Structure of Intermediate C^*-subalgebras of discrete group actions

by
Tattwamasi Amrutam

A dissertation submitted to the Department of Mathematics,

College of Natural Sciences and Mathematics

in partial fulfillment of the requirements for the degree of

Doctor of Philosophy
in Mathematics

Chair of Committee: Mehrdad Kalantar

Committee Member: David Blecher

Committee Member: William Ott

Committee Member: Adam Skalski

University of Houston
May 2021

DEDICATION

To my (late) Grandparents:

Their stories taught us (me and my brother) persistence and enriched our childhood.

ABSTRACT

This dissertation deals with the structure of intermediate C^*-sub-algebras $\mathcal{B}$, either of the form $C^*_\lambda(\Gamma) \subseteq \mathcal{B} \subseteq \mathcal{A} \rtimes_r \Gamma$ or of the type $C(Y) \rtimes_r \Gamma \subseteq \mathcal{B} \subseteq C(X) \rtimes_r \Gamma$. We begin by investigating the ideal structure of intermediate C^*-sub-algebras $\mathcal{B}$ of the form $C^*_\lambda(\Gamma) \subseteq \mathcal{B} \subseteq \mathcal{A} \rtimes_r \Gamma$ for commutative unital Γ-simple Γ-C^*-algebras $\mathcal{A}$. In particular, we show that if Γ is a C^*-simple group, then every such intermediate C^*-sub-algebra $\mathcal{B}$ is simple. Continuing our perusal, we find examples of inclusions $C^*_\lambda(\Gamma) \subseteq \mathcal{A} \rtimes_r \Gamma$ for which every intermediate C^*-sub-algebra $\mathcal{B}$ of the form $C^*_\lambda(\Gamma) \subseteq \mathcal{B} \subseteq \mathcal{A} \rtimes_r \Gamma$ is a crossed product. We show that for a large class of actions $\Gamma \curvearrowright \mathcal{A}$ of C^*-simple groups Γ on unital C^*-algebras $\mathcal{A}$, including any non-faithful action of a hyperbolic group with trivial amenable radical, every intermediate C^*-sub-algebra $\mathcal{B}$, $C^*_\lambda(\Gamma) \subseteq \mathcal{B} \subseteq \mathcal{A} \rtimes_r \Gamma$, is a crossed product. On the von Neumann algebraic side, we show that for every non-faithful action of a acylindrically hyperbolic C^*-simple group Γ on a von Neumann algebra $\mathcal{M}$ with separable predual, every intermediate vNa $\mathcal{N}$, $L(\Gamma) \subseteq \mathcal{N} \subseteq \mathcal{M} \rtimes \Gamma$ is a crossed product vNa. Finally, we inquire into the ideal structure of intermediate C^*-sub-algebras $\mathcal{B}$ of the form $C(Y) \rtimes_r \Gamma \subseteq \mathcal{B} \subseteq C(X) \rtimes_r \Gamma$ for an inclusion of unital Γ-simple Γ-C^*-algebras $C(Y) \subset C(X)$. We introduce a notion of generalized Powers averaging and show that it is equivalent to the simplicity of the crossed product $C(X) \rtimes_r \Gamma$. As an application, we show that every intermediate C^*-sub-algebras $\mathcal{B}$, $C(Y) \rtimes_r \Gamma \subseteq \mathcal{B} \subseteq C(X) \rtimes_r \Gamma$ is simple whenever $C(Y) \rtimes_r \Gamma$ is simple.

Declaration

I declare that the dissertation has been composed by myself and that the work has not be submitted for any other degree or professional qualification. I confirm that the work submitted is my own, except where work which has formed part of jointly-authored publications has been included. My contribution and those of the other authors to this work have been explicitly indicated below. I confirm that appropriate credit has been given within this dissertation where reference has been made to the work of others. The work presented in Chapter two was previously published in Ergodic Theory and Dynamical Systems [3] as “On Simplicity of Intermediate C^*-algebras” (joint work with my advisor Mehrdad Kalantar). The work presented in Chapter three is going to appear in International Mathematics Research Notices [2] as “On Intermediate Subalgebras of C^*-simple Group Actions”. Chapter four is based on a joint work with Dan Ursu [4] and is available in arxiv as “Generalized Powers averaging for commutative crossed products”.

Tattwamasi Amrutam

Houston, Jan 2021

Acknowledgements

When we express our gratitude, it grows.

Richie Norton

In the beginning, I bow down to Goddess Durga (form of divine strength) for giving me strength and for blessing me with the life force flowing in me. I bow down to my spiritual masters, especially Paramahamsa Prajnananda, and seek their blessings.

There are no words to express my gratitude towards my advsior, Dr. Mehrdad Kalantar. I consider myself very fortunate to have him as my teacher, mentor and friend. He showed me the path, I walked on it. In every possible way, whatever I have achieved is because of his blessings and guidance. Whenever his door was open, I would walk in and we would start our mathematical conversations. The conversations would continue in our walks to coffee shops, food places and every other place in the campus. Needless to say that these discussions significantly enhanced my mathematical knowledge and abilities. It is a privilege to have someone like him as my advisor. He taught me everything about research. We ran a lot of learning seminars together and I learnt a lot from the seminars. I am also grateful to him for funding my teaching assistantship which gave me a lot of time to focus on research. He also funded my research visits to Spain, Greece, France and Canada and I thank him for that. These research visits broadened my horizons. I bow down to him in reverence.

I would like to thank my family, my parents and my brother (Dr. Zombie) for their unconditional love and support. They believed in me when even I didn't believe in myself. I bow down to my parents and seek their blessings.

I would like to thank Dr. David Blecher, Dr. Adam Skalski and Dr. William Ott for agreeing to be on my committee. Dr. Blecher and Dr. Ott also taught me various courses during my time here at UH and I am thankful to them for that. I will miss my mathematical duels with Dr. Ott. He took the time to preside over our weekly meetings on topology and manifolds during the summer of 2017, when I prepared for my prelims. I thank him for his kindness and willingness to do so.

I would like to thank Dr. Alan Haynes for giving an excellent course on Topological groups and Adeles. I learnt a lot during that course. I would also like to thank Dr. Vaughn Climenhaga for his many advices and for being understanding of my predicament when I was applying for post-doc positions. He gave me the liberty to grade at my own pace and time.

When I joined UH, I was Mo's apprentice. I started reading C^*-algebras with him. Even when he graduated from UH, he would come from time to time to attend my learning seminars. He taught me a lot and he has become an older brother to me. I also read along side Dylan and Sarah for quite some amount of time. Working with them was great fun. I would also like to thank Kazem and Wilfredo for offering their hands in friendship. Kazem, in particular was a great office mate by virtue of being not there.

You can take an Indian out of India but not India out of an Indian. As such, I am grateful to Prajakta, Priyam (who is my roommate), Amudhan, Dipanwita and Monali for becoming my friends. I can rely on them for anything. I would also like to thank Akshat, Jasmine, Deepjyoti, Oshin, Rajshree, Anuradha and Nilamani for making a part of Houston feel like India. I also offer my gratitude to the friends in India, especially Arpita, Arnav, Maitreyi, Padmaja, Padmaja (Pati), Puspita, Rahul, Kanchan and Ayushman for their support and belief. Their very presence is an assurance.

I cherish the bond of brotherhood I share with Priyam, Navid, Nikos (the third, now of course he has become the first) and Chris. Discussions and readings with them have helped me to "go above and beyond". I consider myself fortunate to have the friendship of Monika, Priyadarshi and Shreya. We have run many learning seminars together (online) and have had many reading sessions on philosophy and life. No amount of thank-you would be enough to justify the importance of these people in my life and their influences on me.

I started the journey of mathematics with Prayag, Pratik and Pratyush. We are still in it together and we shall be in it until the very end. They have contributed to my knowledge in a humongous way and continue to do so. We have run many (online) learning seminars together and will continue to do so in the future as well. Their success is mine and my success is theirs. They are a part of my family. I am blessed to have them in my life. Kalpana aunty and Pradipta uncle (Prayag's parents) have always loved me like their own son. I have been aided on my journey by their prayers and blessings. I bow down to them and seek their blessings.

I am indebted to Prof. Swadhin Pattanayak, who mentored us (myself, Prayag, Pratik and Pratyush) during our undergraduate days. He played a big role in motivating us to pursue higher studies in mathematics. I bow down at his feet.

I bow down to all the teachers who have taught me over the years, especially Binayak Sir and Priya Sir. Binayak Sir (or, Babuli Sir as we fondly call him) and Priya Sir taught me mathematics when I was in school. I seek their blessings. I express my sincere thanks to all my teachers in IIT-Bombay who taught me and made my foundation strong. I especially acknowledge the love and support I received from Prof. B. V. Limaye, Prof. S. G. Dani, Prof. J. K. Verma, Prof. Santanu Dey and Prof. Prachi Mahajan. During my time at IIT-Bombay, I learnt a lot during my discussions with Jaykant, Bappa, Basudev, Monika and Nilakantha. I thank them for the same.

I am also thankful to Dr. Narutaka Ozawa, Dr. Matthew Kennedy, Dr. Yair Hartman, Dr. Darren Creutz and Dr. Adam Skalski for taking the time to discuss with me at various stages of my PhD. I also thank Zahra Naghavi, Yongle Jiang, Shirley Geffen and Dan Ursu for many helpful discussions on various mathematical topics.

I would like to thank the people working in Cougar-grounds Cafe for making awesome coffee and for welcoming me during the early mornings with smiles on their faces. I enjoyed studying there during the early morning hours. Similarly, the people at Nook Cafe welcomed me with open arms during the afternoon hours. Studying under the tree in their backyard is a memory I will cherish forever. I am also deeply indebted to Gaby (from Common Bond Cafe) for her welcoming smile.

Finally, I would like to thank the Department of Mathematics at UH for its support during these many years. It has been a privilege.

TABLE OF CONTENTS

1 Chapter Zero

> Who sees further a dwarf or a giant? Surely a giant for his eyes are situated at a higher level than those of the dwarf. But if the dwarf is placed on the shoulders of the giant who sees further? ... So too we are dwarfs astride the shoulders of giants. We master their wisdom and move beyond it. Due to their wisdom we grow wise and are able to say all that we say, but not because we are greater than they.
>
> A Wise Philosopher

In 1972, Furstenberg [16] introduced "boundary" to study certain properties of lattices of semisimple lie groups (e.g., think of $\mathbb{SL}_n(\mathbb{Z})$ inside $\mathbb{SL}_n(\mathbb{R})$). Around the same time, Powers [34] showed that $C^*_\lambda(\mathbb{F}_2)$ is simple. He used a form of averaging (more famously known as Powers averaging) to show that every non-zero closed ideal of $C^*_\lambda(\mathbb{F}_2)$ must contain an invertible element, hence must be all of $C^*_\lambda(\mathbb{F}_2)$. Until 2014, research on C^*-simplicity was dominated by combinatorial variations of Powers averaging property.

In 2014, Kalantar and Kennedy [22] gave a dynamical characterization of C^*-simplicity. In particular, they showed that a group Γ is C^*-simple if and only if the action $\Gamma \curvearrowright \partial_F\Gamma$ on the Furstenberg boundary $\partial_F\Gamma$ is (topologically) free. We use this characterization in the later chapters for C^*-simple group actions.

For a unital Γ-C^*-algebra $\mathcal{A}$, the reduced crossed product $\mathcal{A} \rtimes_r \Gamma$ encodes the information of $\mathcal{A}$ and the group Γ (much similar to the construction of $G \ltimes H$, a semi-direct product of two groups G and H). Moreover, by construction, $\mathcal{A} \rtimes_r \Gamma$ contains $C^*_\lambda(\Gamma)$ as a C^*-subalgebra. Naturally, people began to investigate if the properties of $C^*_\lambda(\Gamma)$ are reflected in the bigger C^*-algebra $\mathcal{A} \rtimes_r \Gamma$. In particular, they began to study the ideal structure of the crossed product $\mathcal{A} \rtimes_r \Gamma$, e.g., see

[3, 10, 12, 13, 21, 24, 35, 38]. In this dissertation, we enquire into the structure of intermediate sub-algebras of crossed products.

Building on the characterization provided in [22], Kalantar and Kennedy along with their co-authors Breuillard and Ozawa solved many important problems in [7] which were open for several years. In particular, they showed that $\mathcal{A} \rtimes_r \Gamma$ is simple when $\mathcal{A}$ is Γ-simple and $C^*_\lambda(\Gamma)$ is simple (and in the process answered a question asked in [12] in the affirmative). In many ways, this was a starting point for us.

Initially, we investigate the structure of intermediate C^*-sub-algebras $\mathcal{B}$, $C^*_\lambda(\Gamma) \subseteq \mathcal{B} \subseteq \mathcal{A} \rtimes_r \Gamma$. In the final chapter, we enquire into the ideal structure of intermediate C^*-subalgebra $\mathcal{B}$ of the form $C(Y) \rtimes_r \Gamma \subseteq \mathcal{B} \subseteq C(X) \rtimes_r \Gamma$ for an inclusion $C(Y) \subset C(X)$ of unital Γ-C^*-algebras.

This dissertation is divided into two parts. In the first part, we describe the objects of our interest. Since we primarily deal with C^*-simple groups (except for the final chapter), we mention some of the outstanding works done in understanding such structures (i.e., [7, 9, 19, 20, 22, 26, 27, 30]). Since "boundary actions" play a significant role for us, we illustrate two examples of such actions where we can explicitly get hold of a "boundary".

We prove the main results in the second part. In the second chapter, we take the first step towards understanding the structure of intermediate C^*-subalgebras. In this chapter, we deal with the ideal structure of intermediate C^*-subalgebras $\mathcal{B}$ of the form $C^*_\lambda(\Gamma) \subseteq \mathcal{B} \subseteq C(X) \rtimes_r \Gamma$ for C^*-simple groups Γ acting minimally on a compact Hausdorff space X. In particular, we show that if Γ is C^*-simple and X is a minimal Γ-space (in which case $C(X) \rtimes_r \Gamma$ is simple), every intermediate C^*-subalgebra $\mathcal{B}$ of the form $C^*_\lambda(\Gamma) \subseteq \mathcal{B} \subseteq C(X) \rtimes_r \Gamma$ is simple.

In the third chapter, we move further ahead and give examples of C^*-simple group actions $\Gamma \curvearrowright \mathcal{A}$ for which every intermediate C^*-subalgebra $\mathcal{B}$ of the form $C^*_\lambda(\Gamma) \subseteq \mathcal{B} \subseteq \mathcal{A} \rtimes_r \Gamma$ is a crossed product. We prove a similar result in the context of von Neumann algebras as well. For this, we introduce the notion of "plump subgroups" and show that the class of such subgroups is huge. We also give various dynamical characterizations of such groups in the process.

Finally, in the fourth and last chapter, we generalize the well known "Powers averaging" to the level of commutative crossed products and show that such an averaging is equivalent to the simplicity of the crossed product $C(X) \rtimes_r \Gamma$. As an application, we generalize the result of the second chapter to the setting of commutative reduced crossed products.

Part I

Introducing the tools.

2 Introduction

> Young man, in mathematics you don't understand things. You just get used to them.
>
> John von Neumann

In this chapter, we describe the construction of the objects we work with. In particular, we describe the structure of reduced C^*-algebra, reduced crossed products and associated maps. Moreover, we also briefly recall the important works which have been done in understanding the structure of these objects (mostly without proofs). The contents of the first section are mostly taken from [8].

