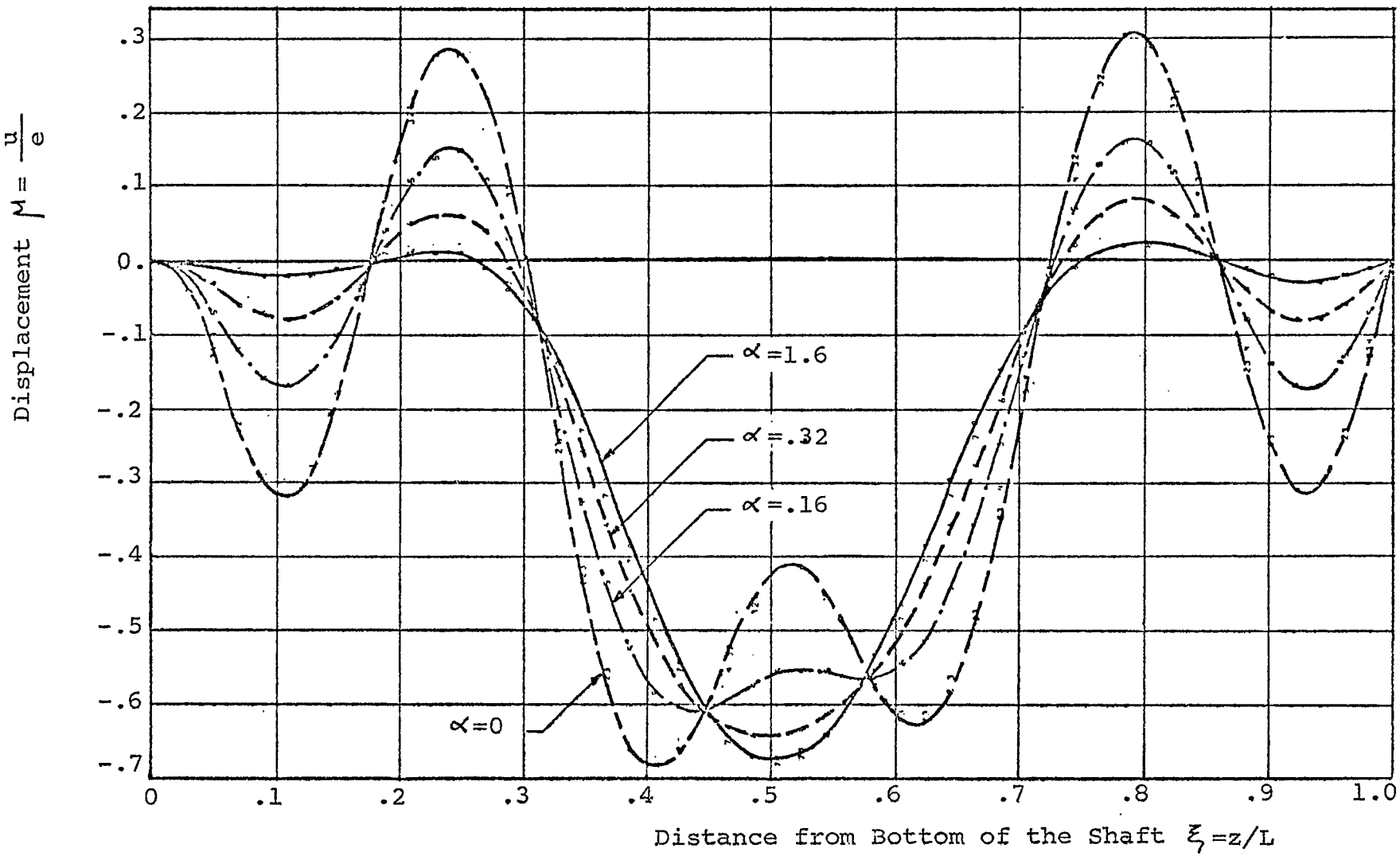


Fig. 11 Example Problem No. 1 - Displacement of the Shaft on the UZ Plane

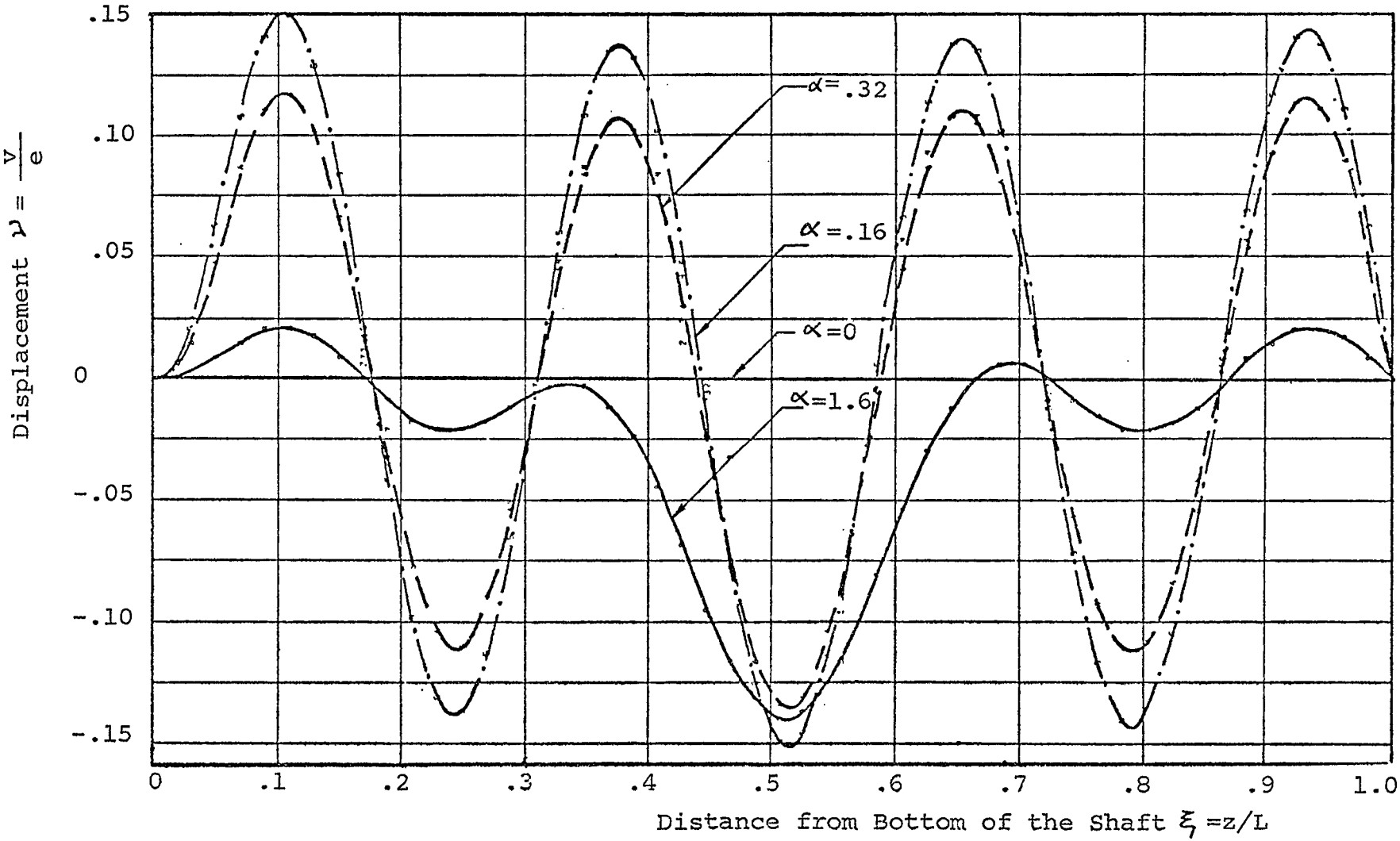


Fig. 12 Example Problem No. 1 - Displacement of the Shaft on the VZ Plane

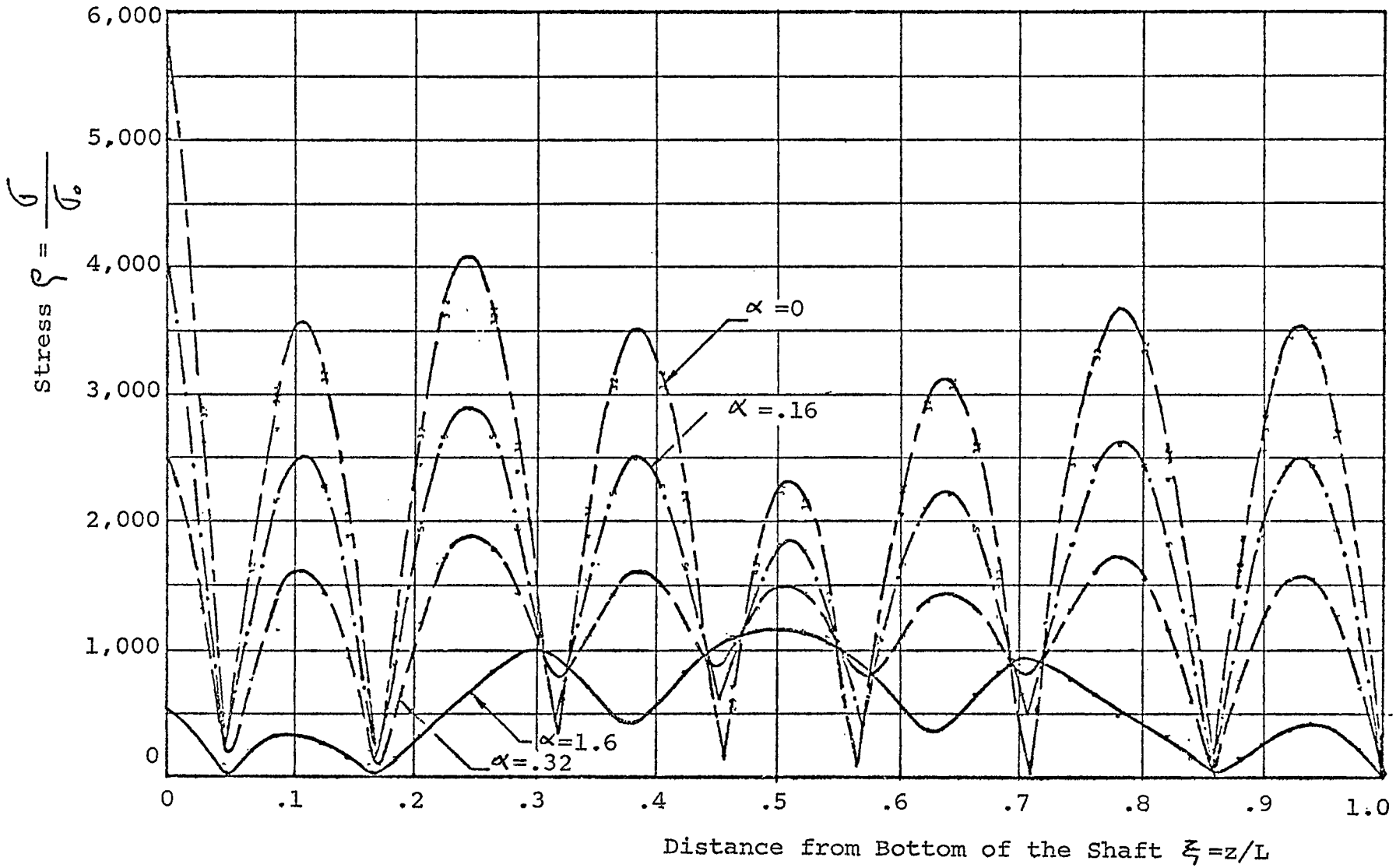


Fig. 13 Example Problem No. 1 - Total Bending Stress along the Shaft

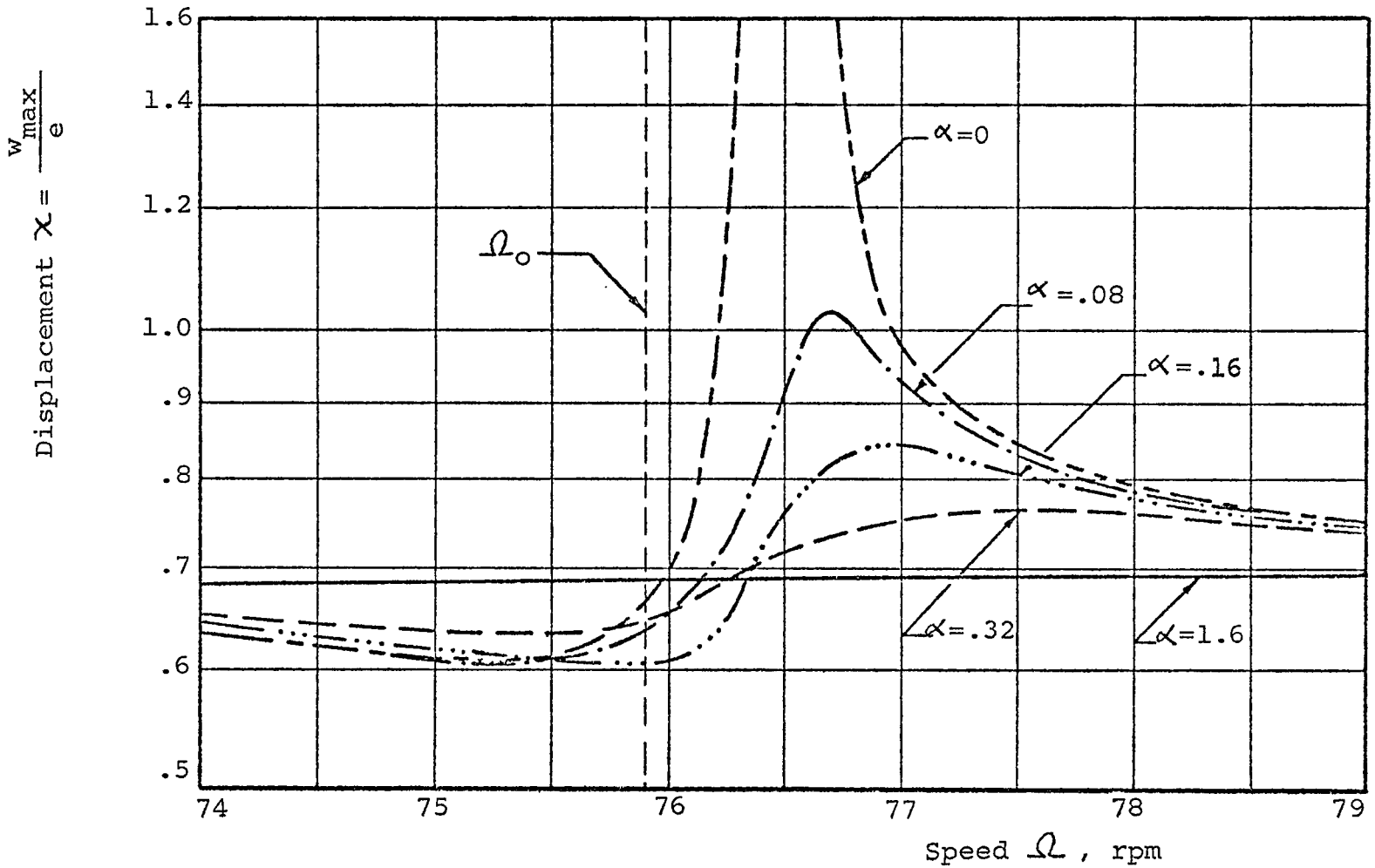


Fig. 14 Example Problem No. 1 - Maximum Displacement of the Shaft vs. Speed of Rotation

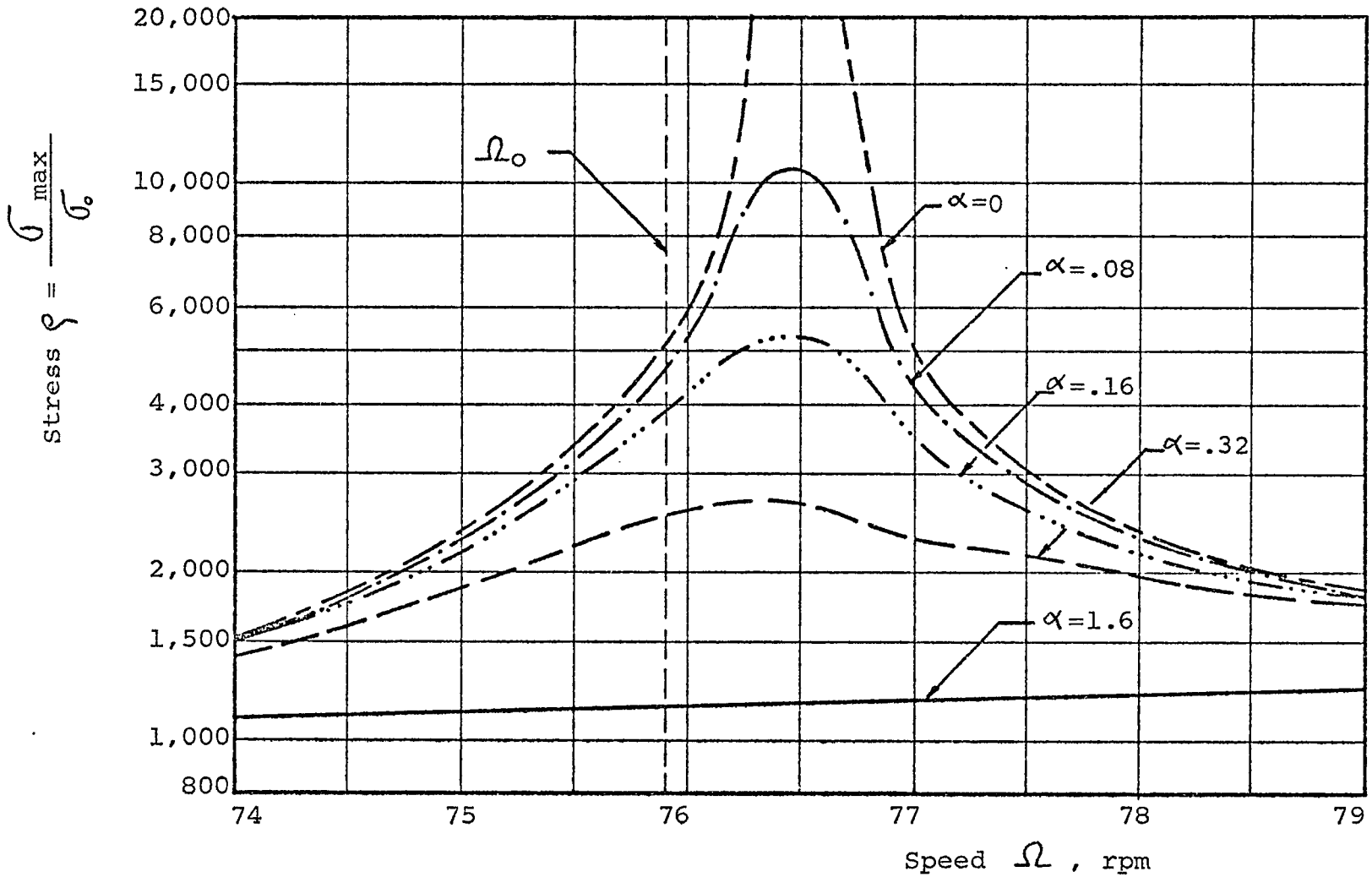


Fig. 15 Example Problem No. 1 - Maximum Bending Stress on the Shaft vs. Speed of Rotation

