THE END PROBLEM FOR A TORSIONLESS HOLLOW CIRCULAR ELASTIC CYLINDER

A Dissertation
Presented to
the Faculty of the Department of Mechanical Engineering
University of Houston

In Partial Fulfillment
of the Requirements for the Degree
Doctor of Philosophy

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by

Guilherme Mauricio de La Penha

August, 1968

'Mine is a long and sad tale'
 said the Mouse, turning to Alice, and sighing
'It is a long tail certainly'
 said Alice, looking down with wonder at
 the Mouse tail;
'but why do you call it sad?'

Lewis Carrol
Alice's Adventures in Wonderland

to my mother for the past
to my wife for the present
to my children for the future

ACKNOWLEDGMENT

Papyrus of Ramses III

I am deeply indebted to Dr. S. B. Childs for guidance and support received during the course of this investigation; his numerous hints and discussions as well as doubts over and about large portions of the derivations have left an indelible mark on this work. In particular, the programing of numerical results rests solely in his skill. His help is acknowledged with sincere thanks.

I am under great obligations to Professor D. Muster in ways more than one. I take this opportunity to express my gratitude for his continual willingness during two years to aid me not only by putting at my disposition his personal library but also for assistance in both academic and private matters.

I am fortunate to have as both friends and colleagues Messrs. A. Mossé and A. P. Ripper who played a decisive role connected with my work. They not only encouraged it at all stages but also let me comment on their dissertations (in subjects foreign to me) while I could not proceed with mine.

A list of works which have been of service to me is appended and for the valuable assistance in obtaining copies of a majority of them I am indebted to Mrs. E. Roberson and C. Conrath for earlier help and to Mrs. R. Banks for more recent ones.

The financial support of the Coordenação do Aperfeiçoamento do Pessoal de Nível Superior (CAPES, Brazil) is gratefully acknowledged.

Thanks are also due to Mrs. C. Callahan who typed a difficult manuscript with patience and ability.

Finally, I am indebted most to my wife who suffered it all (if not altogether in silence).

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ABSTRACT

A class of axisymmetric boundary value problems for a torsionless semi-infinite hollow circular cylinder is considered; the lateral surface of the cylinder is assumed to be traction free, whereas its end-section is subjected to given self-equilibrated loads, given displacements or to mixed boundary conditions. The solution utilizes Love's stress representation - - known to be complete - - to generate an aggregate of biorthogonal eigenfunctions in the interval $a \leq r \leq b$. The problem is formally reduced to an infinite system of linear algebraic equations; explicit expressions being given in the case of mixed boundary conditions.

The close association of the problem with two classical ones, namely, Saint-Venant's problems and Saint-Venant's principle is discussed and supplemented with substantial references.

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NOMENCLATURE

Symbols are defined where they first appear; this list includes only the more important ones.

E	3d Euclidian space
$\mathcal{B}$	set in E
$\partial\mathcal{B}$	boundary of $\mathcal{B}$
$C^{k}(\mathcal{B})$	class of continuous fields over $\mathcal{B}$ with continuous partial derivatives up to the order k
$\underset{\sim}{x}$	position vector, $x = \lvert \underset{\sim}{x} \rvert$
(x_1, x_2, x_3)	Cartesian coordinates
(r, ϑ, z)	cylindrical coordinates
$\underset{\sim}{u}$	displacement vector
u_i	Cartesian components of $\underset{\sim}{u}$
u_r, u_ϑ, u_z	physical components of $\underset{\sim}{u}$ in cylindrical coordinates
$\underset{\sim}{t}$	stress tensor
t_{ij}	Cartesian components of
$t_{rr}, t_{\vartheta\vartheta}, \ldots$	physical components of $\underset{\sim}{t}$ in cylindrical coodinates
$\underset{\sim}{T}$	traction vector
$\bar{\underset{\sim}{u}}, \bar{\underset{\sim}{T}}$	prescribed boundary values for the displacement and traction
$\underset{\sim}{n}$	unit exterior normal
$\underset{\sim}{f}$	body force (per unit volume) vector

$\underset{\sim}{S} = \langle \underset{\sim}{u}, \underset{\sim}{t} \rangle$, elastic state

μ shear modulus

σ Poisson's ratio

$\tau = 2(1 - \sigma)$

$\mathcal{B}, \Gamma_1, \Gamma_2, \Sigma, \Pi_\zeta$; see eq. (1.11)

$\underset{\sim}{\mathfrak{F}}$ resultant force on Π_0

$\underset{\sim}{\mathfrak{M}}$ resultant moment about the centroid of Π_0

δ_{ij} Kronecker's delta $(i, j = 1,2,3)$

$\epsilon_{\alpha\beta}$ 2d - permutation symbol $(\alpha, \beta = 1,2)$

ϵ_{ijk} 3d - permutation symbol $(i, j, k = 1,2,3)$

$,i = \dfrac{\partial}{\partial x_i}$, partial derivatives in Cartesian coordinates

$\Delta = \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial}{\partial r}\right) + \dfrac{\partial^2}{\partial z^2}$

$\Delta^2 = \Delta\Delta$, axisymmetric biharmonic operator

$\Delta_3 = \partial^2/\partial x_i \partial x_i$, 3d - Laplace operator

$\Delta_2 = \partial^2/\partial x_\alpha \partial x_\alpha$, 2d - Laplace operator

χ Love's stress function

a, b internal and external radii

$\rho = a, b$

$\alpha_j, \beta_k, \gamma_j, \lambda_k, \mu_j$ parameters defined in the text which are restricted to certain values.

$J_\nu(\beta_k r), Y_\nu(\beta_k r)$ Bessel and Weber - C. Neumann functions of order $\nu = 0,1$.

$I_\nu(\alpha_j r), K_\nu(\alpha_j r)$ modified Bessel (Basset) and MacDonald functions of order $\nu = 0,1$.

$\mathcal{C}_\nu(\beta_k r) = J_\nu(\beta_k r) + \lambda_k Y_\nu(\beta_k r)$, cylinder function of order $\nu = 0,1$.

$\mathcal{Z}_\nu(\alpha_j r) = I_\nu(\alpha_j r) + \mu_j K_\nu(\alpha_j r)$, modified cylinder function of order $\nu = 0,1$.

$H_\nu^{(2)}(\gamma_j r) = J_\nu(\gamma_j r) - i Y_\nu(\gamma_j r)$, second Hankel function of order $\nu = 0,1$.

$\mathcal{C}_{\rho\nu}(\gamma_j r) = J_\nu(\gamma_j r) Y_1(\gamma_j \rho) - Y_\nu(\gamma_j r) J_1(\gamma_j \rho)$ cylinder function of order $\nu = 0,1$, with $\rho = a,b$.

$\ni_{ab}$ transposition symbol defined by eq. (3.12).

det determinant function

The following trivial set - theoretic notations are also used in a few instances.

$\in$ (is) and element (of)

$\cup$ union of sets

$\{e \mid P(e)\}$ set of all elements e satisfying the property $P(e)$.

$\overset{d}{=}$ equal by definition

$\langle a,b,\rangle$ ordered pair of elements a and b

$f(a \mid b)$ function of the ordered pair $\langle a,b\rangle$

$\mathcal{O}(x^\alpha)$ function that satisfies the inequality

$$|\mathcal{O}(x^\alpha)| \leq M x^\alpha$$

where M is a constant and $x \rightarrow \infty$.

$o(x^\alpha)$ function for which the ratio $o(x^\alpha)/x^\alpha \rightarrow 0$ uniformly with respect to the position vector $\underset{\sim}{x}$.

$\| \quad \|$ norm operator

CHAPTER I

INTRODUCTION

"Of what is passed, or
passing or to come."
William B. Yeats

1.1 SAINT-VENANT'S PRINCIPLE

TOUPIN [1965] has recently obtained a strong result on the Principle of SAINT-VENANT in linear elasticity. Loosely, SAINT-VENANT'S Principle is usually taken to mean that if a self equilibrated stress distribution (resultant force and moment equal to zero) is applied to a section of the surface of a given body, the stresses and strains produced at points far removed from the stressed area will be negligibly small. As stated, the principle is ambiguous and, in many cases, false; cf. HOFF [1945], DOU [1964], TOUPIN [1965] and FILONENKO-BORODICH [1965, par. 31] for counter examples. A rigorous SAINT-VENANT Principle should give sufficient conditions under which the internal effects (stress, strain, etc.) will decrease in some specified sense with the distance from the stressed part of the boundary.

TOUPIN has shown that for a cylinder of arbitrary length and cross-section, with zero body force which has a self-equilibrated loading on one end but is otherwise stress free,

$$\frac{U(z)}{U(0)} \leq e^{-\left(\frac{z-\ell}{S_c(\ell)}\right)} \tag{1.1}$$

where

(i) $U(z)$ is the stored elastic energy in that part of the cylinder whose distance from the stressed end is greater than z,

(ii) $U(o)$ is the total elastic energy in the cylinder,

(iii) $S_c(\ell)$ is a "decay length" which depends on the physical constants of the cylinder and the smallest characteristic frequency of free vibration of the cylinder of length ℓ,

(iv) ℓ is an arbitrary positive parameter which may be chosen so as to provide the smallest possible value for $S_c(\ell)$.

The question of comparing stress distributions produced by statically equivalent loads first arose in connection with the problem of the deformation of a cylinder by prescribed surface tractions distributed over its plane ends. SAINT VENANT [1855, 1856] constructed a solution to the relevant boundary value problem in the theory of elasticity which corresponds to a particular set of end loads. The principle which bears his name was originally enunciated in order to justify the use of his result as an approximation in cases where the end loads are statically equivalent to, but not identical with, the loads for which his solution is rigorously valid.

A generalized statement of SAINT-VENANT'S Principle, intended to apply to elastic bodies of arbitrary shape, was apparently first introduced by BOUSSINESQ [1885; p. 298], whose version of the principle became traditional in the literature, LOVE [1927; par.89]. BOUSSINESQ supported his version of the principle by analyzing an elastic half-space subjected to concentrated forces applied normal to its boundary.

FILON [1902] (cf. also LOVE [1927;par.226], PICKETT [1944]) constructed, in essence, a large but not exhaustive class of solutions for circular cylinders (namely the 'antiplane class', cf. MILNE-THOMSON [1962; p. 42]). Simply by examination of solutions in this class he perceived a rapid decay in the strain induced in a circular rod by self-equilibrated forces applied to one end, but no common feature of all solutions can be easily deduced from his analysis. The remarks of Love, loc. cit., do not constitute a proof of the "exponential decay" of the energy even for this restricted class of loadings of circular cylinders.

Two other classes of general theorems have been proved in connection with the SAINT-VENANT Principle and put forward as having some bearing on the original question posed by SAINT-VENANT'S remarks. The first of these are due to v. MISES [1945] and STERNBERG [1958]. These theorems concern a representation of the strain at an interior point of a given elastic body which is caused by a sequence of loads on a sequence of regions of its boundary. The second class of theorems is due to ZANABONI $[1937]_{1,2}$ and concerns estimates for the total energy of a sequence of bodies under the action of a fixed system of loads on a given common portion of the boundary; this work has been discussed in the treatise by BIEZENO and GRAMMEL [1954]. The efforts of SOUTHWELL [1923] and GOODIER [1937] can also be included in this second class.

