at $r' = 0$ that is $\frac{1}{1000}$ th the incident intensity. Such a result would not ordinarily be distinguishable from black (zero intensity) by the human eye. The surface with negative curvature focuses the light into a bright ring. Thus a Rankine vortex with a small central core would have an intensity distribution quite similar to that of a hyperbolic surface with the important exception that the central disk is gray not black.

Figure 4 shows the measured intensity as a function of r' for a vortex image. It was obtained by digitizing a radial distance of an image of the vortex from a single frame of videotape. In order to compensate for the light scattered from the foreground, data from an image of an opaque disk of similar size, sitting on the surface of the water, was subtracted. The qualitative features of this figure are relevant, showing the monotonic decrease in intensity with increasing radius, characteristic of a parabolic core. Also shown is the bright ring at larger r' and the background incident intensity at still larger r'. Currently experiments are underway to evaluate the quantitative aspects of this technique.

This analysis has shown that the images of vortices can be qualitatively explained by the application of Snell's law to surfaces of negative curvature. The dark disks are due to the shape of the surface at small r, paraboloids of small diameter. The intensity of the dark disk is a measure of that curvature. The diameter of the dark disk is determined by the bright ring that is determined by the shape of the surface of negative curvature.

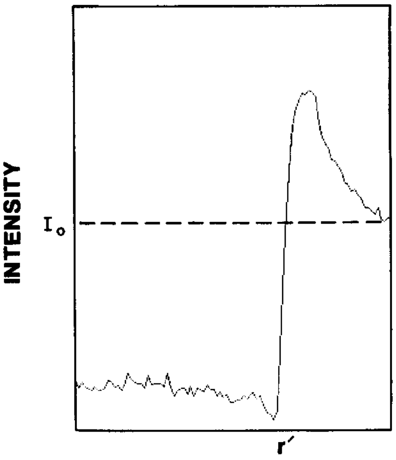

FIG. 4. The measured intensity, obtained from digitization of a frame of videotape, along a radius of vortex image. See text for details.

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The equation of bubble dynamics in a compressible liquid

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An equation accounting to lowest order for the effects of liquid compressibility on the radial motion of a spherical bubble is deduced from the statement of energy conservation for the liquid.

The Rayleigh–Plesset equation describing the dynamics of a spherical bubble in an incompressible liquid is

$$R\ddot{R} + \tfrac{3}{2}\dot{R}^2 = (p_B - p_\infty)/\rho, \tag{1}$$

where R is the bubble radius, ρ the (constant) liquid density, $p_B(t) = p(r = R(t),t)$ the pressure exerted by the liquid on the ("wet" side of the) bubble interface, and p_∞ the pressure at infinity. A number of papers (a list of which can be found in Ref. 1) have been devoted to obtaining generalizations of (1) approximately applicable to a compressible liquid and several different equations have been proposed. In two recent studies[1,2] the same objective has been pursued on the basis of a rigorous singular perturbation expansion that has demonstrated the essential nonuniqueness, in a sense to be specified later, of the approximate compressible equation. It is the purpose of this Brief Communication to present a derivation of this equation based on the principle of conservation of energy. This alternative approach will shed a different light on the more abstract derivations given in Refs. 1 and 2 and may be useful for other problems involving the motion of a body in a slightly compressible fluid.

The one-parameter family of equations derived in Refs.

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1 and 2, called the general Keller–Herring equation, is

$$[1+(\lambda+1)(\dot{R}/c)]R\ddot{R}+\tfrac{3}{2}[1-\tfrac{1}{3}(3\lambda+1)(\dot{R}/c)]\dot{R}^2$$
$$=\frac{1}{\rho}\Big(1+(\lambda-1)\frac{\dot{R}}{c}+\frac{R}{c}\frac{d}{dt}\Big)(p_B-p_\infty)$$
$$+O(c^{-2}), \quad (2)$$

where c is the speed of sound, approximated by its value in the undisturbed liquid, and λ is an arbitrary parameter which does not seem to have any physical meaning. The error term included in this equation indicates its degree of approximation with respect to an "exact" theory. The form of Eq. (2) given in Refs. 1 and 2 is in terms of the enthalpy rather than the pressure, but to $O(c^{-2})$ Eq. (2) represents an equivalent statement.[1] By taking $\lambda=0$, Eq. (2) becomes identical to the equation proposed by Keller,[3] while the value $\lambda=1$ brings it into the form suggested by Herring.[4] It will be noted that, by dropping terms in c^{-1}, Eq. (2) reduces to Eq. (1), which is therefore seen to have an error of order c^{-1}.