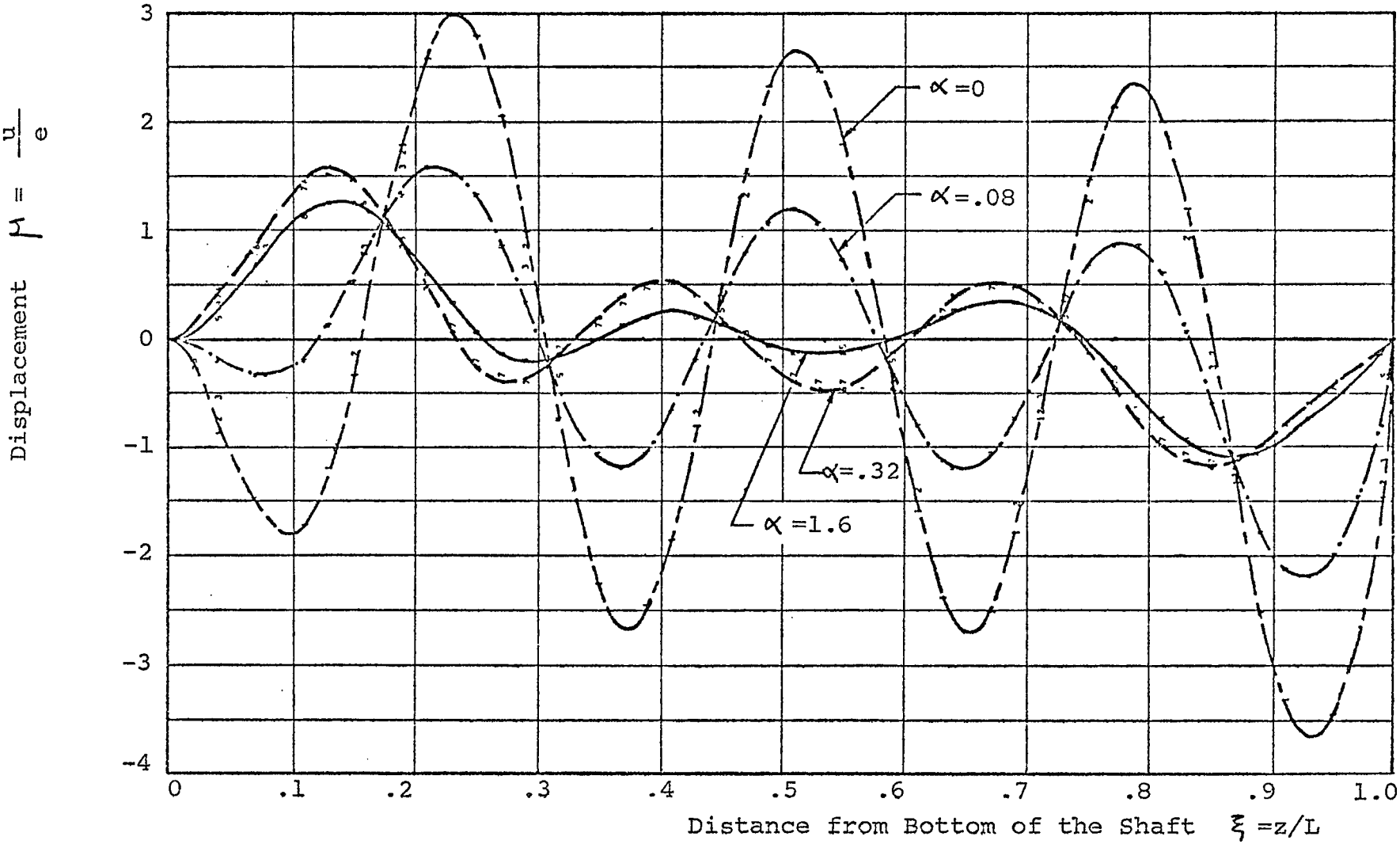


Fig. 16 Example Problem No. 2 - Displacement of the Shaft on the UZ Plane

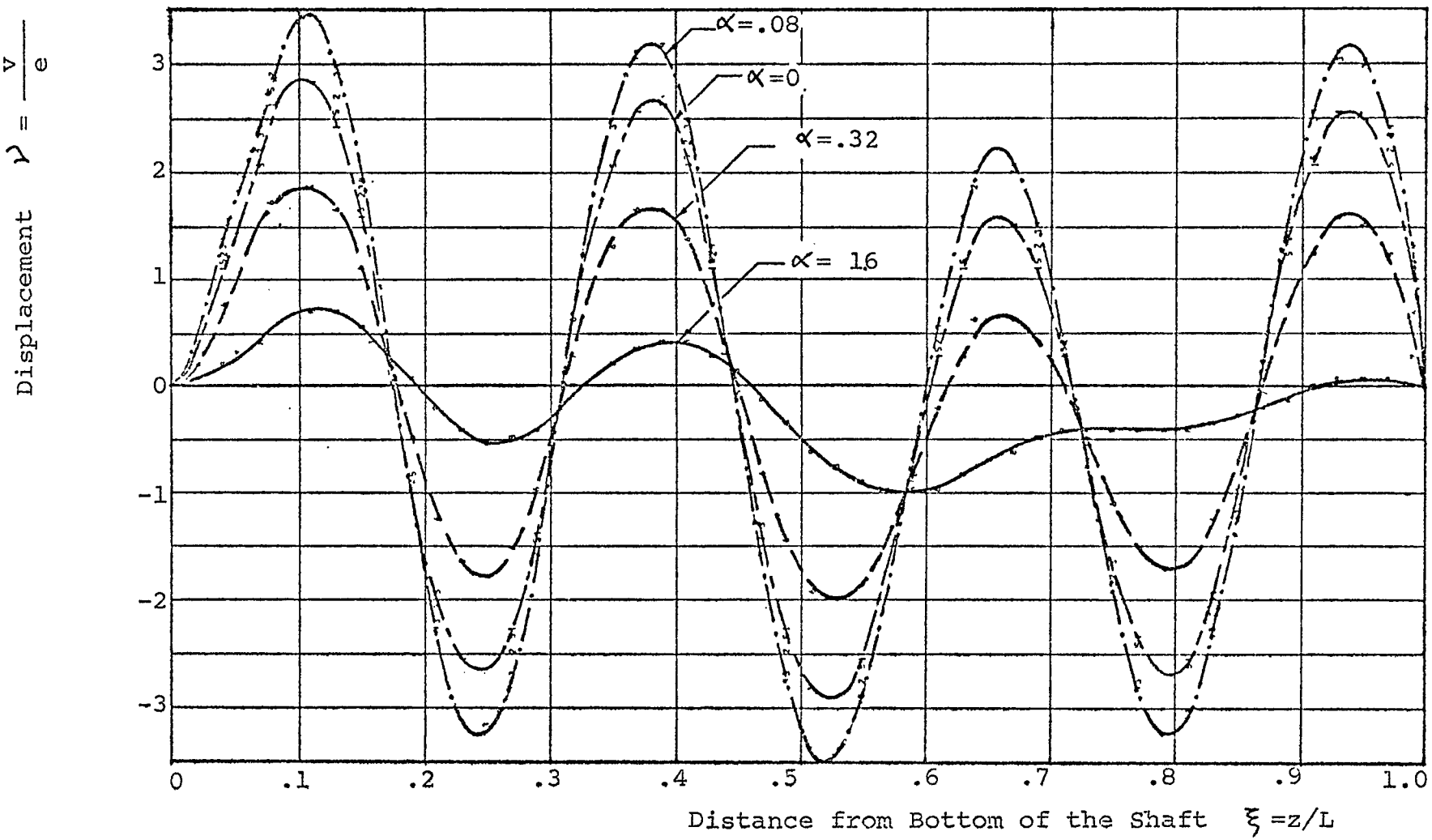


Fig. 17 Example Problem No. 2 - Displacement of the Shaft on the VZ Plane

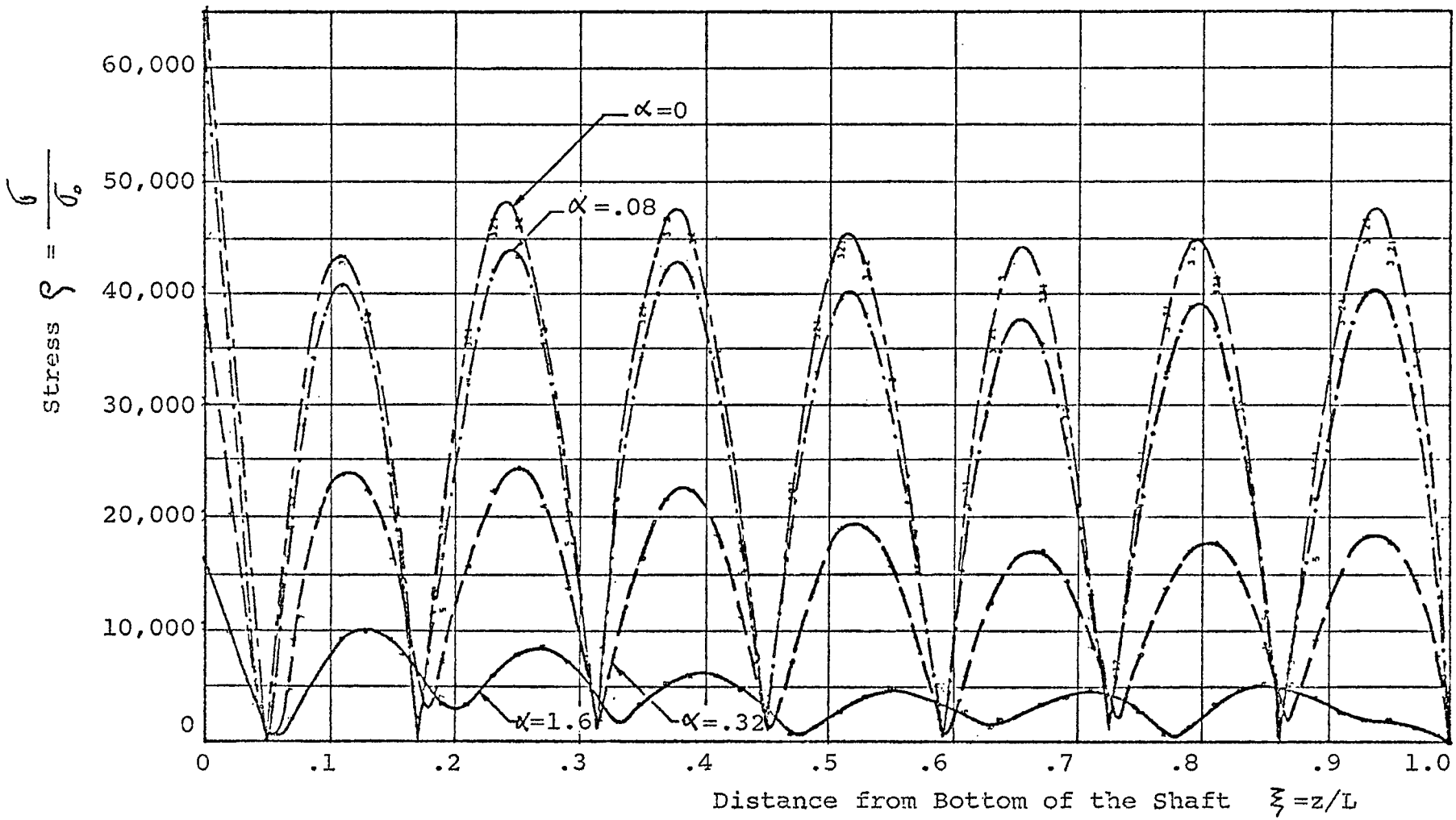


Fig. 18 Example Problem No. 2 - Total Bending Stress along the Shaft

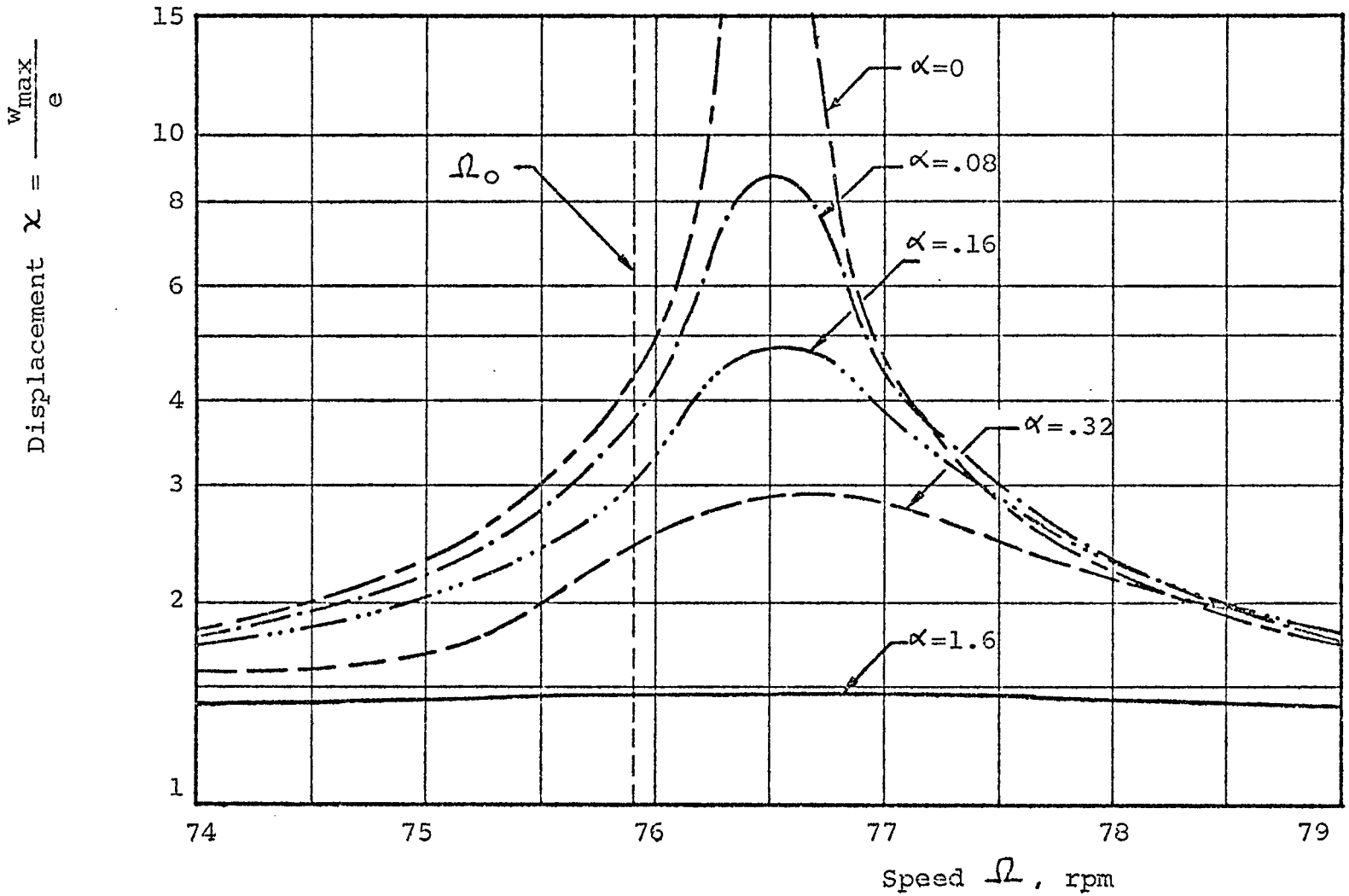


Fig. 19 Example Problem No. 2 - Maximum Displacement of the Shaft vs. Speed of Rotation

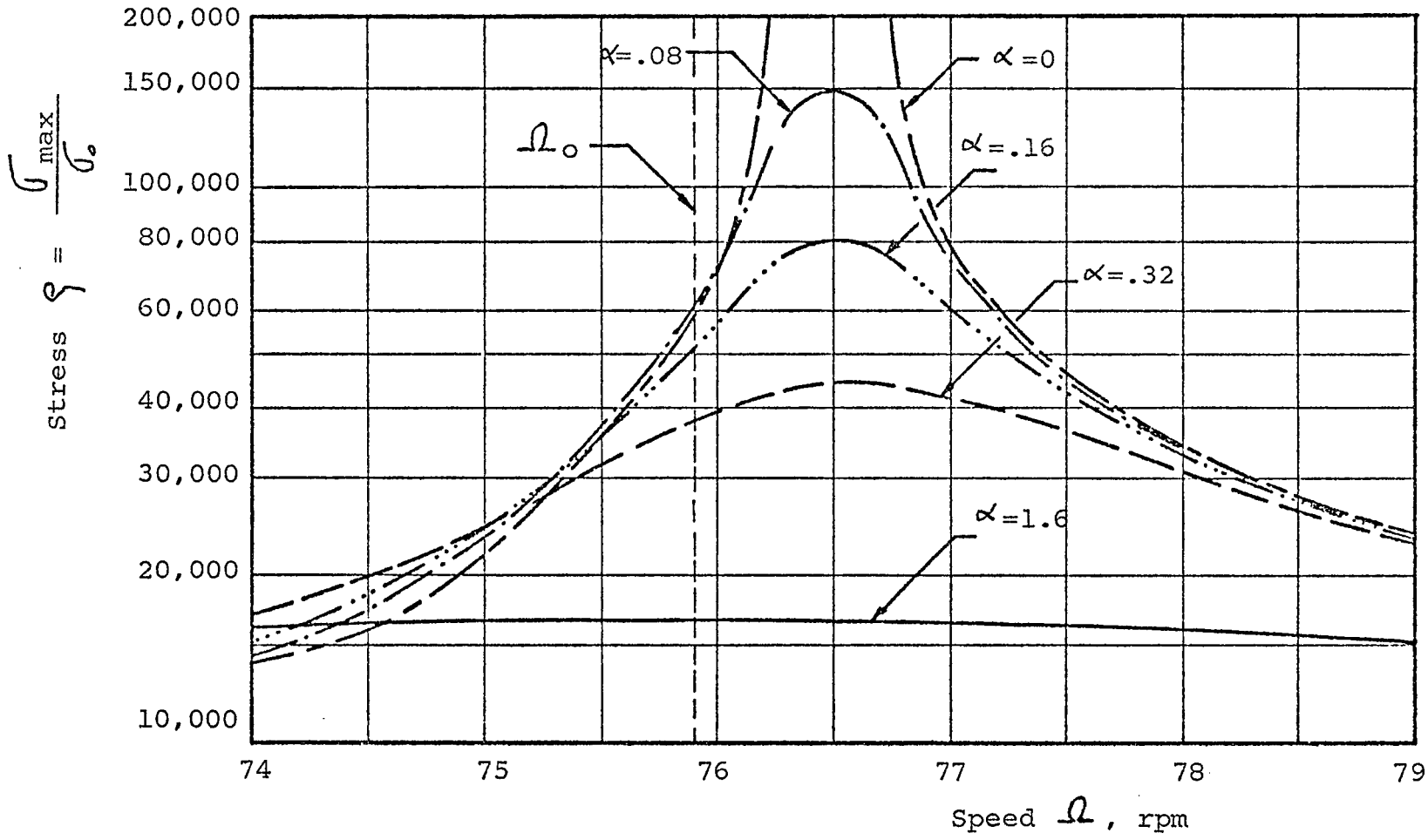


Fig. 20 Example Problem No. 2 - Maximum Bending Stress of the Shaft vs. Speed of Rotation