2.1 Group C^*-algebras

We begin by discussing an important class of C^*-algebras which is primary to our interest. Let Γ be a discrete group. Let $\lambda : \Gamma \to \mathbb{B}(\ell^2(\Gamma))$ denote the left regular representation:

$$\lambda_s(\delta_t) = \delta_{st},\ \ s, t \in \Gamma$$

Note that $\{\delta_t : t \in \Gamma\}$ is an orthonormal basis for $\ell^2(\Gamma)$. For $f \in \ell^\infty(\Gamma)$ and $s \in \Gamma$, we define $s.f \in \ell^\infty(\Gamma)$ by $s.f(t) = f(s^{-1}t), s, t \in \Gamma$. We view $\ell^\infty(\Gamma)$ as multiplication operators on $\mathbb{B}(\ell^2(\Gamma))$, i.e.,

$$M_f(\delta_t) = f(t)\delta_t,\ \ f \in \ell^\infty(\Gamma),\ \ t \in \Gamma$$

An easy calculation then shows that

$$\lambda_s M_f \lambda_s^* = s.f,\ \ \forall f \in \ell^\infty(\Gamma),\ \ s \in \Gamma$$

Definition 2.1. The reduced C^*-algebra of Γ, denoted by $C^*_\lambda(\Gamma)$, is defined as

$$C^*_\lambda(\Gamma) = \overline{\operatorname{Span}}^{\|\cdot\|_{\mathbb{B}\ell^2(\Gamma)}} \{\lambda_t : t \in \Gamma\}$$

A state on φ a unital C^*-algebra $\mathcal{A}$ is a positive linear functional of norm 1. Moreover, a state φ is called a trace if

$$\varphi(ab) = \varphi(ba), \ \forall a, b \in \mathcal{A}.$$

The reduced C^*-algebra comes equipped with a canonical trace which will play an important role for us later.

Proposition 2.2. [8, Proposition 2.5.3] *The map $\tau_0 : C^*_\lambda(\Gamma) \to \mathbb{C}$ defined by $\tau_0(x) = \langle x\delta_e, \delta_e\rangle$ is a faithful trace.*

We introduce the notion of amenable groups below and refer the reader to [8, Theorem 2.6.8] for characterizations of such groups. We do not deal with them in the later chapters but they serve as important non-examples for us.

Definition 2.3. A group Γ is called amenable if there exists a state μ on $\ell^\infty(\Gamma)$ which is invariant under the left translation action, i.e., for all $s \in \Gamma$ and $f \in \ell^\infty(\Gamma)$, $\mu(s.f) = \mu(f)$.

There are plenty of amenable groups around, e.g., finite groups, abelian groups. In fact, all solvable (hence, nilpotent) groups are amenable.

Example 2.4 (Non-abelian free groups)**.** *The free group $\mathbb{F}_2$ of rank two is not amenable. See* [8, Example 2.6.7] *for a proof.*

We shall see later that free groups belong to a class of groups called C^*-simple groups.

Definition 2.5. A group Γ is called C^*-simple if $C^*_\lambda(\Gamma)$ doesnot have any non-trivial closed ideals.

2.2 Reduced crossed products

Definition 2.6. Let Γ be a discrete group and $\mathcal{A}$ be a C^*-algebra. An action of Γ on $\mathcal{A}$ is a group homomorphism α from Γ into the group of $*$-automorphisms on $\mathcal{A}$. A C^*-algebra equipped with a

Γ-action is called a Γ-C^*-algebra.

Suppose that $\mathcal{A}$ is a unital Γ-C^*-algebra. Let $\pi : \mathcal{A} \to \mathbb{B}(\mathcal{H})$ be a faithful $*$-representation. Let $\ell^2(\Gamma, \mathcal{H})$ be the space of square summable $\mathcal{H}$-valued functions on Γ, i.e.,

$$\ell^2(\Gamma, \mathcal{H}) = \left\{ \xi : \Gamma \to \mathcal{H} \text{ such that } \sum_{t \in \Gamma} \|\xi(t)\|_{\mathcal{H}}^2 < \infty. \right\}$$

There is an action $\Gamma \curvearrowright \ell^2(\Gamma, H)$ by left translation:

$$\lambda_s \xi(t) := \xi(s^{-1}t), \xi \in \ell^2(\Gamma, \mathcal{H}), s, t \in \Gamma$$

Let σ be a $*$-representation

$$\sigma : \mathcal{A} \to B(\ell^2(\Gamma, \mathcal{H}))$$

defined by

$$\sigma(a)(\xi)(t) := \pi(t^{-1}a)\xi(t), a \in \mathcal{A}$$

where $\xi \in \ell^2(\Gamma, \mathcal{H})$, $t \in \Gamma$. The reduced crossed product C^*-algebra $\mathcal{A} \rtimes_{\alpha,r} \Gamma$ is the closure in $B(\ell^2(\Gamma, \mathcal{H}))$ of the subalgebra generated by the operators $\sigma(a)$ and λ_s. Note that $\lambda_s \sigma(a) \lambda_{s^{-1}} = \sigma(s.a)$ for all $s \in \Gamma$ and $a \in \mathcal{A}$.

Remark 2.7. It is easy to see that $\mathcal{A} \rtimes_r \Gamma$ contains $C^*_\lambda(\Gamma)$ as as a C^*-subalgebra.

The reduced crossed product comes equipped with a projection from $C(X) \rtimes_r \Gamma$ onto $C(X)$ denoted by $\mathbb{E}$. It is called a "conditional expectation". Since it will play crucial role for us in the later chapters, we take some time to develop the intuition. We do not go into the detailed proofs (for which we refer the reader to [8]). Recall the definition of an operator system.

Definition 2.8. An operator system E is a closed self-adjoint subspace of a unital C^*-algebra $\mathcal{A}$ such that $\mathbf{1}_{\mathcal{A}} \in E$. The $n \times n$ matrices over E, $\mathbb{M}_n(E)$, inherit an order structure from $\mathbb{M}_n(\mathcal{A})$: an element in $\mathbb{M}_n(E)$ is positive if and only if it is positive in $\mathbb{M}_n(\mathcal{A})$.

Towards conditional expectation

A map $\varphi : E \to \mathcal{B}$ from an operator system E into a C^*-algebra $\mathcal{B}$ is called completely positive if $\varphi_n : \mathbb{M}_n(E) \to \mathbb{M}_n(\mathcal{B})$, defined by

$$\varphi_n\left([a_{i,j}]\right) = \left[\varphi(a_{i,j})\right],$$

is positive (i.e., maps positive matrices to positive matrices) for every n. In addition, if $\varphi\left(\mathbf{1}_E\right) = \mathbf{1}_{\mathcal{A}}$, we say that φ is a unital completely positive map. We write u.c.p for "unital completely positive" maps. An important class of such maps are $*$-homomorphisms π between two unital C^*-algebras.

Proposition 2.9. [8, Proposition 1.5.7] *Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras and $\varphi : \mathcal{A} \to \mathcal{B}$ be a u.c.p map.*

1. *(Schwarz Inequality) The inequality $\varphi(a)^*\varphi(a) \leq \varphi(a^*a)$ holds for every $a \in \mathcal{A}$.*

2. *(Bimodule Property) Given $a \in \mathcal{A}$, if $\varphi(a)^*\varphi(a) = \varphi(a^*a)$ and $\varphi(aa^*) = \varphi(a)\varphi(a)^*$, then $\varphi(ba) = \varphi(b)\varphi(a)$ and $\varphi(ab) = \varphi(a)\varphi(b)$ for every $b \in \mathcal{A}$.*

3. *The subspace $\mathcal{A}_\varphi$ defined as*

$$\{a \in \mathcal{A} : \varphi(a)^*\varphi(a) = \varphi(a^*a) \text{ and } \varphi(aa^*) = \varphi(a)\varphi(a)^*\}$$

is a C^*-subalgebra of $\mathcal{A}$.

Definition 2.10. Let $\varphi : \mathcal{A} \to \mathcal{B}$ be a u.c.p map. The C^*-subalgebra $\mathcal{A}_\varphi$ in Proposition 2.9 is called the multiplicative domain of φ.

We shall derive many important results by showing that a C^*-subalgebra falls in multiplicative domain of states but that is for later. It turns out that conditional expectations are important examples of u.c.p maps.

Definition 2.11. Let $\mathcal{B} \subset \mathcal{A}$ be an inclusion of unital C^*-algebras. A projection from $\mathcal{A}$ onto $\mathcal{B}$ is a linear map $E : \mathcal{A} \to \mathcal{B}$ such that $E(b) = b$ for all $b \in \mathcal{B}$. A conditional expectation from $\mathcal{A}$ onto $\mathcal{B}$ is a u.c.p projection E from $\mathcal{A}$ onto $\mathcal{B}$ such that $E(bxb') = bE(x)b'$ for every $x \in \mathcal{A}$ and $b, b' \in \mathcal{B}$.

Theorem 2.12 (Tomiyama)**.** [8, Theorem 1.5.10] *Let $\mathcal{B} \subset \mathcal{A}$ be an inclusion of unital C^*-algebras. Let E be a projection from $\mathcal{A}$ onto $\mathcal{B}$ with $\mathbf{1}_{\mathcal{A}} \in \mathcal{B}$. The following are equivalent:*

1. *E is a conditional expectation.*

2. *E is u.c.p.*

3. *E is contractive.*

The reduced crossed product $\mathcal{A} \rtimes_r \Gamma$ comes equipped with a canonical conditional expectation $\mathbb{E} : \mathcal{A} \rtimes_r \Gamma \to \mathcal{A}$ defined by

$$\mathbb{E}\left(\sigma(a_s)\lambda_s\right) = \left\{ \begin{array}{ll} 0 & \text{if } s \neq e \\ \sigma(a_e) & \text{otherwise} \end{array} \right\}$$

It follows from [8, Proposition 4.1.9] that $\mathbb{E}$ extends to a faithful conditional expectation from $\mathcal{A} \rtimes_r \Gamma$ onto $\mathcal{A}$. Observe that the map $\mathbb{E}$ is Γ-equivariant, i.e.,

$$\mathbb{E}\left(\lambda_s x \lambda_{s^{-1}}\right) = \alpha_s\left(\mathbb{E}(x)\right), x \in \mathcal{A} \rtimes_r \Gamma, s \in \Gamma$$

Sometimes, we will be seeing the reduced crossed product either inside $\mathbb{B}(\ell^2(\Gamma))$. We illustrate how below.

Inside our favorite Hilbert space.

By a compact Γ-space, we mean a compact Hausdorff space X on which Γ acts by homeomorphisms, i.e., there is a homomorphsim $\phi : \Gamma \to \mathrm{Homeo}(X)$ from the group Γ to $\mathrm{Homeo}(X)$, the group of homeomorphisms on X. For an element $s \in \Gamma$, we denote the action $s \curvearrowright X$ by $s.x$ instead of $\phi(s)x$ by a slight abuse of notation. The action $\Gamma \curvearrowright X$ is said to be minimal if the only non-empty Γ-invariant closed subset of X is X itself. For compact minimal Γ-spaces X ($\Gamma \curvearrowright X$ by homeomorphisms), we will view $C(X) \rtimes_r \Gamma$ inside $\mathbb{B}(\ell^2(\Gamma))$. For a probability measure ν on X, we denote by P_ν the corresponding Poisson map, i.e., the corresponding unital positive Γ-equivariant map $P_\nu : C(X) \to \ell^\infty(\Gamma)$ defined by

$$P_\nu(f)(s) = \int_X f(sx)d\nu(x),\ s \in \Gamma,\ f \in C(X).$$

Define a map $T : \ell^\infty(\Gamma) \to \mathbb{B}\ell^2(\Gamma)$ by $T(g) = M_g$, where $M_g(\xi) = g\xi$, $\xi \in \ell^2(\Gamma)$. Define $\pi : C(X) \to \mathbb{B}\ell^2(\Gamma)$ by $\pi(f) = T \circ P_\nu(f)$. In particular, for a point measure $\nu = \delta_{x_0}$, $x_0 \in X$, the Poisson map corresponding to δ_{x_0} turns out to be just the evaluation at x_0, i.e., $P_\nu(f)(s) = f(s.x_0)$. Moreover, in this case, $\pi : C(X) \to \mathbb{B}\ell^2(\Gamma)$ is an injective $*$-homomorphism. Note that $\{\delta_t : t \in \Gamma\}$ is an orthonormal basis for $\ell^2(\Gamma)$. Moreover,

$$\pi(f)(\delta_t) = T \circ P_\nu(f)(\delta_t) = M_{P_\nu(f)}(\delta_t) = f(t.x_0)\delta_t.$$

Now,

$$C(X) \rtimes_r \Gamma = \overline{\text{Span}\,\{\pi(f)\lambda_s : f \in C(X), s \in \Gamma\}}^{\|\cdot\|_{\mathbb{B}(\ell^2(\Gamma))}}$$

We will use this representation when we deal with minimal Γ-spaces X in the later chapters.

2.3 Simplicity of group C^*-algebras and their relation to boundaries

Since Powers proof [34] in 1975 that the free group on two generators is both C^*-simple and has the unique trace property, it had been a major open problem to characterize groups with either of these properties, and in particular to determine whether they are equivalent (see, e.g., [18] for this fact, and for a nice general survey of the subject matter). Recall that a discrete group Γ is called C^*-simple if the reduced C^*-algebra $C^*_r(\Gamma)$ is simple. Using the characterization provided in the pioneering work of Kalantar and Kennedy ([22]), the problem of characterizing groups with the unique trace property was completely settled in [7]. Finally, le Boudec [28] exhibited an example of a group with unique trace property but not C^*-simple, thereby establishing that the unique trace property and C^*-simplicity are not equivalent properties. In all these significant works, the key was the dynamical characterization of C^*-simplicity of Γ in terms of its action on the

Furstenberg boundary $\partial_F\Gamma$ (see [22, Theorem 1.5]), developed by Furstenberg [16] (also see [15]). We define boundary actions and provide two explicit examples of boundary actions before stating the celebrated result of [22].

Definition 2.13 (Boundary Action)**.** The action $\Gamma \curvearrowright X$ is called a boundary action if

$$\{\delta_x : x \in X\} \subset \overline{\Gamma\nu}^{\text{weak}^*}$$

for $\nu \in \text{Prob}(X)$.

Proposition 2.14. [15,16] *The Furstenberg boundary of Γ, $\partial_F\Gamma$ is a Γ boundary which is universal in the sense that every every other Γ-boundary Y is a Γ-equivariant continuous image of $\partial_F\Gamma$. Moreover, such a maximal Γ-boundary exists.*

The existence of such a space is a consequence of a product argument involving representatives of all boundaries (see, e.g., [16, P. 199]). Moreover, Furstenberg showed that it is unique upto Γ-equivariant homeomorphism. The uniqueness is a consequence of the following important property.

Proposition 2.15. [16, Proposition 4.2] *Every Γ-map from a compact Γ-space Y into $\mathcal{P}(X)$, where X is a Γ-boundary, must contain X in its range. Moreover, if Y is minimal, then there is at most one such map.*

2.4 Examples

2.4.1 Boundaries of free groups

We give an explicit example of a boundary action for $\mathbb{F}_2$, the free group of two generators $\{a, b\}$.

Definition 2.16. $\partial\mathbb{F}_2$ is the set of all infinite reduced words made from $\{a, b, a^{-1}, b^{-1}\}$. $\mathbb{F}_2$ acts on $\partial\mathbb{F}_2$ by adjoining the finite string on the left of the infinite word and then reducing it further.

It is not hard to see the following.

Proposition 2.17. *The following statements are true:*

1. $\partial\mathbb{F}_2$ is compact.

2. Every group element of $\mathbb{F}_2$ defines a homeomorphism from $\partial\mathbb{F}_2$ to $\partial\mathbb{F}_2$.

The following proposition is straightforward from the definitions.

Proposition 2.18. *The following hold true for $\mathbb{F}_2 \curvearrowright \partial\mathbb{F}_2$.*

1. For any $x \in \partial\mathbb{F}_2$, the orbit of x denoted by O_x is dense in $\partial\mathbb{F}_2$. Hence, the action $\mathbb{F}_2 \curvearrowright \partial\mathbb{F}_2$ is minimal.

2. Let $g \in \mathbb{F}_2$. Then there are two fixed points for g in $\partial\mathbb{F}_2$, i.e., there exist x_g^+ and x_g^{-1} in $\partial\mathbb{F}_2$ such that $gx_g^+ = x_g^+$ and $gx_g^{-1} = x_g^{-1}$. For $x \in \mathbb{F}_2 \setminus \{x_g^{-1}\}$, $\{g^n x\}_n \xrightarrow{n\to\infty} x_g^+$.

