IMPROVING TEACHERS' CONCEPTUAL KNOWLEDGE OF FRACTIONS THROUGH ONLINE SUBJECT-SPECIFIC PROFESSIONAL DEVELOPMENT

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Dedication

To my mom, Maureen, who has always been my number one supporter and biggest fan. My successes are your successes.

To my husband, Charlie, who has encouraged me throughout this journey. I thank you for your patience, love, and support during this process. Without you, this dream would never have come to fruition. Now that we are finished, I'm ready to write our next chapter. I have a feeling it is going to be an adventure!

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Abstract

Background: Mathematics teachers must possess a thorough understanding of the mathematics content they teach. Content knowledge is particularly essential for teachers responsible for teaching fractions, a principal foundation of algebraic reasoning and structure. Research shows that students who do not grasp the conceptual understanding of fractions in elementary school continue to struggle with these concepts in middle and high school. Therefore, successful instruction of high-leverage concepts that are cardinal in developing advanced mathematics is required for overall student success. Teachers who participate in professional development (PD) opportunities grounded in Specialized-Content Knowledge (SCK) can increase their understanding of subject matter knowledge and build the conceptual fluency necessary for mathematics teaching. However, teaching methods must mirror best practices for learning fractions, such as using manipulatives rather than relying on procedural or memory-based approaches. By including PD opportunities that utilize research-based design principles for adult learners, teachers in this study received instruction on vertical alignment and conceptual fluency to effectively teach fraction concepts. **Purpose:** This study was prompted by the need to develop a relevant online professional development series for elementary and middle school teachers that supported their understanding and instruction of fractions through virtual manipulatives. The study describes the impact of an SCK professional development series on a group of mathematics teachers. **Methods:** This small-scale study included eight participants, all mathematics instructors at a private Prek-8th school in southeast Texas. The participants completed six sessions of professional development, each lasting one hour in duration. The synchronous virtual sessions followed a sequence designed to build both conceptual and procedural fraction understanding in comparison, equivalency, addition, subtraction, multiplication, and division through the use of virtual manipulatives. The evaluation study utilized quantitative methods that included a pre-test and post-test design and analysis to determine participants' change in mathematics knowledge of fractions expected for mastery in the elementary and middle school years. The pre-tests and post-tests were evaluated for participants' use of conceptual representations as justification for their responses. Additionally, descriptive analysis of participants' responses in the form of exit tickets captured participants' ability to apply their learning accurately and navigate the virtual manipulatives employed during the PD sessions. **Results:** The results from the pre-tests (M=61.2250, SD=19.99605) and post-tests (M=92.8750, SD=4.86819) covering fractional knowledge indicated that there was statistical significance in the findings of participants' increased scores upon completion of the PD series, $t(7)=-4.668$, $p = .002$. Furthermore, the evaluation of specific content categories confirmed that all eight participants improved their fractional knowledge. The most significant improvement was seen in the fraction division category, with a 47.95% increase in correct responses. Additionally, the combined exit tickets and evaluation of the use of representations on the pre-tests and post-tests confirmed that participants utilized virtual manipulatives to show their thinking, with an overall increase of 73.75% in the use of representations. **Conclusion:** The pre-test and post-test data showed an increase in fractional knowledge among the participants at the conclusion of the PD. The exit tickets and evaluation of representations used on the pre-test and post-test suggests that participants could apply the virtual manipulatives to create models that showed participants' conceptual understanding of fractions. Overall, subject-specific professional

development increased the participants' understanding of fractional knowledge. Further study will serve to clarify whether the gains are sustainable and impact student learning of fractions.

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Introduction

In a time of increasingly rigorous state standards and a call for students to show proficient gains in mathematics, there is an expectation that teachers must be capable of delivering quality instruction that leads to a deeper understanding of concepts. Studies from national and international data continue to reflect a need for improvement in U.S students' mathematics achievement. According to the National Center for Educational Statistics (2015), 60% of grade 4 students and 77% of grade 8 students scored below proficiency on the National Assessment of Educational Progress. This deficit holds steady, as the 2017 and 2019 results reveal that there has been no significant change in the national average of mathematics scores. The cross-national test, Programme for International Student Assessment (PISA), places the U.S 38th out of 71 countries in mathematics achievement (Desilver, 2017). These alarming numbers have led to severe inquiries concerning the quality of mathematics instruction students receive in the classroom. The attention centers on the teacher's role in the classroom regarding the effect that their subject matter knowledge can have on student outcomes (Darling-Hammond et al., 2019; Hill & Ball, 2004).

The benefit of quality teaching on student achievement has been well documented (Boonen et al., 2014; Hanusheck 2014; Harris & Sass 2011; Jackson et al., 2014; National Science Board, 2016; Stronge et al., 2011) and recognized as the most important school-based factor for student outcomes (Hightower et al., 2011; McCaffrey et al., 2004; Opper, 2019; Rowan et al., 2002; Tucker & Stronge, 2005). Teacher quality can affect student success rates two to three times more than any other school-based factor (Opper, 2019). Furthermore, students placed with highly effective teachers for three consecutive years outscore their counterparts placed with low-performing teachers by over 50 percentile points (Sanders et al., 1977). Studies have reinforced the importance of teacher effects by identifying them as many and perpetual for up to four years (Kupermintz, 2003). These findings have led many educational researchers to agree that the aggregated consequence of ineffective teaching is detrimental to student learning (Jong et al., 2010).

Many classrooms across the U.S are not staffed with mathematically knowledgeable teachers (Hill & Chin, 2018). In turn, this leads to a lack of quality teaching, which is a significant disruption to students' ability to learn (Garcia & Weiss. 2019). Recent studies have found that students achieve higher mathematics success levels with instructors who have a more robust understanding of mathematics content (Blazer & Kraft, 2016; Hill et al., 2008). This finding is contrary to teachers who lack experience and confidence in their abilities, which may contribute to anxiety, having adverse effects on their teaching practices (Manning & Payne, 1993). Furthermore, by interacting with content-based materials, models, and curriculum, teachers' can improve their content knowledge and classroom practices (Ebby, 2000; Sherin, 2002; Newton et al., 2012).

Participation in impactful professional development (PD) can offer teachers the opportunity to provide a curriculum grounded in higher-order thinking skills. The PD sessions should focus on the alignment of teaching methods, multiple representations, and students' thought processes (Darling-Hammond et al., 2019). These findings, coupled with increased attention on teachers' general mathematics content knowledge, have solidified the need for active content-based PD in the area of mathematics (Baumert et al., 2010; Darling-Hammond & McLaughlin, 1995; Phelps et al., 2016). As a result, stakeholders are taking notice and placing their resources in PD opportunities that increase teachers' mathematics knowledge and skills.

Successful mathematics teachers perform operations and construct learning opportunities that lead students to develop the conceptual and procedural fluency required for mathematical understanding. By goal-setting, analyzing, formulating, and differentiating curriculum, effective teachers, use their knowledge to demonstrate and teach complex mathematics concepts (Manning & Payne, 1993). Their awareness of research-based best practices fosters an understanding of mathematical knowledge for teaching (MKT), characterized by the skills, habits, and ability to analyze student thinking and use that information to drive instruction (Hill et al., 2008). Introduced by Deborah Ball (2008) and her colleagues, as a practice-based theory that not only identifies the mathematics content that teachers hold, the model is also concerned with linking that knowledge to the practice of teaching mathematics "through perspectives, habits of mind, and sensibilities." MKT is characterized by interactive mathematics teaching activities within the classroom that include planning, evaluating, explaining, scaffolding, and assessing (Ball et al., 2005). MKT is a critical component for acquiring the insight and understanding needed to identify useful and efficient content delivery models. Therefore, PD that is grounded in the improvement of MKT could lead to increased teacher knowledge.

MKT attempts to extend the framework set forth by Shulman (1986) by categorizing and defining the knowledge that is required for teaching into two general knowledge facets, subject matter knowledge, and pedagogical knowledge. These two categories are further divided into six domains; Common Content Knowledge (CCK), Horizon Content Knowledge (HCK), Specialized Content Knowledge (SCK), Knowledge of Content and Students (KCS), Knowledge of Content and Teaching (KCT), and Knowledge of Content and Curriculum (KCC) (Ball et al., 2008). These domains reflect the knowledge and skills that teachers should maintain to teach mathematics content successfully. The domains represent mathematical reasoning and "demands of teaching" that are typically pertinent only within the classroom. One essential component of this useful teaching model is Specialized Content Knowledge (SCK), mathematical knowledge of the skills exclusive to teaching. This type of knowledge pertains to "unpacking" the methods and solution plans that help teachers decipher and analyze students' mathematical thinking (Ball et al., 2008; Selling et al., 2016). The knowledge includes understanding the tasks required for teaching, such as making sense of students' work, comparing various delivery methods, looking for validity in students' reasoning, and scaffolding mathematics topics (Selling et al., 2016). Unlike its related sub-domains, CCK, and HCK, SCK offers exposure to the knowledge indirectly used for student learning, yet it is not part of the student curriculum (Markworth et al., 2016).

Research has established MKT as a leading conjecture of student success (Hill et al., 2008; Rayner et al., 2009). Additionally, The National Council of Teachers of Mathematics, NCTM, (2000) promotes student support through teaching methods that encourage and challenge students, often differentiating the modes of delivery and employing innovative strategies that are opposite of the traditional approaches to mathematics instruction (Turner & Rowland, 2010). One of the founding principles of NCTM (2000) that is stated in the publication, *The Principles and Standards in School*

Mathematics confirms that quality teaching requires an understanding of the types of content knowledge students hold, what they need to master, and a vast knowledge base of the ways to challenge and support them in their journey to mathematical proficiency. Teachers who become versed in these core understandings will be more capable of applying lessons grounded in MKT. Furthermore, participating in PD that provides development of the specific domains may give teachers the experience and tools necessary to address unpredictable classroom circumstances (Rayner et al., 2009).

The repeated subpar achievement of U.S students, coupled with the absence of qualified teachers, shows a pattern of inadequacy that may stem from teachers' educational experiences and lack of adequate PD opportunities. These weak experiences contribute to the fragmented understanding of valuable mathematics content. The scattered learning, coupled with previous years of insufficient training, has led to over half of U.S teacher candidates failing to qualify on content exams for mathematics on their first attempt (Shuls, 2017). This deficit in training leaves districts tasked with rethinking how teachers obtain the knowledge and understanding necessary for effective teaching while simultaneously delivering content in the classroom (Greenberg et al., 2013). Analyzing PD experiences that intend to increase teachers' understanding of content by improving their subject-specific knowledge, familiarity with alignment, and awareness of current best practices may prove to be the assistance that teachers need to grasp central ideas in mathematics.

Statement of Purpose

A mathematical concept that has proven to be significant and incredibly difficult for teaching and learning is fractions (Behr et al., 1983; Charalambous & Pitta-Pantazi, 2006; Hansen et al., 2015; Lewis & Perry, 2017; Moss & Case, 1999). Labeled in several studies as the most challenging topic in mathematics for elementary school students, failure to master these values can have adverse effects on student learning and future academic success (Behr & Post, 1992; Lamon, 2007; Siegler and Pyke, 2013). Fractions are amid the most compounded mathematical concepts introduced in elementary school (Newstead & Murray, 1998). Similarly, they are also known as the "most intricate" numbers introduced in the early grades (Bulgar, 2003). Furthermore, research shows that a deficit of fractional learning early on extends throughout schooling, with Algebra 1 teachers reporting that students are continually entering the classroom with inadequate knowledge of fraction concepts (Hoffer et al., 2007).

The lack of fractional understanding that U.S students hold has diminished their ability to apply classroom learning to novel situations, even after the arduous task of studying the concept for years. This issue is further exacerbated by the limited number of U.S students who possess the mathematics knowledge essential to continue their studies in fields that require a strong knowledge of mathematics (National Science Board, 2016). These realities have placed policymakers in a position to evaluate, develop, and deliver PD programming to close the gap.

Fractional learning can be viewed as a "house of cards," with mastery relying heavily on the understanding and fluency of several interrelated concepts taught in previous grade levels (Eichhorn, 2018; Rouselle & Noel, 2008). Successful mathematics students possess a strong number sense, understand value and quantity, and connect those relationships with other values. These principal understandings of interconnectivity are the essential building blocks for subsequent computation and application (Yang, 2007;

Witzel & Little, 2016; Tanenbaum et al., 2017). For this understanding to develop, students first need a solid foundation of prerequisite skills (Doabler et al., 2015; Lock & Gurganus, 2004). Therefore, a guiding principle is that teachers need to understand the subject-matter content and the concept of vertical alignment as it pertains to fractions to apply sophisticated strategies in the classroom (Ball et al., 2005; Kaminski et al., 2008). Maintaining an extensive body of knowledge that leads to a deep understanding of the concepts and skills required to deliver learning opportunities beyond surface-level acquisition will promote a successful alignment of standards. Understanding the complete picture of required curriculum objectives is beneficial for promoting mathematics achievement through the consistency of instructional strategies across grade levels (Tanenbaum et al., 2017).

Classified as complex to learn, many students experience tremendous difficulty with fractions (Seigler & Pyke, 2013). However, fractions are also promoted as essential building blocks for students' future success in mathematics (Gabriel et al., 2013; Lamon, 2007; Witzel & Little, 2016). These foundational pieces of the larger puzzle must be in place for learning to extend into the following years of study. Lester (1994) suggested that students' discomfort with these values may stem from their teacher's lack of understanding of the subject. Moreover, pre-service and in-service teachers traditionally struggle with the same issues as their students regarding knowledge and understanding of fractions (Park et al., 2012). Additionally, research shows that teachers with weak math competencies often emphasize procedural steps, resulting in monotonous, detached, and often rote instruction methods (Stigler & Hiebert 1997; Ball et al., 2008). Therefore, the need for teacher PD opportunities that cover an array of standards expected for mastery in the elementary and middle school years of instruction is essential for improving school mathematics.

PD that is linked to classroom activities, established in research-based best practices, sustained in duration, systematically designed, and organized to follow a format that engages adult learners, has proven to be valuable for the improvement of teacher knowledge (Darling-Hammond, 2000; Loucks-Horsley et al., 2010). Targeted PD that strengthens teachers' SCK can encourage instructional best practices, leading to appropriate concrete and pictorial representations when communicating mathematical ideas (Ball, 1993). This improvement in PD can contribute to increased exposure to effective strategies, leading teachers to form a solid foundation of the fundamental principles that support the specific areas required for positive change in the classroom. By identifying possible areas of weakness, policymakers can place themselves in a position to reform future PD opportunities.

Research Question

In what ways does online professional development on specialized content knowledge of mathematics increase teacher knowledge?

Personal Relationship

As I begin my 20th year as a professional educator, one apparent reality stands out: the view on the discipline of mathematics as unfavorable. This problem manifests in the aversion and insecurity of mathematics understanding among teachers, specifically those in elementary and middle schools. This view is troubling since these individuals play a significant role in students' initial mathematics instructional experiences. Despite their mathematics discomfort, these teachers set the stage for how students will view and relate to mathematics for years to come. I have heard more teachers than not making comments about "not being a math person" or talking about how they do not need to understand the reasoning behind specific rules and procedures. This pessimistic view that surrounds the subject leaves little room for students to view mathematics positively, indirectly denying students access to the benefits of a strong mathematics foundation. Mathematics plays a significant role in science, technology, and engineering, all of which are specializations that carry a stigma of difficulty. Students need teachers to guide their development and understanding through practical, real-world problems, with varying representations and authentic discourse. In turn, this calls for teachers to be well-versed in subject matter, best practices, and multiple learning theories that can bring a new balance of breadth and depth to the classroom.

Definitions

Algorithm

A process or set of rules to be followed in calculations or other problem-solving operations. In mathematics, an algorithm is a finite sequence of instructions.

Fractions

A fraction is a number that represents a part of a whole.

Conceptual Understanding. "Learning mathematics with understanding, actively building new knowledge from experience and prior knowledge" (NCTM, 2000). Procedural

Fluency

"The ability to apply procedures accurately, efficiently, and flexibly; to transfer procedures to different problems and contexts; to build or modify procedure from other procedures, and to recognize when one strategy or procedure is more appropriate to apply than another" (NCTM, 2000).

Vertical Alignment

The alignment of lessons and courses prepares learners for the next step in acquiring the knowledge needed to master concepts.

Pre-service teacher

A teacher that participates in a guided teaching program that is supervised by mentor teachers and college professors.

Specialized Content Knowledge (SCK)

Specialized content knowledge is the knowledge and information that students are expected to learn in a content area. It pertains to the ideas, facts, concepts, and theories behind a given subject.

Pedagogical knowledge

Pedagogical knowledge is the type of knowledge that is unique to teaching. It is the working knowledge of how to teach Pre-service teachers. Mentor teachers and college professors supervise a teacher that participates in a guided teaching period.

Methodology

This study seeks to evaluate a SCK professional development program's effectiveness, which focuses on the vertical alignment of fractional concepts found in grades 3-8th as determined by the Texas Essential Knowledge and Skills (Texas Education Agency, 1997). The one-group pre-test and post-test is part of a program evaluation that examines a group of elementary and middle school teachers' professional development efficacy. The Statistical Package for the Social Sciences (SPSS) allowed the researcher to employ a dependent t-test to determine if participants experienced an increase in mathematics knowledge upon completing the PD series. Additionally, the pre-tests and post-tests determined if participants were successful in employing representations to answer questions. The researcher analyzed the exit tickets that participants submitted at the end of each session to determine their fluency with the virtual models employed during the PD. The mathematics content explored during the series consisted of foundational material required for scaffolding fractions in subsequent schooling years. Included were partitioning lengths, equivalency, comparing and reasoning, translating between multiple fractional representations, and employing the four mathematical operations of fractions (addition, subtraction, multiplication, and division). The eight participants from a private school in southeast Texas completed six professional development sessions, each lasting one hour in duration. The program evaluation attempted to determine if teacher participation in the sessions increased their knowledge of factional concepts. As teachers who possess elevated experience and knowledge of skills and methods are better equipped to practice those strategies, therefore improving student-learning opportunities (Creswell & Clark, 2007; Goe 2007; Wiswall, 2013).

Limitations of the Study

As with any study conducted, limitations must be identified. The limitations of the study were the lack of qualitative data collected to address teacher learning trends. The researcher examined the pre-test and post-test scores but could not verify emerging themes from qualitative data that links teachers' areas of struggle during the PD sessions and the post-tests' results. This study was also limited in its generalizability by the absence of data regarding the PD sessions' direct effect on student outcomes. Furthermore, not all of the participants included in the study hold certification or degrees in mathematics. Most of their knowledge is extrapolated from their own experiences as students in basic mathematics classes and their previous teaching experiences. Over half of the participants, 57%, have taught primarily in private schools where there is a higher chance of leniency on hiring certification requirements. Finally, the subjects were not required to obtain any subject-specific PD hours before this experience.

Summary

The ongoing studies that show U.S students are unable to meet the standards required for mathematics proficiency have led to a call for examining current teaching practices. The chronic difficulty that students are experiencing with mathematics, namely fractions, has become commonplace in elementary and middle school classrooms (Hecht & Vagi, 2010; Tsai & Li, 2016). To ensure that this cycle does not continue, schools must offer quality PD opportunities that consider the content taught and the skills and methods required for teachers to succeed in the classroom. PD opportunities that aim to improve MKT through experience and familiarity with SCK activities can give teachers a much-needed understanding of student thinking and research-based best practices. The methods used for an inquiry were the dependent t-test, comparison of the use of representations from the pre-tests and post-tests, coupled with the results of the assessed exit tickets. The difference between the tests' mean scores determined if the participants experienced a change in their mathematical content knowledge.

Chapter II

Literature Review

This study aims to explore if a subject-specific professional development series based on relevant fractional learning can improve teachers' specialized content knowledge. This chapter will review preliminary background research concerning fractions and effective professional development strategies. Included are the following sections: (1) importance of fractions; (2) U.S teacher and student proficiency with fractions; (3) teacher preparation; (4) effective professional development models; (5) virtual manipulatives; (6) mathematical content knowledge for teaching; and (7) social-constructivism learning theory.

Importance of Fractions

Fractions are often the target of cynical math humor. As the joke goes, "three out of two people understand fractions." (Gabriel et al., 2013). Consistently proven to be an area of difficulty for both teachers and students, the concept of fractions has long evoked feelings of angst and anxiety within the education community (Tirosh, 2000; Moss & Case, 1999). Fractions are viewed as among the most challenging and abundant concepts within school mathematics, yet necessary for overall student success (Ball, 1993; Harvey & Averill, 2012; Lamon, 2007, 2012; Ma, 1999; Newstead & Murray, 1998). Fractions are included in the curriculum as early as 2nd grade in most state standards and are compulsory for success throughout middle school (Siegler et al., 2011; Van de Walle et al., 2016). Understanding foundational skills, such as fractions, are crucial for students' ability to build connections and scaffold learning during the elementary and middle school years (Booth & Newton, 2012).

Fractions are embedded in multiple strands of mathematics, including probability, proportional thinking, and algebraic reasoning. The value of understanding rational numbers is beneficial not only within mathematics, but also in biology, engineering, sociology, and psychology (Lortie-Forgues et al., 2015). Fractional understanding is included in various study fields, prioritizing mastery and fluency as the concept reaches far beyond the mathematics classroom. Further confirming this idea is the prevailing view of Algebra I as a gatekeeper to advanced learning opportunities in mathematics and science (Laughbaum, 2017; Rech & Harrington, 2000). Success in Algebra 1 is recognized as a requirement for acceptance to classes such as Pre-calculus, Calculus, and Chemistry. As a determining factor in college acceptance, a lack of strength with fractions places restrictions on possible career options post-graduation (Booth & Newton, 2012).

The National Mathematics Advisory Panel asserts that proficiency with fractions is a central objective for K-8 mathematics education (U.S Department of Education, 2008). The cyclical nature of the concept leads to repeated appearances in state and national standards, therefore rendering them an invaluable component for student success in mathematics (Association of Mathematics Teacher Educators, 2017; National Governors Association Center for Best Practices, 2010). Additionally, findings show that students' experience with fractions directly affects their overall academic achievement patterns (Hill & Ball, 2004; Swanbrow, 2012). Thus, mathematics educators worldwide continue to acknowledge the importance of rational number concepts, specifically fractions, as one of the key learning objectives for measurable success (Behr et al., 1983; Litwiller & Bright, 2002; Mack, 1990).

A vital part of understanding fractions lies within the idea of fraction magnitude (Fuchs et al., 2016; Schneider & Siegler, 2010). Fraction magnitude refers to the amount of the given unit that a fraction represents. Studies indicate that understanding fraction magnitude, shown by correctly ordering fractions using linear representations and estimation, is heavily correlated to overall mathematics success (Fazio et al., 2016). The focus on linear coherence with fractions is essential to mastering grade-level fraction concepts and students' subsequent mathematics fluency. Moreover, students' understanding of number line tasks in first-grade acts as a precursor to mathematics achievement through the fifth grade (Geary, 2004). Evidence links numerical knowledge at age four as a predictor of mathematics achievement at age fifteen (Siegler et al., 2012; Watts et al. 2014). Consistent with the importance of mastering fraction magnitude, a study that included students from eighth grade and community colleges found that when asked to identify the larger of two fractions, their accuracy was measured at 70% (Schneider & Siegler, 2010; Siegler & Pyke, 2013).

Individuals with a strong understanding of fraction magnitude are more capable of succeeding in working with fraction arithmetic (Brynes & Wasik, 1991; Jordan et al., 2010; Siegler et al., 2011). These results indicate that a student's mastery of fraction magnitude at a young age may be a mirror of mathematical comprehension in middle school and beyond. Early understanding of fraction magnitude offers students an underlying framework for acquiring various mathematics concepts presented in grades 6th-8th, such as fraction arithmetic and algebra (Bailey et al., 2012; Siegler & Pyke, 2013). Therefore, mastery of fraction magnitude is necessary for students' continued progress within numerical development (Siegler et al., 2011).

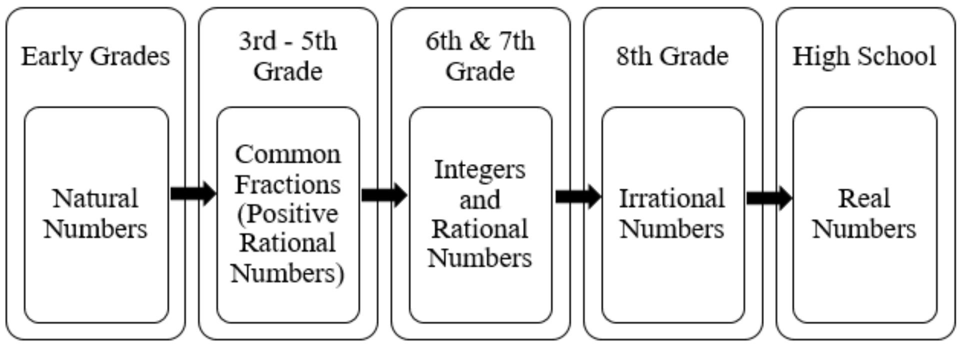
The introduction of fractions into the mathematics curriculum has proved incredibly difficult for elementary school students (DeWolf & Vosniadou, 2015; Siegler et al., 2011; Torbeyns et al., 2015). One of the first signs of confusion can be found in the misconception of executing the same rules and properties of whole numbers onto fractions (Torbeyns et al., 2015). One such example is the misconception that the operation of multiplication always increases the value of the product, which is not the case with fractions (DeWolf & Vosniadou, 2015). Furthermore, The Theory of Numerical Development suggests that children who have not solidified fractional concepts view them as whole numbers (Siegler et al., 2011). Studies have referred to this concept as whole number bias (Mack, 1995; Ni & Zhou, 2005; Siebert & Gaskin, 2006).

Additionally, students continue to view fractions as two whole numbers with a line separating the values, counteracting students' ability to view fractions as values. Similarly, students frequently think of fractions as countable and discrete, the way they do with whole numbers (Vamvakoussi and Vosniadou, 2010). This way of thinking does not work within fractions since they can carry several forms to express a single magnitude (DeWolf & Vosniadou, 2011).

There are many common misconceptions concerning fractional learning's correct trajectory and concepts introduced throughout the specific grade-levels. The Real Number System's learning progression shows the expected movement from natural numbers to real numbers, originating in the early grades and culminating in H.S (Neagoy, 2017). This progression lays an essential framework for all teachers to understand and include in curriculum planning, regardless of their current grade level assignment. As seen in Figure 1, the progression is essential when attempting to reinforce foundational skills that act as a scaffold to subsequent learning.

Figure 1

Grade-by-Grade Progression from Natural to Real Numbers



Note. Recreated from “Unpacking fractions: Classroom-tested strategies to build students’ mathematical understanding”, by M. Neagoy, 2017.

The abstract nature of fractions often results in students' confusion with the subject. (Behr et al., 1983: Charalambous & Pitra-Pantazi, 2006; Goldstone & Son, 2005; Moss & Case, 1999; Newstead & Murray, 1998). When learners do not understand the structure of fractions and what value they represent in relation to previously learned mathematics concepts, such as whole numbers, the result is often a weak connection of material and increased mathematical errors (Askoy & Yazlik, 2017; Aksu, 1997). With fractions expressed by the quotient, a/b of integers, where the denominator, b, is non-zero, understanding these values becomes less intuitive. Fractions represent two types of values, local and global; meanwhile, whole numbers hold only a global value (Gabriel et al., 2013). When considering fraction comparison, the brain has to deal with at least two local values and the global value; this is opposed to working with whole numbers that require only one value to process (Gabriel et al., 2013). Evidence of this thought process is apparent when students explore cardinality principles or aim to identify the quantity of many objects. Cardinality refers to counting objects, with the last number being the total number of objects present (Starkey & Cooper, 1995; Van de Walle et al., 2014). While a fraction does not represent a specific whole number amount; instead, it is a portion of another amount. This transition calls for students to switch their thinking from an additive perspective of combining the absolute values to multiplicative thinking, where they are not provided with the exact value of the whole (Clark & Kamii, 1996; Sieman et al., 2006). Furthermore, the absolute number is different from the relative number, since it is dependent on other values. Fractions have a multifaceted nature that requires learners to be cognizant of how fractions are applied to many situations (Charalambous & Pitta-Pantazi, 2006; Brousseau et al., 2004; Kieren, 1993).

Fractions, first organized into four interrelated categories by Kieren (1976), act as a ratio, operator, quotient, and measure. These categories have been subject to various revisions and extensions throughout the years. Kieren interpreted fractions as a ratio in terms of a comparison between two quantities. For example, when there are two cats for every four dogs, cats' ratio to dogs is 2:4. The resulting fractions would then be cats representing 2/6 of the group and dogs 4/6 of the group. As an operator, a fraction can increase or decrease the quantity, as in multiplication or division of fractions. The quotient category applies to the result of the fraction division. The final category, measure, is associated with two connected ideas: fractions as numbers related to specific amounts and the length of intervals (Gabriel et al., 2013; Kieren, 1976). Behr et al. (1983) revised Kieren's model to include the critical part-whole model as a fifth category. The part-whole model refers to the continuous partitioning of a quantity into equal-sized parts, having the same effect on a set of discrete objects. Once mastered, these categories become the basis for all other interactions of fractional computation. Furthermore, a limited understanding of fractions' various meanings affects students' ability to generalize fraction concepts, make connections, and build fluency within the categories (Hackenberg & Lee, 2015)

U.S Teacher and Student Proficiency with Fractions

The 1983 report A Nation at Risk highlighted the necessity of U.S students' educational success, linking aptitude with "safety and prosperity" (National Commission on Excellence in Education, 1983; Siegler et al., 2010). More than 35 years later, U.S students' mathematical progress is still lagging behind many other industrialized countries, including East Asia and Europe (Organisation for Economic Co-operation and Development, 2019). The insufficient progress has made it difficult for U.S students to compete globally within mathematics (Hossain & Robinson, 2012; National Center for Science Education, 2007).

The overall mathematics deficit in the U.S can be attributed to a lack of knowledge in foundational skills. Fractions, being a cornerstone of mathematics, are an essential scaffold to future academic success. (Hecht et al., 2003; Van Steenbrugge et al., 2014; Van Steenbrugge et al., 2010). Unfortunately, the proficiency levels of U.S students' fractional knowledge has continued to fall short. With an increased percentage of students lacking the conceptual understanding necessary for mathematical competence, the learning gap is widening, creating crippling effects for students as they move through their academic careers. Poor knowledge of rational numbers, specifically operations involving fractions, have been recognized by 1,000 algebra teachers as a leading barrier in students' ability to master algebra content (Fazio et al., 2016; Hoffer et al., 2007). As reported in a national poll, students enter with inadequate preparation of rational numbers and operations that require basic knowledge of fractions and decimals (Hoffer et al., 2007). Even with the early introduction of fractions within state standards and the number of grade-level content strands that incorporate fractional concepts, students face challenges with fractional understanding (Asku, 1997; Behr et al., 1992).

The substandard mathematics education in the U.S is apparent, especially when considering achievement rates in fractions. Many studies have found that U.S students are either falling behind or stagnating in their understanding of fractional concepts. The 2013 NAEP found that 40% of fourth-grade students could not establish that the fractional unit of thirds was greater than the fractional units of fourths, fifths, and sixths (Fuchs et al., 2016). Correspondingly, in a nationally representative sample of 20,000 eighth-grade students, only 24% successfully estimated a fractional sum. Furthermore, when the same questions were given 40 years later, the average percent of correct answers was only raised by a dismal 3%, moving competence from 24% to 27% (Siegler et al., 2017). The National Center for Education Statistics (2009) revealed that U.S students experienced difficulty ordering fractions from least to greatest (2/7, 1/12, 5/9), with only 49% of eighth-grade students answering correctly. This deficit is also seen with the concept of division of fractions, with only 55% of eighth-grade students able to solve division word problems. The lack of mastery in the early years of instruction continues to impede learning acquisition throughout high school, with less than 30% of eleventh-grade students displaying the ability to convert decimals into fractions (Kloosterman, 2010).

These staggering statistics show that difficulties with fractions have a long-term stability pattern, which includes severe consequences for those unable to successfully master at an early age (Hecht & Vagi, 2010; Mazzocco & Devlin, 2008; Siegler & Pyke, 2013). Therefore, U.S students' weak understanding of fractions consistently shows the need for reform in the way fractions are taught throughout the elementary and middle school years.

U.S. Lower and Middle School Teachers' Knowledge of Fractions

The repeated findings that point to U.S students lacking in foundational skills that are required for advancement in mathematics has been a troubling topic for decades (Behr & Post, 1992; Klein, 2003; Lamon, 2005; Siegler et al., 2001; Van de Walle & Lovin, 2006). Fractions are a content strand within the mathematics curriculum that educators at nearly all levels find to be more complicated than expected, often reporting that their students have an initial resistance to the concept (Van de Walle et al., 2016). Studies have shown that students experience difficulty making sense of fractions and cannot navigate word problems that include embedded fractional concepts (Neagoy, 2017). As fluency and knowledge of fractions are considered essential for success in mathematics, educational institutions are turning their attention to teachers' understanding of the concept and that way in which they deliver instruction (Hill & Ball, 2004; Lortie-Fugues et al., 2015; Van Steenbrugge et al., 2013).

It is well-documented that the school-based factor that holds the most influence over students' academic progress is the quality of the teacher in the classroom (Goe, 2007; Sanders & Horn, 1998). Their teaching habits have an opportunity to produce cumulative and enduring effects on students' learning patterns (Boaler & Zoido, 2016;

Darling-Hammond, 2000; McCaffery et al., 2004; Rice, 2003; Rivkin et al., 2000; Tucker & Stronge, 2005). Therefore, the classroom teacher should hold an extensive understanding of concepts, be fluent with the scaffolding curriculum, and clearly understand student learning patterns. In addition to establishing this body of knowledge, teachers must also be skilled in the ways students think about mathematical processes, the common misconceptions they hold, and how to navigate their creative pathways when building content coherence (Darling-Hammond et al., 2020). This practical knowledge then lays the infrastructure for classroom interactions that promote deep conceptual learning while positively developing students' disposition toward mathematics. Therefore, powerful instruction begins with a knowledgeable teacher that can significantly influence student learning through effective teaching practices.

A documented area of struggle for U. S teachers is found in their understanding of fractions' meaning and how efficiently they carry out the specific steps required in fractional computation (Ball, 1990; Lin et al., 2013). Correspondingly, research indicates that many elementary school teachers exhibit a limited understanding of concepts and procedures when dealing with fractions (Garet et al., 2010; Ma, 1999). The lack of knowledge may stem from the limited number of grade-level standards that U.S teachers are exposed to, as it is restrictive compared to the extensive training teachers receive in other countries. Additionally, U.S teachers frequently remain at the same grade level for extended periods, which results in familiarity in a single set of benchmark standards (Moseley et al., 2006). With training that does not include an overall vertical focus on the exposure of concepts, U.S teachers do not gain insight on how their limited teaching can negatively exacerbate future academic success. Therefore, teachers need to fully understand the learning requirements found across grade-levels and the pedagogical best practices associated with applying those standards (National Council of Teachers of Mathematics, 2014). To teach advanced concepts such as fraction multiplication and division, teachers themselves must build a core understanding of how the processes work from a conceptual standpoint.

The inability to recognize appropriate paths and scaffolds to advanced concepts shows a deficit of mathematics knowledge in U.S teachers (Ma, 1999). Correspondingly, the studies have revealed several instances of U.S students receiving inadequate mathematics instruction from the early elementary grade levels through the middle school years (Martin & Herrera, 2007). The deficiency of knowledge is often attributed to mathematics teachers that are lacking fluency and knowledge of the skills necessary to convey mathematics and demonstrate its "elegance and power" (Association of Mathematics Teacher Educators, 2017).

A survey conducted by Ball (1990) found that U. S elementary and middle school teachers had difficulty explaining basic mathematical ideas, such as division of fractions, which is considered a 5th-grade benchmark in most U.S curricula. The report found that the minimal focus on covering the spectrum of essential understanding resulted in teachers weakened subject-matter knowledge. Consequently, teachers who do not understand mathematical concepts typically focus on algorithms and memorization tools rather than crucial underlying concepts and pictorial representations. In contrast, teachers who deeply understand mathematics concepts are more likely to teach conceptually, using multiple representations to help students understand concepts. When the teacher is aware of the vertical alignment, they are more equipped to identify the critical strategies in organizing and tying together a solid mathematical base.

Teachers that understand mathematical ideas are more apt to relate concepts found in student work to familiar contexts while linking the material to students' prior learning (Pasley, 2011). These teachers tend to be versed in collaboratively solving students' problems while uncovering the origins of student misconceptions. The process of understanding and addressing student difficulties is vital to students' mathematical growth, as teachers can clarify and connect the learning (Behr et al., 1983; Moss & Case, 1999). Furthermore, when the same situation is presented to less knowledgeable mathematics teachers, they rely on correct answers in the teacher manual to respond to students' questions. Moreover, these teachers with limited understanding of the mathematics concepts have been found to stray from the provided instructional material, such as curriculum guides and textbooks, which can lead to misrepresenting or distorting the core learning that students are expected to master. Teachers' discomfort with the concepts affects their confidence in effectively teaching conceptual understanding (Hudson, 2017). Furthermore, studies have found that pre-service teachers' mathematical understanding is narrow and rule-bound, often limited to the procedural aspects of mathematics (Ball, 1990).

Conceptual and Procedural Knowledge

Success with delivering any mathematical domain includes both conceptual and procedural knowledge. Aptitude with the two is required for teachers to guide students in developing the understanding that will allow advancement to upper-level mathematics (Geary, 2004; Hiebert, 1986). Conceptual knowledge refers to understanding more than just isolated facts and procedures; instead, it builds a framework composed of interwoven relationships, ideas, and patterns (National Research Council, 2001; Newton, 2008). Furthermore, Hiebert and Lefevie (1986) found that conceptual knowledge is characterized most clearly as a rich process in relationships. When knowledge is connected through a web of familiar interconnected work, connections and patterns start to play a significant role. These connections allow the learner to link ideas and build a schema that supports the application of information to novel situations and creates a path that establishes proficiency in mathematics (Hiebert, 1986; Schneider & Stern, 2010). Additionally, conceptual knowledge is concerned with the quality of the knowledge gained, paying particular attention to the abundance of the connections built (Star & Stylianides, 2013).

In comparison, procedural knowledge refers to correctly and successfully applying procedures to mathematics problems while recognizing appropriate strategies (National Council of Teachers of mathematics, 2000). Procedural understanding is the ability to automatically solve problems while using few cognitive resources (Gabriel et al., 2013; Schneider & Stern, 2010). Hiebert and Carpenter (1992) referred to procedural learning as identifying the patterns and systems of mathematics composed of skills and step-by-step procedures. These steps are presented without explicit reference to mathematical ideas; therefore, the meaning of the process is lost, since the learner cannot relate a mathematical construct to the procedure. There is a greater chance of disconnect that can happen when procedural learning is the sole means of instruction (Carpenter et al., 1996). Deficits are seen in cases where students try to add with multi-digit numbers, starting with the ones place-value, working from right to left is the most conventional algorithm in use and is seen as efficient. However, without the understanding that each place value only holds a single digit, with the additional values being re-grouped, students begin to merely follow a procedure with no knowledge of why it produces a correct answer. Similar to Skemp's (1978) sense of Instrumental Understanding, the student is regurgitating a memorized process without mathematical reasoning behind their actions. Moreover, the steps employed are characterized by constructs that bind the knowledge to one particular situation, such as a skill or strategy, making it difficult to transfer that learning to different situations (Byrnes & Wasik, 1991).

Conceptual and procedural knowledge are mutually supportive, as studies have shown that competence in one leads to a more solid skill set in the other (Hecht & Vagi, 2010, 2012; Rittle-Johnston & Siegler, 1998; Schneider et al., 2011). Students need to participate in activities geared toward understanding mathematical procedures before the procedures are introduced. Evidence suggests that reversing the order makes it more difficult for learners to circle back and relearn the concept (Mack, 1990; Wearne & Hiebert, 1988). Furthermore, studies have shown that instructional programs that initially highlight students' ability to understand conceptually can lead to mathematical learning without forgoing the understanding of procedures (Hiebert & Lefevre, 1986). Therefore, a teacher's understanding of both classroom elements is essential in building instructional plans and delivery methods for the appropriate progression of fractional development.

Conceptual understanding, being a necessary component of building a sufficient mathematics experience for students, is crucial for teachers' mathematics understanding. However, concurring results show that many mathematics teachers are unable to communicate the required level of conceptual understanding when dealing with the concept of fractions (Ball, 1990; Depaepe et al., 2015; Li & Kulm, 2008; Lortie-Forgues et al., 2015; Ma, 1999). Furthermore, U.S pre-service teachers experience difficulty explaining the meaning of fraction division, with the average percentage of correct responses being 26% (Ball, 1990). Recent studies have pointed to U.S teachers' deficit of conceptual understanding compared with teachers from other nations, such as China (Li & Kulum, 2008). These studies confirm that U.S teachers tend to opt for the algorithm of invert and multiply when given fraction division problems instead of producing reasoning for how they arrived at a correct answer. Simultaneously, the Chinese teachers were able to give sufficient explanation with ease, justifying their work through solution plans rooted in conceptual based understanding (Li & Kulm, 2008; Lortie-Forgues et al., 2015 Ma, 1999).

Further confirming this idea of U.S teachers lacking knowledge is found within their deficit of reasoning behind solution plans, that lack conceptual understanding. U.S middle school teachers have difficulty extending coherent reasoning for why the invert and multiply algorithm works in division of fractions (Borko et al., 1992). Moreover, in a comparison of American and Japanese fourth-grade teachers, over their understanding of fractional operations, American teachers were only able to employ the part-whole construct, where fractions are viewed as a comparison of two values, the number of equal parts and the number of total parts (Moseley et al., 2006). The American teachers continued to display that same sub-construct, even when it was not appropriate. Simultaneously, the Japanese teachers proved their thinking through rational, referencing underlying sub-constructs far beyond what was seen with the American teachers' explanations.

There is a wide range of expectations set on teachers to understand their grade-level curriculum and recognize the importance of mathematical reasoning and sequencing strategies that lend themselves to review and reinforcement in later years of instruction. Successful teachers must be versed in delivery modes, maintain awareness of practical procedures, and connect ideas far beyond the knowledge that most receive in their formal training (Ball, 1990; Hill & Ball, 2004; Ma, 1999). Studies indicate an increasing number of educators are entering the teaching field without the requisite mathematical knowledge base necessary for vertical familiarity or the deep conceptual understanding required to effectively convey mathematics subject-matter (Darling-Hammond et al., 2019; Garcia & Weiss, 2019). The mastering of underlying processes and operations is essential for learners to obtain a clear trajectory of the learning that is the basis for delivering quality instruction, leading to improved student learning (Fennema & Franke 1992; Fennema et al., 1996). These elements are especially relevant at the formative elementary level and middle school levels where teachers' understanding of fractions is not as stable as is required to deliver effective instruction (Ball, 1990; Cramer et al., 2002; Luo et al., 2011; Reeder & Utley, 2017; Tobias, 2012; Utley & Reeder, 2012). Mathematics is structured much like a ladder with concepts building from previous encounters; a students' understanding is subject to how well their ladder is constructed. Otherwise, seen as how precise their teachers are with scaffolding content and delivering information.

Developing a conceptual understanding of content is more complicated than memorizing route procedures. The process typically involves more of a focus on the experience of applying ideas (Loucks-Horsley at el., 2010). Those who understand the underlying conceptual understanding of a discipline can access critical facts and principles that can be used to refine, justify, and extend additional questioning models. They are aware of the *"language"* of the discipline and are able to use that understanding as a scaffolding tool to create learning opportunities extending beyond the conventional processes taught in the classroom (Duschl & Osborne, 2002).

Skemp Relational and Instrumental Learning

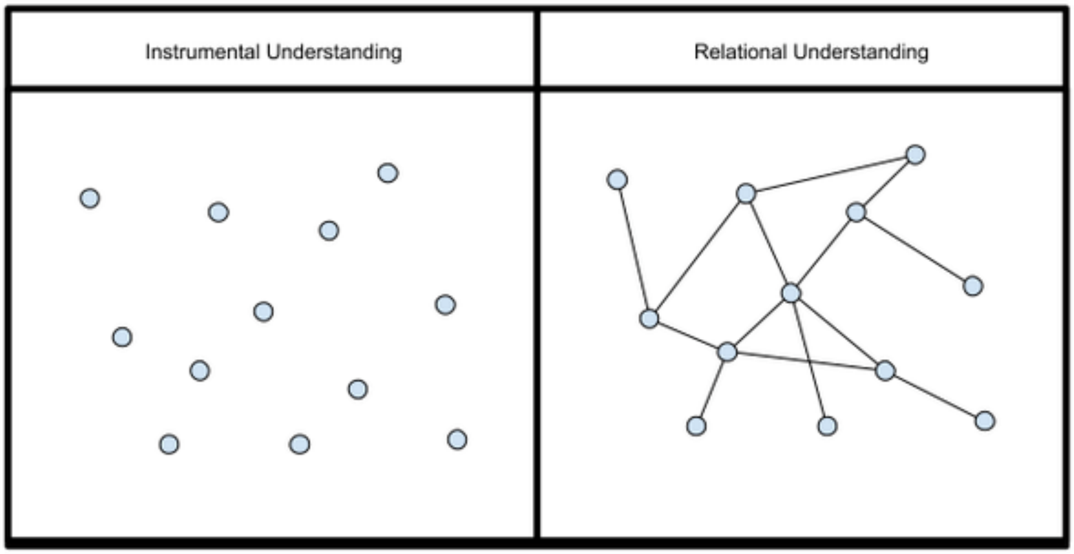
Skemp (1989) categorized teachers' mathematical knowledge into two distinct categories: relational understanding and instrumental understanding. Instrumental understanding can be thought of as knowing the rules and procedures of a problem without understanding the reasoning behind why those methods generate the correct answer. This type of learning is similar to a rote style, where students mostly memorize facts and algorithms by way of repetition (Chubb, 2020). Moreover, the constant review of the same process is thought to help students recall information. Instrumental learning is concentrated on the student completing a specific task using formulas and plans without the forward-thinking of how the task fits into the larger picture of mathematics or how it is similar to other mathematical procedures (Wees, 2011). Furthermore, proficiency in instrumental understanding often results in high performance on calculations and a student's ability to apply the learned procedures in a similar context. However, learning does not transfer to other situations (Skemp, 1989).