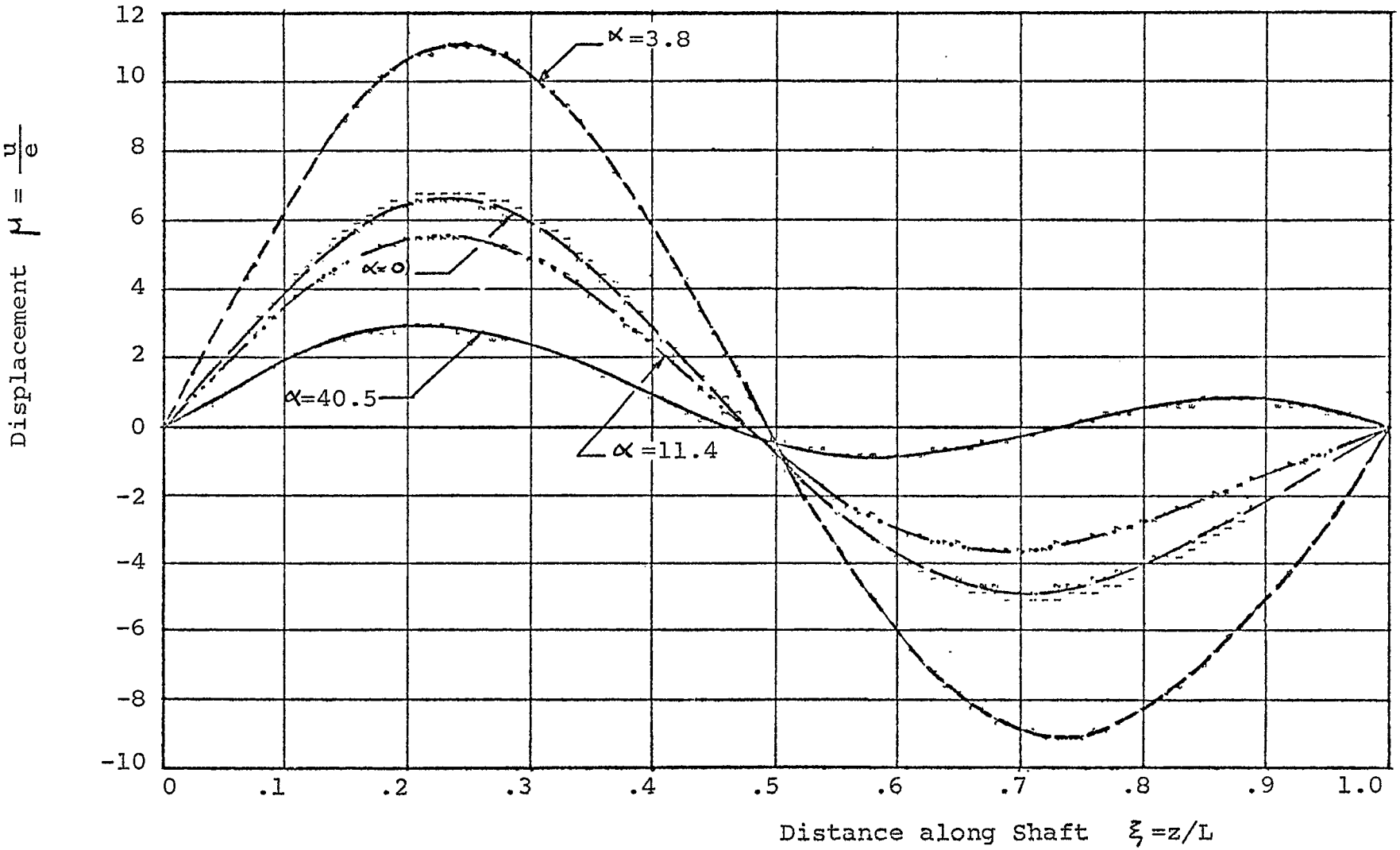


Fig. 21 Example Problem No. 3 - Displacement of the Shaft on the UZ Plane for h = 2.16

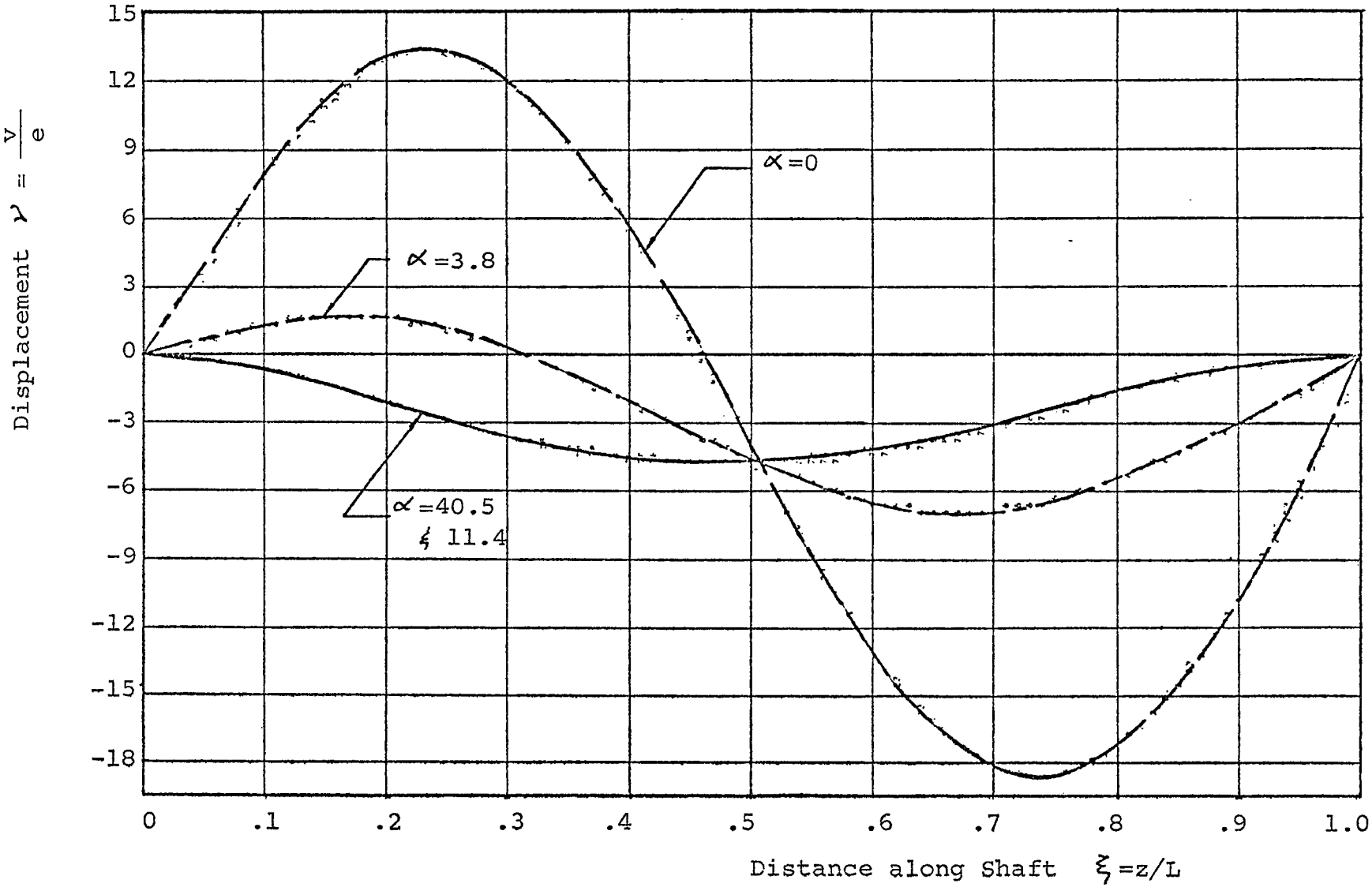


Fig. 22 Example Problem No. 3 - Displacement of the Shaft on the VZ Plane for h = 2.16

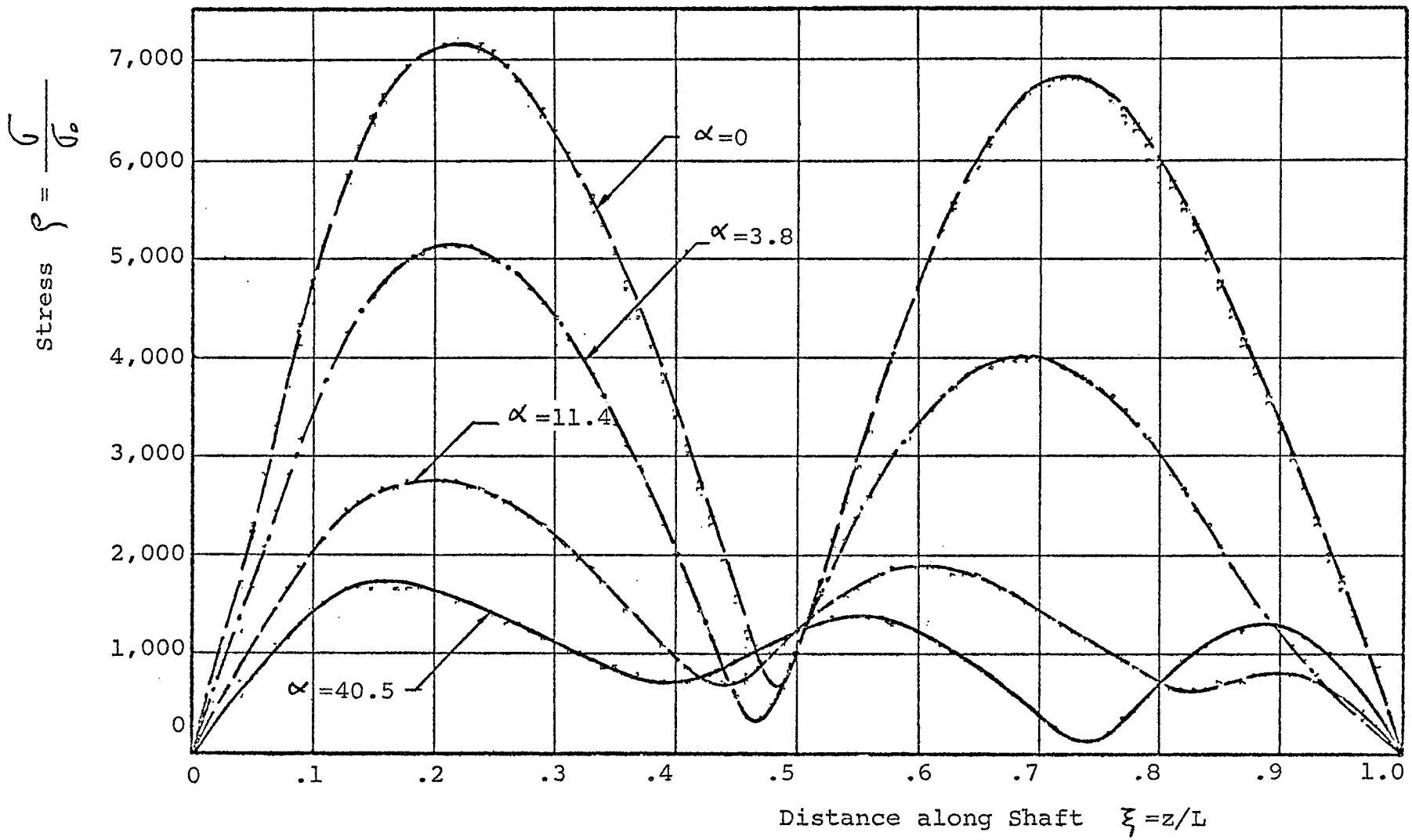


Fig. 23 Example Problem No. 3 - Total Bending Stress along the Shaft for h = 2.16

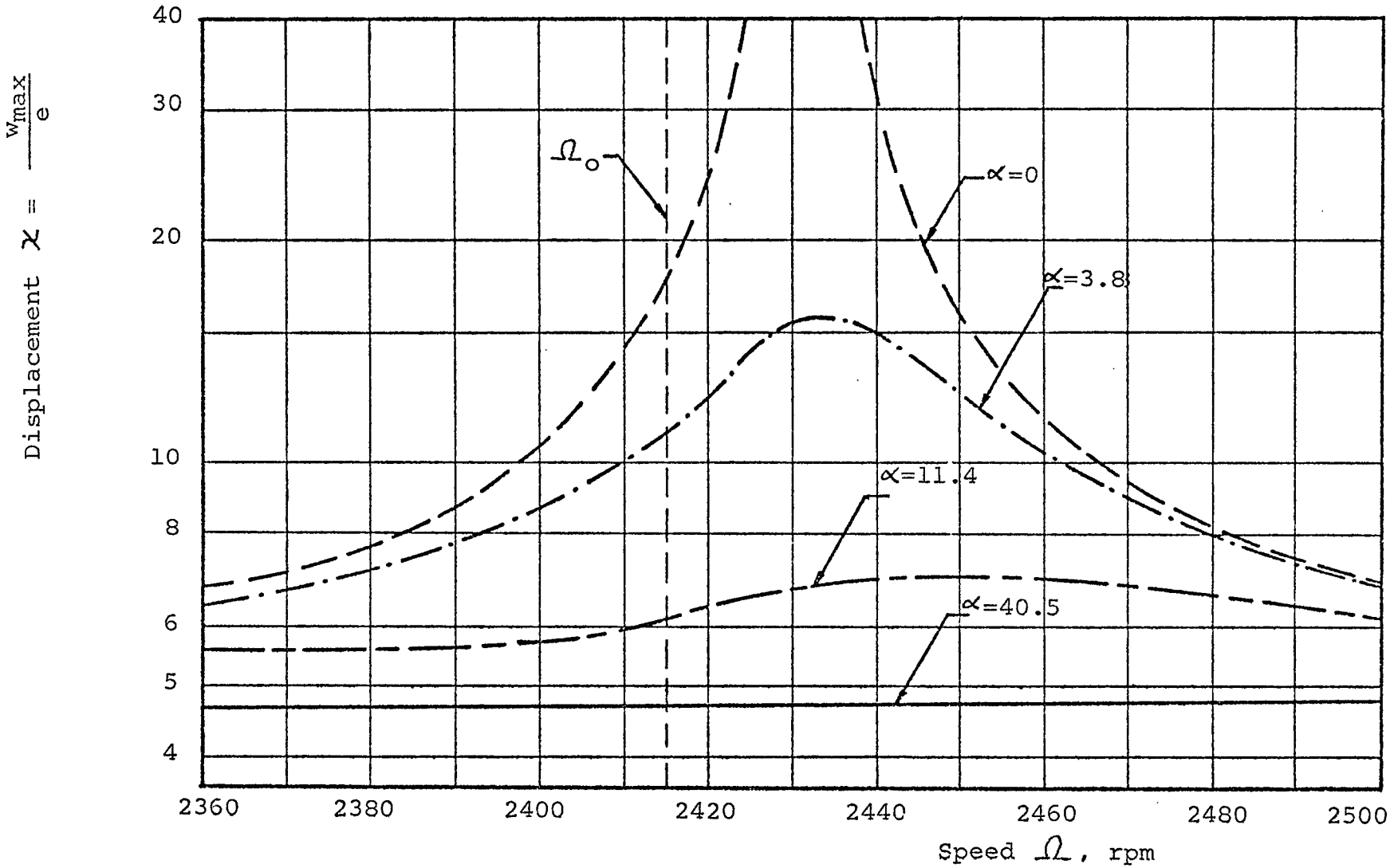


Fig. 24 Example Problem No. 3 - Maximum Displacement of the Shaft vs. Speed of Rotation for h = 2.16

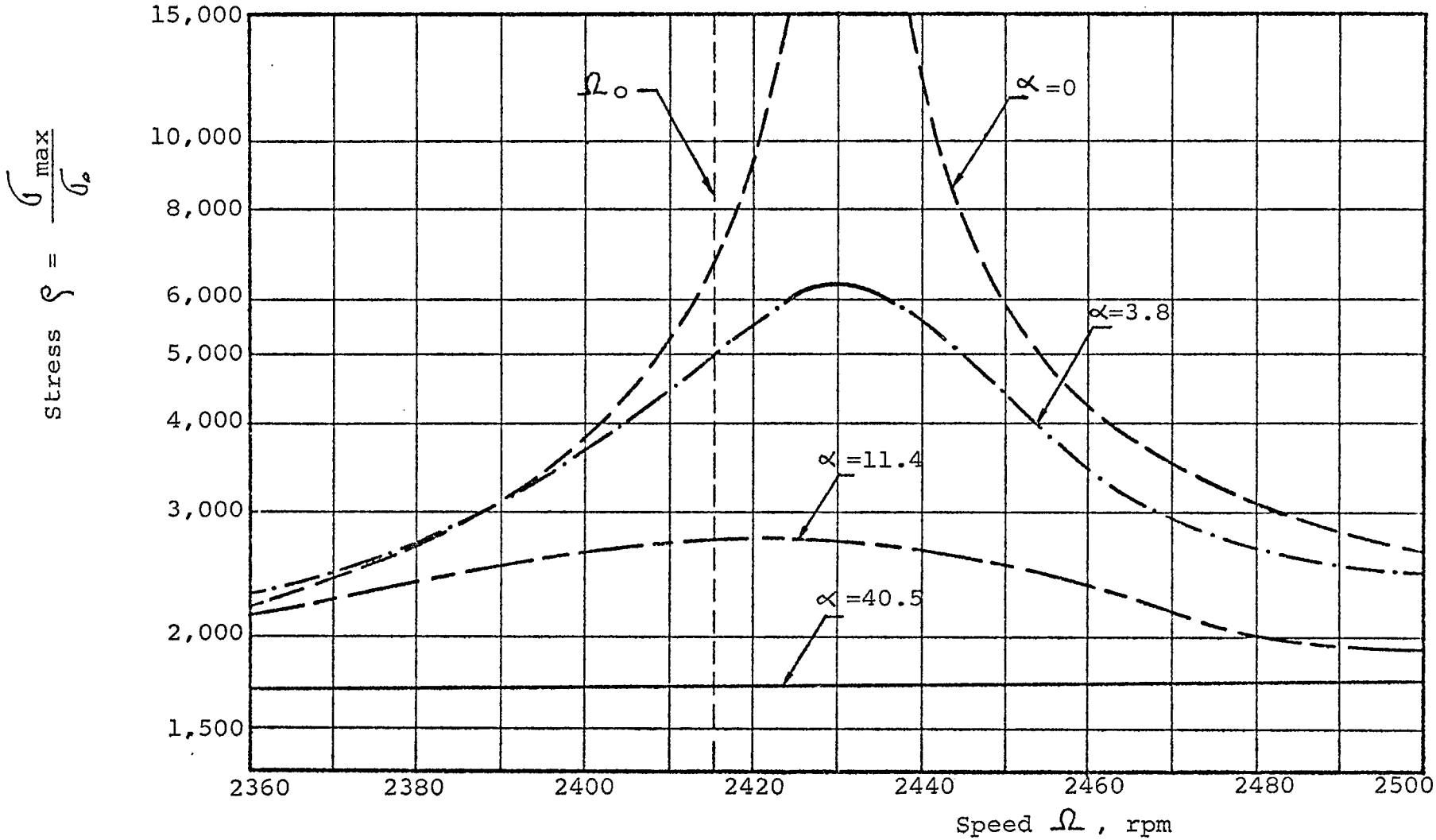


Fig. 25 Example Problem No. 3 - Maximum Bending Stress on the Shaft vs. Speed of Rotation for h = 2.16

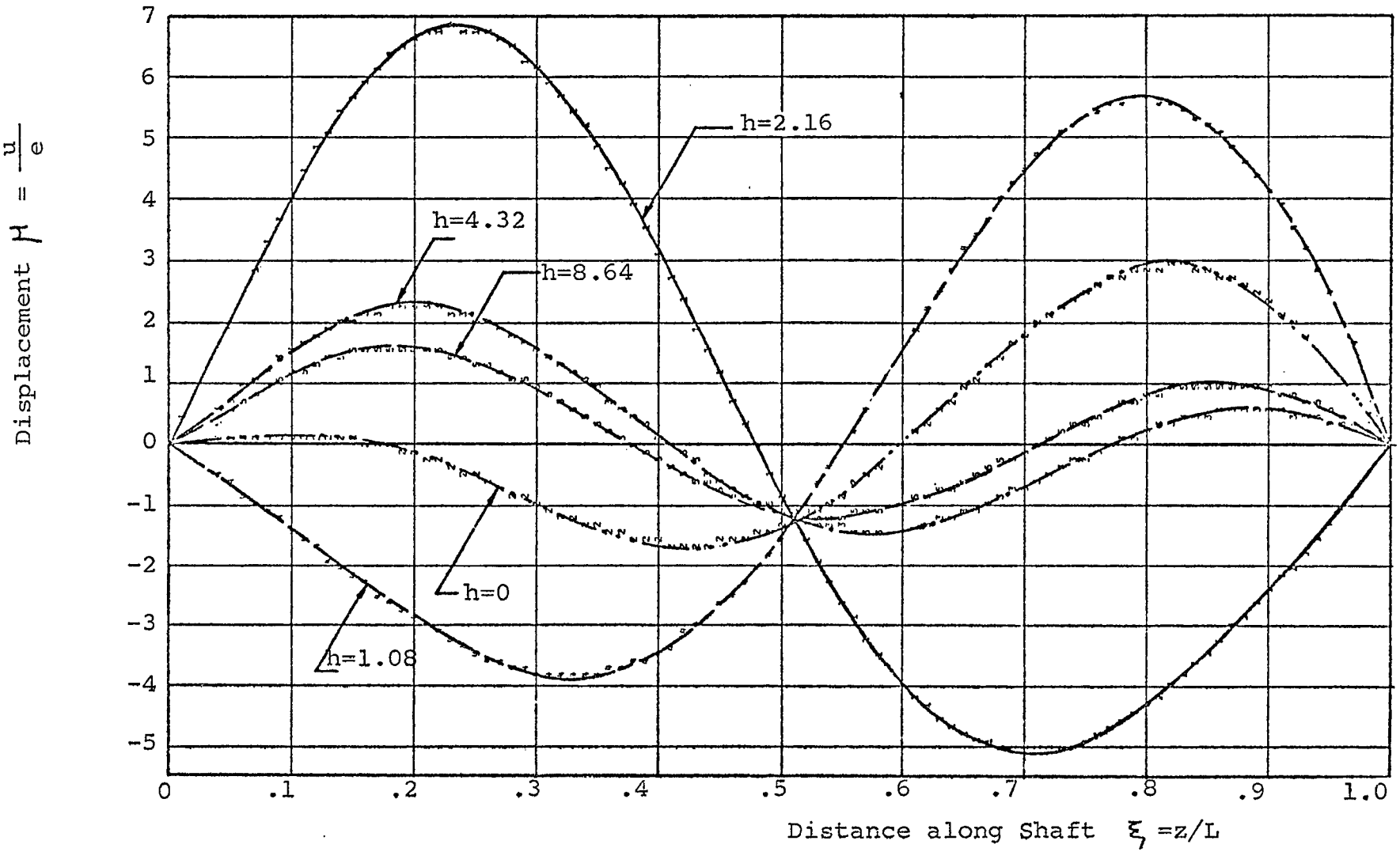


Fig. 26 Example Problem No. 3 - Displacement of the Shaft on the UZ Plane for $\alpha = .05$