3. Fix $x_0 \in \partial\mathbb{F}_2$. Let $\nu \in Prob(\partial\mathbb{F}_2)$. Then there exists $g_n \in \mathbb{F}_2$ such that $g_n.\nu \to \delta_{x_0}$ in weak-topology. In particular, $\mathbb{F}_2 \curvearrowright \partial\mathbb{F}_2$ is a boundary action.*

4. There is no invariant probability measure on $\partial\mathbb{F}_2$.

2.4.2 Linear group acting on the projective plane

Recall that the real projective plane $\mathbb{RP}^1$ is formed by taking the quotient of $\mathbb{R}^2 \setminus \{0\}$ under the equivalence relation $x \sim \lambda x$ for all real numbers $\lambda \neq 0$. We now consider another example of a boundary action, i.e.,

$$\mathbb{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{RP}^1.$$

Here, $\mathbb{SL}_2(\mathbb{Z})$ acts on $\mathbb{RP}^1$ by left multiplication, i.e., $A \cdot x = Ax$, where $A \in \mathbb{SL}_2(\mathbb{Z})$ and $x = [x_1 : x_2] \in \mathbb{RP}^1$. It is not difficult to verify the following facts.

Proposition 2.19. *Suppose that $\mathbb{SL}_2(\mathbb{Z})$ acts on $\mathbb{RP}^1$ by left multiplication, i.e., $A \cdot x = Ax$, where $A \in \mathbb{SL}_2(\mathbb{Z})$ and $x = [x_1 : x_2] \in \mathbb{RP}^1$. Then the following hold true.*

1. For any upper triangular matrix $A \in \mathbb{SL}_2(\mathbb{Z})$, for any $[x_1 : x_2] \in \mathbb{RP}^1$, the sequence $A^n[x_1 : x_2] \to [1, 0]$ as $n \to \infty$.

2. *For each $A \in \mathbb{SL}_2(\mathbb{Z})$ with real eigenvalues, there exists $x_A \in \mathbb{RP}^1$ such that $Ax_A = x_A$ and $A^n y \to x_A$ for all $y \in \mathbb{RP}^1$ except possibly one point.*

3. *$O_x = \{A \cdot x : A \in SL_2(\mathbb{Z})\}$ is dense for any $x \in \mathbb{RP}^1$. Hence, the action $SL_2(\mathbb{Z}) \curvearrowright \mathbb{RP}^1$ is minimal.*

4. *Let ν be a probability measure on $X = \mathbb{RP}^2$. Let $y = [y_1, y_2] \in \mathbb{RP}^2$ be fixed. Then there exists a net $A_i \in \mathbb{SL}_2(\mathbb{Z})$ such that $A_i.\nu \to \delta_y$ in weak-* topology. In particular, the action $SL_2(\mathbb{Z}) \curvearrowright \mathbb{RP}^1$ is a boundary action.*

Using Hamana's theory of Γ-injective envelopes, Kalantar-Kennedy [22] gave the following characterization of C^*-simplicity, which paved the way for many important future works.

Theorem 2.20. *Let Γ be a discrete group. The following are equivalent:*

1. *Γ is C^*-simple.*

2. *Γ acts freely on its Furstenberg boundary $C(\partial_F \Gamma)$*

3. *there exists a topologically free Γ-boundary.*

However, the key insight in Power's proof of the C^*-simplicity is that the left regular representation of $\mathbb{F}_2$ satisfies a certain averaging property.

Definition 2.21. A discrete group Γ is said to have the Powers averaging property if the following holds: for every element a in the reduced C^*-algebra $C_r^*(\Gamma)$ and every $\epsilon > 0$ there are $s_1, \ldots, s_m \in \Gamma$ such that

$$\left\| \frac{1}{m} \sum_{j=1}^{m} \lambda_{s_j}(a - \tau_0(a))\lambda_{s_j^{-1}} \right\| < \epsilon,$$

where τ_0 is the canonical trace on $C_r^*(\Gamma)$.

It is easy to see that any group satisfying the Powers averaging property is C^*-simple. In fact, prior to the publication of [22] and [7], essentially the only method available to prove the C^*-simplicity of discrete groups was to show that the group Γ satisfies some variant of the Powers averaging property. It was shown in [19] and [26] independently that the C^*-simplicity of the group Γ is equivalent to the group having the Powers averaging property.

2.5 Simplicity of reduced crossed products

A unital C^*-algebra $\mathcal{A}$ is called simple if it doesn't have any non-trivial two sided closed ideals. It has long been recognized that the simplicity of the reduced crossed product $C(X) \rtimes_r \Gamma$ is related to the topological dynamics of the Γ-action on X (see, e.g., the work of Kawamura and Tomiyama [25], and the work of Archbold and Spielberg [5]). While the literature has a lot of papers on simplicity of C^*-algebras, a part of the motivation comes from a problem posed by de la Harpe and Skandalis. The Powers averaging property for the reduced crossed product $\mathcal{A} \rtimes_r \Gamma$ of the action of a Powers group Γ on a unital C^*-algebra $\mathcal{A}$, was proved by de la Harpe and Skandalis in [12] (Recently, Bryder and Kennedy [9] studied the ideal structure of (twisted) crossed products over C^*-simple groups. In particular, they showed that the reduced crossed product over a C^*-simple group has the Powers averaging property).

Definition 2.22. The reduced crossed product $\mathcal{A} \rtimes_r \Gamma$ is said to have the Powers averaging property if for every element a in the reduced crossed product $\mathcal{A} \rtimes_r \Gamma$ and every $\epsilon > 0$, there are $s_1, \ldots, s_m \in \Gamma$ such that

$$\left\| \frac{1}{m} \sum_{j=1}^{m} \lambda_{s_j} (a - \mathbb{E}(a)) \lambda_{s_j^{-1}} \right\| < \epsilon.$$

A unital Γ-C^*-algebra $\mathcal{A}$ is called Γ-simple if $\mathcal{A}$ does not have any non-trivial two sided closed Γ-invariant ideals. They used the above definition to prove the simplicity of $\mathcal{A} \rtimes_r \Gamma$ for a Powers group Γ and unital Γ-simple, Γ-C^*-algebra $\mathcal{A}$. The idea is to use this averaging on a non-zero element of an ideal to get it close enough to an invertible element of the reduced crossed product. We employ a variant of this technique in later chapters and we elaborate on this idea then.

Theorem 2.23. [12] *Suppose that $\mathcal{A}$ is a unital Γ-simple Γ-C^*-algebra. If Γ is a Powers group, then $\mathcal{A} \rtimes_r \Gamma$ is simple.*

Moreover, they asked if the above theorem holds true, when Powers group Γ is replaced by C^*-simple groups? And the answer turned out to be yes, which was established in [7].

Theorem 2.24. [7] *Suppose that $\mathcal{A}$ is a unital Γ-simple Γ-C^*-algebra. If Γ is C^*-simple, then $\mathcal{A} \rtimes_r \Gamma$ is simple.*

The idea of the proof for the above theorem is somehow inspired by the techniques in [5]. For an ideal I of $\mathcal{A} \rtimes_r \Gamma$, they show that the ideal J generated by I in $(\mathcal{A} \otimes C(\partial_F \Gamma)) \rtimes_r \Gamma$ is non-trivial. And, then the authors show that $J \cap \mathcal{A}$ is a non-trivial ideal of $\mathcal{A}$ which is impossible since $\mathcal{A}$ is Γ-simple. Hence, $\mathcal{A} \rtimes_r \Gamma$ must be simple. We do not elaborate on the details for it is not necessary to do so for our purposes. The interested reader is advised to look in [7] for details.

After all this, one may ask the following:

Question 2.25. *What can one say about the Γ-simplicity of the Γ-C^*-sub-algebras of the reduced crossed product $A \rtimes_{\alpha,r} \Gamma$, when A is Γ-simple and Γ is C^*-simple?*

The answer to this question makes up the content of the next chapter. Before we go on to the next chapter, we talk briefly about the works of Kawabe [23] which is a generalization of [22] to the situation of reduced crossed product $C(X) \rtimes_r \Gamma$ (see [27] for a generalization to the non-commutative setup) and Naghavi [30]. Kawabe ([23]) and Naghavi ([30]) independently, introduced the notion of generalized Furstenberg boundary for commutative C^*-algebras.

For every Γ-space X, we denote by $\tilde{X}$ the Gelfand spectrum of the Γ-injective envelope of $C(X)$ (see [22] or [30] for more on injective envelopes) i.e., $C(\tilde{X}) = I_\Gamma(C(X))$. We refer the reader to [23] for more details on these. Note that when X is a single point, we recover the Furstenberg boundary $\partial_F \Gamma$ via this process. Kawabe showed that the simplicity of the reduced crossed product $C(X) \rtimes_r \Gamma$ is equivalent to the action $\Gamma \curvearrowright \tilde{X}$ being free. This is a generalization of Theorem 2.20. We state the theorem for minimal Γ-spaces X for this is the situation we are most interested in. Kawabe has shown that the theorem is true in a much more general framework.

Theorem 2.26. [23] *Let X be a minimal Γ-space. The following are equivalent:*

1. *$C(X) \rtimes_r \Gamma$ is simple.*

2. *$C(\tilde{X}) \rtimes_r \Gamma$ is simple.*

3. *$\Gamma \curvearrowright \tilde{X}$ is (topologically) free.*

While the work of Kawabe [23] planted the seed for our work in [4] (of which we speak in Chapter 3), the proofs in [4] used the characterization given in [30]. Naghavi [30] gave a dynamical characterization of the generalised Furstenberg boundary. In particular, she gave the definition of a generalized boundary action and showed that the action $\Gamma \curvearrowright \partial_F(\Gamma, X)$ (which is $\tilde{X}$ in the notation of Kawabe) is a generalized boundary action, where X is a minimal Γ-space.

Definition 2.27. [17, P. 163] Let X be a Γ-space and $\varphi : Y \to X$ be an extension of X.

1. (Y, φ) is called a minimal extension if Y is minimal.

2. (Y, φ) is called a strongly proximal extension if the following holds: Let $\nu \in \text{Prob}(Y)$ be such that $\text{supp}(\nu) \subseteq \varphi^{-1}(x)$, for some $x \in X$. Then, there exists some $y \in Y$ such that $\delta_y \in \overline{\Gamma\nu}^{\text{weak}^*}$, i.e., there exists a net $s_i \in \Gamma$ such that $s_i\nu \xrightarrow{\text{weak}^*} \delta_y$ for some $y \in Y$.

Recall that a measure $\nu \in \text{Prob}(X)$ is called contractible if $\{\delta_x : x \in X\} \subset \overline{\Gamma\nu}^{\text{weak}^*}$. We say that (Y, φ) is an extension of X if $\varphi : Y \to X$ is a continuous surjective map along with the property that $\varphi(s.x) = s.\varphi(x)$ for all $x \in X$ and for all $s \in \Gamma$. Moreover, such an extension induces an injective $*$-homomorphism $\tilde{\varphi} : C(X) \to C(Y)$ defined by $\tilde{\varphi}(f) = f \circ \varphi$.

Theorem 2.28. [30, Theorem 3.2] *For a countable discrete group Γ, let X be a minimal Γ-space and (Y, φ) be an extension of X, inducing an extension $(C(Y), \tilde{\varphi})$ of $C(X)$. The following are equivalent:*

1. *$(C(Y), \tilde{\varphi})$ is a Γ-essential extension of $C(X)$.*

2. *Y is minimal and, for every $\nu \in Prob(Y)$, if the restriction of the Poisson map $P_\nu : C(Y) \to \ell^\infty(\Gamma)$ to $C(X)$ via $\tilde{\varphi}$ is isometric, then P_ν is isometric on $C(Y)$.*

3. *Y is minimal and for every $\nu \in Prob(Y)$, if the push forward of ν on X via φ is contractible, then ν is contractible.*

4. *(Y, φ) is a minimal strongly proximal extension of X.*

Definition 2.29. [30, Definition 3.3] We say that (Y, φ) is a (Γ, X)-boundary, if (Y, φ) satisfies any of the above equivalent conditions.

For a minimal Γ-space X, Naghavi shows that $I_\Gamma(C(X))$ is the maximal Γ-essential extension of $C(X)$. It also follows from [30, Theorem 3.2] that the spectrum of $I_\Gamma(C(X))$ is a (Γ, X)-boundary. We denote this Γ-space, which is unique up to homeomorphism, by $\partial_F(\Gamma, X)$ and write $I_\Gamma(C(X)) = C(\partial_F(\Gamma, X))$. It is shown in [30] that $\partial_F(\Gamma, X)$ is the universal (Γ, X)-boundary.

Part II

Our results

3 Simplicity of intermediate C^*-subalgebras associated with C^*-simple group actions

The man who moves a mountain begins by carrying away small stones.

Confucius

In this chapter, we consider the simplicity problem for intermediate C^*-subalgebras of crossed products of C^*-simple group actions, and more generally, the Γ-simplicity of their unital Γ-invariant C^*-subalgebras. The contents of this chapter are from [3].

3.1 The Problem

The ideal structure of the reduced crossed product has been explored by many (see, e.g., [10, 12, 13, 21, 24, 35, 38] and the references there in). A part of our motivation comes from [12], where the authors proved the following:

Theorem 3.1. [12, Proposition 10] *Suppose that $\mathcal{A}$ is a unital Γ-simple Γ-C^*-algebra. If Γ is a Powers group, then $\mathcal{A} \rtimes_r \Gamma$ is simple.*

Moreover, they left it as an open problem if the above theorem holds true, when the Powers group Γ is replaced by C^*-simple groups. The answer turns out to be yes, which was established in [7].

Theorem 3.2. [7, Theorem 1.8] *Suppose that $\mathcal{A}$ is a unital Γ-simple Γ-C^*-algebra. If Γ is C^*-simple, then $\mathcal{A} \rtimes_r \Gamma$ is simple.*

In this chapter, we seek an answer to the following question which is the first step towards understanding the structure of intermediate C^*-sub-algebras.

Question 3.3. *What can one say about the Γ-simplicity of the Γ-C^*-sub-algebras of the reduced crossed product $\mathcal{A} \rtimes_r \Gamma$, when $\mathcal{A}$ is Γ-simple and Γ is C^*-simple?*

We begin by observing that such a result is very far from being true in general. For example, let $\mathcal{A}$ be a non-trivial simple C^*-algebra and let $\Gamma \curvearrowright \mathcal{A}$ be the trivial action of a Powers group Γ. Then $\mathcal{A} \rtimes_r \Gamma = C_r^*(\Gamma) \otimes \mathcal{A}$ is simple. However, if $\mathcal{B}$ is a non-simple unital C^*-subalgebra of $\mathcal{A}$, then $C_r^*(\Gamma) \subset \mathcal{B} \rtimes_r \Gamma \subset \mathcal{A} \rtimes_r \Gamma$, and $\mathcal{B} \rtimes_r \Gamma = C_r^*(\Gamma) \otimes \mathcal{B}$ is not simple. The main reason that simplicity for invariant subalgebras fail in these situations is that, in general, Γ-simplicity does not pass to subalgebras.

However, for a unital commutative C^*-algebra $\mathcal{A} = C(X)$, it is well-known by Gelfand's theory that closed ideals in $\mathcal{A}$ are in bijection with closed subsets of X. So if $\mathcal{A} = C(X)$ is Γ-simple, then it follows from the above mentioned bijection that $\Gamma \curvearrowright X$ is minimal. Now, suppose that Γ is a C^*-simple group and $\mathcal{B} \subset \mathcal{A}$ is a Γ-invariant unital sub-algebra. Since $\mathcal{B}$ is of the form $C(Y)$ via a Γ-equivariant factor of X (i.e., there exists a continuous surjective map $\pi : X \to Y$) and since minimality passes to factors, it follows by [7, Theorem 7.1] that $\mathcal{B} \rtimes_r \Gamma$ is simple.

From the above discussion, we observe that if Γ is a C^*-simple group and $\Gamma \curvearrowright X$ is minimal, then any intermediate C^*-subalgebra $\mathcal{B}$ with $C_r^*(\Gamma) \subset \mathcal{B} \subset C(X) \rtimes_r \Gamma$ is simple if $\mathcal{B}$ is a crossed product itself. Hence, it is only natural to expect that Γ-simplicity should pass to every intermediate C^*-sub-algebra. In particular, we prove that the following holds true.