The second category of learning is relational understanding, linking the web of conceptual connections that sustain mathematical knowledge as a whole (Holmes, 2013; Skemp, 1978; Wees, 2012). This learning is focused on the cognitive relationships that exist between background knowledge and the new information acquired. Once that meaningful connection is built, problem-solving strategies can develop by exploring how and why procedures work. Learners can participate in practices that emphasize reasoning and explaining processes through the use of multiple representations, uncovering how the content is related and embedded into other strands of mathematics (Leinwand & Fleischman, 2004). Furthermore, students that are initially taught instrumentally have a lower chance of choosing correct solution paths due to their inability to associate new information with prior knowledge (Mack, 1990; Wearne & Hiebert, 1988).

Students often have difficulty transferring their learning of area and perimeter to subsequent practices when taught with instrumental methods (Pesek & Kirshner, 2000). Although, when initial leaning is done with relational methods, the students can show their understanding using conceptual and flexible methods to model their thoughts (Leinwand & Fleischman, 2004). Learning that highlights concept acquisition as a web of ideas creates an anchor for the new concepts, constructing meaning through a deeper connection with the material. Moreover, relational understanding can be viewed as meaningful because it builds relevant mathematical connections that are essential for future learning opportunities (Turmundi 2012).

Figure 2

Teachers' Mathematical Knowledge



Note: From "Ineffective professional development – The Reflective Educator" by D. Wees, (2011). *The Reflective Educator*.

Fraction Instruction in the U.S.

A comprehensive and conceptual based curriculum is imperative to student success. It is a core component that sets the foundation of mathematics learning, yet this has not historically been the case in U.S classrooms (Murata, 2008; Schmidt, 2012). Inquiry into K-12 mathematics instruction shows that the U.S mathematics curriculum prioritizes procedures and memorization of strategies, guiding students to view mathematics as a compilation of rules and facts that must be committed to memory (Darling-Hammond et al., 2020; Richland et al., 2012). U.S educators have a long-established history of employing algorithms from the onset of formal instruction and continuing with that mode of delivery. The method leaves little room for students to experience why those processes work (Huinker, 2002; Vanhille & Baroody, 2002). This tendency to elect rote based procedures has been observed in studies where students were asked what it meant to be good at mathematics. Only 13% replied with answers that would be considered conceptually-based (Rattan et al., 2012). This view was consistent as those same students proceeded to answer questions using algorithms, even when the researchers purposefully presented problems that did not require rote procedures.

Fractions appear in the U.S mathematics curriculum as one of the initial encounters children have with values beyond whole number arithmetic (Siegler et al., 2010). The quality of the introduction of fractions contributes significantly to students' progress or difficulty with the concept (Chan, et al., 2007; Cramer et al., 2002; Paik & Mix, 2003). Researchers Van de Walle and Lovin (2006) stress the importance of offering students' abundant experiences to develop number sense related to fractions. Their belief is that the learner should first acquire a knowledge base built through multiple conceptually-based activities, leading to a full understanding of the concept. As accomplished learners first develop a conceptual understanding that can aid in their working knowledge of procedural structures. This transition offers students the ability to flexibly apply the knowledge and transfer that understanding to multiple situations (Bransford et al., 1999). In comparison, students who are taught procedures and rules of mathematics during the initial stages of discovery fail to understand connections and fluency with fraction magnitude (Fazio & Siegler, 2010). For many students, initial experiences with classroom instruction of fractions are a framework that places value on procedures. A strong focus has historically been placed on memorization and rote processes that generate an answer to specific situations or problems (Darling-Hammond et al., 2019). With many students' misconceptions of fractions stemming from weak conceptual knowledge, there is a call for instruction that favors practices geared at improving student understanding and fluency with mathematics (Fazio & Siegler, 2010).

Student exposure to fraction algorithms and procedures early in their development can be detrimental to numerical reasoning (Asku, 1997; Kamii & Clark, 1995). The focus on computation favors memorization over building a clear awareness of mathematical concepts and connections, leading students to perform algorithms without understanding the reasoning behind choices (Kerslake, 1986). This issue is present when students can execute the steps involved in fractional operations, yet cannot articulate the conceptual reasoning for those actions (Byrnes & Wasik, 1991). In this way, these learners do not employ efficient strategies, such as estimation, to justify their answer (Pantziara & Philippou, 2012). Moreover, the student's inability to link new fractional learning with prior knowledge of mathematics may lead to further misinterpretations, affecting the learner's opportunity to scaffold content (Amato, 2005).

Students' can employ their conceptual understanding of situations to make meaning of procedures and adapt them to new tasks (Halford, 1993; Gelman & William, 1998). In presenting procedures first, students work with mathematical symbols and procedures without adequate familiarity with their meaning (Gabriel et al., 2013). Furthermore, children in the U. S are more likely than not to practice rote procedures repetitively, such as with worksheets or flashcards. These practices can amplify the misunderstanding of mathematical symbols (Byrnes & Wasik, 1991). Saenz-Ludlow (1995) found that in order for the process to hold meaning for students, it must be represented by mental operations and conventional notations. Therefore, students who learn how to apply procedures before conceptually understanding, may struggle considerably when asked to apply them to abstract concepts (Duzenli-Goklap & Sharma, 2010). Currently, the literature surrounding the idea of teaching with a heavy emphasis on algorithms has shifted. It is no longer sustainable to offer fraction instruction that favors procedures over conceptual understanding. The call for more in-depth learning of foundational skills requires mathematical arguments and rationales tied to procedures and conceptual understanding (Yackel & Cobb, 1996).

Fractions in U.S. Textbooks

The format and description of fractions within U.S textbooks support the lack of understanding that we see in the classroom, with many problems being procedurally based (Ozer & Sezer, 2014). In a study of 8th-grade textbooks from the U.S, it was found that 81% of the questions asked focused on procedural knowledge, with only 9% established in conceptual understanding. Furthermore, the textbooks seldom use models and manipulatives when introducing new curricula (Van de Walle et al., 2014) even though considerable evidence indicates the importance of visual models as powerful modes of delivery. Visual models aid in the formation of mental images that can lead to increased understanding of fractional concepts (Cramer et al., 2008; Siebert & Gaskin, 2006; Skemp, 1989).

U.S textbooks frequently employ a stereotypical area model that amplifies the part-whole relationship (Cady et al., 2015; Zhang et al., 2015). This model is often represented by edible items such as pizzas, cookies, pies, and brownies (Freeman & Jorgensen, 2015; Eichhorn, 2018). The dependence on area-related models favors the understanding directly linked to the context, proving challenging to transfer understanding of fractions to other representations or contexts. Moreover, students answering fraction related questions through area models have difficulty applying that understanding to real-life contexts. Fifth-grade students in the U.S that had been exclusively instructed using the area-model were found to be unable to represent similar thinking through alternate models (Zhang et al., 2015). Therefore, the copious reliance on the area model proves difficult in transferring to visualizations of other models, impeding student capacity to form mathematical connections.

In contrast to relying on one model set for the acquisition of fraction knowledge, a body of research is forming that promotes the use of multiples models when teaching fractions (Moss & Case, 1999; Siegler et al., 2010; Van de Walle et al., 2016; Zhang et al., 2015). Focus on instructing students through several visual models, including measurement activities, number lines, and manipulatives such as geoboards, pattern blocks, Cuisenaire Rods, and partitioning activities, can foster the deep understanding that is required for students to obtain mathematical success (Eichhorn, 2018; Moss & Case, 1999; Siegler et al., 2010; Fazio & Siegler, 2011). When presented with the limited option of experiencing fractions through the area-model, learners see that model as the only visual representation of fractional relationships (Clarke et al., 2008). To expand knowledge of fractions, students must become mathematically versed in the alternate models. Students who are given ample opportunities to explore fractions through multiple representations can become more fluent in the abstract structural similarities between them, promoting a deeper conceptual understanding (Zhang et al., 2015).

Concrete, Pictorial, and Abstract (CPA) and Manipulatives

The use of concrete objects to introduce and scaffold mathematics understanding is a topic that has been well documented (Browder et al., 2010; Jimenez & Stanger, 2017; McNeil & Jarvin, 2007; Siegler et al., 2010). Many leading educational researchers and organizations have identified manipulatives' use as a powerful representation in understanding mathematics concepts (Maccini & Hughes, 2000; Mercer & Miller, 1992). Furthermore, the NCTM standards continually emphasize the importance of employing various mathematical models, specifically with fraction learning, to connect mathematics learning to real-life situations (National Council of Mathematics Teachers, 2000). The addition of manipulatives to the classroom offers students an alternative representation of challenging mathematical concepts.

The period of concrete operations is a time frame where students explore and demonstrate their understanding of concepts through relevant use of concrete objects and associated symbols (Piaget, 1954). Additionally, the use of concrete objects to model mathematical thinking spans decades, with Skemp (1989) stating that there is significance in utilizing physical objects to build foundational understanding that can lead to fluency with abstract ideas. Manipulatives are seen as dynamic learning tools that can guide students' conceptual understanding of mathematics topics and help retain material covered (Sowell, 1989). This idea is reflected by the American cognitive psychologist Jerome Bruner, who believed that laying the foundation of concrete learning is essential in progressing to an abstract understanding of concepts. Bruner (1966) proposed three binding modes of representations that students must pass through; enactive, iconic, and symbolic. Enactive is the action-based stage, where students interact physically with concrete objects; the iconic stage is when images are stored as sensory models or pictorial representations. The symbolic stage encompasses the ability to order, classify, and work within symbolic representations. This sequence's trajectory allows students to build understanding, grasp connections, and understand mathematics concepts. This format ensures that students interact with the content and identify the relationships presented, instead of just producing an answer.

Grounded in the findings of Jerome Brunner, the CPA (Concrete-Pictorial-Abstract) model for mathematics instruction is a framework that allows teachers to guide and track students' conceptual development of mathematics (Chang et al., 2017). The CPA model is synonymous with the Singapore Math method. This learning progression utilizes an intentional sequencing of topics to reinforce critical connections required for a conceptual understanding of mathematics topics. The model utilizes existing knowledge to scaffold new learning through concrete manipulatives that intend to bridge mathematics learning within the pictorial and abstract stages (Hoe & Jeremy, 2014). The CPA sequence of teaching has proven to enhance mathematical connections, guiding students to increased mathematics achievement (Witzel & Little, 2016).

Additionally, this type of instruction has shown to be effective with students who struggle with mathematics, specifically when displaying deficits in understanding basic computation strategies (Sousa, 2008). Therefore, a teacher's ability to provide students with meaningful representations within the curriculum can significantly influence their continued success in the classroom (Kang & Liu, 2016). The modeling acts as a foundation for students to develop a more robust understanding of the content strands and bridge the relationships between kinesthetic and visual experiences (Tran et al., 2017).

With findings that show the use of the CPA sequence as more effective than the traditional abstract-level instruction, it is essential that educators understand the model and plan lessons that reflect this type of valuable instruction (Butler et al., 2003; Maccinni & Ruhl, 2000).

Teacher Preparation

There is a considerable number of current U.S mathematics teachers that lack proficiency with foundational procedures concerning fractions. This deficit contributes to their inability to articulate the alignment necessary in demonstrating a conceptual understanding of how fractions function within mathematical operations (Osana & Royea, 2011; Ball et al., 2008; Thompson & Saldanha, 2003; Tirosh & Graeber, 1990). The Association of Mathematics Teacher Educators (2017) found that teaching is often viewed as an autonomous skill set that does not require training in specific content areas expected to be taught in the classroom. This perception overlooks the importance of subject-specific training regarding the knowledge, skills, and methods required for effective delivery. Teachers need to understand their grade-level content and be cognizant of the preceding and subsequent material that their students will encounter. Practical training should include access to mathematics opportunities that lead to a deeper understanding of the methods, skills, and relationships that are often excluded from basic standards alignment and curriculum planning (Hill et al., 2008)

It is essential to understand how fractions have been taught as that history impacts how teachers present and explore fractions with their students. Studies confirm that fraction content mastery is not as prevalent as it should be in U. S teachers (Ball, 1990; Cramer et al., 2002; Luo et al., 2011). This lack of deep understanding leads to challenges when explaining fractions to students (Chinnappan, 2000). Furthermore, the issue stems from the teachers' misunderstandings from their previous learning. The lack of formal instruction and time spent exploring the concepts with depth makes it difficult for teachers to convey the connections required in mastering fractional concepts (Reeder & Utley, 2017).

Mounting evidence suggests that U.S mathematics teachers tend to favor rules and procedures when teaching the concept of fractions. The narrow instruction can be attributed to the absence of conceptual knowledge and reasoning skills presented in their formal training (Reid & Reid, 2017; Ma, 1999). The inadequate preparation and lack of clear trajectory in understanding standards addressed in the classroom contribute to the shortfall of deeply conceptualized knowledge required to teach mathematics content effectively (Association of Mathematics Teacher Educators, 2017).

The average U.S elementary school teacher must only complete approximately 1.3 math courses during their undergraduate preparation (National Center for Education Statistics, 2018). The minimal measure of subject-specific training does not promote the necessary in-depth study required to master the critical mathematics elements needed to develop a solid understanding of mathematical relationships and patterns. A deficit of learning can be seen in recent studies that indicate a minuscule 20% of U.S elementary school teachers rate their knowledge of fractions as strong (Ward & Thomas, 2007).

Furthermore, the U.S Department of Education found that roughly two-thirds of elementary school teachers state that they hold a less than adequate understanding of subjects they teach (Greenberg et al., 2014). With inapt teacher preparation, students' can become more susceptible to acquiring misconceptions resulting from teachers' weak knowledge (Newton, 2008). Findings show that when teachers experience misunderstandings on foundational mathematics concepts, such as fractions, they tend to rely on algorithms and shortcuts without having the ability to make justifiable reasoning for their choices (Ball et al., 2009; Holm & Kajander, 2011; Tirosh & Graeber, 1989). Documented accounts of teachers' limited knowledge of fractional understanding chronicle how their limitations are passed onto students through the simplified definition of functions and novice examples that do not provide a solid foundation for the subsequent development of functions (Stein et al.,1990). Therefore, mathematics teachers may not possess a rich enough understanding of fraction content to encourage students to engage deeply with abstract concepts (Ma, 1999).

Teachers are the essential factor in students' learning, as their influence and quality of teaching methods can heavily impact mathematical learning (Darling-Hammond, 2000; Eichorn, 2018). Moreover, studies indicate that a teacher's coursework can contribute significantly to the quality of teaching that they display in the classroom (Baumert et al., 2010; Campbell et al., 2014; Goe, 2007; Seidel & Shavelson, 2007). Strong correlations have been found between teacher content knowledge and mathematics scores on standardized tests for first and third-grade students (Charalambous et al., 2019). Additionally, the type and amount of mathematics coursework that a teacher engages in often leads to a direct and consistent relationship to their students' success in mathematics (Rice, 2003). Unfortunately, patterns of lacking instruction and surface-level understanding ultimately pervade classroom instruction and, therefore, student learning. These findings may suggest that poor student performance results from inadequate teacher preparation and professional development opportunities that U.S teachers experience (Ma, 1999).

Teachers need to develop an in-depth understanding of the content required for guiding students through a rigorous curriculum. The evidence lies within studies that continually show the significance of teacher knowledge over student understanding (Seidel & Shavelson, 2007). This recurring idea of teacher knowledge has shifted educational stakeholders' attention to the quality of U.S teacher preparation programs and the professional development (PD) opportunities that are available once in the field (Hill et al., 2004; Shulman, 1986). U.S teachers are the result of a flawed system that does not require candidates to possess thorough mathematical knowledge before entering the classroom (Ball et al., 2005). With the vast structuring of teacher preparation programs, findings point to a limited experience in mathematics content training, especially when compared to other industrialized countries (Ma, 1999; Stigler & Hiebert, 1999). Currently, the limited exposure to methods courses does not include practice with the demanding curriculum needed for U.S. students to compete internationally (Schmidt, 2012). When considering fractions, U.S teachers' understanding and application show a deficit compared to China, Japan, and Germany (Carnoy & Rothstein, 2013). The current system provides candidates with an inadequate number of mathematics methods courses and professional learning experiences, which has led to an urgent need for a more focused professional development model to address the absence of vital mathematics content knowledge (Association of Mathematics Teacher Educators, 2017).

The restructuring of teacher certification programs redefines pre-service teachers' criteria, showing that these programs are evolving. This change can also be seen with PD offerings that fluctuate in program design, framework, and principles (Darling-Hammond et al., 2009; Guskey & Yoon, 2009). With credentials varying from state to state and even district to district, there is a lack of uniformity that determines the content and benchmarks for teacher excellence. Currently, the number of in-field teachers, those who teach subjects matching their training or education, for mathematics is low. Studies have found that the percentage of elementary and middle school mathematics teachers holding subject-specific credentials was far less than in other disciplines (Rice, 2003; Darling-Hammond, & Carver-Thomas, 2016).

Additionally, only 5% of elementary school teachers have obtained sufficient mathematics endorsements. The shortage is further represented in the 2018 findings from the National Survey of Science and Math Education that show that 3% of elementary school teachers surveyed held a degree in mathematics. Meanwhile, 45% of middle school teachers and 79% of high school teachers completed a degree in mathematics education (Durrance, 2019).

The exiguous percentage of teachers who can't demonstrate the basic mathematics proficiency standards is a contributing factor in the continuous cycle of ill-prepared mathematics students. Therefore, a shift is occurring within districts to analyze the current frameworks while searching for PD opportunities to close teacher learning gaps. Improving student learning will only come from improving teachers' education (Tatto & Senk, 2011). Those stakeholders involved in preparing mathematics teachers must establish learning opportunities that resolve to intensify the acquisition of understanding, proficiency, and dispositions necessary to provide all students with equitable mathematics practices (Association of Mathematics Teacher Educators, 2017).

The field of mathematics education has continued to encounter a decreasing number of qualified mathematics teachers exiting pre-service programs. Reports from The Education Commission of the States show that teacher shortages are habitually confined to specific subject areas such as math and science (Aragon, 2016). Furthermore, teachers who have not received remediation for lacking skills or subject-specific training resort to teaching in the same sequence and way they have been taught (Borg, 2004). Researchers refer to this as the "apprenticeship of observation," the idea that teachers, especially novice teachers, rely on the thousands of hours they spent in the classroom as students to model their current teaching practices (Kennedy, 1991). Often this is seen in classrooms that favor direct instruction over student-centered learning. In theory, these teachers know what differentiation is, how to create small groups, and are aware of appropriate times to engage students in discourse. However, they resort back to the rote algorithms, and teacher-centered strategies they believe are teaching hallmarks.

Furthermore, the conversation circles back to teacher quality, in terms of the ability to deeply understand and teach the necessary components (Newton et al., 2012). Increasing overall student outcomes requires teacher preparation models to be reconfigured, raising the quality of mathematics in schools through districts reexamining and expanding their professional development offerings to mathematics teachers (Even, 2014). Creating professional development that improves teachers' understanding of content may be the key element to improved classroom instruction.

Professional Development

With billions of dollars invested each year in PD programs that intend to improve teaching and learning within school districts across the country, stakeholders are looking for the value that these programs bring to education (Darling-Hammond et al., 2017: Desimone & Garet, 2015; Garet et al., 2011). Many are disappointed to find out that there is little evidence to link professional development and positive student learning outcomes (Guskey & Yoon, 2009). In a study published by the American Institutes for Research, of over 1,300 professional development studies, only nine met the standards of credible evidence set by the What Works Clearinghouse, a sector of the U.S Department of Education responsible for providing scientific evidence for policymakers (Yoon et al., 2007). With PD currently viewed as a general subject that can encompass various delivery modes, it is difficult for school leaders to pinpoint their institution's most pronounced choice (Desimone, 2009). The National Mathematics Advisory Panel's report (2008) found that most of the professional development offered is descriptive and lacks the methodological rigor needed to connect the opportunities to outcomes that can be seen as valid. This trend is even more pronounced for mathematics-focused PD aimed at elementary school teachers (Polly et al., 2013). Teacher preparation and continued PD at the elementary school level is often approached as an overview of topics and procedures. This general structure does not allow for explicit teaching on the way students' think about numbers, the instructional approaches that support mathematical learning, or how the scaffolding curriculum strands connect to current content (Association of Mathematics Teacher Educators, 2017). Often the focus of these sessions concentrates on data analysis of student test scores, implementation of a new curriculum, and understanding the scripted textbook, leaving little time to deepen teachers' understanding of content (Hill & Ball, 2004; Hill & Grossman., 2013;). The focus on all-encompassing professional development sessions has led to a "mile-wide and inch-deep" preparation, which supports a system that produces educators who cannot access the demanding curriculum essential in today's classrooms.

Teachers often perceive PD opportunities as limited and ineffective. In a report from the Economic Policy Institute, Garcia and Weiss (2019) found ample room for improvement of PD offerings that aid in retaining quality teachers and help educators expand their repertoire of subject-matter knowledge. Moreover, numerous PD approaches are available that intend to increase teacher knowledge, but are not specific enough to transform instructional improvement (Fullan, 2007). These approaches are not designed around maximizing teacher learning. Instead, they are a quick fix created to satisfy a need, usually having a short duration and no formal evaluation to reference (Hill & Grossman., 2013). Findings have detailed how teachers can apply mathematical concepts such as fractions using rote procedures and algorithms. Although when asked about solution plans, teachers cannot communicate why the algorithm worked or how that same procedure is applicable in other instances (Borko at el., 1992). Yang (2007) drew similar findings when exploring number sense strategies employed by pre-service teachers in understanding fractions. The problems presented called for teachers to work with the four arithmetic operations for real numbers. The data collected showed that one-third of the participants were able to employ efficient number sense strategies. The remaining two-thirds resorted to rule-based strategies that were highly dependent on standard written algorithms. Many participants struggled with basic mental math concepts such as estimation, choosing to provide an exact answer instead of following through with the directions. This lack of awareness with essential mathematical knowledge is disturbing when considering that fractions are identified as one of the central concepts that serve as a precursor to mathematical success in subsequent years of schooling (Ball at el., 2005; Siegler at el., 2012).

It is a long-standing belief that teachers must understand a subject thoroughly in order to deliver effective teaching. This idea is so prevalent that it has been included in federal mandates, such as Every Student Succeeds Act (ESSA) as a primary contributing requirement in being considered profession ready (U.S Department of Education, 2017). Mathematics teachers are expected to enter the classroom with specialized content knowledge (SCK), the knowledge strictly related to mathematical content, and essential for competent instruction (Ball et al., 2008). This knowledge is required to help teachers understand the framework that builds within the curriculum and is essential when mathematical issues arise in the classroom. Teachers who have a solid understanding of SCK are better equipped to understand the complexities of the content and map the interconnected strands for students. Teachers' deep understanding of subject matter makes it possible to engage in more advanced discussions and evaluations of their students thinking processes. Furthermore, studies have found that teachers' knowledge of key mathematical concepts was enhanced through professional development programs, resulting in the addition of an increased number of open-ended questions and more targeted classroom discussions with students (Swafford et al., 1997). Experiences such as these give an insider's view of the content knowledge that ultimately influences the available teaching depth. It is no secret that good teaching provides a gateway to future success. Teachers are currently juggling a continually changing curriculum, classroom management issues, state and federal mandates, overcrowding, standardized testing, and many other behind-the-scenes issues that make it increasingly difficult to focus on effectively teaching students.

Furthermore, many teachers are placed in situations that have to lead to their professional development becoming stagnant or non-existent, which leads them to fall back on methods and strategies that are not conducive to the changing landscape of education. Targeted PD opportunities should include the skills necessary to promote student learning, affording the contingency to identify the commonalities and related strands that exist (Hill et al., 2008; Evens, 2014).

Professional development in mathematics structured to reflect the discipline's nature has proven effective (Loucks-Horsley et al., 2010). Employing a model that offers teachers an opportunity to think and reason mathematically in a discovery-oriented atmosphere provides opportunities to discuss, model, and engage in reflective practices vital in constructing new meaning and formulating ideas (Darling-Hammond et al., 2009; Yoon et al., 2007). Empirical research on characteristics of learner-centered PD call for real-world tasks that require teachers to pose questions, work with multiple problem-solving strategies, and cultivate useful models that display their learning (Bransford et al., 1999; Polly & Hannafin, 2011). Securing effective PD includes building structured sessions to guide new teacher understanding, allowing for experience in applying newly learned skills (Stigler & Hiebert, 1997).

The Learning Policy Institute's paper *Effective Teacher Professional Development* addresses the need for a sophisticated teaching model that is required to obtain a deep mastery of rigorous curriculum through problem-solving, communication, and critical-thinking activities (Darling-Hammond et al., 2017). The paper provides evidence of a positive and direct relationship between teacher PD and valuable classroom practices. The reviews of 35 studies covering PD show that there are seven commonly used features of effective PD: content-focused, sustained duration, active learning, collaborative environment, use of models, expert support, and ample time for feedback and reflection.

Delivering PD that will alter instructional practices requires a framework that provides ample time to translate changes into practice (Darling-Hammond et al., 2009; Loucks-Horsley, 2010). The often fragmented and sporadic PD opportunities in the form of one-shot workshops have dominated the field of mathematics education (Darling-Hammond et al., 2017). These surface-level opportunities do not offer the time necessary for a multifaceted approach that intends to change teachers' skills and competencies (Darling-Hammond et al., 2009; Garet et al., 2001). Various studies have indicated that mathematics PD is most effective when sustained, allowing for experience in deepening teachers' knowledge while promoting transformative practice (Gore et al., 2017; Hill & Ball, 2004; Kennedy, 2016).

Effective PD is commonly characterized by a heavy emphasis on collaboration (Dash et al., 2012). Collaborating is the act of purposefully building interpersonal relationships while working towards healthy interdependence, which occurs when teachers are comfortable giving and receiving assistance without forfeiting accountability (Bannister, 2018). A collaborative PD series begins with common goals and is continually changing. Sessions are improved upon by practice, developing a shared vision that establishes group norms and expectations while fostering a sense of school community. The shared process is enhanced by the strong social networks created within the experience. Engaging in a collaborative platform that facilitates discovery and best practices provides educators with support and learning opportunities that can foster a student-centered culture (Friedlaender et al., 2014).

Unfortunately, there are plenty of PD opportunities that fall into the traditional single conference or workshop model that does not sustain the essential components to extend and reflect on ideas learned. Many of the sessions tailored to educators do not encompass the connections required to scaffold; instead, they are unrelated snippets of information that leave educators uninspired.

Criticism over the state of teachers' PD points to opportunities that are "shallow, fragmented, and unfocused" (Hawley & Valli, 1999). Inadequate offerings have consistently been prevalent in schools (Nieto, 2009), including minimal professional development activities offered as a snapshot of what is needed. Working from a top-down model, administrators are often the decision-maker on the chosen topic, with teachers acting as an audience rather than engaging in any critical learning that will impact their practice. Furthermore, The Center for Policy Research labeled nearly all PD as "pedagogically naive," referring to the lack of knowledge teachers obtain from their participation (Gigante & Firestone, 2008).

Furthermore, teachers are looking for PD that provides understanding on how to use the acquired information post-session (Darling-Hammond & McLaughlin, 1995) Clear indicators in current research have called for teacher development in the area of subject-specific content knowledge. The content focused developmental pieces provide teachers with specialized knowledge, relevant practices, and opportunities for interaction with different instructional strategies. Professional development should allow for reflection opportunities, both individually and collectively, and time to connect learning to what is relevant for classroom instruction (Van Driel & Berry, 2012). Providing authentic professional development that targets teachers' learning deficiencies by modeling effective techniques, coherent content scaffolding, and reflective space for professional collaboration can improve teacher fluency and understanding of concepts (Butler et al., 2003; Harris & Sass, 2011).

Online Professional Development

The vast array of online opportunities currently available to teachers has continued to improve and increase over the years, as their convenience and adaptability make them viable options for many (Bates et al., 2016; Fishman et al., 2013). In deciding on an online PD option, one can choose between asynchronous programs that offer learning on the participants' schedule or synchronous, remote learning that requires participants to attend real-time sessions (Bates et al. 2016). Providing effective options in either of the formats mentioned above depends on the design and execution of maintaining a viable program that offers tenets similar to that of in-person experiences (Darling-Hammond & McLaughlin, 1995; Fetzner, 2013). Employing technology, online resources, and tools can offer new ways to support and enhance learning. Furthermore, participants have a first-hand view of the types of learning experiences they are expected to bring to their students (Desimone, 2009)

Online PD offers facilitators the option to customize and tailor learning to the individual needs of the participants. Through flexible and open pacing of offerings, participants have the option to learn at their own pace and revisit material when they deem necessary (Wynants & Dennis, 2018). The platform encourages collaboration through chat rooms, direct and whole group messaging strings, virtual manipulatives to collaboratively solve mathematics problems in real-time, and the capability to share screens with other participants (Francis & Jacobsen, 2013). Furthermore, technology makes it possible to track participant input and participation, improving accountability, and future outcomes in the classroom.

Virtual Manipulatives

The use of virtual manipulatives is fast becoming a part of the mathematics landscape in K-12 education. These models have emerged as a promising tool for students to explore and develop a deeper understanding of mathematics content (Moyer-Packenham & Westenskow, 2013; Sarama & Clements, 2009). A virtual manipulative, is defined as "an interactive, web-based visual representation of a dynamic object that presents opportunities for constructing mathematical knowledge" (Moyer et al., 2002, p. 373). As a dynamic representation of traditional concrete manipulatives, the virtual manipulatives replace the typical classroom model and improves educators' and learners' experience.

Virtual manipulative use provides an interactive climate that fosters engagement through unique options and additional features exclusive to the nature of virtual opportunities (Sarama & Clements, 2009). The evolving digital platform allows for collaboration through the possibility of offering immediate feedback and extensions that enable learners to represent their thinking in ways that have previously not been possible with traditional concrete manipulatives (Steen et al., 2006). The expanding offerings of virtual manipulatives allow modifying the shape, color, amount, and size of the models selected. The dynamic representations are offered in both 2-D and 3-D options with the potential to slide, flip, turn, redirect, and rotate the models according to the learner's needs. Furthermore, students can create side-by-side comparisons of models that enhance and bridge their awareness and understanding of the relationship between abstract values and visual models (Lee & Chen, 2005). Features of virtual manipulatives highlight the iconic and symbolic connection of mathematics learning by offering pictorial representations connected to numerical content (Reimer & Moyer-Packenham, 2005). The versatility can be seen in several offerings, such as the fraction application from The Math Learning Center. The fraction application gives learners an option to hide or publish numeric labels, color in sections of the fraction bars, and superimpose fractional pieces on each other, showing comparison and equivalency. Additionally, virtual manipulatives are offered in unlimited supply, making it cost-effective for school districts and less time consuming for teachers to prepare lessons (Moyer et al., 2002). The abundant options available for educators make it possible for students to not only access these scaffolding materials in the classroom but wherever there is internet access, making them invaluable for distance and hybrid learning.

As virtual manipulatives are becoming more prevalent in classroom instruction, studies have begun to emerge that suggest there is success with teaching fractional concepts through the use of virtual manipulatives, especially compared to traditional paper-pencil instruction that tends to rely heavily on abstract models (Reimer & Moyer-Packenham, 2005). One such study from Moyer-Pakenham and Suh (2005) examined fifth-grade students' understanding of fraction equivalency and addition. The researchers established that the virtual manipulatives supported the prevention of common mathematical errors, allowing students to engage in discovery learning through experimentation and identification of mathematical relationships. Furthermore, using virtual manipulatives enhances fractional learning by providing the option to create fractional pieces that have not been readily available (i.e., 7ths or 11th). (Bouck & Flanagan, 2010). Currently, there is limited research on how virtual manipulatives can effectively be incorporated into teacher PD. However, research does indicate that the flexibility of virtual manipulatives makes them powerful resources for mathematics learning (Reimer & Moyer-Packenham, 2005).

Virtual manipulatives promote learning acquisition that is often equal or more enhanced than traditional concrete manipulatives (Dorward & Heal, 1999). Students interact with the virtual manipulatives similarly to the concrete models, accurately utilizing technology to communicate their thinking (Burris, 2013). Furthermore, the landmark publication, *Principals to Action* (2014), NCTM, requires "successful mathematics teachers to engage students in activities that enhance mathematical knowledge through various modes of delivery." Therefore, offering students at all levels the assistance that visual models can provide. (National Council of Teachers of Mathematics, 2014).

In order for students to experience the full benefit of virtual manipulatives, teachers must be versed in the various features of the platform and how they can include the mode of delivery into the classroom (Uttal et al., 1997). A teacher's understanding and fluency with manipulatives is a vital piece for successfully integrating them into the curriculum, as studies indicate a direct correlation with a teacher's experience with models and their ability to effectively bridge the idea of manipulative use with mathematics (Moyer 2001). Furthermore, employing a practice alone does not guarantee that learners will be able to extract conceptual understanding. Teachers must participate in professional development to experience and understand how the models can contribute to mathematics learning objectives.

Mathematical Content Knowledge for Teaching

The practice-based theory of Mathematical Knowledge for Teaching (MKT) is defined as the mathematical knowledge, skills, and dispositions required for teaching (Ball et al., 2008). The theory identifies the teaching domains based on the Learning Mathematics for Teaching (LMT) project. The LMT project studied the mathematical knowledge required for teaching; specifically, the types of fluency that led to successful student instruction. The MKT model has been utilized extensively with educators, both in the US and abroad, as it acts as a framework to assess a teachers' quality of instruction (Santagata & Lee, 2019; Delaney et al., 2008). The model, which is grounded in Shulman's (1978) idea of the seven forms of knowledge as a basis for teaching, has been continually refined and extended by several educators and researchers.

Additionally, the MKT model highlights the particular knowledge base required for teaching. It takes into consideration the interconnected relationship between knowing mathematics and teaching mathematics. Ball et al. (2008) referenced Shulman's categories, placing focus on subject matter knowledge and pedagogical content knowledge. Additionally, MKT addresses the tasks that teachers complete daily, such as explaining mathematical processes and procedures, assessing students' work, addressing curriculum alignment, and various other functions that are the underlying reasoning behind teachers' decision-making (Ball et al., 2008).

MKT blends the mathematical understanding common to individuals working in diverse professions with the mathematical knowledge exclusively found in education

(Hill et al., 2019). This standard of mathematical learning places emphasis on classroom management concerning how activities are facilitated. Therefore, teachers must be versed in several domains that only a classroom teacher would need. Such as the specifications of time restrictions allotted for assignments, the structuring of assessments, types of praise offered, differentiation of questioning, lesson planning, monitoring classroom behavior, and ways to formulate explanations and clarifications in student-friendly contexts (Shulman, 1986).

The MKT model comprises the two main types of knowledge required to successfully teach mathematical content for classroom purposes; subject matter knowledge and pedagogical content knowledge (Ball et al., 2008). Housed under the domain of subject matter knowledge is Common Content Knowledge (CCK), Horizon Content Knowledge (HCK), and Specialized Content Knowledge (SCK). CCK refers to the knowledge that is only used for teaching, specifically the knowledge of mathematics that other professions would not need to access (Hill et al., 2008). Horizon Content Knowledge is the awareness of the overarching ideas in mathematics education and how they are connected. Specialized Content Knowledge is a unique domain with relevance that lies within the comprehension of a robust conceptual understanding required for teachers to know more than merely how to "do" mathematics. SCK directly aids in developing a mathematical understanding that applies to teaching (Ball et al., 2008).

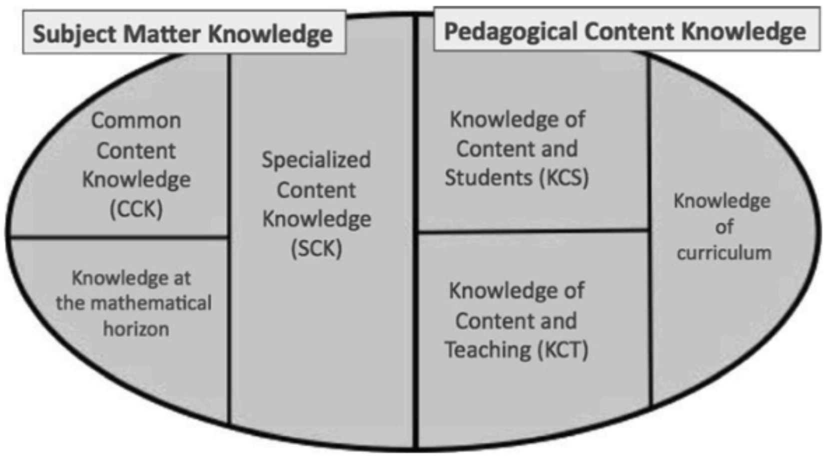
Additionally, there is more than just skill that is required to teach mathematics effectively. Instead, there is a type of understanding that enables one to provide students with explanations to understand how to analyze student responses for accuracy and employ appropriate pictorial models when representing concepts (Ball at. el., 2008;

Rittle-Johnson et al. 2001). The overall deficit in teachers' MKT can deter their ability to discern and evaluate students' understanding, which can hinder productive conversations and scaffolding opportunities (Goldsmith et al., 2013). Furthermore, gains in student achievement, specifically in the elementary grades, have been linked to a teacher's MKT (Hill et al., 2008; Selling et al., 2016). Therefore, effective teaching must include SCK to build students' understanding of concepts.

SCK is essential in understanding how to transform a text into effective instruction that students can access. In drawing from the unique connections between content knowledge and pedagogical knowledge, teachers can better structure learning situations and articulate flexible responses. Teachers must also know how to translate their thinking into relevant terms to students, as there are instances of experienced middle school teachers unable to present content effectively (Borko et al., 1992). Additionally, research shows that mathematics teachers experience difficulty when drawing representations to explain their algorithms, unable to select an appropriate operation to correctly answer problem-solving exercises (Izsak et al., 2012). Therefore, teachers need the depth and breadth of the content that reaches far beyond the skill set needed to mimic algorithms. Instead, they must be capable of connecting their current learning to overarching concepts that extend meaning (Ma, 1999). This concept of a need to expand ideas is also referred to as the Profound Understanding of Fundamental Mathematics (PUFM) (Ma, 1999). Teachers need this wide-spread knowledge to build fluency and conceptual understanding of content as evidence of their proficiency (Schoenfeld & Kilpatrick, 2013).

Figure 3

Domains of Mathematical Knowledge for Teaching



Note: From “Content Knowledge for Teaching: What Makes it Special?” by Ball et al., 2008, *Journal of Teacher Education*, *59*(5), p. 403 (https://doi.org/10.1177/0022487108324554).

The increasing standards alignment efforts in states and districts have influenced growing concern about teachers' understanding of fractions and their ability to effectively teach students the conceptual and procedural knowledge required to succeed in mathematics. Many teachers hold a limited knowledge of scaffolding concepts necessary for students to move through the content. It is expected that current and prospective teachers have acquired these skills through pre-service and on-going PD sessions. However, this is not common practice in many programs offered. Studies have shown that teachers' mathematical knowledge is often rooted in their own educational experience with mathematics. Additionally, teachers who have an inadequate understanding of fractions must experience a more rigorous and comprehensive mathematics education if they are expected to be effective in the classroom (Park et al., 2012).

Social-Constructivism Learning Theory

The Mathematical Knowledge for Teaching model stresses the importance of both subject-matter and pedagogical knowledge by addressing the connection between teachers' mathematical knowledge and practice. A goal of any effective PD model includes the presentation of content in a way that values social interaction as a means to create a collaborative workspace (Hurst et al., 2013). As a model for teacher learning, social interaction encourages sharing strategies and emphasizes mathematics through collaborative problem solving (Darling-Hammond & McLaughlin, 1995). Furthermore, this type of model emphasizes the view of teaching as a social activity that requires learners to actively participate in creating an understanding of new content through meaningful dialogue and activities that mimic real-life situations (Dewey, 1963; Routman, 2005; Vacca et al., 2011).

The social constructivist framework supports the idea of learning as a social activity that occurs within purposeful contexts while favoring interaction over abstract learning methods (Dewey, 1938). A central component is that the process of learning happens over time, with initial knowledge adjusting and reconfiguring into new pathways that confirm and challenge the prior knowledge attained (Brooks & Brooks, 1993). Moreover, Vygotsky (1978) highlights the fundamental role of social interaction in the development of cognition. His theory strongly suggests that community function is imperative when learners are constructing meaning. The act of obtaining knowledge is a collaborative process of assembling understanding as the learner interacts with culture and society. Vygotsky (1978) believed that everything learned could be categorized into two levels; the social and individual level. The latter is heavily reliant on the students being taught at their zone of proximal development. This zone lies within the space of exploration, a place where the student is mentally capable of working with the concepts but requires assistance through social interaction to develop the meaning and further categorize it within their previously learned knowledge (Briner, 1999). The Zone of Proximal development is based on the notion that individuals learn best when interacting with what would be considered a more skilled person, be it a teacher or peer, to facilitate learning through support (Shabani et al., 2010; Vygotsky, 1978). Quality learning should occur through interactions that promote discourse, debate, and rigorous problem-solving (Polly et al., 2017). A collaborative environment that encourages learners to think critically through authentic discourse and apply the newly taught knowledge is central to sociocultural learning theory. Classrooms that promote this learning model are required to staff teachers that have mastered the depth of the concepts presented as well as an understanding of the practices that are beneficial to the learning development patterns of their students.

Chapter III

Methodology

Purpose of the Study

The study intended to determine if subject-specific professional development of fractions can increase teachers' mathematical knowledge. With mathematics scores from the 2019 National Assessment of Educational Progress test leaving the U.S student achievement rate at a standstill since 2009, the education community is looking at causes of this stagnation (National Center for Education Statistics, 2019). One thought that has gained traction in the past few decades is the focus on the content knowledge and effectiveness of the teacher in the classroom, thus calling for educators to acquire and maintain a deep and concrete understanding of the subject matter they teach (Ball et al., 2005; Kahan et al., 2003; Ma, 1999). Furthermore, research has continually shown that teachers often lack the understanding necessary to lead students through the required curriculum successfully. Evidence of this deficit is found in studies that reflect U. S students displaying inadequacies in conceptual understanding of key mathematical ideas that are critical in advancing within grade-level targets (Aksoy & Yazlik, 2017). Exploring these shortcomings has led to findings in the *Psychological Science Journal* that point to a correlation between success in algebra and overall mathematics achievement in high school, linked to a student's understanding of fractions at age ten (Seigler et al., 2012). For students to be versed with the resources required for future success, they must be prepared by teachers with a profound and fluent understanding of the curriculum, skills, and models necessary for effective instruction.

This study's results are included to establish a design for evaluating and modifying future teacher learning opportunities. The hypothesis was that teacher participation in a specialized content knowledge professional development series focused on fractional learning through virtual manipulatives would increase their conceptual understanding of fractions' foundational practices. Through the use of concrete models, teachers explored learning similarly to their students, strengthening necessary skills in content and gaining familiarity with how students learn. This experience aimed to increase teachers' understanding of vital fractional concepts while laying a foundation for quality classroom lessons.

A critical tenet of professional development provides participants an experience that aims to improve the existing structure (Hill & Grossman, 2013). Program design elements such as content-focus, inquiry-based questioning, collaborative experiences, and exploration with multiple models have been shown to maximize teacher learning (Garet et al., 2001; Hill & Ball, 2004; Penuel et al., 2007). Furthermore, educators are continually seeking ways to implement adequate PD opportunities that are tied directly to the relevant ongoing instruction that happens in the classroom while adhering to research-based best practices within the field (Aelterman et al., 2013; Darling-Hammond & McLaughlin, 1995; Grossman et al., 2001; Kennedy, 2005). The focus on job-embedded inquiry, or problems of practice, allows teachers to develop a skill set relevant to their current and future practices (Aelterman et al., 2013; Darling-Hammond & McLaughlin, 1995; Wei et al., 2009). Moreover, offering a content-focused PD model supports real-time solutions for practice problems, as it sets out to augment a specific instructional practice. The cyclical nature of the model promotes continuous improvement within the classroom by identifying areas of weakness. (Hawley & Valli, 1999).

Participants

The eight participants in this study were female, seven Caucasian, and one African-American. The participants' average age was 39.5, with the mean number of years of experience at 16.25. Each participant held a Bachelor's degree, with two individuals completing a Master's degree in education. Neither of the Master's degrees focused on mathematics. The participants in this study were not Texas State certified in mathematics, and all except one were first career teachers. Their average number of years employed at the school was 3.75. Table 1 provides detailed participant information.