The indeterminacy present in Eq. (2) can be removed, at a price, by applying the operator $c^{-1}[(\lambda-1)\dot{R}+R\,d/dt]$ to Eq. (1) and subtracting from (2). Since (1) is $O(c^{-1})$, the quantity subtracted from (2) according to this procedure is $O(c^{-2})$ as Eq. (2) itself. The result is therefore consistent and is

$$R\ddot{R}+\tfrac{3}{2}\dot{R}^2-(1/c)(R^2\dddot{R}+6R\dot{R}\ddot{R}+2\dot{R}^3)$$
$$=(p_B-p_\infty)/\rho+O(c^{-2}). \quad (3)$$

The striking appearance of a third-order time derivative of the radius in this equation can be explained by noting that a precise theory would involve the retarded time $t-R/c$. A Taylor series expansion of a quantity such as $\ddot{R}(t-R/c)$ would then lead to $\dddot{R}$. As shown in Ref. 1, this quantity can be eliminated in more than one way by using the lower-order result (1), which leads to the indeterminacy present in (2). A similar situation arises, for analogous reasons, in the derivation of the equation of motion for a classical charged particle, as pioneered by Lorentz.[5,6] In that case, however, contrary to the procedure adopted here, the form with the third-order derivative was retained, which gave rise to considerable confusion[7] before the root of the problem was understood and the issue clarified.[8]

Because of the presence of the third time derivative of the radius, the form (3) of the radial equation is hardly more attractive than (2), if for nothing else than for the need to prescribe an initial condition for $\ddot{R}$.[9] However, in view of its uniqueness, it is perhaps proper to consider Eq. (3) the fundamental form of the $O(c^{-1})$ radial equation. It is this equation that will be derived in the following.

Consider the liquid contained in the volume $\mathscr{V}$ comprised between the bubble surface S_B and a large, concentric spherical surface S_∞. If $\mathbf{u}$ and p denote the local velocity and pressure, conservation of energy demands that

$$\frac{d}{dt}\frac{1}{2}\int_{\mathscr{V}}\rho\mathbf{u}\cdot\mathbf{u}\,d\mathscr{V}=\int_{\mathscr{V}}p\nabla\cdot\mathbf{u}\,d\mathscr{V}-\int_{S_B+S_\infty}p\mathbf{u}\cdot\mathbf{n}\,dS, \quad (4)$$

where $\mathbf{n}$ is the outward normal and viscous effects have been neglected. As first proposed by Keller,[3] and rigorously justified in Refs. 1 and 2, a mathematical model consistent to order c^{-1} can be built upon the wave equation satisfied by the velocity potential ϕ:

$$\nabla^2\phi-\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}=0, \quad (5)$$

and the incompressible form of the Bernoulli integral

$$p=p_\infty-\rho\Big(\frac{\partial\phi}{\partial t}+\frac{1}{2}\mathbf{u}\cdot\mathbf{u}\Big). \quad (6)$$

If we consider for simplicity the case of free motion under a constant pressure at infinity, then the relevant solution of (5) has the form

$$\phi=f(t-r/c)/r, \quad (7)$$

where r is the distance from the bubble center.