Theorem 3.4. [3, Theorem 1.3] *Let Γ be a countable discrete C^*-simple group, and let $\Gamma \curvearrowright X$ be a minimal action of Γ on a compact space X. Then, any unital Γ-invariant C^*-subalgebra of $C(X) \rtimes_r \Gamma$ is Γ-simple. In particular, any intermediate C^*-subalgebra $C_r^*(\Gamma) \subseteq \mathcal{B} \subseteq C(X) \rtimes_r \Gamma$ is simple.*

3.2 Simplicity of Γ-invariant subalgebras of $C(X) \rtimes_r \Gamma$

This section is devoted to the proof of Theorem 3.4. We begin by proving an estimation that will allow us to lift an averaging scheme from the reduced C^*-algebra to the reduced crossed product.

Let $\mathcal{A}$ be a unital Γ-C^*-algebra. For $a \in \mathcal{A}$ and a measure $\mu' \in \mathrm{Prob}(\Gamma)$, we denote $\mu' * a = \sum_{s\in\Gamma} \mu'(s) s^{-1} a$ for the convolution of a by μ'. Such convolution actions in the case of C^*-dynamical systems have been introduced and studied in [20], where applications to several rigidity problems in ergodic theory and operator algebras were given. We refer the reader to [20] for more details on these.

Fix a faithful $*$-representation $\pi : \mathcal{A} \to \mathbb{B}(H)$ of $\mathcal{A}$ into the space of bounded operators on the Hilbert space H. Denote by $\ell^2(\Gamma, H)$ the space of square summable H-valued functions on Γ. Recall the construction of $\mathcal{A} \rtimes_r \Gamma$ from Section 2.2.

Lemma 3.5. *Let Γ be a discrete group, let $\mu \in \mathrm{Prob}(\Gamma)$, and let $\mathcal{A}$ be a Γ-C^*-algebra. Then for any $t \in \Gamma$ and $a \in \mathcal{A}$ we have*

$$\left\| \mu * (\sigma(a)\tilde{\lambda}_t) \right\|_{\mathbb{B}(\ell^2(\Gamma,H))} \leq \|a\|_{\mathcal{A}} \|\mu * \lambda_t\|_{\mathbb{B}(\ell^2(\Gamma))} . \tag{1}$$

Proof. For $\xi \in \ell^2(\Gamma, H)$, observe that for each $t' \in \Gamma$ we have

$$\begin{aligned}
&\left([\mu * (\sigma(a)\tilde{\lambda}_t)](\xi)\right)(t') \\
&= \sum_{s\in\Gamma} \mu(s)[\tilde{\lambda}_s \sigma(a)\tilde{\lambda}_t \tilde{\lambda}_{s^{-1}}(\xi)](t') \\
&= \sum_{s\in\Gamma} \mu(s)[\sigma(sa)\tilde{\lambda}_{sts^{-1}}\xi](t') \\
&= \sum_{s\in\Gamma} \mu(s)\pi(t'^{-1}sa)[\xi(st^{-1}s^{-1}t')].
\end{aligned}$$

Define the function $\xi_1(t') = \|\xi(t')\|_H$, $t' \in \Gamma$. Then $\xi_1 \in \ell^2(\Gamma)$, and $\|\xi_1\|_{\ell^2(\Gamma)} = \|\xi\|_{\ell^2(\Gamma,H)}$. We have

$$\begin{aligned}
&\left\| [\mu * (\sigma(a)\tilde{\lambda}_t)](\xi) \right\|^2_{\ell^2(\Gamma,H)} \\
&= \sum_{t'\in\Gamma} \left\| \left([\mu * (\sigma(a)\tilde{\lambda}_t)](\xi) \right)(t') \right\|^2_H \\
&= \sum_{t'\in\Gamma} \left\| \sum_{s\in\Gamma} \mu(s)\pi(t'^{-1}sa)[\xi(st^{-1}s^{-1}t')] \right\|^2_H \\
&\leq \|a\|^2_{\mathcal{A}} \sum_{t'\in\Gamma} \left(\sum_{s\in\Gamma} \mu(s) \left\|\xi(st^{-1}s^{-1}t')\right\|_H \right)^2 \\
&= \|a\|^2_{\mathcal{A}} \left\| \sum_{s\in\Gamma} \mu(s)\lambda_{sts^{-1}}(\xi_1) \right\|^2_{\ell^2(\Gamma)} \\
&\leq \|a\|^2_{\mathcal{A}} \left\| \sum_{s\in\Gamma} \mu(s)\lambda_{sts^{-1}} \right\|^2_{\mathbb{B}(\ell^2(\Gamma))} \|\xi_1\|^2_{\ell^2(\Gamma)},
\end{aligned}$$

and since $\|\xi_1\|_{\ell^2(\Gamma)} = \|\xi\|_{\ell^2(\Gamma,H)}$, the inequality (1) follows. □

It follows, in particular, from Lemma 3.5 that if $C^*_r(\Gamma)$ has the Powers averaging property, then so does the reduced crossed product $\mathcal{A} \rtimes_r \Gamma$ for any action $\Gamma \curvearrowright \mathcal{A}$. Now, recall the following easy exercise from Functional Analysis.

Lemma 3.6. *Let $T \in B(H)$ be a linear operator. Suppose that $\|I - T\| < 1$. Then, T is invertible.*

We also record the following easy observation about a non-negative function $f \in C(X)$ and its convolution $\mu * f$ for a full support measure μ.

Lemma 3.7. *Let X be a minimal Γ-space. Choose $\mu \in Prob(\Gamma)$ with full support. Let $f \in C(X)$ be a non-negative non-zero function. Then, there exists a $\delta > 0$ such that $\mu * f > \delta$.*

Proof. Since X is compact, it is enough to show that $(\mu * f)(x) > 0$ for all $x \in X$. Suppose not. Then, there exists $x \in X$ such that $\mu * f(x) = 0$. Therefore,

$$\mu * f(x) = \sum_{s\in\Gamma} \mu(s) f(s^{-1}x) = 0$$

Since μ has full support, this implies that $f(s^{-1}x) = 0$ for all $s \in \Gamma$. From the minimality of $\Gamma \curvearrowright X$, it follows that $f(y) = 0$ for all $y \in X$. This is a contradiction. Hence, the claim holds. $\square$

Using the Powers averaging on $C(X) \rtimes_r \Gamma$, we show that every non-zero Γ-invariant closed ideal of Γ-invariant C^*-subalgebra of $C(X) \rtimes_r \Gamma$ must contain an invertible element.

Proof of Theorem 3.4. Let I be a non-zero Γ-invariant closed two sided ideal of a Γ-invariant C^*-subalgebra of $C(X) \rtimes_r \Gamma$. Let $a \in I$ and replacing a by a^*a and dividing by an appropriate scalar, we may assume that $a > 0$ and $\|a\| < 1$. Since $\mathbb{E}$ is faithful, we see that $\mathbb{E}(a) \geq 0$. Choose $\mu \in \mathrm{Prob}(\Gamma)$ such that μ has full support. From Lemma 3.7, it follows that there exists a $\delta > 0$ such that $\mathbb{E}(\mu * a) = \mu * \mathbb{E}(a) > \delta$. Since Γ is C^*-simple, it follows from Lemma 3.5 that $C(X) \rtimes_r \Gamma$ has the Powers averaging property. Hence, for $0 < \epsilon < 1$, we can find $s_1, s_2, \ldots, s_m \in \Gamma$ such that

$$\left\| \frac{1}{m} \sum_{j=1}^{m} \lambda_{s_j} \left(\mu * a - \mathbb{E}(\mu * a)\right) \lambda_{s_j^{-1}} \right\| < \delta\epsilon$$

which further implies that

$$\left\| \frac{1}{m\delta} \sum_{j=1}^{m} \lambda_{s_j} \left(\mu * a - \mathbb{E}(\mu * a)\right) \lambda_{s_j^{-1}} \right\| < \epsilon$$

Letting $F = \frac{1}{m\delta} \sum_{j=1}^{m} \lambda_{s_j} \mathbb{E}(\mu * a) \lambda_{s_j^{-1}} = \frac{1}{m\delta} \sum_{j=1}^{m} s_j . \mathbb{E}(\mu * a)$, we see that $F(x) \geq 1$ for $x \in X$, making it invertible. Therefore, $F^{-1}(x) \leq 1$ for all $x \in X$ and hence, $\|F^{-1}\| < 1$. Letting $\tilde{a} = \frac{1}{m\delta} \sum_{j=1}^{m} \lambda_{s_j} (\mu * a) \lambda_{s_j^{-1}}$, we see that $\|\tilde{a} - F\| < \epsilon < 1$. Also, observe that $\|\tilde{a}\| \leq \|a\| < 1$. Therefore,

$$\left\|\tilde{a}F^{-1} - I\right\| = \left\|(\tilde{a} - F)F^{-1}\right\| \leq \|\tilde{a} - F\| \|F^{-1}\| < \epsilon < 1$$

Moreover, $\left\|\tilde{a}F^{-1}\right\| < 1$. Lemma 3.6 tells us that $\tilde{a}F^{-1}$ is invertible making $\tilde{a}$ an invertible element. Since I is Γ-invariant, $\tilde{a} \in I$. The claim follows. $\square$

We remark in passing that the proof of [3, Theorem 1.3] being inspired by the work in [20] uses the notion of stationary states and is completely different from the above mentioned proof.

4 Intermediate C^*-subalgebras as crossed products for C^*-simple group actions

The only journey is the one within.

Rainer Maria Rilke

In this chapter, for a large class of C^*-simple group actions $\Gamma \curvearrowright \mathcal{A}$, we give a complete description of intermediate C^*-subalgebras $\mathcal{B}$ of the form $C^*_\lambda(\Gamma) \subseteq \mathcal{B} \subseteq \mathcal{A} \rtimes_r \Gamma$. We also give a complete description of intermediate vNa's $\mathcal{N}$ of the form $L(\Gamma) \subseteq \mathcal{N} \subseteq \mathcal{M} \rtimes \Gamma$ for C^*-simple group actions $\Gamma \curvearrowright \mathcal{M}$. Namely, we show that if the kernel of the action contains a subgroup with respect to which, Γ has the Powers averaging property, all the intermediate C^*-subalgebras or von Neumann algebras are of the form of crossed products. The idea of the proof is very simple: we show that for every intermediate C^*-algebra $\mathcal{B}$, $\mathbb{E}(\mathcal{B}) \subset \mathcal{B}$. For this, we use the Powers type averaging at the crossed product level to show $\mathbb{E}(\mathcal{B}) \subset \mathcal{B}$ for every intermediate algebra $\mathcal{B}$ from which we conclude the result. The averaging in general gets us close to a convex combination of $E(a)$ by elements from the reduced C^*-algebra but our assumption ensures that we can average over the kernel of the action and hence we get as close to $E(a)$ as we want to be. The whole point is that, the elements which average at the reduced C^*-algebra level, also average at the crossed product level. The contents of this chapter are from [2].

4.1 Conditional expectation and property AP

In order to give a complete description of intermediate C^*-sub-algebras, we need an extra continuity assumption on the group Γ.

Definition 4.1. [Property AP] A discrete group Γ is said to have the approximation property (AP) if there exists a net $(\phi_i)_{i\in I}$ of finitely supported complex valued functions on Γ such that $m_{\phi_i} \otimes \mathrm{id}_{\mathbb{B}}$ converges to the identity map in the pointwise norm topology. Here, for a finitely supported function ϕ on Γ, denote by $m_\phi : C^*_\lambda(\Gamma) \to C^*_\lambda(\Gamma)$ the completely bounded map denoted by the formula $m_\phi\left(\sum_s a_s\lambda_s\right) = \sum_s a_s\phi(s)\lambda_s$. We refer the reader to [8, Chapter 12] for more details on Property AP.

Lemma 4.2. [36, Proposition 3.4] *Let Γ be a group with the AP. Let $\mathcal{A}$ be a $\Gamma - C^*$-algebra and let X be a closed subspace of A. Assume that an element $x \in A \rtimes_{\alpha,r} \Gamma$ satisfies $E_g(x) \in X$ for all $g \in \Gamma$. Then x is contained in the closed subspace*

$$X \rtimes_{\alpha,r} \Gamma := \overline{span}\{x(1 \otimes \lambda_g) : x \in X, g \in \Gamma\}$$

There is an abundance of groups with Property AP. Ozawa [33] shows that all weakly amenable groups have Property AP and all hyperbolic groups are weakly amenable.

4.2 Notion of plump subgroups

Definition. A non-trivial $\Lambda \leq \Gamma$ is called a *plump subgroup*, if the following holds: For every $\epsilon > 0$ and finite subset $F \subset \Gamma \setminus \{e\}$, there exist $s_1, s_2, \ldots, s_m \in \Lambda$ such that

$$\left\| \frac{1}{m} \sum_{j=1}^{m} \lambda_{s_j} \lambda_t \lambda_{s_j^{-1}} \right\| < \epsilon, \quad \forall t \in F.$$

Note that e is the identity element of the group Γ.

First, we give conditions under which a subgroup Λ is plump in Γ.

Lemma 4.3. *Let Λ be a subgroup of Γ. Suppose that there is a free action of Γ on a compact Hausdorff space X such that the action of Γ restricted to Λ is strongly proximal. Then Λ is plump in Γ.*

Proof. First, proceeding exactly as in the proof of [19, Theorem 4.5], one sees that $\tau_0 \in \overline{\{s.\varphi : s \in \Lambda\}}^{\text{weak}*}$ for each state φ on $C^*_\lambda(\Gamma)$. Again, by the same theorem, for any finite collection $t_1, t_2, \ldots, t_n \in \Gamma - \{e\}$ and $\epsilon > 0$, there exist $s_1, s_2, \ldots, s_m \in \Lambda$ such that

$$\left\| \frac{1}{m} \sum_{j=1}^{m} \lambda_{s_j} \lambda_{t_i} \lambda_{s_j^{-1}} \right\| < \epsilon, \text{ for each } i = 1, 2, \ldots, n\ .$$

□

Proposition 4.4 ([7, Lemma 5.3]). *Suppose that Λ is a normal C^*-simple subgroup of Γ with trivial centralizer inside Γ. Then, $\Lambda \curvearrowright \partial_F\Lambda$ extends to a free action of Γ on $\partial_F\Lambda$. In particular, Λ is plump in Γ.*

For a normal subgroup $\Lambda \leq \Gamma$, it turns out that Λ is plump in Γ if and only if $C_\Gamma(\Lambda) = \{e\}$. Ursu in [41] shows the converse by proving the existence of a type of relative Furstenberg boundary with respect to arbitrary (not necessarily normal) subgroups. We give a very short proof of this via different techniques.

Proposition 4.5. [41, Theorem 1.3] *Suppose that Λ is a normal C^*-simple subgroup of Γ. Then, Λ is plump in Γ if and only if $C_\Gamma(\Lambda) = \{e\}$.*

Proof. If $C_\Gamma(\Lambda) = \{e\}$, then it follows from Proposition 4.4 that Λ is plump in Γ. Now, suppose that Λ is plump in Γ. If $e \neq s \in C_\Gamma(N)$, then for $0 < \epsilon < 1$, there are $t_1, t_2, \dots, t_m \in N$ such that

$$\left\| \frac{1}{m} \sum_{j=1}^{m} \lambda_{t_j s t_j^{-1}} \right\| < \epsilon < 1$$

But $s \in C_\Gamma(N)$ implies that $t_j s = s t_j$ for all $j = 1, 2, \dots, m$. Hence, we obtain that

$$1 = \|\lambda_s\| < \epsilon < 1$$

This is a contradiction. Therefore, $C_\Gamma(N) = \{e\}$. □

Below, we give various conditions for which a subgroup $\Lambda \leq \Gamma$ is plump.

Corollary 4.6. *Let Γ be a C^*-simple group and let $\Lambda \leq \Gamma$ be a non-trivial normal subgroup. If Λ contains an element s with amenable centralizer $C_\Gamma(s)$ in Γ, then Λ is plump in Γ.*

Proof. In light of Proposition 4.4, it is enough to show that $C_\Gamma(\Lambda) = \{e\}$. Since Λ is normal, so is $C_\Gamma(\Lambda)$. Moreover, $C_\Gamma(\Lambda) \subset C_\Gamma(s)$ for a non-trivial $s \in \Lambda$ is amenable by the assumption.