Although a comprehensive review of the literature on SAINT-VENANT'S Principle in the linearized equilibrium theory of elastic solids would serve a useful purpose, such a survey is clearly beyond the scope of these introductory remarks; we mention, however, that it has received great attention in recently published literature towards a sharper mathematical formulation. We single out specially the work of KNOWLES [1966] on two-dimensional problems, ROSEMAN [1966] using TOUPIN'S formulation to obtain a pointwise estimate for the stress in simply connected cylindrical bodies, STERNBERG and KNOWLES [1967] on the torsion of solid and hollow cylinders. The earlier statements due to SAINT-VENANT and BOUSSINESQ are surveyed in the classic of TODHUNTER and PEARSON [1893], more recent developments being reviewed in STERNBERG [1958].

1.2 THE END PROBLEM OF A CYLINDER

Determining the state of stress and strain within a homogeneous, isotropic and elastic circular cylinder subjected to prescribed forces and/or displacements at its surfaces, is one of the classical problems of the mathematical theory of elasticity. It has received great attention from various authors. The problem essentially reduces to finding solutions to the equations of elasticity in cylindrical coordinates and then adapting them to the prescribed boundary conditions at the curved and flat surfaces of the cylinder.

Problems of this type arise, for example, in the thermal-stress problem of bonding of a semi-infinite cylinder at its plane end to another cylinder or plate. A knowledge of the stress distribution and how it decays away from the bonded end shows how long a finite cylinder needs to be for its free end to be unaffected by conditions at the bonded end. This problem and similar ones are often referred to by engineers as "the end problem of a cylinder."

The method of series expansions in terms of special functions can be employed successfully to solve the equations involved. Perhaps the first investigations along these lines are those of POCHHAMMER [1876] and CHREE [1886, 1889] using FOURIER-BESSEL series which were restricted in the freedom of prescribing arbitrary stresses on displacements on all surfaces of the cylinder. Later DOUGALL [1914] presented a more extensive and detailed general analytic study of the problem. This paper indicated quite clearly the complexity involved in the mathematical treatment. SYNGE [1945] considered the equilibrium of a homogeneous cylinder with arbitrary cross section which is free from stress on the bounding surface. He examined in some generality the nature of the eigenvalue problem, and, in the case of a circular cross section, he indicated the form of the equation for the eigenvalues, and this is in agreement with the results of DOUGALL (this equation can also be lifted from under the haze notation of CHREE [1886; eq (29)] where the problem studied is a dynamical one).

MURRAY [1945] considered the problem of the thermal stresses and strains in an elastic circular cylinder of finite length which is free from stress on its curved boundary. He obtained expressions for the stresses in terms of solutions of the biharmonic equation and his application of the boundary conditions on the curved surface gave rise to a transcedental equation involving Bessel functions of complex argument. Only the first two eigenvalue solutions of this equation were stated, but no indication was given of how they had been obtained. The boundary conditions on the end face were satisfied in an approximate manner.

A calculus of variations approach to the end problem of solid cylinders has been given by HORVAY and MIRABAL [1958]. They consider the case when self-equilibrating axially-symmetric normal and shear tractions act on the end of a semi-infinite circular cylinder. They stated that in principle a rigorous solution can be obtained if one follows an analysis after the manner of MURRAY [1945]. Because of the complexity of the method of this last paper when presented in detail, they prefer to obtain approximate values for the stresses from a variational approach. They use special representations for the stresses and find the EULER equations. They give approximate values for the first three eigenvalues. For these eigenvalues the product approximations used in this method appear to create large discrepancies in the verification of the conditions of compatability, a point which is noted by the authors.

The variational method of the last paragraph is improved in the article by HORVAY, GIAEVER and MIRABAL [1959]. In particular the values of the displacements are much better than those obtained in HORVAY and MIRABAL, op. cit., and the compatibility requirements are satisfied to a greater accuracy.

Two stress potential functions Φ, Ψ are introduced into the basic equations by HODGKINS [1962] and the stresses are expressed in terms of them. The following equations satisfied by ϕ, ψ are deduced from the equations of equilibrium:

$$\frac{\partial^2\phi}{\partial r^2} - \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{\partial^2\phi}{\partial z^2} = 0$$

$$\frac{\partial^2\psi}{\partial r^2} - \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial z^2} = \frac{\partial^2\phi}{\partial z^2} \tag{1.2}$$

where r, z are cylindrical coordinates. These equations are solved by the method of finite differences using various mesh sizes. Four problems involving a finite cylinder are investigated.

Problems involving a finite hollow cylinder which is free from stress on its inner and outer curved surfaces, and has normal and shear loadings on its end faces are considered in the article of MENDELSON and ROBERTS [1963]. Here the basic equations are written in terms of the four stresses and after systematic elimination, and integration, they yield a partial integro-differential equation for the shear stress in the bases of the cylinder. This equation incorporates all the boundary conditions except for those dependent on prescribed values of the shear stresses on the ends. They write an expression of the form

$$t_{rz} = \sum_n \sum_m P_n(r)\, Q_m(z)\, t_{nm} + \phi(r)\, Q_o(z) \tag{1.3}$$

as an approximation to the shear stress. The quantities P_n, Q_m are station functions such that P_n vanishes at $z=\pm\ell$ but $Q_o(\pm\ell)=1$, so that t_{nz} takes the given value $\phi(r)$ on the ends $z=\pm\ell$. Here t_{mn} is the shear stress at the station (r_n, z_m) and a and b are respectively the internal and external radii of the cylinder and 2ℓ is its length. When the above representation for t_{rz} is substituted in the integro-differential equation, and the technique of double collocation used, a system of simultaneous linear equations is produced.

Two problems are then solved, one for a solid cylinder, and the other for a hollow cylinder, using the end conditions. The values of the longitudinal stress are prescribed on $z=\pm\ell$. By considering various numbers of collocation stations MENDELSON and ROBERTS indicate that their computed results satisfy the basic differential equations. An advantage of the method is that the formulation in terms of the integro-differential equations gives a kind of averaging of values throughout the cylinder, but a serious objection to it is that the above representation to the shear stress certainly does not satisfy the prescribed conditions on the curved surfaces, where in particular t_{rz} ought to be zero. In illustrative examples they choose $\phi(r)$ to be zero so that this difficulty was avoided. This explains why their results are satisfactory.

The more complete problem for the finite solid cylinder, i.e., one in which either the displacements or stresses are prescribed at the end and the stresses or displacements on the lateral surfaces has been analyzed by VALOV [1962] using Papkovich-Neuber representation of the solution of the Navier's equations[1] obtaining an infinite system of equations whose possibility of providing a bounded solution is investigated through a careful analysis of the Fourier-Bessel coefficients.

The following problem for a semi-infinite cylinder is investigated by GRINCHENKO [1963]. On the curved boundary the values of the normal and shear stresses are prescribed as arbitrary functions of z and on the plane end the longitudinal and shear stresses are prescribed as arbitrary functions of r. The Navier equations are used and the radial displacement is written in terms of a Fourier series involving the Bessel function of order zero plus a Fourier integral. A similar representation is taken for the longitudinal

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[1]These are called LAME'S equations in the Russian literature. No solutions of this system of equations other than for the axially symmetric problem are known.

displacement. Application of the boundary conditions yields a system of functional equations from which the unknown coefficients in the above representations are determined. By a systematic process these equations are then changed into an infinite system of simultaneous linear algebraic equations. The regularity of this system is established, which means that it is theoretically possible to solve the system by iteration techniques. A numerical example is concerned with a semi-infinite cylinder with zero stress on its curved boundary, zero shear stress and a prescribed value of the longitudinal stress on $z = 0$. However, no numerical results are quoted for the stresses and displacements, although these appear to have been computed.

The first textbook account known to the present author of problems of the type being discussed here is given by LUR'E [1964] who includes at the end of Chap. 7 a short bibliography with references. For the problem with zero stress on the curved boundary numerical results are presented for the first three complex eigenvalues together with the corresponding values of the modified Bessel functions I_0, I_1. The end of the chapter furnishes an attempt at satisfying the two conditions on the end face of the cylinder, where the normal stress $t_{zz} = F(r)$ and the shear stress $t_{nz} = -\phi(r)$ are prescribed. To obtain the coefficients in the simultaneous representations to $F(r)$ and $\phi(r)$ a least square method is employed and the resulting infinite set of simultaneous linear equations is truncated to give a finite number of equations. Solution of these equations gives the approximate values of the coefficients. Unfortunately, no numerical values of the end stresses are presented, and the author omits the computation of the coefficients in the series expansions and the subsequent evaluation of displacements and stresses at points of the cylinder away from the end.

Following the work of LUR'E, WARREN, ROARK and BICKFORD [1967] and WARREN and ROARK [1967], studied the end effect numerically by expanding the solution into a series of eigenfunctions satisfying stress free boundary conditions on the lateral surface, the end face subjected to given axisymmetric self-equilibrated distribution of normal and shearing stresses. The coefficients are selected so as to minimize the square error between the prescribed boundary conditions on $z = 0$ and the eigenfunction representation with a finite number of terms. Numerical results including up to 40 terms are included.

The stress analysis for a hollow cylinder of finite length is treated by KAEHLER [1965] who formulates a partial integro-differential equation for the shear stress. He allows the normal and shear stresses on the inner and outer curved surfaces to be functions of the axial coordinate z. A representation of the shear stress is written down which satisfies the boundary conditions on $r = a,b$ requiring certain quantities $t_i(z)$, $i = 1, 2, \ldots, N$ to be determined. Substituting this representation into the integro-differential equation, collocating on two radial stations $i = 1,2$ and thence differentiating the result yields two fourth order ordinary nonhomogeneous differential equations with constant coefficients for $t_1(z)$, $t_2(z)$. These are solved for the complementary functions only and eventually the solutions are cast in a form whereby the conditions on t_{rz} at $z = \pm \ell$ can be utilized. As an illustrative example the case of a solid cylinder with a band of shear stress on the curved surface is considered.

At this stage, it is well to point out that the use of Fourier transforms as potentials is by no means new in treating axisymmetric elastic problems. The starting point possibly goes back to DOUGALL [1914] (cf. also GRAY, MATHEWS and MACROBERT [1931; Chap. 15]) which has been followed by FOPPL and FOPPL [1928], and more recently by BARTON [1941], TRANTER and CRAGGS [1945] (cf. also TRANTER [1956;par.3.7]), LING and LEE [1954] and KOGAN and KHRUSTALEV [1958]; all for the case of loading on the curved boundaries. We mention this because it facilitates the justification of the form of the solution used here.