By use of (5) and (6) the first term in the right-hand side of the energy balance (4) may be written

$$\int_{\mathscr{V}}p\nabla\cdot\mathbf{u}\,d\mathscr{V}$$
$$=-\rho\int\Big[\Big(\frac{\partial\phi}{\partial t}+\frac{1}{2}\mathbf{u}\cdot\mathbf{u}\Big)\nabla^2\phi\Big]d\mathscr{V}+p_\infty\int\nabla\cdot\mathbf{u}\,d\mathscr{V}$$
$$=-\frac{\rho}{c^2}\int\frac{\partial\phi}{\partial t}\frac{\partial^2\phi}{\partial t^2}d\mathscr{V}+p_\infty\int_{S_B+S_\infty}\mathbf{u}\cdot\mathbf{n}\,dS$$
$$-\frac{\rho}{2c^2}\int\mathbf{u}\cdot\mathbf{u}\frac{\partial^2\phi}{\partial t^2}d\mathscr{V}. \quad (8)$$

The last term is at least of the second order in the Mach number and can be dropped to the present order of approximation. As for the first one we note that, using the form (7) for ϕ, it may be written

$$\frac{1}{c^2}\int_{\mathscr{V}}\frac{\partial\phi}{\partial t}\frac{\partial^2\phi}{\partial t^2}d\mathscr{V}=\frac{1}{2c^2}\int_{\mathscr{V}}\frac{\partial}{\partial t}\Big(\frac{\partial\phi}{\partial t}\Big)^2d\mathscr{V}$$
$$=\frac{2\pi}{c^2}\int_R^\infty\frac{\partial}{\partial t}\Big(\frac{\partial f}{\partial t}\Big)^2dr$$
$$=-\frac{2\pi}{c}\int_R^\infty\frac{\partial}{\partial r}\Big(\frac{\partial f}{\partial t}\Big)^2dr. \quad (9)$$

In the last step we have used the fact that, for a function of $t-r/c$, $\partial/\partial t$ equals $-c\,\partial/\partial r$. With these manipulations we find, for the first term in the right-hand side of (4),

$$\int_{\mathscr{V}}p\nabla\cdot\mathbf{u}\,d\mathscr{V}=p_\infty\int_{S_B+S_\infty}\mathbf{u}\cdot\mathbf{n}\,dS$$
$$-\frac{2\pi\rho}{c}\Big(\frac{\partial f}{\partial t}\Big)^2_{r=R(t)}+O(c^{-2}).$$

To $O(c^{-2})$ the integrated term in this expression can be evaluated at $r=0$ rather than at $r=R(t)$ and upon substitution into the right-hand side of the energy equation (4) we find

$$\int_{\mathscr{V}}p\nabla\cdot\mathbf{u}\,d\mathscr{V}-\int_{S_B+S_\infty}p\mathbf{u}\cdot\mathbf{n}\,dS$$
$$=\int_{S_B}(p_\infty-p_B)\mathbf{u}\cdot\mathbf{n}\,dS-\frac{2\pi\rho}{c}[f'(t)]^2+O(c^{-2})$$
$$=-4\pi R^2\dot{R}(p_\infty-p_B)$$
$$-(2\pi\rho/c)[f'(t)]^2+O(c^{-2}), \quad (10)$$

since

$$-\mathbf{u}\cdot\mathbf{n} = \frac{\partial\phi}{\partial r} = \dot{R} \tag{11}$$

at the bubble surface $r = R(t)$.

The calculation of the kinetic energy $\mathscr{T}$ in the left-hand side of (4) is similar although slightly more complicated. We start from

$$\mathscr{T} = \frac{1}{2}\rho\int_{\mathscr{V}} \mathbf{u}\cdot\mathbf{u}\, d\mathscr{V} = \frac{1}{2}\rho\int_{\mathscr{V}} [\nabla\cdot(\phi\nabla\phi) - \phi\nabla^2\phi]\, d\mathscr{V}$$

$$= -2\pi\rho R^2\phi(R,t)\dot{R} - \frac{\rho}{2c^2}\int_{\mathscr{V}} \phi\,\frac{\partial^2\phi}{\partial t^2}\, d\mathscr{V}, \tag{12}$$

where the fact that $\phi \to 0$ at infinity and the kinematic boundary condition (11) at the bubble surface have been used. Assuming now the form (7) for ϕ we find for the second term

$$\frac{1}{c^2}\int_{\mathscr{V}} \phi\,\frac{\partial^2\phi}{\partial t^2}\, d\mathscr{V} = \frac{4\pi}{c^2}\int_R^\infty f\frac{\partial^2 f}{\partial t^2}\, dr$$

$$= -\frac{4\pi}{c}\int_R^\infty f\frac{\partial^2 f}{\partial r\,\partial t}\, dr$$

$$= -\frac{4\pi}{c}\int_R^\infty \left[\frac{\partial}{\partial r}\left(f\frac{\partial f}{\partial t}\right) - \frac{\partial f}{\partial r}\frac{\partial f}{\partial t}\right] dr$$

$$= \frac{4\pi}{c}\left(f\frac{\partial f}{\partial t}\right)_{r=R(t)} - \frac{4\pi}{c^2}\int_R^\infty \left(\frac{\partial f}{\partial t}\right)^2 dr.$$