Since Γ is C^*-simple, it does not contain any non-trivial normal amenable subgroups. Hence, $C_\Gamma(\Lambda) = \{e\}$. □

It turns out that finite index subgroups of C^*-simple groups are always plump.

Corollary 4.7. *Let Γ be a C^*-simple group and let Λ be a finite index subgroup of Γ. Then Λ is plump in Γ.*

Proof. Since Γ is C^*-simple, the action $\Gamma \curvearrowright \partial_F\Gamma$ is free [7, Theorem 1.1]. It follows from [18, Chapter-II, Lemma 3.2] that this action restricted to Λ is strongly proximal. Hence, by Proposition 4.4, Λ is plump in Γ. □

The above results allow us to use certain free boundary actions to conclude plumpness of subgroups. But, in practice, many natural examples of boundary actions (e.g., $\mathbb{F}_n \curvearrowright \partial\mathbb{F}_n$) are only topologically free. Below, we prove some results which provide us ways to conclude plumpness of subgroups from existence of certain topologically free boundary actions.

Recall that an action $\Gamma \curvearrowright X$ is topologically free if $X \setminus X^s$ is dense in X for every non-trivial element $s \in \Gamma$, where $X^s = \{x \in X : sx = x\}$.

Lemma 4.8. *Let Γ be a countable discrete group, let Λ be a subgroup of Γ. Suppose that there exists an action $\Gamma \curvearrowright X$, where X is a compact Hausdorff space, such that for each non-trivial element $s \in \Gamma$, the set X^s of fixed points of s, is countable. Further suppose that the action restricted to Λ is strongly proximal and that X does not contain any Λ-fixed point. Then, Λ is plump in Γ.*

Proof. Let φ be a state on $C^*_\lambda(\Gamma)$. Extend φ to a state $\tilde{\varphi}$ on $C(X) \rtimes_r \Gamma$ and let $\tilde{\varphi}|_{C(X)} = d\nu$, where $\nu \in \mathrm{Prob}(\Gamma)$. Since the action restricted to Λ is strongly proximal, there are $s_i \in \Lambda$ such that $s_i\nu \to \delta_{x_0}$ in weak*-topology, for some $x_0 \in X$. Now, we claim that $\overline{\Lambda x_0}$ is an uncountable set. Let Y be a minimal Λ-component of $\overline{\Lambda x_0}$. If $\overline{\Lambda x_0}$ were a countable set, then Y would be a finite set. Since the action restricted to Λ is strongly proximal, we must have that Y is a singleton and hence a Λ-fixed point. This shows that $\overline{\Lambda x_0}$ is an uncountable set. Now, since $\cup_{s\neq e, s\in\Gamma} X^s$ is countable, we can find $y_0 \in \overline{\Lambda x_0} \subset \overline{\Lambda \nu}$ with trivial stabilizer. Now, it follows from the proof of

[19, Theorem 4.5] that $\tau_0 \in \overline{\{s\varphi : s \in \Lambda\}}^{\text{weak}^*}$. Then, by the same theorem, for any finite collection $t_1, t_2, \ldots, t_n \in \Gamma - \{e\}$ and $\epsilon > 0$, there exist $s_1, s_2, \ldots, s_m \in \Lambda$ such that

$$\left\| \frac{1}{m} \sum_{j=1}^{m} \lambda_{s_j} \lambda_{t_i} \lambda_{s_j^{-1}} \right\| < \epsilon, \text{ for each } i = 1, 2, \ldots, n\ .$$

$\square$

4.3 Examples of plump subgroups

Well, you may say "what good are conditions if they don't give examples"? Our reply is an old adage "Hold acylindrically hyperbolic groups". We briefly recall the notion of acylindrically hyperbolic groups and refer the reader to [31] for more details. An action $\Gamma \curvearrowright (X, d)$ on a metrizable space (X, d) is called acylindrical if for every $\epsilon > 0$ there are $\delta, N > 0$ such that for every x,$y \in X$ with $d(x, y) \geq \delta$, the elements $g \in \Gamma$ satisfying $d(x, gx) \leq \epsilon$ and $d(y, gy) \leq \epsilon$ are atmost N in number, i.e., for every $x, y \in X$ satisfying $d(x, y) \geq \delta$,

$$|\{g \in \Gamma : d(x, gx) \leq \epsilon \text{ and } d(y, gy) \leq \epsilon\}| \leq N.$$

A group Γ is called acylindrically hyperbolic if admits a non-elementary acylindrical action on a hyperbolic space. The examples of acylindrically hyperbolic groups include all non-(virtually) cyclic groups hyperbolic relative to proper subgroups, $\mathrm{Out}(F_n)$ for $n > 1$, all but finitely many mapping class groups, non-(virtually cyclic) groups acting properly on proper CAT(0)-spaces and containing rank one elements and many others (see, e.g., [29] and the references therein). We also give the definition of convergence action and classification of some elements there knowing that they will come in handy later. An action $\Gamma \curvearrowright X$ is called a convergence action (in this case, Γ is called a convergence group), if for every infinite sequence of distinct elements $\gamma_n \in \Gamma$, there exist a subsequence γ_{n_k} and points $a, b \in X$ such that $\gamma_{n_k}|_{X \setminus \{a\}}$ converge uniformly on compact subsets to b. A non-torsion element $s \in \Gamma$ is called loxodromic if s has exactly two fixed points and is called parabolic if s fixes exactly one point. A subgroup $\Lambda \leq \Gamma$ is elementary if it is finite, or preserves setwise a nonempty subset of X with at most two elements, and called non-elementary otherwise. We refer the reader to [6, 14, 40] for more details on these.

It turns out that every normal subgroup of a C^*-simple acylindrically hyperbolic group is plump. We thank Ionut Chifan for directing us to acylindrically hyperbolic groups. Recall that a subgroup $\Lambda \leq \Gamma$ is called s-normal in Γ if for every $t \in \Gamma$ one has

$$|\Lambda \cap t^{-1}\Lambda t| = \infty.$$

Proposition 4.9. *Let N be a normal subgroup of a acylindrically hyperbolic group Γ which is C^*-simple. Then, N is plump in Γ.*

Proof. In the light of Proposition 4.4, it is enough to show that $C_\Gamma(N) = \{e\}$. First, observe that N is infinite. Moreover, it follows from [29, Theorem 3.7] that every infinite normal subgroup is s-normal. Since Γ is acylindrically hyperbolic, it admits a non-elementary acylindrical action on a hyperbolic space S. By [31, Lemma 7.1], we get that the action $N \curvearrowright S$ is non-elementary. Again, by [31, Theorem 1.2], we see that N contains at least one loxodromic element, say n. Hence, by [31, Corollary 6.9], $C_\Gamma(n)$ is virtually cyclic, hence amenable. Therefore, $C_\Gamma(N)$ is a normal amenable subgroup of Γ. Since Γ has trivial amenable radical, $C_\Gamma(N) = \{e\}$. This concludes the proof. $\square$

We show that there are many plump subgroups of Convergence groups.

Proposition 4.10. *Let Γ be a non-elementary torsion-free convergence group. Then every non-elementary subgroup $\Lambda \leq \Gamma$ is plump in Γ.*

Proof. Let $\Gamma \curvearrowright X$ be a convergence action. Since Γ is torsion-free, every element in Γ is either parabolic or loxodromic ([40, Theorem 2B]). Thus, for each non-trivial element $s \in \Gamma$, $|X^s| \leq 2$. Since Λ is a non-elementary subgroup of Γ, one sees that the action restricted to Λ is strongly proximal (see, e.g., [32, Example 2]). Since Λ is non-elementary, it follows from [40, Theorem 2S] that $\overline{\Lambda x}$ is non-trivial for every $x \in X$, i.e., $\overline{\Lambda x}$ is not a single point. The claim now follows from Lemma 4.8. $\square$

4.4 Intermediate C^*-subalgebras as crossed products

With enough examples in hand, we proceed to give examples of inclusion $C^*_\lambda(\Gamma) \subset \mathcal{A} \rtimes_r \Gamma$ for which every intermediate C^*-sub-algebra is a crossed product.

Theorem 4.11. *Let Γ be a discrete group with the approximation property(AP), let $\mathcal{A}$ be a unital Γ-C^*-algebra. Suppose that the kernel of the action $\Gamma \curvearrowright \mathcal{A}$ contains a plump subgroup of Γ. Then, every intermediate C^*-subalgebra $\mathcal{B}$, $C^*_\lambda(\Gamma) \subseteq \mathcal{B} \subseteq \mathcal{A} \rtimes_r \Gamma$, is of the form $\mathcal{A}_1 \rtimes_r \Gamma$, for some unital Γ-C^*-subalgebra $\mathcal{A}_1$ of $\mathcal{A}$.*

Proof. Let $\mathcal{A}$ be a unital Γ-C^*-algebra and let $\mathcal{B}$ an intermediate C^*-subalgebra of the form $C^*_\lambda(\Gamma) \subseteq \mathcal{B} \subseteq \mathcal{A} \rtimes_r \Gamma$. Suppose that Λ is a plump subgroup of Γ such that Λ is contained in the kernel of the action $\Gamma \curvearrowright \mathcal{A}$. Fix $b \in \mathcal{B}$. Let $\epsilon > 0$. Then, there are $t_1, t_2, \ldots, t_n \in \Gamma \setminus \{e\}$ such that

$$\left\| b - \left(\sum_{i=1}^{n} a_{t_i} \lambda_{t_i} + \mathbb{E}(b) \right) \right\| < \epsilon.$$

Let $M = \max_{1 \le i \le n} \|a_{t_i}\|_{\mathcal{A}}$. Since Λ is a plump subgroup of Γ, there exist $s_1, s_2, \ldots, s_m \in \Lambda$ such that

$$\left\| \frac{1}{m} \sum_{j=1}^{m} \lambda_{s_j} \lambda_t \lambda_{s_j^{-1}} \right\| < \frac{\epsilon}{nM}, \quad \forall i = 1, 2, \ldots, n.$$

By [3, Lemma 2.1], it follows that

$$\begin{aligned} &\left\| \frac{1}{m} \sum_{j=1}^{m} \lambda_{s_j} \left(\sum_{i=1}^{n} a_{t_i} \lambda_{t_i} \right) \lambda_{s_j^{-1}} \right\| \\ &\le \sum_{i=1}^{n} \|a_{t_i}\|_{\mathcal{A}} \left\| \frac{1}{m} \sum_{j=1}^{m} \lambda_{s_j} \lambda_t \lambda_{s_j^{-1}} \right\| < \epsilon. \end{aligned}$$

Now,

$$\begin{aligned}
&\left\| \frac{1}{m} \sum_{j=1}^{m} \lambda_{s_j} (b - \mathbb{E}(b)) \lambda_{s_j^{-1}} \right\|_{\mathbb{B}(l^2(\Gamma,\mathcal{H}))} \\
&\leq \left\| \frac{1}{m} \sum_{j=1}^{m} \lambda_{s_j} \left(b - \left(\sum_{i=1}^{n} a_{t_i} \lambda_{t_i} + \mathbb{E}(b) \right) \right) \lambda_{s_j^{-1}} \right\| \\
&+ \left\| \frac{1}{m} \sum_{j=1}^{m} \lambda_{s_j} \left(\sum_{i=1}^{n} a_{t_i} \lambda_{t_i} \right) \lambda_{s_j^{-1}} \right\| < 2\epsilon.
\end{aligned}$$

Since Λ acts trivially on $\mathcal{A}$, we get that

$$\begin{aligned}
&\left\| \frac{1}{m} \sum_{j=1}^{m} \lambda_{s_j} b \lambda_{s_j^{-1}} - \mathbb{E}(b) \right\| \\
&= \left\| \frac{1}{m} \sum_{j=1}^{m} \lambda_{s_j} \left(b - \mathbb{E}(b) \right) \lambda_{s_j^{-1}} \right\| < 2\epsilon
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, this shows that $\mathbb{E}(\mathcal{B}) \subset \mathcal{B}$. By [36, Proposition 3.4], it follows that $\mathcal{B} = \mathbb{E}(\mathcal{B}) \rtimes_r \Gamma$. $\square$

Corollary 4.12. *Let Γ be a C^*-simple group such that the centralizer $C_\Gamma(a)$ of any non-trivial element $a \in \Gamma$ is amenable. Then for any non-faithful action $\Gamma \curvearrowright \mathcal{A}$, every intermediate C^*-subalgebra $\mathcal{B}$, $C^*_\lambda(\Gamma) \subseteq \mathcal{B} \subseteq \mathcal{A} \rtimes_r \Gamma$, is of the form $E(\mathcal{B}) \rtimes_r \Gamma$.*

Proof. Let Λ be the kernel of the action $\Gamma \curvearrowright \mathcal{A}$. Note that Λ is non-trivial by the assumption. The claim now follows from Corollary 4.6. $\square$

It turns out that every hyperbolic group with trivial amenable radical is an ideal candidate for such examples.

Theorem 4.13. *Let Γ be a hyperbolic group with trivial amenable radical. For any non-faithful action $\Gamma \curvearrowright \mathcal{A}$, every intermediate C^*-subalgebra $\mathcal{B}$, $C^*_\lambda(\Gamma) \subseteq \mathcal{B} \subseteq A \rtimes_r \Gamma$, is of the form $\mathcal{A}_1 \rtimes_r \Gamma$, where $\mathcal{A}_1$ is a Γ-C^*-subalgebra of $\mathcal{A}$.*

Proposition 4.10 together with Theorem 4.11 imply Theorem 4.13 for all torsion-free hyperbolic groups. Recall that all non-elementary hyperbolic groups with trivial amenable radical are C^*-simple (see, e.g., [11]). We give an alternative proof below which applies to all hyperbolic groups.

Proof of Theorem 4.13. Let Γ be a hyperbolic group with trivial amenable radical, let $\Gamma \curvearrowright \mathcal{A}$ be a non-faithful action. Let Λ be the kernel of the action $\Gamma \curvearrowright \mathcal{A}$. Since Λ is non-amenable, therefore non-elementary, it contains a non-torsion element s. We will show that the centraliser $C_\Gamma(s)$, of s in Γ is amenable, which will imply the theorem by Corollary 4.12. Let $x_s^+, x_s^- \in \partial\Gamma$ be the points of attraction and repulsion of s, i.e., x_s^+ and x_s^- are fixed by s, and $s^n x \xrightarrow{n\to\infty} x_s^+$ for all $x \in \partial\Gamma \setminus \{x_s^-\}$. We claim that $C_\Gamma(s)$ leaves the set $\{x_s^+, x_s^-\}$ invariant. To see this, observe that for any element $t \in C_\Gamma(s)$, $tx_s^\pm = ts^n x_s^\pm = s^n t x_s^\pm$. Letting $n \to \infty$, we get that $tx_s^\pm = x_s^\pm$. Therefore, the kernel of the homomorphism from $C_\Gamma(s)$ to $\mathrm{Sym}(\{x_s^+, x_s^-\})$ has index ≤ 2. Let's denote the kernel of this homomorphism by K. Since $\Gamma \curvearrowright \partial\Gamma$ is topologically amenable [1], the stabilizer $\Gamma_{x_s^+}$ is amenable, hence $K \subset \Gamma_{x_s^+}$ is amenable. Therefore, $C_\Gamma(s)$ is amenable. This completes the proof. □

While all the above examples deal with an arbitrary unital C^*-algebra $\mathcal{A}$ (for which a complete classification of intermediate C^*-algebras problem seems very remote at this moment but I would say we are on our way), it is possible to give a complete description of intermediate C^*-subalgebras $\mathcal{B}$ of the form $C^*_\lambda(\Gamma) \subseteq \mathcal{B} \subseteq C(X) \rtimes_r \Gamma$, where X is a finite Γ-space.