Finally, we come to the work of CHILDS [1966] and LITTLE and CHILDS [1967] who consider the semi-infinite circular elastic cylinder with mixed boundary conditions on the finite end. By using Love's stress functions expressed as an infinite series in the eigenfunctions of the biharmonic equation plus a Fourier integral, and by requiring the latter to satisfy the stress free condition on the curved boundary while the former is chosen so as to generate the boundary conditions on the plane end, they have constructed the general solution of the problem in terms of a biorthogonal family of functions whose basis are the Bessel functions of order zero and one.

The problem is reduced to solving a doubly infinite system of linear equations with a strongly diagonal matrix. Due to the rapid decrease of the modulus of the Fourier-Bessel coefficients, the system can be truncated and solved numerically, yielding good results. They also have provided a table containing the first twenty eigenvalues of the characteristic equation for various values of Poisson's ratio. An example is given for the case of the normal and shearing stresses specified at the plane end; the loading being self-equilibrated there, the exponential decay of both stresses and strains is apparent from their plots.

The study reported here, is a continuation of the last mentioned one and depends strongly on it in the sense that the results contained there were the source for the amount of formulae derived here which would otherwise not be obvious. In this respect a quotation from LOVE [1927] is adequate: "... nothing that has once been discovered ever loses its value or has to be discarded ...".

1.3 SOME DEFINITIONS AND FORMULATION OF THE PROBLEM

Before engaging on the rigorous statement and solution of the problem, we turn now to some preliminary notational agreements and definitions in conformity with the standard practice in modern mechanics of continua.

We employ the letter E for the entire three-dimensional Euclidian space. If $\mathfrak{B}$ is set in E we write $\partial\mathfrak{B}$ for the boundary of $\mathfrak{B}$. The class of continuous fields over $\mathfrak{B}$ which possess continuous partial derivatives up to and including the order k is denoted by $C^k(\mathfrak{B})$. Standard indicial notation is used in connection with Cartesian components of tensors of any order. Subscripts preceeded by a comma indicate partial differentiation, while underlined tildes designate tensors (the non-zero order of which will be clear from the context). We also employ some of the trivial symbols of set theory. The summation convention applies to repeated indices.

Def. 1 (Elastic state). If $\underset{\sim}{u}$ and $\underset{\sim}{t}$ are respectively a vector-valued and a second order tensor valued function defined in a domain $\mathfrak{B}$ in E, we call the ordered pair $\mathcal{S} = \langle \underset{\sim}{u}, \underset{\sim}{t} \rangle$ an elastic state on corresponding to the body force field $\underset{\sim}{f}$, the shear modulus μ and Poisson's ratio σ and write

$$\mathfrak{s} = \langle \underset{\sim}{u}, \underset{\sim}{t} \rangle \in \mathcal{E}(\underset{\sim}{f}, \mu, \sigma, \mathfrak{B}) \tag{1.4}$$

provided

(a) $$\begin{aligned} &\underset{\sim}{u} \in C^2(\mathfrak{B}) \quad , \quad \underset{\sim}{t} \in C^1(\mathfrak{B}) \\ &\underset{\sim}{f} \in C^0(\mathfrak{B}) \quad , \quad \mu > 0 \; , -1 < \sigma < 1/2 \end{aligned} \tag{1.5}$$

μ and σ being constants.

(b) $\underset{\sim}{u}$, $\underset{\sim}{t}$, $\underset{\sim}{f}$, μ and σ on $\mathfrak{B}$ satisfy

$$\begin{aligned} &t_{ji,j} + f_i = 0 \quad , \quad t_{ij} = t_{ji} \\ &t_{ij} = \mu\left[\frac{2\sigma}{1-2\sigma}\delta_{ij} u_{k,k} + u_{i,j} + u_{j,i}\right] \\ &\mu(u_{i,j} + u_{j,i}) = t_{ij} - \frac{\sigma}{1+\sigma}\delta_{ij} t_{kk} \end{aligned} \tag{1.6}$$

the last two equations being equivalent.

(c) If $\mathfrak{B}$ is unbounded

$$\left.\begin{aligned} \underset{\sim}{u}(\underset{\sim}{x}) &= \mathcal{O}(x^{-1}) \\ \underset{\sim}{t}(\underset{\sim}{x}) &= \mathcal{O}(x^{-2}) \\ \underset{\sim}{f}(\underset{\sim}{x}) &= \mathcal{O}(x^{-3}) \end{aligned}\right\} \quad \text{as} \quad x \to \infty \tag{1.7}$$

$\underset{\sim}{x}$ being the position vector and x its magnitude. The symbol $\mathcal{O}(\, . \,)$ having the usual meaning of order of magnitude.

If $\underset{\sim}{f} = \underset{\sim}{0}$ on $\mathfrak{B}$ it has been shown by FICHERA [1950] that (1.4) implies $\underset{\sim}{u} \in C^\infty(\mathfrak{B})$, $\underset{\sim}{t} \in C^\infty(\mathfrak{B})$ a more elaborated proof has been given previously by FRIEDRICHS [1947; p.459 et seqq.]. We recall that the inequalities imposed in (1.5) on the elastic moduli μ and σ are necessary and sufficient for the positive definiteness of the strain energy density. $(1.6)_1$ represents the stress equations of equilibrium (Cauchy), while $(1.6)_{2,3}$ are the stress-displacement relations (constitutive assumptions). If $\underset{\sim}{f} = \underset{\sim}{0}$, the order conditions at infinity (1.7) are implied by

$$\underset{\sim}{u}(\underset{\sim}{x}) = o(1) \quad \text{as} \quad x \to \infty \tag{1.8}$$

a result also due to FICHERA [1950]. If $\mathcal{S} = \langle \underset{\sim}{u}, \underset{\sim}{t} \rangle$ is a state on $\mathcal{B}$ and Σ is one side of a regular surface with the unit outer normal vector $\underset{\sim}{n}$, we call $\underset{\sim}{T}$ the traction vector of $\mathcal{S}$ on Σ if

$$T_i = t_{ij} n_j \tag{1.9}$$

at all regular points of Σ .

CLEBSH [1862] called the determination of the elastic state within a cylinder (or prism) which — in the absence of body forces — is subjected to surface tractions arbitrarily prescribed over its ends and which is free from lateral loading the "Problem of SAINT-VENANT" (cf. WEBSTER [1912; p. 478], MUSKHELISHVIL [1953; Chap. 22] , SYNGE [1945], STERNBERG and KNOWLES [1966]). SAINT-VENANT treatment of the foregoing problem rests on a relaxed formulation in which the detailed assignment of the terminal tractions is abandoned in favor of prescribing merely the appropriate stress resultants.

Here we are concerned with the determination of the elastic state within a semi-infinite hollow circular cylinder unstressed on the lateral surfaces and supporting an axisymmetric self-equilibrated loading on the finite plane end.

Let $\mathcal{B}$ be such a cylinder, $\partial\mathcal{B}$ consists of two coaxial circular cylindrical surfaces and a plane annulus. Let (x_1, x_2, x_3) be rectangular Cartesian and (r, ϑ, z) circular cylindrical coordinates related by

$$\begin{gathered} x_1 = r\cos\vartheta\ , \quad x_2 = r\sin\vartheta\ , \quad x_3 = z \\ 0 \leq r < \infty\ , \quad 0 \leq \vartheta < 2\pi\ , \quad -\infty < z < \infty \end{gathered} \tag{1.10}$$

and suppose the axis of $\mathcal{B}$ coincident with the x_3-axis.

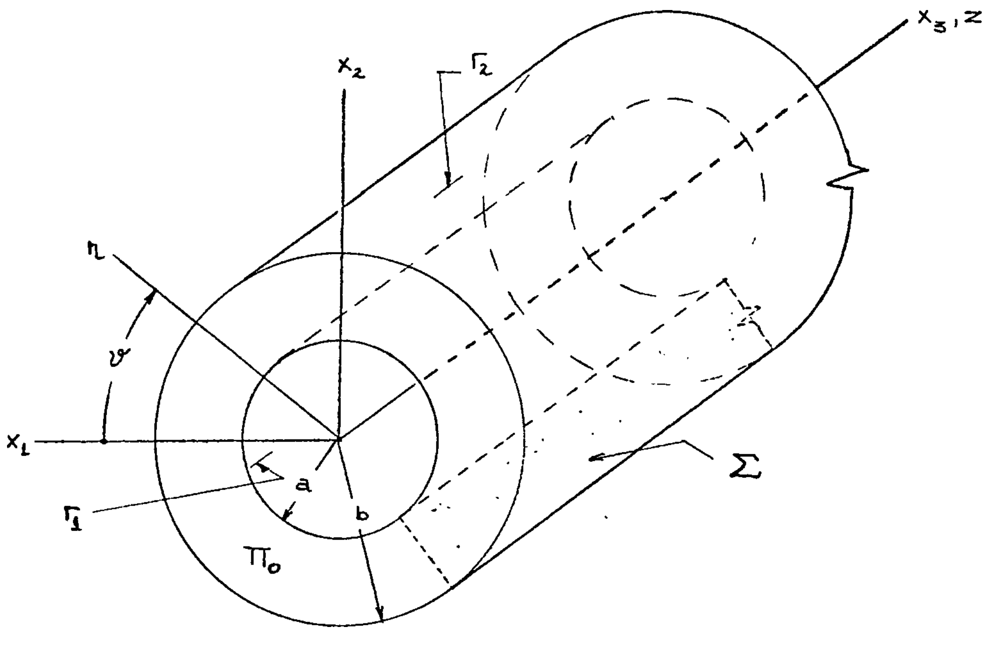

Fig. 1.1. Cylinder geometry

For convenience we define (Fig. 1.1):

$$
\begin{aligned}
\mathcal{B} &\overset{d}{=} \{(r,\vartheta,z) \mid a \leq r \leq b \,, 0 \leq \vartheta < 2\pi \,, \ 0 \leq z < \infty\} \\
\Gamma_1 &\overset{d}{=} \{(r,\vartheta,z) \mid r = a \,, \quad 0 \leq \vartheta < 2\pi \,, \ 0 \leq z < \infty\} \\
\Gamma_2 &\overset{d}{=} \{(r,\vartheta,z) \mid r = b \,, \quad 0 \leq \vartheta < 2\pi \,, \ 0 \leq z < \infty\} \\
\Sigma &\overset{d}{=} \{(r,\vartheta,z) \mid a \leq r \leq b \,, \ \vartheta = \text{const} \,, 0 \leq z < \infty\} \\
\Pi_\zeta &\overset{d}{=} \{(r,\vartheta,z) \mid a \leq r \leq b \,, 0 \leq \vartheta < 2\pi \,, \quad z = \zeta \ \} \\
\partial\mathcal{B} &= \Gamma_1 \cup \Gamma_2 \cup \Pi_0
\end{aligned}
\tag{1.11}
$$

In view of the linearity of the underlying theory, it is clear that to investigate the question with which SAINT-VENANT'S Principle is concerned, it is sufficient to confine our attention to the stresses arising from a surface traction $\underset{\sim}{T}$ which vanishes on $\Gamma_1 \cup \Gamma_2$ and which is self-equilibrated on Π_0.