For the subsequent developments the integral term could be left as written, but it is more interesting to recast it in a form that brings out very clearly its origin rooted in the finite speed of propagation of pressure waves. Let $s = t - r/c$ and note that, for a motion started at $t = 0$, $f(s)$ vanishes for $s < 0$. Then

$$\frac{1}{c^2}\int_R^\infty \left(\frac{\partial f}{\partial t}\right)^2 dr = \frac{1}{c}\int_0^{t-R/c} \left(\frac{\partial f(s)}{\partial t}\right)^2 ds$$

$$= \frac{1}{c}\int_0^t [f'(s)]^2\, ds + O(c^{-2}).$$

Again from (7)

$$\phi(R,t) = \frac{f(t - R/c)}{R}$$

$$= \frac{1}{R}\left(f(t) - \frac{R}{c}f'(t) + O(c^{-2})\right),$$

while

$$\frac{1}{c}\left(f\frac{\partial f}{\partial t}\right)_{r=R(t)} = \frac{1}{c}f(t)f'(t) + O(c^{-2}).$$

Upon substitution of these results into (12) we find the following expression for the kinetic energy:

$$\mathscr{T} = 2\pi\rho\left(-R\dot{R}f(t) + \frac{1}{c}f'(t)[R^2\dot{R} - f(t)]\right.$$

$$\left. + \frac{1}{c}\int_0^t [f'(s)]^2\, ds\right). \tag{13}$$

To complete the derivation an expression for $f(t)$ must be obtained. This task is readily accomplished by substituting the form (7) for the potential into the kinematic boundary condition (11) and carrying out a Taylor series expansion centered at t to find

$$f(t) = -R^2(t)\dot{R}(t) + (R^2/c^2)f''(t), \tag{14}$$

in which the last term can again be dropped. By differentiating (13) with f given by this expression and equating to (10), the form (3) of the radial equation is readily obtained.

In conclusion it may be of some interest to comment on the criteria with which terms have been discarded and retained in the above calculation, criteria that may at first sight appear somewhat arbitrary especially for the intermediate steps. Any ambiguity can be removed by adopting a proper scaling as in Refs. 1 and 2, but the following considerations may furnish an intuitive shortcut.

The precise meaning of the notation $O(c^{-1})$ is that, with respect to those retained, the terms neglected are of the order R_0/cT, where R_0 and T are typical scales for radius and time. In this light, the second term in the wave equation (5), $c^{-2}\,\partial^2\phi/\partial t^2$, cannot be discarded because it is of the same order as the first term at distances $L \sim cT$ from the bubble, as the explicit form (7) makes clear.[10] It also follows from this expression that, at such distances, ϕ itself has a magnitude of the order of

$$\phi \sim \frac{f}{cT} \sim \frac{f}{R_0}\frac{R_0}{cT} = O(c^{-1}),$$

while, by the same argument, $u \sim O(c^{-2})$. With these estimates we recognize that the region $r \sim L$ contributes to the first integral in (8) a term of magnitude $O(c^{-2}) \times O(c^3)$, where the second factor accounts for the volume of integration. The region at distances $r \sim R_0$ gives, on the other hand, a contribution independent of c. After division by c^2, the final magnitude of this term is then $O(c^{-1})$, as the explicit result (9) demonstrates. In the last integral in (8), instead, the region $r \sim L$ gives a contribution of $O(c^{-4}) \times O(c^{-1}) \times O(c^3) = O(c^{-2})$ which, upon division by c^2, is quite negligible. The whole integral can therefore be dropped. Similar arguments may be invoked for the evaluation of $\mathscr{T}$.

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