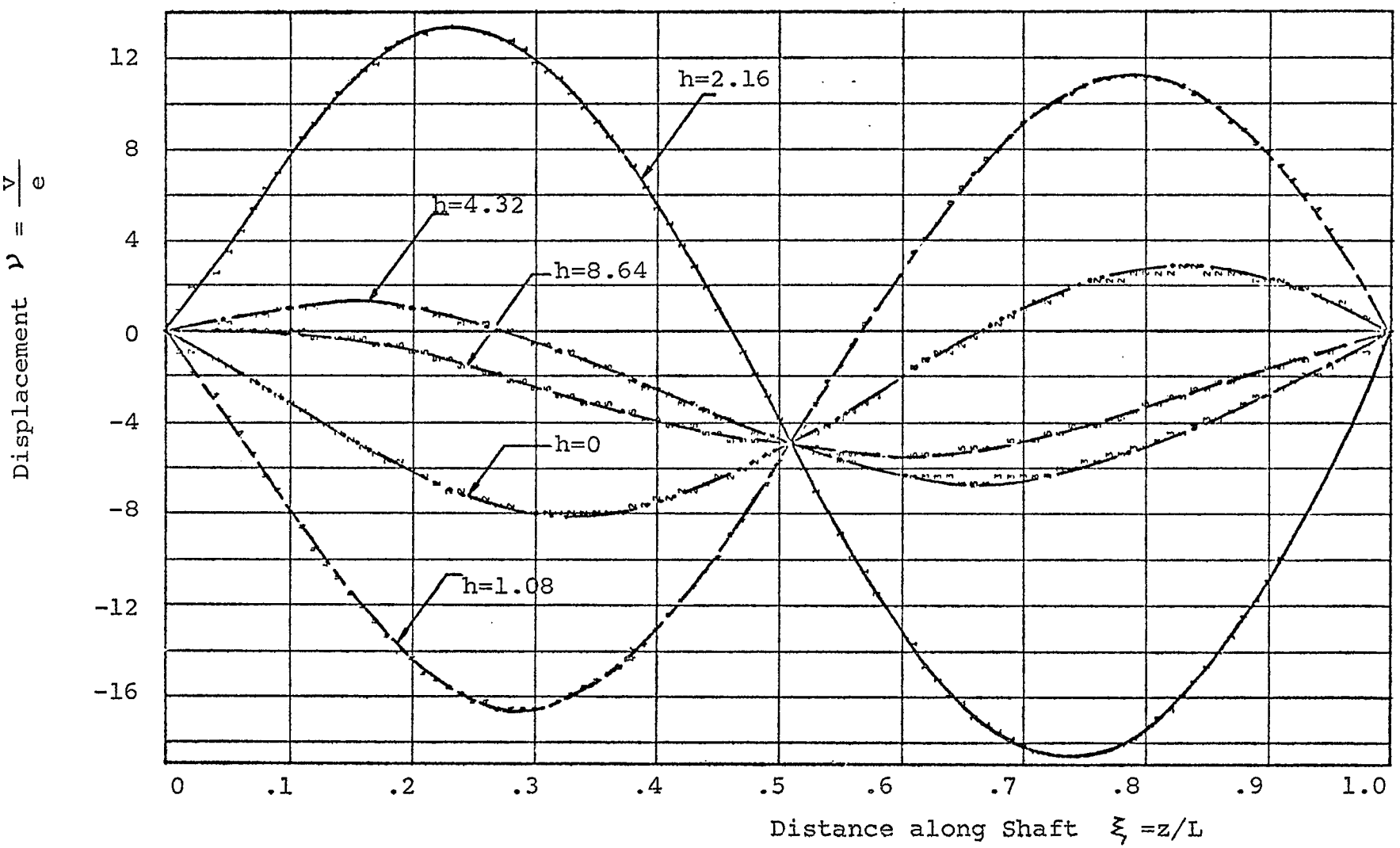


Fig. 27 Example Problem No. 3 - Displacement of the Shaft on the VZ Plane for $\alpha = .05$

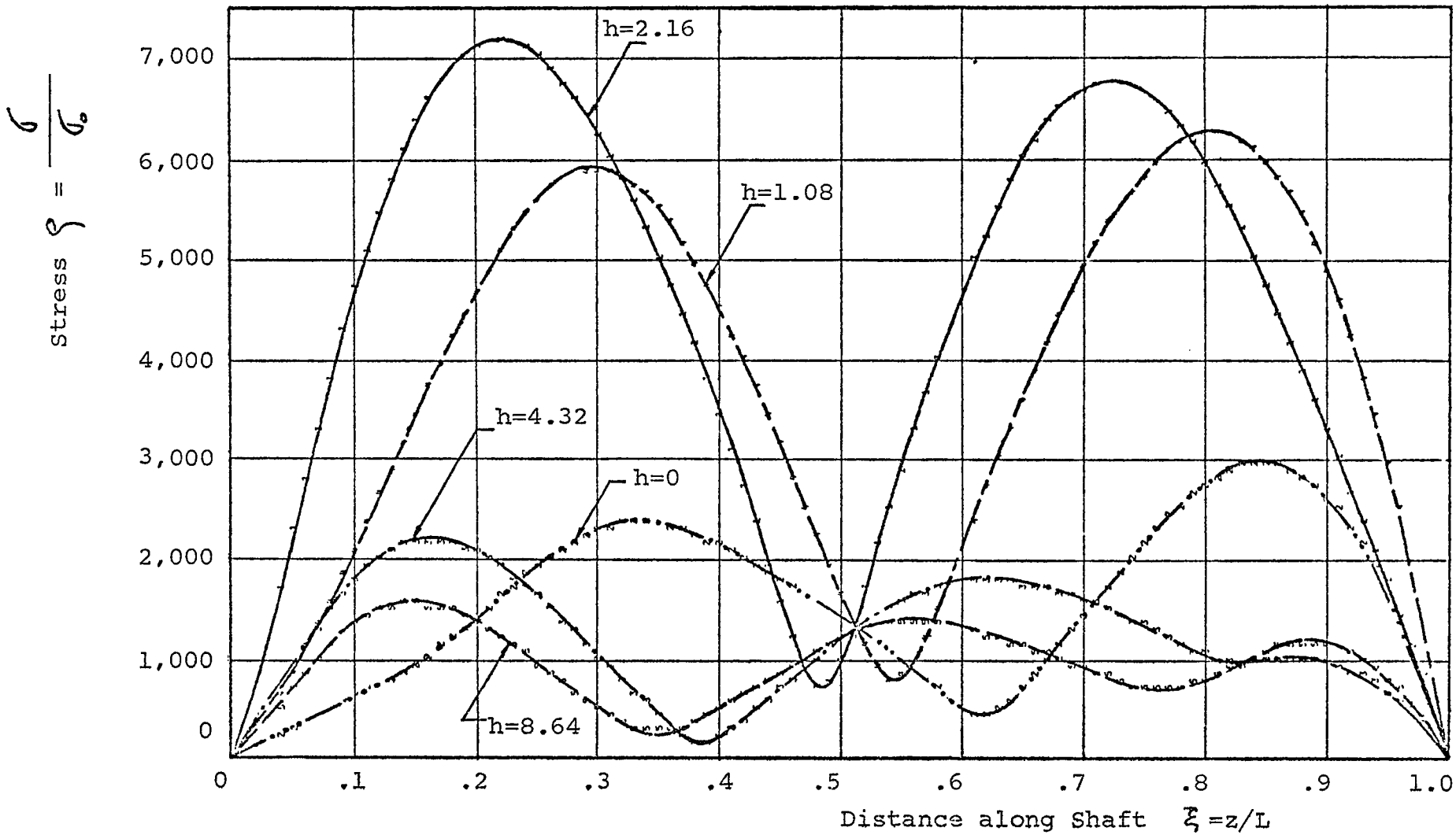


Fig. 28 Example Problem No. 3 - Total Bending Stress along the Shaft for $\alpha = .05$

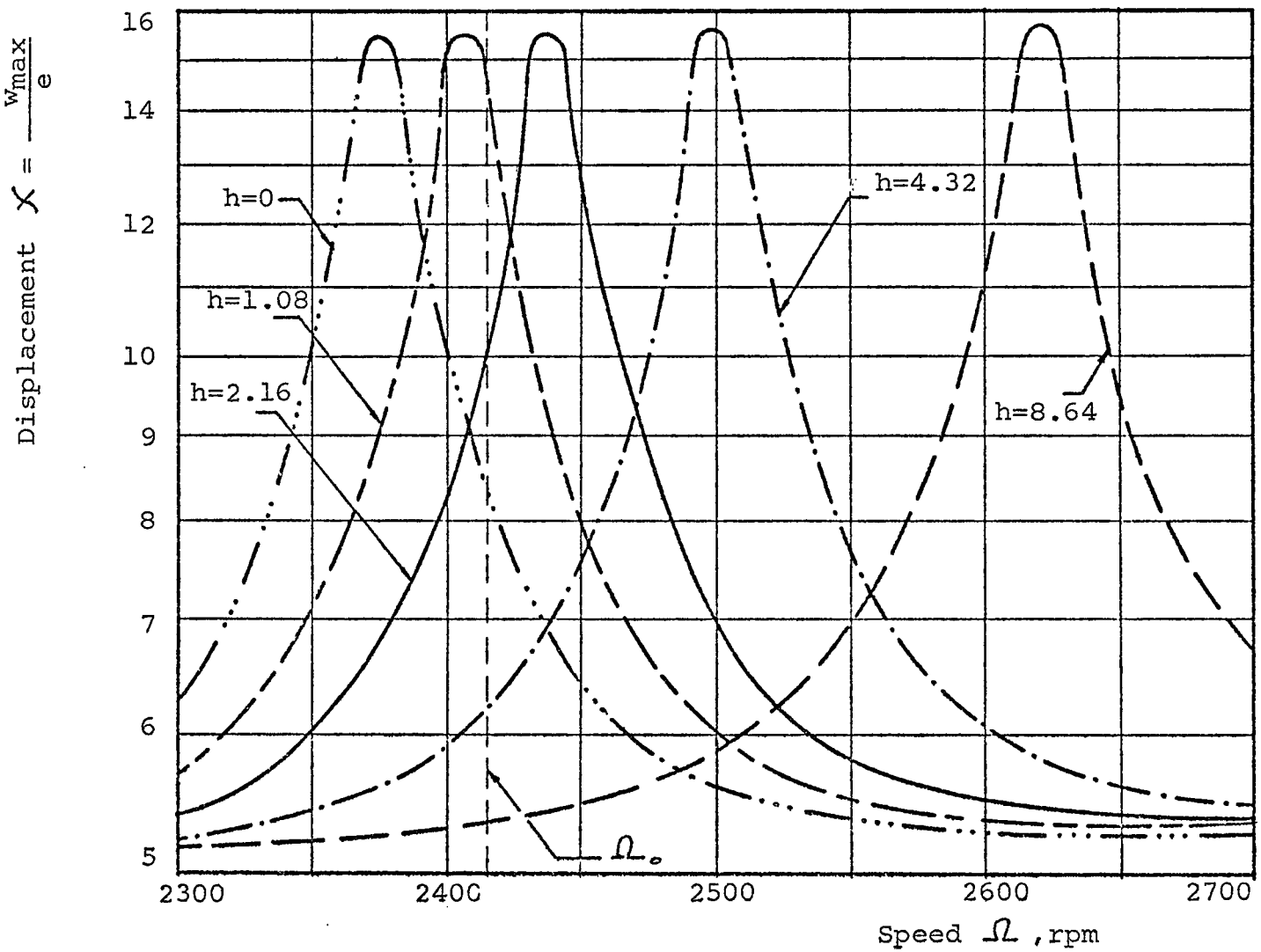


Fig. 29 Example Problem No. 3 - Maximum Displacement of the Shaft vs. Speed of Rotation for α=3.8

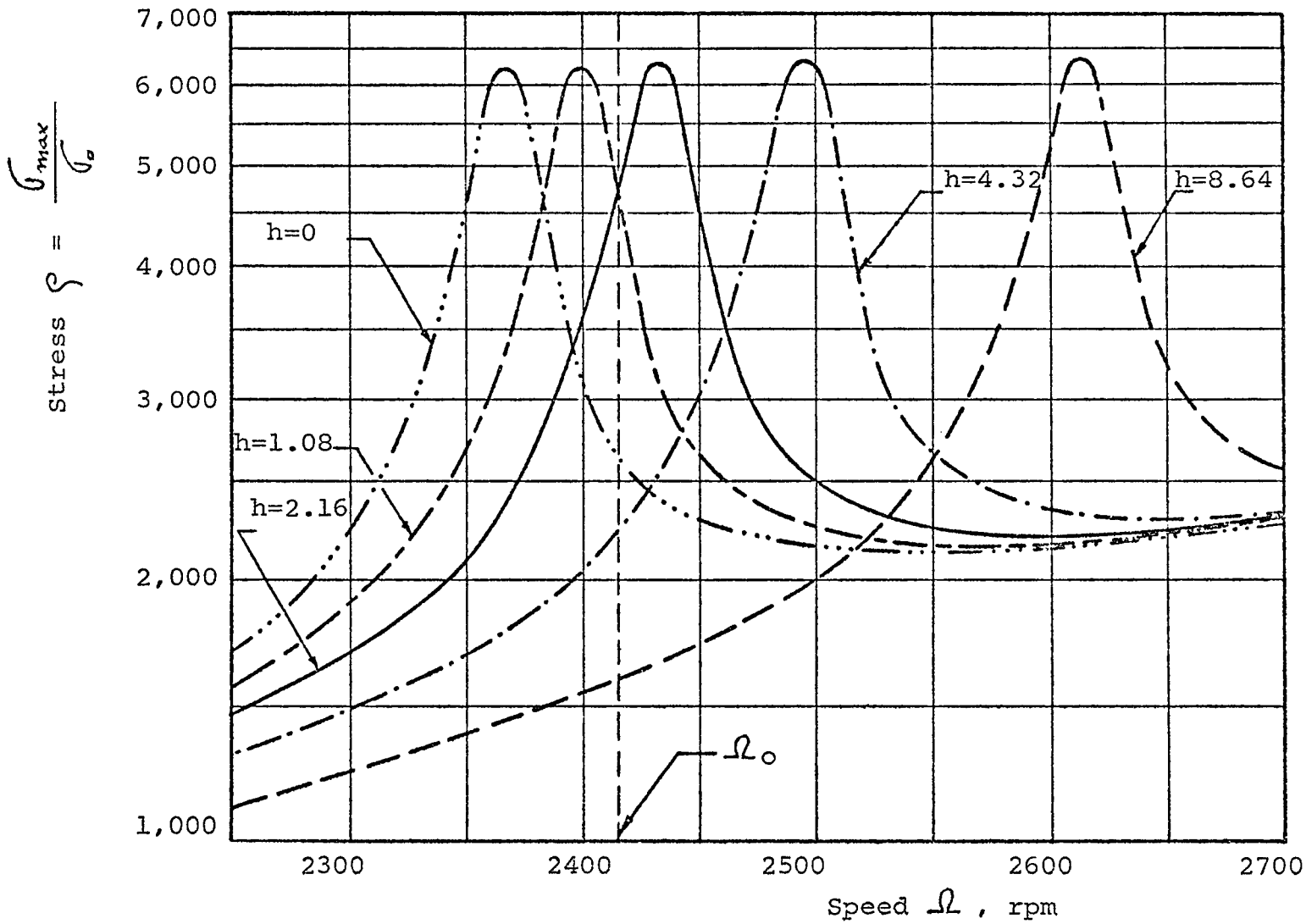


Fig. 30 Example Problem No. 3 - Maximum Bending Stress on the Shaft vs. Speed of Rotation for α =3.8

Chapter 7

SUMMARY AND CONCLUSIONS

This dissertation has presented an analytical investigation of the effect of tension on the dynamics of eccentric shafts rotating in fluid medium. Solutions were constructed using eigenfunction expansions. The eigenfunctions were obtained from the associated free-vibration problem of an ideal shaft (with no eccentricity).

Solutions for two special cases of tension were derived, namely: (1) linearly-varying tension and (2) constant tension. The method of analysis, however, could accommodate any variation of tension with axial distance provided that it is not a function of time.

Displacement and bending stress were computed along the shaft for a specific speed Ω_o. Also maximum values of displacement and stress at each speed were computed for a range of speeds which includes one resonant frequency.

It has been shown that the system is always bounded for a damped motion. For an undamped system it is unbounded at the resonant frequencies.

Comparison of the natural frequencies between a

shaft with linearly-varying tension and an identical shaft with constant tension (average value of the first) shows that they are nearly the same. Better agreement is observed at the higher frequencies.

Besides the change on the resonant frequency due to the damping effect, the surrounding fluid induces a further decrease on the value of the resonant frequency due to the lift force. For a small damping, most of the contribution to the total displacement and stress is due to the eigenfunction (modal shape) corresponding to the nearest resonant frequency. However, for high damping the shaft has a tendency to deflect into a shape similar to the (smoothed) eccentricity function.

Considering a fixed speed of rotation Ω_o, the following statement can be made concerning the effect of a change in tension: in general, there will be a decrease in displacement and bending stress if the change in tension moves the nearest resonant frequency away from the operational speed. Note that this is true for an increase or decrease of the tension.

Numerical results presented were obtained by including up to 11 eigenfuctions. The eigenvalues were calculated by the method of false position (secant method) using double precision mode on a 1108 Univac computer.

REFERENCES

1. Jeffcott, H. H. "The Lateral Vibration of Loaded Shafts in the Neighbourhood of a Whirling Speed. - The Effect of Want of Balance," Phil. Mag., 37 (March, 1919), 304-14.

2. Taylor, H. D. "Shaft Behavior at Critical Speed," Gen. Elect. Rev., 32, 4 (April, 1929), 194-200.

3. Robertson, D. "The Whirling of Shafts," Engineer, Lond., 158 (July-December, 1934), 216-17, 228-31.

4. Johnson, D. C. "Free Vibration of a Rotating Elastic Body," Aircr. Eng., 24 (August, 1952), 234-36.

5. _____. "Forced Vibration of a Rotating Elastic Body," Aircr. Eng., 24 (September, 1952), 271-73.

6. Bishop, R. E. D. "The Vibration of Rotating Shafts," J. Mech. Eng. Sci., 1, 1 (1959), 50-65.

7. Parkinson, A. G. "An Introduction to the Vibration of Rotating Flexible Shafts," Bull. Mech. Eng. Educ., 6 (1967), 47-62.

8. Ariaratnam, S. T. "The Vibration of Unsymmetrical Rotating Shafts," J. Appl. Mech. (March, 1965), 157-62.

9. Graham, R. D., M. A. Frost, III, and J. C. Wilhoit, Jr. "Analysis of the Motion of Deep-Water Drill Strings—Part 1: Forced Lateral Motion; Part 2: Forced Rolling Motion," J. Eng. Ind., Trans. ASME, Paper no. 64—Pet 6,7 (1964).

10. Huang, T., and D. W. Dareing. "Buckling and Lateral Vibration of Drill Pipe," J. Eng. Ind. (November, 1968), 613-619.