Def. 2 (Self-equilibrated loading). Given $\underset{\sim}{T}$ over Π_0, the vector-valued linear functionals $\underset{\sim}{\mathcal{F}}\{\cdot\}$ and $\underset{\sim}{\mathcal{M}}\{\cdot\}$ defined by

$$
\begin{aligned}
\mathcal{F}\{\underset{\sim}{t}\} &\overset{d}{=} \int_{\Pi_0} \underset{\sim}{T} \, dS \\
\underset{\sim}{\mathcal{M}}\{\underset{\sim}{t}\} &\overset{d}{=} \int_{\Pi_0} \underset{\sim}{x} \wedge \underset{\sim}{T} \, dS
\end{aligned}
\tag{1.12}
$$

are called respectively, the resultant force and resultant moment about the centroid of Π_0, dS being an area element in Π_0.

If and only if

$$\underset{\sim}{\mathcal{F}}\{\underset{\sim}{t}\} \equiv \underset{\sim}{0} \quad , \quad \underset{\sim}{\mathcal{M}}\{\underset{\sim}{t}\} \equiv \underset{\sim}{0} \tag{1.13}$$

the traction $\underset{\sim}{T}$ is self-equilibrated over Π_0 .

Boundary conditions are never exactly known in elasticity theory. Even if the two boundary conditions were known everywhere, the corresponding problem may be too difficult to solve, so that it becomes necessary to explore the possibility of using boundary conditions that are statically equivalent (in the sense of having the same force and moment resultants) but simpler. The original boundary conditions and the "relaxed" boundary conditions[2] then differ by a self-equilibrating load. It seems natural that a relaxation of boundary conditions will be justified if the load region is small compared to some characteristic dimension, e.g., distance from the load region; how-

---

[2]It is instructive to note here that the canonical classification of the relaxed problem rests on various assumptions concerning the resultants $\underset{\sim}{\mathcal{F}}$ and $\underset{\sim}{\mathcal{M}}$ namely:

I. Extension: $\mathcal{F}_\alpha = \mathcal{M}_i = 0 \;,\; \mathcal{F}_3 = F$

II. Bending: $\mathcal{F}_i = \mathcal{M}_1 = \mathcal{M}_3 = 0 \;,\; \mathcal{M}_2 = M$

III. Torsion: $\mathcal{F}_i = \mathcal{M}_\alpha = 0 \;,\; \mathcal{M}_3 = M$

IV. Flexure: $\mathcal{F}_2 = \mathcal{F}_3' = \mathcal{M}_i = 0 \;,\; \mathcal{F}_1 = F$

where $\alpha = 1,2$. These problems of course have no unique solution.

ever the shape of the body may be an important factor.

The boundary value problems which arise in this subject can be classified in the following categories; here we denote by the subscripts (n) and (t) projections along the normal and tangent plane to Π_o , bars indicate prescribed values.

PT: Traction problem, defined by the boundary condition

$$\underset{\sim}{T}\,\Big|_{\partial\mathcal{B}} = \overline{\overline{\underset{\sim}{T}}}$$

PM 1: Mixed-mixed problem composed of

(i) Traction problem on $\Gamma_1 \cup \Gamma_2$

$$\underset{\sim}{T}\,\Big|_{\Gamma_1 \cup \Gamma_2} = \underset{\sim}{O}$$

(ii) 'Stick contact' problem on Π_o

$$\underset{\sim}{T}_{(n)}\,\Big|_{\Pi_o} = \overline{\overline{\underset{\sim}{T}}}_{(n)}$$

$$\underset{\sim}{u}_{(t)}\,\Big|_{\Pi_o} = \overline{\underset{\sim}{U}}_{(t)}$$

PM 2: Mixed-mixed problem composed of

(i) Traction problem on $\Gamma_1 \cup \Gamma_2$

$$\underset{\sim}{T}\,\Big|_{\Gamma_1 \cup \Gamma_2} = \underset{\sim}{O}$$

(ii) 'Rigid contact' problem on Π_o

$$\underset{\sim}{T}_{(t)}\,\Big|_{\Pi_o} = \overline{\overline{\underset{\sim}{T}}}_{(t)}$$

$$\underset{\sim}{u}_{(n)}\,\Big|_{\Pi_o} = \overline{\underset{\sim}{U}}_{(n)}$$

PM 3: Mixed problem composed of

(i) Traction problem on $\Gamma_1 \cup \Gamma_2$

$$\underset{\sim}{T}\,\big|_{\Gamma_1 \cup \Gamma_2} = \underset{\sim}{0}$$

(ii) Displacement problem on Π_0

$$\underset{\sim}{u}\,\big|_{\Pi_0} = \underset{\sim}{\overline{U}}$$

The best compact discussion of those problems are to be found in BERGMAN and SCHIFFER [1953; Chap. 4] and MIKHLIN [1965, Chap. 4] .

For the moment we will be concerned mainly with problems PM 1 and PM 2; if the traction $\underset{\sim}{T}$ acting on the boundary is written in terms of the circular cylindrical physical components $t\langle ij\rangle$ of the stress tensor acting at the boundary, our prescribed boundary conditions will be as follows:

$$\text{PM 1:}\quad \text{(i)}\quad \left.\begin{aligned} t\langle 11\rangle &\equiv t_{rr} = 0 \\ t\langle 12\rangle &\equiv t_{r\vartheta} = 0 \\ t\langle 13\rangle &\equiv t_{rz} = 0 \end{aligned}\right\} \text{ on } \Gamma_1 \cup \Gamma_2$$

$$\text{(ii)}\quad \left.\begin{aligned} t\langle 33\rangle &\equiv t_{zz} = e(r) \\ u\langle 1\rangle &\equiv u_r = f(r)/2\mu \end{aligned}\right\} \text{ on } \Pi_0 \tag{1.14}$$

$$\text{PM 2:}\quad \text{(i)}\quad \begin{aligned} t\langle 11\rangle &\equiv t_{rr} = 0 \\ t\langle 12\rangle &\equiv t_{r\vartheta} = 0 \qquad \text{on} \quad \Gamma_1 \cup \Gamma_2 \\ t\langle 13\rangle &\equiv t_{rz} = 0 \end{aligned} \tag{1.15}$$

$$\text{(ii)}\quad \begin{aligned} t\langle 13\rangle &\equiv t_{rz} = g(r) \qquad \text{on} \quad \Pi_0 \\ \underset{\sim}{u}\langle 3\rangle &\equiv u_z = h(r)/2\mu \end{aligned}$$

Necessary for the existence of a solution to the foregoing boundary value problems is that e, f, g, h be continuous on Π_0 and that the given loads meet the overall equilibrium conditions (1.13) namely:

$$\begin{aligned} \mathcal{F}_i\{\underset{\sim}{t}\} &= \int_{\Pi_0} t_{3i}\, dS = 0 \qquad i = 1,2,3 \\ \mathcal{M}_\alpha\{\underset{\sim}{t}\} &= \int_{\Pi_0} t_{33}\, \epsilon_{\alpha\beta}\, x_\beta\, dS = 0 \qquad \alpha,\beta = 1,2 \\ \mathcal{M}_3\{\underset{\sim}{t}\} &= \int_{\Pi_0} t_{3\beta}\, \epsilon_{\alpha\beta}\, x_\alpha\, dS = 0 \end{aligned} \tag{1.16}$$

where $\epsilon_{12} = -\epsilon_{21} = 1$, $\epsilon_{11} = \epsilon_{22} = 0$; the components of $\underset{\sim}{t}$ being expressed in rectangular Cartesian coordinates and the x_i are defined in (1.10).

Further, the solution is unique provided $\mu > 0$, $-1 < \sigma < 1/2$ (cf., e.g., KNOPS [1965]).

Although the principle is not very explicitly mentioned, the problem of the edge-layer effect in the theory of elastic plates studied by FRIEDRICHS [1949] and a decade later by FRIEDRICHS and DRESSLER [1961] is in fact a genuine case of SAINT-VENANT'S Principle. It gave rise to a concept frequently encountered in the literature of SAINT-VENANT problem and which for the sake of nomenclature we define here.

Def. 3 (Elastic boundary layer). Given $\eta > 0$, ($[\eta] = F^2 L^{-4}$), the set $\mathfrak{B}_\delta \subset \mathfrak{B}$ of points P such that

$$\| t_{ij} \|_P = t_{ij}(P)\, t_{ij}(P) < \eta \quad , \quad P \in \mathfrak{B}_\delta$$

is called an elastic boundary layer of $\mathfrak{B}$. δ, the width of the layer, is the distance of the farthest point P from Π_0 for which the above inequality holds.

The "end problem" for cylinders, corresponds thus to the determination of δ .

The analysis to be presented could well be extended to the case of the hollow body, i.e., when $a = a(z)$, $b = b(z)$ however, it would never allow numerical computations due to the degree of complexity of the integrals involved, let alone the determination of the eigenvalues.

A word of caution is also in order: certain displacement boundary conditions prescribed on the end face Π_0 may produce stress singularities at the cylindrical corner. Neither the exact linear elastic analysis nor the approximate methods are capable of adequately treating this problem, see for instance ZAK [1964].

Very recently, FLUGGE and KELKAR [1968], discussed a similar problem under different sets of boundary conditions using Navier's equations. They studied the solutions for (a) $\underset{\sim}{u} = \underset{\sim}{0}$ in $\Gamma_1 \cup \Gamma_2$, $\underset{\sim}{u} = \overline{\underset{\sim}{u}}$ on Π_0, (b) $\underset{\sim}{u} = \overline{\underset{\sim}{u}}$ in $\Gamma_1 \cup \Gamma_2$, $\underset{\sim}{u} = \underset{\sim}{0}$ on Π_0 and claim that by superposition any displacement boundary value problem can be solved. This paper, however, has reached the present author too late to be mastered and properly evaluated; it is based on a method devised by Professor Gordon E. Latta and as of yet unpublished.

CHAPTER II

LOVE'S STRESS FUNCTION AND REPRESENTATION

2.1 LOVE'S FUNCTION

If a particular rule enables us to find finite algebraic combinations of the derivatives of a set of arbitrary functions, possibly supplemented by quantities associated with the geometry of the space in question, such that these combinations when substituted for the stress tensor satisfy the equations of equilibrium or motion identically in the arbitrary functions, the rule is said to furnish a "solution in terms of stress functions".