Table 1

Participant Information

Participant Number	Current Grade Level	Years Taught in Grade Level	Highest Degree Held	Degree Specialization
P#1	Grade 2	4	BS	Interdisciplinary Studies
P#2	Grade 1	1	BA	Educational Studies
P#4	Grades 5 – 6	3	MLIS	Library and Information Science
P#4	Kindergarten	8	BS	Teaching and Learning
P#5	Grade 1	1	BA	Educational Studies
P#6	Grades 7 – 8	2	BBA	Business Administration (Marketing)
P#7	Grade 3	3	BS	Elementary Education
P#8	Grade 4	7	MEd	Administration and Supervision

Instruments

The instruments employed in the study included a pre-test and post-test that spanned fraction content knowledge in grades three through eight and exit tickets generated from content standards presented during the six individual sessions. The pre-test and post-test consisted of thirty questions over fractional knowledge required for success within grades three through eight. The strands explored were comparison, equivalency, addition, subtraction, multiplication, and division of fractions. The assessment questions were generated from the previously released State of Texas Assessments of Academic Readiness, STAAR® questions, 2018 and 2019 mathematics.

STAAR® exams were created to test the most critical content strands in each course subject to ensure access to college and career readiness (Texas Education Agency, 1997). The assessments measure proficiency on the state curriculum standards, also known as Texas Essential Knowledge and Skills Standards, or TEKS, in the core subjects of mathematics, science, reading, and social studies (Texas Education Agency, 2020). Assessments determine the test takers' proficiency level regarding the current grade-level standards set forth by the Texas Education Agency (TEA). The validity and reliability of the assessments have been confirmed by a third party, the Human Resources Research Organization, HumRRO. In 2016, HumRRO provided empirical evidence of the reliability and validity of STAAR®, that centered on three main criteria: the understanding that the items presented on the assessment were correlated with the TEKS it expected to measure, that the test construction and scoring process allowed for validity and reliability in the scores, and the standard error of measurement was consistent (HumRRO, 2016).

The pre-test and post-test scores were analyzed using a paired sample t-test, comparing participants' means before and after their participation in the PD series. Additionally, the researcher analyzed exit tickets to assess teachers' understanding of concepts and the proper use of virtual manipulatives to represent mathematical thinking. The robust research design of this study adds to the body of knowledge concerning teachers' content knowledge for fraction instruction. The study clarified the effects of professional development, identified areas of continued concern, and aided in the design of future professional development learning opportunities.

Limitations

Despite the substantial investment in professional development by districts across the United States, there is little empirical evidence to prove a direct correlation between a specific professional development series and student achievement (Huffman et al., 2003; Loucks-Horsley & Matsumato, 1999; Yoon et al., 2007). This issue may be partly due to the lack of in-depth studies and reliable instruments used to assess the connections. While the current study seeks to address the need for more data regarding the effects of content-specific professional development, the study has several limitations. First, the data collected from this study was generated from a small sample size of educators employed at a private school. Many private schools in Texas do not require that their teaching staff hold a subject-specific certification. Not all private schools in Texas mandate that teaching staff extends their professional learning outside of their pre-service training. Therefore, the participants in this study have not been required to complete any additional content-specific training in the area of mathematics.

Additionally, the participants had not completed advanced college-level mathematics coursework or participated in mathematics professional development during their employment. Furthermore, the intervention period was relatively short, with participants attending a combined six hours of PD, occurring one hour a day for six days. Due to COVID 19 restrictions, the study's length was protracted. The delivery methods switched from in-person to a synchronous online format that was relatively unfamiliar to both the researcher and participants.

Methods

Drawing from Shulman's (1986) thought that teachers must understand the knowledge of the subject they teach and the organizing structures foundational to the discipline, this study emphasized the awareness of underlying principles and forthcoming structures associated with acquiring knowledge of fractions. In order to impart a solid understanding of fractions to their students, teachers must be versed in the way that concepts form as well as conceptually explore the connections through vertical alignment, being cognizant of the current researched-based best practices that allow for fluency and scaffolding (Hiebert et al., 2007). The vertical knowledge included centers on "familiarity with the topics and issues that have been and will be taught in the same subject area during the preceding and later years in school, and the materials that embody them" (Schulman, 1986, p. 10). Crucial to design and assessment opportunities, effective teaching covers the required standards that scaffold future learning opportunities (Squires et al., 2012). Furthermore, the understanding of vertical alignment of strands provides a framework of what is considered "proficient" at each grade level (Tanenbaum et al., 2017). The use of best practices in planning and delivering an effective curriculum can better support the mathematical competency required for success in the classroom and beyond.

This study was conducted as an evaluation to determine if there was a change in teachers' fractional knowledge due to participation in a content-specific professional development series. The quantitative data of test scores and exit tickets were used to assess the learning acquired in each session, allowing the researcher to analyze the impact of the subject-specific professional development on a small group of mathematics teachers. An evaluation study analyzes a program's worth or success, policy, or project (Payne & Payne, 2004). Thus, the current study centered on teachers' capacity of knowledge obtained from attendance in a subject-specific professional development opportunity.

Employing the substratum that Shulman (1986) developed and has been continually extended and revised by educational researchers, effective educators must understand an exclusive domain of subject-specific professional knowledge unique to teachers (Ball et al., 2005). Content included in the domain centers not only the rules and methods of the discipline but the organizing principles and structures that form the framework. The knowledge provided equips teachers with a thorough understanding of the imperative content in discerning why a particular topic strand is central to learning, while others are peripheral (Shulman, 1986).

The exit tickets used as instruments in this study provided a formative assessment piece that allowed the researcher to assess how well the participants understood the material and if they could use the virtual manipulatives to explore and express understanding. Like exit tickets used in mathematics classrooms, the instruments provided a real-time assessment tool that helped determine whether participants had mastered the content presented or remained at a superficial understanding (Miranda & Hermann, 2015). The ongoing and immediate snapshots of learning allowed the researcher to address areas of concern by monitoring and adapting instruction to progress toward intended learning goals (Chappuis & Stiggins, 2002).

Participants in this study worked in collaboration to explore fractions through the use of virtual manipulatives. Drawing from Vygotsky's theory of social development, in which the learner plays an active role in acquiring knowledge, the study highlighted the collaborative interaction of participants to develop skills and strategies (Vygotsky, 1978). The integral role that participants play in the learning process leads to a more extensive understanding of the More Knowledgeable Other (MKO) and that role in constructing their thoughts and understanding of the topic being explored (Abtahi, 2017). The MKO refers to a person with higher ability in a particular subject, acting as a guide or scaffold in understanding. This guidance takes place in the Zone of Proximal Development, where participants can develop higher mental functions concerning the topic being taught (Vygotsky, 1978).

Furthermore, the MKO is considered to have a learning capability level slightly above the learner's current understanding level. The level is attainable when presented with opportunities for learners to collaborate with skilled peers or facilitators in navigating new concepts, skills, and models (Shabani et al., 2010). Learners that work through tasks jointly are typically better able to repeat success due to the ZPD level for that particular task being raised.

Professional Development Sessions

The six PD sessions were delivered using the Zoom platform, each lasting one hour in duration. The sessions were provided in a synchronous framework spanning over two weeks. The researcher facilitated the discussions and presented data that supported participant findings. During each session, the participants were introduced to various virtual manipulatives as options to utilize and were encouraged to access them when working on activities. Sessions included whole group discussion, teacher-led activities, collaborative learning experiences, and summaries of learning. These lessons are summarized below, and the full lesson plan can be found in Appendix D.

Lessons were designed using a 5E model, an inquiry-based teaching sequence that comprises Engage, Explore, Explain, Elaborate, and Evaluate. The design was chosen to maximize student engagement in learning. With each step informing the next, the lessons activated prior knowledge and attempted to scaffold learning and discovery through activities. This method favors active engagement by the students, placing the teacher in the role of facilitator. The Inquiry Model of Teaching acted as a guide in the planning of the six 5E lessons. The Inquiry Model provided the outline for the engagement of participants in activities that required active investigation. The teaching model's investigative nature was designed to increase student involvement and promote ownership over the content while stimulating curiosity (Murskey, 2011).

Session #1

The first professional development session targeted the participants' prior knowledge by exploring fundamental concepts required to understand fractional concepts, including division related to sharing equally. The activities presented provided opportunities to develop a deep conceptual understanding of part-whole relationships. The researcher built on what teachers already understood informally about fractions, specifically sharing equally, and encouraged them to use that understanding as a foundation for formal fraction instruction. The *IES* states that by the time students attend formal schooling, they have acquired a fundamental understanding of sharing that offers insight into dividing a set of objects equally among two or more people (Siegler et al., 2010). Therefore, teachers must also have a strong foundation for sharing sets of objects equally to provide instruction to students. Additionally, the fair share concept highlighted how proportional thinking develops through the set model at an early age. The thinking process set the stage for students to obtain the knowledge required to share an object equally. The researcher aimed to build teachers' understanding of the connection between fair sharing scenarios and students' early understanding of fractions.

The first activity introduced participants to the pictorial representations, manipulatives, and words used to model sharing a set evenly among multiple recipients. The problem presented asked participants to work independently to answer and then engaged them in a group discussion about solution plans. The following question was used:

> Three children want to share 12 cookies so that each child receives the same number of cookies. How many cookies should each child get? Please work independently and show your thinking through pictures, words, or numbers. If you finish before the time allotted, show alternative models for your answer using the provided materials.

Participants then worked on a problem over unit fractions, fractions with a numerator of one, and a positive integer as a denominator. Discussion of the differences between unit and non-unit fractions were based on the pair-share and whole group portion.

The follow-up activity allowed teachers to partition several shapes and number lines, providing an opportunity to reason about the relationship between part and whole. Using the area model, participants viewed various representations of partitioned shapes and number lines representing both equal and unequal partitioning. They were asked to decide if the partitions were equal or not and encouraged to individually model or shade particular parts of shapes $\left(\frac{1}{4}, \frac{1}{3}, \frac{1}{2}\right)$ using virtual manipulatives. Participants compared their solution plans to the whole group, justifying why their solution worked. This activity offered participants a way to explore the relationship between parts and wholes while understanding the idea of equally covered space as the determining factor in equivalency (McNamara & Shaughnessy, 2010). In providing a region dimension activity that included geometric partitions that were easily reconfigured as the starting point of the lesson, the researcher added an extension for participants to engage in a formal assessment that guided the lesson's questioning and flow. The activity gave the researcher insight into struggles and rates of mastery for the participants. The researcher used this as a tool for redirecting learning and focus as the core lessons brought awareness to the critical patterns required for algebra's application-based work. Understanding multiplication, division, and fractions as a reciprocal in geometric understanding were central to the lesson. Next, the researcher presented the following question:

> There are three friends. They want to run a relay race that is one mile. Each friend runs an equal amount of the race. How much of the race would each friend run? How could you partition the shape into three equal parts? Please model your solution plan using the fraction application.

The partitioning activity reinforced the idea of a fractional quantity, as research shows that students often use whole numbers and fractions to represent the identical fractional quantity (Mack, 1995). An example of this is when students believe that they can use the numerator's value to represent the amount. Without referencing the denominator, five is the same as five eights because they see five pieces.

This session's activities allowed participants to work with various delivery modes and begin thinking more flexibly about the relationships between values. Using partitioned shapes, number lines, and manipulatives in this way assisted participants in thinking about how to present fractions as not just counting individual "parts." Rather, fractions represent that "the size of a fractional part is relative to the size of the whole" (National Council of Teachers of Mathematics, 2000). The distinction between unit and non-unit fractions is crucial to fractional fluency since it gives a starting point to introduce formal names of fractions (Empson, 1999; Streefland, 1991). This understanding is essential for fractions placed into the lowest terms, with the non-unit fraction being composed of the unit fractions, but the opposite is not true (Siegler et al., 2010). Understanding these concepts gave the study's participants a solid base of knowledge for understanding equivalency. Since the concept of equivalency was foundational to the subsequent lessons found in the study.

Session #2

The second lesson extended participants' exploration of fractions created through linear models and comparison activities. Number lines and Cuisenaire models were used to compare fractions and discuss common misconceptions that students may hold when thinking about fractional values.

The *IES* recommends that students understand that fractions are numbers with values that can be identified as equivalent, compared, and ordered (Sielger et al., 2010). Numerous studies have found positive correlations with number line representations of mathematical concepts, namely whole numbers and decimals, positively correlated with subsequent classroom success (Booth & Siegler, 2008; Seigler & Booth, 2004; Siegler & Ramani, 2009). One study focused on the knowledge gained after participants took part in a decimal number line intervention and found an increase in their ability to locate fraction values on a number line (Durkin & Rittle-Johnson, 2015). Additionally, the *IES* (2010) confirmed that positive correlations of linear models and mathematics concepts extend to fractions (Siegler et al., 2010).

The first two activities used rods of various lengths to represent parts and wholes (McNamara & Shaughnessy, 2010). These differing lengths allowed participants to understand how a fraction's value depends on its relationship to a whole. The value of a fraction fluctuates as the whole is changed out. Rather than identifying what a ½ is in a visual sense, they explored the descriptions of the relationships between the designated rod and part, not a specific example of the fraction. For example, a white rod is $\frac{1}{10}$ when an orange rod is used as the whole but, the white rod is $\frac{1}{6}$ when a dark green is used. The researcher emphasized how hands-on experience helps the participants instruct their students to understand specific fraction quantities without labeling the fractions or referring to benchmark fractions. The researcher drew participants' attention to describing the relationship between the shorter rod and the one designated as the whole. The activities also highlighted the importance of comparing fractions and the role that the unit-fraction plays in decomposing fraction values. The early use of linear models in obtaining fractional knowledge, specifically with comparing and ordering, is beneficial to students' continued success in acquiring fractional knowledge. This concept is seen in studies such as Niemi (1996), which indicates that student experience with linear models can increase performance on tasks requiring conceptual understanding. The idea is evident when comparing students' assessment scores using the linear model and those exposed only to the area and set models.

The last activities allowed participants to analyze common misconceptions that students hold of fractions and understand the inverse relationship between the number of parts and their size. Participants identified, compared, and ordered equivalent fractions and fractions greater than one by exploring and reasoning with linear models and models of various magnitudes. Providing these conceptually-based opportunities led participants to clear up misconceptions they held due to fractions' multifaceted nature as values with multiple meanings. Additionally, participants gained exposure to possible whole-number bias and confusion over the unique format in which fractions are written (McNamara & Shaughnessy, 2010).

Session #3

The third lesson targeted the understanding of fraction equivalency, as in two values representing the same number and covering the same space. Equivalent fractions are interchangeable as they are part of a set of values that refer to a relative amount (Cathcart et al., 2006; Lamon, 2005). Throughout the lesson, the activities allowed participants to engage in discourse and build representations of equivalency in various forms, such as modeling with virtual manipulatives, pictorial representations, and symbolic notation. Working with these concrete models offered learners a conceptual explanation of equivalency's procedural understanding. The activities were selected to help participants understand why "rules" and prescribed steps work. This awareness can lead to a more robust understanding of the relationship of differing representations (Wong & Evans, 2011).

The activities employed the use of linear and area models to explore the concept of equivalency. The understanding of equivalency is especially critical to master in the early stages of fraction development as it is a foundational concept that builds throughout the learning progression of fractions (Lortie-Forgues et al., 2015). Algebra readiness calls for a deep understanding of equivalent expressions and equalities (Driscoll, 1999). Modeling with Cuisenaire Rods offers learners a measurement perspective of fractions by naming the same quantity in various ways. Participants were encouraged to explore the rods to find how many brown rods create the selected object (marker); this led to the concept of mixed numbers or fractional values greater than one, as only one brown can fit. The remaining pieces were superimposed with red rods, which represented $\frac{1}{2}$ of a brown rod. Participants then lined up the rods to show that four reds are equivalent to one brown; therefore, the fraction was $1\frac{2}{4}$ because the red is $\frac{1}{4}$ of the brown rod. Participants then work with the white rods and the brown rod to create the length of the marker. This model showed that four white rods create the same space as two reds or eight white rods that create the one brown rod's space, which shows the equivalent fractions of $\frac{1}{2} = \frac{2}{4} = \frac{4}{8}$. Therefore, participants were able to identify the equivalent fractions of $\frac{1}{4} = \frac{2}{8}$ and $\frac{3}{4} = \frac{6}{8}$.

The last activity centered on the number line and extended the work with virtual Cuisenaire Rods. Participants were guided in exploring equivalent fractions representing precise points on number lines (McNamara & Shaughnessy, 2010). By cycling through partitions of thirds, sixths, and twelfths, participants placed the values vertically on the number line, showing that the equivalent fractions can be found simultaneously. Participants continued to explore quantities from $\frac{1}{12}$ to $\frac{24}{12}$, showing their understanding of virtual manipulatives with fraction equivalence by sharing their screen images with the whole group.

Session #4

The fourth lesson was designed to help participants understand why procedures for computations with fractions make sense. Teachers of mathematics must acquire a strong conceptual understanding of fraction computation to develop their students' thinking (Siegler et al., 2010). The *IES* panel has recommended using visual representations, concrete models, and real-world contexts to solidify learning. Several studies have confirmed a positive correlation between student success in mathematics and the use of pictorial and concrete models. Additionally, the *IES* report identifies the benefits of including real-world context for learners to support computational procedures when dealing with fractions. There are documented benefits to personalizing word problems when attempting fraction division (Ross and Anand, 1987). Furthermore, a strong understanding of fractional computation based on conceptual knowledge leads to an improved understanding of procedural steps (Rittle-Johnson & Koedinger, 2009).

The activities included visual representations and virtual models to help participants gain insight into the computational procedures of addition and subtraction with fractions that have like and unlike denominators. Students in the early grades must experience opportunities to decompose and recompose fractions in many ways, gaining fluency with strategies and operations while experiencing the inverse relationship between addition and subtraction of fractions (McNamara, 2015). Furthermore, mastery of fraction computation depends on a successful understanding of previous learning, including foundational knowledge such as the part-to-whole concept, equivalence, and fraction magnitude (Petit et al., 2016).

The lessons engage portion introduced participants to a real-world story problem that included fractions with like denominators. The problem presented asked participants to work independently to answer using virtual manipulatives and allowed for engagement of group discussions about solution plans. The following question was used:

> Jack is making a soccer bag for a friend. He uses $\frac{2}{5}$ of a yard of green fabric and $\frac{1}{5}$ of a yard of blue fabric. How much fabric does he use in all? Model with manipulatives, pictures, or words. Be sure to include the number sentence.

The explore activities allowed participants to work with concrete models in identifying equivalent fractions used for the addition and subtraction of fractions with unlike denominators. Instead of relying on the abstract concept of equivalent denominators, the participants used models to explain how they could add fractions' varying magnitude. Participants created their solution plan using fraction tiles and number lines. This experience with visual models showed that one could use equal pieces to add and subtract when joining two different sized quantities, shown in the case of $\frac{1}{3}+\frac{1}{2}$, where using the sixths pieces allows for the addition of the values (Siegler et al. 2010).

> Jack is making another soccer bag for a friend. He uses $\frac{1}{2}$ yard of green fabric and $\frac{1}{4}$ yard of blue fabric. How much fabric does Jack use?
>
> Charlie's brownie recipe requires $1\frac{1}{3}$ cups of milk. Laura's recipe needs $\frac{2}{3}$ of a cup of milk. How much more milk does Charlie's recipe need than Laura's?
>
> Will fills a hummingbird feeder with $\frac{3}{4}$ cup of sugar water on Friday. On Tuesday, Will sees that $\frac{1}{8}$ cup of sugar water is left. What is the difference in sugar water between Friday and Tuesday?

The Cuisenaire activities gave participants experience using estimation when computing fractional values. The estimation activities involved reasoning skills, which have been shown to increase a learner's focus in obtaining fractional knowledge (Starkey et al., 2004). The activities presented supported the idea of recognizing equivalent fractions as different ways to name the same quantity (McNamara & Shaughnessy, 2015). Therefore, bridging the learning from the previous session with the participants labeling a "whole", renaming the fractions greater than one into mixed numbers.

The activities listed in the extend and evaluate sections of Lesson #4 relied on the area model of circles to improve understanding of formal computation strategies. Cramer and Wyberg (2009) found that one of the most common mistakes learners make when working with denominators and computation is adding both the numerator and denominator. The circular model is the most effective concrete model for part-whole and understanding of relative size. These models support students' understanding of the inverse relationship between the fractional piece's size and the denominator (Cramer et al., 2002). The understanding allows students to observe that when the circle is equally partitioned, the more parts there are, the smaller the parts become. Additionally, using these models to create equivalent denominators reinforced participants’ thinking of reasonable answers and justification of solution plans by giving them meaningful mental representations that aided in estimation. Solidifying these representations was a critical component in building participants’ understanding of fraction computation, as the subsequent sessions continued to explore operations through the use of similar models.

Session #5

The fifth lesson explored the conceptual understanding of multiplying fractions. Once again drawing, from the *IES* report, participants worked through problems that required sense-making strategies in determining mathematical situations (Siegler et al., 2010). This work supported learners in identifying which operation to employ while understanding what the answer represented. The engage section allowed participants to use contexts and representations to identify the process of multiplication. Fosnot and Dolk (2006) found that using concrete representations is beneficial when introducing multiplication and division of fractions since learners can transfer that knowledge to a more generalizable understanding that supports mathematical relationships regardless of the context. Additionally, the continued use of manipulatives and number lines within the session supported participants' understanding of fractional multiplication when considering time, distance, and area.

The extend segment required that estimation be used in determining the relative size of the product. These activities supported participants' understanding as they required the learner to think about all that they know of the problem's values. The activity was also relevant because it did not include a story problem that students could reference. Instead, they were expected to be able to solve the problems without associating it to a context. As mathematically proficient individuals can manipulate fractions and perform calculations in abstract problems that do not have a point of reference (McNamara, 2015).

Fluency with the multiplication of fractions, as with whole number values, comes with experience and estimating values while using real-world references to explain the reasoning behind the computation method. Students often carry misconceptions of fraction computation due to their lack of conceptual understanding (Lewis & Mayer, 1987). To address the deficit of understanding, educators must plan lessons that make use of real-world problem solving, include concrete manipulatives, present questions that do not include context, model with multiple representations, and create situations that focus on understanding what the answer represents in the equation (McNamara, 2015).

Session #6

The division of fractions is often referred to as the most complex and misunderstood topic for students to master in the elementary grade levels (Tirosh, 2000). The common issues that learners face can be separated into three distinct categories. These misconceptions include mistakes made in replicating an algorithm, overgeneralizing properties of operations that stem from the division of natural numbers, and assuming that one can use the commutative property in division (McNamara, 2015; Tirosh, 2000). These errors may stem from the teacher's lack of conceptual understanding, as research indicates that even after working with the concept of dividing fractions, many preservice teachers were not fully equipped to explain and model the steps needed for mastery (Ball, 1990; Boroko et al., 1992). The professional development sessions in this PD series focused on the understanding required for students to succeed within elementary and middle school levels of mathematics by ensuring that participants were exposed to fractions' vertical nature.

The engage activity presented participants the opportunity to work with manipulatives on partitive problems (partition problems) that often involve sharing or partitioning an amount or object (Kent et al., 2015; Van de Walle et al., 2016). These problems were set in real-world contexts (cups, hours, gallons) to provide meaning to the fraction quantity. In providing this understanding, the researcher attempted to invoke the participants' fractional problem-solving abilities by providing context to the fractional values (Kamii & Warrington, 1995; Mack, 2001). Additionally, the problems presented focused on conceptually dividing a fraction and a whole number without referring to the "keep, flip, change" trick or explaining that one can rewrite the whole number as a fraction by placing the value over the number one. Instead, participants were required to explore the values through manipulatives, arriving at a representation through virtual manipulatives that explained the process.

The explore section featured work with fraction division through measurement problems, seen as repeated subtraction or an equal group problem. According to Van de Walle et al. (2016), these problems are more easily understood when presented in context. The researcher was aware of the importance of featuring the standard dividing rules of whole numbers before engaging in the activity. This exposure was done so that the researcher could reference those misconceptions after the activity was completed, calling attention to the errors that can precede the division of fractions. Participants were asked to complete two rounds of one-minute jogs, both with and without being told when ¼ of a minute had passed. The participants were then asked if they could determine how many ¼ parts were derived from the one minute and how that would look if placed on a one-unit number line. As an extension of the one-unit number line activity, the participants were introduced to a two-unit number line and asked to model their answers using virtual manipulatives. They looked at the amount of ¼ pieces that would fit on a two-unit number line. Length models, such as Cuisenaire Rods or rectangles, are typically used due to the range of colors that can make it easier to identify various lengths as parts of a whole or designate a specific piece as the whole (Van de Walle et al., 2016).

The extended section of the lesson utilized virtual fraction bars to represent the division of fractions, while showing the participants the importance of identifying the whole unit. Participants were asked to cut various representations of a specific "whole" into eights and sixths while noting the various lengths generated. Participants represented their answers using fraction bar virtual manipulatives and offering pictorial representations to model their thinking. This activity supported the building of reasoning and sense-making skills through work with concrete models. The lesson emphasized the real-world applicability of fraction division through a conceptual understanding that encouraged participants to identify patterns, estimate quotients, justify their reasoning, and examine the connections between multiplication and division of fractions.

Summary

This quantitative study explored the effect of subject-specific mathematics professional development on a group of teachers' fraction content knowledge. In assessing the eight private school teaching professionals before and after their attendance in the six hours of sustained professional development spanning one week, the researcher hypothesized a positive correlation of increased test scores with time spent in the professional development series. This treatment included professional development related to the fraction knowledge required for mathematical success for grades three through eight. The instruments, including released STAAR® questions and exit tickets, were used to measure the effects of the treatment. Exit tickets were analyzed by the researcher to confirm mastery of fraction understanding and guide forthcoming sessions. The findings from this study will inform future planning for subject-specific professional development opportunities in mathematics.

Chapter IV

Results

This chapter presents the quantitative study results of the effects that a subject-specific professional development had on teachers' content knowledge of fractions. The study's purpose was to increase participants' understanding of fractional concepts found within the elementary and middle school grade levels. The program evaluation measured the change in teacher understanding after participation in an online professional development series covering fraction concepts explored in grades three through eight. The study intended to answer the following question:

In what ways does online professional development on specialized content knowledge of mathematics increase teacher knowledge?

This chapter includes (1) results, (2) representations, (3) formative assessments, and a (4) summary. The assessments administered at the beginning and end of the professional development series served as the primary instruments for gathering quantitative data regarding the participants' rate of understanding with fractional concepts. The assessments included thirty questions classified into six categories: comparison, equivalency, addition, subtraction, multiplication, and division. The assessment is located in Appendix C. Additionally, exit tickets were collected as a formative assessment to verify the use of representations, which provided supplemental data to reinforce the results from the assessments; see Appendix D.

Results

A paired-sample t-test was administered to determine the efficacy of the subject-specific PD series. The pre-test and post-test, derived from previously released mathematics assessments from the 2018 and 2019 STAAR®, covered the vertical alignment of fractional knowledge required for success in grades three through eight. The pre-test and post-test results served as the primary source of quantitative data regarding the participants' change in understanding of fractional concepts. The assessments, located in Appendix C, include thirty questions classified into six categories: comparison, equivalency, addition, subtraction, multiplication, and division. Each question was scored at a value of one point for a correct response and zero points for an incorrect response. The results from the t-test showed a statistically significant increase of values, as detailed in Table 2.

Table 2

Analysis Findings

Analysis Performed	Calculated Value
Mean	-31.65000
Standard Deviation	19.17677
Standard Error Mean	6.78001
95% Confidence Interval of the Difference Lower	-47.68218
95% Confidence Interval of the Difference Upper	-15.61782
t-test	-4668
f	7
Sig (2-tailed)	$.002$

Correct Responses

The scores for each participant increased upon completion of the professional development series. Participants experienced a cumulative increase of 31.67% in correct responses. The test was comprised of thirty questions, with each question worth one point, for a total of thirty points per assessment. Table 3 details the participants' point change from the pre-test to the post-test. On average, participants showed a 9.25-point increase following the professional development series. This change in points shows a difference in correct responses equivalent to approximately 30% per participant.

Table 3

Participant Point Change from Pre-Test to Post-Test

Participant	Individual Pre-Test Points	Individual Post-Test Points	Point Change
#1	8	26	+18 pts
#2	23	27	+4 pts.
#3	15	29	+14 pts.
#4	21	26	+5 pts.
#5	12	26	+14 pts
#6	25	28	+3 pts.
#7	21	29	+8 pts.
#8	22	30	+8pts.

The evaluation of content-specific categories confirmed that all eight participants improved their fractional knowledge in each category assessed. The most significant cumulative improvement was seen in the fraction division category, with a 47.95% increase in correct responses. Two of the eight participants experienced their highest gains in this area, as shown in Table 4.

Table 4

Highest Category Increases per Participant

Participant	Highest Increase	Second Highest Increase
P#1	Subtraction 80%	Comparison 75%
P#2	Subtraction 60%	Division 33.4%
P#3	Division 66.7%	Subtraction 60%
P#4	Subtraction 60%	Division 33.4%
P#5	Subtraction 80%	Division 66.7%
P#6	Division 33.4%	Multiplication 33.4%
P#7	Addition 66.7%	Division 33.4%
P#8	Addition 66.7%	Division 66.7%

The subtraction category showed similar gains, with a cumulative increase of participant scores equivalent to 47.5%. Three of the eight participants experienced their highest gains in this area. As seen in Table 5, the remaining categories also showed cumulative percentage increases for all participants. The researcher calculated each participant's change in scores for the categories of multiplication, addition, equivalency, and comparison. Those values were then used to obtain the cumulative increases found within each category after participation in the PD series.

Table 5

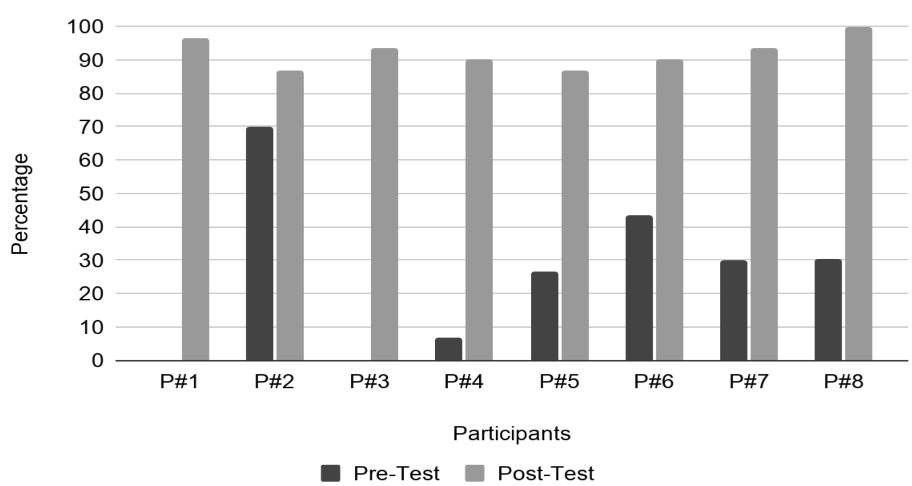
Cumulative Percent Increase per Category

Category	Percent Increase
Multiplication	+25%
Addition	+20.8%
Equivalency	+23.4%
Comparison	+15.6%

Additionally, results revealed an increase from the pre-test to the post-test in the number of representations used by participants to justify answers. Representations are defined as physical or visual models, such as diagrams, concrete objects, number lines, fractions tiles, or equations that demonstrate mathematical relationships and concepts (Goldin, 2014). The cumulative use of representations, including pictorial, abstract, or a combination of pictorial and abstract, was reported as increasing from 18.33% on the pre-test to 92.08% on the post-test. Therefore, suggesting that the participants' employed strategies introduced through the professional development series at a rate of 73.75% upon completion. As shown in Figure 4, there was an increase in representations by all eight of the participants.

Figure 4

Use of Representation per Participant

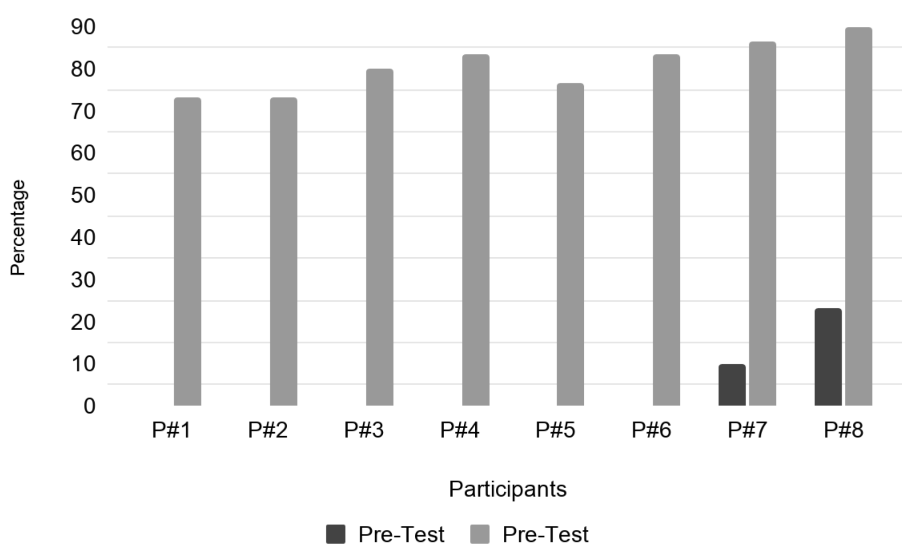


Notes. Figure 4 details the individual percent change that each participant experienced after the treatment. The increased use of representations from the pre-test to the post-test ranged from 16.6%-96.6% per participant.

The increase in representations was explicitly evident in pictorial representations, with a 76.5% increase in pictorial representations from the pre-test to the post-test. A review of the pre-tests revealed that only two of the participants employed pictorial representations, leaving six of the eight participants without pictorial representations on the pre-test. As shown in Figure 5, the post-test confirmed that all eight participants increased their use of pictorial representations.

Figure 5

Pictorial Representations



Notes. Figure 5 represents the individual percentage change in use of pictorial representations per participant from the pre-test to the post-test. The increased scores ranged from 73.3%-83.3% per participant.

In the next section, data are analyzed for each of the eight participants in the study. The researcher looked at the participant's score for correct responses, analyzed items by content strand, and calculated the change in the use of representations on the pre-test and post-test. To calculate the total number of pictorial representations used, the researcher combined all answers that contained these models, including the categories labeled as pictorial and both.

Participant #1

The first participant, P#1, scored 26.6% for correct responses on the pre-test and 86.6% on the post-test results. These findings indicated an increase in teacher knowledge equivalent to 60% or 18 points. As seen in the individual item analysis in Table 6, there were gains in all categories assessed. The most sizable gains were seen within the categories of comparison and subtraction of fractions.

Table 6

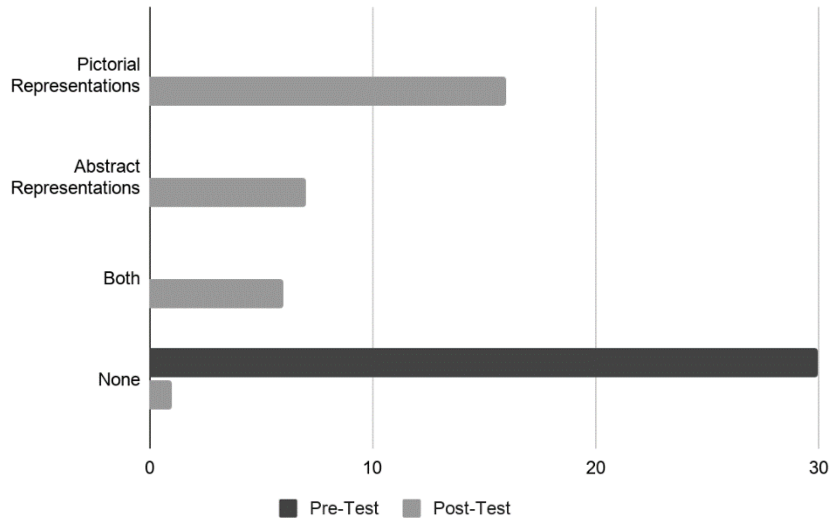
Participant #1 Change per Category

Category	Percent Increase/Decrease	Number of Questions Answered Correctly
Comparison	+75%	1/4 to 4/4
Equivalency	+54%	3/9 to 8/9
Subtraction	+80%	1/5 to 5/5
Addition	+33.3%	1/3 to 2/3
Multiplication	+33.3%	1/3 to 2/3
Division	+66.7%	1/6 to 5/6

P#1 did not include representations on the pre-test. The post-test revealed that P#1 employed representational models to justify answers for 29 of the 30 questions assessed, for a 96.6% increase, as seen in Figure 6. Further analysis confirmed that P#1 favored pictorial representations, using this model for 73.3% of the post-test or 22 of the 30 questions.

Figure 6

Participant #1 Representation Analysis



Participant #2

The second participant, P#2, scored 76.6% for correct responses on the pre-test and 90% on the post-test results. These findings indicated an increase in teacher knowledge, equivalent to 16.6% or four points. As seen in the individual item analysis in Table 7, there were gains in less than half of the categories assessed. The most substantial increases were found in the division and subtraction categories. Additionally, there was a decrease of 14.2% in the category of equivalency.

Table 7

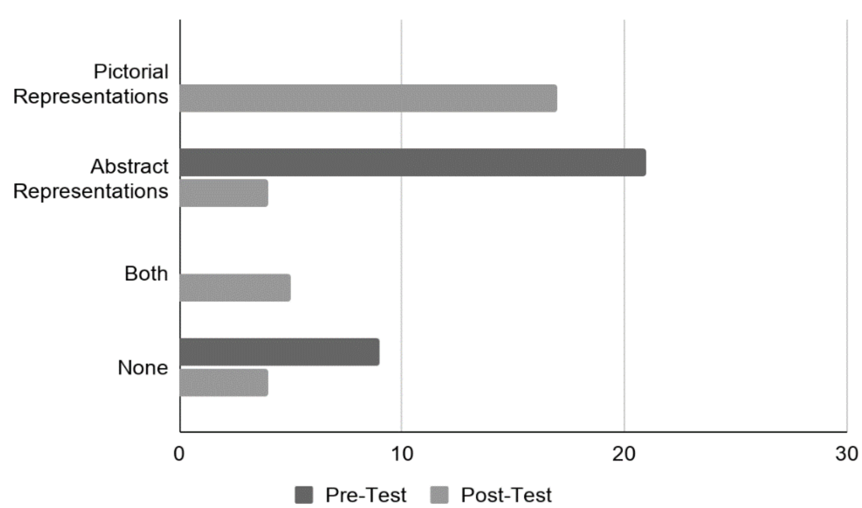
Participant #2 Change per Category

Category	Percent Increase/Decrease	Number of Questions Answered Correctly
Comparison	No Change	4/4 to 4/4
Equivalency	-14.2%	9/9 to 8/9
Subtraction	+60%	2/5 to 5/5
Addition	No Change	2/3 to 2/3
Multiplication	No Change	1/3 to 2/3
Division	+33.4%	4/6 to 6/6

P#2 lacked pictorial representations on the pre-test, providing abstract representations for 70% or 21 questions. The post-test revealed that P#2 employed representational models to justify answers for 26 of the 30 questions assessed, for a 16.67% increase, as seen in Figure 7. Additionally, the number of pictorial representations increased by 73.3%, with 22 of the 30 questions answered using pictorial models.

Figure 7

Participant #2 Representation Analysis



Participant #3

The third participant, P#3, scored 50% for correct responses on the pre-test and 96.6% on the post-test results. These findings indicated an increase in teacher knowledge equivalent to 46.6%, or 14 points. As seen in the individual item analysis in Table 8, there were gains in all categories assessed. The highest increases were found in subtraction and division of fractions.

Table 8

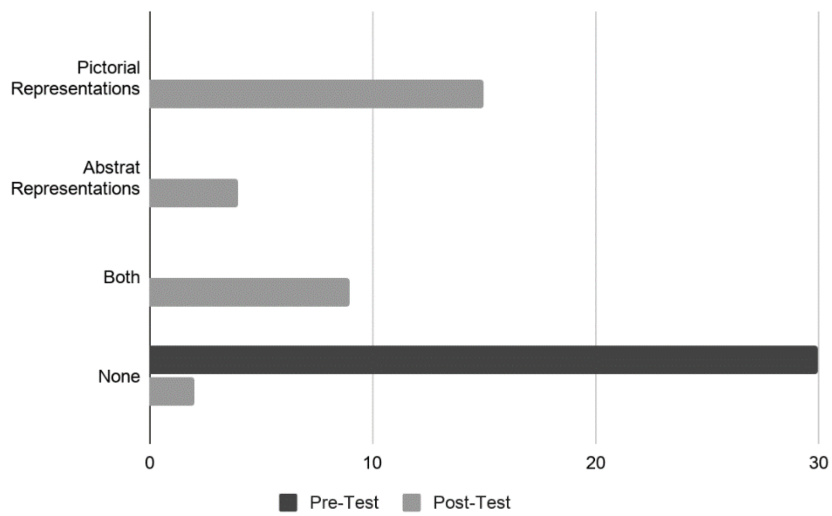
Participant #3 Change per Category

Category	Percent Increase/Decrease	Number of Questions Answered Correctly
Comparison	+25%	3/4 to 4/4
Equivalency	+55.6%	4/9 to 9/9
Subtraction	+60%	2/5 to 5/5
Addition	+33.3%	2/3 to 3/3
Multiplication	+33.3%	2/3 to 3/3
Division	+66.7 %	1/6 to 5/6

P#3 did not include representations on the pre-test. The post-test confirmed that representations were used for 28 of the 30 questions, for a 93.3% increase, as seen in Figure 8. The post-test revealed that the number of pictorial representations increased 80%, with 24 of the 30 questions answered using pictorial models.

Figure 8

Participant #3 Representation Analysis



Participant #4

The fourth participant, P#4, scored 70% for correct responses on the pre-test and 86.6% on the post-test results. These findings indicated an increase in teacher knowledge, equivalent to 16.6% or five points. As seen in Table 9, the highest increases were seen in subtraction and division of fractions.

Table 9

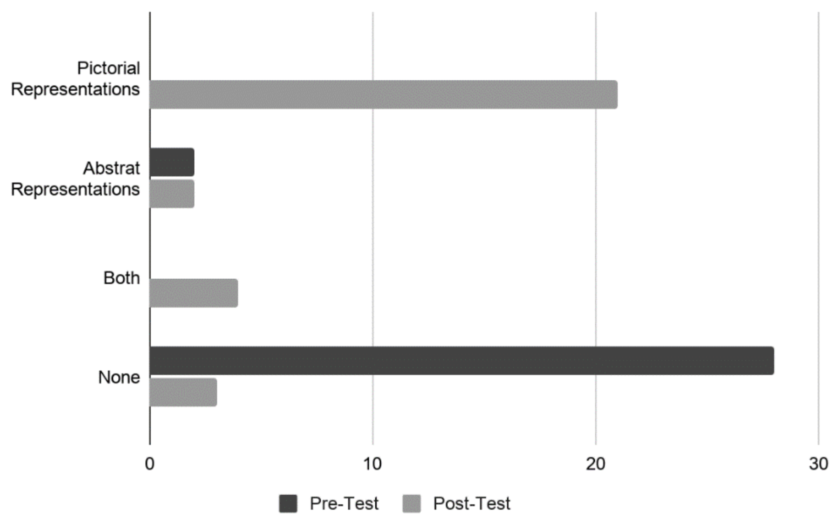
Participant #4 Change per Category

Category	Percent Increase/Decrease	Number of Questions Answered Correctly
Comparison	No Change	8/9 to 8/9
Equivalency	No Change	9/9 to 9/9
Subtraction	+60%	2/5 to 5/5
Addition	No Change	2/3 to 2/3
Multiplication	No Change	2/3 to 2/3
Division	+33.3%	3/6 to 5/6

P#4 included abstract representations for 6.67%, or two questions, on the pre-test. The post-test confirmed that representations were used for 27 of the 30 questions, for an 83.33% increase of representation use, as seen in Figure 9. Further analysis confirmed that P#4 employed pictorial representations for 83.33% of the post-test or 25 of the 30 questions assessed.

Figure 9

Participant #4 Representation Analysis



Participant #5

The fifth participant, P#5, scored 40% for correct responses on the pre-test and 93.3% on the post-test results. These findings indicated an increase in teacher knowledge, equivalent to 53.3% or 16 points. As seen in Table 10, the item analysis confirmed an increase of correct responses in five of the six categories. Findings show that the most significant increase was found within the subtraction and division categories.

Table 10

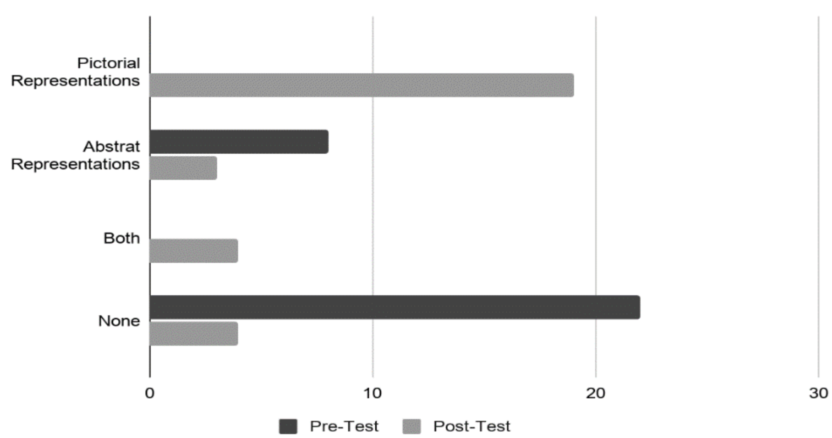
Participant #5 Change per Category

Category	Percent Increase/Decrease	Number of Questions Answered Correctly
Comparison	+25%	2/4 to 3/4
Equivalency	+44.5%	5/9 to 9/9
Subtraction	+80%	1/5 to 5/5
Addition	No Change	3/3 to 3/3
Multiplication	+33.4%	2/3 to 3/3
Division	+66.7%	1/6 to 5/6

P#5 included abstract representations for 26.67%, or eight questions, on the pre-test. The post-test confirmed that representations were used for 26 of the 30 questions, for a 60% increase, as seen in Figure 10. Further analysis confirmed that P#5 chose pictorial representations for 23 of the 30 questions on the post-test, for an increase of 86.67%.