11. Hildebrand, F. B. Advanced Calculus for Applications. Englewood Cliffs, N.J.: Prentice-Hall, Inc., 1965. Pp. 8-13.

12. Hartog, J. P. Den. Mechanical Vibrations. New York: McGraw-Hill Book Company, Inc., 1956. P. 289.

13. Ince, E. L. Ordinary Differential Equations. New York: Dover Publications, Inc., Pp. 284-86, 356-57.

14. Brand, L. Differential and Difference Equations. New York: John Wiley & Sons, Inc., 1966. P. 418.

15. Nowacki, W. Dynamic of Elastic Systems. London: Chapman & Hall Ltd., 1963. Pp. 137-40.

16. Meirovitch, L. Analytical Methods in Vibrations. New York: The MacMillan Company, 1967.

17. Sneddon, I. N. Fourier Transforms. New York: McGraw-Hill Book Company, Inc.,1951.

18. Erdélyi, A. (ed.). Higher Transcendental Functions. Vol. 1. New York: McGraw-Hill Book Company, Inc., 1953.

19. Southworth, R. W. and Deleeuw, S. L. Digital Computation and Numerical Methods. New York: McGraw-Hill Book Company, Inc., 1965.

Appendix 1

STEADY-STATE SOLUTION FOR A SHAFT WITH CONSTANT TENSION USING FOURIER SINE TRANSFORMS

The differential equations governing the steady-state motion of the shaft are obtained from Eqs. (3.11) if one disregards the time-variable terms. Introducing the steady-lift force, given by (3.12), one has for the differential equations governing steady-state motion of the shaft

$$-\Omega^2(u+a) = -\frac{EI}{m}\frac{d^4(u-u_0)}{dz^4} + \frac{1}{m}\frac{d}{dz}\left(T\frac{du}{dz}\right) + \alpha\Omega v + \frac{m_0}{m}\Omega^2 u \tag{A1.1a}$$

$$-\Omega^2(v+b) = -\frac{EI}{m}\frac{d^4(v-v_0)}{dz^4} + \frac{1}{m}\frac{d}{dz}\left(T\frac{dv}{dz}\right) - \alpha\Omega u + \frac{m_0}{m}\Omega^2 v \,. \tag{A1.1b}$$

Define now three complex quantities

$$w = u + i v \tag{A1.2a}$$

$$\epsilon = a + i b \tag{A1.2b}$$

$$w_0 = u_0 + i v_0 \,. \tag{A1.2c}$$

Multiplying Eq. (A1.1b) by the imaginary unit $\underline{i}$ and adding it to Eq. (A1.1a) yields

$$-\Omega^2(w+\epsilon) = -\frac{EI}{m}\frac{d^4(w-w_0)}{dz^4} + \frac{1}{m}\frac{d}{dz}\left(T\frac{dw}{dz}\right) - i\alpha\Omega w + \frac{m_0}{m}\Omega^2 w. \tag{A1.3}$$

Consider only constant tension $T = T_0$. Making the substitution $z = \xi L$ and multiplying both sides of Eq. (A1.3) by $\frac{mL^4}{EI}$, one obtains

$$\frac{d^4}{d\xi^4}(w-w_0) - h\frac{d^2w}{d\xi^2} + iqw - k^4R_m^2 w - k^4\epsilon = 0, \tag{A1.4}$$

where the following non-dimensional constants have been introduced:

$$h = \frac{T_0 L^2}{EI}, \tag{A1.5a}$$

$$q = \frac{\alpha\Omega m L^4}{EI}, \tag{A1.5b}$$

$$k^4 = \frac{m\Omega^2 L^4}{EI}, \tag{A1.5c}$$

$$R_m^2 = 1 + \frac{m_0}{m}. \tag{A1.5d}$$

For a simply-supported shaft, the complex variable w must satisfy the following boundary conditions:

$$w(0) = w(L) = \frac{d^2 w}{d\xi^2}(0) = \frac{d^2 w}{d\xi^2}(L) = 0. \qquad \text{(A1.6 a,b,c,d)}$$

Using the non-dimensional variable ξ, the Finite Fourier Sine Transform is defined as

$$\bar{w} = \int_0^1 w(\xi) \sin r\pi\xi \, d\xi \qquad \text{(A1.7)}$$

with similar expression for $\bar{w}_o$ and $\bar{\epsilon}$. Then, applying the transform to Eq. (A1.4) yields

$$r^4\pi^4\bar{w} - r^4\pi^4\bar{w}_o + h r^2\pi^2\bar{w} + i g\bar{w} - k^4 R_m^2 \bar{w} - k^4\bar{\epsilon} = 0. \qquad \text{(A1.8)}$$

Solving for $\bar{w}$ and rationalizing leads to

$$\bar{w} = \frac{(k^4\bar{\epsilon} + r^4\pi^4\bar{w}_o)\left[(r^4\pi^4 + h r^2\pi^2 - k^4 R_m^2) - i g\right]}{(r^4\pi^4 + h r^2\pi^2 - k^4 R_m^2)^2 + g^2}. \qquad \text{(A1.9)}$$

If the numerator and denominator are divided by k^8, the following final expression for $\bar{w}$ is obtained:

$$\bar{w} = \frac{(\bar{\epsilon} + R_{o,r}^2\bar{w}_o)\left[(R_r^2 - R_m^2) - i\alpha_o\right]}{(R_r^2 - R_m^2)^2 + \alpha_o^2}, \qquad \text{(A1.10)}$$

since, according to Eqs. (A1.5), (5.28), (5.51) and (5.54),

$$\frac{r^4\pi^4}{k^4} = \frac{EI}{m}\frac{r^4\pi^4}{L^4}\frac{1}{\Omega^2} = \frac{\omega_{o,r}^2}{\Omega^2} = R_{o,r}^2 ,$$

$$\frac{r^4\pi^4}{k^4} + \frac{h\, r^2\pi^2}{k^4} = \left[\frac{EI}{m}\cdot\frac{r^4\pi^4}{L^4} + \frac{T_o}{m}\frac{r^2\pi^2}{L^2}\right]\frac{1}{\Omega^2} = \frac{\omega_r^2}{\Omega^2} = R_r^2 ,$$

$$\frac{g}{k^4} = \frac{\alpha}{\Omega} = \alpha_o .$$

Applying the transform to Eqs. (A1.2) and substituting in Eq. (A1.10), a separation of the real and imaginary parts yields the following expressions for $\bar{U}$ and $\bar{V}$:

$$\bar{U} = \frac{(R_r^2 - R_m^2)(\bar{a} + R_{o,r}^2\,\bar{U}_o) + \alpha_o(\bar{b} + R_{o,r}^2\,\bar{V}_o)}{(R_r^2 - R_m^2)^2 + \alpha_o^2}, \tag{A1.11a}$$

$$\bar{V} = \frac{(R_r^2 - R_m^2)(\bar{b} + R_{o,r}^2\,\bar{V}_o) - \alpha_o(\bar{a} + R_{o,r}^2\,U_o)}{(R_r^2 - R_m^2)^2 + \alpha_o^2}. \tag{A1.11b}$$

The inverse Sine Transforms are defined as

$$U(\xi) = 2\sum_{r=1}^{\infty} \bar{U}\, \sin r\pi\xi , \tag{A1.12a}$$

$$V(\xi) = 2\sum_{r=1}^{\infty} \bar{V}\, \sin r\pi\xi . \tag{A1.12b}$$

Substituting Eqs. (A1.11) into (A1.12) one finds for the

nondimensional displacements μ and ν (after dividing by $\underline{e}$)

$$\mu(\xi) = 2\sum_{n=1}^{\infty} \frac{(R_n^2 - R_m^2)(a_{no}^* + R_{o,n}^2 c_{no}^*) + \alpha_o (b_{no}^* + R_{o,n}^2 d_{no}^*)}{(R_n^2 - R_m^2)^2 + \alpha_o^2} \sin n\pi\xi \,,$$

(A1.13a)

$$\nu(\xi) = 2\sum_{n=1}^{\infty} \frac{(R_n^2 - R_m^2)(b_{no}^* + R_{o,n}^2 d_{no}^*) - \alpha_o (a_{no}^* + R_{o,n}^2 c_{no}^*)}{(R_n^2 - R_m^2)^2 + \alpha_o^2} \sin n\pi\xi \,,$$

(A1.13b)

where $a_{no}^* = \dfrac{\bar{a}}{e}$; $b_{no}^* = \dfrac{\bar{b}}{e}$; $c_{no}^* = \dfrac{\bar{u}_o}{e}$; $d_{no}^* = \dfrac{\bar{v}_o}{e}$.

(A1.14 a,b,c,d)

It should be noted that Eqs. (A1.13) are the same as (5.55) which were derived by modal analysis.

Appendix 2

EQUIVALENT VISCOUS DAMPING COEFFICIENT α_o OR LINEARIZATION OF QUADRATIC DAMPING

The damping coefficient used in the governing equations has been assumed to be linear. It's value can be obtained from the quadratic damping coefficient. Equating the energy dissipated by viscous and quadratic damping in one revolution of the shaft, the equivalent viscous damping coefficient can be determined. For steady-state motion the equality just mentioned is

$$\int_0^L (2\pi w)(c\Omega w\,dz) = \int_0^L (2\pi w)(d\Omega^2 w^2\,dz) \tag{A2.1}$$

where w is the total displacement of the shaft, always positive and d is the quadratic damping coefficient. Simplifying the above equation, the value of c can be expressed as

$$c = d\Omega \frac{\int_0^L w^3\,dz}{\int_0^L w^2\,dz} .$$

Using $\alpha_o = c/m\Omega$, $\xi = z/L$ and $w_{\mu\nu} = w/e$ the

non-dimensional form of the linear damping coefficient is obtained.

$$\alpha_o = \frac{de}{m} \frac{\int_0^1 w_{\mu\nu}^3 \, d\xi}{\int_0^1 w_{\mu\nu}^2 \, d\xi} \tag{A2.2}$$

where $$w_{\mu\nu} = \left[\mu^2(\xi) + \nu^2(\xi) \right]^{1/2} . \tag{A2.3}$$

According to Eqs. (5.40) and (5.55), the displacements $\mu(\xi)$ and $\nu(\xi)$ are also functions of α_o. It follows that Eq. (A2.2) has to be solved for α_o by trial and error.

In what follows, a procedure is developed to find an initial value for α_o. For simplicity, a shaft with constant tension and simply supported will be used in the analysis.

Assume the following eccentricity and lack of straightness functions:

$$a(z) = e \sin \frac{p\pi z}{L} \tag{A2.4a}$$

$$b(z) = U_o(z) = V_o(z) = 0 . \tag{A2.4 b,c,d}$$

The characteristic parameters of this problem, given by Eqs. (5.57)-(5.60) together with (A2.4), are

$$a_{ro}^* = \int_0^1 \sin p\pi\xi \quad \sin r\pi\xi \, d\xi \begin{cases} = 1/2 , & \text{for } r = p \\ = 0 , & \text{for } r \neq p, \end{cases}$$

$$b^*_{n0} = c^*_{n0} = d^*_{n0} = 0. \qquad \text{(A2.5 a,b,c,d)}$$

Substituting Eqs. (A2.5) into (5.55) yields

$$\mu(\xi) = \frac{R_p^2 - R_m^2}{(R_p^2 - R_m^2)^2 + \alpha_0^2} \sin p\pi\xi , \qquad \text{(A2.6a)}$$

$$\nu(\xi) = -\frac{\alpha_0}{(R_p^2 - R_m^2)^2 + \alpha_0^2} \sin p\pi\xi . \qquad \text{(A2.6b)}$$

Further substitution of (A2.6) into (A2.3) leads to

$$w_{\mu\nu} = \frac{|\sin p\pi\xi|}{[(R_p^2 - R_m^2)^2 + \alpha_0^2]^{1/2}} , \qquad \text{(A2.7)}$$

where the absolute value of the sine function has been used for agreement with the definition of w .

The integrals involved in Eq. (A2.2) are then

$$\int_0^1 w_{\mu\nu}^3 \, d\xi = \frac{p}{[(R_p^2 - R_m^2)^2 + \alpha_0^2]^{3/2}} \int_0^{1/p} \sin^3 p\pi\xi \, d\xi$$

$$= \frac{4}{3\pi} \frac{1}{[(R_p^2 - R_m^2)^2 + \alpha_0^2]^{3/2}} , \qquad \text{(A2.8a)}$$

$$\int_0^1 w_{\mu\nu}^2 \, d\xi = \frac{1}{(R_p^2 - R_m^2)^2 + \alpha_0^2} \int_0^1 \sin^2 p\pi\xi \, d\xi$$

$$= \frac{1}{2} \frac{1}{(R_p^2 - R_m^2)^2 + \alpha_0^2} . \qquad \text{(A2.8b)}$$

Introduction of these integrals in (A2.2) gives the following equation for α_o :

$$\alpha_o = \frac{8de}{3\pi m} \frac{1}{\left[(R_p^2 - R_m^2)^2 + \alpha_o^2\right]^{1/2}} .$$

This equation can be rewritten as

$$\alpha_o^4 + (R_p^2 - R_m^2)^2 \alpha_o^2 - \left(\frac{8de}{3\pi m}\right)^2 = 0. \tag{A2.9}$$

For a circular shaft, the quadratic damping coefficient $\underline{d}$ is defined as

$$d = \frac{\rho_o D C_D}{2} \tag{A2.10}$$

where ρ_o is the mass per unit volume of the damping fluid and C_D is the drag coefficient for lateral motion of a cylinder. Since $\rho_o = \frac{4m_o}{\pi D^2}$, the last term of (A2.9) becomes

$$\frac{8de}{3\pi m} = \frac{16}{3\pi^2} C_D \frac{e}{D} (R_m^2 - 1) , \tag{A2.11}$$

where use of Eq. (5.24c) has been made.

Finally, if (A2.11) is substituted into (A2.9) the following bi-quadratic equation is obtained for α_o :

$$\alpha_o^4 + (R_p^2 - R_m^2)^2 \alpha_o^2 - \frac{256}{9\pi^4} (R_m^2 - 1)^2 C_D^2 \left(\frac{e}{D}\right)^2 = 0. \tag{A2.12}$$

For example problem No. 1, the eccentricity function is comparable to the first mode, that is $p = 1$. Using the geometric parameters defined for that problem

$$R_p = \frac{\omega_1}{\Omega} = \frac{.67516}{7.9500} = .084926 ,$$

$$R_m^2 = 1 + \frac{m_o}{m} = 1 + \frac{15.352}{40.498} = 1.37908 ,$$

$$C_D = 1.2 ,$$

$$\frac{e}{D} = 0.1 .$$

Substitution of these values in (A2.12) yields the equation

$$\alpha_o^4 + 1.882\ \alpha_o^2 - .0006055 = 0,$$

from which

$$\alpha_o = .01871 .$$

With this value, the dimensional damping coefficient α becomes

$$\alpha = \alpha_o \Omega = .01871 \times 7.9500 = .1487 \text{ sec.}^{-1}$$

Thus, the starting value $\alpha = .15$ should be used in the first problem.

Appendix 3

NON-DIMENSIONAL FORM OF DISPLACEMENTS AND BENDING MOMENTS

The dimensional form of displacements and bending moments were shown to be expressed by

$$u(z) = \sum_{n=1}^{\infty} U_n \phi_n(z), \qquad v(z) = \sum_{n=1}^{\infty} V_n \phi_n(z),$$

$$M_u(z) = EI \sum_{n=1}^{\infty} U_n \frac{d^2\phi_n(z)}{dz^2} - EI\frac{d^2 u_0(z)}{dz^2}, \quad M_v(z) = EI \sum_{n=1}^{\infty} V_n \frac{d^2\phi_n(z)}{dz^2} - EI\frac{d^2 v_0(z)}{dz^2}.$$

Major changes occur in U_n and V_n when these expressions are transformed to a non-dimensional form. But V_n can be easily obtained from U_n. For this reason only $\mu(\xi)$, non-dimensional form of $u(z)$, is derived here. The other three variables, $\nu(\xi)$, $M_\mu(\xi)$ and $M_\nu(\xi)$, can be written by inspection.

A3.1 Linearly-Varying Tension

Eq. (5.24a), repeated here for convenience, is

$$u(z) = \sum_{n=1}^{\infty} \frac{(\omega_n^2 - R_m^2\Omega^2)(\omega_n^2 c_n + \Omega^2 a_n) + \alpha\Omega(\omega_n^2 d_n + \Omega^2 b_n)}{(\omega_n^2 - R_m^2\Omega^2)^2 + \alpha^2\Omega^2}\, \phi_n(z). \tag{A3.1}$$

Dividing the numerator and denominator of the fraction by Ω^4 , yields

$$u(z)=\sum_{n=1}^{\infty}\frac{(R_n^2-R_m^2)(R_n^2c_n+a_n)+\alpha_o(R_n^2d_n+b_n)}{(R_n^2-R_m^2)^2+\alpha_o^2}\phi_n(z)\ , \tag{A3.2}$$

where $R_n=\frac{\omega_n}{\Omega}$ and $\alpha_o=\frac{\alpha}{\Omega}$, Eqs. (5.28) and (5.29), respectively. Dividing now both sides of (A3.2) by the arbitrary unit of eccentricity $\underline{e}$ and introducing ξ , the non-dimensional displacement $\mu(\xi)$ is

$$\mu(\xi)=\sum_{n=1}^{\infty}\frac{(R_n^2-R_m^2)(R_n^2c_{no}+a_{no})+\alpha_o(R_n^2d_{no}+b_{no})}{(R_n^2-R_m^2)^2+\alpha_o^2}\phi_n(\xi) \tag{A3.3}$$

in which $\xi=z/L$, $\mu(\xi)=u(\xi)/e$, Eqs. (5.6) and (5.30), respectively. Since $a_{no}=\frac{a_n}{e}$, from (4.25) one obtains

$$a_{no}=\frac{1}{H_n}\int_0^L\frac{a(z)}{e}\phi_n(z)\,dz=\frac{1}{h_n}\int_0^1\frac{a(\xi)}{e}\phi_n(\xi)\,d\xi \tag{A3.4}$$

where $h_n=\frac{H_n}{L}=\int_0^1\phi_n^2(\xi)\,d\xi.$ (A3.5)

Similarly,

$$b_{no}=\frac{1}{h_n}\int_0^1\frac{b(\xi)}{e}\phi_n(\xi)\,d\xi\,. \tag{A3.6}$$

In the same way, Eq. (4.27) must be used to express c_{no} .

$$c_{ro} = \frac{c_r}{e} = \frac{EI}{m\omega_r^2 H_r} \int_0^L \frac{d^4[u_o(z)/e]}{dz^4} \phi_r(z)\, dz \quad \text{, or}$$

$$c_{ro} = \frac{EI}{m\omega_r^2 L^4} \frac{1}{h_r} \int_0^1 \frac{d^4\mu_o(\xi)}{d\xi^4} \phi_r(\xi)\, d\xi \tag{A3.7}$$

in which Eqs. (A3.5), (5.6) and (5.30) have been used. Similarly,

$$d_{ro} = \frac{EI}{m\omega_r^2 L^4} \frac{1}{h_r} \int_0^1 \frac{d^4\nu_o(\xi)}{d\xi^4} \phi_r(\xi)\, d\xi. \tag{A3.8}$$

Comparison of the expressions derived here with those in Chapter 5, permits one to observe the agreement of the following pairs of equations: (A3.3), (5.40a); (A3.4), (5.43); (A3.6), (5.44); (A3.7), (5.45); (A3.8), (5.46) and (A3.5), (5.47). This establishes Eq. (5.40a) for the non-dimensional displacement $\mu(\xi)$.