Consider an internally free three-dimensional medium in equilibrium undergoing a constant and uniform velocity and with $\underset{\sim}{f} = \underset{\sim}{0}$. The stress tensor satisfies Cauchy's laws of local balance of linear momentum and moment of momentum respectively, $(i, j = 1,2,3)$

$$t_{ij,j} = 0 \quad , \quad t_{ij} = t_{ji} \tag{2.1}$$

In order that $t_{ij,j} = 0$ in a Euclidian space, application of the classical theorem of the vector potential shows it to be necessary and sufficient that

$$t_{ij} = b_{ijk,k} \quad , \quad b_{ijk} = - b_{ikj} \tag{2.2}$$

the condition $(2.1)_2$ may now be written in the form

$$(b_{ijk} - b_{jik})_{,k} = 0 \tag{2.3}$$

and this is equivalent to the existence of a tensor $\underset{\sim}{v}$ such that

$$b_{ijk} - b_{jik} \overset{d}{=} v_{ijkm,m} \tag{2.4}$$

where

$$v_{ijkm} = - v_{ijmk} = - v_{jikm} \tag{2.5}$$

Therefore

$$2\, b_{ijk} = (v_{kijm} + v_{jkmi} + v_{ijkm})_{,m} \tag{2.6}$$

so that (2.1) becomes

$$t_{ij} = f_{kijm,mk} \tag{2.7}$$

where

$$\begin{aligned} f_{kijm} &\overset{d}{=} \tfrac{1}{2}(v_{kijm} + v_{jmki}) \\ f_{kijm} = - f_{ikjm} &= - f_{kimj} = f_{jmki} \end{aligned} \tag{2.8}$$

The elegant foregoing derivation, given by DORN and SCHILD [1956][3], shows that (2.7) furnishes the

[3]We are working with Cartesian coordinates just for simplicity's sake; in reality the above result is valid in a flat space of any dimension.

general solution of Cauchy's laws for equilibrium of an internally free body. If we set

$$\alpha_{pq} \stackrel{d}{=} \frac{1}{4} \epsilon_{pki} \epsilon_{qmj} f_{kijm} \tag{2.9}$$

so that

$$f_{kijm} = \epsilon_{ipk} \epsilon_{jqm} \alpha_{pq}, \quad \alpha_{pq} = \alpha_{qp} \tag{2.10}$$

Then (2.7) becomes

$$t_{ij} = \epsilon_{ipk} \epsilon_{jqm} \alpha_{pq,km} \tag{2.11}$$

which is the general solution of GWYTHER [1912] and FINZI [1934]. If we write out (2.11) explicitly in cylindrical polar coordinates, at the same time supposing that all derivatives with respect to the azimuth angle are zero, for the physical components $t_{<ij>}$ of $\underset{\sim}{t}$ we obtain

$$\begin{aligned}
t_{<11>} &\equiv t_{rr} = a^2{}_{,zz} + \frac{1}{r} a^3{}_{,r} - \frac{2}{r} a^5{}_{,z} \\
t_{<22>} &\equiv t_{\vartheta\vartheta} = a^3{}_{,rr} + a^2{}_{,zz} - 2 a^5{}_{,rz} \\
t_{<33>} &\equiv t_{zz} = a^2{}_{,rr} + \frac{2}{r} a^2{}_{,r} - \frac{1}{r} a^1{}_{,r} \\
t_{<13>} &\equiv t_{rz} = -\left(a^2{}_{,r} + \frac{1}{r} a^2 - \frac{1}{r} a^1 \right)_{,z} \\
t_{<12>} &\equiv t_{r\vartheta} = \left(a^4{}_{,r} - \frac{1}{r} a^4 - a^6{}_{,z} \right)_{,z} \\
t_{<23>} &\equiv - a^4{}_{,rr} - \frac{1}{r} a^4{}_{,r} + \frac{1}{r^2} a^4 + a^6{}_{,zr} + \frac{2}{r} a^6{}_{,z}
\end{aligned} \tag{2.12}$$

which were derived by BRDIČKA [1957]. Here we have set $a^1 \equiv \mathfrak{A}_{rr}$, $a^2 \equiv \mathfrak{A}_{\vartheta\vartheta}/r$, $a^3 = \mathfrak{A}_{zz}$, $a^4 \equiv \mathfrak{A}_{\vartheta z}/r$, $a^5 \equiv \mathfrak{A}_{zr}$, $a^6 \equiv \mathfrak{A}_{r\vartheta}/r$. Axially symmetric stress distributions in which $t_{r\vartheta} = 0 = t_{\vartheta z}$ are often called "torsionless" (TIMPE [1948]). Since only a^4 and a^6 appear in the expressions of these stress components, the most general torsionless system is obtained by setting $a^4 = 0 = a^6$ in (2.12). In the general case, the six potentials may be reduced to three in a variety of ways, and in the particular torsionless case to only two. For instance when we set, (TRUESDELL [1959])

$$
\begin{aligned}
L_{,r} &\overset{d}{=} a^2_{,r} + \frac{1}{r} a^2 - \frac{1}{r} a^1 \\
M &\overset{d}{=} a^2_{,zz} + \frac{1}{r} a^3_{,r} - \frac{2}{r} a^5_{,z} - L_{,zz}
\end{aligned}
\tag{2.13}
$$

the first four members of (2.12) become

$$
\begin{aligned}
t_{rr} &= L_{,zz} + M \quad ; \quad t_{\vartheta\vartheta} = (rM)_{,r} + L_{,zz} \\
t_{zz} &= L_{,rr} + \frac{1}{r} L_{,r} \quad ; \quad t_{rz} = - L_{,zr}
\end{aligned}
\tag{2.14}
$$

variants are given by BRDIČKA[4], op. cit. For an elastic material obeying the generalized Hooke's

[4]In particular the Boussinesq-Papkovich-Neuber system.

constitutive laws, these potentials can be reduced to one by letting

$$
\begin{aligned}
L_{,z} &= (\sigma-1)\,\Delta\chi + \chi_{,zz} \\
M &= \frac{1}{r}\,\chi_{,rz}
\end{aligned}
\tag{2.15}
$$

where

$$\Delta \stackrel{d}{=} \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{\partial^2}{\partial z^2} \tag{2.16}$$

We then arrive at

$$
\begin{aligned}
t_{rr} &= [\sigma\,\Delta\chi - \chi_{,rr}]_{,z} \\
t_{\vartheta\vartheta} &= [\sigma\,\Delta\chi - \tfrac{1}{r}\,\chi_{,r}]_{,z} \\
t_{zz} &= [(2-\sigma)\,\Delta\chi - \chi_{,zz}]_{,z} \\
t_{rz} &= [(1-\sigma)\,\Delta\chi - \chi_{,zz}]_{,r}
\end{aligned}
\tag{2.17}
$$

the Beltrami-Michell stress compatibility equations in cylindrical coordinates (BARREKETTE [1968]) impose on $\chi(r,z)$ the following restriction

$$\Delta^2\,\chi \equiv 0 \tag{2.18}$$

i.e., that χ be a biharmonic function. The function χ is Love's stress function for torsionless axially symmetric stress fields, having been introduced in the second edition (1906) of Love [1927;par. 188] where its completeness was also asserted[5]. The reduction (2.15) is not to be read off from any work known to the author.

Expressed in terms of Love's function, from integration of $(1.6)_3$, the nonzero displacements have the representations

$$
\begin{aligned}
u_r &= -\frac{1}{2\mu}\,\chi_{,rz} \\
u_z &= \frac{1}{2\mu}\left[2(1-\sigma)\Delta\chi - \chi_{,zz}\right]
\end{aligned}
\tag{2.19}
$$

The equations $t_{r\vartheta} = 0 = t_{\vartheta z}$ are therefore identities by using the above representation, likewise $u_\vartheta = 0$.

---

[5] First fully satisfactory treatment, including forms in general curvilinear coordinate systems, existence and generalization to elastodynamics: NOLL [1957] . Other references: WESTERGAARD [1952] , MARGUERRE [1955] , SNEDDON and BERRY [1958] , YU [1962] , FUNG [1965] . TRUESDELL [1959] gives an exaustive bibliography on works dealing with stress functions that are very valuable for research in this area.

2.2 BIHARMONIC FUNCTIONS IN TERMS OF HARMONIC FUNCTIONS

The biharmonic equation (2.18) is classified as a non-degenerated elliptic equation (MIKHLIN [1967; p. 124]). Sometimes a complete solution of it can be expressed as a combination of appropriate potential functions. Let us assume that a biharmonic scalar function χ, considered in a tri-dimensional domain can be represented in the form of a product of two scalar functions Ψ and Ω, which must be of class C^4 in this domain:

$$\chi = \Psi \, \Omega \tag{2.20}$$

BLOKH [1958] in a not widely known paper has shown that the most general expression for Ψ and Ω are

$$\Psi = a + \underset{\sim}{b} \cdot \underset{\sim}{x} + c\, \underset{\sim}{x}^2 \quad , \qquad \Omega = \Phi \tag{2.21}$$

where a and c are scalar constants, $\underset{\sim}{x}$ is the position vector of the point under consideration, $\underset{\sim}{b}$ is a constant vector and Φ is a harmonic function. This result seems more explicit for applications than that of Almansi (cf. e.g. EUBANKS and STERNBERG [1954], FUNG [1965; p. 207]) although less general.

Obviously a more expanded representation may be obtained by adding such representations as (2.21) in which the constants and the value of the harmonic function Φ are changed.

Designating such variables, constants and functions by the subscript κ we introduce harmonic scalar functions and a harmonic vector function

$$H_1 = \sum_{\kappa} a_\kappa \Phi_\kappa \;, \quad H_2 = \sum_{\kappa} c_\kappa \Phi_\kappa \;, \quad \underset{\sim}{\mathcal{H}} = \sum_{\kappa} \underset{\sim}{b}_\kappa \Phi_\kappa \tag{2.22}$$

the formal biharmonic function χ may thus be written as

$$\chi = H_1 + \underset{\sim}{x} \cdot \underset{\sim}{\mathcal{H}} + \underset{\sim}{x}^2 H_2 \tag{2.23}$$

2.3 A REPRESENTATION FOR LOVE'S FUNCTION IN THE REGION Σ

The general solution to equation (2.18) can always be expressed in the form of a sum consisting of a 'particular' solution χ_0 and a 'complementary' function χ_1 which is biharmonic, i.e.,

$$\chi = \chi_0 + \chi_1 \tag{2.24}$$

A 'particular' solution which satisfies in Σ, as defined in (1.11), homogeneous boundary conditions for equation (2.18) along $\Gamma_1 \cup \Gamma_2$ is readily found to be as suggested by the work of LING and LEE [1954].

$$\begin{pmatrix} \chi_0^1 \\ \\ \chi_0^2 \end{pmatrix} = \int_0^\infty \left\{ \begin{matrix} C(\alpha)\, I_0(\alpha r) + D(\alpha)\, \alpha r\, I_1(\alpha r) \\ + E(\alpha)\, K_0(\alpha r) + F(\alpha)\, \alpha r\, K_1(\alpha r) \end{matrix} \right\} \begin{pmatrix} \cos \alpha z \\ \sin \alpha z \end{pmatrix} d\alpha \tag{2.25}$$

where $I_\nu(\alpha r)$, $K_\nu(\alpha r)$ are modified Bessel functions[6] of order $\nu = 0, 1$; the expressions for the parameters C, D, E, F and the restriction on α are developed in the next chapter. We call attention to the fact that χ_0^1 is an odd function in z while χ_0^2 is even in this variable.

The integrals in (2.25) are supposed to be of class $C^4(\Sigma)$ and required to converge absolutely and uniformly in the region Σ.