Figure 10

Participant #5 Representation Analysis



Participant #6

The sixth participant, P#6, scored 83.3% on the pre-test and 93.3% on the post-test results. These findings indicated an increase in teacher knowledge equivalent to 10%, or three points. As seen in Table 11, the item analysis confirmed an increase of correct responses in 50% of the categories. The most significant increases were found within multiplication and division of fractions. Additionally, P#6 experienced a decrease of 33.4%, equivalent to one item, in the addition category.

Table 11

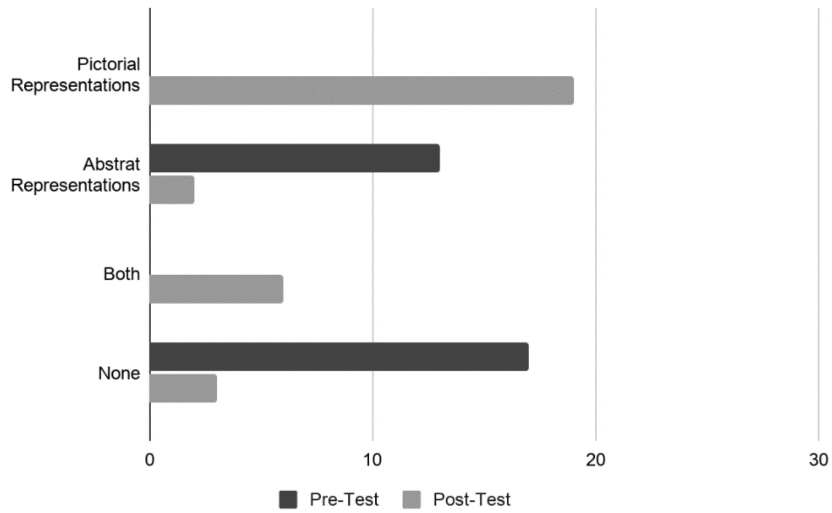
Participant #6 Change per Category

Category	Percent Increase/Decrease	Number of Questions Answered Correctly
Comparison	No Change	4/4 to 4/4
Equivalency	+11.2%	8/9 to 9/9
Subtraction	No Change	4/5 to 4/5
Addition	-33.4%	3/3 to 2/3
Multiplication	+33.4%	2/3 to 3/3
Division	+33.4%	4/6 to 6/6

P#6 included abstract representations for 43.3%, or 13 items, on the pre-test. The post-test confirmed that representations were used for 27 of the 30 questions, for a 46.7% increase, as seen in Figure 11. Further analysis confirmed that P#6 chose pictorial representations for 25 of the 30 questions on the post-test, for an increase of 83.3%.

Figure 11

Participant #6 Representation Analysis



Participant #7

The seventh participant, P#7, scored 70% on the pre-test and 96.6% on the post-test results. These findings indicated an increase in teacher knowledge, equivalent to 26.6% or eight points. As seen in Table 12, the item analysis confirmed an increase of correct responses in five of the six categories. The most significant increases were found in addition and division of fractions.

Table 12

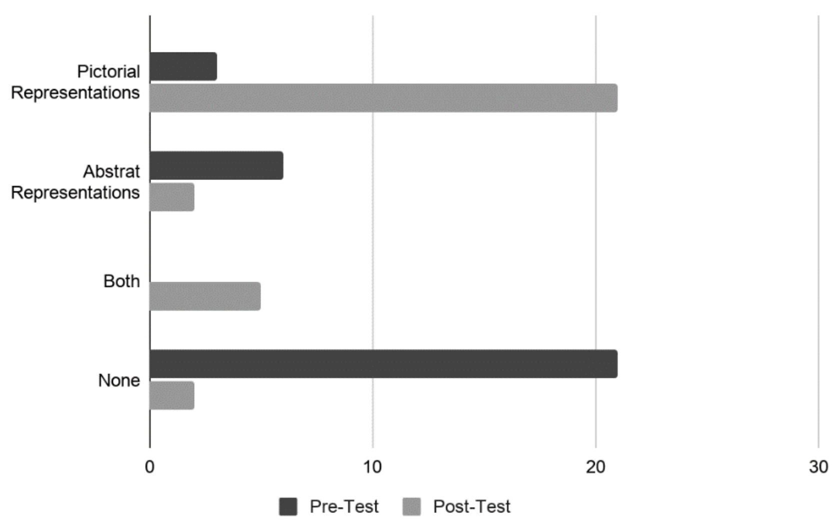
Participant #7 Change per Category

Category	Percent Increase/Decrease	Number of Questions Answered Correctly
Comparison	No Change	4/4 to 4/4
Equivalency	+11.2%	8/9 to 9/9
Subtraction	+20%	3/5 to 4/5
Addition	+66.7%	1/3 to 3/3
Multiplication	+33.3%	2/3 to 3/3
Division	+33.4%	4/6 to 6/6

P#7 included abstract representations for 20%, or 6 items, and pictorial representations for 10%, or 3 items on the pre-test. Therefore, P#7 employed representations for 30% of the pre-test. The post-test indicated that representations were used for 28 of the 30 questions, for a 60% increase, as seen in Figure 12. Further analysis confirmed that P#7 chose pictorial representations for 27 of the 30 questions on the post-test, for an increase of 80%.

Figure 12

Participant #7 Representation Analysis



Participant #8

The eighth participant, P#8, scored 73.3% on the pre-test and 100% on the post-test results. These findings indicated an increase in teacher knowledge equivalent to 26.7%, or eight points. As seen in Table 13, the item analysis confirmed an increase of correct responses in five of the six categories. The most significant gains were found in addition and division of fractions.

Table 13

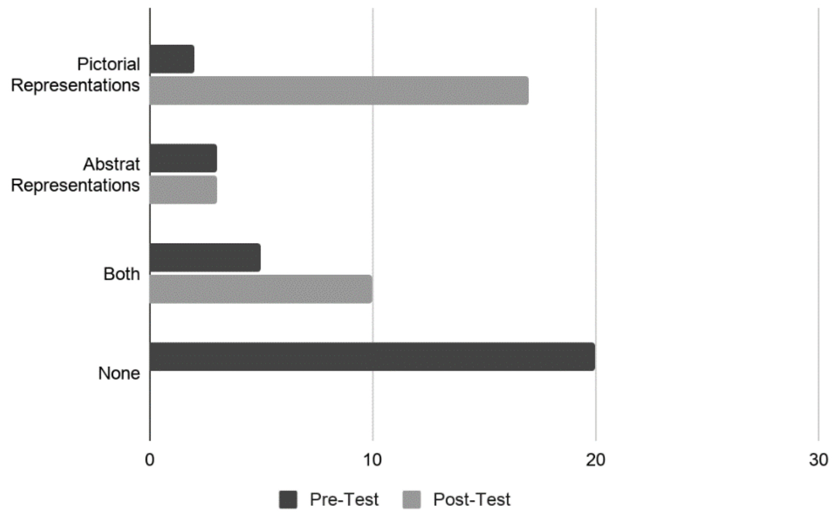
Participant #8 Change per Category

Category	Percent Increase/Decrease	Number of Questions Answered Correctly
Comparison	No Change	4/4 to 4/4
Equivalency	+11.2%	8/9 to 9/9
Subtraction	+20%	4/5 to 5/5
Addition	+66.7%	1/3 to 3/3
Multiplication	+33.3%	2/3 to 3/3
Division	+50%	3/6 to 6/6

P#8 included representations for 33.3%, or 10 items on the pre-test. The pre-test showed that pictorial representations were employed for 6.67%, or 2 items, abstract representations for 10%, or 3 items, and a combination of both for 16.67%, or 5 items. The post-test confirmed that representations were used for all 30 questions, for an increase of 66.7%, as seen in Figure 12. Further analysis confirmed that P#8 chose pictorial representations for 27 of the 30 questions on the post-test, for an increase of 66.7%.

Figure 13

Participant #8 Representation Analysis



Exit Tickets

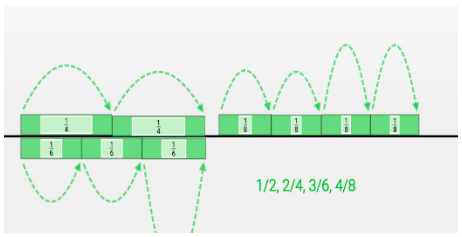
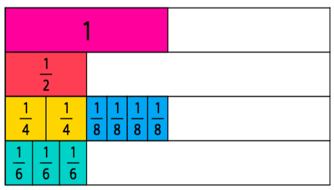
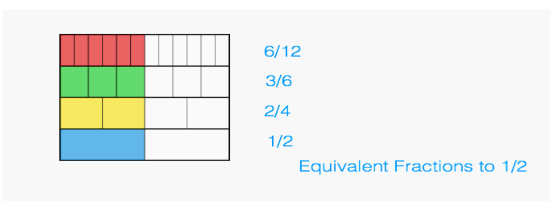
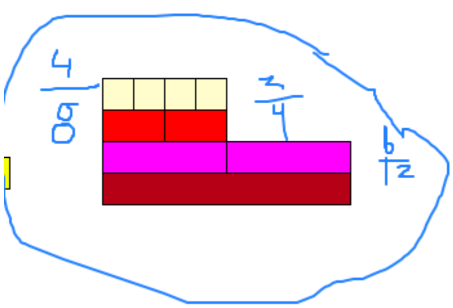
The submission of exit tickets confirmed participant use of the virtual manipulatives for justifying answers. Exit tickets were completed by the participants at the end of each of the six sessions. The results located in Table 14 and Table 15 document a sample of the digitally collected responses from the eight participants. The remaining exit tickets can be found in Appendix D. The combined exit tickets and evaluation of the use of representations on the pre-test and post-test confirmed that participants correctly utilized virtual manipulatives to show their thinking, with an overall increase of 73.75% in the use of representations.

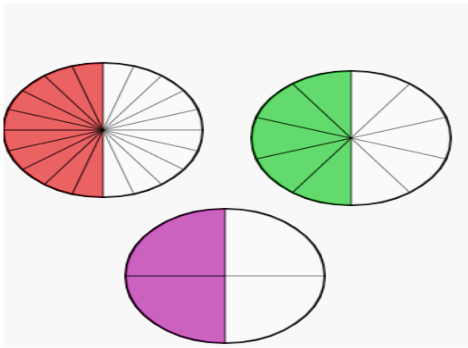
Exit Tickets from Session #3-A

Table 14 provides participant responses to the following exit ticket question: Can you show three fractions that are equivalent to ½ with manipulatives?

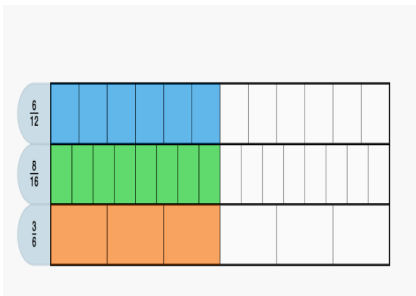
Table 14

Participant Exit Ticket Responses, Session #3-A

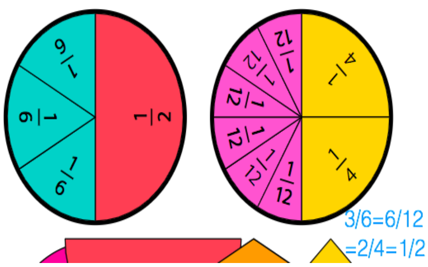
Visual Representation	Participant Response
	P#1: Participant one was able to represent their thinking using an area model and abstract representation. This is a clear pictorial representation of equivalency. The participant used the selected VM to model their thinking.
	P#2: Participant two was able to represent their thinking using an area model with reference to part-whole thinking. This is a clear pictorial representation of equivalency. The participant used the selected VM to model their thinking.
	P#3: Participant three was able to represent their thinking using an area model with reference to part-whole thinking. This is a clear pictorial representation of equivalency. The participant used the selected VM to model their thinking.
	P#4: Participant four was able to represent their thinking using an area model and abstract representation with reference to part-whole thinking. This is a clear pictorial representation of equivalency. The participant used the selected VM to model their thinking.



P#5: Participant five was able to represent their thinking using a circular area model. This is a clear pictorial representation of equivalency. The participant used the selected VM to model their thinking.



P#6: Participant six was able to represent their thinking using an area model and abstract representation. This is a clear pictorial representation of equivalency. The participant used the selected VM to model their thinking.



P#7: Participant seven was able to represent their thinking using a circular area model and abstract representations. To further show equivalency, the participant could have used an alternate virtual model that offered the ability to superimpose the images, for a more effective representation. The participant used the selected VM to model their thinking.

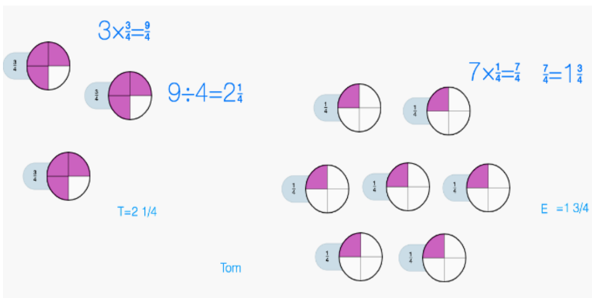
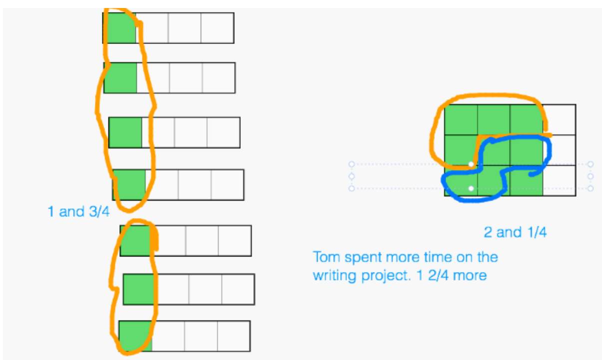
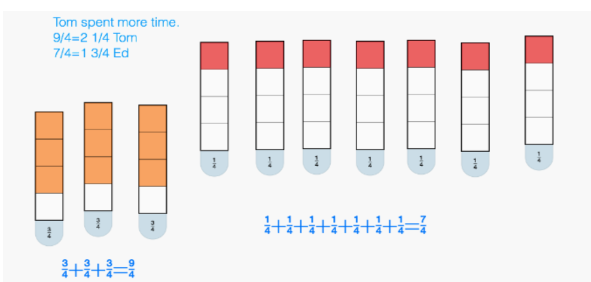
P#8: Participant eight was able to represent their thinking using an area model and abstract representation. This is a clear pictorial representation of equivalency. The participant used the selected VM to model their thinking.

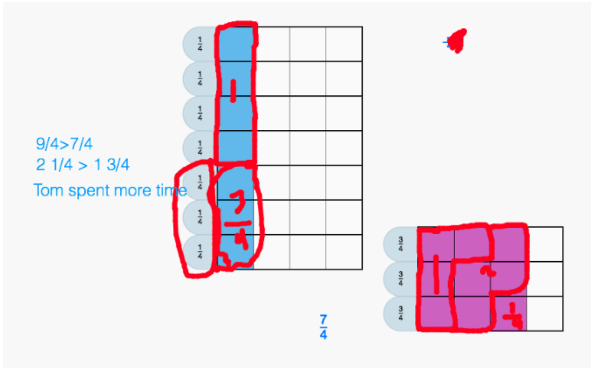
Exit Tickets from Session #5

Table 15 provides participant responses to the following exit ticket question: Tom spent ¾ of an hour each day for 3 days working on his writing project. Ed spent ¼ of an hour each day for 7 days working on his writing project. Who spent more time in total working on their writing project?

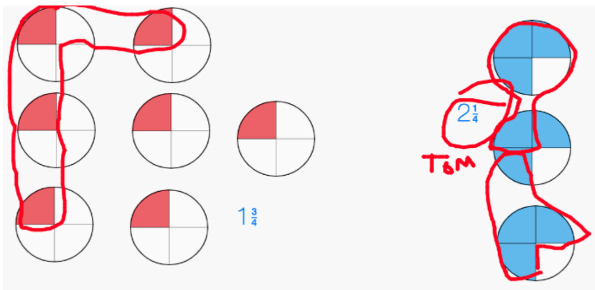
Table 15

Participant Exit Ticket Responses, Session #5

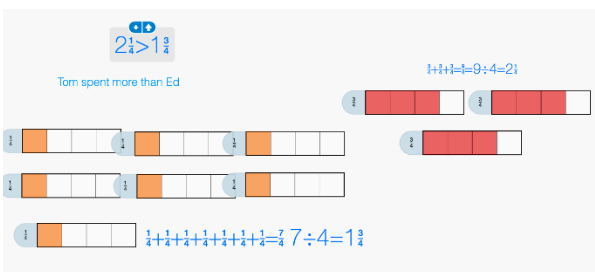
Visual Representation	Participant Response
	P#1: Participant one was able to represent their thinking using a circular model and abstract representation. The pictorial representations are labeled and the answer is correct. P#1 was able to further show mathematical thinking by displaying the equivalent mixed numbers for fractions greater than one. The participant used the selected VM to model their thinking.
	P#2: Participant two was able to represent their thinking using an area model and abstract representation. The pictorial representations are labeled and the answer is correct. P#2 displayed mathematical thinking by providing the equivalent mixed numbers for the improper fractions. The participant used the selected VM to model their thinking.
	P#3: Participant three was able to represent their thinking using an area model and abstract representation. The pictorial representations are labeled and the answer is correct. P#3 displayed mathematical thinking by providing the equivalent mixed numbers for the improper fractions. The participant used the selected VM to model their thinking.



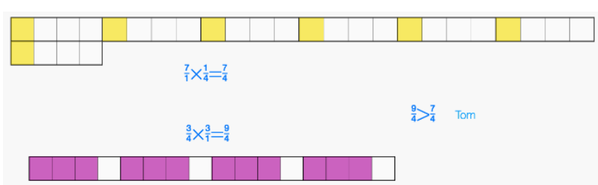
P#4: Participant four was able to represent their thinking using an area model and an abstract representation. The pictorial representations are labeled and the answer is correct. P#3 displayed mathematical thinking by providing the equivalent mixed numbers for the improper fractions. The participant used the selected VM to model their thinking.



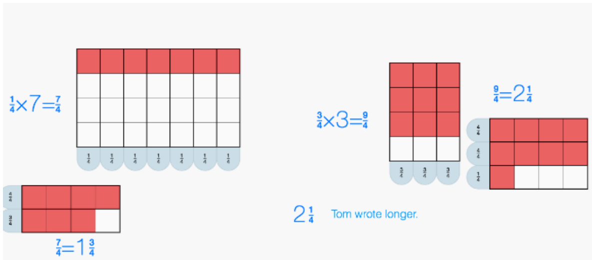
P#5: Participant five was able to represent their thinking using a circular model and abstract representation. The answer is correct. P#5 displayed mathematical thinking by providing the equivalent mixed numbers for the improper fractions. The participant used the selected VM to model their thinking.



P#6 Participant six was able to represent their thinking using an area model and abstract representation. The pictorial representations are labeled and the answer is correct. P#6 displayed mathematical thinking by providing the equivalent mixed numbers for the improper fractions. The participant used the selected VM to model their thinking.



P#7: Participant seven was able to represent their thinking using an area model and abstract representation. The pictorial representations are labeled and the answer is correct. P#3 displayed mathematical thinking by providing the equivalent mixed numbers for the improper fractions. The participant used the selected VM to model their thinking.



P#8: Participant eight was able to represent their thinking using an area model and abstract representation. The pictorial representations are labeled and the answer is correct. P#8 displayed mathematical thinking by providing the equivalent mixed numbers for the improper fraction. The participant used the selected VM to model their thinking.

Chapter V

Discussion

The purpose of this study was to determine if participation in an online subject-specific professional development series could increase teachers' knowledge of foundational fractional concepts. This chapter summarizes the evaluative methodology study that explored the following research question:

> In what ways does online professional development on specialized content knowledge of mathematics increase teacher knowledge?

The researcher utilized quantitative data to identify the change in participants' fractional knowledge after their attendance in a subject-specific online professional development (PD) series. The researcher included a descriptive analysis of exit tickets to identify the rate of proficiency with the tools employed during the PD sessions. Participants completed a pre-test and post-test that covered fractional knowledge required for success in grades three through eight. The questions were obtained from previously released State of Texas Assessments of Academic Readiness, STAAR®, from 2018 and 2019. The quantitative data yielded information on the overall percent change of correct responses. The researcher calculated the average percent change to determine the PD sessions' impact on teachers' subject-specific mathematics knowledge. The data was further analyzed by examining the six core categories of fractional knowledge in the assessments to statistically determine the percent change within each strand. The pre-test and post-test were further assessed to determine the percentage of representations used to justify answers. Additionally, participants submitted exit tickets to confirm their understanding of the content and show their ability to navigate the virtual manipulatives introduced throughout the PD sessions.

This chapter includes (1) results (2) fraction understanding, (3) representations, (4) formative assessments, (5) limitations, (6) online professional development, (7) recommendations, and a (8) summary.

Data Analysis

The assessments administered at the beginning and end of the PD series served as the primary instrument for gathering quantitative data regarding the participants' rate of understanding fractional concepts. The assessments included thirty questions classified into six categories: comparison, equivalency, addition, subtraction, multiplication, and division. The assessment is located in Appendix C. Exit tickets were collected to verify the use of representations, providing supplemental data to reinforce the assessments' results. The following section describes the results of data collection for the pre-test and post-test.

Fractional Knowledge Assessment

The researcher administered the assessment before the start of the PD series. Due to national social distancing restrictions from the global pandemic, the assessments were delivered through the participants' school email. Participants' also completed and returned the assessments through school email. The researcher administered the pre-test to determine a baseline of teachers' understanding of topics covered in the PD.

As discussed in Chapter 3, the assessment was derived from the previously released State of Texas Assessments of Academic Readiness, STAAR® questions, from 2018 and 2019. The assessment included the foundational concepts required for mastery in grades three through eight. The six categories presented spanned the scaffolding concepts that build the framework for fractional knowledge. The assessment consisted of nine equivalency, six division, five subtraction, four comparison, three addition, and three multiplication questions. The questions were chosen due to their ability to assess fractional concepts' flexible development, as fluency with these strands aid in building a solid mathematics foundation (Francis & Rowan, 2001). The emphasis on equivalency and division prioritized the importance of the two areas as critical in fully mastering the conceptual understanding of fractional concepts required in subsequent years of schooling (Ball, 1990; Cramer et al., 2008). The concept of equivalency in fractions plays an integral role in reasoning and scaffolding mathematics knowledge (Cramer et al., 2008; Van de Walle et al., 2016). Securing a definite number sense within fractional concepts requires that students develop an intuitive feel about the relative size of fractions, strengthening their ability to compare, estimate, add and subtract rational parts (Clarke & Roche, 2009; Van de Walle et al., 2016). This study solidified the importance of fluency with fractional size through concrete exploration of fractional values in a variety of situations. Activities that highlighted common misconceptions related to fractions, use of multiple models, and group discussions over the relationships identified, all acted to deepen participants knowledge of the underlying importance of equivalency in fractions. As success with fractions requires a substantial understanding of fraction equivalence.

Additionally, the division of fractions calls for a considerable understanding of the organization of a variety of interdependent relationships (Thompson, 1993; Zembat, 2015). The group discussion in Session 6, found in Appendix B, highlighted the relationship between multiplication and division. Additionally, the participants experienced fractional concepts that were real-world applicable, such as intervals of time and partitioning lengths. Fluency within the fractional quantities' relationships is essential to understanding multiplicative thinking, which acts as the basis for proportionality (Philipp, 2000). Furthermore, the researcher used the assessment as a framework for the PD sessions, ensuring that the content explored addressed the participants' needs. As research suggests that PD opportunities are more effective when the focus is on current teaching challenges instead of abstract concepts that are not immediately applicable in the classroom setting (Darling-Hammond et al., 2009).

Results of Pre-Test Correct Responses (CR)

The pre-test was used to identify the strengths and weaknesses of the participants' knowledge base of fractional concepts covered in grades three-eight. As presented in Table 16, three of the eight participants, 37.5%, did not have a passing score on the assessment. Of the six content categories presented, subtraction and division of fractions were the most significant areas of struggle for participants, with 83.3% of the questions answered in those categories coming in below 50% proficiency. These findings are in line with research that suggests that elementary and middle school teachers have not obtained a deep enough understanding of fractional concepts needed to successfully teach required concepts at the elementary and middle school levels (Ball, 1990; Ball et al., 2005). The heavy reliance on algorithms and rote methods of solving fraction problems can leave teachers unable to effectively teach students the conceptual understanding behind the steps employed (Ball 1998; Schneider & Stern, 2010). Teachers' understanding of fractional division has been found to specifically correlate with their ability to recall the precise algorithm of invert and multiply, which is often taught abstractly, without reference to models for conceptual understanding, leading to a shallow knowledge base of the concept (Ball, 1990; Borko et al., 1992; Li & Kulm, 2008; Simon & Tzur, 1999). This deficit was evident throughout the study, as participants initially tried using abstract models to answer the questions, creating corresponding pictorial models to match. The researcher found that participants could explain the algorithm they used yet struggled when asked why that process worked or how they decided on which pictorial model to apply. Although as participants become fluent with the concept of fraction scaling, they were able to estimate the fractions' sizes, identify reasoning behind why a reciprocal is used, and construct equivalent fractions greater than one. The activities clarified participants' misconceptions and improved their understanding of fraction operations, as seen in the exit tickets. The information gained can further aid participants in recognizing meaningful scaffolding opportunities and situations that require their students to activate prior knowledge (Garet et al., 2010; Ma, 1999).

Table 16

Pre-Test CR Scores

Participant	Pre-Test CR Scores
P#1	26.6%
P#2	76.6%
P#3	50%
P#4	70%
P#5	40%
P#6	83.3%
P#7	70%
P#8	73.3%

Results of Post-Test Correct Responses (CR)

The post-test scores indicate that all eight participants increased their understanding of fractional concepts presented in the PD series. As seen in Table 17, the increases range from 10%-53.3%. The Paired T-Test confirmed that the pre-test (M=61.2250, SD=19.99605) and post-test (M=92.8750, SD=4.86819) covering fractional knowledge indicated a statistical significance in the participants' increased scores upon completion of the PD series, $t(7)=-4.668$, $p = .002$

Table 17

Changes in Participants' Pre-Tests and Post-Tests

Participant	Pre-Test	Post-Test	Percent Change
P#1	26.6%	86.6%	+20%
P#2	76.6%	90%	+13.4%
P#3	50%	96.6%	+46.6%
P#4	70%	86.6%	+16.6%
P#5	40%	93.3%	+53.3%
P#6	83.3%	93.3%	+10%
P#7	70%	96.6%	+26.6%
P#8	73.3%	100%	+26.7%

The results suggest that participants can improve their understanding of mathematical concepts through subject-specific PD opportunities that intend to bridge current understanding with research-based knowledge (Lewis & Perry, 2017). Improving efficacy with content-focused PD can build teachers' repertoire of best practices through frameworks that aid in constructing foundational teaching practices, further supporting the delivery of authentic learning experiences (Charalambous et al., 2007). Results on the post-test suggest that all participants were able to utilize the understanding gained during the PD and apply it to the assessment. The PD sessions intentional sequencing of scaffolding concepts led participants in building connections and proficiency with the schema that supports fractional knowledge required to provide quality teaching (Hiebert, 1986; Schneider & Stern, 2010).

The focus on subject-specific understanding is most effective when it aligns with school-wide goals and connects to the curriculum (Adler & Venkat, 2014). This study was part of a school-wide initiative to strengthen standardized testing scores in grades three through eight. By identifying the areas of concern for teachers through the pre-test results, the researcher was able to present topics that were relevant to the needs of group. Therefore, the content and best practices explored could be immediately applied within the classroom. This understanding was imperative as the concept of fractions is a significant part of the curriculum in grades three through five and lays the foundation for ratios, rate, and probability in the middle school years. In strengthening teachers' knowledge and fluency with rational numbers, they are better equipped to guide their students' in forming a concrete understanding of pertinent concepts through engaging lessons that are guided by conceptual understanding. The impact of quality PD is that it exposes teachers to exemplary teaching methods, skills, and content that can bring the depth of knowledge required to successfully teach the topic (Hill & Ball, 2009); Fennema

& Franke, 1992; Shulman, 1986). Additionally, continued participation in relevant PD that is aligned with standards and goals has been shown to significantly influence teachers' knowledge and practice (Darling-Hammond et al., 2019; Garet et al., 2010; Penuel et al., 2007).

Increase in Specific Categories

The participants experienced a cumulative increase of 31.67 % for correct responses. Referenced in Ch. 4, the point increase for individual participants' correct responses ranged from eight to eighteen points, equivalent to a 9.25 increase per participant. The percentage change for each category was then calculated to determine which content strands experienced the most significant gains from time spent in the PD series. Analysis of the data according to specific problems, found that the greatest growth could be seen within the subtraction and division categories.

Division of fractions rates as a continued area of difficulty for many elementary, secondary, and preservice teachers, especially in their ability to justify the procedures they employ (Ball, 1990; Borko et al., 1992). Meanwhile, students often struggle with the subtraction of fractions due to number bias, the idea that fractional values hold the same properties as whole numbers (Mack, 1995; Ni & Zhou, 2005). Research has continued to show that teachers' understanding in these areas is instrumental in creating content rich lessons and activities. The participants in this study were able to analyze and compare solution plans by dissecting the content and participating in activities that aided in identifying and describing mathematical relationships and underlying patterns (Chinnappan, 2000; Skemp, 1978).

The focused sessions covering subtraction and division of fractions were based on the understanding that the participants were entering the PD with deficits in their understanding. The researcher addressed the inadequacies through research-based best practices for teaching of operations. Drawing from the IES (2010) report, session four focused on circular representations to address the need to build conceptual fluency with fraction subtraction problems, both with like and unlike denominators (Siegler et al, 2010). When this concept is taught solely through a procedural lens, the idea of a common denominator leaves the learner confused about why the denominator contains a different digit. The participants in this study found that the virtual manipulatives were helpful in quickly identifying equivalent fractions. The visual models took away the need to find the least common multiple. The study results also suggest that the visual model of the circular shape was a powerful conceptual model for participants to build representations of both addition and subtraction of fractions (Cramer et al., 2008). The researcher made continual references to view the shift in denominators as equivalent fractions instead of fractions with a "common denominator." This differentiation allowed participants to conceptually build a fraction with the same value in the denominator, identifying the relationship between the two numbers. The intentional focus on fraction equivalency highlighted the importance of the foundational concept. Mastering equivalency in the early elementary and middle school grades is vital since equivalency is found throughout the learning progression of fractions. The process of constructing equivalent fractions can be seen as one of the first experiences students have with separating fraction values from whole numbers (Lamon, 2005). Conceptually based lessons, that rely on inquiry-based activities, such as the Cuisenaire rod and fraction tile lessons in the third session of the PD series, found in Appendix B, allow the participants to investigate and use their reasoning skills to construct equivalent fractions. Furthermore, identifying equivalent fractions is essential when working with fraction arithmetic, algebra, and fraction word problems.

Session six concentrated on the real-world applicability of the division of fractions. (Siegler et al., 2010). The researcher was aware that most participants had never taught the concept, nor expected to, as it is not included in the TEKS for Kindergarten through 4th grade. Additionally, the participants' accuracy on the pretest and their lack of exposure to the content revealed that the participants would benefit from contexts that were real-world applicable, showing that division of fractions is an integral part of life that can be found outside the classroom (Lortie-Forgues et al., 2015). Participants engaged in introductory activities that linked active problem-solving with situations experienced in daily life, such as determining a fraction of time, weight, and distance. The results from the study suggest that content delivered in meaningful contexts contributes to improved understanding and is more easily recalled by the learner (Rittle-Johnson & Koedinger, 2009; Siegler et al., 2010). Group discussions showed that participants could associate the lesson’s main ideas with issues that they experienced in prior classroom situations and could generate ways to modify those lessons. The changes switched the main focus to conceptual understanding of an operation.

Participants made improvements in all six of the categories assessed. The increases in the remaining content knowledge categories were as follows: multiplication, 25%; addition, 20.8%, equivalency, 23.4%, and comparison, 15.6%. The increase in overall scores can be attributed to the program design elements used to maximize participants' acquisition of material. The PD series incorporated the tenets of effective professional development to improve learning outcomes for the participants. The series was content-focused, incorporated active learning, supported collaboration, made use of various models of instruction, provided support, offered feedback, and was of sustained duration (Darling-Hammond et al., 2017).

Additionally, the study was structured using the IES (2010) report that prioritized "improving teachers' understanding of fractions and how to teach them" (Siegler et al, 2010, P.42). In building participants' depth of understanding with fractions and computational procedures through concrete and pictorial representations, the study laid the foundation for participants to assess fractional content and identify common misconceptions typically found in grades three through eight. The researcher focused on experiences that participants needed to build conceptual knowledge within each strand. After actively participating in inquiry-based opening activities that addressed standards, collaborating with other participants, and spending dedicated time exploring ways to implement the virtual manipulatives, participants showed improvement in all six areas of fraction knowledge (Hill & Grossman, 2013; Penuel et al., 2007).

Pictorial Representations Change

The use of representations to justify solution plans was significantly increased from the pre-test to the post-test, with all eight participants employing representations on the post-test. The pre-test results indicated that 25% of the participants did not use any models to explain their reasoning, and 50% used only abstract representations. The post-test results showed a cumulative increase of 73.75% in the overall use of representations.

The PD series focused on using virtual manipulatives to conceptually teach the fractional concepts found in grades three through eight. The virtual manipulatives were used in place of concrete models as a way for participants to identify the relationships between visual and symbolic representations. Technology-based tools, such as virtual manipulatives, offered participants an alternate way to discover, visualize, construct, and organize mathematical ideas through hands-on experiences that go beyond the traditional pencil and paper activities (Moyer-Packenham et al., 2013). After first learning, how to access and use the virtual manipulatives, participants were able to use them to model their thinking, justify answers, and complete exit tickets with accuracy.

Therefore, the most relevant finding was the participants' increase in pictorial models employed from the pre-test to the post-test. As seen in Table 18, there was a substantial increase in the participants' use of pictorial models. The representations were not optional for the PD sessions; instead, they were presented as mandatory components of the learning process. The non-negotiable use of representations mirrors the beliefs of the NCTM (2000), in that teachers should hold representations as "essential elements in supporting students' understanding of mathematical concepts and relationships" (p. 67).

Table 18

Changes in the use of Pictorial Representation from Pre-Test to Post-Test

Participant	Pictorial Representations Pre-Test	Pictorial Representations Post-Test
P#1	0%	73.3%
P#2	0%	73.3%
P#3	0%	80%
P#4	0%	83.3%
P#5	0%	76.6%
P#6	0%	83.3%
P#7	10%	86.6%
P#8	23.3%	90%

The PD highlighted scaffolding experiences through activities that required participants to engage with virtual manipulatives and have in-depth discussions over the inverse relationships and underlying patterns that they saw in work. Effective teaching of fractions offers learners opportunities to experience learning and actively engage in the process of forming their knowledge, as these higher-order concepts cannot be mastered through lecture or direct-instruction (Van de Walle et al., 2016). Furthermore, teachers can more easily employ strategies and skills that have been comprehensively developed in PD settings (Darling-Hammond et al., 2019). Upon completing the PD, participants showed strong mental representations for the fractional values. They could refer back to their experiences within the PD to employ models in justifying subsequent solution plans during sessions and they provided accurate exit tickets at the conclusion of the sessions.

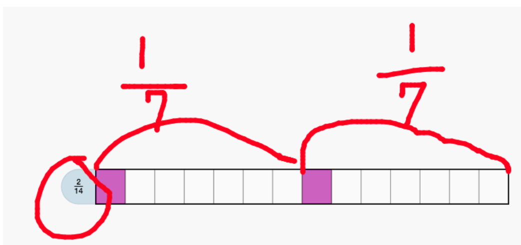
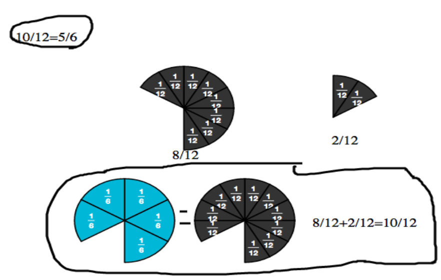
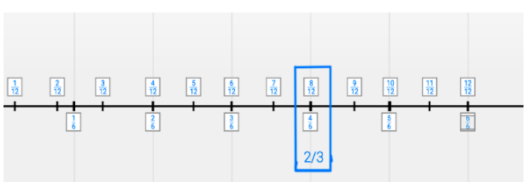
Exit Tickets

To guide participants in developing an understanding of how to teach with virtual manipulatives and navigate the wide variety of options, they explored working with the technology throughout the PD series. Each session concluded with a formative assessment, an exit ticket, a piece that served as a quick snapshot of the participants' daily learning (Sterrett et al., 2010). All of the exit tickets required that participants used a virtual manipulative to display their thinking. Participants were able to choose from any of the options introduced during the PD and were encouraged to employ various models, including linear, area, and set models.

The exit tickets nurtured subsequent mathematics conversations over the strategies employed (Humphreys & Parker, 2015). Each session was tied back to the previous experiences to activate participants' prior knowledge and organize their thinking in a way that would be beneficial for future teaching opportunities (Harbour et al., 2014). The results from the exit tickets, as seen in Table 19, showed that participants gained fluency with successfully demonstrating their thinking through the use of various virtual manipulatives. This use of diversified models is essential in developing a deeper understanding of fractions (Reeder & Utley, 2017).

Table 19

Sample of Participant Exit Ticket Responses

Visual Representation	Participant Response
	P#2: Participant two was able to represent their thinking using an area model and abstract representations. Placed side-by-side, the pictorial model represents the amount of ribbon each child would receive from the two ribbons. This representation includes clear partitions of the models and is labeled with both the original sum and the equivalent fraction. The participant used the selected VM to model their thinking.
	P#4: Participant four was able to represent their thinking using a circular area model and abstract representation. This is a pictorial representation of addition with like denominators. There is reference to the whole being 12 units. The participant did extend to show the equivalent fraction of five-sixths. The participant used the selected VM to model their thinking.
	P#6: Participant six was able to represent their thinking using an area model and abstract representations. The participant was able to show equivalency with four-sixths and eight-twelfths. The model would be more effective if given the reference to two-thirds in a pictorial form.

Limitations

This study's limitations include the results not being generalizable to larger public-school districts since the PD content was tailored to meet the individual participants' needs. The sessions' format was created with a small sample size as a primary influence in the activities chosen. The small group discussion, ability to share solution plans as a group, and focus on specific content strands could pose issues for replicating this study with larger sample sizes. In this study, the researcher was an employee at the testing site, in a mentor and coach's role to the participants. The researcher had already established familiarity, as well as motives for administering the pre-test and post-test. The researcher was able to address individual questions and discuss real-world examples that included situations that occur within the school and classrooms. This type of connection between researcher and participant may be more difficult to mirror in a larger district.

Additionally, the participants were private school instructors, not required to participate in mathematics PD since the start of their employment at the testing site. The participants did not hold degrees or certification in mathematics, while public school teachers must attain specific certifications to teach mathematics. Therefore, their knowledge base of the items assessed on the pre-test may generate contrasting results that lead to a different lesson format.

Online Professional Development

This study suggests that online PD that focuses on Specialized Content Knowledge in the area of fractions could increase teachers' conceptual understanding of the topic. As seen with the increase in scores from the pre-test to the post-test, meaningful exposure to vertically aligned fraction concepts through virtual manipulatives can improve teachers' knowledge of online platforms and foundational mathematics concepts. Like the series presented in this study, online PD offers participants developmentally grounded, personalized, and targeted training (Darling-Hammond et al., 2009). When adequately executed, online PD has the opportunity to change teachers' practices and ultimately increase student outcomes (National Research Council, 2007).

The online PD format lends itself to versatility and flexibility in allowing teachers to participate in sessions on their schedule, through asynchronous and synchronous options. The convenience of completing classes from school or home meets the varying needs of teachers unable to make the time commitment that traditional PD requires. The platform's constant availability supports the idea of community by allowing teachers to interact in real-time or post comments to receive feedback and engage in group discussions.

Online PD focused on foundational concepts can begin to alleviate teachers' deficit of fraction mastery and aid in effectively employing best practices (Ball, 1990). Additionally, online PD participation builds the common language required to teach and learn content-specific topics. Online PD can offer participants an opportunity to gain insight, apply that information in the classroom, and receive feedback on classroom practices. This idea is contrary to a one-time workshop model that does not provide the same guidance after receiving the information (National Research Council, 2007). Furthermore, evidence of teachers' continued misconceptions concerning foundational mathematics concepts, seen explicitly in their inability to provide meaningful explanations, calls for PD grounded in content-specific training (Ball et al., 2008; Ma, 1999). This PD directly responded to the participants' need to increase fractional knowledge and present that information conceptually.

Dependence on rote methods was observed in this study. The participants did not have a solid familiarity with the content strands explored and struggled at the onset with displaying pictorial representations of their thought processes. The reliance on scaffolding opportunities and cognitively demanding tasks in the lesson design helped participants understand the scope and sequence necessary to diagnose student thinking and successfully engage in conceptually based lessons. During the study, algorithms' use was explicitly evident when working with multiplication and division of fractions. Over half of the participants required the researcher to model problems similar to the opening questions due to their confusion over using virtual manipulatives and their lack of understanding of fractional values. Participants who answered the multiplication and division problems did so using abstract representations without knowing why the processes worked.

The rapid onset of remote learning becoming more mainstream in education has opened the door for teachers to enhance and improve their understanding of the challenges present in their classrooms. As Linda Darling-Hammond (2000) advises, PD should focus on the challenges teachers face in the classroom. The online platform is convenient, offers synchronous and asynchronous options, is specific in its objectives, and offers teachers space to work with the ideas presented before taking them back to the classroom. Furthermore, online PD offers real-time solutions for exploring curriculum and learning how to navigate new content delivery systems.

Recommendations for Online Professional Development

The participants in this study benefited from the whole group discussion, specifically in hearing how others thought of their solution plans and the similarities and differences that could be found between individuals' paths. Initially, the participants were hesitant to share their thinking on the online platform. The abundance of time allocated for sharing ideas and mathematical discourse required participants to display their work for the group. Many of them voiced that they were not comfortable with everyone analyzing their work, fearing that the methods employed were not sophisticated enough. This occurrence was amplified during the fifth and sixth sessions, multiplication and division of fractions, and several participants were vocal about their discomfort with the topic. The researcher emphasized that the PD series was a learning opportunity for all. Just as their students struggle at times, it was expected that they would find difficulty in one if not more of the activities presented. The researcher highlighted the benefits of celebrating growth, analyzing mistakes, utilizing virtual manipulatives, and the need for experience with multiple solution paths. Drawing from previously learned content strands, such as multiplication, the researcher discussed the significance of multiple models through partial products, standard algorithm, and the box method. Participants understood that when learning a concept, there is value in identifying multiple solution paths. Exposure to virtual manipulatives to build conceptual understanding acted as the base for many conversations as most of the participants had never included them in their lessons. Participants began to understand how virtual manipulatives could develop conceptual understanding in themselves and their students. Participants became more open to sharing and were able to work together, tying in learning from previous sessions to answer questions through building concrete models, drawing pictorial representations, and explaining their thinking.

The extended time offered for discussion allowed participants to collaborate and engage in a safe and accepting environment to strengthen their understanding of relationships and patterns in their work. The focus on a vertically aligned curricular framework depends on teachers looking beyond their classroom to collaborate with other faculty members to create a more organized and focused curriculum. PD should build foundational skills required for scaffolding and be designed around the prerequisite skills identified for success. Teachers will walk away with a better understanding of why they need to teach their grade level concepts in a particular order and with a tremendous amount of depth when they have access to the scope and sequence of what students need to be successful.

As participants shared and the tone of the conversation was positive and encouraging, the real issues that they faced in the classroom came to the surface. Participants were able to share real-life examples of issues they had in developing and presenting conceptually based fraction lessons without fear that it would somehow be used against them in an upcoming goals meeting. The conversations' open and transparent nature switched the atmosphere of the PD sessions from a presentation style to more of a "think tank" situation where individuals came together to problem-solve and support each other in obtaining new methods and strategies. Online PD has an incredible opportunity to build learning communities focused on improving the overall trajectory of learning in a school or district. Through authentic discourse that prioritizes growth and learning in an encouraging and uplifting environment, participants can come together to explore content and strategies. The PD should be focused on relevant issues that teachers face in the classroom and allow teachers flexibility in how they wish to share their ideas.