A3.2 Constant Tension

The dimensional form of $u(z)$ is given by Eq. (5.52a), repeated here for convenience.

$$u(z) = \sum_{r=1}^{\infty} \frac{(\omega_r^2 - R_m^2\Omega^2)(\omega_r^2 c_r + \Omega^2 a_r) + \alpha\Omega(\omega_r^2 d_r + \Omega^2 b_r)}{(\omega_r^2 - R_m^2\Omega^2)^2 + \alpha^2\Omega^2} A_r \sin\frac{r\pi z}{L} \tag{A3.9}$$

The expression for $\mu(\xi)$ is obtained by dividing the numerator and denominator of the fraction in (A3.9) by Ω^4 and then both sides of (A3.9) by $\underline{e}$. This yields

$$\mu(z) = \sum_{r=1}^{\infty} \frac{(R_r^2 - R_m^2)(R_r^2 c_{ro} + a_{ro}) + \alpha_o (R_r^2 d_{ro} + b_{ro})}{(R_r^2 - R_m^2)^2 + \alpha_o^2} A_r \sin \frac{r\pi z}{L} \tag{A3.10}$$

in which use has been made of Eqs. (5.28) and (5.29), and

$$a_{ro} = \frac{a_r}{e}, \quad b_{ro} = \frac{b_r}{e}, \quad c_{ro} = \frac{c_r}{e}, \quad d_{ro} = \frac{d_r}{e}. \tag{A3.11 a,b,c,d}$$

As the modal shapes are sine functions, for this case, the expression for $\mu(z)$ can be modified. Using Eq. (4.25) a_{ro} can be written as

$$a_{ro} = \frac{1}{H_r} \int_0^L \frac{a(z)}{e} A_r \sin \frac{r\pi z}{L} \, dz.$$

But

$$H_r = L h_r = L \int_0^1 A_r^2 \sin^2 r\pi\xi \, d\xi = \frac{L}{2} A_r^2, \tag{A3.12}$$

which yields

$$a_{ro} = \frac{2}{L A_r^2} L \int_0^1 \frac{a(\xi)}{e} A_r \sin r\pi\xi \, d\xi,$$

or

$$a_{ro} = \frac{2}{A_r} a_{ro}^* \tag{A3.13}$$

where

$$a^*_{ro} = \int_0^1 \frac{a(\xi)}{e} \sin r\pi\xi \, d\xi \, . \tag{A3.14}$$

Similarly,

$$b_{ro} = \frac{2}{A_r} b^*_{ro} \tag{A3.15}$$

where

$$b^*_{ro} = \int_0^1 \frac{b(\xi)}{e} \sin r\pi\xi \, d\xi \, . \tag{A3.16}$$

Eqs. (5.28), (A3.11c) and (4.27) will now be used to evaluate $R_r^2 c_{ro}$, as follows:

$$R_r^2 c_{ro} = \frac{\omega_r^2}{\Omega^2} \frac{EI}{m\omega_r^2 H_r} \int_0^L \frac{d^4[v_o(z)/e]}{dz^4} A_r \sin\frac{r\pi z}{L} \, dz$$

$$= \frac{2EI}{m\Omega^2 L A_r} \frac{L}{L^4} \int_0^1 \frac{d^4\mu_o(\xi)}{d\xi^4} A_r \sin\frac{r\pi\xi}{L} \, d\xi \, ,$$

where Eqs. (A3.12), (5.6) and (5.32) were used in this derivation. Integrating by parts 4 times and simplifying yields

$$R_r^2 c_{ro} = \frac{2}{A_r} \frac{1}{\Omega^2} \frac{EI r^4 \pi^4}{m L^4} \int_0^1 \mu_o(\xi) \sin r\pi\xi \, d\xi \, , \tag{A3.17}$$

since the function $\mu_o(\xi)$ must satisfy the boundary

conditions of the problem. The third fraction in (A3.17) is the square of the natural frequency for a shaft with zero tension, i.e.

$$\omega_{o,r}^2 = \frac{EI}{m} \frac{r^4 \pi^4}{L^4},$$

which can be obtained from (5.51). If the notation of (5.54) is also introduced into (A3.17), one finally obtains

$$R_r^2 c_{ro} = \frac{2}{A_r} R_{o,r}^2 c_{ro}^*, \tag{A3.18}$$

where
$$c_{ro}^* = \int_0^1 \mu_o(\xi) \sin r\pi\xi \, d\xi. \tag{A3.19}$$

Similarly,

$$R_r^2 d_{ro} = \frac{2}{A_r} R_{o,r}^2 d_{ro}^*, \tag{A3.20}$$

$$d_{ro}^* = \int_0^1 \nu_o(\xi) \sin r\pi\xi \, d\xi. \tag{A3.21}$$

Substituting Eqs. (A3.13), (A3.15), (A3.18) and (A3.20) into (A3.10) and using $\xi = z/L$, the A_r simplifies and the constant 2 can be moved in front of the summation sign. The result is

$$\mu(\xi) = 2\sum_{n=1}^{\infty} \frac{(R_n^2 - R_m^2)(R_{o,n}^2 c_{no}^* + a_{no}^*) + \alpha_o(R_{o,n}^2 d_{no}^* + b_{no}^*)}{(R_n^2 - R_m^2)^2 + \alpha_o^2} \sin n\pi\xi \, . \tag{A3.22}$$

This last equation is the same as (5.55a). The agreement can be observed by comparison of the following pairs of equations: (A3.14), (5.57); (A3.16), (5.58); (A3.19), (5.59) and (A3.21), (5.60). Eq. (5.55a) is thus established.

Appendix 4

LISTING OF THE COMPUTER PROGRAMS

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C     ECCENTRIC ROTATING SHAFT SUBJECT TO LINEARLY VARYING TENSION.     MAIN  50
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C                KCASE=1,COMPUTES DISPLACEMENTS AND STRESSES FOR SEVERALMAIN 160
C                        VALUES OF AFO, AT A SPECIFIC SPEED.            MAIN 170
C                KCASE=2,COMPUTES MAXIMUM DISPLACEMENTS AND STRESSES ON MAIN 180
C                        THE SHAFT FOR SEVERAL AFO AT SEVERAL SPEEDS.   MAIN 190
C         KPART -INDICATES PART OF THE RESPONSE COMPUTED.               MAIN 200
C                KPART=1, ECCENTRICITY OF THE SHAFT IS CONSIDERED ONLY. MAIN 210
C                KPART=2,ONLY LACK OF STRAIGHTNESS IS CONSIDERED.       MAIN 220
C                KPART=3, BOTH CASES ABOVE ARE CONSIDERED.              MAIN 230
C         RL    -LENGHT OF THE SHAFT                                    MAIN 240
C         GM    -WEIGHT PER UNIT LENGHT OF THE SHAFT, IN FLUID MEDIUM.  MAIN 250
C         T     -TENSION AT THE BOTTOM OF SHAFT                         MAIN 260
C         RI    -INERTIA MOMENT OF CROSS SECTION AREA ABOUT A DIAMETER  MAIN 270
C         RM    -MASS PER UNIT LENGHT PARTICIPATING IN MOTION           MAIN 280
C         OMG   -SPEED OF ROTATION OF SHAFT                             MAIN 290
C         DO    -OUTSIDE DIAMETER OF SHAFT                              MAIN 300
C         DI    -INSIDE DIAMETER OF SHAFT                               MAIN 310
C         E     -YOUNG'S MODULUS                                        MAIN 320
C         ECC   -ARBITRARY UNIT OF ECCENTRICITY.                        MAIN 330
C         CSI   -NONDIMENSION DISTANCE OF A SECTION.  MATRIX.           MAIN 340
C         RIUZ  -MATRIX, CONTAINS THE BENDING MOMENTS ON UOZ PLANE, DUE MAIN 350
C                TO LACK OF STRAIGHTNESS.  SUBTRACTED FROM FINAL MOMENTSMAIN 360
C         RIVZ  -SAME AS ABOVE, ON VOZ PLANE                            MAIN 370
C         CU    -THREE-DIMENSION MATRIX.  DISPLACEMENTS ON UZ PLANE.    MAIN 380
C                FIRST DIMENSION POINTS DISTANCE ALONG SHAFT            MAIN 390
C                SECOND DIMENSION INDICATES AMOUNT OF DAMPING.          MAIN 400
C                THIRD DIMENSION SPECIFIES THE SPEED OMG.               MAIN 410
C         CV    -SAME AS ABOVE, ON VZ PLANE.                            MAIN 420
C         CRU   -THREE-DIMENSION MATRIX.  BENDING MOMENTS ON UZ PLANE.  MAIN 430
C                SAME DIMENSIONS AS CU.                                 MAIN 440
C         CRV   -SAME AS ABOVE, ON VZ PLANE.                            MAIN 450
C         N     -NUMBER OF TERMS USED IN THE SERIES SOLUTION OF THE BEAMMAIN 460
C         OMGR  -RTH NATURAL FREQUENCY.                                 MAIN 470
C         QR    -PARAMETER PROPORTIONAL TO OMGR                         MAIN 480
C         HR    -NORMALIZING CONSTANT.  INTEGRAL OF SQUARE OF THE MODAL MAIN 490
C                SHAPE                                                  MAIN 500
C         AR    -RTH WEIGHTING PARAMETER,DUE TO ECCENTRICITY ON UZ PLANEMAIN 510
C         BR    -SAME AS ABOVE FOR VZ PLANE.                            MAIN 520
C         CR    -RTH WEIGHTING PARAMETER, DUE TO LACK OF STRAIGHTNESS ONMAIN 530
C                UZ PLANE.                                              MAIN 540
C         DR    -SAME AS ABOVE FOR VZ PLANE.                            MAIN 550
C         AFO   -AMOUNT OF DAMPING.  MATRIX.                            MAIN 560
C         TM    -MATRIX.  TOTAL BENDING MOMENT AT A SECTION.            MAIN 570
C         SIGONE-MATRIX. MAXIMUM STRESS AT A SECTION DUE TO TM.         MAIN 580
C         SIGTWO-MATRIX. STRESS AT A SECTION DUE TO THE TENSION         MAIN 590
C         SIGMAZ-MATRIX. MAXIMUM STRESS AT A SECTION.  IT IS EQUAL TO   MAIN 600
C                SIGONE + SIGTWO.                                       MAIN 610
C         PHI   -MATRIX, CONTAINS THE MODAL SHAPE OF THE BEAM           MAIN 620
C         PSIKZ -MATRIX, CONTAINS THE MODAL MOMENT OF THE BEAM          MAIN 630
C         YREL  -MATRIX. NONDIMENSIONAL MODAL SHAPE.  RATIO OF ACTUAL   MAIN 640
C                AND MAXIMUM DISPLACEMENTS                              MAIN 650
C         PSIREL-MATRIX.  NONDIMENSIONAL MODAL MOMENT. RATIO OF ACTUAL  MAIN 660
C                AND MAXIMUM BENDING MOMENT                             MAIN 670
C         OMGTAU-THREE DIMENSION MATRIX. TABLE OF RESULTS. MAXIMUM DISPLMAIN 680
C                AND STRESSES FOR SEVERAL SPEEDS. THIRD DIMENSION POINTSMAIN 690
C                AMOUNT OF DAMPING.                                     MAIN 700
C                                                                       MAIN 710
C     DESCRIPTIONS OF PARAMETERS IN SUBROUTINES OR FUNCTIONS ARE THE    MAIN 720
C     SAME AS IN MAIN PROGRAM                                           MAIN 730
C                                                                       MAIN 740
C.......................................................................MAIN 750
C                                                                       MAIN 760
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)                               MAIN 770
      DIMENSION A(150),B(150),RA(150),RB(150),RIUZ(51),RIVZ(51)         MAIN 780
      DIMENSION PHI(51), PSIKZ(51),YREL(51),PSIREL(51),AFO(8),CSI(51)   MAIN 790
      REAL CA(51,9),CB(51,9),CC(51,9),CF(30,6),CG(30,6)                 MAIN 800
C                                                                       MAIN 810
      COMMON CU(51,5,15),CV(51,5,15),CRU(51,5,15)-CRV(51,5,15),TM(51),  MAIN 820
     1TOPL(51),SIGONE(51),SIGTWO(51),SIGMAZ(51),OMGTAU(15,5,8)          MAIN 830
C                                                                       MAIN 840
      KEX=0                                                             MAIN 850
    5 READ (5,8) KZ,KA,KS,   KMODE,KCASE,KPART,INS,IDS                  MAIN 860
    8 FORMAT(8I5)                                                       MAIN 870
      WRITE (6,9) KZ,KA,KS,   KMODE,KCASE,KPART,INS,IDS                 MAIN 880
    9 FORMAT('1KZ    =',I4,//,1X,'KA    =',I4,//,1X,'KS    =',I4,//,1X, MAIN 890
     1              'KMODE=',I4,//,1X,'KCASE=',I4,//,1X,'KPART=',I4,//, MAIN 900
     21X,'INS  =',I4,//,1X,'IDS  =',I4)                                 MAIN 910
      READ (5,10) GM,RL,T,RI,RM,DO,DI,E,JSTOP,CMG,RMO                   MAIN 920
   10 FORMAT(7D10.4,/,D)5.4,I5,2D10.4)                                  MAIN 930
      KEX=KEX+1                                                         MAIN 940
      ECC=DO/10                                                         MAIN 950
      WRITE (6,20) RL,T,RI,RM,CMG,DO,DI,E,ECC,JSTOP,RMO,GM              MAIN 960
   20 FORMAT(////,1X,'RL=',D26.18,//,1X,'T =',D26.18,//,1X,'RI=',D26.18,MAIN 970
     1//,1X,'RM=',D26.18,//,1X,'OMG=',D25.18,//,1X,'DO=',D26.18,//,1X,'DMAIN 980
     21=',D26.18,//,1X,'E =',D26.18,//,1X,'ECC=',D25.18,//,1X,'JSTOP=',IMAIN 990
     33,//,1X,'RHO=',D25.18,//,1X,'GM=',D26.18)                          MAIN1000
      IF(JSTOP.EQ.0) STOP                                               MAIN1010
C                                                                       MAIN1020
      PI=3.14159265358979324                                            MAIN1030
      AF=(GM*RL**3)/(E*RI)                                              MAIN1040
      BT=(T*RL**2)/(E*RI)                                               MAIN1050
      RMMS=1*RMO/RM                                                     MAIN1060
      WRITE (6,30) AF,BT,RMMS                                           MAIN1070
   30 FORMAT('0',//,1X,'AF=',D26.18,//,1X,'BT=',D26.18,//,1X,'RMMS=',D24MAIN1080
     1.18)                                                              MAIN1090
      AFO(1)=0                                                          MAIN1100
      AFO(2)=8D-2                                                       MAIN1110
      AFO(3)=16D-2                                                      MAIN1120
      AFO(4)=32D-2                                                      MAIN1130
      AFO(5)=16D-1                                                      MAIN1140
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115:     34 DO 35 I=1,KA                                                MAIN1150
116:     35 WRITE(6,36) I, AFO(I)                                       MAIN1160
117:     36 FORMAT(/ ,1X,'AFO(',I1,')=',D16.8)                          MAIN1170
118: C                                                                  MAIN1180
119: C          INITIAL MOMENTS, DUE TO LACK OF STRAIGHTNESS            MAIN1190
120: C                                                                  MAIN1200
121:        IF(KPART.EQ.1) GO TO 45                                     MAIN1210
122:        CONST=-E*RI*ECC/RL**2                                       MAIN1220
123:        WRITE (6,40)                                                MAIN1230
124:     40 FORMAT('1   Z',6X,'INITIAL MOMENTS ON UOZ PLANE',4X,'INITIAL MOMENMAIN1240
125:       1TS ON VOZ PLANE',/)                                         MAIN1250
126:     45 DO 60 I=1,KZ                                                MAIN1260
127:        IREL=(I-1)*2                                                MAIN1270
128:        CSI(I)=IREL/100                                             MAIN1280
129:        IF(KPART.EQ.1) GO TO 55                                     MAIN1290
130:        RIUZ(I)=CONST*(20*CSI(I)**3+12*CSI(I)**2-42*CSI(I)+10)      MAIN1300
131:        RIVZ(I)=CONST*(1000*CSI(I)**3-1650*CSI(I)**2+1425/2*CSI(I)-125/2) MAIN1310
132:        WRITE (6,50) IREL,RIUZ(I),RIVZ(I)                           MAIN1320
133:     50 FORMAT(I5,2D32.18)                                          MAIN1330
134:        GO TO 60                                                    MAIN1340
135:     55 RIUZ(I)=0                                                   MAIN1350
136:        RIVZ(I)=0                                                   MAIN1360
137:     60 CONTINUE                                                    MAIN1370
138: C                                                                  MAIN1380
139: C          INITIALIZATION OF DISPLACEMENTS CU,CV, AND MOMENTS CRU,CRV MAIN1390
140: C                                                                  MAIN1400
141:        DO 78 K=1,KS                                                MAIN1410
142:        DO 76 J=1,KA                                                MAIN1420
143:        DO 74 I=1,KZ                                                MAIN1430
144:        CU(I,J,K)=0                                                 MAIN1440
145:        CV(I,J,K)=0                                                 MAIN1450
146:        CRU(I,J,K)=RIUZ(I)                                          MAIN1460
147:     74 CRV(I,J,K)=RIVZ(I)                                          MAIN1470
148:     76 CONTINUE                                                    MAIN1480
149:     78 CONTINUE                                                    MAIN1490
150: C                                                                  MAIN1500
151: C          TERMS IN THE SERIES                                     MAIN1510
152: C                                                                  MAIN1520
153:        DO 300 L=1,KMODE                                            MAIN1530
154:        READ (5,90) N,OR                                            MAIN1540
155:     90 FORMAT(I5,D25.10)                                           MAIN1550
156:        OMGR=OR/(RL**2)*DSQRT(E/RM*RI)                              MAIN1560
157:        OMGN=OMGR/PI*30                                             MAIN1570
158:        WRITE(6,92) L,N,OR,OMGR,OMGN                                MAIN1580
159:     92 FORMAT(//,1X,'MODE NO.=',I2,4X,'N=',I3,4X,'OR=',D26.18,4X,'OMGR=',MAIN1590
160:       1D12.6,4X,'OMGN=',D12.6)                                     MAIN1600
161: C                                                                  MAIN1610
162: C          CALL FOR SUBROUTINES AND FUNCTIONS                      MAIN1620
163: C                                                                  MAIN1630
164:        CALL COEFFC(N,AF,BT,OR,AO,A,BO,B,RAO,RA,RBO,RB,JMAX)        MAIN1640
165:        CALL FREQUTIN,AO,A,BO,B,RAO,RA,RBO,RB,SGASGB,OR)            MAIN1650
166:        CALL MOOSHP(N,AO,A,BO,B,SGASGB,RL,L,PHI,YO,YREL,KZ,CSI)     MAIN1660
167:        CALL MOOMOM(N,RAO,KA,RBO,RB,SGASGB,RL,L,PSIKZ,YO,PSIREL,KZ,CSI) MAIN1670
168:        HR=HFCTR(RL,PHI,KZ)                                         MAIN1680
169:        IF(KCASE.EQ.2)  GO TO 81                                    MAIN1690
170:        WRITE(6,880)                                                MAIN1700
171:    880 FORMAT(//,1X,'  CSI   ',11X,'PHI',15X,'PHIREL',12X,'PSIKZ',12X,'PSIMAIN1710
172:       1REL',/)                                                     MAIN1720
173:        DO 885 I=1,KZ                                               MAIN1730
174:    885 WRITE(6,890) CSI(I),PHI(I),  YREL(I),PSIKZ(I),PSIREL(I)     MAIN1740
175:    890 FORMAT(D10.4,4D18.8)                                        MAIN1750
176:     81 GO TO (82,84,86), KPART                                     MAIN1760
177:     82 AR=AFCTR(N,AO,A,BO,B,SGASGB,ECC,RL,HR)                      MAIN1770
178:        BR=0                                                        MAIN1780
179:        CR=0                                                        MAIN1790
180:        DR=0                                                        MAIN1800
181:        GO TO 100                                                   MAIN1810
182:     84 AR=0                                                        MAIN1820
183:        BR=0                                                        MAIN1830
184:        CR=CFCTR(N,AO,A,BO,B,SGASGB,RL,ECC,E,RI,RM,OMGR,HR)         MAIN1840
185:        DR=DFCTR(N,AO,A,BO,B,SGASGB,RL,ECC,E,RI,RM,OMGR,HR)         MAIN1850
186:        GO TO 100                                                   MAIN1860
187:     86 AR=AFCTR(N,AO,A,BO,B,SGASGB,ECC,RL,HR)                      MAIN1870
188:        BR=0                                                        MAIN1880
189:        CR=CFCTR(N,AO,A,BO,B,SGASGB,RL,ECC,E,RI,RM,OMGR,HR)         MAIN1890
190:        DR=DFCTR(N,AO,A,BO,B,SGASGB,RL,ECC,E,RI,RM,OMGR,HR)         MAIN1900
191: C                                                                  MAIN1910
192: C          CUMULATIVE DISPLACEMENTS CU,CV AND MOMENTS CRU,CRV      MAIN1920
193: C                                                                  MAIN1930
194:    100 DO 190 K=1,KS                                               MAIN1940
195:        IF(KS.EQ.1) GO TO 107                                       MAIN1950
196:        OMG=(INS+K)/IDS                                             MAIN1960
197:    107 DO 180 J=1,KA                                               MAIN1970
198:        PSS=((OMGR**2-RMMS*OMG**2)*(OMGR**2*CR+OMG**2*AR)+AFO(J)*OMG*(OMGRMAIN1980
199:       1**2*DR+OMG**2*BR))/((OMGR**2-RMMS*OMG**2)**2+AFO(J)**2*OMG**2)  MAIN1990
200:        QSS=((OMGR**2-RMMS*OMG**2)*(OMGR**2*DR+OMG**2*BR)-AFO(J)*OMG*(OMGRMAIN2000
201:       1**2*CR+OMG**2*AR))/((OMGR**2-RMMS*OMG**2)**2+AFO(J)**2*OMG**2)  MAIN2010
202:        IF(KS.NE.1) GO TO 110                                       MAIN2020
203:        WRITE(6,109) J,AR,BR,CR,DR,PSS,QSS                          MAIN2030
204:    109 FORMAT(12X,'J=',I1,2X,'AR=',D11.5,2X,'BR=',D11.5,2X,'CR=',D11.5,2XMAIN2040
205:       1,'DR=',D11.5,2X,'PSS=',D11.5,2X,'QSS=',D11.5)               MAIN2050
206:    110 DO 140 I=1,KZ                                               MAIN2060
207:        CU(I,J,K)=CU(I,J,K)+PSS*PHI(I)                              MAIN2070
208:        CV(I,J,K)=CV(I,J,K)+QSS*PHI(I)                              MAIN2080
209:        CRU(I,J,K)=CRU(I,J,K)+E*RI*PSS*PSIKZ(I)                     MAIN2090
210:        CRV(I,J,K)=CRV(I,J,K)+E*RI*QSS*PSIKZ(I)                     MAIN2100
211:    140 CONTINUE                                                    MAIN2110
212:        IF(L.LT.KMODE) GO TO 180                                    MAIN2120
213: C                                                                  MAIN2130
214:        CALL STRESS(RL,J,K,DO,DI,T,GM,AFO,OMG,PI,ECC,KZ,KCASE,CF,CG,KEX, MAIN2140
215:       1KS,CSI,RI,E)                                                MAIN2150
216: C                                                                  MAIN2160
217:        GO TO (151,180), KCASE                                      MAIN2170
218:    151 WRITE(6,152) J,AFO(J)                                       MAIN2180
219:    152 FORMAT(//  ,1X,'AFO(',I1,')=',D10.4)                        MAIN2190
220:        WRITE (6,154)                                               MAIN2200
221:        IREL=(I-1)*2                                                MAIN2210
222:    154 FORMAT(//,3X,'Z',6X,'CU(Z)',11X,'CV(Z)',11X,'CRU(Z)',10X,'CRV(Z)',MAIN2220
223:       17X,'BENDING MOMENT',2X,'BENDING STRESS',2X,'TOTAL STRESS',/)  MAIN2230
224:        DO 158 I=1,KZ                                               MAIN2240
225:        WRITE(6,156)IREL,CU(I,J,K  ),CV(I,J,K  ),CRU(I,J,K  ),CRV(I,J,K)  MAIN2250
226:       1,TM(I),SIGONE(I),SIGMAZ(I)                                  MAIN2260
227:    156 FORMAT(I4,7D16.8)                                           MAIN2270
228:    158 CONTINUE                                                    MAIN2280
229:        JP1=J+1                                                     MAIN2290
230:        DO 160 I=1,KZ                                               MAIN2300
231:        CA(I,1)=CSI(I)                                              MAIN2310
232:        CB(I,1)=CSI(I)                                              MAIN2320
233:        CC(I,1)=CSI(I)                                              MAIN2330
234:        IF(DABS(CU(I,J,K)).LT.1D-36)  CU(I,J,K)=0                   MAIN2340
235:        IF(DABS(CV(I,J,K)).LT.1D-36)  CV(I,J,K)=0                   MAIN2350
236:        IF(SIGONE(I).LT.1D-36)  SIGONE(I)=0                         MAIN2360
237:        CA(I,JP1)=CU(I,J,K)/ECC                                     MAIN2370
238:        CB(I,JP1)=CV(I,J,K)/ECC                                     MAIN2380
239:    160 CC(I,JP1)=SIGONE(I)/(E/RL**2*RI/DO**3*ECC)                  MAIN2390
240: C                                                                  MAIN2400
241:        CALL ALFAOT(KZ,TOPL,RM,PI,AFO,ECC,DO,J,RMO,OMG)             MAIN2410
242: C                                                                  MAIN2420
243:    180 CONTINUE                                                    MAIN2430
244: C                                                                  MAIN2440
245:    190 CONTINUE                                                    MAIN2450
246: C                                                                  MAIN2460
247:    300 CONTINUE                                                    MAIN2470
248:        GO TO (310,320), KCASE                                      MAIN2480
249:    310 KACOL=KA+1                                                  MAIN2490
250:        CALL PLOT (10,CA,KZ,KACOL,101,0)                            MAIN2500
251:        CALL PLOT (20,CB,KZ,KACOL,101,0)                            MAIN2510
252:        CALL PLOT (30,CC,KZ,KACOL,101,0)                            MAIN2520
253:        GO TO 5                                                     MAIN2530
254:    320 DO 328 J=1,KA                                               MAIN2540
255:        WRITE (6,322) J,AFO(J)                                      MAIN2550
256:    322 FORMAT('1  AFO(',I1,')=',D10.4)                             MAIN2560
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      WRITE (6,324)                                                     MAIN2570
  324 FORMAT(///,6X,'OMG',8X,'ZBEND',4X,'BENDMAX',6X,'ZDPL',8X,'DPLMAX',MAIN2580
     I/)                                                                MAIN2590
      DO 326 I=1,KS                                                     MAIN2600
      WRITE(6,325) OMGTAU(I,1,J),OMGTAU(I,2,J),OMGTAU(I,3,J),OMGTAU(I,4,MAIN2610
     1J),OMGTAU(I,5,J)                                                  MAIN2620
  325 FORMAT(5D12.5)                                                    MAIN2630
  326 CONTINUE                                                          MAIN2640
  328 CONTINUE                                                          MAIN2650
      IF(KEX.NE.KEX/2*2)  GO TO 5                                       MAIN2660
      KACOL=KA+1                                                        MAIN2670
      KST2=KS*2                                                         MAIN2680
      CALL PLOT (40,CF,KST2,KACOL,88,0)                                 MAIN2690
      CALL PLOT (50,CG,KST2,KACOL,88,0)                                 MAIN2700
      CALL PLOT (400,CF,KST2,KACOL,120,0)                               MAIN2710
      CALL PLOT (500,CG,KST2,KACOL,120,0)                               MAIN2720
      GO TO 5                                                           MAIN2730
      END                                                               MAIN2740
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      SUBROUTINE FREQUT(N,AO,A,BO,B,RAO,RA,RBO,RB,SGASGB,QR)            FREQ  10
C                                                                       FREQ  20
C     THIS SUBROUTINE CHECKS THE FREQUENCY EQUATION FOR THE VALUE QR    FREQ  30
C                                                                       FREQ  40
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)                               FREQ  50
      DIMENSION A(150),B(150),RA(150),RB(150)                           FREQ  60
      SUMA=AO                                                           FREQ  70
      SUMB=BO                                                           FREQ  80
      SUMODA=6*AO                                                       FREQ  90
      SUMODB=2*BO                                                       FREQ 100
      DO 310 I=1,N                                                      FREQ 110
      SUMA=SUMA+A(I)                                                    FREQ 120
      SUMB=SUMB+B(I)                                                    FREQ 130
      SUMODA=SUMODA+RA(I)                                               FREQ 140
      SUMODB=SUMODB+RB(I)                                               FREQ 150
  310 CONTINUE                                                          FREQ 160
      SGASGB=SUMA/SUMB                                                  FREQ 170
      FCTOR=SUMODA-SGASGB*SUMODB                                        FREQ 180
      RETURN                                                            FREQ 190
      END                                                               FREQ 200
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      SUBROUTINE COEFFC(N,AF,BT,Q,AO,A,BO,B,RAO,RA,RBO,RB,JMAX)         COEF  10
C                                                                       COEF  20
C     THIS SUBROUTINE COMPUTES THE COEFFICIENTS A(K) AND B(K) OF THE    COEF  30
C     SERIES SOLUTION OF A VIBRATING BEAM                               COEF  40
C     ALSO COMPUTES RA(K)=(K+3)*(K+2)*A(K) AND RB(K)=(K+2)*(K+1)*B(K)   COEF  50
C                                                                       COEF  60
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)                               COEF  70
      DIMENSION A(150),B(150),RA(150),RB(150)                           COEF  80
  210 AO=1                                                              COEF  90
      A(1)=0                                                            COEF 100
      A(2)=BT/(5*4)                                                     COEF 110
      A(3)=(3*AF)/(6*5*4)                                               COEF 120
      A(4)=(BT*A(2))/(7*6)+(4*AF*A(1))/(7*6*5)+(Q**2*AO)/(7*6*5*4)      COEF 130
      RAO=6                                                             COEF 140
      RA(1)=12*A(1)                                                     COEF 150
      RA(2)=20*A(2)                                                     COEF 160
      RA(3)=30*A(3)                                                     COEF 170
      RA(4)=42*A(4)                                                     COEF 180
      BO=1                                                              COEF 190
      B(1)=0                                                            COEF 200
      B(2)=BT/(4*3)                                                     COEF 210
      B(3)=(2*AF)/(5*4*3)                                               COEF 220
      B(4)=(BT*B(2))/(6*5)+(3*AF*B(1))/(6*5*4)+(Q**2*BO)/(6*5*4*3)      COEF 230
      RBO=2                                                             COEF 240
      RB(1)=6*B(1)                                                      COEF 250
      RB(2)=12*B(2)                                                     COEF 260
      RB(3)=20*B(3)                                                     COEF 270
      RB(4)=30*B(4)                                                     COEF 280
  225 DO 289 K=5,N                                                      COEF 290
      A(K)=A(K-2)/(K+3)*BT/(K+2)+A(K-3)/(K+3)*AF/(K+2)*K/(K+1)+A(K-4)*Q/COEF 300
     1((K+3)*(K+2))*Q/((K+1)*K)                                         COEF 310
      IF (A(K).LT..1D-24)  A(K)=0                                       COEF 320
      B(K)=B(K-2)/(K+2)*BT/(K+1)+B(K-3)/(K+2)*AF/(K+1)*(K-1)/K+B(K-4)*Q/COEF 330
     1((K+2)*(K+1))*Q/(K*(K-1))                                         COEF 340
      IF(B(K).LT..1D-24)  B(K)=0                                        COEF 350
      RA(K)=(K+3)*(K+2)*A(K)                                            COEF 360
      RB(K)=(K+2)*(K+1)*B(K)                                            COEF 370
  289 CONTINUE                                                          COEF 380
      JMAX=2                                                            COEF 390
      AMAX=A(2)                                                         COEF 400
      DO 270 KK=3,N                                                     COEF 410
      IF(AMAX-A(KK))  260,270,270                                       COEF 420
  260 AMAX=A(KK)                                                        COEF 430
      JMAX=KK                                                           COEF 440
  270 CONTINUE                                                          COEF 450
      RETURN                                                            COEF 460
      END                                                               COEF 470
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      SUBROUTINE MODSHP(N,AO,A,BO,B,SGASGB,RL,L,PHI,YO,YREL,KZ,CSI)     MODS  10
C                                                                       MODS  20
C     THIS SUBROUTINE OBTAINS THE MODAL SHAPE CORRESPONDING TO FREQ.OMGRMODS  30
C                                                                       MODS  40
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)                               MODS  50
      DIMENSION A(150),B(150),PHI( 51),YREL( 51), YABS( 51),CSI(51)     MODS  60
      DO 480 I=1,KZ                                                     MODS  70
      PHI(I)=AO*CSI(I)**3-SGASGB*BO*CSI(I)**2                           MODS  80
      DO 450 M=1,N                                                      MODS  90
      PHI(I)=PHI(I)+A(M)*CSI(I)**(M+3)-SGASGB*B(M)*CSI(I)**(M+2)        MODS 100
  450 CONTINUE                                                          MODS 110
  480 CONTINUE                                                          MODS 120
      DO 485 I=1,KZ                                                     MODS 130
  485 YABS(I)=DABS(PHI(I))                                              MODS 140
      YO=YABS(1)                                                        MODS 150
      DO 486 I=2,KZ                                                     MODS 160
  486 YO=DMAX1(YO,YABS(I))                                              MODS 170
      DO 491 I=1,KZ                                                     MODS 180
      YREL(I)=PHI(I)/YO                                                 MODS 190
  491 CONTINUE                                                          MODS 200
      RETURN                                                            MODS 210
      END                                                               MODS 220
```