The 'complementary' function χ_1 is required for the moment, only to satisfy (2.18) and is here chosen as the following infinite series of biharmonic eigenfunctions

$$\chi_1 = \sum_K [A_K(\beta_K) + B_K(\beta_K)\beta_K z]\, \mathscr{C}_0(\beta_K r)\, e^{-\beta_K z} \tag{2.26}$$

obtaining from (2.23) by putting $\Phi_K = \mathscr{C}_0(\beta_K r)\, e^{-\beta_K z}$ (cf. BIEZENO and GRAMMEL [1954; p. 160], MOON and SPENCER [1961; pp 12-17]), $\underset{\sim}{b}_K = B_K \underset{\sim}{e}_3$, $a_K = A_K$; where

$$\mathscr{C}_\nu(\beta_K r) \stackrel{d}{=} J_\nu(\beta_K r) + \lambda_K Y_\nu(\beta_K r) \tag{2.27}$$

is a cylinder function of order ν whose singularity

[6] The notation used here for solutions of Bessel's equations is the one set forth by WATSON [1944] and adopted in ABRAMOWITZ and STEGUN [1965].

is avoided intrinsically by the geometry of Π_{ζ} . It is expected that there will be some kind of decay of the stress and displacement as we move away from Π_0 according to (1.1) and (1.7), a fact that is called the "exponential condition", and this constitutes the justification of the exponential term in (2.26). The parameters λ_k, β_k are as yet undetermined; the summation in (2.26) is taken over the integral values of k .

The crucial question of whether or not this aggregate of solutions is complete, remains open. It will be seen that χ_0 and χ_1 are so closely mingled that they do not admit a distinction, this being done here, solely for operational advantages. Remarks on completeness will then be left for the last chapter.

The original main problems (1.14) and (1.15) are now reduced to the determination of the biharmonic function $\chi(r,z)$ which satisfies:

PM 1: (i) $$\left.\begin{aligned}&[\sigma\Delta\chi - \chi_{,rr}]_{,z} = 0\\&[(1-\sigma)\Delta\chi - \chi_{,zz}]_{,r} = 0\end{aligned}\right\}\quad \text{on } \Gamma_1 \cup \Gamma_2$$

$$\text{(ii)}\quad \left.\begin{aligned}&[(2-\sigma)\Delta\chi - \chi_{,zz}]_{,z} = e(r)\\&\chi_{,rz} = -f(r)\end{aligned}\right\}\quad \text{on } \Pi_0 \qquad (2.28)$$

For this problem we will choose the representation (2.24) to be of the form

$$\chi = \chi_0^1 + \chi_1 \tag{2.29}$$

PM 2: (i) as in PM 1

(ii)
$$\left.\begin{aligned} \left[(1-\sigma)\Delta\chi - \chi_{,zz}\right]_{,r} &= g(r) \\ 2(1-\sigma)\Delta\chi - \chi_{,zz} &= h(r) \end{aligned}\right\} \text{ on } \Pi_0 \tag{2.30}$$

and in this case we choose to represent χ as

$$\chi = \chi_0^2 + \chi_1 \tag{2.31}$$

We note in passing that both problems are non self-adjoint and as such neither the eigenvalues are restricted to the real field (in fact, we show in Appendix I that they are all complex) nor are the eigenfunctions orthogonal. That the problems are physically well posed is obvious, and mathematically this can be corollated from the discussion of SOBOLEV [1963; pars. 14-15].

$\mathfrak{s}$ as given by (1.4) is uniquely characterized in each PM 1 and PM 2 except for an additive rigid displacement field. To avoid repetitious qualifications we agree to call a displacement field uniquely determined if it is unique within the unessential indeterminacy just mentioned (again, this fact is also elaborated in Appendix I).

CHAPTER III

THE MIXED-MIXED PROBLEMS

3.1 PM 1: THE COEFFICIENTS C, D, E, F

We analyze first the mixed-mixed problem PM 1 as stated in (1.14) and (2.28). To this end we take

$$\chi = \chi_0^1 + \chi_1 \tag{3.1}$$

where χ_0^1, χ_1 have been defined in (2.25) and (2.26).

When the representation (3.1) is substituted into $(2.17)_4$ we obtain:

$$t_{rz} = \sum_K \mathscr{C}_1(\beta_K r)[A_K + B_K(\beta_K z - 2\sigma)]\beta_K^3 \, e^{-\beta_K z} + \int_0^\infty \{C\, I_1(\alpha r) + D[2(1-\sigma) I_1(\alpha r) + \alpha r\, I_0(\alpha r)] - E\, K_1(\alpha r) + F[2(1-\sigma)K_1(\alpha r) - \alpha r\, K_0(\alpha r)]\}\, \alpha^3 \cos \alpha z \, d\alpha \tag{3.2}$$

while using $(2.17)_1$ we have

$$t_{rr} = \sum_K \{ \mathscr{C}_0(\beta_K r)[-A_K + B_K(1 + 2\sigma - \beta_K z)] + \frac{1}{\beta_K r} \mathscr{C}_1(\beta_K r)[A_K + B_K(1 - \beta_K z)]\} \beta_K^3 \, e^{-\beta_K z} +$$

$$\int_0^\infty \Big\{ C\left[I_0(\alpha r) - \frac{1}{\alpha r} I_1(\alpha r)\right] + D\left[(1-2\sigma) I_0(\alpha r) + \alpha r\, I_1(\alpha r)\right] +$$

$$E\left[K_0(\alpha r) + \frac{1}{\alpha r} K_1(\alpha r)\right] - F\left[(1-2\sigma) K_0(\alpha r) - \alpha r\, K_1(\alpha r)\right]\Big\} \tag{3.3}$$

$$\alpha^3 \sin \alpha z \, d\alpha$$

If we are to satisfy the stress free boundary conditions along $\Gamma_1 \cup \Gamma_2$, we start by letting $\mathscr{C}_1(\beta_K r)$ be zero at these boundaries. This implies then:

$$\begin{aligned} J_1(\beta_K a) + \lambda_K\, Y_1(\beta_K a) &= 0 \\ J_1(\beta_K b) + \lambda_K\, Y_1(\beta_K b) &= 0 \end{aligned} \tag{3.4}$$

which is a homogeneous linear system in λ_K. For a solution to exist we define:

$\beta_K \stackrel{d}{=}$ roots of the equation

$$J_1(\beta_K a)\, Y_1(\beta_K b) - J_1(\beta_K b)\, Y_1(\beta_K a) = 0 \tag{3.5}$$

then readily

$$\lambda_K = -\frac{J_1(\beta_K a)}{Y_1(\beta_K a)} \equiv -\frac{J_1(\beta_K b)}{Y_1(\beta_K b)} \tag{3.6}$$

It is known that the equation in (3.5) admits an infinite number of roots all of which are real and simple. Such equation plays a preponderant role concerning the orthogonality of the cylinder function $\mathscr{C}_\nu(\beta_K r)$

in the finite interval [a,b] .

To complete the requirements $t_{rz} \equiv 0 \equiv t_{rr}$ along $\Gamma_1 \cup \Gamma_2$ we are left with:

$$C\, I_1(\alpha\rho) + D\,[2(1-\sigma)\, I_1(\alpha\rho) + \alpha\rho\, I_0(\alpha\rho)] - E\, K_1(\alpha\rho) + F[2(1-\sigma)\, K_1(\alpha\rho) - \alpha\rho\, K_0(\alpha\rho)] = 0$$

$$\begin{aligned}&\int_0^\infty \Big\{ C[I_0(\alpha\rho) - \frac{1}{\alpha\rho} I_1(\alpha\rho)] + D[(1-2\sigma) I_0(\alpha\rho) + \alpha\rho I_1(\alpha\rho)] + \\ &E[K_0(\alpha\rho) + \frac{1}{\alpha\rho} K_1(\alpha\rho)] - F[(1-2\sigma) K_0(\alpha\rho) - \alpha\rho K_1(\alpha\rho)]\Big\} \\ &\alpha^3 \sin\alpha z\, d\alpha = \sum_K \mathscr{C}_0(\beta_K\rho)[A_K + B_K(\beta_K z - 1 - 2\sigma)]\, \beta_K^3\, e^{-\beta_K z}\end{aligned} \tag{3.7}$$

where from now on we use $\rho = a, b$.

If we restrict our set of roots β_K to the positive ones, equations $(3.7)_2$ can be reversed by applying the inverse Fourier transform (see e.g. SNEDDON [1951; p. 18], TRANTER [1956; p. 15]). We obtain thus the system:

$$
\begin{bmatrix}
I_1(\alpha a) & \alpha a I_0(\alpha a) + \tau I_1(\alpha a) & -K_1(\alpha a) & -[\alpha a K_0(\alpha a) - \tau K_1(\alpha a)] \\
I_0(\alpha a) - \frac{1}{\alpha a} I_1(\alpha a) & \tau I_0(\alpha a) + \alpha a I_1(\alpha a) & K_0(\alpha a) + \frac{1}{\alpha a} K_1(\alpha a) & -[\tau K_0(\alpha a) - \alpha a K_1(\alpha a)] \\
I_1(\alpha b) & \alpha b I_0(\alpha b) + \tau I_1(\alpha b) & -K_1(\alpha b) & -[\alpha b K_0(\alpha b) - \tau K_1(\alpha b)] \\
I_0(\alpha b) - \frac{1}{\alpha b} I_1(\alpha b) & \tau I_0(\alpha b) + \alpha b I_1(\alpha b) & K_0(\alpha b) + \frac{1}{\alpha b} K_1(\alpha b) & -[\tau K_0(\alpha b) - \alpha b K_1(\alpha b)]
\end{bmatrix}
\begin{bmatrix} C \\ D \\ E \\ F \end{bmatrix}
=
\begin{bmatrix} 0 \\ \mathscr{b}(a) \\ 0 \\ \mathscr{b}(b) \end{bmatrix}
\qquad (3.8)
$$

Where we have defined the following symbols:

$$\mathscr{C}(\varrho) \stackrel{d}{=} \frac{2}{\pi} \sum_{K} \mathscr{C}_o(\beta_K \varrho) \{ A_K (\alpha^2+\beta_K^2) + B_K [(\tau-1)(\alpha^2+\beta_K^2) - 2\alpha^2] \} \frac{\beta_K^3}{\alpha^2(\alpha^2+\beta_K^2)^2} \tag{3.9}$$

$$\tau \stackrel{d}{=} 2(1-\sigma) \tag{3.10}$$

we define further,

Def. 4: The <u>matrix</u> of the coefficients in equation (3.8) will be denoted by $\sqrt{i\alpha}\ \mathscr{Z}(\alpha)$ and we call

$$\det \mathscr{Z}(\alpha) = 0 \tag{3.11}$$

the "associate characteristic equation".