Recommendations for Future Research

Future research recommendations include observations of participants' classroom teaching methods and assessments to determine if they successfully transferred their learning into effective teaching strategies. This study offered quantitative data that suggested an increase in the participants' fraction knowledge concerning all six categories assessed. The next steps would include gathering qualitative and quantitative data to determine if the participants can use the PD knowledge to inform their practice. Data collected from student samples, testing, and interviews can verify if student outcomes are positively affected by their teacher's participation in subject-specific PD. Last, this conceptually based online PD model is flexible in that it should be extended to other content strands designated as areas of concern.

Summary

The PD series was successful in that participants experienced increased scores in all six categories assessed. Additionally, the study confirmed that participants successfully improved their understanding of concrete models through virtual manipulatives. The vertical alignment proved beneficial in allowing participants to build their knowledge and understand the previous work that goes into building a schema for fractional knowledge. As research suggests, there is a benefit to mathematics PD that encompasses topics spanning the curricula (Siegler et al., 2010). Teachers at all levels benefit from understanding how each grade level coherently layers the curriculum, scaffolding concepts to create authentic learning opportunities, and bringing awareness to common misconceptions. Teachers that do not understand the scope and sequence of the standards are unlikely to diagnose students' thinking, plan relevant lessons that include cognitively demanding questions, or identify scaffolding opportunities (Ma, 1999; Polly & Hannafin, 2011). Through the use of research-based practices that include inquiry-based learning opportunities, coherent sequencing of curricula, and the use of multiple representations, participants improved their understanding of fractional concepts (Ball et al., 2005).

The continued opportunities to engage in specialized content knowledge-based PD can offer elementary and middle school teachers the understanding and fluency needed to facilitate effective learning opportunities in their classrooms. Virtual manipulatives as an alternate for hands-on, concrete models in conceptually presenting mathematics content can prove beneficial for increasing conceptual understanding of high-leverage mathematics concepts as schools continue to employ more advanced technology in the classroom. Further research is needed to assess the direct impact that specialized content knowledge PD has on students.

References

Abtahi, Y. (2017). The "more knowledgeable other": A necessity in the Zone of Proximal Development. *For the Learning of Mathematics*, *37*(1), 35–39. https://flm-journal.org/Articles/5BBB35871F125602623D27A6B963AB.pdf

Adler, J., & Venkat, H. (2012). Coherence and connections in teachers' mathematical discourses in instruction: Original research. Pythagoras, 33(3).

https://doi.org/10.4102/pythagoras.v33i3.188

Aelterman, N., Vansteenkiste, M., Van Keer, H., De Meyer, J., Van den Berghe, L., & Haerens, L. (2013). Development and evaluation of a training on need-supportive teaching in physical education: Qualitative and quantitative findings. *Teaching and Teacher Education*, *29*(1), 64–75. https://doi.org/10.1016/j.tate.2012.09.001

Akiba, M., Murata, A., Howard, C. C., & Wilkinson, B. (2019). Lesson study design features for supporting collaborative teacher learning. *Teaching and Teacher Education*, *77*, 352–365. https://doi.org/10.1016/j.tate.2018.10.012

Aksoy, N. C., & Yazlik, D. O. (2017). Student Errors in Fractions and Possible Causes of These Errors. *Journal of Education and Training Studies*, *5*(11), 219. https://doi.org/10.11114/jets.v5i11.2679

Aksu, M. (1997). Student Performance in Dealing with Fractions. *The Journal of Educational Research*, *90*(6), 375–380. https://doi.org/10.1080/00220671.1997.10544595

Amato, S. A. (2005). Developing students' understanding of the concept of fractions as numbers. Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education, Melbourne, Australia. https://www.emis.de/proceedings/PME29/PME29RRPapers/PME29Vol2Amato.pdf

Association of Mathematics Teacher Educators. (2017). *Standards of preparing teachers of mathematics*. https://amte.net/standards

Bailey, D. H., Hoard, M. K., Nugent, L., & Geary, D. C. (2012). Competence with fractions predicts gains in mathematics achievement. *Journal of Experimental Child Psychology*, 113(3), 447–455. https://doi.org/10.1016/j.jecp.2012.06.004

Ball, D. L. (1990). The mathematical understandings that prospective teachers bring to teacher education. *The Elementary School Journal*, *90*(4), 449–466. https://doi.org/10.1086/461626

Ball, D. L. (1993). With an eye on the mathematical horizon: Dilemmas of teaching elementary school mathematics. *The Elementary School Journal*, *93*(4), 373–397. https://doi.org/10.1086/461730

Ball, D., Hill, H., & Bass, H. (2005). Knowing mathematics for teaching: Who knows mathematics well enough to teach third grade, and how can we decide? *American Educator*, *29*(1), 14–17, 20–22, 43–46. https://deepblue.lib.umich.edu/bitstream/handle/2027.42/65072/Ball_F05.pdf?sequence=4&isAllowed=y

Ball, D., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special?. *Journal of Teacher Education*, *59*(5), 389–407. https://doi.org/10.1177/0022487108324554

Bannister, N.A. (2018). Theorizing collaborative mathematics teacher learning in communities of practice. *Journal for Research in Mathematics Education*, *49*(2), 125. https://doi.org/10.5951/jresematheduc.49.2.0125

Baumert, J., Kunter, M., Blum, W., Brunner, M., Voss, T., Jordan, A., Klusmann, U., Krauss, S., Neubrand, M., & Tsai, Y.-M. (2010). Teachers' mathematical knowledge, cognitive activation in the classroom, and student progress. *American Educational Research Journal*, *47*(1), 133–180. https://doi.org/10.3102/0002831209345157

Behr, M., & Post, T. R. (1992). Teaching rational number and decimal concepts: Research based methods. In T. R. Post (Ed.), *Teaching mathematics in grades K-8: Research based methods* (2nd Sub Edition ed., pp. 201-248).

Behr, M.J., Lesh, R., Post, T.R. & Silver, E.A. (1983) Rational number concepts. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematics concepts and processes* (pp. 91-126). New York: Academic Press.

Blazar, D., & Kraft, M. A. (2016). Teacher and teaching effects on students' attitudes and behaviors. *Educational Evaluation and Policy Analysis*, *39*(1), 146–170. https://doi.org/10.3102/0162373716670260

Boaler, J., & Zoido, P. (2016). Why math education in the U.S. doesn't add up. *Scientific American Mind*, *27*(6), 18–19. https://doi.org/10.1038/scientificamericanmind1116-18

Boonen, A. J. H., van Wesel, F., Jolles, J., & van der Schoot, M. (2014). The role of visual representation type, spatial ability, and reading comprehension in word problem solving: An item-level analysis in elementary school children. *International Journal of Educational Research*, *68*(4), 15–26. https://doi.org/10.1016/j.ijer.2014.08.001

Booth, J. L., & Newton, K. J. (2012). Fractions: Could they really be the gatekeeper's doorman? *Contemporary Educational Psychology*, *37*(4), 247–253. https://doi.org/10.1016/j.cedpsych.2012.07.001

Booth, J. L., & Siegler, R. S. (2008). Numerical magnitude representations influence arithmetic learning. *Child Development*, *79*(4), 1016–1031. https://doi.org/10.1111/j.1467-8624.2008.01173.x

Borko, H., Eisenhart, M., Brown, C. A., Underhill, R. G., Jones,tD., & Agard, P. C. (1992). Learning to teach hard mathematics: Do novice teachers and their instructors give up too easily? *Journal for Research in Mathematics Education*, *23*(3), 194. https://doi.org/10.2307/749118

Bouck, E. C., & Flanagan, S. M. (2009). Virtual manipulatives. *Intervention in School and Clinic*, *45*(3), 186–191. https://doi.org/10.1177/1053451209349530

Bransford, J., Brown, A., & Cocking, R. (1999*). How people learn: Brain, mind, experience and school*. National Acad. Press. https://ebookcentral.proquest.com/lib/uh/detail.action?docID=3375627

Bright, G., & Litwiller, B. (2002) *Making sense of fractions, ratios, and proportions: 2002 yearbook.* National Council of Teachers of Mathematics.

Browder, D. M., Trela, K., Courtade, G. R., Jimenez, B. A., Knight, V., & Flowers, C. (2010). Teaching mathematics and science standards to students with moderate and severe developmental disabilities. *The Journal of Special Education*, *46*(1), 26–35. https://doi.org/10.1177/0022466910369942

Brousseau, G., Brousseau, N., & Warfield, V. (2004). Rationals and decimals as required in the school curriculum: Part 1: Rationals as measurement. *The Journal of Mathematical Behavior*, *23*(1), 1–20. https://doi.org/10.1016/j.jmathb.2003.12.001

Brown, M., Askew, M., Millett, A., & Rhodes, V. (2003). The key role of educational research in the development and evaluation of the National Numeracy Strategy. *British Educational Research Journal*, *29*(5), 655–672. http://www.jstor.org/stable/1502116

Bruner, J. S. (1966). *Toward a theory of instruction.* Harvard University Press.

Bulgar, S. (2003). Children's sense-making of division of fractions. *The Journal of Mathematical Behavior*, *22*(3), 319–334. https://doi.org/10.1016/s0732-3123(03)00024-5

Burris, J.T. (2013). Virtual place value. *Teaching Children Mathematics*, *20*(4), 228. https://doi.org/10.5951/teacchilmath.20.4.0228

Butler, F. M., Miller, S. P., Crehan, K., Babbitt, B., & Pierce, T. (2003). Fraction instruction for students with mathematics disabilities: Comparing two teaching sequences. *Learning Disabilities Research and Practice*, *18*(2), 99–111. https://doi.org/10.1111/1540-5826.00066

Byrnes, J. P. (2003). Factors predictive of mathematics achievement in White, Black, and Hispanic 12th graders. *Journal of Educational Psychology*, *95*(2), 316–326. https://doi.org/10.1037/0022-0663.95.2.316

Byrnes, J. P., & Wasik, B. A. (1991). Role of conceptual knowledge in mathematical procedural learning. *Developmental Psychology*, *27*(5), 777–786. https://doi.org/10.1037/0012-1649.27.5.777

Cady, J. A., Hodges, T. E., & Collins, R. L. (2015). A comparison of textbooks' presentation of fractions. *School Science and Mathematics*, *115*(3), 105–116. https://doi.org/10.1111/ssm.12108

Carpenter, T. P., Fennema, E., & Franke, M. L. (1996). Cognitively guided instruction: A knowledge base for reform in primary mathematics instruction. *The Elementary School Journal*, *97*(1), 3–20. http://www.jstor.org/stable/1001789

Cathcart, G., Pothier, Y. M., Vance, J., Bezuk, N. S., & Al, E. (2006). *Learning mathematics in elementary and middle school: A learner-centered approach*. Allyn and Bacon.

Chan, W. H., Leu, Y.-C., & Chen, C.-M. (2007). Exploring group-wise conceptual deficiencies of fractions for fifth and sixth graders in Taiwan. *The Journal of Experimental Education*, *76*(1), 26–57. https://doi.org/10.3200/jexe.76.1.26-58

Chang, S., Lee, N., & Koay, P. (2017). Teaching and learning with concrete-pictorial-abstract sequence: A proposed model. *Association of Mathematics Educators*, *17*(1), 1–28. https://repository.nie.edu.sg/bitstream/10497/18838/3/ME-17-1-1.pdf

Chappuis, S., & Stiggins, R. J. (2002). Classroom assessment for learning. *Educational Leadership*, *60*(1), 40–43. http://www.ascd.org/publications/educational-leadership/sept02/vol60/num01/Classroom-Assessment-for-Learning.aspx

Charalambous, C. Y., Philippou, G. N., & Kyriakides, L. (2007). Tracing the development of preservice teachers' efficacy beliefs in teaching mathematics during fieldwork. *Educational Studies in Mathematics*, *67*(2), 125–142. https://doi.org/10.1007/s10649-007-9084-2

Charalambous, C. Y., & Pitta-Pantazi, D. (2006). Drawing on a theoretical model to study students' understandings of fractions. *Educational Studies in Mathematics*, *64*(3), 293–316. https://doi.org/10.1007/s10649-006-9036-2

Chinnappan, M. (2000). Preservice teachers' understanding and representation of fractions in a JavaBars environment. *Mathematics Education Research Journal*, *12*(3), 234–253. https://doi.org/10.1007/bf03217087

Chubb, M. (2020, August 22). *Mark Chubb*. Thinking mathematically. https://buildingmathematicians.wordpress.com/author/markchubb3/

Clark, F. B., & Kamii, C. (1996). Identification of multiplicative thinking in children in grades 1-5. *Journal for Research in Mathematics Education*, *27*(1), 41. https://doi.org/10.2307/749196

Clarke, D. M., & Roche, A. (2009). Students' fraction comparison strategies as a window into robust understanding and possible pointers for instruction. *Educational Studies in Mathematics*, *72*(1), 127–138. https://doi.org/10.1007/s10649-009-9198-9

Clarke, D. M., Roche, A., & Mitchell, A. (2008). Ten practical tips for making fractions come alive and make sense. *Mathematics Teaching in the Middle School*, *13*(7), 372–380. http://www.promaco.com.au/events/MERGA/Ten%20tips%20to%20make%20fractions%20come%20alive.pdf

Cogan, L., & Schmidt, W. (1999). Middle school math reform. *Middle Matters*, *8*(1). https://www.researchgate.net/profile/Leland_Cogan/publication/235979472_Middle_school_math_reform/links/00b7d529c9450e9fde000000/Middle-school-math-reform.pdf

Cramer, K. A., Post, T. R., & delMas, R. C. (2002). Initial fraction learning by fourth- and fifth-grade students: A comparison of the effects of using commercial curricula with the effects of using the Rational Number Project curriculum. *Journal for Research in Mathematics Education*, *33*(2), 111. https://doi.org/10.2307/749646

Cramer, K., & Wyberg, T. (2009). Efficacy of different concrete models for teaching the part-whole construct for fractions. *Mathematical Thinking and Learning*, *11*(4), 226–257. https://doi.org/10.1080/10986060903246479

Cramer, K., Wyberg, T., & Leavitt, S. (2008). The role of representations in fraction addition and subtraction. *Mathematics Teaching in the Middle School*, *13*(8), 490–496. http://cunycci.pbworks.com/f/role%20of%20representation%20in%20fraction%20addition%20and%20subtraction.pdf

Creswell, J. W., & Clark, V. L. P. (2007). *Designing and conducting mixed methods research.* Sage Publications, Inc.

Darling-Hammond, L. (2000). Teacher quality and student achievement. *Education Policy Analysis Archives*, *8*, 1. https://doi.org/10.14507/epaa.v8n1.2000

Darling-Hammond, L., Chung Wei, R., Andree, A., Richardson, N., & Orphanos, S. (2009). Professional learning in the learning profession: A status report on teacher development in the United States and abroad. National Staff Development Council. https://www.researchgate.net/publication/237327162_Professional_Learning_in_t he_Learning_Profession_A_Status_Report_on_Teacher_Development_in_the_United_States_and_Abroad

Darling-Hammond, L., Flook, L., Cook-Harvey, C., Barron, B., & Osher, D. (2019). Implications for educational practice of the science of learning and development. *Applied Developmental Science*, *24*(2), 1–44. https://doi.org/10.1080/10888691.2018.1537791

Darling-Hammond, L., & McLaughlin, M. W. (1995). Policies that support professional development in an era of reform. Phi Delta Kappan, 92(6), 81–92. https://doi.org/10.1177/003172171109200622

Darling-Hammond, L., Schachner, A., Edgerton, A. K., Badrinarayan, A., Cardichon, J., Cookson Jr., P. W., Griffin, M., Klevan, S., Maier, A., Martinez, M., Melnick, H., Truong, N., & Wojcikiewicz, S. (2020). *Restarting and reinventing school: Learning in the time of Covid and beyond*. Learning Policy Institute. https://restart-reinvent.learningpolicyinstitute.org

Delaney, S., Ball, D. L., Hill, H. C., Schilling, S. G., & Zopf, D. (2008). "Mathematical knowledge for teaching": adapting U.S. measures for use in Ireland. *Journal of Mathematics Teacher Education*, *11*(3), 171–197. https://doi.org/10.1007/s10857-008-9072-1

Depaepe, F., Torbeyns, J., Vermeersch, N., Janssens, D., Janssen, R., Kelchtermans, G., Verschaffel, L., & Van Dooren, W. (2015). Teachers' content and pedagogical content knowledge on rational numbers: A comparison of prospective elementary and lower secondary school teachers. *Teaching and Teacher Education*, *47*, 82–92. https://doi.org/10.1016/j.tate.2014.12.009

Dewey, J. (1938). *Experience and Education.* New York: Macmillan Company.

DeWolf, M., & Vosniadou, S. (2011). The whole number bias in fraction magnitude comparisons with adults. *Cognitive Science Society*, P. 1751-1756. https://www.semanticscholar.org/paper/The-Whole-Number-Bias-in-Fraction-Magnitude-with-Dewolf-Vosniadou/0eddc0d4ddcc24e11a5f85e046ab2de0b35ad774

DeWolf, M., & Vosniadou, S. (2015). The representation of fraction magnitudes and the whole number bias reconsidered. *Learning and Instruction*, *37*, 39–49. https://doi.org/10.1016/j.learninstruc.2014.07.002

Desilver, D. (2017, February 15). U.S. students' academic achievement still lags that of their peers in many other countries. *Pew Research Center*. https://www.pewresearch.org/fact-tank/2017/02/15/u-s-students-internationally-math-science/

Doabler, C. T., Baker, S. K., Kosty, D. B., Smolkowski, K., Clarke, B., Miller, S. J., & Fien, H. (2015). Examining the association between explicit mathematics instruction and student mathematics achievement. *The Elementary School Journal*, *115*(3), 303–333. https://doi.org/10.1086/679969

Dorward, J., & Heal, R. (1999). *National Library of Virtual Manipulatives for*

Elementary and Middle Level Mathematics. Www.Learntechlib.org; Association for the Advancement of Computing in Education (AACE). https://www.learntechlib.org/primary/p/7655/

Driscoll, M. J. (1999). *Fostering algebraic thinking: a guide for teachers, grades 6-10*. Heinemann.

Durkin, K., & Rittle-Johnson, B. (2015). Diagnosing misconceptions: Revealing changing decimal fraction knowledge. *Learning and Instruction*, *37*, 21–29. https://doi.org/10.1016/j.learninstruc.2014.08.003

Duschl, R. A., & Osborne, J. (2002). Supporting and promoting argumentation discourse in science education. *Studies in Science Education*, *38*(1), 39–72. https://doi.org/10.1080/03057260208560187

Duzenli-Gokalp, N., & Sharma, M. D. (2010). A study on addition and subtraction of fractions: The use of Pirie and Kieren model and hands-on activities. *Procedia - Social and Behavioral Sciences*, *2*(2), 5168–5171. https://doi.org/10.1016/j.sbspro.2010.03.840

Dyson, N., Jordan, N., & Gutting, J. (2011). A number sense intervention for low-income kindergartners at risk for mathematics difficulties. *Journal of Learning Disabilities*, *46*(2), 166–181. https://doi.org/https://doi.org/10.1177/0022219411410233

Ebby, C. B. (2000). Learning to teach mathematics differently: The interaction between coursework and fieldwork for preservice teachers. *Journal of Mathematics Teacher Education*, *3*(1), 69–97. https://doi.org/10.1023/a:1009969527157

Eichhorn, M. S. (2018). When the fractional cookie begins to crumble: Conceptual understanding of fractions in the fifth grade. *International Journal of Research in Education and Science*, *4*(1), 39–54. https://doi.org/10.21890/ijres.382933

Empson, S. B. (1999). Equal sharing and shared meaning: The development of fraction concepts in a first-grade classroom. *Cognition and Instruction*, *17*(3), 283–342. https://doi.org/10.1207/s1532690xci1703_3

Fazio, L. K., Kennedy, C. A., & Siegler, R. S. (2016). Improving children's knowledge of fraction magnitudes. *PLOS ONE*, *11*(10), e0165243. https://doi.org/10.1371/journal.pone.0165243

Fazio, L., & Siegler, R. (2010). Teaching fractions. *International Academy of Education*, *22*. http://www.eklavya.org/edu-practices_22_eng.pdf

Francis (Skip), F., & Rowan, T. (2001). Representation: An important process for teaching and learning mathematics. *Teaching Children Mathematics*, *7*(4), 288–292.

Fennema, E., & Carpenter, T. P. (1998). New perspectives on gender differences in mathematics: An introduction. *Educational Researcher*, *27*(5), 4. https://doi.org/10.2307/1176732

Fennema, E., Carpenter, T. P., Franke, M. L., Levi, L., Jacobs, V. R., & Empson, S. B. (1996). A longitudinal study of learning to use children's thinking in mathematics instruction. *Journal for Research in Mathematics Education*, *27*(4), 403. https://doi.org/10.2307/749875

Fennema, E., & Franke, M. L. (1992). Teachers' knowledge and its impact. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning: A* *project of the National Council of Teachers of Mathematics*. 147–164. Macmillan Publishing Co, Inc.

Fosnot, C. T., & Dolk, M. (2006). *Young mathematicians at work. Constructing fractions, decimals, and percents. Sharing submarine sandwiches, grades 5-8 : a context for fractions : facilitator's guide* (3rd ed.). Heinemann.

Freeman, D. W., & Jorgensen, T. A. (2015). Moving beyond brownies and pizza. *Teaching Children Mathematics*, *21*(7), 412. https://doi.org/10.5951/teacchilmath.21.7.0412

Fuchs, L. S., Schumacher, R. F., Long, J., Namkung, J., Hamlett, C. L., Cirino, P. T., Jordan, N. C., Siegler, R., Gersten, R., & Changas, P. (2013). Improving at-risk learners' understanding of fractions. *Journal of Educational Psychology*, *105*(3), 683–700. https://doi.org/10.1037/a0032446

Fuchs, L. S., Fuchs, D., Compton, D. L., Wehby, J., Schumacher, R. F., Gersten, R., & Jordan, N. C. (2015). Inclusion versus specialized intervention for very-low-performing students. *Exceptional Children*, *81*(2), 134–157. https://doi.org/10.1177/0014402914551743

Fuchs, L. S., Malone, A. S., Schumacher, R. F., Namkung, J., & Wang, A. (2016). Fraction intervention for students with mathematics difficulties: Lessons learned from five randomized controlled trials. *Journal of Learning Disabilities*, *50*(6), 631–639. https://doi.org/10.1177/0022219416677249

Gabriel, F., Coché, F., Szucs, D., Carette, V., Rey, B., & Content, A. (2012). Developing children's understanding of fractions: An intervention study. *Mind, Brain, and Education*, *6*(3), 137–146. https://doi.org/10.1111/j.1751-228x.2012.01149.x

Gabriel, F., Coché, F., Szucs, D., Carette, V., Rey, B., & Content, A. (2013). A componential view of children's difficulties in learning fractions. *Frontiers in psychology*, *4*, 715. https://doi.org/10.3389/fpsyg.2013.00715

Garcia, E., & Weiss, E. (2019). *The teacher shortage is real, large and growing, and worse than we thought: The first report in "The Perfect Storm in the Teacher Labor Market" series*. Economic Policy Institute. https://www.epi.org/publication/the-teacher-shortage-is-real-large-and-growing-and-worse-than-we-thought-the-first-report-in-the-perfect-storm-in-the-teacher-labor-market-series

Garet, M. S., Porter, A. C., Desimone, L., Birman, B. F., & Yoon, K. S. (2001). What makes professional development effective? Results from a national sample of teachers. *American Educational Research Journal*, *38*(4), 915–945. https://doi.org/10.3102/00028312038004915

Garet, M., Wayne, A. J., Stancavage, F., Taylor, J., Walters, K., Song, M., & Doolittle, F. (2011). *Middle school mathematics professional development impact study: Findings after the first year of implementation*. National Center for Education Evaluation and Regional Assistance. https://ies.ed.gov/ncee/pubs/20104009/pdf/20104010.pdf

Geary, D. C. (2004). Mathematics and learning disabilities. *Journal of Learning Disabilities*, *37*(1), 4–15. https://doi.org/10.1177/00222194040370010201

Gelman, R. & Williams, E. (1998). Enabling constraints for cognitive development and learning: Domain specificity and epigenesis. Handbook of Child Psychology. https://www.researchgate.net/publication/232508159_Enabling_constraints_for_cognitive_development_and_learning_Domain_specificity_and_epigenesis

Goe, L. (2007). The link between teacher quality and student outcomes: A research synthesis. *National Comprehensive Center for Teacher Quality*. https://files.eric.ed.gov/fulltext/ED521219.pdf

Goldin, G. A. (2014). Mathematical representations. In *Encyclopedia of Mathematics Education* (409–413). Springer, Dordrecht. https://doi.org/10.1007/978-94-007-4978-8_103

Goldsmith, L. T., Doerr, H. M., & Lewis, C. C. (2013). Mathematics teachers' learning: a conceptual framework and synthesis of research. *Journal of Mathematics Teacher Education*, *17*(1), 5–36. https://doi.org/10.1007/s10857-013-9245-4

Goldstone, R. L., & Son, J. Y. (2005). The transfer of scientific principles using concrete and idealized simulations. *Journal of the Learning Sciences*, *14*(1), 69–110. https://doi.org/10.1207/s15327809jls1401_4

Greenberg, J., McKee, A., & Walsh, K. (2013). Teacher prep review: A review of the nation's teacher preparation programs. *SSRN Electronic Journal*. https://doi.org/10.2139/ssrn.2353894

Grossman, P., And, S. W., & Woolworth, S. (2001). Toward a theory of teacher community. *Teachers College Record*, *103*(6), 942–1012. https://doi.org/10.1111/0161-4681.00140

Hackenberg, A. J., & Lee, M. Y. (2015). Relationships between students' fractional knowledge and equation writing. *Journal for Research in Mathematics Education*, *46*(2), 196–243. https://doi.org/10.5951/jresematheduc.46.2.0196

Halford, G. S. (1993). *Children's Understanding: The Development of Mental Models.* Lawrence Erlbaum Associates, Inc.

Hansen, N., Jordan, N. C., Fernandez, E., Siegler, R. S., Fuchs, L., Gersten, R., & Micklos, D. (2015). General and math-specific predictors of sixth-graders' knowledge of fractions. *Cognitive Development*, *35*, 34–49. https://doi.org/10.1016/j.cogdev.2015.02.001

Hanusheck, E., Rivkin, S., & Kain, J. (2014). Teachers, schools, and academic achievement. *Econometrica*, *82*(4), 1543–1543. https://doi.org/10.3982/ecta12211

Harbour, K. E., Evanovich, L. L., Sweigart, C. A., & Hughes, L. E. (2014). A brief review of effective teaching practices that maximize student engagement. *Preventing School Failure: Alternative Education for Children and Youth*, *59*(1), 5–13. https://doi.org/10.1080/1045988x.2014.919136

Harris, D., & Sass, T. (2011). Teacher training, teacher quality and student achievement. *Journal of Public Economics*, *95*(7), 798–812. https://ideas.repec.org/a/eee/pubeco/v95y2011i7-8p798-812.html

Harvey, R., & Averill, R. (2012). A lesson based on the use of contexts: An example of effective practice in secondary school mathematics. *Mathematics Teacher Education and Development*, *14*(1), 41–59. https://eric.ed.gov/?id=EJ991857

Hawley, W., & Valli, L. (1999). The essentials of effective professional development: A new consensus. *Teaching as the Learning Profession: Handbook of Policy and Practice*, 127–150.

Hecht, S. A., & Vagi, K. J. (2010). Sources of group and individual differences in emerging fraction skills. *Journal of Educational Psychology*, *102*(4), 843–859. https://doi.org/10.1037/a0019824

Hecht, S. A., & Vagi, K. J. (2012). Patterns of strengths and weaknesses in children's knowledge about fractions. *Journal of Experimental Child Psychology*, *111*(2), 212–229. https://doi.org/10.1016/j.jecp.2011.08.012

Heitin, L. (2015, May 13). Science education. *Education Week*, *34*(30), 5.

https://link.gale.com/apps/doc/A414843066/OVIC?u=txshracd2588&sid=OVIC&xid=e6671840

Hiebert, J. (1986). *Conceptual and procedural knowledge: The case of mathematics*. Erlbaum.

Hiebert, J., & Carpenter, T. P. (1992). Learning and teaching with understanding. In *Handbook of Research on Mathematics Teaching and Learning: A Project of the National Council of Teachers of Mathematics*. (p. 65–97). Macmillan Publishing.

Hiebert, J., Carpenter, T. P., Fennema, E., Fuson, K., Human, P., Murray, H., Olivier, A., & Wearne, D. (1997). Making mathematics problematic: A rejoinder to Prawat and Smith. *Educational Researcher*, *26*(2), 24–26. https://doi.org/10.3102/0013189x026002024

Hiebert, J, & Lefevre, P. (1986). Conceptual and procedural knowledge in mathematics: An introductory analysis. In *Conceptual and procedural knowledge: The case of mathematics* (pp. 1–27). Lawrence Erlbaum Associates, Inc.

Hiebert, J., Morris, A. K., Berk, D., & Jansen, A. (2007). Preparing teachers to learn from teaching. *Journal of Teacher Education*, *58*(1), 47–61. https://doi.org/10.1177/0022487106295726

Hightower, A., Delgado, R., Llyod, S., Wittenstein, R., Sellers, K., & Swanson, C. (2011). Improving student learning by supporting quality teaching: Key issues, effective strategies. *Editorial Projects in Education.* https://www.edweek.org/media/eperc_qualityteaching_12.11.pdf

Hill, H. C., & Ball, D. L. (2004). Learning mathematics for teaching: Results from California's mathematics professional development institutes. *Journal for Research in Mathematics Education*, *35*(5), 330. https://doi.org/10.2307/30034819

Hill, H., & Ball, D. L. (2009). The curious — and crucial — case of mathematical knowledge for teaching. *Phi Delta Kappan*, *91*(2), 68–71. https://doi.org/10.1177/003172170909100215

Hill, H. C., Blunk, M. L., Charalambous, C. Y., Lewis, J. M., Phelps, G. C., Sleep, L., & Ball, D. L. (2008). Mathematical knowledge for teaching and the mathematical quality of instruction: An exploratory study. *Cognition and Instruction*, *26*(4), 430–511. https://doi.org/10.1080/07370000802177235

Hill, H., & Chin, M. (2018). Connections between teachers' knowledge of students, instruction, and achievement outcomes. *American Educational Research Journal*, *55*(5), 1076–1112. https://doi.org/10.3102/0002831218769614

Hill, H., & Grossman, P. (2013). Learning from teacher observations: Challenges and opportunities posed by new teacher evaluation systems. *Harvard Educational Review*, *83*(2), 371–384. https://doi.org/10.17763/haer.83.2.d11511403715u376

Hill, H. C., Lovison, V., & Kelley-Kemple, T. (2019). Mathematics teacher and curriculum quality, 2005 and 2016. *AERA Open*, *5*(4), 233285841988052. https://doi.org/10.1177/2332858419880521

Hoe, L. N., & Jeremy, T. B. L. (2014). The role of virtual manipulatives on the Concrete-Pictorial-Abstract approach in teaching primary mathematics. *The Electronic Journal of Mathematics and Technology*, *8*(2), 102–121. https://repository.nie.edu.sg/bitstream/10497/18917/1/TEJMT-8-2-102.pdf

Hofer, M., & Grandgenett, N. (2012). TPACK development in teacher education. *Journal of Research on Technology in Education*, *45*(1), 83–106. https://doi.org/10.1080/15391523.2012.10782598

Hoffer, T., Venkataraman, L., Hedberg, E. C., & Shagle, S. (2007). *Final report on the national survey of algebra teachers for the national math panel*. Washington, DC, U.S. Department of Education. Conducted by the National Opinion Research Center (NORC) at the University of Chicago.

https://www2.ed.gov/about/bdscomm/list/mathpanel/final-report-algebra-teachers.pdf

Holmes, A. B. (2013). *Effects of manipulative use on PK-12 mathematics achievement: A meta-analysis*. [Poster presentation]. Society for Research in Educational Effectiveness, Washington, DC. https://files.eric.ed.gov/fulltext/ED563072.pdf

Houssain, M., & Robinson, M. (2012). How to motivate US students to pursue STEM (Science, Technology, Engineering, and Mathematics). *US-China Education Review*, 442–451. https://files.eric.ed.gov/fulltext/ED533548.pdf

Hudson, T. (2017, April 17). *Math teachers need better professional development. Here's a personalized approach.* EdSurge; EdSurge. https://www.edsurge.com/news/2017-04-17-math-teachers-need-better-professional-development-here-s-a-personalized-approach

Huinker, D. (2002). Examining dimensions of fraction operation sense. In *2002 Yearbook of the National Council of Teachers of Mathematics* (pp. 72–78). National Council of Teachers of Mathematics. https://eric.ed.gov/?id=ED474863

Huffman, D., Thomas, K., & Lawrenz, F. (2003). Relationship between professional development, teachers' instructional practices, and the achievement of students in science and mathematics. *School Science and Mathematics*, *103*(8), 378–387. https://doi.org/10.1111/j.1949-8594.2003.tb18123.x

HUMRRO. (2016). *Independent Evaluation of the Validity and Reliability of STAAR Grades 3-8 Assessment Scores: Part 2 Final Report Texas Education Agency*. https://tea.texas.gov/sites/default/files/Independent%20Evaluation%20of%20the

%20Validity%20and%20Reliability%20of%20STAAR%20Grades%203-8%20Assessments_Part2.pdf

Humphreys, C., & Parker, R. E. (2015). *Making Number Talks Matter: Developing Mathematical Practices and Deepening Understanding, Grades 4-10*. Stenhouse Publishers.

Hurst, B., Nixon, R., & Wallace, B. (2013). Reading Horizons: The impact of social interaction on student learning. *The Impact of Social Interaction on Student Learning. Reading Horizons*, *52*(4), 375–398. https://scholarworks.wmich.edu/cgi/viewcontent.cgi?article=3105&context=reading_horizons

Iszak, A., Jacobson, E., & de Araujo, Z. Orrill, C. (2012). Measuring mathematical knowledge for teaching fractions with drawn quantities. *Journal for Research in Mathematics Education*, *43*(4), 391–427.

Jackson, C. K., Rockoff, J. E., & Staiger, D. O. (2014). Teacher rffects and teacher-related policies. *Annual Review of Economics*, *6*(1), 801–825. https://doi.org/10.1146/annurev-economics-080213-040845

Jimenez, B. A., & Stanger, C. (2017). Math manipulatives for students with severe intellectual disability: A survey of special education teachers. *Physical Disabilities: Education and Related Services*, *36*(1), 1–12. https://doi.org/10.14434/pders.v36i1.22172

Jong, C., Pedulla, J. J., Reagan, E. M., Salomon-Fernandez, Y., & Cochran-Smith, M.

(2010). Exploring the link between reformed teaching practices and pupil learning in elementary school mathematics. *School Science and Mathematics*, *110*(6), 309–326. https://doi.org/10.1111/j.1949-8594.2010.00039.x

Jordan, N. C., Glutting, J., & Ramineni, C. (2010). The importance of number sense to mathematics achievement in first and third grades. *Learning and Individual Differences*, *20*(2), 82–88. https://doi.org/10.1016/j.lindif.2009.07.004

Jordan, N. C., Kaplan, D., Nabors Olah, L., & Locuniak, M. N. (2006). Number sense growth in kindergarten: A longitudinal investigation of children at risk for mathematics difficulties. *Child Development*, *77*(1), 153–175. https://doi.org/10.1111/j.1467-8624.2006.00862.x

Kahan, J. A., Cooper, D. A., & Bethea, K. A. (2003). The role of mathematics teachers' content knowledge in their teaching: A framework for research applied to a study of student teachers. *Journal of Mathematics Teacher Education*, *6*(3), 223–252. https://doi.org/10.1023/a:1025175812582

Kamii, C., & Clark, F. B. (1995). Equivalent fractions: Their difficulty and educational implications. *The Journal of Mathematical Behavior*, *14*(4), 365–378. https://doi.org/10.1016/0732-3123(95)90035-7

Kamii, C., & Warrington, M. A. (1995). Division with fractions: A Piagetian, Constructivist approach. *Hiroshima Journal of Mathematics Education*, *3*(1), 53–62.

Kaminski, J. A., Sloutsky, V. M., & Heckler, A. F. (2008). Learning Theory: The advantage of abstract examples in learning math. Science, 320(5875), 454–455. https://doi.org/10.1126/science.1154659

Kang, R., & Liu, D. (2016). The importance of multiple representations of mathematical problems: Evidence from Chinese preservice elementary teachers' analysis of a learning goal. *International Journal of Science and Mathematics Education*, *16*(1), 125–143. https://doi.org/10.1007/s10763-016-9760-8

Kennedy, A. (2005). Models of continuing professional development: a framework for analysis. *Journal of In-Service Education*, *31*(2), 235–250. https://doi.org/10.1080/13674580500200277

Kent, L. B., Epsom, S. B., & Neilsen, L. (2015). The richness of children's fraction strategies. *Teaching Children Mathematics*, *22*(2), 84–90.

Kerslake, D. (1986). Fractions: Children's strategies and errors. A repot of the strategies and errors in secondary mathematics project. https://files.eric.ed.gov/fulltext/ED295826.pdf

Kieren, T. E. (1976). On the mathematical, cognitive, and instructional foundations of rational numbers. *Number and Measurement: Papers from a Research Workshop*, 101–144.

Kieren, T. E. (1993). Rational and fractional numbers: From quotient to recursive understanding. In *Studies in Mathematical Thinking and Learning.* Lawrence Erlbaum Associates, Inc.

Klein, D. (2003). A brief history of American K-12 mathematics education in the 20th century. In J. Royer (Ed.), *Mathematical Cognition*. Information Age Publishing. http://www.csun.edu/~vcmth00m/AHistory.html

Kloosterman, P. (2010). Mathematics skills of 17-Year-Olds in the United States: 1978 to 2004. *Journal for Research in Mathematics Education*, *41*(1), 20–51. https://doi.org/10.5951/jresematheduc.41.1.0020

Kupermintz, H. (2003). Teacher effects and teacher effectiveness: A validity investigation of the Tennessee value added assessment system. *Educational Evaluation and Policy Analysis*, *25*(3), 287–298. http://www.jstor.org/stable/3699496

Lai, M. Y., & Clark, J. (2018). Extending the notion of specialized content knowledge: proposing constructs for SCK. *Mathematics Teacher Education and Development*, *20*(2), 75–95. https://files.eric.ed.gov/fulltext/EJ1186000.pdf

Lamon, S. J. (2005). *Teaching fractions and ratios for understanding : essential content knowledge and instructional strategies for teachers*. Routledge.

Lamon, S. J. (2007). Rational numbers and proportional reasoning: Toward a theoretical framework. In F. Lester (Ed.), Second handbook of research on mathematics teaching and learning (pp. 629-668). Charlotte, NC: Information Age Publishing.

Lamon, S. J. (2012). Proportional reasoning: An overview. *Teaching Fractions and Ratios for Understanding*, 1-19. https://doi.org/10.4324/9780203803165-1

Laughbaum, Edward. (2017). Why is algebra a gatekeeper? ResearchGate.

https://doi.org/10.13140/RG.2.2.30770.61129

Leinwand, S., & Fleischman, S. (2004). Teach mathematics right the first time. *Educational Leadership*, *62*(1), 88–89.

Lester, F. K. (1994). Musings about mathematical problem-solving research: 1970-1994. *Journal for Research in Mathematics Education*, *25*(6), 660–675. https://doi.org/10.5951/jresematheduc.25.6.0660

Lewis, A. B., & Mayer, R. E. (1987). Students' miscomprehension of relational statements in arithmetic word problems. *Journal of Educational Psychology, 79*, 363–371.

Lewis, C., & Perry, R. (2017). Lesson study to scale up research-based knowledge: A randomized, controlled trial of fractions learning. *Journal for Research in Mathematics Education*, *48*(3), 261–299. https://doi.org/10.5951/jresematheduc.48.3.0261

Li, Y., & Kulm, G. (2008). Knowledge and confidence of pre-service mathematics teachers: the case of fraction division. *ZDM*, *40*(5), 833–843. https://doi.org/10.1007/s11858-008-0148-2

Lin, C.-Y., Becker, J., Byun, M.-R., Yang, D.-C., & Huang, T.-W. (2013). Preservice teachers' conceptual and procedural knowledge of fraction operations: A comparative study of the United States and Taiwan. *School Science and Mathematics*, *113*(1), 41–51. https://doi.org/10.1111/j.1949-8594.2012.00173.x

Litwiller, B. H., & Bright, G. W. (2002). *Making sense of fractions, ratios, and proportions : 2002 yearbook*. National Council Of Teachers Of Mathematics.

Lock, R. H., & Gurganus, S. (2004). Promote number sense. *Intervention in School and Clinic*, *40*(1), 55–58. https://doi.org/10.1177/10534512040400010501

Loucks-Horsley, S., & Matsumoto, C. (1999). Research on professional development for teachers of mathematics and science: The state of the scene. *School Science and Mathematics*, *99*(5), 258–271. https://doi.org/10.1111/j.1949-8594.1999.tb17484.x

Loucks-Horsley, S., Stiles, K. E., Mundry, S., Love, N., & Hewson, P. W. (2010). *Designing professional development for teachers of science and mathematics*. Corwin Press, Cop.

Lortie-Forgues, H., Tian, J., & Siegler, R. S. (2015). Why is learning fraction and decimal arithmetic so difficult? *Developmental Review*, *38*, 201–221. https://doi.org/10.1016/j.dr.2015.07.008

Luo, W., Paris, S. G., Hogan, D., & Luo, Z. (2011). Do performance goals promote learning? A pattern analysis of Singapore students' achievement goals. *Contemporary Educational Psychology*, *36*(2), 165–176. https://doi.org/10.1016/j.cedpsych.2011.02.003

Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United states.* Routledge.

Maccini, P., & Hughes, C. A. (2000). Effects of a problem-solving strategy on the introductory algebra performance of secondary students with learning disabilities. *Learning Disabilities Research and Practice*, *15*(1), 10–21. https://doi.org/10.1207/sldrp1501_2

Maccini, P., & Ruhl, K. L. (2000). Effects of a graduated instructional sequence on the algebraic subtraction of integers by secondary students with learning disabilities. Education and Treatment of Children, 23(4).

Mack, N. K. (1990). Learning fractions with understanding: Building on informal knowledge. *Journal for Research in Mathematics Education*, *21*(1), 16. https://doi.org/10.2307/749454

Mack, N. K. (1995). Confounding whole-number and fraction concepts when building on informal knowledge. *Journal for Research in Mathematics Education*, *26*(5), 422. https://doi.org/10.2307/749431

Manning, B. H., & Payne, B. D. (1993). A Vygotskian-based theory of teacher cognition: Toward the acquisition of mental reflection and self-regulation. *Teaching and Teacher Education*, *9*(4), 361–371. https://doi.org/10.1016/0742-051x(93)90003-y

Markworth, K. A., Brobst, J., Ohana, C., & Parker, R. (2016). Elementary content specialization: models, affordances, and constraints. *International Journal of STEM Education*, *3*(1). https://doi.org/10.1186/s40594-016-0049-9

Martin, T. S., & Herrera, T. (2007). *Mathematics teaching today: Improving practice, improving student learning*. National Council of Teachers of Mathematics.

Mazzocco, M. M. M., & Devlin, K. T. (2008). Parts and ‘holes’: gaps in rational number sense among children with vs. without mathematical learning disabilities.

Developmental Science, *11*(5), 681–691. https://doi.org/10.1111/j.1467-7687.2008.00717.x

McCaffrey, D. F., Hamilton, L. S., Stecher, B. M., Klein, S. P., Bugliari, D., & Robyn, A. (2001). Interactions among instructional practices, curriculum, and student achievement: The case of standards-based high school mathematics. *Journal for Research in Mathematics Education*, *32*(5), 493. https://doi.org/10.2307/74980

McCaffrey, D. F., Koretz, D. M., Lockwood, J. R., Hamilton, L. S., Rand Corporation, Rand Education. Institute, & Carnegie Corporation Of New York. (2004). *Evaluating value-added models for teacher accountability*. Rand.

McNamara, J. (2015). *Beyond invert & multiply : making sense of fraction computation, grades 3-6 : a multimedia professional learning resource*. Math Solutions.

McNamara, J., & Shaughnessy, M. M. (2010). *Beyond pizzas & pies: 10 essential strategies for supporting fraction sense, grades 3-5.* Math Solutions.