Proposition 4.14. *Suppose that Γ is a C^*-simple group acting on a finite space X. Assume that Γ has property-AP. Then, intermediate C^*-subalgebras $\mathcal{B}$ with $C^*_\lambda(\Gamma) \subseteq \mathcal{B} \subseteq C(X) \rtimes_r \Gamma$ is of the form $\mathbb{E}(\mathcal{B}) \rtimes_r \Gamma$.*

Proof. We show that the $\mathrm{Ker}(\Gamma \curvearrowright X)$ has finite index in Γ, hence plump in Γ by Theorem 4.11. For a C^*-simple group Γ acting on a finite space X, first observe that Γ can't be finite. Since the group action leads to a homomorphism $\varphi : \Gamma \to S_{\{|X|\}}$, the homomorphism can't be injective for that would imply that Γ is finite. So, it has a non-trivial kernel which is not finite. Now, observe that $|G/N| < \infty$, where N=Ker(φ). Therefore, N has finite index in Γ, hence plump (Corollary 4.7). □

All these examples deal with non-faithful actions. And then, one begins to wonder if there are examples of faithful actions for which we can completely classify the intermediate C^*-subalgebras. In passing, we remark that there is a class of such examples. Together with Yongle Jiang, we were able to find such a class. We mention it below and refer the reader to [2] for details of the proof and definitions.

Proposition 4.15. *Let Γ be a residually finite, C^*-simple group. If the action $\Gamma \curvearrowright X$ is a free exact Γ-odometer, then every intermediate C^*-algebra $\mathcal{B}$, $C^*_\lambda(\Gamma) \subseteq \mathcal{B} \subseteq C(X) \rtimes_r \Gamma$, is of the form $\mathcal{A}_1 \rtimes_r \Gamma$, where $\mathcal{A}_1$ is a Γ-C^*-subalgebra of $\mathcal{A}$.*

4.5 Intermediate von Neumann-subalgebras

It is possible to give examples of inclusions $L(\Gamma) \subset \mathcal{M} \rtimes_r \Gamma$ for a Γ-von Neumann algebra $\mathcal{M}$ for which every intermediate von Neumann-subalgebra $L(\Gamma) \subset \mathcal{N} \subset \mathcal{M} \rtimes_r \Gamma$ is a crossed product von Neumann algebra. We thank Yongle Jiang for pointing out to us that the same proof as in Theorem 4.11 works in this setting as well.

Theorem 4.16. *Let Γ be a discrete group, $\mathcal{M}$ be a Γ-von Neumann-algebra with a separable predual. Suppose that Λ is a plump subgroup of Γ such that Λ is contained in the kernel of the action $\Gamma \curvearrowright \mathcal{M}$. Then every intermediate von Neumann-subalgebra $\mathcal{N}$, $L(\Gamma) \subseteq \mathcal{N} \subseteq \mathcal{M} \rtimes \Gamma$ is of the form $\mathcal{M}_1 \rtimes \Gamma$, where $\mathcal{M}_1$ is a Γ-von Neumann subalgebra of $\mathcal{M}$.*

Proof. Let φ be a faithful normal state on $\mathcal{M}$ and let $\tilde{\varphi}(a) = \varphi(\mathbb{E}(a)), a \in \mathcal{M} \rtimes \Gamma$, where $\mathbb{E}$ is the canonical conditional expectation from $\mathcal{M} \rtimes \Gamma$ onto $\mathcal{M}$. Then $\tilde{\varphi}$ is a faithful normal state on $\mathcal{M} \rtimes \Gamma$. Consider the $\|.\|_2$-norm on $\mathcal{M} \rtimes \Gamma$ associated to $\tilde{\varphi}$, defined by

$$\|a\|_2 := \sqrt{\tilde{\varphi}(a^*a)} \quad \text{for } a \in \mathcal{M} \rtimes \Gamma.$$

Let $b \in \mathcal{N}$ and let $\epsilon > 0$ be given. Then there are $t_1, t_2, \dots, t_n \in \Gamma \setminus e$ such that

$$\left\| b - \left(\sum_{i=1}^{n} a_{t_i} \lambda_{t_i} + a_e \right) \right\|_2 < \frac{\epsilon}{2}.$$

Since $\tilde{\varphi}$ is $\mathbb{E}$-invariant, $\mathbb{E}$ is continuous with respect to the $\|.\|_2$-norm and hence by the triangle inequality, we see that

$$\left\| b - \left(\sum_{i=1}^{n} a_{t_i} \lambda_{t_i} + \mathbb{E}(b) \right) \right\|_2 < \epsilon.$$

Since the $\|.\|_2$ is dominated by the operator norm, by proceeding exactly as in the proof of Theorem 4.11, we see that $\mathbb{E}(b) \in \mathcal{N}$. By [37, Corollary 3.4], the proof is complete. $\square$

Since we do not need the extra assumption of Property AP on the side of von Neumann algebras, we are able to give a complete classification of intermediate von Neumann algebras for non-faithful actions of C^*-simple acylindrically hyperbolic groups. Recall that every non-elementary subgroup of a word-hyperbolic group is acylindrically hyperbolic. Similarly, every non-elementary relatively hyperbolic group is acylindrically hyperbolic. The mapping class group $MCG(S_{g,p})$ of a connected oriented surface of genus $g \geq 0$ with $p \geq 0$ punctures is acylindrically hyperbolic. For $n \geq 2$ the group $\mathrm{Out}(F_n)$ is acylindrically hyperbolic. So, it includes all the C^*-simple groups for which we have shown that every normal subgroup is plump.

Theorem 4.17. *Let Γ be a C^*-simple acylindrically hyperbolic group. Let $\mathcal{M}$ be a Γ-von Neumann algebra with separable predual. For any non-faithful action $\Gamma \curvearrowright \mathcal{M}$, every intermediate von Neumann-subalgebra $\mathcal{N}$, $L(\Gamma) \subseteq \mathcal{N} \subseteq \mathcal{M} \rtimes \Gamma$, is of the form $\mathcal{M}_1 \rtimes \Gamma$, where $\mathcal{M}_1$ is a Γ-vN-subalgebra of $\mathcal{M}$.*

Proof. Using Proposition 4.9, we see that $\mathrm{Ker}(\Gamma \curvearrowright \mathcal{M})$ is a plump subgroup of Γ. The claim now follows from Theorem 4.16. $\square$

5 Generalized Powers averaging for crossed products

> There are others. There will be others. Other heroes, other heroines. Other prophecies to fulfill, other adversaries to despise. There will be stories told and forgotten, and reinvented anew until one day, perhaps, the oldest are remembered, and the beginning may end, and the ending begin.
>
> Banewreaker

The Powers averaging property for the reduced crossed product $\mathcal{A} \rtimes_r \Gamma$ of the action of a Powers group Γ on a unital C^*-algebra $\mathcal{A}$, was proved by de la Harpe and Skandalis in [12]. In [7], it was established that reduced crossed product over a C^*-simple group is simple whenever the underlying C^*-algebra has no Γ-invariant closed ideals. Bryder and Kennedy [9] studied the ideal structure of (twisted) crossed products over C^*-simple groups. In particular, they showed that the reduced crossed product over a C^*-simple group has the Powers averaging property (Definition 2.22).

A similar averaging was shown in [3] except the difference was that in [3], it was shown that the elements which average at the level of $C^*_\lambda(\Gamma)$ lift upstairs to average at the level of $\mathcal{A} \rtimes_r \Gamma$.

5.1 Need for Powers averaging

It is natural to ask if the simplicity of the reduced crossed product $C(X) \rtimes_r \Gamma$ is equivalent to some form of the Powers averaging. It is not difficult to see that an averaging of the form definition 2.22 would imply that the group is C^*-simple, thereby restricting us to the purview of C^*-simple groups. It is also futile to expect that we can average by elements from $C(X)$ for non-free actions as the following proposition demonstrates.

Recall that we have a canonical conditional expectation $\mathbb{E}_x : C(X) \rtimes_r \Gamma \to C^*_\lambda(\Gamma_x)$ defined by

$$\mathbb{E}_x(f\lambda_s) = f(x)\mathbb{E}_{\Gamma_x}(\lambda_s)$$

Note that for any subgroup $\Lambda \leq \Gamma$, $\mathbb{E}_\Lambda : C^*_\lambda(\Gamma) \to C^*_\lambda(\Lambda)$ is the canonical conditional expectation defined by sending every element $s \in \Lambda$ to itself and every other element to 0. Also recall that a unit character τ on a group Γ is a map $\tau : \Gamma \to \mathbb{C}$ which sends every group element $s \to 1$. While it may not always extend to a continuous map from $C^*_\lambda(\Gamma) \to \mathbb{C}$, it does so if and only if Γ is amenable (see, e.g., [8, Theorem 2.6.8]).

Proposition 5.1. *Suppose that X is a Γ-space. The following are equivalent:*

1. *The action $\Gamma \curvearrowright X$ is free.*

2. *For every $a \in C(X) \rtimes_r \Gamma$ and $\epsilon > 0$, we can find unitaries $u_1, u_2, \dots, u_n$ of $C(X)$ such that*

$$\left\| \frac{1}{n} \sum_{i=1}^{n} u_i a u_i^* - \mathbb{E}(a) \right\| < \epsilon.$$

3. *Every $C(X)$-central state τ on $C(X) \rtimes_r \Gamma$ is of the form $\tau \circ \mathbb{E}$.*

4. *The inclusion $C(X) \subset C(X) \rtimes_r \Gamma$ has pure extension property, i.e., every pure state on $C(X)$ extends uniquely to a state on $C(X) \rtimes_r \Gamma$.*

Proof. ((1) $\implies$ (2)): [39, Proposition 11.1.9].

((2) $\implies$ (3)): Let τ be a $C(X)$-central state. Let $a \in C(X) \rtimes_r \Gamma$ and $\epsilon > 0$. We can find unitaries $u_1, u_2, \dots, u_n$ of $C(X)$ such that equation in condition (2) holds. Applying τ we obtain that

$$\left\| \tau \left(\frac{1}{n} \sum_{i=1}^{n} u_i a u_i^* - \mathbb{E}(a) \right) \right\| < \epsilon$$

Since τ is $C(X)$-central, we see that

$$\tau \left(\frac{1}{n} \sum_{i=1}^{n} u_i a u_i^* \right) = \frac{1}{n} \sum_{i=1}^{n} \tau(u_i a u_i^*) = \frac{1}{n} \sum_{i=1}^{n} \tau(a u_i^* u) = \tau(a).$$

Therefore,

$$\|\tau(a)-\tau(\mathbb{E}(a))\| = \left\|\tau\left(\frac{1}{n}\sum_{i=1}^{n} u_i a u_i^* - \mathbb{E}(a)\right)\right\| < \epsilon$$

Since $\epsilon > 0$ is arbitrary, the claim follows.

((3) $\implies$ (4)): Let φ be a state on $C(X) \rtimes_r \Gamma$ such that $\varphi|_{C(X)}$ is a pure state. Since every pure state on $C(X)$ is of the form δ_x for some $x \in X$, $C(X)$ falls in the multiplicative domain of φ. Hence, φ is $C(X)$-central and therefore, is of the form $\varphi \circ \mathbb{E}$.

(4) $\implies$ (1): Suppose that $\Gamma \curvearrowright X$ is not free. Then, there exists $x \in X$ such that Γ_x is non-trivial. Let $e \neq s \in \Gamma_x$. Let $\Lambda = \langle s \rangle$. Let $\varphi = \tau_0 \circ \mathbb{E}_\Lambda \circ \mathbb{E}_x$ where τ_0 is the unit character on $C^*_\lambda(\Lambda)$, which is continuous since Λ is amenable. Observe that $\varphi|_{C(X)} = \delta_x$, hence, $\varphi|_{C(X)}$ is a pure state and $C(X)$ falls in the multiplicative domain of φ. Moreover, $\varphi(\lambda_s) = 1$. Hence, φ is not of the form $\varphi \circ \mathbb{E}$. Therefore, $\varphi|_{C(X)}$ doesn't extend uniquely to a state on $C(X) \rtimes_r \Gamma$. $\square$

In [4], we introduce a generalized version of the Powers averaging which does turn out to be equivalent to simplicity in the end.

5.2 Arriving upon the right notion of the Powers averaging for crossed products

It is clear that the averaging we want must involve some combination of group elements and functions from $C(X)$ as an averaging involving just the group elements or just the functions from $C(X)$ won't suffice. After some introspection, one realizes that if an averaging of the form

$$\left\|\sum_{j=1}^{m} g_j \lambda_{t_j}(a - \mathbb{E}(a))\lambda_{t_j^{-1}} g_j\right\| < \epsilon.$$

could be proved for a given $a \in C(X) \rtimes_r \Gamma$ and $\epsilon > 0$, then we can show that such an averaging implies simplicity of $C(X) \rtimes_r \Gamma$ for a minimal Γ-space X. We first prove that $C(X) \rtimes_r \Gamma$ satisfies such an averaging if $\Gamma \curvearrowright X$ is topologically free and minimal. We begin with the following Lemma.

Lemma 5.2. *Let Y be a compact Hausdorff Γ-space. Suppose that $\Gamma \curvearrowright Y$ is minimal and topologically free. Let $F \subset \Gamma \setminus \{e\}$ be a finite set. Then, for each $y \in Y$, there exists $t_y \in \Gamma$ such that*

$st_y y \neq t_y y$ *for all* $s \in F$.

Proof. Suppose not. Then, there exists $y \in Y$ such that for all $t \in \Gamma$, there exists $s \in F$ such that $sty = ty$. In particular, we obtain that $\{ty : t \in \Gamma\} \subset \cup_{s \in F} X^s$. Since X^s is closed and $\Gamma \curvearrowright Y$ is minimal, taking closure on both sides, we obtain that

$$Y = \overline{\{ty : t \in \Gamma\}} \subset \cup_{s \in F} X^s$$

From the Baire-category theorem, it follows that there must exist $s \in F$ such that X^s has non-empty interior, which is a contradiction since $\Gamma \curvearrowright Y$ is topologically free. Hence, the claim holds. $\square$

Proposition 5.3. *Suppose that X is a minimal Γ-space with the action $\Gamma \curvearrowright X$ topologically free. Then, given $\epsilon > 0$ and $a \in C(X) \rtimes_r \Gamma$, we can find $s_1, s_2, \ldots, s_m \in \Gamma$ and $g_1, g_2, \ldots, g_m \in C(X)$ such that*

$$\left\| \sum_{j=1}^{m} g_j \lambda_{s_j} (a - \mathbb{E}(a)) \lambda_{s_j^{-1}} g_j \right\| < \epsilon, \text{ where } \sum_{j=1}^{m} g_j^2 = 1.$$

Proof. Let $a \in C(X) \rtimes_r \Gamma$ and $\epsilon > 0$. Then, there are $t_1, t_2, \ldots, t_n \in \Gamma \backslash \{e\}$ and $f_1, f_2, \ldots, f_n \in C(X)$ such that

$$\left\| a - \left(\sum_{i=1}^{n} f_i \lambda_{t_i} + \mathbb{E}(a) \right) \right\| < \epsilon$$

Let $F = \{t_1, t_2, \ldots, t_n\}$. Then, for each $x \in X$, using Lemma 5.2, we can find $t_x \in \Gamma$ such that $t_i t_x x \neq t_x x$ for each $i = 1, 2, \ldots, n$. Hence, we can find a neighborhood $U_x \ni x$ such that $t_x^{-1} t_i t_x U_x \cap U_x = \emptyset$ for each $i = 1, 2, \ldots, n$. Note that this is equivalent to saying that $t_i t_x U_x \cap t_x U_x = \emptyset$ for all $i = 1, 2, \ldots, n$. Now, using compactness of X, there are $x_1, x_2, \ldots, x_m \in X$ such that $X \subset \cup_{j=1}^{m} U_{x_j}$ and there are $t_{x_1}, t_{x_2}, \ldots, t_{x_m} \in \Gamma$ such that $t_i t_{x_j} U_{x_j} \cap t_{x_j} U_{x_j} = \emptyset$ for all $i = 1, 2, \ldots, n$ and each $j = 1, 2, \ldots, m$. Let $\{g_j\}_{j=1}^{m}$ be a partition of unit subordinate to $\{U_j\}_{j=1}^{m}$, i.e., for each