Def. 5. We define a "transposition symbol" $\ni_{ab} \equiv \ni_{ba}$ to be such that, if $f(a|b)$ is a function of the ordered pair $\langle a, b \rangle$ then

$$\ni_{ab} f(a|b) = f(b|a) \tag{3.12}$$

For operational purposes we postulate the linearity of $\ni_{ab}$. Also

$$\ni_{ab}(\ni_{ab}) = 1$$

Solving (3.8) we obtain a suitable expression for the 'particular' solution χ_o^1, namely:

$$\chi_o^1 = \frac{2}{\pi}(1+\partial_{ab})\, b r \sum_K \mathcal{C}_o(\beta_K a)\beta_K^3 \,.$$

$$\int_0^\infty \Big[C(\alpha a|\alpha b)\frac{I_o(\alpha r)}{\alpha r} - D(\alpha a|\alpha b)\, I_1(\alpha r) +$$

$$E(\alpha a|\alpha b)\frac{K_o(\alpha r)}{\alpha r} - F(\alpha a|\alpha b)\, K_1(\alpha r)\Big] \tag{3.13}$$

$$\frac{\mathcal{A}_K \cos \alpha z}{\alpha^2\,(\alpha^2+\beta_K^2)^2 \det \mathcal{F}(\alpha)}\, d\alpha$$

where

$$\mathcal{A}_K = [A_K + (\tau-1)B_K](\alpha^2+\beta_K^2) - 2B_K\alpha^2 \tag{3.14}$$

For convenience, the expressions for the coefficients $C(\alpha a|\alpha b)$, etc. have been listed in Appendix II.

In (3.13) the real part of the integrand is an even function of α, so that the path of integration can be deformed in a semi-circle of infinite radius on the upper semi-plane together with the real axis. This being done, we may use the residue theorem to evaluate the integral. The following isolated singularities exist:

(a) pole of order two at the origin: the residue at this singularity does not contribute to the solution since we are assuming self-equilibrated loads (see the discussion in Appendix I; cf. also BUCHWALD [1964] and CHILDS [1966]).

(b) poles of order two at $\alpha = \pm i\beta_k$: the sum of residues at these poles add up to $-\chi_1$ and cancel out the series part of the solution in (3.1), (cf. KOITER and ALBLAS [1954], JOHNSON and LITTLE [1965], CHILDS [1966]). This fact shows that in reality, our representations χ_0 and χ_1 are so closely mingled together that they do not admit distinction and have only been used so as to facilitate operational expansions.

(c) zeros of order one at the roots α_j of the characteristic equation (3.11): the complete solution depends essentially on these roots. From now on the symbol α_j will stand for the roots of (3.11) with $0 < \arg \alpha_j < \pi$, they will be called the eigenvalues of the problem as explained in Appendix I.

Love's function thus reduces in this case to the double series representation:

$$\chi = (1+\partial_{ab})\sum_{K}\sum_{j} \frac{i\beta_K^3\ \mathscr{C}_0(\beta_K a)\ \mathcal{A}_{Kj}}{(\alpha_j^2+\beta_K^2)^2\ \ \mathcal{U}_j}\,[\,b r\ \mathcal{M}(\alpha_j a|\alpha_j b)]\,e^{i\alpha_j z} \tag{3.15}$$

where

$$\mathcal{A}_{Kj} \overset{d}{=} \mathcal{A}_K \quad \text{evaluated at} \quad \alpha_j$$

$$\mathcal{M}(\alpha_j a|\alpha_j b) \overset{d}{=} C(\alpha_j a|\alpha_j b)\ \frac{I_0(\alpha_j r)}{\alpha_j r} -$$

$$D(\alpha_j a|\alpha_j b)\, I_1(\alpha_j r) + E(\alpha_j a|\alpha_j b)\ \frac{K_0(\alpha_j r)}{\alpha_j r} - F(\alpha_j a|\alpha_j b)\, K_1(\alpha_j r) \tag{3.16}$$

$$\mathcal{U}_j \overset{d}{=} \frac{\alpha_j^2}{2}\ \frac{d}{d\alpha_j}\,[\det \mathcal{F}(\alpha_j)]$$

3.2 PM 1, THE COEFFICIENTS A_K, B_K

At the plane end Π_0 , $z = 0$ and condition $(2.28)_{(ii)}$ yield respectively using (3.1); the integral part being zero there:

$$\sum_K \mathscr{C}_0(\beta_K r)\,[A_K + (\tau-1)B_K]\,\beta_K^3 = e(r)$$
$$\sum_K \mathscr{C}_1(\beta_K r)\,[A_K - B_K]\ \ \beta_K^2 = -f(r) \tag{3.17}$$

which are coupled Dini's series for the determination of A_K, B_K . Since $\mathscr{C}_1(\beta_K a) = 0 = \mathscr{C}_1(\beta_K b)$, Lommel's formula yields (WATSON [1944; p. 134]):

$$\int_a^b \mathcal{C}_\nu(\beta_\kappa r)\, \mathcal{C}_\nu(\beta_\ell r)\, r\, dr = \delta_{\kappa\ell}\, N_\kappa \qquad \nu = 0, 1 \tag{3.18}$$

$$N_\kappa = \frac{1}{2} r^2\, \mathcal{C}_0^2(\beta_\kappa r) \Big|_a^b$$

Using this property we determine

$$A_\kappa = B_\kappa - \frac{\int_a^b e(r)\, \mathcal{C}_1(\beta_\kappa r)\, r\, dr}{\beta_\kappa^2\, N_\kappa}$$

$$B_\kappa = \frac{\int_a^b e(r)\, \mathcal{C}_0(\beta_\kappa r)\, r\, dr + \beta_\kappa \int_a^b f(r)\, \mathcal{C}_1(\beta_\kappa r)\, r\, dr}{\tau\, \beta_\kappa^3\, N_\kappa} \tag{3.19}$$

In (3.15) we define

$$c_j(a) \stackrel{d}{=} i \sum_\kappa \frac{\mathcal{A}_{\kappa j}\, \beta_\kappa^3\, \mathcal{C}_0(\beta_\kappa a)}{(\alpha_j^2 + \beta_\kappa^2)^2} \tag{3.20}$$

and substituting (3.19)

$$c_j(a) = i \int_a^b \frac{e(r)}{\tau} \left[\tau \sum_\kappa \frac{\mathcal{C}_0(\beta_\kappa a)\, \mathcal{C}_0(\beta_\kappa r)}{(\alpha_j^2 + \beta_\kappa^2)\, N_\kappa} - 2\alpha_j^2 \sum_\kappa \frac{\mathcal{C}_0(\beta_\kappa a)\, \mathcal{C}_0(\beta_\kappa r)}{(\alpha_j^2 + \beta_\kappa^2)^2\, N_\kappa} \right] r\, dr$$

$$- i \int_a^b \frac{f(r)}{\tau} \left[2\alpha_j^2 \sum_\kappa \frac{\beta_\kappa\, \mathcal{C}_0(\beta_\kappa a)\, \mathcal{C}_1(\beta_\kappa r)}{(\alpha_j^2 + \beta_\kappa^2)^2\, N_\kappa} \right] r\, dr \tag{3.21}$$

The kernels of these integrals can be simplified by determining the convergence value of the series involved in (3.21); this is shown in Appendix III.

The final solution to PM 1 has thus the form

$$\chi = (1-\partial_{ab}) \sum_j \frac{b r \; \mathcal{M}(\alpha_j a | \alpha_j b)}{\mathcal{U}_j} c_j(a) e^{i\alpha_j z} \tag{3.22}$$

where

$$c_j(a) = i \int_a^b -\frac{e(r)}{\tau} \left[\tau \frac{\mathcal{Z}_{b0}(\alpha_j r)}{\alpha_j a \; \mathcal{Z}_{b1}(\alpha_j a)} + \frac{r \, \mathcal{Z}_{b1}(\alpha_j r)}{a \; \mathcal{Z}_{b1}(\alpha_j a)} - \frac{b \, \mathcal{Z}_{b0}(\alpha_j b) \mathcal{Z}_{a0}(\alpha_j r)}{a \, \mathcal{Z}_{a1}(\alpha_j b) \, \mathcal{Z}_{b1}(\alpha_j a)} - \frac{\mathcal{Z}_{b0}(\alpha_j a) \mathcal{Z}_{b0}(\alpha_j r)}{\mathcal{Z}_{b1}^2(\alpha_j a)} \right] r \, dr$$

$$+ i \alpha_j \int_a^b \frac{f(r)}{\tau} \left[\frac{r \, \mathcal{Z}_{b0}(\alpha_j r)}{a \; \mathcal{Z}_{b1}(\alpha_j a)} - \frac{b \, \mathcal{Z}_{b0}(\alpha_j b) \, \mathcal{Z}_{a1}(\alpha_j r)}{a \; \mathcal{Z}_{a1}(\alpha_j b) \mathcal{Z}_{b1}(\alpha_j a)} - \frac{\mathcal{Z}_{b0}(\alpha_j a) \, \mathcal{Z}_{b1}(\alpha_j r)}{\mathcal{Z}_{b1}^2(\alpha_j a)} \right] r \, dr \tag{3.23}$$

and

$$\mathcal{Z}_{\rho\nu}(\alpha_j r) = I_\nu(\alpha_j r) + (-1)^\nu \frac{I_1(\alpha_j \rho)}{K_1(\alpha_j \rho)} K_\nu(\alpha_j r) \tag{3.24}$$

$$\rho = a, b \quad ; \quad \nu = 0, 1$$

Using (2.17) and (2.19), $S = \langle \underset{\sim}{u}, \underset{\sim}{t} \rangle$ is fully determined in the 'stick' contact problem on Π_o .

3.3 THE SECOND MIXED-MIXED PROBLEM: PM 2

To analyze PM 2 as stated in (1.15) and (2.30) we use

$$\chi = \chi_o^2 + \chi_1 \tag{3.25}$$

and develop along parallel lines to PM 1. Here the boundary conditions $(2.30)_{(ii)}$ involve second order derivatives with respect to z so that at Π_o the integral part of the solution is zero. The boundary values $g(r)$ and $h(r)$ can thus be expressed in terms of Dini series.

The final result is then found to be of the form

$$\chi = (1-\partial_{ab})\sum_j \frac{b\,r\,\mathcal{M}(\alpha_j a|\alpha_j b)}{\mathcal{N}_j}\, d_j(a)\, e^{i\alpha_j z} \tag{3.26}$$

where

$$d_j(a) = \int_a^b \frac{g(r)}{\tau}\Bigg[\frac{r\,\mathcal{Z}_{bo}(\alpha_j r)}{a\,\mathcal{Z}_{b1}(\alpha_j a)} - \Bigg(\frac{\tau}{\alpha_j a\ \mathcal{Z}_{b1}(\alpha_j r)} +$$

$$\frac{\mathcal{Z}_{bo}(\alpha_j a)}{\mathcal{Z}_{b1}^2(\alpha_j a)}\Bigg)\mathcal{Z}_{b1}(\alpha_j r) - \frac{b\,\mathcal{Z}_{bo}(\alpha_j b)\mathcal{Z}_{a1}(\alpha_j r)}{a\,\mathcal{Z}_{a1}(\alpha_j b)\mathcal{Z}_{b1}(\alpha_j a)}\Bigg]\, r\,dr +$$

$$+\int_a^b \frac{h(r)}{\tau}\left[\frac{\alpha_j r\, \mathcal{Z}_{b1}(\alpha_j r)}{a\, \mathcal{Z}_{b1}(\alpha_j a)} + \left(\frac{2}{\tau}\quad \frac{\tau}{\alpha_j a\, \mathcal{Z}_{b1}(\alpha_j a)} - \frac{\mathcal{Z}_{bo}(\alpha_j a)}{\mathcal{Z}_{b1}^2(\alpha_j a)}\right)\alpha_j\, \mathcal{Z}_{bo}(\alpha_j r) - \frac{\alpha_j\, b\, \mathcal{Z}_{bo}(\alpha_j b)\, \mathcal{Z}_{ao}(\alpha_j r)}{a\, \mathcal{Z}_{a1}(\alpha_j b)\, \mathcal{Z}_{b1}(\alpha_j a)}\right] r\, dr \tag{3.27}$$

all symbols involved in (3.26) and (3.27) having the same significance as in PM 1. We find $\mathcal{S} = \langle \underset{\sim}{u}, \underset{\sim}{t} \rangle$ for the rigid contact problem by using as before (2.17) and (2.19).