```
      SUBROUTINE MOOMOM(N,RAO,RA,RBO,RO,SGASGB,RL,L,PSIKZ,YO,PSIREL,KZ,CMOOM  10
     1SI)                                                               MOOM  20
C                                                                       MOOM  30
C     THIS SUBROUTINE OBTAINS THE MODAL MOMENT CORRESPONDING TO FR. ONGRMOOM  40
C                                                                       MOOM  50
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)                               MOOM  60
      DIMENSION RA(150),RB(150),PSIKZ( 51),PSIREL( 51),CSI(51)          MOOM  70
      PSIKZ(1)= -SGASGB*RBO/RL**2                                       MOOM  80
      PSIREL(1)=PSIKZ(1)*RL**2/YO                                       MOOM  90
      DO 580 I=2,KZ                                                     MOOM 100
      PSIKZ(I)=RAO*CSI(I)-SGASGB*RBO                                    MOOM 110
      DO 550 M=1,N                                                      MOOM 120
      PSIKZ(I)=PSIKZ(I)+RA(M)*CSI(I)**(M+1)-SGASGB*RB(M)*CSI(I)**M      MOOM 130
  550 CONTINUE                                                          MOOM 140
  560 PSIKZ(I)=PSIKZ(I)/RL**2                                           MOOM 150
      PSIREL(I)=PSIKZ(I)*RL**2/YO                                       MOOM 160
  580 CONTINUE                                                          MOOM 170
      RETURN                                                            MOOM 180
      END                                                               MOOM 190
```

```
      DOUBLE PRECISION FUNCTION AFCTR(N,AO,A,BO,B,SGASGB,ECC,RL,HR)
C
C     AFCTR IS A WEIGHTING FUNCTION, DUE TO THE ECCENTRICITY ON PLANE
C     UOZ.
C
C     FIRST EXAMPLE
C
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DIMENSION A(150),B(150)
      AFCTR=(3**5-2**6+1)/(5*4*4**4)*AO-SGASGB*(3**4-2**5+1)/(4*3*4**3)*
     180
      DO 710 KN=1,N
      AFCTR=AFCTR+(3**(KN+5)-2**(KN+6)+1)/((KN+5)*(KN+4)*4**(KN+4))*A(KN
     1)-SGASGB*(3**(KN+4)-2**(KN+5)+1)/((KN+4)*(KN+3)*4**(KN+3))*B(KN)
  710 CONTINUE
      AFCTR=AFCTR*ECC*RL/HR
      WRITE (6,720)  AFCTR
  720 FORMAT('0  AFCTR=',D26.18)
      RETURN
      END
```

```
      DOUBLE PRECISION FUNCTION CFCTR(N,AO,A,BO,B,SGASGB,RL,ECC,E,RI,RM,
     1OMGR,HR)
C     CFCTR IS A WEIGHTING FUNCTION, DUE TO THE LACK OF STRAIGHTENESS ON
C     PLANE UOZ.
C
C     THIRD EXAMPLE
C
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DIMENSION A(150), B(150)
      CFCTR=25/20*AO-SGASGB*19/12*BO
      DO 910 KN=1,N
      CFCTR=CFCTR+(6*KN+25)/((KN+5)*(KN+4))*A(KN)-SGASGB*(6*KN+19)/((KN+
     14)*(KN+3))*B(KN)
  910 CONTINUE
      CFCTR=CFCTR*24/RM*E/RL**3*RI/OMGR**2*ECC/HR
      WRITE (6,920)   CFCTR
  920 FORMAT('0  CFCTR=',D26.18)
      RETURN
      END
```

```
      DOUBLE PRECISION FUNCTION DFCTR(N,AO,A,BO,B,SGASGB,RL,ECC,E,RI,RM,
     1OMGR,HR)
C
C     DFCTR IS A WEIGHTING FUNCTION DUE TO THE LACK OF STRAIGHTNESS ON
C     PLANE VOZ.
C
C     THIRD EXAMPLE
C
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DIMENSION A(150), B(150)
      DFCTR=75/20*AO-SGASGB*48/12*BO
      DO 1010 KN=1,N
      DFCTR=DFCTR+(27*KN+75)/((KN+5)*(KN+4))*A(KN)-SGASGB*(27*KN+48)/((K
     1N+4)*(KN+3))*B(KN)
 1010 CONTINUE
      DFCTR=DFCTR*100/RM*E/RL**3*RI/OMGR**2*ECC/HR
      WRITE (6,1020)  DFCTR
 1020 FORMAT('0  DFCTR=',D26.18)
      RETURN
      END
```

```
      DOUBLE PRECISION FUNCTION HFCTR (RL,PHI,KZ)
C
C     THIS FUNCTION COMPUTES THE INTEGRAL(0,L) OF THE SQUARE OF MODAL
C     SHAPE.   HFCTR=INT/(PHI(Z))**2*DZ/.
C
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DIMENSION                 PHI( 51), PHISQR( 51)
      DO 605 I=1,KZ
  605 PHISQR(I)=(PHI(I))**2
      SUMA=19*(PHISQR(1)+PHISQR(KZ))
      SUMB=0
      KZM5=KZ-5
      DO 630 KI=6,KZM5,5
  630 SUMB=SUMB+38*PHISQR(KI)
      SUMC=0
      KZM3=KZ-3
      DO 635 KJ=3,KZM3,5
      KK=KJ+1
  635 SUMC=SUMC+50*(PHISQR(KJ)+PHISQR(KK))
      SUMD=0
      KZM4=KZ-4
      DO 640 KL=2,KZM4,5
      KM=KL+3
  640 SUMD=SUMD+75*(PHISQR(KL)+PHISQR(KM))
      HFCTR=RL/2880*(SUMA+SUMB+SUMC+SUMD)
      WRITE (6,650)  HFCTR
  650 FORMAT(//,1X,'HFCTR=',D26.18)
      RETURN
      END
```

```
      SUBROUTINE ALFAOT(KZ,TDPL,RM,PI,AFO,ECC,DO,J,RMO,OMG)
C
C     THIS SUBROUTINE CHECKS THE ASSUMED VALUE ASSIGNED TO THE DAMPING
C     COEFFICIENT AFO
C
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DIMENSION TDPL( 51), FCT( 51), W(3),AFO(8)
C
      DO 660 M=2,3
      DO 610 I=1,KZ
  610 FCT(I)=(TDPL(I))**M
  620 SUMA=19*(FCT(1)+FCT(KZ))
      SUMB=0
      KZM5=KZ-5
      DO 630 I=6,KZM5,5
  630 SUMB=SUMB+38*FCT(I)
      SUMC=0
      KZM3=KZ-3
      DO 635 I=3,KZM3,5
      IP1=I+1
  635 SUMC=SUMC+50*(FCT(I)+FCT(IP1))
      SUMD=0
      KZM4=KZ-4
      DO 640 K=2,KZM4,5
      KP3=K+3
  640 SUMD=SUMD+75*(FCT(K)+FCT(KP3))
      W(M)=SUMA+SUMB+SUMC+SUMD
  660 CONTINUE
C
C     NEW AFO FOR SHAFT ROTATING IN SEA-WATER
C     ECC/DO=.1 AND CO=1.2
C
      RAFO=2/PI*RMO/RM*1.2/DO*OMG/W(2)*W(3)
      WRITE (6,670) AFO(J), RAFO
  670 FORMAT(///,1X,'AFO(',D10.4,')=',D10.4)
      RETURN
      END
```