$j = 1, 2, \ldots, m$, $g_j \geq 0$, $\mathrm{Supp}(g_j) \subset U_j$ and $\sum_{j=1}^m g_j^2 = 1$. Now, for each fixed j, observe that

$$\begin{aligned} g_j \lambda_{t_{x_j}}^{-1} f_i \lambda_{t_i} \lambda_{t_{x_j}} g_j &= g_j \left(t_{x_j}^{-1}.f_i\right) \lambda_{t_{x_j}^{-1} t_i} \left(t_{x_j}.g_j\right) \lambda_{t_{x_j}} \\ &= \left(t_{x_j}^{-1}.f_i\right) g_j \lambda_{t_{x_j}^{-1} t_i} \left(t_{x_j}.g_j\right) \lambda_{t_{x_j}} \\ &= \left(t_{x_j}^{-1}.f_i\right) \lambda_{t_{x_j}^{-1} t_i} \left(t_i^{-1} t_{x_j}.g_j\right) \left(t_{x_j}.g_j\right) \lambda_{t_{x_j}} \end{aligned}$$

Now, note that $g_j(t_{x_j}^{-1}x) \neq 0$ and $g_j(t_{x_j}^{-1} t_i x) \neq 0$ iff $t_{x_j}^{-1} x \in U_{x_j}$ and $t_{x_j}^{-1} t_i x \in U_{x_j}$iff $x \in t_{x_j} U_{x_j} \cap t_i^{-1} t_{x_j} U_{x_j} = \emptyset$, for all $i = 1, 2, \ldots, n$. Hence, for each fixed j and for all $i = 1, 2, \ldots, n$, $g_j \lambda_{t_{x_j}}^{-1} f_i \lambda_{t_i} \lambda_{t_{x_j}} g_j = 0$. Therefore,

$$\sum_{j=1}^m g_j \lambda_{t_{x_j}}^{-1} \left(\sum_{i=1}^n f_i \lambda_{t_i}\right) \lambda_{t_{x_j}} g_j = \sum_{j=1}^m \sum_{i=1}^n g_j \lambda_{t_{x_j}}^{-1} f_i \lambda_{t_i} \lambda_{t_{x_j}} g_j = 0.$$

Moreover, it is easy to check that the map $\varphi : C(X) \rtimes_r \Gamma \to C(X) \rtimes_r \Gamma$ defined by $\varphi(a) = \sum_{j=1}^m g_j \lambda_{t_{x_j}^{-1}} a \lambda_{t_{x_j}} g_j$ is a unital completely positive map. Hence,

$$\begin{aligned} &\left\| \sum_{j=1}^m g_j \lambda_{t_{x_j}^{-1}} (a - \mathbb{E}(a)) \lambda_{t_{x_j}} g_j \right\| \\ &\leq \left\| \sum_{j=1}^m g_j \lambda_{t_{x_j}^{-1}} \left(a - \left(\sum_{i=1}^n f_i \lambda_{s_i} + \mathbb{E}(a)\right)\right) \lambda_{t_{x_j}} g_j \right\| \\ &+ \left\| \sum_{j=1}^m g_j \lambda_{t_{x_j}}^{-1} \left(\sum_{i=1}^n f_i \lambda_{t_i}\right) \lambda_{t_{x_j}} g_j \right\| \\ &= \left\| \varphi\left(a - \left(\sum_{i=1}^n f_i \lambda_{s_i} + \mathbb{E}(a)\right)\right) \right\| + \left\| \sum_{j=1}^m g_j \lambda_{t_{x_j}}^{-1} \left(\sum_{i=1}^n f_i \lambda_{t_i}\right) \lambda_{t_{x_j}} g_j \right\| \\ &\leq \left\| a - \left(\sum_{i=1}^n f_i \lambda_{s_i} + \mathbb{E}(a)\right) \right\| + \left\| \sum_{j=1}^m g_j \lambda_{t_{x_j}}^{-1} \left(\sum_{i=1}^n f_i \lambda_{t_i}\right) \lambda_{t_{x_j}} g_j \right\| \\ &< \epsilon. \end{aligned}$$

Renaming $t_{x_j}^{-1}$ as s_j, we obtain the claim. $\square$

5.3 Simplicity and generalized Powers averaging

In order to prove such an averaging for simple crossed products, we use the notion of generalized Furstenberg boundary introduced by Naghavi [30] and Kawabe [23] independently and various characterizations provided there. In this section, we introduce a generalized version of the Powers averaging and show that it is equivalent to simplicity of the reduced crossed product. The contents of this section and the one following it are taken from [4].

Definition 5.4 ([4])**.** A C^*-dynamical system $(C(X), \Gamma)$ is said to have generalized Powers averaging property if for every element a in the reduced crossed product $C(X) \rtimes_r \Gamma$ and every $\epsilon > 0$, there are $s_1, \dots, s_m \in \Gamma$ and $f_1, f_2, \dots, f_m \in C(X)$ with $f_j \geq 0$ for each $j = 1, 2, \dots, m$, $\sum_{j=1}^m f_j^2 = 1$ such that

$$\left\| \sum_{j=1}^{m} f_j \lambda_{s_j} (a - \mathbb{E}(a)) \lambda_{s_j^{-1}} f_j \right\| < \epsilon.$$

Moreover, it is said to have strong generalized Powers averaging property if for every element a in the reduced crossed product $C(X) \rtimes_r \Gamma$ and every $\epsilon > 0$, there are $s_1, \dots, s_m \in \Gamma$ and $f_1, f_2, \dots, f_m \in C(X)$ with $f_j \geq 0$ for each $j = 1, 2, \dots, m$, $\sum_{j=1}^m f_j^2 = 1$ such that

$$\left\| \sum_{j=1}^{m} f_j \lambda_{s_j} a \lambda_{s_j^{-1}} f_j - \mathbb{E}(a) \right\| < \epsilon.$$

Theorem 5.5 ([4])**.** *Let Γ be a discrete group acting on a compact Hausdorff space X by homeomorphisms, and assume that the action is minimal. The following are equivalent:*

1. *$C(X) \rtimes_r \Gamma$ is simple.*
2. *$C(X) \rtimes_r \Gamma$ has the generalized Powers averaging property.*
3. *$C(X) \rtimes_r \Gamma$ has the strong generalized Powers averaging property.*

One possible way of proving this theorem is to follow in the footsteps of [26] and show that there is a one-to-one correspondence between boundaries in $\{\nu \circ \mathbb{E} : \nu \in \mathrm{Prob}(X)\}$ and conditional expectations $\mathbb{E} : C(X) \rtimes_r \Gamma \to C(X)$. We do not do that!! Instead, we argue similarly as in

[19]. The key to establishing our theorem is the following lemma which allows us to send arbitrary measures to δ-measures.

For a state φ on $C(X) \rtimes_r \Gamma$, $f_j \in C(X)$ with $f_j \geq 0$, $\sum_{j=1}^m f_j^2 = 1$ and $s_j \in \Gamma$, we define a new state on $C(X) \rtimes_r \Gamma$: For $a \in C(X) \rtimes_r \Gamma$,

$$\varphi\left(\sum_{j=1}^m f_j \lambda_{s_j} (\cdot) \lambda_{s_j^{-1}} f_j\right)(a) = \varphi\left(\sum_{j=1}^m f_j \lambda_{s_j} a \lambda_{s_j^{-1}} f_j\right).$$

Moreover, for $f_j \geq 0$ with $\sum_{j=1}^m f_j^2 = 1$, let

$$K = \left\{\Phi : C(X) \rtimes_r \Gamma \to C(X) \rtimes_r \Gamma : \Phi(a) = \sum_{j=1}^m f_j \lambda_{s_j} a \lambda_{s_j^{-1}} f_j\right\}$$

Lemma 5.6. *Suppose that X is a minimal Γ-space. Then, for every $x \in X$, there exists a net $\Phi_i \in K$ such that $\Phi_i|_{C(X)} \xrightarrow{\|\cdot\|} \delta_x$. Moreover, for any state φ on $C(X) \rtimes_r \Gamma$, $\varphi(\Phi_i)|_{C(X)} \xrightarrow{w^*} \delta_x$.*

Proof. Let $x_0 \in X$. Let V be a neighborhood around x_0. Since $\Gamma \curvearrowright X$ is minimal, for every $x \in X$ there exists $t_x \in \Gamma$ such that $t_x x \in V$. Then, there exists an open set $U_x \ni x$ such that $t_x U_x \subset V$ (we can choose $U_x = t_x^{-1} V$). Using the compactness of X, we can find $x_1, x_2, \ldots, x_n \in X$ such that $X \subset \cup_{i=1}^n U_{x_i}$ and $t_{x_1}, t_{x_2}, \ldots, t_{x_n} \in \Gamma$ such that $t_{x_i} U_{x_i} \subset V$. Let $\{f_j\}_{j=1}^n$ be a partition of unity subordinate to $\{U_{x_j}\}_{j=1}^n$. Define a u.c.p map $\Phi_V : C(X) \rtimes_r \Gamma \to C(X) \rtimes_r \Gamma$ by

$$\Phi_V(a) = \sum_{j=1}^n f_j^{\frac{1}{2}} \lambda_{t_j} a \lambda_{t_j^{-1}} f_j^{\frac{1}{2}}, a \in C(X) \rtimes_r \Gamma$$

(Here, we rename t_{x_j} as t_j.) We claim that

$$\{\Phi_V : V \text{ open}, V \ni x_0, \Phi_V \leq \Phi_W \iff W \subset V\} \xrightarrow{\|\cdot\|} \delta_{x_0}$$

Let $f \in C(X)$. Let $\epsilon > 0$. Since f is continuous at x_0, there exists an open neighborhood $U \ni x_0$ such that $|f(y) - f(x_0)| < \epsilon$ for all $Y \in U$. By the above construction, there exists a u.c.p map Φ_U

on $C(X) \rtimes_r \Gamma$ defined by

$$\Phi_U(a) = \sum_{j=1}^{m} g_j^{\frac{1}{2}} \lambda_{t_j^{-1}} a \lambda_{t_j} g_j^{\frac{1}{2}}, a \in C(X) \rtimes_r \Gamma$$

such that $t_j U_j \subset U$ and $\mathrm{Supp}(g_j) \subset U_j$ with $X = \cup_{j=1}^m U_j$. Now, for $x \in X$ observe that

$$\begin{aligned}
&\left| \sum_{j=1}^{m} g_j(x) t_j^{-1}.f(x) - f(x_0) \right| \\
&= \left| \sum_{j=1}^{m} g_j(x) t_j^{-1}.f(x) - \sum_{j=1}^{m} g_j(x) f(x_0) \right| \\
&\leq \sum_{j=1}^{m} g_j(x) |f(t_j x) - f(x_0)| \\
&< \epsilon \qquad\qquad (\text{for } x \in U_j, t_j x \in U)
\end{aligned}$$

Moreover, for any state φ,

$$\begin{aligned}
|\varphi_U(f) - f(x_0)| &= \left| \varphi\left(\sum_{j=1}^{m} g_j t_j^{-1}.f - f(x_0) \right) \right| \\
&\leq \left\| \sum_{j=1}^{m} g_j t_j^{-1}.f - f(x_0) \right\| < \epsilon.
\end{aligned}$$

The claim follows. □

Proof of Theorem 5.5. We give an outline of the proof and refer the reader to [4] for details. (3) $\implies$ (1) is the easy direction and it really is a mere modification of Theorem 3.4. We leave it as an exercise.

For (1) $\implies$ (2), arguing similarly as in [9, Lemma 3.3], it is enough to show that given any state $\varphi \in S(C(X) \rtimes_\lambda \Gamma)$, that

$$\{\nu \circ \mathbb{E} : \nu \in \mathrm{Prob}(X)\} \subseteq \overline{\{\varphi(\mu), \mu \in K\}}^{\text{weak}^*}.$$

We begin by observing that $\tilde{K} = \overline{\{\varphi(\mu), \mu \in K\}}^{\text{weak}^*}$ is a w^*-compact convex Γ-invariant subset of $S(C(X) \rtimes_r \Gamma)$. Let $x \in X$. Using Lemma 5.6, we can find a net $\{\mu_i\} \in K$ such that $\varphi \circ \mu_i|_{C(X)} = \varphi_i|_{C(X)} \xrightarrow{w^*} \delta_x$. Upon passing to a subnet, we may assume that $\varphi_i \xrightarrow{w^*} \tilde{\psi} \in \tilde{K}$, where $\tilde{\psi}$ is a state on $C(X) \rtimes_r \Gamma$. Extend $\tilde{\psi}$ to a state $\tilde{\varphi}$ on $C(\tilde{X}) \rtimes_r \Gamma$, where $C(\tilde{X}) = I_\Gamma(C(X))$. Note that $\tilde{\varphi}|_{C(X)} = \delta_x$. Using [30, Theorem 3.2], we can find a net $s_j \in \Gamma$ such that $s_j.\tilde{\varphi}|_{C(\tilde{X})} \xrightarrow{w^*} \delta_y$ for some $y \in \tilde{X}$. Upon passing to a subnet, we may assume that $s_j\tilde{\varphi} \xrightarrow{w^*} \psi \in S(C(\tilde{X}) \rtimes_r \Gamma)$. Moreover, $C(\tilde{X})$ falls in the multiplicative domain of ψ since $\psi|_{C(\tilde{X})} = \delta_y$. Since $C(X) \rtimes_r \Gamma$ is simple, it follows from [23, Theorem 3.4] that $C(\tilde{X}) \rtimes_r \Gamma$ is simple, hence by the same theorem again, the action $\Gamma \curvearrowright \tilde{X}$ is free. Arguing similarly as in [19, Lemma 3.1], we conclude that $\psi = \psi \circ \mathbb{E}$. The claim follows by observing that

$$\psi|_{C(X)\rtimes_r\Gamma} = \lim_j s_j\tilde{\varphi}|_{C(X)\rtimes_r\Gamma} = \lim_j s_j\tilde{\psi} \in \tilde{K}.$$

For (2) $\implies$ (3): Arguing similarly as above, given $a \in C(X) \rtimes_r \Gamma$, we can show that for each $x_0 \in X$, there exists a net $\mu_i^{x_0} \in K$ such that $\mu_i^{x_0}(a) \xrightarrow{\|.\|} \mathbb{E}(a)(x_0)$. Since $\mathbb{E}(a)$ is continuous, for each $x \in X$, there exists a neighborhood $U_x \ni x$ such that $|\mathbb{E}(a)(x) - \mathbb{E}(a)(y)| < \epsilon$ for all $y \in U_x$. Using the compactness of X, we can find finitely many elements $x_1, x_2, \ldots, x_n \in X$ such that $X \subset \cup_{i=1}^n U_{x_i}$. Let $\{f_i\}$ be a partition of unity corresponding to $\{U_{x_i}\}_{i=1}^n$ with $f_i \geq 0$ and $\sum_{i=1}^n f_i^2 = 1$. For each $i = 1, 2, \ldots, n$, it follows from our earlier argument that we can find u.c.p maps $\mu_i \in K$ such that

$$\|\mu_i(a) - \mathbb{E}(a)(x_i)\| < \epsilon, \ \forall i = 1, 2, \ldots, n.$$

Let $\mu \in K$ be defined by $\mu(.) = \sum_{i=1}^n f_i\mu_i(.)f_i$. Then,

$$\begin{aligned}
\|\mu(a) - \mathbb{E}(a)\| &= \left\|\sum_{i=1}^n f_i\mu_i(a)f_i - \sum_{i=1}^n f_i\mathbb{E}(a)f_i\right\| \\
&\leq \left\|\sum_{i=1}^n f_i\mu_i(a)f_i - \sum_{i=1}^n f_i\mathbb{E}(a)(x_i)f_i\right\| \\
&+ \left\|\sum_{i=1}^n f_i\mathbb{E}(a)(x_i)f_i - \sum_{i=1}^n f_i\mathbb{E}(a)f_i\right\|
\end{aligned}$$

For each $x \in \mathrm{Supp}(f_i) \subset U_i$, $|\mathbb{E}(a)(x_i) - \mathbb{E}(a)(x)| < \epsilon$ by construction. Hence,

$$
\begin{aligned}
&\left\| \sum_{i=1}^{n} f_i \mathbb{E}(a)(x_i) f_i - \sum_{i=1}^{n} f_i \mathbb{E}(a) f_i \right\| \\
&= \sup_{x \in X} \left| \sum_{i=1}^{n} f_i(x) \mathbb{E}(a)(x_i) f_i(x) - \sum_{i=1}^{n} f_i(x) \mathbb{E}(a)(x) f_i(x) \right| \\
&\leq \sup_{x \in X} \left(\sum_{i=1}^{n} f_i(x) \left| \mathbb{E}(a)(x_i) - \mathbb{E}(a)(x) \right| f_i(x) \right) \\
&\leq \epsilon.
\end{aligned}
$$

The claim follows. $\square$

5.4 Application of generalized Powers averaging

We give a generalization of Theorem 3.4 as an application of the generalized Powers averaging. The motivation for this problem originated from the work of Kawabe [23], especially [23, Theorem 3.4] and [23, Theorem 6.1]. The aim of this section is to prove the following proposition. We start by saying how we arrived upon this proposition.