McNeil, N., & Jarvin, L. (2007). When theories don't add up: Disentangling the manipulatives debate. *Theory into Practice*, *46*(4), 309–316. https://doi.org/10.1080/00405840701593899

Mercer, C. D., & Miller, S. P. (1992). Teaching students with learning problems in math to acquire, understand, and apply basic math facts. *Remedial and Special Education*, *13*(3), 19–35. https://doi.org/10.1177/074193259201300303

Miranda, R., & Hermann, R. (2015). Methods and strategies: Teaching in real time. *Science and Children*, *53*(1), 80–85. https://my.nsta.org/resource/?id=10.2505/4/sc15_053_01_80

Murata, A. (2008). Mathematics teaching and learning as a mediating process: The case of tape diagrams. *Mathematical Thinking and Learning*, *10*(4), 374–406. https://doi.org/10.1080/10986060802291642

Murskey, L. (2011). Restorative practices: Giving everyone a voice to create safer saner school communities. *The Prevention Researcher, 18*(Supplement), 58–64. https://www.iirp.edu/images/pdf/Mirsky-Prevention-Researcher-2011.pdf

Moseley, B. J., Okamoto, Y., & Ishida, J. (2006). Comparing US and Japanese elementary school teachers' facility for linking rational number representations. *International Journal of Science and Mathematics Education*, *5*(1), 165–185. https://doi.org/10.1007/s10763-006-9040-0

Moseley, C., & Utley, J. (2006). The effect of an integrated science and mathematics content-based course on science and mathematics teaching efficacy of preservice elementary teachers. *Journal of Elementary Science Education*, *18*(2), 1–12. https://doi.org/10.1007/bf03174684

Moss, J., & Case, R. (1999). Developing children's understanding of the rational numbers: A new model and an experimental curriculum. *Journal for Research in Mathematics Education*, *30*(2), 122. https://doi.org/10.2307/749607

Moyer, P. S. (2001). Are we having fun yet? How teachers use manipulatives to teach mathematics. *Educational Studies in Mathematics*, *47*(2), 175–197. https://doi.org/10.1023/a:1014596316942

Moyer-Packenham, P., Baker, J. M., Westenskow, A., & Anderson-Pence, K. L. (2013). A study of comparing virtual manipulatives with other instructional treatments in third- and fourth- grade classrooms. *Journal of Education*, *193*(2), 25–39. https://doi.org/10.1177/002205741319300204

Moyer-Packenham, P., & Suh, J. (2012). Learning mathematics with technology: The influence of virtual manipulatives on different achievement groups. https://digitalcommons.usu.edu/cgi/viewcontent.cgi?article=1278&context=teal_facpub

National Center for Education Statistics. (2009). The nation's report card: Mathematics 2009. In *U.S Department of Education*. https://nces.ed.gov/nationsreportcard/pdf/main2009/2010451.pdf

National Center for Education Statistics. (2015). The nation's report card: 2015: mathematics and reading assessments. In *The Nation's Report Card*. https://www.nationsreportcard.gov/reading_math_2015/#mathematics?grade=

National Center for Education Statistics. (2018, April 10). *2017 NAEP mathematics and reading assessments: Highlighted results at grades 4 and 8 for the nation, states, and districts*. Nces.Ed.Gov. https://nces.ed.gov/pubsearch/pubsinfo.asp?pubid=2018037

National Center for Education Statistics (NCES). (2019). *NAEP mathematics 2019 highlights grade 12*. Www.Nationsreportcard.Gov. https://www.nationsreportcard.gov/highlights/mathematics/2019/g12/

National Center for Science Education. (2007). *National Academy of Sciences (2007) | National Center for Science Education*. Ncse.Ngo. https://ncse.ngo/national-academy-sciences-2007

National Commission on Excellence in Education. (1993). A nation at risk: The imperative for educational reform. https://edreform.com/wp-content/uploads/2013/02/A_Nation_At_Risk_1983.pdf

National Council of Teachers of Mathematics. (2000, 2014). Principles and Standards for School Mathematics. Reston, VA: The National Council of Teachers of Mathematics, Inc.

National Governors Association Center for Best Practices. (2010). Council of chief state school officers title: Common core state standards (mathematics). *National Governors Association Center for Best Practices*. http://www.corestandards.org

National Research Council. (2007). Enhancing professional development for teachers. National Academies Press. https://doi.org/10.17226/11995

National Science Board. (2016). S&E indicators 2016 | NSF - National Science Foundation. Www.Nsf.Gov. https://www.nsf.gov/statistics/2016/nsb20161/#/report/chapter-1/student-learning-in-mathematics-and-science

National Science Foundation. (2010). *MSP - Knowledge management and dissemination*. www.mspkmd.net. http://www.mspkmd.net

Neagoy, M. (2017). *Unpacking fractions : Classroom-tested strategies to build students' mathematical understanding*. Ascd.

Newstead, K., & Murray, H. (1998). Young students' constructions of fractions. In A. Olivier, & K. Newstead (Eds.), *Proceedings of the 22nd Conference of the*

International Group for the Psychology of Mathematics Education, 3, 295-303. Stellenbosch, South Africa.

Newton, K. J. (2008). An extensive analysis of preservice elementary teachers' knowledge of fractions. *American Educational Research Journal*, *45*(4), 1080–1110. https://doi.org/10.3102/0002831208320851

Newton, K. J., Leonard, J., Evans, B. R., & Eastburn, J. A. (2012). Preservice elementary teachers' mathematics content knowledge and teacher efficacy. *School Science and Mathematics*, *112*(5), 289–299. https://doi.org/10.1111/j.1949-8594.2012.00145.x

Ni, Y., & Zhou, Y.-D. (2005). Teaching and learning fraction and rational numbers: The origins and implications of whole number bias. *Educational Psychologist*, *40*(1), 27–52. https://doi.org/10.1207/s15326985ep4001_3

Olivier, A., & Newstead, K. (1998). *Proceedings of the Conference of the International Group for the Psychology of Mathematics Education. 22nd, Stellenbosch, South Africa, July 12-17, 1998). Volume 3*.

Organisation for Economic Co-operation and Development. (2019). International student assessment (PISA) - Mathematics performance (PISA) - *OECD Data*. TheOECD. https://data.oecd.org/pisa/mathematics-performance-pisa.htm

Özer, E., & Sezer, R. (2014). A Comparative analysis of questions in American, Singaporean, and Turkish mathematics textbooks based on the topics covered in

8th grade in Turkey. *Educational Sciences: Theory & Practice*. https://doi.org/10.12738/estp.2014.1.1688

Paik, J. H., & Mix, K. S. (2003). U.S. and Korean children's comprehension of faction names: A reexamination of cross-national differences. *Child Development*, *74*(1), 144–154. https://doi.org/10.1111/1467-8624.t01-1-00526

Pantziara, M., & Philippou, G. (2012). Levels of students' "conception" of fractions. *Educational Studies in Mathematics*, *79*(1), 61–83. https://doi.org/10.2307/41413098

Park, J., Güçler, B., & McCrory, R. (2012). Teaching prospective teachers about fractions: historical and pedagogical perspectives. *Educational Studies in Mathematics*, *82*(3), 455–479. https://doi.org/10.1007/s10649-012-9440-8

Pasley, J. (2011). MSP knowledge management and dissemination perspectives on deepening teachers' mathematics and science content knowledge. http://www.mspkmd.net/cases/tck/perspectives/introduction.pdf

Payne, G., & Payne, J. (2004). *Key Concepts in Social Research*. Sage Publications. https://doi.org/10.4135/9781849209397

Penuel, W. R., Fishman, B. J., Yamaguchi, R., & Gallagher, L. P. (2007). What Makes professional development effective? Strategies that foster curriculum implementation. *American Educational Research Journal*, *44*(4), 921–958. https://doi.org/10.3102/0002831207308221

Pesek, D. D., & Kirshner, D. (2000). Interference of instrumental instruction in subsequent relational learning. *Journal for Research in Mathematics Education*, *31*(5), 524–540. https://doi.org/10.2307/749885

Petit, M. M., Laird, R. E., Marsden, E. L., & Ebby, C. B. (2016). *A focus on fractions : bringing research to the classroom*. Routledge.

Phelps, G., Kelcey, B., Jones, N., & Liu, S. (2016). Informing estimates of program effects for studies of mathematics professional development using teacher content knowledge outcomes. *Evaluation Review*, *40*(5), 383–409. https://doi.org/10.1177/0193841x16665024

Philipp, R. A. (2000). Unpacking a conceptual lesson: The case of dividing fractions. *Center for Research in Mathematics and Science Education.* http://www.sci.sdsu.edu/CRMSE/IMAP/pubs/UnpackConceptualLessonDivFracs..pdf

Piaget, J. (1954). *The construction of reality in the child.* Basic Books. https://doi.org/10.1037/11168-000

Polly, D., & Hannafin, M. J. (2011). Examining how learner-centered professional development influences teachers espoused and enacted practices. *The Journal of Educational Research*, *104*(2), 120–130. https://doi.org/10.1080/00220671003636737

Rattan, A., Good, C., & Dweck, C. S. (2012). “It’s ok — Not everyone can be good at math”: Instructors with an entity theory comfort (and demotivate) students. *Journal of Experimental Social Psychology*, *48*(3), 731–737. https://doi.org/10.1016/j.jesp.2011.12.012

Rayner, V., Pitsolantis, N., & Osana, H. (2009). Mathematics anxiety in preservice teachers: Its relationship to their conceptual and procedural knowledge of fractions. *Mathematics Education Research Journal*, *21*(3), 60–85. https://doi.org/10.1007/bf03217553

Rech, J. F., & Harrington, J. (2000). Algebra as a gatekeeper: A descriptive study at an urban university. *Journal of African American Men*, *4*(4), 63–71. https://doi.org/10.1007/s12111-000-1022-7

Reeder, S., & Utley, J. (2017). What is a fraction? Developing fraction understanding in prospective elementary teachers. *School Science and Mathematics*, *117*(7–8), 307–316. https://doi.org/10.1111/ssm.12248

Reimer, K., & Moyer-Packenham, P. S. (2005). Third graders learn about fractions using virtual manipulatives: A classroom study. *Journal of Computers in Mathematics and Science Teaching*, *24*(1), 5–25.

Rice, J. (2003). *Understanding the Effectiveness of Teacher Attributes*. Economic Policy Institute.

Richland, L. E., Stigler, J. W., & Holyoak, K. J. (2012). Teaching the conceptual structure of mathematics. *Educational Psychologist*, *47*(3), 189–203. https://doi.org/10.1080/00461520.2012.667065

Rittle-Johnson, B., & Koedinger, K. (2009). Iterating between lessons on concepts and procedures can improve mathematics knowledge. *British Journal of Educational Psychology*, *79*(3), 483–500. https://doi.org/10.1348/000709908x398106

Rittle-Johnson, B., & Siegler, R. S. (1998). The relation between conceptual and procedural knowledge in learning mathematics: A review. In *The Development of Mathematical Skills* (pp. 75–100). Psychology Press.

Rittle-Johnson, B., Siegler, R. S., & Alibali, M. W. (2001). Developing conceptual understanding and procedural skill in mathematics: An iterative process. *Journal of Educational Psychology*, *93*(2), 346–362. https://doi.org/10.1037//0022-0663.93.2.346

Ross, S. M., & Anand, P. G. A. (1987). A computer-based strategy for personalizing verbal problems in teaching mathematics. *Educational Technology Research and Development*, *35*(1), 151–162. https://doi.org/https://doi.org/10.1007/BF02793843

Rousselle, L., & Noël, M.-P. (2008). Mental arithmetic in children with mathematics learning disabilities. *Journal of Learning Disabilities*, *41*(6), 498–513. https://doi.org/10.1177/0022219408315638

Rowan, B., Correnti, R., & Miller, R. J. (2002). What large-scale, survey research tells us about teacher effects on student achievement: Insights from the prospects study of elementary schools. *Teachers College Record*, *104*(8), 1525–1567. https://doi.org/10.1111/1467-9620.00212

Saenz-Ludlow, A. (1995). Ann's fraction schemes. *Educational Studies in Mathematics*, *28*(2), 101–132. https://doi.org/10.1007/bf01295789

Sanders, W. L., & Horn, S. P. (1998). Research findings from the Tennessee value-added assessment system (TVAAS) database: Implications for educational evaluation and research. *Journal of Personnel Evaluation in Education*, *12*(3), 247–256. https://doi.org/10.1023/a:1008067210518

Sanders, W. L., Wright, P. S., & Horn, S. P. (1997). Teacher and classroom context effects on student achievement: Implications for teacher evaluation. *Journal of Personnel Evaluation in Education*, *11*(1), 57–67. https://doi.org/10.1023/a:1007999204543

Santagata, R., & Lee, J. (2019). Mathematical knowledge for teaching and the mathematical quality of instruction: a study of novice elementary school teachers. *Journal of Mathematics Teacher Education*. https://doi.org/10.1007/s10857-019-09447-y

Schmidt, W. H. (2012). At the Precipice: The story of mathematics education in the United States. *Peabody Journal of Education*, *87*(1), 133–156. https://doi.org/10.1080/0161956x.2012.642280

Schneider, M., Beeres, K., Coban, L., Merz, S., Susan Schmidt, S., Stricker, J., & De Smedt, B. (2016). Associations of non-symbolic and symbolic numerical magnitude processing with mathematical competence: a meta-analysis. Developmental Science, 20(3), e12372. https://doi.org/10.1111/desc.12372

Schneider, B., Ehrhart, M. G., & Macey, W. H. (2011). Perspectives on organizational climate and culture. *APA Handbook of Industrial and Organizational Psychology, Vol 1: Building and Developing the Organization.*, 373–414. https://doi.org/10.1037/12169-012

Schneider, M., & Siegler, R. S. (2010). Representations of the magnitudes of fractions. *Journal of Experimental Psychology: Human Perception and Performance*, *36*(5), 1227–1238. https://doi.org/10.1037/a0018170

Schneider, M., & Stern, E. (2010). The developmental relations between conceptual and procedural knowledge: A multimethod approach. *Developmental Psychology*, *46*(1), 178–192. https://doi.org/10.1037/a0016701

Schoenfeld, A. H., & Kilpatrick, J. (2013). A US perspective on the implementation of inquiry-based learning in mathematics. *ZDM: The International Journal on Mathematics Education*, *45*(6), 901–909. https://doi.org/10.1007/s11858-013-0531-5

Selling, S., Garcia, N., & Ball, D. (2016). What does it take to develop assessments of mathematical knowledge for teaching?: Unpacking the mathematical work of teaching. *The Mathematics Enthusiast*, *13*(1), 35–51. https://scholarworks.umt.edu/tme/vol13/iss1/4

Shabani, K., Khatib, M., & Ebadi, S. (2010). Vygotsky's Zone of Proximal Development: Instructional implications and teachers' professional development. *English Language Teaching*, *3*(4), 238. https://doi.org/10.5539/elt.v3n4p237

Sherin, M. G. (2002). When teaching becomes learning. *Cognition and Instruction*, *20*(2), 119–150. https://doi.org/10.1207/s1532690xci2002_1

Shulman, L. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, *15*(2), 4–14. https://doi.org/10.3102/0013189x015002004

Shuls, J. V. (2017). Raising the bar on teacher quality: Assessing the impact of increasing licensure exam cut-scores. *Educational Policy*, *32*(7), 969–992. https://doi.org/10.1177/0895904816682315

Siebert, D., & Gaskin, N. (2006). Creating, naming, and justifying fractions. *Teaching Children Mathematics*, *12*(8), 394–400. http://www.jstor.org/stable/41198803

Siegler, R. S. (2017). Fractions: Where it all goes wrong. Scientific American.

https://www.scientificamerican.com/article/fractions-where-it-all-goes-wrong/

Siegler, R. S., & Booth, J. L. (2004). Development of numerical estimation in young children. *Child Development*, *75*(2), 428–444. https://doi.org/10.1111/j.1467-8624.2004.00684.x

Siegler, R., Carpenter, T., Fennell, F., Geary, D., Lewis, J., Okamoto, Y., Thompson, L., & Wray, J. (2010). *Developing effective fractions instruction for kindergarten through 8th grade: A practice guide* (NCEE #2010-4039). Washington, DC: National Center for Education Evaluation and Regional Assistance, Institute of Education Sciences, U.S. Department of Education. Retrieved from whatworks.ed.gov/ publications/practiceguides.

Siegler, R. S., Duncan, G. J., Davis-Kean, P. E., Duckworth, K., Claessens, A., Engel, M., Susperreguy, M. I., & Chen, M. (2012). Early predictors of high school mathematics achievement. *Psychological Science*, *23*(7), 691–697. https://doi.org/10.1177/0956797612440101

Siegler, R. S., & Pyke, A. A. (2013). Developmental and individual differences in understanding of fractions. *Developmental Psychology*, *49*(10), 1994–2004. https://doi.org/10.1037/a0031200

Siegler, R. S., & Ramani, G. B. (2009). Playing linear number board games—but not circular ones—improves low-income preschoolers' numerical understanding. *Journal of Educational Psychology*, *101*(3), 545–560. https://doi.org/10.1037/a0014239

Siegler, R. S., Thompson, C. A., & Schneider, M. (2011). An integrated theory of whole number and fractions development. *Cognitive Psychology*, *62*(4), 273–296. https://doi.org/10.1016/j.cogpsych.2011.03.001

Siemon, D., Breed, M., Virgona, J. (2006). *From additive to multiplicative thinking-The big challenge of the middle years.* The scaffolding numeracy in the middle years project (SNMY). Victoria Department of Education and Tasmanian Education Department. https://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.123.6773&rep=rep1&type=pdf

Simon, M. A., & Tzur, R. (1999). Explicating the teacher's perspective from the researchers' perspectives: Generating accounts of mathematics teachers' practice. *Journal for Research in Mathematics Education*, *30*(3), 252. https://doi.org/10.2307/749835

Skemp, R. R. (1978). Relational understanding and instrumental understanding. *The Arithmetic Teacher*, *26*(3), 9–15. https://doi.org/10.5951/at.26.3.0009

Skemp, R. R. (1989). *Mathematics in the primary school.* Routledge.

Sousa, D. A. (2008). *How the brain learns mathematics*. Corwin Press.

Sowell, E. J. (1989). Effects of manipulative materials in mathematics instruction. *Journal for Research in Mathematics Education*, *20*(5), 498–505. https://doi.org/10.5951/jresematheduc.20.5.0498

Squires, J., Pribble, L., Chen, C.-I., & Pomés, M. (2012). Early childhood education: Improving outcomes for young children and families. *Handbook of Psychology, Second Edition*, *7*. https://doi.org/10.1002/9781118133880.hop207011

Star, J. R., & Stylianides, G. J. (2013). Procedural and conceptual knowledge: Exploring the gap between knowledge type and knowledge quality. *Canadian Journal of Science, Mathematics and Technology Education*, *13*(2), 169–181. https://doi.org/10.1080/14926156.2013.784828

Starkey, P., Klein, A., & Wakeley, A. (2004). Enhancing young children's mathematical knowledge through a pre-kindergarten mathematics intervention. *Early Childhood Research Quarterly*, *19*(1), 99–120. https://doi.org/10.1016/j.ecresq.2004.01.002

Sterrett, W., Fiddner, P., Gilman, C., & Fiddner, P. (2010). A reflective look at formative assessment: Using math exit slips to check for understanding. *Consortium for Research on Educational Accountability and Teacher Evaluation.* https://files.eric.ed.gov/fulltext/ED512809.pdf

Stigler, J., & Hiebert, J. (1997). Understanding and improving classroom mathematics instruction: An overview of the TIMSS video study. *Phi Delta Kappan*, *79*(1), 14–21.

Stigler, J., & Hiebert, J. (2000). The teaching Gap: Best ideas from the world's teachers for improving education in the classroom. *Journal of Curriculum Studies*, *32*(6), 867–874. https://doi.org/10.1080/00220270050167215

Streefland, L. (1991). *Fractions in Realistic Mathematics Education*. Springer Netherlands. https://doi.org/10.1007/978-94-011-3168-1

Stronge, J. H., Ward, T. J., & Grant, L. W. (2011). What makes good teachers good? A cross-case analysis of the connection between teacher effectiveness and student achievement. *Journal of Teacher Education*, *62*(4), 339–355. https://doi.org/10.1177/0022487111404241

Swafford, J. O., Jones, G. A., & Thornton, C. A. (1997). Increased knowledge in geometry and instructional practice. *Journal for Research in Mathematics Education*, *28*(4), 467. https://doi.org/10.2307/749683

Swanbrow, D. (2012, 18). Fractions are the key to math success, new study shows. *University of Michigan. Institute for Social Research*. https://isr.umich.edu/news-events/news-releases/fractions-are-the-key-to-math-success-new-study-shows/

Tanenbaum, C., Walters, K., Freels, A., & Smith, T. (2017). Vertically aligned curricular framework: Profile of practice brief. American Institute for Research. https://www2.ed.gov/programs/dropout/verticalalignmentprofile.pdf

Texas Education Agency. (1997). *Texas Education Agency - 19 TAC Chapter 111*. Ritter.Tea.State.Tx.Us. http://ritter.tea.state.tx.us/rules/tac/chapter111/index.html

Texas Education Agency. (2020). *Texas Education Agency - 19 TAC Chapter 111*. Ritter.Tea.State.Tx.Us. https://tea.texas.gov/student-assessment/testing/staar/staar-resources

Tirosh, D. (2000). Enhancing prospective teachers' knowledge of children's conceptions: The case of division of fractions. *Journal for Research in Mathematics Education*, *31*(1), 5. https://doi.org/10.2307/749817

Thompson, P. W. (1993). Quantitative reasoning, complexity, and additive structures. *Educational Studies in Mathematics*, *25*(3), 165–208. https://doi.org/10.1007/bf01273861

Tobias, J. M. (2012). Prospective elementary teachers' development of fraction language for defining the whole. *Journal of Mathematics Teacher Education*, *16*(2), 85–103. https://doi.org/10.1007/s10857-012-9212-5

Torbeyns, J., Schneider, M., Xin, Z., & Siegler, R. S. (2015). Bridging the gap: Fraction understanding is central to mathematics achievement in students from three different continents. *Learning and Instruction*, *37*, 5–13. https://doi.org/10.1016/j.learninstruc.2014.03.002

Tran, C., Smith, B., & Buschkuehl, M. (2017). Support of mathematical thinking through embodied cognition: Non-digital and digital approaches. *Cognitive Research: Principles and Implications*, *2*(1). https://doi.org/10.1186/s41235-017-0053-8

Tsai, T.-L., & Li, H.-C. (2016). Towards a framework for developing students' fraction proficiency. *International Journal of Mathematical Education in Science and Technology*, *48*(2), 244–255. https://doi.org/10.1080/0020739x.2016.1238520

Tucker, P. D., & Stronge, J. H. (2005). *Linking teacher evaluation and student learning*. Association For Supervision And Curriculum Development, Cop.

Turmundi. (2012). Teachers' perception toward mathematics teaching innovation in Indonesian junior high school: An exploratory factor analysis. *Journal of Mathematics Education*, *5*(1), 97–120. http://educationforatoz.net/images/7_Turmudi.pdf

Turner, F., & Rowland, T. (2010). The knowledge quartet as an organising framework for developing and deepening teachers' mathematics knowledge. *Mathematical Knowledge in Teaching*, *11*(2), 195–212. https://doi.org/10.1007/978-90-481-9766-8_12

U.S Department of Education. (2008). *The final report of the national mathematics advisory panel*. Www.Ed.Gov. http://ed.gov/

U.S. Department of Education. (2017). Every Student Succeeds Act (ESSA) | U.S. Department of Education. In *Ed.gov*. https://www.ed.gov/essa?src=rn

Utley, J., & Reeder, S. (2012). Prospective elementary teachers' development of fraction number sense. *Investigations in Mathematics Learning*, *5*(2), 1–13. https://doi.org/10.1080/24727466.2012.11790320

Uttal, D. H., Scudder, K. V., & DeLoache, J. S. (1997). Manipulatives as symbols: A new perspective on the use of concrete objects to teach mathematics. *Journal of Applied Developmental Psychology*, *18*(1), 37–54. https://doi.org/10.1016/s0193-3973(97)90013-7

Vamvakoussi, X., & Vosniadou, S. (2010). How many decimals are there between two fractions? Aspects of secondary school students' understanding of rational numbers and their notation. *Cognition and Instruction*, *28*(2), 181–209. https://doi.org/10.1080/07370001003676603

Van de Walle, J. A., Karp, K. S., & Bay-Williams, J. M. (2014). *Elementary and middle school mathematics : teaching developmentally.* Pearson.

Van de Walle, J. A., Bay-Williams, J. M., Mcgarvey, L. M., & Karp, K. S. (2016). *Elementary and middle school mathematics + Enhanced Pearson Etext Access Card : teaching developmentally*. Pearson.

Van de Walle, J. A., & Lovin, L. H. (2006). *Teaching student-centered mathematics. Grades K-3*. Pearson/Allyn And Bacon.

Van Steenbrugge, H., Lesage, E., Valcke, M., & Desoete, A. (2013). Preservice elementary school teachers' knowledge of fractions: a mirror of students' knowledge? *Journal of Curriculum Studies*, *46*(1), 138–161. https://doi.org/10.1080/00220272.2013.839003

Van Steenbrugge, H., Valcke, M., & Desoete, A. (2010). Mathematics learning difficulties in primary education: teachers' professional knowledge and the use of commercially available learning packages. *Educational Studies*, *36*(1), 59–71. https://doi.org/10.1080/03055690903148639

Vanhille, L., & Baroody, A. J. (2002). Fraction instruction that fosters multiplicative reasoning.

Vygotsky, L. (1978). Interaction between learning and development. *Readings on the Development of Children*, *23*(3).

Watts, T. W., Duncan, G. J., Siegler, R. S., & Davis-Kean, P. E. (2014). What's past is prologue: Relations between early mathematics knowledge and high school achievement. *Educational Researcher*, 43(7), 352-360 https://doi.org/10.3102/0013189X14553660

Wearne, D., & Hiebert, J. (1988). A cognitive approach to meaningful mathematics instruction: Testing a local theory using decimal numbers. *Journal for Research in Mathematics Education*, *19*(5), 371. https://doi.org/10.2307/749172

Wees, D. (2011, August 17). *Ineffective professional development – The Reflective Educator*. *The Reflective Educator*. https://davidwees.com/content/ineffective-professional-development/

Wei, R. C., Darling-Hammond, L., Andree, A., Richardson, N., & Orphanos, S. (2009). Professional learning in the learning profession: A status report on teacher development in the U.S. and abroad technical report. https://edpolicy.stanford.edu/sites/default/files/publications/professional-learning-learning-profession-status-report-teacher-development-us-and-abroad.pdf

Wexler, N. (2019, March 13). *Why So Many Aspiring Teachers Can't Pass A Licensing Test--And Why It Matters*. Forbes. https://www.forbes.com/sites/nataliewexler/2019/03/13/why-so-many-aspiring-teachers-cant-pass-a-licensing-test--and-why-it-matters/?sh=60b9ef03321a

Wiswall, M. (2013). The dynamics of teacher quality. *Journal of Public Economics*, *100*(C), 61–78. https://doi.org/10.1016/j.jpubeco.2013.01.006

Witzel, B. S., & Little, M. (2016). *Teaching elementary mathematics to struggling learners*. The Guilford Press.

Wong, M. M. D., & Evans, D. (2011). Assessing students' understanding of fraction equivalence. In *Fractions: Teaching for Understanding* (pp. 81–90). Australian Association of Mathematics Teachers.

Yackel, E., & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education*, *27*(4), 458. https://doi.org/10.2307/749877

Yang, D.-C. (2007). Investigating the strategies used by pre-service teachers in Taiwan when responding to number sense questions. *School Science and Mathematics*, *107*(7), 293–301. https://doi.org/10.1111/j.1949-8594.2007.tb17790.x

Yoon, K. S., Duncan, T., Lee, S. W.-Y., Scarloss, B., & Shapley, K. (2007). Reviewing the evidence on how teacher professional development affects student achievement (Issues & Answers Report, REL 2007–No. 033). Washington, DC: *U.S. Department of Education, Institute of Education Sciences, National Center for Education Evaluation and Regional Assistance, Regional Educational Laboratory Southwest*. Retrieved from http://ies.ed.gov/ncee/edlabs

Zembat, İ. Ö. (2015). An alternative route to teaching fraction division: Abstraction of common denominator algorithm. *International Electronic Journal of Elementary Education, 7*(3), 399-422. https://files.eric.ed.gov/fulltext/EJ1068041.pdf

Zhang, X., Clements, M. A. (Ken), & Ellerton, N. F. (2015). Engaging students with multiple models of fractions. *Teaching Children Mathematics*, *22*(3), 138. https://doi.org/10.5951/teacchilmath.22.3.0138

Appendix A

Session Objectives and Assessed Items

Virtual Models
The following virtual models were used for all sessions one through 6.

Free Math Apps: https://www.mathlearningcenter.org/apps
MathsBot.com - Tools for Maths Teachers
https://www.didax.com/math/virtual-manipulatives.html
https://toytheater.com/category/teacher-tools/virtual-manipulatives/
https://www.sadlier.com/school/mathematics
Fraction Squares
Fraction Circles

Session #1
Introduction to Fractions through Partitioning and Concrete Models

- Partitioning Equally
- Unit Fraction
- Non-Unit
- Represent fractions with denominators of 2, 3, 4, 6, 8

Session 1 Objective(s) – TEKS Addressed

Second Grade: 2.3.A, B, C, and D

(3) Number and operations. The student applies mathematical process standards to recognize and represent fractional units and communicates how they are used to name parts of a whole. The student is expected to:

> (A) partition objects into equal parts and name the parts, including halves, fourths, and eighths, using words;
> (B) explain that the more fractional parts used to make a whole, the smaller the part; and the fewer the fractional parts, the larger the part;
> (C) use concrete models to count fractional parts beyond one whole using words and recognize how many parts it takes to equal one whole;
> (D) identify examples and nonexamples of halves, fourths, and eighths.

Third Grade: 3.3.A, B, C, D, and E

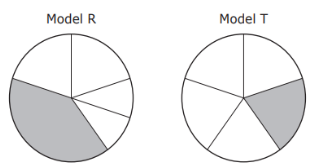
(3) Number and operations. The student applies mathematical process standards to represent and explain fractional units. The student is expected to:

> (A) represent fractions greater than zero and less than or equal to one with denominators of 2, 3, 4, 6, and 8 using concrete objects and pictorial models,

including strip diagrams and number lines;
(B) determine the corresponding fraction greater than zero and less than or equal to one with denominators of 2, 3, 4, 6, and 8 given a specified point on a number line;
(C) explain that the unit fraction 1/*b* represents the quantity formed by one part of a whole that has been partitioned into *b* equal parts where *b* is a non-zero whole number;
(D) compose and decompose a fraction *a*/*b* with a numerator greater than zero and less than or equal to *b* as a sum of parts 1/*b;*
(E) solve problems involving partitioning an object or a set of objects among two or more recipients using pictorial representations of fractions with denominators of 2, 3, 4, 6, and 8.

Correlated Assessment Items

17 Models R and T are shown.



Which statement is true?

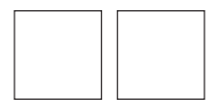
A The shaded parts of Model R and Model T are different sizes, but each model represents the same fraction of the whole.

B The shaded part of Model R cannot be written as the fraction $\frac{1}{5}$, because the parts are not all equal in size.

C The shaded part of Model T is $\frac{1}{4}$, because the parts are all equal in size.

D The total number of parts in Model R is 5, so $\frac{1}{5}$ of Model R is shaded.

8 Brandon drew the two congruent squares shown.



- He divided one square into 2 congruent triangular parts.
- He divided the other square into 2 congruent rectangular parts.

Which statement is true?

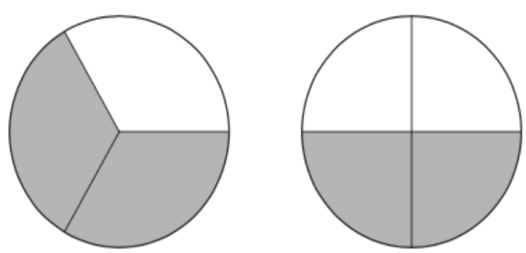
F Each triangular part and each rectangular part represents $\frac{1}{2}$ the area of one square.

G Each triangular part has an area that is greater than the area of each rectangular part.

H Each triangular part and each rectangular part represents $\frac{1}{4}$ the area of one square.

J Each rectangular part has an area that is greater than the area of each triangular part.

The models shown are the same size and are each divided into equal-size parts.
The models are shaded to represent two fractions.



Which statement is true?

F $\frac{2}{3} > \frac{2}{4}$, because thirds are larger than fourths.

G $\frac{2}{3} = \frac{2}{4}$, because each model has 2 parts shaded.

H $\frac{1}{3} < \frac{1}{4}$, because 3 is less than 4.

J $\frac{1}{3} = \frac{1}{4}$, because each model shows 1 whole.

Session #2
Fraction Concepts Continued: (Number Lines & Comparing Fractions)

- Partitioning Equally
- Unit Fraction
- Non-Unit

Session 2 Objective(s) – TEKS Addressed

Second Grade: 2.3.A, B, C, and D

(3) Number and operations. The student applies mathematical process standards to recognize and represent fractional units and communicates how they are used to name parts of a whole. The student is expected to:

(A) partition objects into equal parts and name the parts, including halves, fourths, and eighths, using words;
(B) explain that the more fractional parts used to make a whole, the smaller the part; and the fewer the fractional parts, the larger the part;
(C) use concrete models to count fractional parts beyond one whole using words and recognize how many parts it takes to equal one whole;
(D) identify examples and nonexamples of halves, fourths, and eighths.

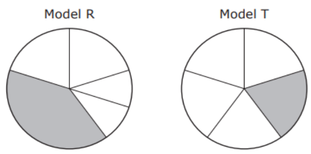
Third Grade: 3.3.A, B, C, D, and E

(3) Number and operations. The student applies mathematical process standards to represent and explain fractional units. The student is expected to:

(A) represent fractions greater than zero and less than or equal to one with denominators of 2, 3, 4, 6, and 8 using concrete objects and pictorial models, including strip diagrams and number lines;
(B) determine the corresponding fraction greater than zero and less than or equal to one with denominators of 2, 3, 4, 6, and 8 given a specified point on a number line;
(C) explain that the unit fraction $1/b$ represents the quantity formed by one part of a whole that has been partitioned into b equal parts where b is a non-zero whole number;
(D) compose and decompose a fraction a/b with a numerator greater than zero and less than or equal to b as a sum of parts $1/b$;
(E) solve problems involving partitioning an object or a set of objects among two or more recipients using pictorial representations of fractions with denominators of 2, 3, 4, 6, and 8.

Correlated Assessment Items

17 Models R and T are shown.



Which statement is true?

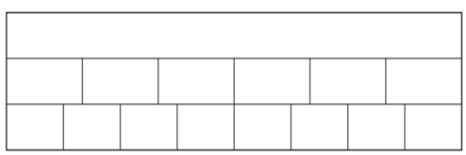
A The shaded parts of Model R and Model T are different sizes, but each model represents the same fraction of the whole.

B The shaded part of Model R cannot be written as the fraction $\frac{1}{5}$, because the parts are not all equal in size.

C The shaded part of Model T is $\frac{1}{4}$, because the parts are all equal in size.

D The total number of parts in Model R is 5, so $\frac{1}{5}$ of Model R is shaded.

13 Fraction strips are shown.



Which comparison and explanation are true?

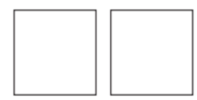
A $\frac{5}{6} < \frac{5}{8}$, because eighths are larger than sixths

B $\frac{5}{6} < \frac{5}{8}$, because sixths are larger than eighths

C $\frac{5}{6} > \frac{5}{8}$, because eighths are larger than sixths

D $\frac{5}{6} > \frac{5}{8}$, because sixths are larger than eighths

8 Brandon drew the two congruent squares shown.



- He divided one square into 2 congruent triangular parts.
- He divided the other square into 2 congruent rectangular parts.

Which statement is true?

F Each triangular part and each rectangular part represents $\frac{1}{2}$ the area of one square.

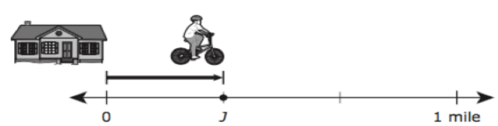
G Each triangular part has an area that is greater than the area of each rectangular part.

H Each triangular part and each rectangular part represents $\frac{1}{4}$ the area of one square.

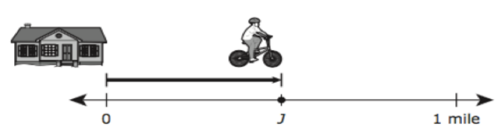
J Each rectangular part has an area that is greater than the area of each triangular part.

Javier rode his bike a distance of $\frac{1}{2}$ mile from his house. On which number line does point *J* represent Javier's position after riding his bike?

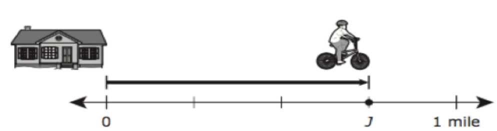
A



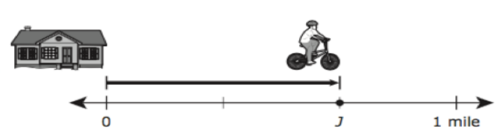
B



C



D



Session #3
Fraction Equivalence

- Equivalence
- Comparing with like denominators
- Comparing with unlike denominators

Session 3 Objective(s) – TEKS Addressed

Third Grade: 3.3.F, G, and H

(3) Number and operations. The student applies mathematical process standards to represent and explain fractional units. The student is expected to:

> (F) represent equivalent fractions with denominators of 2, 3, 4, 6, and 8 using a variety of objects and pictorial models, including number lines;
> (G) explain that two fractions are equivalent if and only if they are both represented by the same point on the number line or represent the same portion of a same size whole for an area model;
> (H) compare two fractions having the same numerator or denominator in problems by reasoning about their sizes and justifying the conclusion using symbols, words, objects, and pictorial models.

Fourth Grade: 4.3.A, B, C, D, and F

(3) Number and operations. The student applies mathematical process standards to represent and generate fractions to solve problems. The student is expected to:

> (A) represent a fraction a/b as a sum of fractions 1/b, where a and b are whole numbers and b > 0, including when a > b;
> (B) decompose a fraction in more than one way into a sum of fractions with the same denominator using concrete and pictorial models and recording results with symbolic representations;
> (C) determine if two given fractions are equivalent using a variety of methods;
> (D) compare two fractions with different numerators and different denominators and represent the comparison using the symbols >, =, or <;
> (F) evaluate the reasonableness of sums and differences of fractions using benchmark fractions 0, 1/4, 1/2, 3/4, and 1, referring to the same whole.

Correlated Assessment Items

9 Ms. Thompson needs $\frac{15}{2}$ yards of red fabric and $7\frac{1}{2}$ yards of silver fabric. Which comparison is true?

A $\frac{15}{2} > 7\frac{1}{2}$

B $\frac{15}{2} = 7\frac{1}{2}$

C $\frac{15}{2} < 7\frac{1}{2}$

D None of these

32 The table shows the fractions of the bulletin boards in four classrooms that will be used to display artwork.

Artwork on Bulletin Boards

Teacher	Fraction for Artwork
Ms. Brady	$\frac{5}{10}$
Mr. Chang	$\frac{2}{4}$
Ms. Gupta	$\frac{5}{6}$
Mr. Taylor	$\frac{4}{8}$

Which comparison is true?

F $\frac{2}{4} > \frac{4}{8}$

G $\frac{4}{8} < \frac{5}{10}$

H $\frac{5}{6} > \frac{4}{8}$

J $\frac{5}{6} < \frac{5}{10}$

A number cube with faces labeled from 1 to 6 was rolled 20 times. Each time the number cube was rolled, the number showing on the top face was recorded. The table shows the results.

Results

Number Showing on Top Face	Frequency
1	0
2	3
3	3
4	6
5	3
6	5

Based on these results, what is the experimental probability that the next time the number cube is rolled it will land with 5 or 6 showing on the top face?

A $\frac{2}{5}$

B $\frac{3}{20}$

C $\frac{1}{3}$

D $\frac{3}{5}$

19 Point *P* on the number line represents two equivalent fractions.

Which two equivalent fractions can point *P* represent?

A $\frac{1}{4}$ and $\frac{1}{8}$

B $\frac{1}{3}$ and $\frac{2}{6}$

C $\frac{1}{4}$ and $\frac{2}{8}$

D $\frac{1}{4}$ and $\frac{3}{4}$

Session #4
Adding and Subtracting Fractions

- Like denominators
- Unlike denominators

Session 4 Objective(s) – TEKS Addressed

Second Grade: 2.3.C

(3) Number and operations. The student applies mathematical process standards to recognize and represent fractional units and communicates how they are used to name parts of a whole. The student is expected to:

> (C) use concrete models to count fractional parts beyond one whole using words and recognize how many parts it takes to equal one whole;

Third Grade: 3.3.D

(3) Number and operations. The student applies mathematical process standards to represent and explain fractional units. The student is expected to:

> (D) compare two fractions with different numerators and different denominators and represent the comparison using the symbols >, =, or <;.

Fourth Grade: 4.3.A, B, E, and F.

(3) Number and operations. The student applies mathematical process standards to represent and generate fractions to solve problems. The student is expected to:

> (A) represent a fraction a/b as a sum of fractions 1/b, where a and b are whole numbers and b > 0, including when a > b;
> (B) decompose a fraction in more than one way into a sum of fractions with the same denominator using concrete and pictorial models and recording results with symbolic representations;
> (E) represent and solve addition and subtraction of fractions with equal denominators using objects and pictorial models that build to the number line and properties of operations;
> (F) evaluate the reasonableness of sums and differences of fractions using benchmark fractions 0, 1/4, 1/2, 3/4, and 1, referring to the same whole.

Fifth Grade: 5.3.H

(3) Number and operations. The student applies mathematical process standards to develop and use strategies and methods for positive rational number computations in order to solve problems with efficiency and accuracy. The student is expected to:

> (H) represent and solve addition and subtraction of fractions with unequal

denominators referring to the same whole using objects and pictorial models and properties of operations.

Sixth Grade: 6.2.C, and D

(2) Number and operations. The student applies mathematical process standards to represent and use rational numbers in a variety of forms. The student is expected to:

(C) locate, compare, and order integers and rational numbers using a number line;
(D) order a set of rational numbers arising from mathematical and real-world contexts.

Correlated Assessment Items

2 Ignacio and Elaine read the same book. The shaded part of each model represents the fraction of the book that each student read.

Which expression can be used to find the difference between the fraction of the book Elaine read and the fraction of the book Ignacio read?

F $\frac{16}{4} - \frac{13}{7}$

G $\frac{7}{13} - \frac{4}{16}$

H $\frac{16}{20} - \frac{13}{20}$

J $\frac{20}{16} - \frac{20}{13}$

Zeke used $\frac{3}{4}$ cup white sugar, $\frac{3}{4}$ cup brown sugar, and $2\frac{1}{4}$ cups of flour to bake some cookies.

What was the difference between the amount of flour and the combined amount of sugar Zeke used?

F $3\frac{3}{4}$ cups

G $1\frac{2}{4}$ cups

H $\frac{2}{4}$ cup

J $\frac{3}{4}$ cup

Which expression is equivalent to $\frac{9}{8}$?

F $\frac{3}{8} + \frac{3}{8}$

G $\frac{1}{2} + \frac{2}{3} + \frac{6}{3}$

H $\frac{1}{9} + \frac{1}{9} + \frac{1}{9} + \frac{1}{9} + \frac{1}{9} + \frac{1}{9} + \frac{1}{9} + \frac{1}{9}$

J $\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}$

Session #5
Multiplying Fractions

- Multiplying a whole number by a fraction
- Multiplying a fraction by a fraction

Session 5 Objective(s) – TEKS Addressed

Prekindergarten: V.B.3

Child uses informal strategies to separate up to 10 items into equal group

Second Grade: 2.6.A*

(6) Number and operations. The student applies mathematical process standards to connect repeated addition and subtraction to multiplication and division situations that involve equal groupings and shares. The student is expected to:

> (A) model, create, and describe contextual multiplication situations in which equivalent sets of concrete objects are joined.

Fifth Grade: 5.3.I

(3) Number and operations. The student applies mathematical process standards to develop and use strategies and methods for positive rational number computations in order to solve problems with efficiency and accuracy. The student is expected to:

> (I) represent and solve multiplication of a whole number and a fraction that refers to the same whole using objects and pictorial models, including area models.

Sixth Grade: 6.3.E

(3) Number and operations. The student applies mathematical process standards to represent addition, subtraction, multiplication, and division while solving problems and justifying solutions. The student is expected to:

> (E) multiply and divide positive rational numbers fluently.

Correlated Assessment Items

Tommy bought 3 cups of blueberries. He will eat $\frac{1}{2}$ cup of blueberries each day.

How many days can Tommy eat the blueberries before they are all gone?

F 6

G 2

H 5

J 4

This week Andres will practice with his band for $1\frac{1}{2}$ hours on Monday, $1\frac{3}{4}$ hours on Tuesday, and 2 hours on Wednesday. Next week Andres will practice with his band for the same number of hours on Monday, Tuesday, and Wednesday.

What is the total number of hours Andres will practice with his band over these 6 days?

A $5\frac{1}{4}$ hours

B $10\frac{1}{2}$ hours

C $4\frac{1}{4}$ hours

D $8\frac{1}{2}$ hours

Maya has 120 caramel apples to sell. Each caramel apple is covered with one topping.

- $\frac{1}{5}$ of the caramel apples are covered with peanuts.
- $\frac{1}{3}$ are covered with chocolate chips.
- $\frac{3}{10}$ are covered with coconut.
- The rest are covered with sprinkles.

How many caramel apples are covered with sprinkles?

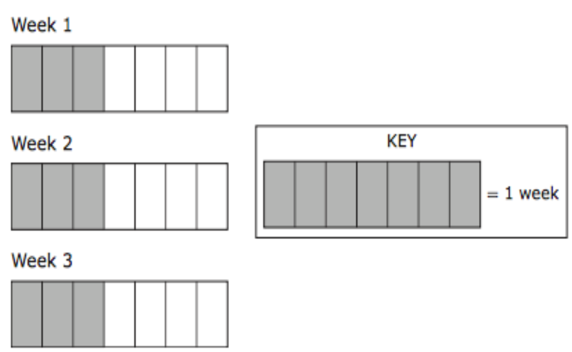
A 100

B 33

C 25

D 20

5 Darenda worked for 3 weeks. The shaded parts of the model represent the fraction of each week she worked from her home office.