```
      SUBROUTINE STRESS(RL,J,K,DO,DI,T,GM,AFO,OMG,PI,ECC,KZ,KCASE,CF,
     1CG,KEX,KS,CSI,RI,E)
C
C     THIS SUBROUTINE COMPUTES THE STRESSES (MAXIMUM AT A SECTION) ALONG
C     THE SHAFT. ALSO POINTS THE MAXIMUM STRESS AND MAXIMUM DISPLACEMENT
C     ON THE SHAFT.
C
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      REAL CF(30,6),CG(30,6)
      DIMENSION AFO(8),CSI(51)
C
      COMMON CU(51,5,15),CV(51,5,15),CRU(51,5,15),CRV(51,5,15),TM(51),
     1TDPL(51),SIGONE(51),SIGTWO(51),SIGMAZ(51),OMGTAU(15,5,8)
C
      DO 1110 I=1,KZ
 1110 SIGTWO(I)=4/PI*(T+GM*CSI(I)*RL)/(DO**2-DI**2)
      SIGMX=0
      ZZMX=0
      DO 1160 I=1,KZ
      TM(I)=DSQRT((CRU(I,J,K))**2+(CRV(I,J,K))**2)
      SIGONE(I)=32*DO*TM(I)/(PI*(DO**4-DI**4))
      SIGMAZ(I)=SIGONE(I)+SIGTWO(I)
      IF(SIGMX-SIGMAZ(I))  1150,1160,1160
 1150 SIGMX=SIGMAZ(I)
      ZZMX=(I-1)/50
 1160 CONTINUE
      BENDMX=SIGONE(1)
      ZBEND=0
      DO 1180 I=2,KZ
      C=BENDMX
      BENDMX=DMAX1(BENDMX,SIGONE(I))
      IF(C.LT.BENDMX)  ZBEND=(I-1)/50
 1180 CONTINUE
      DO 1190 I=1,KZ
 1190 TDPL(I)=DSQRT((CU(I,J,K))**2+(CV(I,J,K))**2)
      DPLMX=TDPL(1)
      ZDPL=0
      DO 1198 I=2,KZ
      D=DPLMX
      DPLMX=DMAX1(DPLMX,TDPL(I))
      IF(D.LT.DPLMX)  ZDPL=(I-1)/50
 1198 CONTINUE
C
      GO TO (1130,1140),KCASE
 1130 WRITE (6,1135) AFO(J),ZBEND,BENDMX,ZDPL,DPLMX,ZZMX,SIGMX
 1135 FORMAT(//,1X,'AFO=',D10.4,/,1X,'MAX. STRESS    ZBEND=',D10.4,4X,'B
     1ENDMAX=',D12.6,4X,'MAX. DISPL.',4X,'ZDPL=',D10.4,4X,'DPLMAX=',D12.
     26,/,1X,'TOTAL MAX. STRESS    ZZMX=',D10.4,4X,'SIGMX=',D12.6)
      GO TO 1280
 1140 OMGTAU(K,1,J)=OMG
      OMGTAU(K,2,J)=ZBEND
      OMGTAU(K,3,J)=BENDMX
      OMGTAU(K,4,J)=ZDPL
      OMGTAU(K,5,J)=DPLMX
      IF(KEX.EQ.KEX/2*2)   GO TO 1145
      CF(K,1)=OMG/PI*30
      CG(K,1)=OMG/PI*30
      JP1=J+1
      CF(K,JP1)=DLOG10(DPLMX/ECC)
      CG(K,JP1)=DLOG10(BENDMX/(E*RI*ECC/(RL**2*DO**3)))
      GO TO 1280
 1145 KSPK=KS+K
      CF(KSPK,1)=OMG/PI*30
      CG(KSPK,1)=OMG/PI*30
      JP1=J+1
      CF(KSPK,JP1)=DLOG10(DPLMX/ECC)
      CG(KSPK,JP1)=DLOG10(BENDMX/(E*RI*ECC/(RL**2*DO**3)))
 1280 RETURN
      END
```

```
      DOUBLE PRECISION FUNCTION AFCTR(N,AO,A,BO,B,SGASGB,ECC,RL,HR)
C
C     AFCTR IS A WEIGHTING FUNCTION, DUE TO THE ECCENTRICITY ON PLANE
C     UOZ.
C
C     SECOND EXAMPLE
C
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DIMENSION A(150),B(150)
      AFCTR=(4**4-3**4-1)/(4**4**4)*AO-SGASGB*(4**3-3**3-1)/(3*4**3)*BO
      DO 710 KN=1,N
      AFCTR=AFCTR+(4**(KN+4)-3**(KN+4)-1)/((KN+4)*4**(KN+4))*A(KN)-SGASGB
     1B*(4**(KN+3)-3**(KN+3)-1)/((KN+3)*4**(KN+3))*B(KN)
  710 CONTINUE
      AFCTR=AFCTR*ECC*RL/HR
      RETURN
      END
```

```
      DOUBLE PRECISION FUNCTION BFCTR(N,AO,A,BO,B,SGASGB,ECC,RL,HR)
C
C     BFCTR IS A WEIGHTING FUNCTION, DUE TO THE ECCENTRICITY ON PLANE
C     UOZ.
C
C     SECOND EXAMPLE
C
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DIMENSION A(150), B(150)
      BFCTR=(3**4-2**4)/(4*4**4)*AO-SGASGB*(3**3-2**3)/(3*4**3)*BO
      DO 810 KN=1,N
      BFCTR=BFCTR+(3**(KN+4)-2**(KN+4))/((KN+4)*4**(KN+4))*A(KN)-SGASGB*
     1(3**(KN+3)-2**(KN+3))/((KN+3)*4**(KN+3))*B(KN)
  810 CONTINUE
      BFCTR=BFCTR*ECC*RL/HR
      RETURN
      END
```

Note: The subroutine PLOT is available in the IBM 360 Scientific Subroutine Package.

```
C-----------------------------------------------------------------------MAIN  10
C     THIS PROGRAM CALCULATES THE NATURAL FREQUENCIES OF A BEAM SUBJECT MAIN  20
C     TO LINEARLY VARYING TENSION.                                      MAIN  30
C     THE MAIN PROGRAM PLOTS THE FREQUENCY EQUATION AS A FUNCTION OF Q. MAIN  40
C     THE SUBROUTINE NATFRQ DETERMINES THE ROOTS OF THE FREQUENCY       MAIN  50
C     EQUATION FROM TWO CONSECUTIVES VALUES WITH OPPOSITE SIGNS OF THE  MAIN  60
C     EQUATION.                                                         MAIN  70
C-----------------------------------------------------------------------MAIN  80
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)                               MAIN  90
      DIMENSION A(250),B(250),RA(250),RB(250)                           MAIN 100
      READ 5,GM,RL,T,RI,E,RM                                            MAIN 110
    5 FORMAT(4F10.4,F15.4,F10.4)                                        MAIN 120
      PRINT 20,GM,RL,T,RI,E,RM                                          MAIN 130
   20 FORMAT('0',//,1X,'GM=',D26.18,//,1X,'RL=',D26.18,//,1X,'T =',D26.1MAIN 140
     18,//,1X,'RI=',D26.18,//,1X,'E =',D26.18,//,1X,'RM=',D26.18)      MAIN 150
   25 READ 30,N,MIT                                                     MAIN 160
   30 FORMAT(2I4)                                                       MAIN 170
      PRINT 40,N,MIT                                                    MAIN 180
   40 FORMAT('0','N =',I4,//,1X,'MIT=',I4)                              MAIN 190
      IF(N.EQ.0) STOP                                                   MAIN 200
      AF=(GM*RL**3)/(E*RI)                                              MAIN 210
      BT=(T*RL**2)/(E*RI)                                               MAIN 220
      PRINT 45,AF,BT                                                    MAIN 230
   45 FORMAT('0',//,1X,'AF=',D26.18,//,1X,'BT=',D26.18)                 MAIN 240
      Q1=-.101                                                          MAIN 250
      F1=.1002                                                          MAIN 260
      PRINT 42,Q1,F1                                                    MAIN 270
   42 FORMAT('1 VALUES OF Q1 AND F1 FOR INITIALIZATION ONLY   Q1=',D26.1MAIN 280
     18,6X,'F1=',D26.18)                                                MAIN 290
      JJ=1                                                              MAIN 300
      JK=101                                                            MAIN 310
      JL=2                                                              MAIN 320
      NCOUNT=1                                                          MAIN 330
      GO TO 48                                                          MAIN 340
   46 JJ=103                                                            MAIN 350
      JK=201                                                            MAIN 360
      JL=4                                                              MAIN 370
      GO TO 48                                                          MAIN 380
   47 JJ=205                                                            MAIN 390
      JK=300                                                            MAIN 400
      JL=8                                                              MAIN 410
   48 DO 99 J=JJ,JK,JL                                                  MAIN 420
      Q2=J-1                                                            MAIN 430
      JPRINT=1                                                          MAIN 440
      CALL COEFFC(N,AF,BT,Q2,AO,A,BO,B,RAO,RA,RBO,RB,J,JPRINT)          MAIN 450
      JTEST=(J-1)/100*100                                               MAIN 460
      SUMA=AO                                                           MAIN 470
      SUMB=BO                                                           MAIN 480
      SUMODA=6*AO                                                       MAIN 490
      SUMODB=2*BO                                                       MAIN 500
      IF(JTEST.NE.J-1)  GO TO 51                                        MAIN 510
      IF(JPRINT.EQ.2)  GO TO 51                                         MAIN 520
      PRINT 50,SUMA,SUMB,SUMODA,SUMODB                                  MAIN 530
   50 FORMAT('0   I',13X,'SUMA',24X,'SUMB',23X,'SUMODA',22X,'SUMODB',//,MAIN 540
     14X,'0',4D28.18)                                                   MAIN 550
   51 DO 89 I=1,N                                                       MAIN 560
      SUMA=SUMA+A(I)                                                    MAIN 570
      SUMB=SUMB+B(I)                                                    MAIN 580
      SUMODA=SUMODA+RA(I)                                               MAIN 590
      SUMODB=SUMODB+RB(I)                                               MAIN 600
      IF(JTEST.NE.J-1)  GO TO 89                                        MAIN 610
      IF(JPRINT.EQ.2)  GO TO 89                                         MAIN 620
      PRINT 60, I,SUMA,SUMB,SUMODA,SUMODB                               MAIN 630
   60 FORMAT(I5,4D28.18)                                                MAIN 640
   89 CONTINUE                                                          MAIN 650
      F2=SUMODA-SUMA/SUMB*SUMODB                                        MAIN 660
      PRINT 70,Q2,F2                                                    MAIN 670
   70 FORMAT('0  Q=',D26.18,6X,'FCTQ=',D26.18)                          MAIN 680
      F12=F1*F2                                                         MAIN 690
      IF(F12)  110,120,130                                              MAIN 700
  110 CALL NATFRQ(N,MIT,AF,BT,Q1,F1,Q2,F2,J,KL,E,RI,RM)                 MAIN 710
      GO TO 130                                                         MAIN 720
  120 IF(F2*10)  130,123,130                                            MAIN 730
  123 PRINT 125,Q2,F2                                                   MAIN 740
  125 FORMAT('0  ROOT LOCATED,  Q=',D26.18,6X,'FCTQ=',D26.18)           MAIN 750
      OMEGAR=Q2/(RL**2)*DSQRT(E/RM*RI)                                  MAIN 760
      RPM=OMEGAR/(2*3.141592653589793)*60                               MAIN 770
      PRINT 126,OMEGAR,RPM                                              MAIN 780
  126 FORMAT('0',12X,'OMEGAR=',D26.18,//,16X,'RPM=',D26.18)             MAIN 790
  130 Q1=Q2                                                             MAIN 800
      F1=F2                                                             MAIN 810
   99 CONTINUE                                                          MAIN 820
      NCOUNT=NCOUNT+1                                                   MAIN 830
      IF(NCOUNT.EQ.2)  GO TO 46                                         MAIN 840
      IF(NCOUNT.EQ.3)  GO TO 47                                         MAIN 850
      PRINT 100                                                         MAIN 860
  100 FORMAT('0  VALUES OF Q EXAUSTED')                                 MAIN 870
      GO TO 25                                                          MAIN 880
      END                                                               MAIN 890
```

```
      SUBROUTINE COEFFC(N,AF,BT,Q,AO,A,BO,B,RAO,RA,RBO,RB,J,JPRINT)     COEF  10
C     THIS SUBROUTINE CALCULATES THE COEFFICIENTS A(K) AND B(K)         COEF  20
C     ALSO CALCULATES RA(K)=(K+3)*(K+2)*A(K) AND RB(K)=(K+2)*(K+1)*B(K) COEF  30
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)                               COEF  40
      DIMENSION A(250),B(250),RA(250),RB(250)                           COEF  50
      JTEST=(J-1)/100*100                                               COEF  60
      IF(JTEST.NE.J-1)  GO TO 210                                       COEF  70
      IF(JPRINT.EQ.2) GO TO 210                                         COEF  80
      PRINT 200,Q                                                       COEF  90
  200 FORMAT('1 ENTERING SUBROUTINE COEFFC FOR Q=',D26.18,//,4X,'K',13X,COEF 100
     1'A(K)',24X,'B(K)',23X,'RA(K)',23X,'RB(K)')                        COEF 110
  210 AO=1                                                              COEF 120
      A(1)=0                                                            COEF 130
      A(2)=BT/(5*4)                                                     COEF 140
      A(3)=(3*AF)/(6*5*4)                                               COEF 150
      A(4)=(BT*A(2))/(7*6)+(4*AF*A(1))/(7*6*5)+(Q**2*AO)/(7*6*5*4)      COEF 160
      RAO=6                                                             COEF 170
      RA(1)=12*A(1)                                                     COEF 180
      RA(2)=20*A(2)                                                     COEF 190
      RA(3)=30*A(3)                                                     COEF 200
      RA(4)=42*A(4)                                                     COEF 210
      BO=1                                                              COEF 220
      B(1)=0                                                            COEF 230
      B(2)=BT/(4*3)                                                     COEF 240
      B(3)=(2*AF)/(5*4*3)                                               COEF 250
      B(4)=(BT*B(2))/(6*5)+(3*AF*B(1))/(6*5*4)+(Q**2*BO)/(6*5*4*3)      COEF 260
      RBO=2                                                             COEF 270
      RB(1)=6*B(1)                                                      COEF 280
      RB(2)=12*B(2)                                                     COEF 290
      RB(3)=20*B(3)                                                     COEF 300
      RB(4)=30*B(4)                                                     COEF 310
      IF(JTEST.NE.J-1)  GO TO 225                                       COEF 320
      IF(JPRINT.EQ.2) GO TO 225                                         COEF 330
      PRINT 220,AO,BO,RAO,RBO,(K,A(K),B(K),RA(K),RB(K),K=1,4)           COEF 340
  220 FORMAT('0   0',4D28.18,/,(I5,4D28.18))                            COEF 350
```

```
  225 DO 289 K=5,N                                                      COEF 360
      A(K)=A(K-2)/(K+3)*BT/(K+2)+A(K-3)/(K+3)*AF/(K+2)*K/(K+1)+A(K-4)*Q/COEF 370
     1((K+3)*(K+2))*Q/((K+1)*K)                                         COEF 380
      IF (A(K).LT..1D-24)  A(K)=0                                       COEF 390
      B(K)=B(K-2)/(K+2)*BT/(K+1)+B(K-3)/(K+2)*AF/(K+1)*(K-1)/K+B(K-4)*Q/COEF 400
     1((K+2)*(K+1))*Q/(K*(K-1))                                         COEF 410
      IF(B(K).LT..1D-24)  B(K)=0                                        COEF 420
      RA(K)=(K+3)*(K+2)*A(K)                                            COEF 430
      RB(K)=(K+2)*(K+1)*B(K)                                            COEF 440
      IF(JTEST.NE.J-1)  GO TO 289                                       COEF 450
      IF(JPRINT.EQ.2)  GO TO 289                                        COEF 460
      PRINT 240,K,A(K),B(K),RA(K),RB(K)                                 COEF 470
  240 FORMAT(I5,4D28.18)                                                COEF 480
  289 CONTINUE                                                          COEF 490
      IF(JTEST.NE.J-1)  RETURN                                          CO.F 500
      IF(JPRINT.EQ.2)  RETURN                                           COEF 510
      PRINT 230                                                         COEF 520
  230 FORMAT('0  LEAVING SUBROUTINE COEFFC')                            COEF 530
      RETURN                                                            COEF 540
      END                                                               COEF 550
```

```
      SUBROUTINE NATFRQ(N,MIT,AF,BT,Q1,F1,Q2,F2,J,RL,E,RI,RM)           NATF  10
C     THIS SUBROUTINE DETERMINES THE ROOTS OF THE FREQUENCY EQUATION    NATF  20
C     TWO CONSECUTIVES VALUES OF OPPOSITE SIGNS OF THE FREQ EQUT ARE    NATF  30
C     GIVEN AND THE SUBROUTINE MAKES AN ITERATION TO FIND THE ROOTS     NATF  40
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)                               NATF  50
      DIMENSION A(250),B(250),RA(250),RB(250)                           NATF  60
      QN1=Q1                                                            NATF  70
      QN2=Q2                                                            NATF  80
      FN1=F1                                                            NATF  90
      FN2=F2                                                            NATF 100
      PRINT 310,QN1,FN1,QN2,FN2                                         NATF 110
  310 FORMAT('1  NATURAL FREQUENCY IN THE INTERVAL   QN1=',D26.18,8X,'FNNATF 120
     11=',D26.18,//,39X,'QN2=',D26.18,8X,'FN2=',D26.18)                 NATF 130
      EPS1=QN1**6                                                       NATF 140
      EPS2=1.D-26                                                       NATF 150
      EPS=EPS1*EPS2                                                     NATF 160
      PRINT 311,EPS                                                     NATF 170
  311 FORMAT('0  EPS=',D26.18)                                          NATF 180
      DO 379 L=1,MIT                                                    NATF 190
      DELTAQ=((QN2-QN1)*DABS(FN1))/(DABS(FN1)+DABS(FN2))                NATF 200
      Q3=QN1+DELTAQ                                                     NATF 210
      JPRINT=2                                                          NATF 220
      CALL COEFFC(N,AF,BT,Q3,AO,A,BO,B,RAO,RA,RBO,RB,J,JPRINT)          NATF 230
      SUMA=AO                                                           NATF 240
      SUMB=BO                                                           NATF 250
      SUMODA=6*AO                                                       NATF 260
      SUMODB=2*BO                                                       NATF 270
      DO 369 II=1,N                                                     NATF 280
      SUMA=SUMA+A(II)                                                   NATF 290
      SUMB=SUMB+B(II)                                                   NATF 300
      SUMODA=SUMODA+RA(II)                                              NATF 310
      SUMODB=SUMODB+RB(II)                                              NATF 320
  369 CONTINUE                                                          NATF 330
      F3=SUMODA-SUMA/SUMB*SUMODB                                        NATF 340
      PRINT 320,L,Q3,F3                                                 NATF 350
  320 FORMAT('0 ITR=',I3,6X,'Q3=',D26.18,6X,'F3=',D26.18)               NATF 360
      IF(DABS(F3).LT.EPS)  GO TO 340                                    NATF 370
      F13=FN1*F3                                                        NATF 380
      IF(F13)      330,340,350                                          NATF 390
  330 QN2=Q3                                                            NATF 400
      FN2=F3                                                            NATF 410
      GO TO 379                                                         NATF 420
  340 PRINT 360,Q3,F3                                                   NATF 430
  360 FORMAT('0',//,2X,'ROOT LOCATED,  Q3=',D26.18,6X,'F3=',D26.18)     NATF 440
      GO TO 380                                                         NATF 450
  350 QN1=Q3                                                            NATF 460
      FN1=F3                                                            NATF 470
  379 CONTINUE                                                          NATF 480
      PRINT 370                                                         NATF 490
  370 FORMAT('0  NO CONVERGENCE')                                       NATF 500
  380 OMEGAR=Q3/(RL**2)*DSQRT(E/RM*RI)                                  NATF 510
      RPM=OMEGAR/(2*3.1415926535897 93)*60                              NATF 520
      PRINT 326,OMEGAR,RPM                                              NATF 530
  326 FORMAT('0',12X,'OMEGAR=',D26.18,//,16X,'RPM=',D26.18,/,'1')       NATF 540
      RETURN                                                            NATF 550
      END                                                               NATF 560
```