As mentioned before, Γ-simplicity doesn't necessarily pass to sub-algebras and therefore, simplicity for invariant sub-algebras of simple crossed products shouldn't be expected to hold in general. Consider, for example, any simple C^*-algebra A, any C^*-simple group Γ acting on $\mathcal{A}$ trivially, and any abelian C^*-subalgebra $\mathcal{B} \subseteq \mathcal{A}$. However, given an inclusion of unital Γ-C^*-algebras $C(Y) \subset C(X)$ (via a factor map $\pi : X \to Y$), every intermediate crossed product $C(Z) \rtimes_r \Gamma$ sitting between $C(Y) \rtimes_r \Gamma$ and $C(X) \rtimes_r \Gamma$ is simple whenever $C(Y) \rtimes_r \Gamma$ is simple. For this, we use the characterization of simplicity of the reduced crossed product provided in [23, Theorem 6.1].

Towards that end, let $z \in Z$ and Λ be an amenable subgroup of Γ_z. Let $\rho : Z \to Y$ be the factor map coming from the inclusion of $C(Y) \subset C(Z)$. Observe that $\Gamma_z \leq \Gamma_{\rho(z)}$ and therefore, Λ is an amenable subgroup of $\Gamma_{\rho(z)}$. Hence, by [23, Theorem 6.1] we can find a net $\{g_i\} \in \Gamma$ such that

$g_i\Lambda g_i^{-1} \to \{e\}$ in the Chabauty topology. Again, by [23, Theorem 6.1] it follows that $C(Z) \rtimes_r \Gamma$ is simple. After this, it is only natural to expect that this should hold for every intermediate C^*-subalgebra.

Proposition 5.7. *Let $\pi : X \to Y$ be a continuous Γ-equivariant onto map, where X, Y are compact Hausdorff Γ-spaces. Suppose that $\Gamma \curvearrowright X$ is minimal and $C(Y) \rtimes_r \Gamma$ is simple. Then, every intermediate C^*-algebra $\mathcal{A}$ with $C(Y) \rtimes_r \Gamma \subseteq \mathcal{A} \subseteq C(X) \rtimes_r \Gamma$, is simple.*

The following lemma allows us to lift the averaging scheme to higher crossed products. The spirit is similar to that of Lemma 3.5 and so is its proof. We denote the u.c.p map $\Phi(\cdot) = \sum_{i=1}^n f_i\lambda_{s_i} \cdot \lambda_{s_i^{-1}} f_i$ by $\mu(\cdot)$ for convinience of notation (we can view it as a generalized probability measure but it is not needed for the proof we propose below. The interested reader can look at our recent preprint [4] for more on generalized measures).

Lemma 5.8. *Let Γ be a discrete group, let μ be a u.c.p map of the form $\sum_{i=1}^n f_i\lambda_{s_i} \cdot \lambda_{s_i^{-1}} f_i$, where $f_i \geq 0$ and $\sum_{i=1}^n f_i^2 = 1$. Let $\pi : X \to Y$ be a Γ-equivariant surjective continuous map with $\Gamma \curvearrowright X$ minimal. Then for any $t \in \Gamma$ and $f \in C(X)$ we have*

$$\|\mu * (f\lambda_t)\|_{\mathbb{B}(\ell^2(\Gamma))} \leq \|f\| \, \|\mu * \lambda_t\|_{\mathbb{B}(\ell^2(\Gamma))} .$$

Proof. For $\xi \in \ell^2(\Gamma)$, observe that for each $t' \in \Gamma$ we have

$$\begin{aligned}
&([\mu * (f\lambda_t)](\xi))\,(t') \\
&= \sum_{s\in\Gamma} [g_s\lambda_s f\lambda_t\lambda_{s^{-1}} g_s(\xi)](t') \\
&= \sum_{s\in\Gamma} [g_s s.f\lambda_{sts^{-1}} g_s(\xi)](t') \\
&= \sum_{s\in\Gamma} [g_s(s.f)(sts^{-1}.g_s)\lambda_{sts^{-1}}(\xi)](t') \\
&= \sum_{s\in\Gamma} g_s(t'.x_0) f(s^{-1}t'.x_0) g_s(st^{-1}s^{-1}t'.x_0)\xi(st^{-1}s^{-1}t').
\end{aligned}$$

Define the function $\xi_1(t') = |\xi(t')|$, $t' \in \Gamma$. Then $\xi_1 \in \ell^2(\Gamma)$, and $\|\xi_1\|_{\ell^2(\Gamma)} = \|\xi\|_{\ell^2(\Gamma)}$. We have

$$\begin{aligned}
&\|[\mu * (f\lambda_t)](\xi)\|^2_{\ell^2(\Gamma)} \\
&= \sum_{t'\in\Gamma} \left|\left([\mu * (f\lambda_t)](\xi)\right)(t')\right|^2 \\
&= \sum_{t'\in\Gamma} \left|\sum_{s\in\Gamma} g_s(t'.x_0) f(s^{-1}t'.x_0) g_s(st^{-1}s^{-1}t'.x_0)\xi(st^{-1}s^{-1}t')\right|^2 \\
&\leq \|f\|^2 \sum_{t'\in\Gamma} \left(\sum_{s\in\Gamma} g_s(t'.x_0) g_s(st^{-1}s^{-1}t'.x_0)|\xi(st^{-1}s^{-1}t')|\right)^2 \\
&= \|f\|^2 \left\|\sum_{s\in\Gamma} g_s\lambda_{sts^{-1}} g_s(\xi_1)\right\|^2_{\ell^2(\Gamma)} \\
&\leq \|f\|^2 \left\|\sum_{s\in\Gamma} g_s\lambda_{sts^{-1}} g_s\right\|^2_{\mathbb{B}(\ell^2(\Gamma))} \|\xi_1\|^2_{\ell^2(\Gamma)} \\
&= \|f\|^2 \|\mu * (\lambda_t)\|^2_{\mathbb{B}(\ell^2(\Gamma))} \|\xi_1\|^2_{\ell^2(\Gamma)},
\end{aligned}$$

and since $\|\xi_1\|_{\ell^2(\Gamma)} = \|\xi\|_{\ell^2(\Gamma)}$, the inequality follows. $\square$

We follow the strategy employed in the proof of Theorem 3.4.

Proof of Proposition 5.7. Let $\mathcal{A}$ be an intermediate C^*-sub-algebra of the form $C(Y) \rtimes_r \Gamma \subseteq \mathcal{A} \subseteq C(X) \rtimes_r \Gamma$. Let I be a non-zero ideal of $\mathcal{A}$. Let $a \in I$. Replacing a by a^*a and dividing by an appropriate scalar if required, we can assume that $a \geq 0$ and $\|a\| < 1$. Let $\nu \in \mathrm{Prob}(\Gamma)$ be such that $\mathrm{Supp}(\nu) = \Gamma$. Then, it is a consequence of Lemma 3.7 that there exists $\delta > 0$ such that $\nu * \mathbb{E}(a)(x) > \delta$ for all $x \in X$. Now, let $0 < \epsilon < 1$. Then, using Lemma 5.8, we can find $g_1, g_2, \ldots, g_m \in C(Y)$ with $0 \leq g_j \leq 1$, $\sum_{j=1}^m g_j^2 = 1$ and $s_1, s_2, \ldots, s_m \in \Gamma$ such that

$$\left\|\frac{1}{\delta}\sum_{j=1}^m g_j\lambda_{s_j}\left(\nu * a - \mathbb{E}(\nu * a)\right)\lambda_{s_j^{-1}} g_j\right\| < \epsilon.$$

Now, for every $x \in X$,

$$\begin{aligned}
&\frac{1}{\delta}\left(\sum_{j=1}^{m} g_j \lambda_{s_j} \mathbb{E}(\nu * a) \lambda_{s_j^{-1}} g_j\right)(x) \\
&= \frac{1}{\delta}\left(\sum_{j=1}^{m} g_j^2(x) \mathbb{E}(\nu * a)(s_j^{-1}x)\right) \\
&= \frac{1}{\delta}\left(\sum_{j=1}^{m} g_j^2(x) \left(\nu * \mathbb{E}(a)\right)(s_j^{-1}x)\right) \\
&\geq \frac{1}{\delta}\sum_{j=1}^{m} g_j^2(x)(\delta) = 1
\end{aligned}$$

Now, let $S = \frac{1}{\delta}\sum_{j=1}^{m} g_j \lambda_{s_j} (\nu * a) \lambda_{s_j^{-1}} g_j$ and $T = \frac{1}{\delta}\sum_{j=1}^{m} g_j \lambda_{s_j} \mathbb{E}(\nu * a) \lambda_{s_j^{-1}} g_j$. Observe that $\|S - T\| < 1$. Moreover, T is invertible, $\|T^{-1}\| < 1$ and $\|S\| < 1$. Therefore, using Lemma 3.6 we see that S is invertible. Since $S \in I$, $I = \mathcal{A}$. This completes the proof. $\square$

Bibliography

[1] S. Adams, *Boundary amenability for word hyperbolic groups and an application to smooth dynamics of simple groups*, Topology. **33** (1994), 765–783.

[2] T. Amrutam, *On intermediate subalgebras of C*-simple group actions*, International Mathematics Research Notices (2019).

[3] T. Amrutam and M. Kalantar, *On simplicity of intermediate C*-algebras*, Ergodic Theory and Dynamical Systems **40** (2020), no. 12, 3181–3187.

[4] T. Amrutam and D. Ursu, *Generalized powers averaging for commutative crossed products* (2021), 22 pp., available at `arxiv:2101.02853`.

[5] R. J. Archbold and J. S. Spielberg, *Topologically free actions and ideals in discrete C*-dynamical systems*, Proceedings of the Edinburgh Mathematical Society **37** (1994), no. 1, 119–124.

[6] B. H. Bowditch, *Convergence groups and configuration spaces*, Geometric group theory down under (1999), 23–54.

[7] E. Breuillard, M. Kalantar, M. Kennedy, and N. Ozawa, *C*-simplicity and the unique trace property for discrete groups*, Publications Mathématiques De L'IHÉS **1** (2017), 35–71.

[8] N. P. Brown and N. Ozawa, *C^*-algebras and finite-dimensional approximations*, Graduate Studies in Mathematics, vol. 88, American Mathematical Society, Providence, RI, 2008.

[9] R. S Bryder and M. Kennedy, *Reduced twisted crossed products over C*-Simple groups*, International Mathematics Research Notices **2018** (201612), no. 6, 1638–1655.

[10] E. Bédos, *Discrete groups and simple C*-algebras*, Mathematical Proceedings of the Cambridge Philosophical Society **109** (1991), no. 3, 521–537.

[11] P. de la Harpe, *On simplicity of reduced C*-algebras of groups*, Bulletin of the London Mathematical Society **39** (2006), no. 1, 1–26.

[12] P. de La Harpe and G. Skandalis, *Powers property and simple C*-algebras*, Mathematische Annalen (1986), 241–250.

[13] G. Elliott, *Some simple C*-algebras constructed as crossed products with discrete outer automorphism groups*, Publications of The Research Institute for Mathematical Sciences **16** (1980), 299–311.

[14] E. M. Freden, *Properties of convergence groups and spaces*, Conform. Geom. Dyn. **1** (1997), 13–23 (electronic).

[15] H. Furstenberg, *A poisson formula for semi-simple lie groups*, Annals of Mathematics **77** (1963), 335–386.

[16] ———, *Boundary theory and stochastic processes on homogeneous spaces*, Proceedings of Symposia in Pure Mathematics **26** (1973), 193–229.

[17] S. Glasner, *Compressibility properties in topological dynamics*, American Journal of Mathematics **97** (1975), 148–171.

[18] ———, *Proximal flows*, Lecture Notes in Mathematics, Vol. 517, Springer-Verlag, Berlin-New York, 1976.

[19] U. Haagerup, *A new look at C*-simplicity and the unique trace property of a group*, Operator Algebras and Applications (2016), 167–176.

[20] Y. Hartman and M. Kalantar, *Stationary C*-dynamical systems*, arXiv e-prints (2017), arXiv:1712.10133, available at `1712.10133`. to appear in Journal of the European Mathematical Society.

[21] S. Y. Jang and S. G. Lee, *Simplicity of crossed products of C*-algebras*, Proceedings of the American Mathematical Society **118** (1993), no. 3, 823–826.

[22] M. Kalantar and M. Kennedy, *Boundaries of reduced C*-algebras of discrete groups*, Journal Für Die Reine Und Angewandte Mathematik **727** (2017), 247–267.

[23] T. Kawabe, *Uniformly recurrent subgroups and the ideal structure of reduced crossed products*, arXiv e-prints (2017), arXiv:1701.03413, available at `1701.03413`.

[24] S. Kawamura and J. Tomiyama, *Properties of topological dynamical systems and corresponding C*-algebras*, Tokyo J. Math. **13** (199012), no. 2, 251–257.

[25] ———, *Properties of topological dynamical systems and corresponding C*-algebras*, Tokyo J. Math. **13** (1990), no. 2, 251–257.

[26] M. Kennedy, *An intrinsic characterization of C^*-simplicity*, Ann. Sci. Éc. Norm. Supér. (4) **53** (2020), no. 5, 1105–1119.

[27] M. Kennedy and C. Schafhauser, *Noncommutative boundaries and the ideal structure of reduced crossed products*, Duke Mathematical Journal **168** (2019), no. 17, 3215–3260.

[28] A. Le Boudec, *C*-simplicity and the amenable radical*, Inventiones Mathematicae **209** (2017), no. 1, 159–174.

[29] A. Minasyan and D. Osin, *Acylindrical hyperbolicity of groups acting on trees*, Math. Ann. **362** (2015), no. 3-4, 1055–1105.

[30] Z. Naghavi, *Furstenberg boundary of minimal actions*, Integral Equations and Operator Theory **92** (2020), no. 2.

[31] D. Osin, *Acylindrically hyperbolic groups*, Trans. Amer. Math. Soc. **368** (2016), no. 2, 851–888.

[32] N. Ozawa, *Lecture on the furstenberg boundary and C*-simplicity* (2014), available at `http://www.kurims.kyoto-u.ac.jp/~narutaka/notes/yokou2014.pdf`.

[33] ———, *Weak amenability of hyperbolic groups*, Groups Geom. Dyn. **2** (2008), no. 2, 271–280.

[34] R. T. Powers, *Simplicity of the c^* -algebra associated with the free group on two generators*, Duke Math. Journal **42** (1975), no. 1, 151–156.

[35] A. Sierakowski, *The ideal structure of reduced crossed products*, arXiv: Operator Algebras (2010), 237.

[36] Y. Suzuki, *Group C*-algebras as decreasing intersection of nuclear C*-algebras*, American Journal of Mathematics **3** (2017), 681–705.

[37] ———, *Complete descriptions of intermediate operator algebras by intermediate extensions of dynamical systems*, Comm. Math. Phys. **375** (2020), no. 2, 1273–1297.

[38] M. Takesaki, *On the cross-norm of the direct product of C*-algebras*, Tohoku Math. J. (2) **16** (1964), no. 1, 111–122.

[39] T.Giordano, David Kerr, N. C. Phillips, and A. Toms, *Crossed products of C^*-algebras, topological dynamics and classification, advanced courses in mathematics. crm barcelona*, Lecture Notes Based On The Course Held At The Centre de Recerca Matemàtica (CRM), Barcelona, June 14–23, 2011 (2018), x+498.

[40] P. Tukia, *Convergence groups and Gromov's metric hyperbolic spaces*, New Zealand J. Math. **23** (1994), no. 2, 157–187.

[41] D. Ursu, *Relative C*-simplicity and characterizations for normal subgroups*, Journal of Operator Theory (to appear.)