Which expression can be used to determine the number of weeks Darenda worked from her home office over these 3 weeks?

A $3 + \frac{3}{4}$

B $3 + \frac{3}{7}$

C $3 \times \frac{3}{4}$

D $3 \times \frac{3}{7}$

Session #6
Dividing Fractions

- Fraction by a fraction
- Fraction by a whole number

Session 5 Objective(s) – TEKS Addressed

Prekindergarten: V.B.3

Child uses informal strategies to separate up to 10 items into equal group

Second Grade: 2.6.B*

(6) Number and operations. The student applies mathematical process standards to connect repeated addition and subtraction to multiplication and division situations that involve equal groupings and shares. The student is expected to:

> (B) model, create, and describe contextual division situations in which a set of concrete objects is separated into equivalent sets.

Fifth Grade: 5.3.J, and L

(3) Number and operations. The student applies mathematical process standards to develop and use strategies and methods for positive rational number computations in order to solve problems with efficiency and accuracy. The student is expected to:

> (J) represent division of a unit fraction by a whole number and the division of a whole number by a unit fraction such as 1/3 ÷ 7 and 7 ÷ 1/3 using objects and pictorial models, including area models;
> (L) divide whole numbers by unit fractions and unit fractions by whole numbers.

Sixth Grade: 6.3.E

(3) Number and operations. The student applies mathematical process standards to represent addition, subtraction, multiplication, and division while solving problems and justifying solutions. The student is expected to:

> (E) multiply and divide positive rational numbers fluently.

Correlated Assessment Items

There are 16 pies on a picnic table.

- Each pie is cut into pieces.
- Each piece is $\frac{1}{8}$ of a pie.

How many pieces of pie are on the picnic table?

A 2

B 88

C 24

D 128

What is the value of $\frac{4}{15} \div \frac{2}{3}$?

F $\frac{8}{45}$

G $\frac{14}{15}$

H $\frac{5}{2}$

J $\frac{2}{5}$

9 The owner of a snow-cone stand used $\frac{1}{4}$ gallon of syrup to make 16 cherry snow cones. She used the same amount of syrup in each snow cone.

How much syrup in gallons was used in each cherry snow cone?

A $\frac{1}{4}$ gal

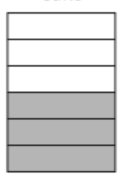
B 4 gal

C $\frac{1}{64}$ gal

D 64 gal

The model is shaded to represent the remaining one-half of a cake. Three friends will each receive an equal amount of the remaining cake until it is all gone.

Cake



Which equation can be used to determine the fraction of the whole cake each friend will receive?

F $\frac{1}{2} \times 3 = \frac{3}{2}$

G $\frac{1}{2} \times 6 = \frac{6}{2}$

H $\frac{1}{2} \div 3 = \frac{1}{6}$

J $\frac{1}{2} \div 6 = \frac{1}{12}$

A worker uses 450 inches of steel wire to make 300 springs of the same size. At this rate how many inches of steel wire are needed to make 1 spring?

F $\frac{1}{3}$ in.

G $\frac{1}{15}$ in.

H $\frac{2}{3}$ in.

J $1\frac{1}{2}$ in.

Appendix B

Session 5E Lesson Plans

Lesson #1 Introduction to Fractions through Partitioning and Concrete Models.
Standards & Objectives: TEKS & Vertical Alignment
2.3 (**A**) partition objects into equal parts and name the parts, including halves, fourths, and eighths, using words;
2.3 (B) explain that the more fractional parts used to make a whole, the smaller the part; and the fewer the fractional parts, the larger the part;
2.3 (**C**) use concrete models to count fractional parts beyond one whole using words and recognize how many parts it takes to equal one whole; and
2.3(D) identify examples and non-examples of halves, fourths, and eighths
3.3 (**A**) represent fractions greater than zero and less than or equal to one with denominators of 2, 3, 4, 6, and 8 using concrete objects and pictorial models, including strip diagrams and number lines
3.3(C) explain that the unit fraction 1/b represents the quantity formed by one part of a whole that has been partitioned into b equal parts where b is a non-zero whole number;
3.3(E) solve problems involving partitioning an object or a set of objects among two or more recipients using pictorial representations of fractions with denominators of 2, 3, 4, 6, and 8

Materials
Paper/pencil
Computer Access
Virtual manipulatives

Essential Questions

- What prior knowledge would be beneficial to students' in understanding partitioning and the initial steps in identifying fractional parts?
- Do we often see lessons that mirror those understandings?
- What real-life situations could children have already experienced that would act as a foundation to fractional learning?
- From your experience, what type of delivery is most commonly used to introduce fractions? Manipulatives, videos, direct instruction, models?
- How does understanding what a unit fraction is aid in building fractional fluency?
- What is the difference between a unit and non-unit fraction?

Procedures
Engage:
Sharing Activity #1:
Participants will explore sharing a set of objects among multiple individuals through the IES activity *Sharing a Set of Objects Evenly Among Recipients*. Participants will use the number frames application from The Math Learning Center to model their thinking. They

will share a personalized link with the researcher that contains their representation of the answer.

The following question will be presented:
Three children want to share 12 cookies so that each child receives the same number of cookies. How many cookies should each child get? Please show your thinking by answering through pictures, words, numbers, or modeling on the number frames application. [Give teachers time to explore the virtual manipulative.]
https://apps.mathlearningcenter.org/number-frames/
One person will share their screen and explain their thinking. Whole group discussion over similarities and differences of solution plans.

Sharing Activity #2: Unit and Non-Unit Fractions
Working as a whole group, participants will be given a second sharing problem that focuses on the idea of a unit fraction.
Researcher introduces the manipulative and allows time for teachers to work within the application.
https://apps.mathlearningcenter.org/fractions/

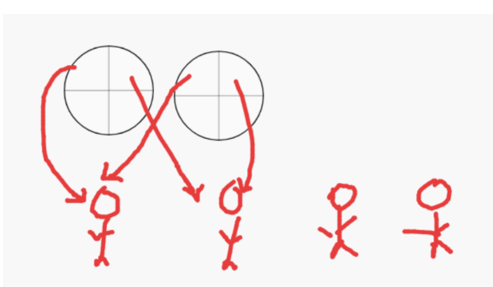
Question:
There are 3 friends. They want to run a relay race that is one mile. Each friend runs an equal amount of the race. How much of the race would each friend run?

How could you partition the shape into three equal parts? Please model your solution plan using the fraction application.

One person will share their screen and explain justification/thinking.

Four children want to share two apples. Each child wants to have an equal amount of the apples. What fraction of the apple will each kid get? Show *all the ways* you know that can represent the answer with fractions? Use virtual manipulative to model.
https://apps.mathlearningcenter.org/fractions/

[Caution, teachers will partition this into halves. Kids will partition into fourths because that's how many children are in the problem.]



Equivalent fraction
Are there any other fractional parts that cover the same area as the pieces you chose?

$$\frac{1}{4}+\frac{1}{4}$$

Explore:
Include additional problem here for clarification:
Five friends want to share 3 apples that are the same size. Each friend will have the same amount. Draw a picture to show what each child should get. (Non-unit Fraction).

Explain:
Sharing Activity #1
Why are we focusing on this type of problem? (Sharing equally)
Research shared about the importance of sharing equally.
Partitive and Measurement Review

Sharing Activity #2
How is division or sharing equally similar or different to this problem. Share/record answers. Equal parts, equal sharing, "one for you, one for you"

Explain Unit-Fractions - sharing one object with multiple friends or people. Partitioning one object into equal parts. Present research and define Unit Fraction.

1 shared equally with 3 friends. Each friend will run 1 equal part of 3 or $\frac{1}{3}$

Elaborate/Extend:
Discussion over the Inquiry-based learning and the value of concrete models in the classroom.

Evaluate:
Exit Ticket:

1. There are four children wrapping presents. They have one yard of ribbon to wrap all of the presents. If each child receives an equal amount of ribbon, what fraction would they get? Use a model to explain.
2. There are 7 friends. The friends have two yards of ribbon to wrap presents. If each child receives an equal amount of ribbon, what fraction would they get? Use a model to explain.

Lesson #2 Fraction Concepts Continued (Number Lines & Comparing Fractions)

<u>Standards & Objectives: TEKS & Vertical Alignment</u>
2.3(A) partition objects into equal parts and name the parts, including halves, fourths, and eighths, using words;
2.3(B) explain that the more fractional parts used to make a whole, the smaller the part; and the fewer the fractional parts, the larger the part;
2.3(C) use concrete models to count fractional parts beyond one whole using words and recognize how many parts it takes to equal one whole; and
2.3(D) identify examples and non-examples of halves, fourths, and eighths.

3.3(A) represent fractions greater than zero and less than or equal to one with denominators of 2, 3, 4, 6, and 8 using concrete objects and pictorial models, including strip diagrams and number lines;
3.3(B) determine the corresponding fraction greater than zero and less than or equal to one with denominators of 2, 3, 4, 6, and 8 given a specified point on a number line;
3.3(C) explain that the unit fraction 1/b represents the quantity formed by one part of a whole that has been partitioned into b equal parts where b is a non-zero whole number;
3.3(D) compose and decompose a fraction a/b with a numerator greater than zero and less than or equal to b as a sum of parts 1/b;
3.3(E) solve problems involving partitioning an object or a set of objects among two or more recipients using pictorial representations of fractions with denominators of 2, 3, 4, 6, and 8

<u>Materials</u>
Paper/pencil
Math Center Apps
Computer Accessibility

<u>Essential Questions</u>
- Use the word "number" to identify the fractions, to further solidify that fractions are numbers.

Examples:
- What number is halfway between zero and one?
- What other numbers represent one-half?
- When we share the same whole in various ways (halves, thirs, fourths), what happens to the size of the pieces?
- Are there pieces that can be combined to create the same area coverage as one piece?

<u>Procedures</u>
Engage:

Activity #1 Number Line Activities with Cuisenaire Rods *http://pbs.panda-prod.cdn.s3.amazonaws.com/media/assets/wgbh/rttt12/rttt12_int_cuisenaire/index.html*

Participants will work with virtual Cuisenaire rods. Looking at the comparison between the whole and parts/fractional measurement. Developing the understanding that fractions are descriptions of relationships between the rod deemed as the whole and the rod designated as the part.

Make note of relationships by having teachers put their thinking into words. “It takes three red rods to make a green rod”. This will set the foundations for reciprocal/multiplicative inverse relationships (½ and 2)

Participants will partition and record the fractions according to the rods; halves, thirds, fourths, sixths, eights, and 12ths.

Discussion questions:

1. What number one-fourth more than one-half? One-sixth more than one-half?
2. What number is one-sixth less than one?
3. What number is one-third more than one?
4. What number is halfway between one-twelfth and three-twelfths?

Which number is closest to one?

Explore:

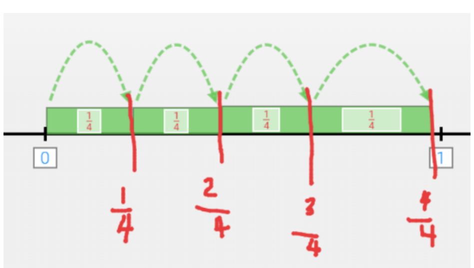
Activity #2 Connecting Cuisenaire Rods with One-Unit Number Lines

Make a connection between and the Cuisenaire rods and the number lines.

Exploring the idea that the smaller the number in the denominator, dictates a larger fraction is only true when the numerator is the same.

Present number lines without tick marks, just 0 and 1.

Participants will create number lines either with virtual manipulatives or with pictorial models. They will partition the number line and compare the fractional values while providing reasoning for comparisons.



Present Research:

When grade students were given the fractions $\frac{5}{6}$ and $\frac{7}{8}$, 40% of 4th grade students and 34% of 6th grade students said that $\frac{5}{6}$ was bigger because sixths are bigger than eights. (McNamara & Shaughnessey, 2010).

Number Lines are valuable for identifying fractional values, as they provide a more exact means of measurement (Siegler et al, 2010).

Activity #3 Connecting Cuisenaire Working with the Two-Unit Number Line:

Extending from the work with number lines to one, the researcher will introduce a number line that includes two whole numbers. This time focusing on the idea of mixed numbers.

Participants will be asked to partition the two-unit number lines into halves, thirds, fourths, sixths, eights, and twelfths.

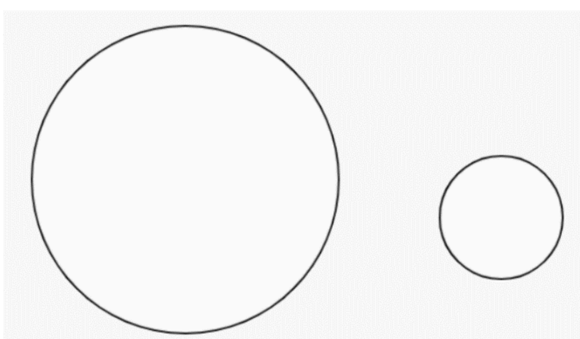
Participants will answer the same questions as in the prior activity. As a whole group, the participants will discuss their reasoning about fractions greater than one compared to those that are less than one.

Activity #4: Comparing Fractions

Jen has $\frac{1}{2}$ of a pizza.

Katrina has $\frac{1}{2}$ of a pizza.

Katrina has more pizza. Is this possible?



Malik has run two thirds of the marathon. Angela has run four sixths of the same marathon. Who has run the least distance of the marathon?

Use a number line.(Find a virtual manipulative that has multiple number lines.)

Explain:
There are many misconceptions surrounding fractions, especially when it comes to "rules". This session brings to light those misconceptions and generalizations through use

of conceptual models. Understanding the why behind the operation encourages participants and students to not simply apply a rule because it is a rule, rather it supports their understanding of fractions by giving them experience with the foundation of fractional concepts.

Elaborate/Extend:
The group will generate a list of "rules" and misconceptions about fractions. These may include;

- Smaller the denominator, bigger the fraction. This should connect back to...the more people we share with...the smaller the part or piece. SHARING EQUALLY *****
- When comparing fraction values, you only have to look at one part of the numerator or denominator
- Fractions are always less than 1.

Participants will individually decide if the rules are sometimes true, always true, or never true.

By focusing the participants' attention on the common misconceptions, they are able to see why the rule is incorrect and start identifying the core mathematical ideas related to fractions..

Evaluate:
Exit Ticket:

Answer the following by adding an inequality sign and show model.

$\frac{1}{3}$ ____________ $\frac{1}{2}$

Lesson #3 Fraction Equivalence

Standards & Objectives: TEKS & Vertical Alignment

3.3 (F) represent equivalent fractions with denominators of 2, 3, 4, 6, and 8 using a variety of objects and pictorial models
3.3(G) explain that two fractions are equivalent if and only if they are both represented by the same point on the number line or represent the same portion of a same size whole for an area model
3.3(H) compare two fractions having the same numerator or denominator in problems by reasoning about their sizes and justifying the conclusion using symbols, words, objects, and pictorial models.
4.3(A) represent a fraction a/b as a sum of fractions 1/b, where a and b are whole numbers and $b > 0$, including when $a > b$
4.3(B) decompose a fraction in more than one way into a sum of fractions with the same denominator using concrete and pictorial models and recording results with symbolic representations
4.3(C) determine if two given fractions are equivalent using a variety of methods

Materials

Choice of virtual manipulatives: fraction squares, fraction rectangles, fraction circles, number lines, two-unit number lines, and Cuisenaire rods.

Essential Questions

- How do you know which students can articulate the relationships between fractions? Can you think of any appropriate activities for students who have not yet developed this understanding?
- How can teachers adjust instruction continually in ways that support and extend learning?
- When we share the same whole in various ways (halves, thirds, fourths), what happens to the size of the pieces?
- Are there pieces that can be combined to create the same area coverage as one piece?

Procedures

Engage:

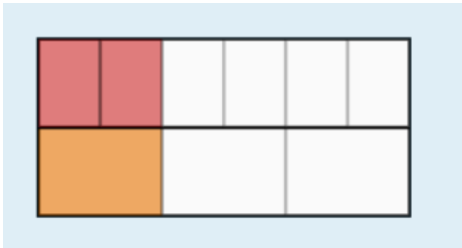
Participants will be given the following question:

Decide whether you agree or disagree with the following statement

$$\frac{2}{6} \text{ is the same as } \frac{1}{3}$$

Justify your reasoning by modeling with virtual manipulatives.

Discuss answers/solution paths.



Individuals share their thinking in a round table format.

Explore:

Modeling with Cuisenaire Rods

Participants will use the Cuisenaire rods from a measurement perspective, to identify the distances of different length rods to realize that equivalency is naming the same piece in a different way.

They will be asked to find how many of a specified color (brown/purple/red) rods are needed to create the same length as the marker. Participants will then work with the brown rod, which is half of the identified object, looking for colors that are equivalent. Calling on the connection to division and equal sharing from the previous PD lesson, the researcher will guide students in understanding the relationship between the rods.

Teachers will share their created images.

Participants will continue to measure other items (pencil, book, ruler, etc.) with the rods, recording their findings.

Explain:
This lesson explores the idea of fraction equivalency through the use of various models; including pictorial representations, Cuisenaire rods, and fraction tiles.

Teachers will share their equivalent fraction models with the class through screen-share.

The goal of the lesson is for participants to build fluency with fraction equivalency, gain understanding of strategies to use in the classroom and recognition that fractions can be named in different ways..

Elaborate/Extend:
<u>*Connecting Cuisenaire Measurement to the Number Line:*</u>

Students will use number lines and double number lines to identify that equivalent fractions describe the same magnitude.

The lesson aims to show that fractions are identified by the size of the partition in relation to the whole and that the distance from zero is important in determining the value.

Participants will be provided with a number line that has intervals of 12 centimeters between each whole number.

Participants will use virtual manipulatives to locate and mark $\frac{1}{2}$ and $\frac{3}{2}$, also identifying $\frac{0}{2}, \frac{2}{2}, 1\frac{1}{2}$, and $\frac{4}{2}$.

Participants will use virtual manipulatives to partition the same line into fourths, aligning $\frac{2}{4}$ with $\frac{1}{2}$ and $1\frac{2}{4}$ with $1\frac{1}{2}$.

Participants will continue to partition the line into thirds, sixths, and twelfths. This helps to show that as one continues to divide the segments, they continue to make smaller pieces.

Participants will complete an Equivalent Fractions on the Number Line recording sheet. The sheet details the number that are being considered and equivalent values on the number line.

Using these segments is a great transition into adding fractions with like denominators, as the participants start to see the fractions that can be combined to make the larger fractional segment.

Evaluate:
Exit Ticket:

1. Can you show three fractions that are equivalent to $\frac{1}{2}$ with manipulatives?
2. Can you show two equivalent fractions for $\frac{2}{3}$ using a number line?

Lesson Plan #4 Adding and Subtracting Fractions

Standards & Objectives: TEKS & Vertical Alignment
2.3(C) use concrete models to count fractional parts beyond one whole using words and recognize how many parts it takes to equal one whole

3.3(D) compose and decompose a fraction a/b with a numerator greater than zero and less than or equal to b as a sum of parts 1/b

4.2 (H) compare two fractions having the same numerator or denominator in problems by reasoning about their sizes and justifying the conclusion using symbols, words, objects, and pictorial models.
4.3(A) represent a fraction a/b as a sum of fractions 1/b, where a and b are whole numbers and b > 0, including when a > b
4.3(B) decompose a fraction in more than one way into a sum of fractions with the same denominator using concrete and pictorial models and recording results with symbolic representations
4.3(C) determine if two given fractions are equivalent using a variety of methods
4.3(D) compare two fractions with different numerators and different denominators and represent the comparison using the symbols >, =, or <
4.3 (E) Represent and solve addition and subtraction of fractions with equal denominators using objects and pictorial models that build to the number line and properties of operations
4.3(F) represent equivalent fractions with denominators of 2, 3, 4, 6, and 8 using a variety of objects and pictorial models, including number lines;
4.3(G) explain that two fractions are equivalent if and only if they are both represented by the same point on the number line or represent the same portion of a same size whole for an area model;

5.3(H) represent and solve addition and subtraction of fractions with unequal denominators referring to the same whole using objects and pictorial models and properties of operations

6.2(C) locate, compare, and order integers and rational numbers using a number line.
6.2(D) order a set of rational numbers arising from mathematical and real-world contexts

Materials
Cuisenaire Rods
Fraction strips
https://apps.mathlearningcenter.org/fractions/

Essential Questions

- How does decomposing and recomposing fractions contribute to understanding the concept of adding and subtracting fractions?
- Why do we add only the numerator when adding fractions?
- What is important to remember when adding fractions with like-denominators?

- What is a fraction called when the numerator and denominator are the same number?
- How is a common denominator similar to an equivalent fraction?
- What happens when students find a common denominator, not the Least Common Denominator (LCM)?
- How can models aid in representing fraction computation?
- Does the relationships and rules that apply to whole numbers also apply to fractions?

Procedures

Engage:

Jack is making a soccer bag for a friend. He uses 2/5 of a yard of green fabric and ⅕ of a yard of blue fabric. How much fabric does he use in all? Model with manipulatives, pictures, or words. Be sure to include the number sentence.

Participants will share their solution plan with the group.

The researcher will highlight the idea of finding equivalent fractions as opposed to a common denominator for addition of fractions. Making note of the specific parts of a whole being added.

Explore:

The researcher will then present the following problem:

Jack is making another soccer bag for a friend. He uses 1/2 yard of green fabric and 1/4 yard of blue fabric. How much fabric does Jack use?

https://apps.mathlearningcenter.org/fractions/
https://mathsbot.com

Participants will create models using virtual manipulatives and justify their answers.

What's the Difference?

Charlie's brownie recipe requires 1 ⅓ cups of milk. Laura's recipe needs ⅔ of a cup of milk. How much more milk does Charlie's recipe need than Laura's?

The researcher will display a number line and work through the problem with the participants, showing the time spent on Monday night as intervals on the top of the number line and the time spent Tuesday night in intervals at the bottom of the number line.

Participants will use the inverse operation, showing that by combining the difference and Laura's milk needed, one can generate the larger value of milk needed by Charlie.

The researcher will then present the following question as an independent activity:

Will fills a hummingbird feeder with 3/4 cup of sugar water on Friday. On Tuesday, Will sees that $\frac{1}{8}$ cup of sugar water is left. What is the difference in sugar water between Friday and Tuesday?

The participants will be encouraged to find $\frac{3}{4} - \frac{1}{8}$ by placing three $\frac{1}{4}$ strips beneath the 1-whole strip. Then place a $\frac{1}{8}$ strip under the $\frac{1}{4}$ strips. Finding all of the fraction strips with the same denominator, that can fit beneath the difference of $\frac{3}{4} - \frac{1}{8}$.

Participants will work in pairs using a virtual number line to solve problems that include the difference of two fractions that are generated by the whole group.

<u>*Addition with Cuisenaire Rods*</u>

Building on participants' reasoning and sense-making skills by working with manipulatives, they will use Cuisenaire rods to answer the following question:

$\frac{1}{2}$ of a brown rod + $\frac{1}{2}$ of a brown rod = _____________brown rods.

The researcher will encourage participants to justify their answers using various color rods. Circling back to what they understand about the relationship of a part to a whole and identify that by adding two rods together, they are essentially showing an equivalent length. Focus on estimation strategies and justification for possible solution plans.

The researcher will display the following problem:

$\frac{1}{4}$ brown rod + $\frac{1}{4}$ brown rod = _________brown rod(s).

This will further justify their answers using rods.

The researcher will next display the following problem:

$\frac{1}{2}$ brown rod + $\frac{1}{4}$ brown rod = _________ brown rod (s)

Using the information already gained in the activity as a scaffold, they should be able to identify that the answer is $\frac{3}{4}$. The researcher will remind the participants to include the concept of equivalency in their justification.

Participants will continue exploring the concept while changing the unit considered to be the "whole" Moving into mixed numbers, such as in the example $\frac{2}{3}$ of an orange rod added to $\frac{4}{5}$ of an orange rod.

Subtraction with Cuisenaire Rods

The researcher will display the following problem:

$\frac{3}{4}$ of a brown rod - $\frac{1}{4}$ of a brown rod = ___________brown rod.

Encouraging participants to display the problem with their rods, the researcher will engage the participants in identifying the rod that shows the fractional relationship between $\frac{1}{4}$ and $\frac{4}{4} = 1$ whole. Participants will use estimation strategies to identify the reasonableness of their solution plan and explain strategies.

The researcher will then display $1\frac{3}{2}$ brown rod $-\frac{1}{3}$ brown rod = __________brown rods.

Explain:
The activities are centered on participants becoming familiar with the foundational ideas of part-to-whole relationships and connection between conceptual and procedural fluency of adding fractions.

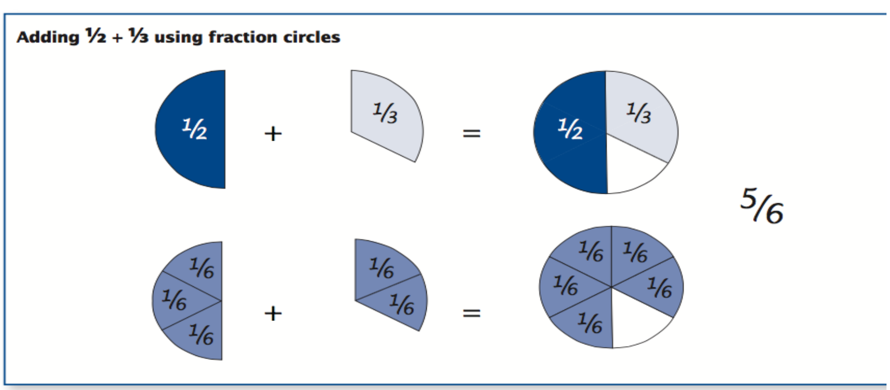
The extend section helps learners to see the need for common denominators.

Elaborate/Extend:
Drawing from the IES (2010) report on using visual representations to improve student learning of formal computational practices, the participants will be asked to use fraction circles to complete the number sentence $\frac{1}{2} + \frac{1}{3}$.

The researcher will ask the participants to find the fractional pieces (all the same color/size) that cover the combined circular pieces of $\frac{1}{2}$ and $\frac{1}{3}$.

The researcher will discuss why 6th and 12th's work through use of equivalent fractions.

Figure 6. Fraction circles for addition and subtraction



Source: Adapted from Cramer and Wyberg (2009).

Participants will continue exploring the need for Equivalent Fractions.

Addition
Ed rides ¼ mile from his house to this friend's house. Together they ride $\frac{2}{8}$ of a mile to school. How far does Ed ride to school?

Subtraction
Elliott is cutting pieces of wood to construct a dog house. The piece of wood he cuts for the base is $\frac{1}{3}$ foot long. The pieces of wood he cuts for the sides are $\frac{1}{2}$ foot tall. How much longer, in feet, are the pieces for the sides than the pieces for the base?

Questions for researcher to include:

- Using manipulatives, how can we rename the fractions so that they have like denominators?
- How would this change if Ed lived $\frac{1}{3}$ mile away from his friend's house?

Create 2 more scenarios where you would change the fractions to create more unlike denominators. Model with virtual manipulatives; Cuisenaire Rods and fraction strips.

Evaluate:
Participants will use fraction models to show addition with like denominators and subtraction with unlike denominators. Answers should include a visual representation.

Jen ate 2/12 of the candy in the morning and another 8/12 of the candy in the afternoon. How much candy did she consume for the entire day?

On Monday night Leighton spent ½ an hour on her math work. On Tuesday night, she spent ¾ of an hour on her math work. How much more time did Leighton spend on math work Tuesday night than on Monday night? Show your work using manipulatives, pictures, numbers, or words.

Lesson Plan #5 Multiplying Fractions

Standards & Objectives: TEKS & Vertical Alignment
V.B.3. child uses informal strategies to separate up to 10 items into equal group
2.6(A) model, create, and describe contextual multiplication situations in which equivalent sets of concrete objects are joined
5.3(I) represent and solve multiplication of a whole number and a fraction that refers to the same whole using objects and pictorial models, including area models
6.3(E) multiply and divide positive rational numbers fluently

Materials
Virtual Manipulatives

Essential Questions

- Does multiplication make everything bigger?
- What is an alternative for the repeated addition model in fractions?
- When a product unknown fraction problem is presented, how can this be represented using equal groups, arrays, and comparing?

Procedures

Engage:

Participants will be asked to answer the following question using a virtual concrete representation (area model).

> Catherine is icing a cake. She knows that 1 cup of icing will cover $\frac{2}{3}$ of a cake. How much cake can she cover with $\frac{1}{4}$ cup of icing?

Participants will share their solution plan with the whole group.
The researcher will model the solution using a number line model (linear) and scaling.

Explore:

Explain:
The researcher will discuss the importance of using linear models, measurement approaches, and area models when presenting real-world problems. Utilizing the connections between the fraction notation and the problem presented.

Participants will work on the following problem:

> Cindy had $\frac{3}{4}$ of a candy bar. She ate $\frac{1}{3}$ of it after lunch. What part of the candy bar did Cindy eat after lunch?

Researcher will highlight the following questions:

- What are we looking for in this problem? (We are looking for $\frac{1}{3}$ of $\frac{3}{4}$ not what is left)
- How does this look in a pictorial model (real world)?

Working with two-unit number lines, display the problem $1\times\frac{1}{2}$.

- How would you explain it using 4×2 instead of a $\frac{1}{2}\times2$? As $4\times2$is four groups of two, $\frac{1}{2}\times2$ is two groups of a half or $\frac{1}{2}$ of a group

The researcher will highlight the misconception of multiplying just one part of the fractions (multiplying across only the numerator or only the denominator).

Elaborate/Extend:
Tell me all you Can, Multiplication

The researcher will give the following problem to the participants:

Do not give an exact answer, instead I want to know all that you can tell me about the answer.

$$6\times2\frac{1}{2}$$

Starters will be posted before participants begin to explore

The answer will be more than ____________________ because _____________________.
The answer will be less that ____________________ because _____________________.
The answer will be between _______________ and __________________ because ______.
The answer is ______________________.

Whole group discussion over individual findings and reasoning. Participants will be asked to create a similar problem, one will be chosen to share their problems, while the others work out the solution.

Whole group: Participants will be asked to complete the following, showing their thinking with concrete models:

$$\frac{1}{3}\times\frac{2}{4}$$

Evaluate:
Participants will answer the following questions using virtual manipulatives:

1. Tom spent $\frac{3}{4}$ of an hour each day for 3 days working on his writing project.
2. Ed spent $\frac{1}{4}$ of an hour each day for 7 days working on his writing project.

3. Who spent more time in total working on their writing project?

Lesson Plan #6 Dividing Fractions

Standards & Objectives: TEKS & Vertical Alignment
V.B.3. Child uses informal strategies to separate up to 10 items into equal group
2.6(B)* model, create, and describe contextual division situations in which a set of concrete objects is separated into equivalent sets
5.3(L) divide whole numbers by unit fractions and unit fractions by whole numbers
6.3(E) multiply and divide positive rational numbers fluently

5.3 (J) represent division of a unit fraction by a whole number and the division of a whole number by a unit fraction such as $\frac{1}{3} \div 7$ and $7 \div \frac{1}{3}$ using objects and pictorial models, including area models; Supporting Standard

Materials
Virtual Manipulatives
Stop Watch

Essential Questions

- What is the best approach to use when dividing fractions?
- What is the connection between multiplication and division?
- Why do people use the invert and multiply method?

Procedures

Engage:

Work on the following problems using concrete models (choice of fraction tiles, number line, Cuisenaire Rods, or fractions circles).

The researcher will highlight;

- Unit fractions for composing a whole
- Multiplying by fractions reciprocal results in a product of one
- Dividing any number by one leaves the number unchanged

How many quarter-hours are in three hours?
How many thirds of a cup are in two cups?
How many half-meters are in five meters?
How many eighths of a mile are in four miles?
How many sixths of a yard are in two yards?
How many fifths of a gallon are in three gallons?
How many fourths of an ounce are in two ounces?

How many tenths of a pound are in four pounds?
https://apps.mathlearningcenter.org/fractions/
https://mathsbot.com/
When we divide does the number always get smaller?
Whole-groups discussion of how these models look as invert and multiply.

Explore:
The researcher will lead a discussion over what students know about division.

Participants will be instructed to jog in place beside their desk as the researcher times them for 1 minute.

Repeat the action above, this time calling out when ¼ of the time has passed (15 second intervals)

Discuss the difference between the first jog and the second jog. Highlighting the fact that they completed 60 seconds of running through 15 second intervals. The researcher will ask the participants what division problem was represented. The researcher will further question, leading them to realize that they actually figured out 1 divided by ¼ as opposed to 60 divided by 15.

The researcher will then model this on the one-unit number line using Cuisenaire rods.

Next, the researcher will present a two-unit number line and ask how many fourths would they jog if they had to jog for two minutes? (2 Divided by ¼)

Whole-group discussion over dividing a whole number by a fraction.

Explain:
Using the contexts of time and distance aids in making the connection of what one already knows about the process of division with whole numbers and how that can be used for the division of fractions.

By supporting students in investigations, assessments and redefinition of their ideas of division of fractions, they will have more experience with reasoning, justification and sense-making skills. Additionally, the continued use of manipulatives gives learners a concrete model to visualize when encountering division of fractions.

Elaborate/Extend:
Math Competition

A school is creating ribbons to give away at a Math Competition. Students were asked to donate ribbon scraps to help make the ribbons. The ribbons collected were cut into the following lengths:

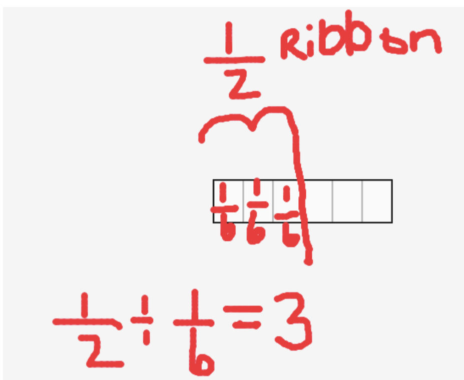
3 yards
2 yards
1 yard

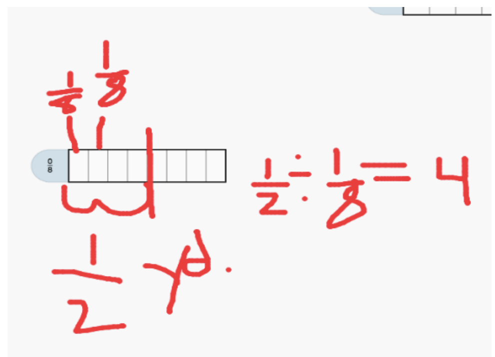
$\frac{3}{4}$ yard
$\frac{2}{3}$ yard
$\frac{1}{3}$ yard
$\frac{1}{4}$ yard

The Math Competition ribbons will be $\frac{1}{6}$ and $\frac{1}{3}$ long.

How many of each type of ribbon can be cut from each length of scrap ribbon collected?

The researcher will demonstrate how to partition a ribbon that is $\frac{1}{2}$ a yard long into sixths and into eights.





Participants will work on the problems independently and share their findings with the group.

Evaluate:
Participants will answer the following questions using virtual manipulatives:

$\frac{3}{4}$ divided by $\frac{1}{2}$

Appendix C

Assessment

Name: ________________________________

Complete the following problems. Give justification for your answers.

The number lines model two different fractions.

Which comparison of these fractions is true?

F $\frac{1}{2} > \frac{1}{1}$

G $\frac{2}{8} > \frac{1}{8}$

H $\frac{1}{8} = \frac{2}{8}$

J $\frac{2}{8} < \frac{1}{8}$

Point *P* on the number line represents two equivalent fractions.

Which two equivalent fractions can point *P* represent?

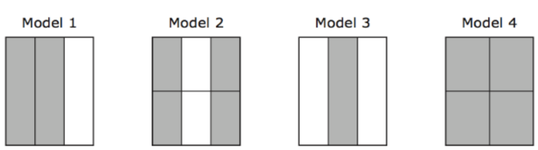
A $\frac{1}{4}$ and $\frac{1}{8}$

B $\frac{1}{3}$ and $\frac{2}{6}$

C $\frac{1}{4}$ and $\frac{2}{8}$

D $\frac{1}{4}$ and $\frac{3}{4}$

Four fraction models are shown.



Which two models are shaded to show equivalent fractions?

F Models 1 and 2

G Models 1 and 3

H Models 2 and 4

J Models 2 and 3

Which fraction is equivalent to 1.5?

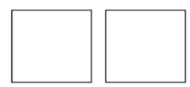
F $\frac{15}{10}$

G $\frac{15}{100}$

H $\frac{100}{15}$

J $\frac{10}{15}$

Brandon drew the two congruent squares shown.



- He divided one square into 2 congruent triangular parts.
- He divided the other square into 2 congruent rectangular parts.

Which statement is true?

F Each triangular part and each rectangular part represents $\frac{1}{2}$ the area of one square.

G Each triangular part has an area that is greater than the area of each rectangular part.

H Each triangular part and each rectangular part represents $\frac{1}{4}$ the area of one square.

J Each rectangular part has an area that is greater than the area of each triangular part.

Ella finished a bike race in 37.6 minutes. Miranda finished the race $9\frac{1}{10}$ minutes sooner than Ella finished it. How many minutes did it take Miranda to finish the race?

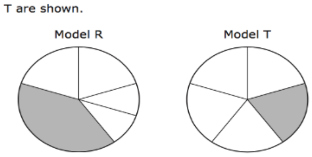
F 32.5 minutes

G 46.7 minutes

H 28.59 minutes

J Not here

Models R and T are shown.



Which statement is true?

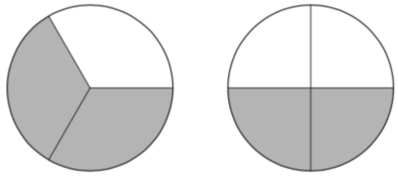
A The shaded parts of Model R and Model T are different sizes, but each model represents the same fraction of the whole.

B The shaded part of Model R cannot be written as the fraction $\frac{1}{5}$, because the parts are not all equal in size.

C The shaded part of Model T is $\frac{1}{4}$, because the parts are all equal in size.

D The total number of parts in Model R is 5, so $\frac{1}{5}$ of Model R is shaded.

The models shown are the same size and are each divided into equal-size parts. The models are shaded to represent two fractions.



Which statement is true?

F $\frac{2}{3} > \frac{2}{4}$, because thirds are larger than fourths.

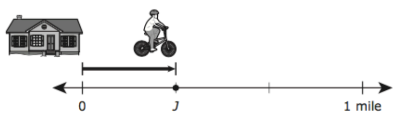
G $\frac{2}{3} = \frac{2}{4}$, because each model has 2 parts shaded.

H $\frac{1}{3} < \frac{1}{4}$, because 3 is less than 4.

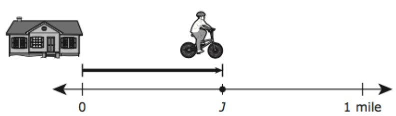
J $\frac{1}{3} = \frac{1}{4}$, because each model shows 1 whole.

Javier rode his bike a distance of $\frac{1}{2}$ mile from his house. On which number line does point *J* represent Javier's position after riding his bike?

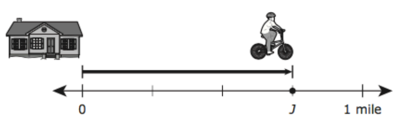
A



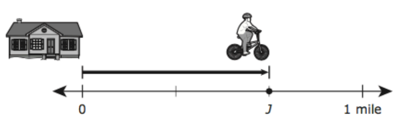
B



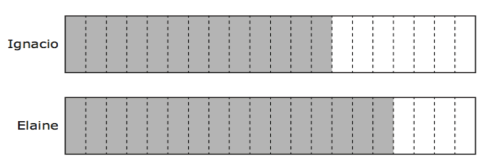
C



D



Ignacio and Elaine read the same book. The shaded part of each model represents the fraction of the book that each student read.



Which expression can be used to find the difference between the fraction of the book Elaine read and the fraction of the book Ignacio read?

F $\frac{16}{4} - \frac{13}{7}$

G $\frac{7}{13} - \frac{4}{16}$

H $\frac{16}{20} - \frac{13}{20}$

J $\frac{20}{16} - \frac{20}{13}$

Ms. Thompson needs $\frac{15}{2}$ yards of red fabric and $7\frac{1}{2}$ yards of silver fabric. Which comparison is true?

A $\frac{15}{2} > 7\frac{1}{2}$

B $\frac{15}{2} = 7\frac{1}{2}$

C $\frac{15}{2} < 7\frac{1}{2}$

D None of these

Martha bought a new box of cereal. In one week she ate $\frac{4}{9}$ of the cereal. Which is closest to the fraction of the cereal she had left?

F Less than $\frac{1}{4}$ of the cereal was left.

G Less than $\frac{1}{2}$ of the cereal was left.

H About $\frac{1}{2}$ of the cereal was left.

J About $\frac{1}{4}$ of the cereal was left.

Which expression is equivalent to $\frac{9}{8}$?

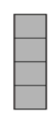
F $\frac{3}{8} + \frac{3}{8}$

G $\frac{1}{2} + \frac{2}{3} + \frac{6}{3}$

H $\frac{1}{9} + \frac{1}{9} + \frac{1}{9} + \frac{1}{9} + \frac{1}{9} + \frac{1}{9} + \frac{1}{9} + \frac{1}{9}$

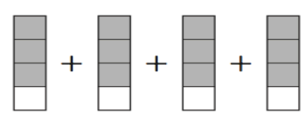
J $\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}$

The model is shaded to represent one whole.



Model Y is shaded to represent a number greater than one.

Model Y:



Which expression CANNOT be used to represent this number?

F $\frac{4}{4} + \frac{4}{4} + \frac{4}{4}$

G $\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4}$

H $\frac{4}{4} + \frac{4}{4} + \frac{3}{4} + \frac{1}{4}$

J $\frac{3}{4} + \frac{3}{4} + \frac{3}{4} + \frac{3}{4}$

Zeke used $\frac{3}{4}$ cup white sugar, $\frac{3}{4}$ cup brown sugar, and $2\frac{1}{4}$ cups of flour to bake some cookies.

What was the difference between the amount of flour and the combined amount of sugar Zeke used?

F $3\frac{3}{4}$ cups

G $1\frac{2}{4}$ cups

H $\frac{2}{4}$ cup

J $\frac{3}{4}$ cup

Nerissa has 5 pink bows, 1 blue bow, and 4 purple bows in a box. She will randomly choose 1 bow from the box.

What is the probability Nerissa will choose a purple bow?

F $\frac{1}{2}$

G $\frac{2}{5}$

H $\frac{1}{10}$

J $\frac{3}{5}$

The table shows the fractions of the bulletin boards in four classrooms that will be used to display artwork.

Artwork on Bulletin Boards

Teacher	Fraction for Artwork
Ms. Brady	$\frac{5}{10}$
Mr. Chang	$\frac{2}{4}$
Ms. Gupta	$\frac{5}{6}$
Mr. Taylor	$\frac{4}{8}$

Which comparison is true?

F $\frac{2}{4} > \frac{4}{8}$

G $\frac{4}{8} < \frac{5}{10}$

H $\frac{5}{6} > \frac{4}{8}$

J $\frac{5}{6} < \frac{5}{10}$

Justin has 50 pictures in an album. Of these pictures, 30 show his friends, 12 show his fam and 8 show only Justin. Justin is in $\frac{1}{2}$ of the pictures that show his friends and $\frac{1}{2}$ of the pictures that show his family.

Based on this information, which statement is true?

A The probability of randomly selecting a picture that shows Justin with his friends is grea than the probability of randomly selecting a picture that shows Justin with his family.

B The probability of randomly selecting a picture that shows Justin is 8%.

C The probability of randomly selecting a picture that shows Justin with his family is 5 tim the probability of randomly selecting a picture that shows only his friends.

D The probability of randomly selecting a picture that does not show Justin is 21%.

The owner of a snow-cone stand used $\frac{1}{4}$ gallon of syrup to make 16 cherry snow cones. She used the same amount of syrup in each snow cone.

How much syrup in gallons was used in each cherry snow cone?

A $\frac{1}{4}$ gal

B 4 gal

C $\frac{1}{64}$ gal

D 64 gal

What is the value of $\frac{4}{15} \div \frac{2}{3}$?

F $\frac{8}{45}$

G $\frac{14}{15}$

H $\frac{5}{2}$

J $\frac{2}{5}$

Tommy bought 3 cups of blueberries. He will eat $\frac{1}{2}$ cup of blueberries each day.

How many days can Tommy eat the blueberries before they are all gone?

F 6

G 2

H 5

J 4

A worker uses 450 inches of steel wire to make 300 springs of the same size. At this rate how many inches of steel wire are needed to make 1 spring?

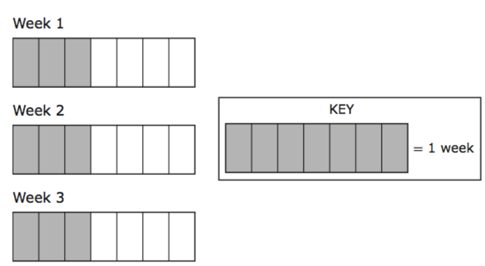
F $\frac{1}{3}$ in.

G $\frac{1}{15}$ in.

H $\frac{2}{3}$ in.

J $1\frac{1}{2}$ in.

Darenda worked for 3 weeks. The shaded parts of the model represent the fraction of each week she worked from her home office.



Which expression can be used to determine the number of weeks Darenda worked from her home office over these 3 weeks?

A $3 + \frac{3}{4}$

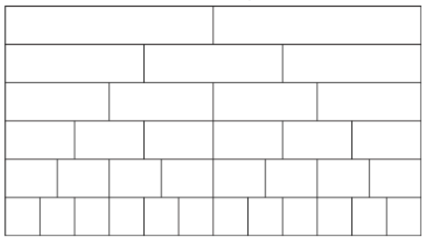
B $3 + \frac{3}{7}$

C $3 \times \frac{3}{4}$

D $3 \times \frac{3}{7}$

Vanna used the fraction strips shown to help her determine the difference between $\frac{5}{6}$ and $\frac{1}{4}$.

Fraction Strips



What is $\frac{5}{6} - \frac{1}{4}$?

A $\frac{1}{5}$

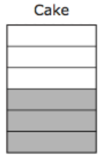
B $\frac{7}{12}$

C $\frac{1}{2}$

D $\frac{5}{8}$

The model is shaded to represent the remaining one-half of a cake. Three friends will each receive an equal amount of the remaining cake until it is all gone.

Cake



Which equation can be used to determine the fraction of the whole cake each friend will receive?

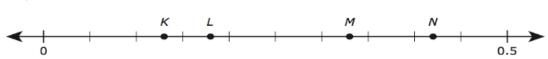
F $\frac{1}{2} \times 3 = \frac{3}{2}$

G $\frac{1}{2} \times 6 = \frac{6}{2}$

H $\frac{1}{2} \div 3 = \frac{1}{6}$

J $\frac{1}{2} \div 6 = \frac{1}{12}$

Four points are labeled on the number line.



Which point best represents $\frac{1}{3}$?

F Point *K*

G Point *L*

H Point *M*

J Point *N*

Maya has 120 caramel apples to sell. Each caramel apple is covered with one topping.

- $\frac{1}{5}$ of the caramel apples are covered with peanuts.
- $\frac{1}{3}$ are covered with chocolate chips.
- $\frac{3}{10}$ are covered with coconut.
- The rest are covered with sprinkles.

How many caramel apples are covered with sprinkles?

A 100

B 33

C 25

D 20

This week Andres will practice with his band for $1\frac{1}{2}$ hours on Monday, $1\frac{3}{4}$ hours on Tuesday, and 2 hours on Wednesday. Next week Andres will practice with his band for the same number of hours on Monday, Tuesday, and Wednesday.

What is the total number of hours Andres will practice with his band over these 6 days?

A $5\frac{1}{4}$ hours

B $10\frac{1}{2}$ hours

C $4\frac{1}{4}$ hours

D $8\frac{1}{2}$ hours

There are 16 pies on a picnic table.

- Each pie is cut into pieces.
- Each piece is $\frac{1}{8}$ of a pie.

How many pieces of pie are on the picnic table?

A 2

B 88

C 24

D 128

A number cube with faces labeled from 1 to 6 was rolled 20 times. Each time the number cube was rolled, the number showing on the top face was recorded. The table shows the results.

Results

Number Showing on Top Face	Frequency
1	0
2	3
3	3
4	6
5	3
6	5

Based on these results, what is the experimental probability that the next time the number cube is rolled it will land with 5 or 6 showing on the top face?

A $\frac{2}{5}$

B $\frac{3}{20}$

C $\frac{1}{3}$

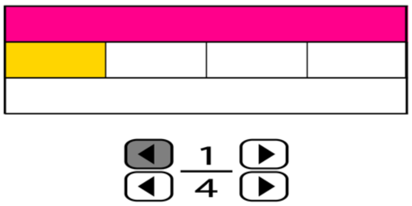
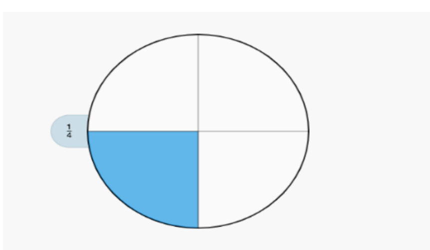
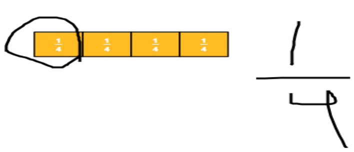
D $\frac{3}{5}$

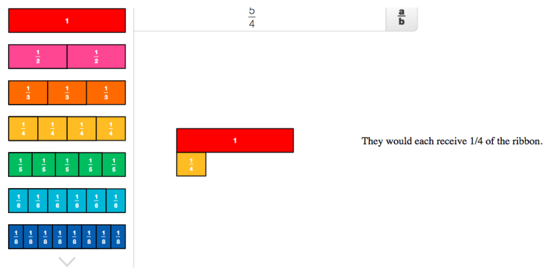
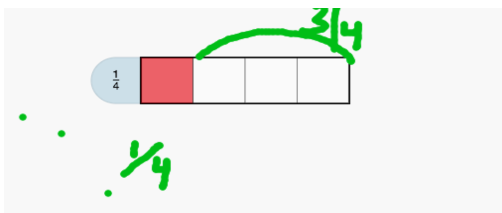
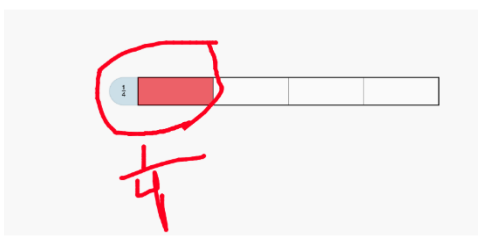
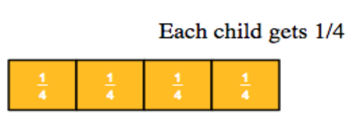
Appendix D

Exit Tickets

Session #1-A Exit Tickets

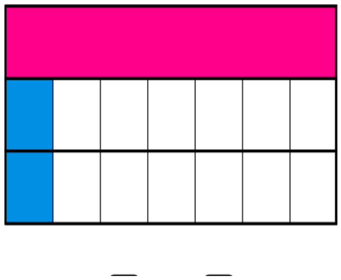
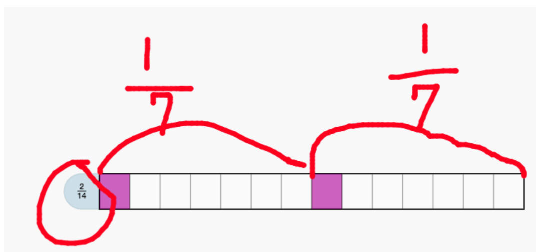
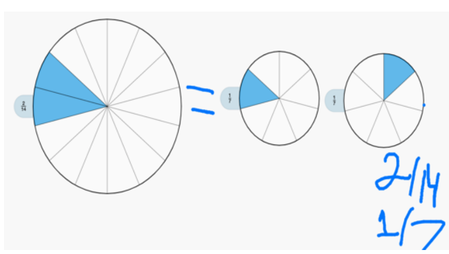
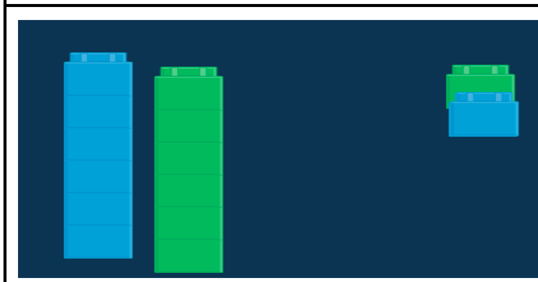
There are four children wrapping presents. They have one yard of ribbon to wrap all of the presents. If each child receives an equal amount of ribbon, what **fraction** would they get? Use a model to explain.

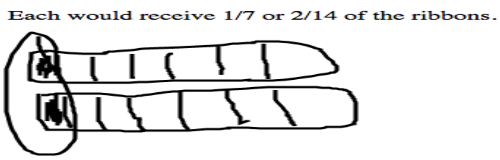
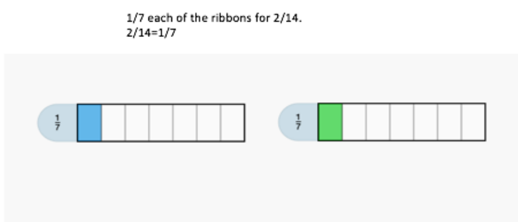
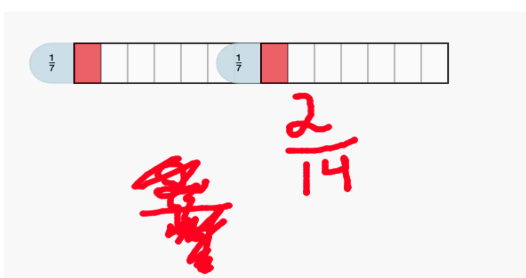
	P#1: Participant one was able to represent their thinking using an area model. They employed both abstract and pictorial models. This representation includes the partitions that define each child receiving one-fourth of the piece of ribbon. There is also a representation of the whole, showing reference to part-whole thinking. The participant used the selected VM to model their thinking.
	P#2: Participant two was able to represent their thinking using an area model. They employed both an abstract and pictorial model. This representation includes the partitions that define each child receiving one-fourth of the ribbon. The participant used the selected VM to model their thinking.
	P#3: Participant three was able to represent their thinking using a circular area model divided into four equal sections. The representation included an abstract representation to name the fractional piece. The participant used the selected VM to model their thinking
	P#4: Participant four was able to represent their thinking using an area model. They employed both an abstract and pictorial model. This representation includes the partitions that define each child receiving one-fourth of the ribbon. The part that each child received was clearly labeled. The participant used the selected VM to model their thinking.

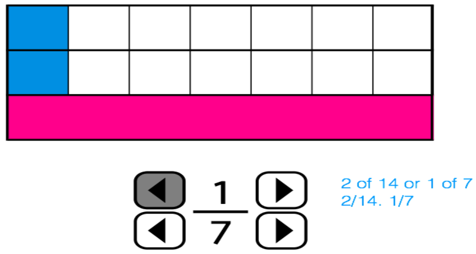
	P#5: Participant five was able to represent their thinking using an area model. They employed both an abstract and pictorial model. This representation includes the partitions that define each child receiving one-fourth of the ribbon. There is a representation of the whole, showing reference to part-whole thinking. The participant used the selected VM to model their thinking.
	P#6: Participant six was able to represent their thinking using an area model. They employed both an abstract and pictorial model. This representation includes the partitions that define each child receiving one-fourth of the ribbon with a remaining three-fourths left over. The participant used the selected VM to model their thinking.
	P#7: Participant seven was able to represent their thinking using an area model. They employed both an abstract and pictorial model. This representation includes the partitions that define each child receiving one-fourth of the ribbon. The part that each child received was clearly labeled. The participant used the selected VM to model their thinking.
	P#8: Participant eight was able to represent their thinking using an area model. They employed both an abstract and pictorial model. This representation includes the partitions that define each child receiving one-fourth of the ribbon. The part that each child received was clearly labeled. The participant used the selected VM to model their thinking.

Session #1-B Exit Tickets

There are 7 friends. The friends have two yards of ribbon to wrap presents. If each child receives an equal amount of ribbon, what **fraction** would they get? Use a model to explain.

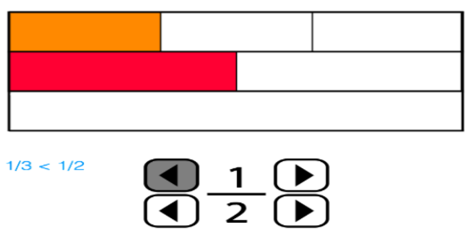
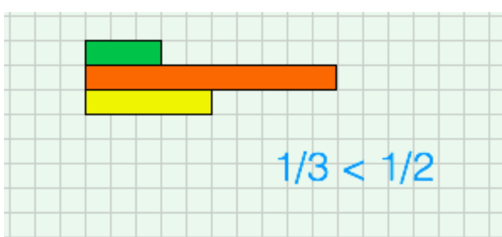
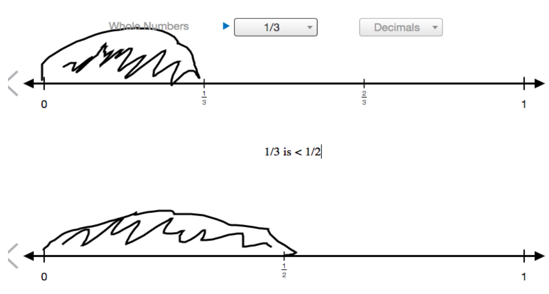
	P#1: Participant one was able to represent their thinking using an area model. The pictorial model represents the amount of ribbon each child would receive from the two ribbons, although the absence of an abstract representation leaves questioning as to what fraction the child would receive, two-fourteenths or one-sevenths. This representation includes clear partitions of both models. The whole is represented, showing reference to part-whole thinking. The participant used the selected VM to model their thinking.
	P#2 : Participant two was able to represent their thinking using an area model and abstract representations. Placed side-by-side, the pictorial model represents the amount of ribbon each child would receive from the two ribbons. This representation includes clear partitions of the models and is labeled with both the original sum and the equivalent fraction. The participant used the selected VM to model their thinking.
	P#3 : Participant three was able to represent their thinking using a circular area model and abstract representations. The pictorial model represents the amount of ribbon each child would receive from the combined two ribbons and the equivalent fraction. This representation includes clear partitions of both models. The participant used the selected VM to model their thinking.
	P#4 Participant four was able to represent their thinking using virtual unifix cubes. The pictorial model represents the amount of ribbon each child would receive from the two ribbons, although it lacks an abstract representation to explain that the cubes should be considered “fraction bars” when linked. The participant was able to represent the idea of a set model, with one out of each seven being pulled away.

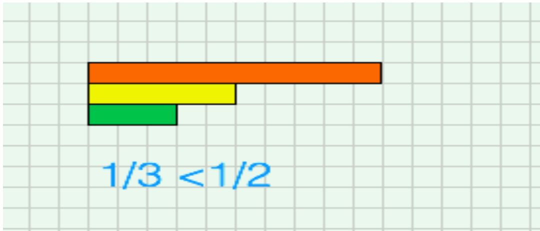
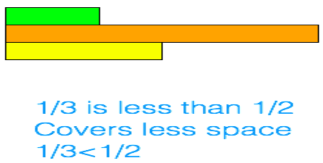
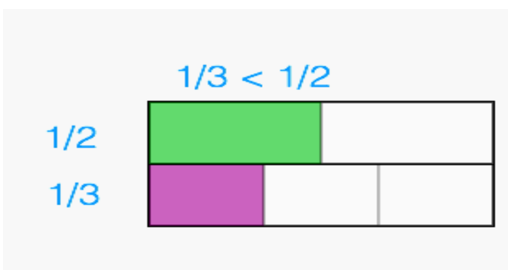
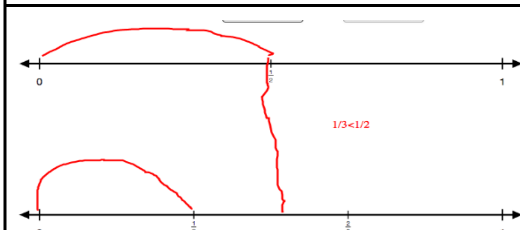
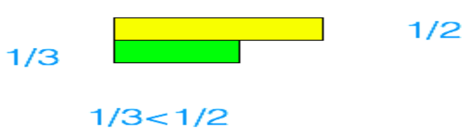
	However, the representation could benefit from additional information.
	P#5: Participant five was able to represent their thinking using an area model and abstract representations. The pictorial model represents the amount of ribbon each child would receive from the two ribbons. The participant distinctly separated the two models to indicate separate ribbons. This representation includes clear partitions of the models and is labeled with the sum and the equivalent fraction. The participant did not employ the actual virtual manipulative, rather they chose the drawing feature, which is also a component of online leanring.
	P#6: Participant six was able to represent their thinking using an area model and abstract representations. Placed side-by-side, the pictorial model represents the amount of ribbon each child would receive from the two ribbons. This representation includes clear partitions of the models and is labeled with both the sum and the equivalent fraction. The participant used the selected VM to model their thinking.
	P#7: Participant seven was able to represent their thinking using an area model and abstract representations. Placed side-by-side, the pictorial model represents the amount of ribbon each child would receive from the two ribbons. This representation includes clear partitions of the models and is labeled with both the sum and the equivalent fraction. The participant used the selected VM to model their thinking.

	P#8: Participant eight was able to represent their thinking using both an area model and an abstract representation. The pictorial model represents the amount of ribbon each child would receive from the two ribbons. This representation includes clear partitions of both models and is labeled with the sum and the equivalent fraction. The whole is represented, showing reference to part-whole thinking. The participant used the selected VM to model their thinking.

Session #2 Exit Tickets

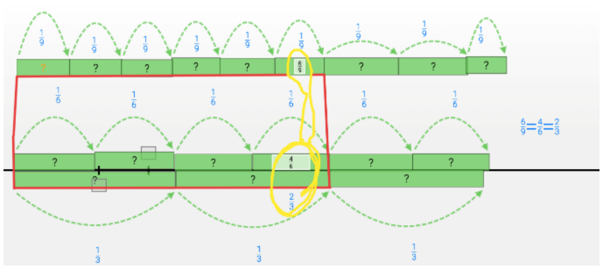
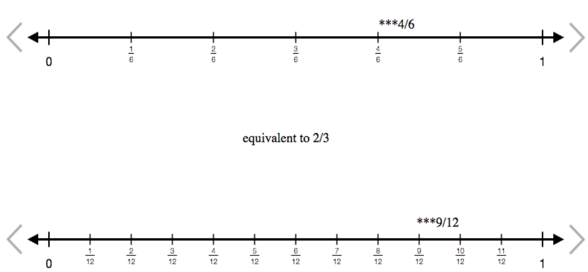
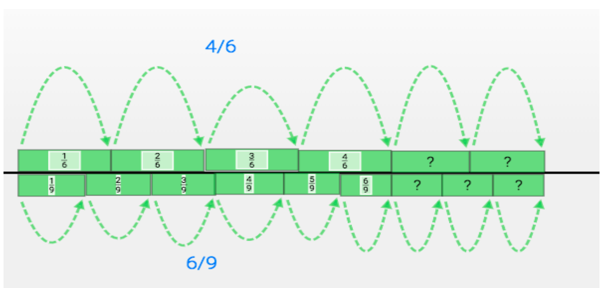
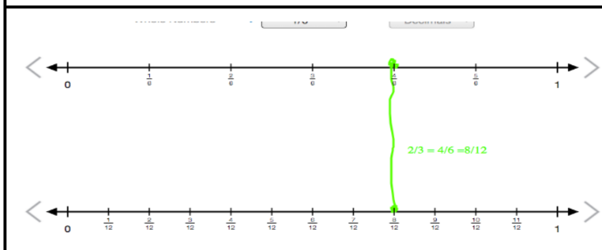
Answer the following by adding an inequality sign and show model. $\frac{1}{3}$ ______ $\frac{1}{2}$

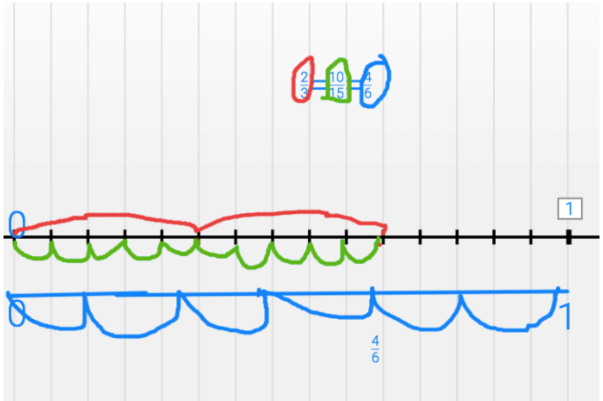
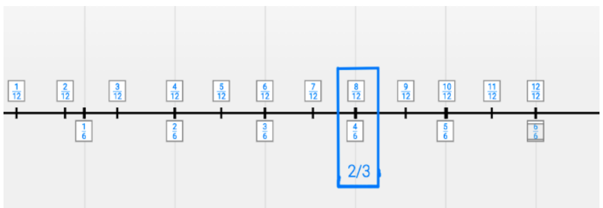
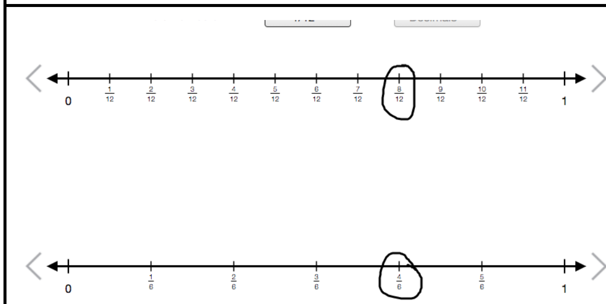
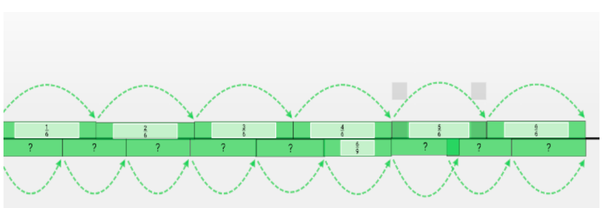
	P#1: Participant one was able to represent their thinking using an area model and abstract representation. This representation includes clear partitions of the models and an inequality sign to show the comparison. There is reference to the whole. The participant used the selected VM to model their thinking.
	P#2: Participant two was able to represent their thinking using an area model and abstract representation. This is a clear pictorial representation of comparison and includes an inequality sign. There is reference to the whole. The participant used the selected VM to model their thinking.
	P#3: Participant three was able to represent their thinking using a linear model and abstract representation. This is a pictorial representation of comparison and includes an inequality sign. The participant used the selected VM to model their thinking.

 	P#4: Participant four was able to represent their thinking using an area model and abstract representation. This is a clear pictorial representation of comparison and includes an inequality sign. There is reference to the whole. The participant used the selected VM to model their thinking.
 	P#5: Participant five was able to represent their thinking using an area model and abstract representation. This is a clear pictorial representation of comparison and includes an inequality sign. There is reference to the whole. The participant used the selected VM to model their thinking.
 	P#6: Participant six was able to represent their thinking using an area model and abstract representations. This is a clear pictorial representation of comparison and includes an inequality sign. There is reference to the whole and the representations are labeled with their value. The participant used the selected VM to model their thinking.
 	P#7: Participant seven was able to represent their thinking using a linear model and abstract representation. This is a clear pictorial representation of comparison and includes an inequality sign. The participant used the selected VM to model their thinking.
 	P#8: Participant eight was able to represent their thinking using an area model and abstract representation. This is a clear pictorial representation of comparison and includes an inequality sign. The participant used the selected VM to model their thinking.

Session #3-B Exit Tickets

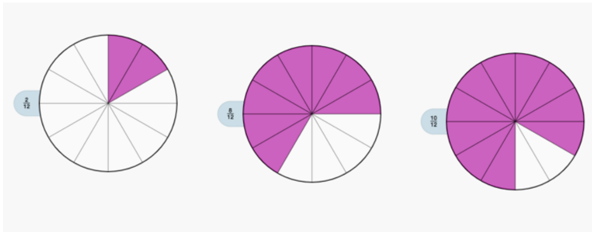
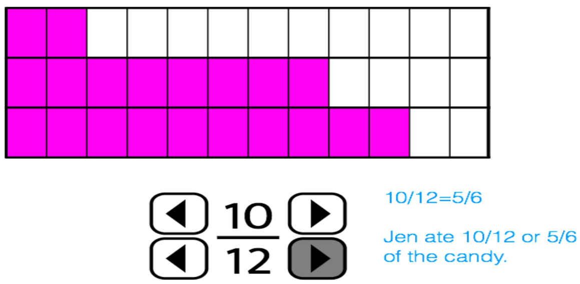
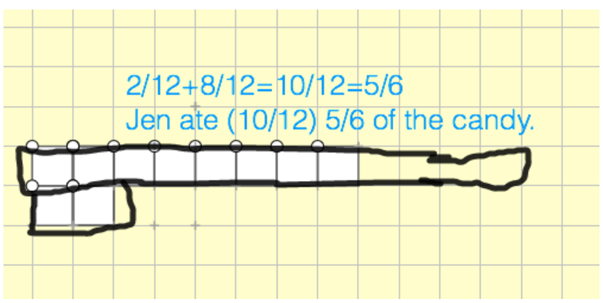
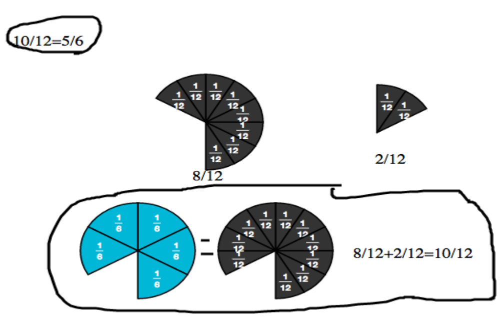
Can you show two equivalent fractions for ⅔ using a number line?

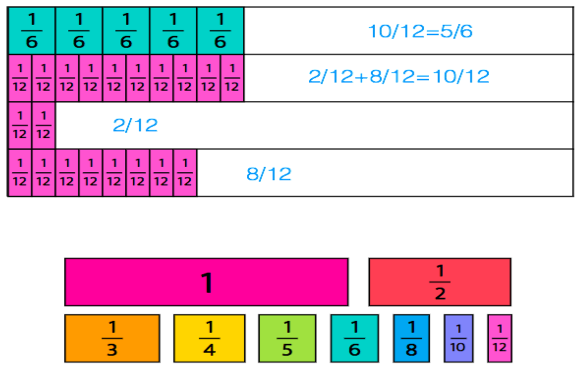
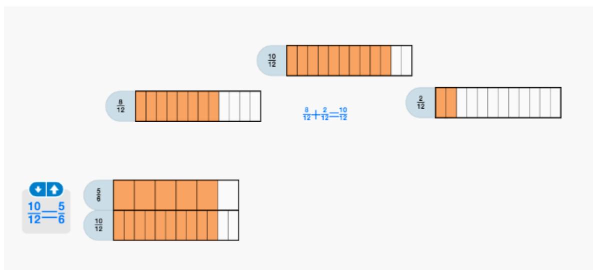
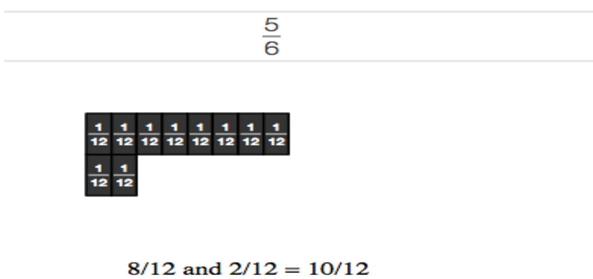
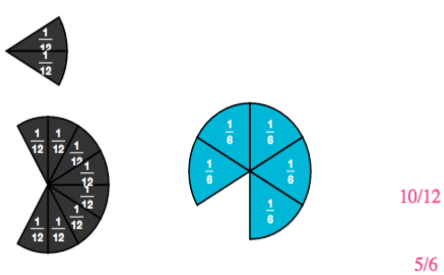
	P#1: Participant one was able to represent their thinking using area and abstract representations. This is a clear pictorial representation of equivalency for two-thirds, four-sixths and six-ninths. The participant did not label their number line with the whole.
	P#2: Participant two represented their thinking using linear and abstract representations. The models four-sixths and nine-twelfths are not equivalent. The answer was not correct.
	P#3: Participant three was able to represent their thinking using area and abstract representations. This is a clear pictorial representation of equivalency for four-sixths and six-ninths. The model would be more effective if given the reference to two-thirds in a pictorial form. The participant did not label their number line with the whole.
	P#4: Participant four was able to represent their thinking using linear and abstract representations. This is a clear pictorial representation of equivalency for four-sixths and eight-twelfths. The model would be more effective if given the reference to two-thirds in a pictorial form.

	P#5: Participant five was able to represent their thinking using linear and abstract representations. This is a clear pictorial representation of equivalency for two-thirds, four-sixths and ten-fifteenths. The participant used the selected VM to model their thinking.
	P#6: Participant six was able to represent their thinking using an area model and abstract representations. The participant was able to show equivalency with four-sixths and eight-twelfths. The model would be more effective if given the reference to two-thirds in a pictorial form.
	P#7: Participant seven was able to represent their thinking using linear and abstract representations. The participant was able to show equivalency with four-sixths and eight-twelfths. The model would be more effective if given the reference to two-thirds in a pictorial form.
	P#8 Participant eight was able to represent their thinking using linear and abstract representations. The participant was able to show equivalency with four-sixths and six-ninths. The model would be more effective if given the reference to two-thirds in a pictorial form.

Session #4-A Exit Tickets

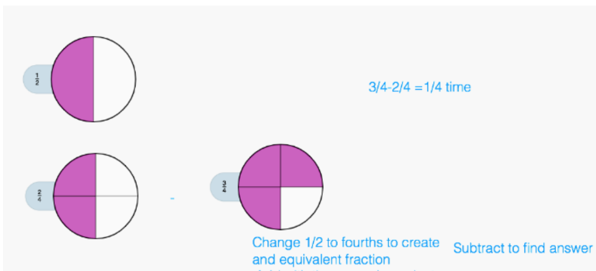
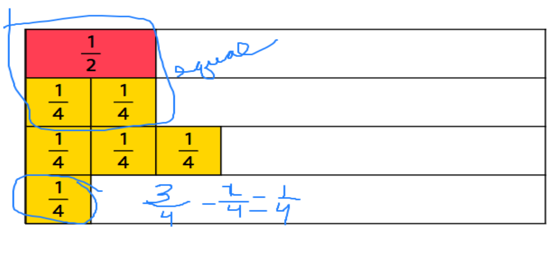
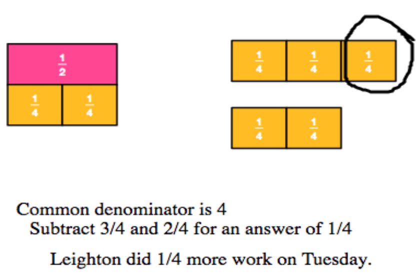
Jen ate 2/12 of the candy in the morning and another 8/12 of the candy in the afternoon. How much candy did she consume for the entire day?

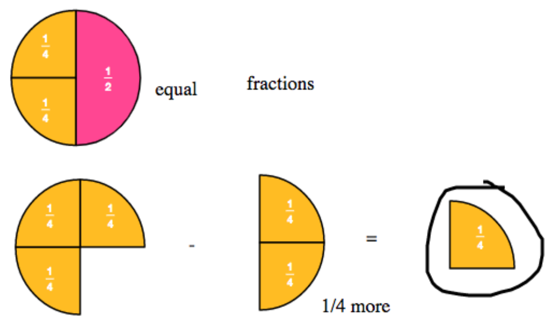
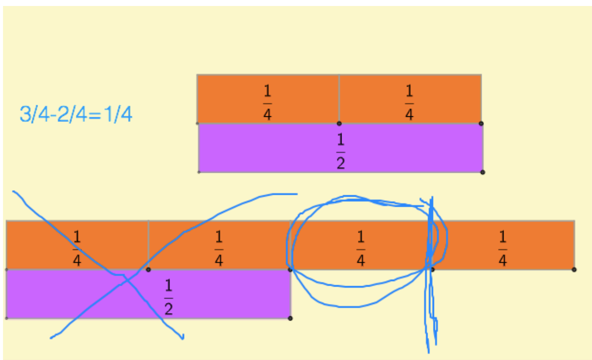
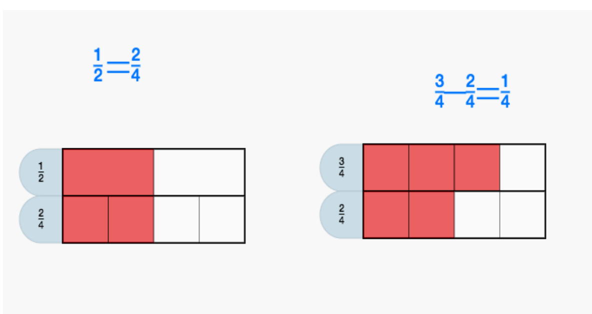
	P#1: Participant one was able to represent their thinking using a circular area model and abstract representation. This is a pictorial representation of addition with like denominators. There is reference to the whole being 12 units. The participant did not extend to show the equivalent fraction of five-sixths. The participant used the selected VM to model their thinking.
	P#2: Participant two was able to represent their thinking using an area model and abstract representation. This is a pictorial representation of addition with like denominators. There is reference to the whole being 12 units. The participant did show the equivalent fraction of five-sixths. The participant used the selected VM to model their thinking.
	P#3: Participant two was able to represent their thinking using an abstract representation. The participant did show reference to the equivalent fraction of five-sixths. There are pictorial representations of eight and two units, although they are not attached to a model that would represent a fraction bar.
	P#4: Participant four was able to represent their thinking using a circular area model and abstract representation. This is a pictorial representation of addition with like denominators. There is reference to the whole being 12 units. The participant did extend to show the equivalent fraction of five-sixths. The participant used the selected VM to model their thinking.

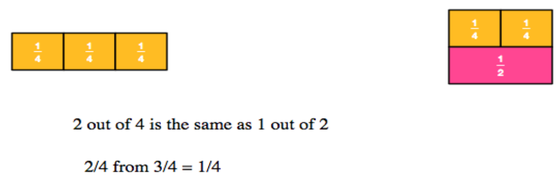
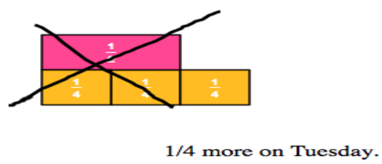
	P#5: Participant five was able to represent their thinking using an area model and abstract representation. The participant used the selected VM to model their thinking.
	P#6: Participant six was able to represent their thinking using an area model and abstract representation. This is a pictorial representation of addition with like denominators. There is reference to the whole being 12 units. The participant did extend to show the equivalent fraction of five-sixths. The participant used the selected VM to model their thinking.
	P#7: Participant seven was able to represent their thinking using an area model and abstract representation. The participant did extend to show the equivalent fraction of five-sixths. The participant did not show the pictorial representation for five-sixths.
	P#8: Participant eight was able to represent their thinking using a circular area model and abstract representation. There is a reference to the whole as 12 units and that five-sixths covers the same area as ten-twelfths. The models containing the twelfths pieces should be combined to effectively represent the equivalency shown by the abstract representation.

Session #4-B Exit Tickets

On Monday night Leighton spent ½ an hour on her math work. On Tuesday night she spent ¾ of an hour on her math work**.** How much more time did Leighton spend on math work Tuesday night than on Monday night? Show your work using manipulatives, pictures, numbers, or words.

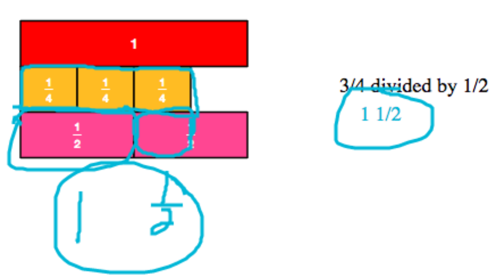
	P#1: Participant one was able to represent their thinking using a circular model and abstract representations. The pictorial representation references the comparison of the fractions and the representations are labeled with their values. There is evidence of fraction equivalence and the answer is correct. The participant used the selected VM to model their thinking.
	P#2: Participant two was able to represent their thinking using an area model and abstract representations. The pictorial representation references the comparison of the fractions (one-half and two-fourths). The representations are labeled and there is evidence of fraction equivalence. The model includes a correct number sentence. The participant used the selected VM to model their thinking.
	P#3: Participant three was able to represent their thinking using an area model and abstract representation. The pictorial representation references the comparison of the fractions (one-half and two-fourths). The representations are labeled and there is evidence of fraction equivalence. The model includes a correct answer. The participant used the selected VM to model their thinking.

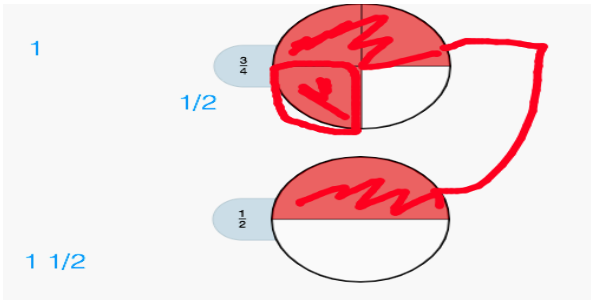
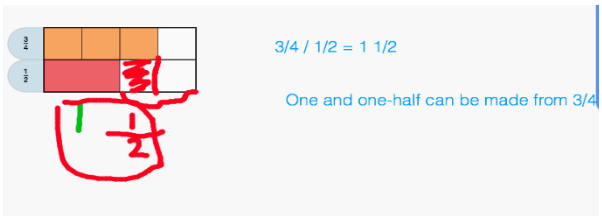
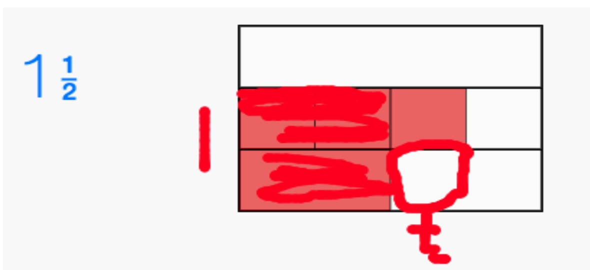
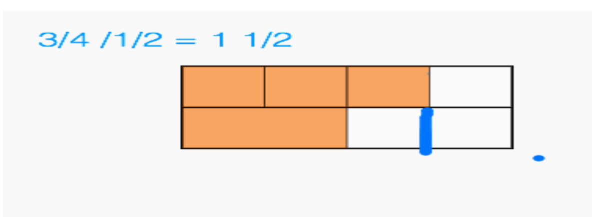
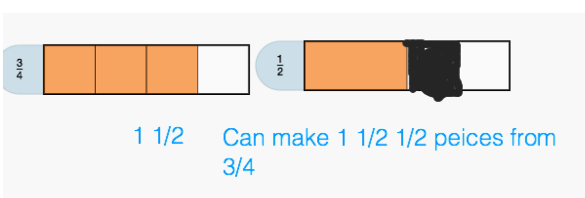
	P#4: Participant four was able to represent their thinking using a circular model and abstract representation. The pictorial representation references the comparison of the fractions (one-half and two-fourths). The representations are labeled and there is evidence of fraction equivalence. The model includes a correct answer. The participant used the selected VM to model their thinking.
	P#5: Participant five was able to represent their thinking using an area model and abstract representation. The pictorial representation references the comparison of the fractions (one-half and two-fourths). The representations are labeled and there is evidence of fraction equivalence. The model includes a correct answer. The participant used the selected VM to model their thinking.
	P#6: Participant six was able to represent their thinking using an area model and abstract representation. The pictorial representation references the comparison of the fractions (one-half and two-fourths). The representations are labeled and there is evidence of fraction equivalence. The model includes a correct answer. The participant used the selected VM to model their thinking.

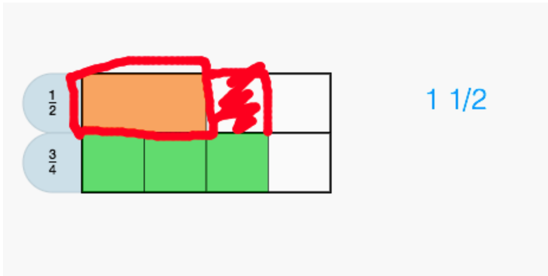
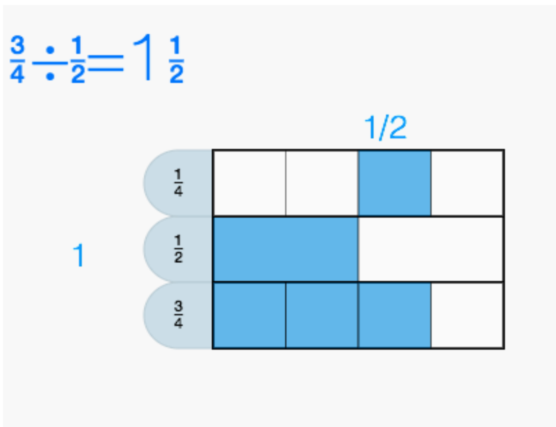
	P#7: Participant seven was able to represent their thinking using an area model and abstract representation. The pictorial representation references the comparison of the fractions (one-half and two-fourths). The representations are labeled and there is evidence of fraction equivalence. The model is lacking a reference to the whole as four-fourths. The model includes a correct answer. The participant used the selected VM to model their thinking.
	P#8: Participant eight was able to represent their thinking using an area model. The pictorial representation references the comparison of the fractions (one-half and two-fourths). The model is lacking a reference to the whole being four-fourths. The model includes a correct answer. The participant used the selected VM to model their thinking.

Session #6 Exit Tickets

Show with virtual manipulatives. $\frac{3}{4}$ divided by $\frac{1}{2}$

	P#1 Participant one displayed an area model and abstract representation of the correct answer. P#1 gave reference to the whole, showing the portion that was to be divided. The participant used the selected VM to model partitive thinking.

	P#2: Participant two displayed a circular model and abstract representation of the correct answer. The participant used the selected VM to model partitive thinking.
	P#3: Participant three displayed an area model and abstract representation of the correct answer. P#3 was able to show the partitions that represent one and one-half. The participant used the selected VM to model partitive thinking.
	P#4: Participant four displayed an area model and abstract representation of the correct answer. P#4 gave reference to the whole, showing the portion that was to be divided. The participant used the selected VM to model partitive thinking.
	P#5: Participant five displayed an area model and abstract representation of the correct answer. P#1 gave reference to the whole, showing the portion that was to be divided. The participant used the selected VM to model partitive thinking.
	P#6: Participant six displayed an area model and abstract representation of the correct answer. P#6 gave reference to the whole, showing the portion that was to be divided. The participant used the selected VM to model partitive thinking.

	P#7: Participant seven displayed an area model and abstract representation of the correct answer. P#7 gave reference to the whole, showing the portion that was to be divided. The participant used the selected VM to model partitive thinking.
	P#8: Participant eight displayed an area model and abstract representation of the correct answer. P#8 gave reference to the whole, showing the portion that was to be divided. The participant used the selected VM to model partitive thinking.

Appendix E

University of Houston IRB Concent

UNIVERSITY of HOUSTON
DIVISION OF RESEARCH
Institutional Review Boards

APPROVAL OF SUBMISSION

June 4, 2020

Laura Lopez

lalopez12@uh.edu

Dear Laura Lopez:

On June 4, 2020, the IRB reviewed the following submission:

Type of Review:	Initial Study
Title of Study:	Improving Teachers' Conceptual Knowledge of Fractions Through Subject-Specific Professional Development.
Investigator:	Laura Lopez
IRB ID:	STUDY00002185
Funding/ Proposed Funding:	Name: University of Houston
Award ID:	
Award Title:	
IND, IDE, or HDE:	None
Documents Reviewed:	• IRB LOPEZ, Category: IRB Protocol; • IRB CONFIRMATION LETTER LOPEZ, Category: Letters of Cooperation / Permission;
Review Category:	Exempt
Committee Name:	Not Applicable
IRB Coordinator:	Sandra Arntz

The IRB approved the study on June 4, 2020 ; recruitment and procedures detailed within the approved protocol may now be initiated.

As this study was approved under an exempt or expedited process, recently revised regulatory requirements do not require the submission of annual continuing review documentation. However, it is critical that the following submissions are made to the IRB to ensure continued compliance:

- Modifications to the protocol prior to initiating any changes (for example, the addition of study personnel, updated recruitment materials, change in study design, requests for additional subjects)

UNIVERSITY of HOUSTON
DIVISION OF RESEARCH
Institutional Review Boards

- Reportable New Information/Unanticipated Problems Involving Risks to Subjects or Others
- Study Closure

Unless a waiver has been granted by the IRB, use the stamped consent form approved by the IRB to document consent. The approved version may be downloaded from the documents tab.

In conducting this study, you are required to follow the requirements listed in the Investigator Manual (HRP-103), which can be found by navigating to the IRB Library within the IRB system.

Sincerely,

Research Integrity and Oversight (RIO) Office
University of Houston, Division of Research
713 743 9204
cphs@central.uh.edu
http://www.uh.edu/research/compliance/irb-cphs/

Appendix F

IRB Consent

REGIS
SCHOOL

March 11, 2020

IRB Review Board,

The purpose of this letter is to inform you that I give Laura Lopez permission to conduct the research titled **Improving Teachers' Conceptual Knowledge of Fractions Through Subject-Specific Professional Development** at The Regis School of the Sacred Heart. We have agreed to the following study procedures of: Mrs. Lopez analyzing teacher's knowledge of fractional learning through the administered pre-and post-assessment and review of the content presented in the vertically aligned professional development series. The job-embedded professional development series consisted of 360 minutes of content. The questions and professional development were derived from the TEKS standards (Texas Essential Knowledge and Skills). The released de-identified data will be used to determine if the content-specific sessions contributed to an increase in teacher learning. The results will inform future professional development in the area of mathematics education.

Sincerely,

Dennis P. Phillips

Head of School

Regis School of the Sacred Heart