3.4 A PARTIAL SUMMARY

To shorten forthcoming expressions, we define the following symbols

$$\begin{aligned} P_{ja} &\stackrel{d}{=} 1 \,/\, a\, \mathcal{Z}_{b1}(\alpha_j a) \\ Q_{ja} &\stackrel{d}{=} \mathcal{Z}_{bo}(\alpha_j a) \,/\, \mathcal{Z}_{b1}^2(\alpha_j a) \\ R_{ja} &\stackrel{d}{=} \tau \,/\, \alpha_j a\, \mathcal{Z}_{b1}(\alpha_j a) \\ S_{ja} &\stackrel{d}{=} b\, \mathcal{Z}_{bo}(\alpha_j b) \,/\, a\, \mathcal{Z}_{a1}(\alpha_j b)\, \mathcal{Z}_{b1}(\alpha_j a) \end{aligned} \tag{3.28}$$

and

$$
{}_a\underset{\sim}{\mathcal{M}}_j \stackrel{d}{=} \begin{bmatrix} Q_{ja}-R_{ja} & -r P_{ja} & S_{ja} & 0 \\ r P_{ja} & -(Q_{ja}+R_{ja}) & 0 & -S_{ja} \\ \alpha_j r P_{ja} & -\alpha_j Q_{ja} & 0 & -\alpha_j S_{ja} \\ -\alpha_j(Q_{ja}-\frac{2}{\tau}R_{ja}) & \alpha_j r P_{ja} & -\alpha_j S_{ja} & 0 \end{bmatrix} \begin{bmatrix} \mathcal{Z}_{b0}(\alpha_j r) \\ \mathcal{Z}_{b1}(\alpha_j r) \\ \mathcal{Z}_{a0}(\alpha_j r) \\ \mathcal{Z}_{a1}(\alpha_j r) \end{bmatrix} \tag{3.29}
$$

The results obtained so far may then be summarized as:

<u>PM 1</u>: Love's function is of the form

$$
\begin{aligned} \chi &= (1-\ni_{ab})\sum_j \frac{b r\, \mathcal{M}(\alpha_j a|\alpha_j b)}{\mathcal{U}_j}\; c_j(a)\, e^{i\alpha_j z} \\ c_j(a) &= \frac{i}{\tau}\int_a^b [\, e(r)\, {}_a\mathcal{M}_{1j} + f(r)\, {}_a\mathcal{M}_{3j}\,]\, r\, dr \end{aligned} \tag{3.30}
$$

PM 2: Love's function is

$$\chi = (1 - \ni_{ab}) \sum_j \frac{b r \,\mathcal{M}(\alpha_j a | \alpha_j b)}{\mathcal{V}_j} d_j(a)\, e^{i\alpha_j z}$$

$$d_j(a) = \frac{1}{\tau} \int_a^b [g(r)\, {}_a\mathcal{N}_{2j} + h(r)\, {}_a\mathcal{N}_{4j}]\, r\, dr \qquad (3.31)$$

in both solutions, α_j are the eigenvalues computed from (3.11) which have positive imaginary part.

CHAPTER IV

THE GENERAL APPROACH

4.1 NOTATIONAL AGREEMENTS

It is convenient to express all the results obtained so far in terms of the Bessel functions J_ν and Y_ν. To achieve this, the complex parameters α_j will be replaced by an equivalent one denoted by $i\gamma_j$, i.e., we rotate the domain of the eigenvalues by $-(\pi/2)$.

We symbolize by

$$\begin{aligned} H_\nu^{(1)} &\overset{d}{=} J_\nu + i\,Y_\nu \\ H_\nu^{(2)} &\overset{d}{=} J_\nu - i\,Y_\nu \end{aligned} \tag{4.1}$$

the Hankel functions of order ν. For convenience in writing the complex mathematical expressions, we define:

$$\hat{\ell}_\rho \overset{d}{=} \gamma_j\rho\, J_0(\gamma_j\rho) + \tau\, J_1(\gamma_j\rho) \tag{4.2}$$

$$\hat{m}_\rho \overset{d}{=} J_0^2(\gamma_j\rho) + \left(1 - \frac{\tau}{\gamma_j^2\rho^2}\right) J_1^2(\gamma_j\rho) \tag{4.3}$$

$$\hat{n}_\rho \overset{d}{=} \frac{2i}{\pi}\left\{\frac{\tau a}{\gamma_j b^2}\, J_0(\gamma_j a) - \left(1 - \frac{\tau}{\gamma_j^2 b^2}\right) J_1(\gamma_j a)\right\} \tag{4.4}$$

$$\hat{p}_\rho \stackrel{d}{=} \gamma_j \rho\, H_0^{(2)}(\gamma_j \rho) + \tau\, H_1^{(2)}(\gamma_j \rho) \tag{4.5}$$

$$\hat{q}_\rho \stackrel{d}{=} [H_0^{(2)}(\gamma_j\rho)]^2 + \left(1 - \frac{\tau}{\gamma_j^2\rho^2}\right)[H_1^{(2)}(\gamma_j\rho)]^2 \tag{4.6}$$

$$\hat{r} \stackrel{d}{=} \frac{2i}{\pi}\left\{\frac{\tau a}{\gamma_j b^2} H_0^{(2)}(\gamma_j a) - \left(1 - \frac{\tau}{\gamma_j^2 b^2}\right) H_1^{(2)}(\gamma_j a)\right\} \tag{4.7}$$

$$\hat{s}_\rho \stackrel{d}{=} J_0(\gamma_j\rho) H_0^{(2)}(\gamma_j\rho) + \left(1 - \frac{\tau}{\gamma_j^2\rho^2}\right) J_1(\gamma_j\rho)\, H_1^{(2)}(\gamma_j\rho) \tag{4.8}$$

We also recall the Wronskian result

$$H_0^{(1)}(\gamma_j\rho)\, H_1^{(2)}(\gamma_j\rho) - H_1^{(1)}(\gamma_j\rho)\, H_0^{(2)}(\gamma_j\rho) = 4i/\pi\gamma_j\rho \tag{4.9}$$

With these notations, we then have

$$\mathfrak{M}(\alpha_j a|\alpha_j b) = -(\pi^2/4)\ \mathcal{m}(\gamma_j a|\gamma_j b) \tag{4.10}$$

$$\begin{aligned} C(\alpha_j a|\alpha_j b) &\stackrel{d}{=} -(\pi^2/4)\, i\ c_j(a|b) \\ D(\alpha_j a|\alpha_j b) &\stackrel{d}{=} -(\pi^2/4)\, i\ d_j(a|b) \\ E(\alpha_j a|\alpha_j b) &\stackrel{d}{=} (\pi/2)\ e_j(a|b) \\ F(\alpha_j a|\alpha_j b) &\stackrel{d}{=} -(\pi/2)\ f_j(a|b) \end{aligned} \tag{4.11}$$

where

$$\mathcal{M}(\gamma_j a|\gamma_j b) \stackrel{d}{=} c_j(a|b)\ J_0(\gamma_j r)/(\gamma_j r) + d_j(a|b)\ J_1(\gamma_j r)$$
$$+ e_j(a|b)\ H_0^{(2)}(\gamma_j r)/(\gamma_j r) + f_j(a|b)\ H_1^{(2)}(\gamma_j r) \tag{4.12}$$

and

$$c_j(a|b) = \hat{l}_a\,\hat{q}_b - \hat{p}_a\,\hat{s}_b + \hat{r}$$
$$d_j(a|b) = J_1(\gamma_j a)\,\hat{q}_b - H_1^{(2)}(\gamma_j a)\,\hat{s}_b + \frac{2i}{\pi}\,\frac{a}{\gamma_j b^2}\,H_0^{(2)}(\gamma_j a)$$
$$e_j(a|b) = \hat{p}_a\,\hat{m}_b - \hat{l}_a\,\hat{s}_b - \hat{n} \tag{4.13}$$
$$f_j(a|b) = H_1^{(2)}(\gamma_j a)\,\hat{m}_b - J_1(\gamma_j a)\,\hat{s}_b - \frac{2i}{\pi}\,\frac{a}{\gamma_j b^2}\,J_0(\gamma_j a)$$

Likewise

$$\begin{aligned} \mathcal{V}_j &= i\ \frac{\gamma_j^2}{2}\,\frac{d}{d\gamma_j}\,[\det \mathcal{A}] \\ &\stackrel{d}{=} -\frac{\pi^2}{4}\,i\ \frac{\gamma_j^2}{2}\ \frac{d}{d\gamma_j}\ [\hat{\mathcal{A}}] \\ &\stackrel{d}{=} -\frac{\pi^2}{4}\,i\ \hat{\mathcal{V}}_j \end{aligned} \tag{4.14}$$

and here we have

$$\det \mathcal{A} = -\frac{\pi^2}{4}\,ab\,\Big\{\hat{m}_a\,\hat{q}_b + \hat{m}_b\,\hat{q}_a - 2\,\hat{s}_a\,\hat{s}_b$$
$$+ \frac{4}{\pi^2}\,\frac{1}{(\gamma_j ab)^2}\Big[a^2+b^2 - \frac{2\tau}{\gamma_j^2}\Big]\Big\} \tag{4.15}$$

the use of

$$\det \mathcal{A} \doteq -\frac{\pi^2}{4}\hat{\mathcal{A}} \tag{4.16}$$

will avoid numerical multipliers in subsequent formulae.

The eigenvalues are now the roots of

$$\hat{\mathcal{A}} = 0 \tag{4.17}$$

which are tabulated on Appendix IV. It is interesting to remark that the corresponding eigenvalues on the case of the solid circular cylinder, are simply the roots of

$$\hat{m}_b = 0 \tag{4.18}$$

In obtaining the above formulae, we made use of the transformation laws:

$$\begin{aligned} I_\nu(\alpha_j r) &= (i)^\nu J_\nu(\gamma_j r) \\ K_\nu(\alpha_j r) &= -(\pi/2)(i)^{(1-\nu)} H_\nu^{(2)}(\gamma_j r) \end{aligned} \tag{4.19}$$

we also need the following one

$$\mathcal{Z}_{\rho\nu}(\alpha_j r) = (-i)^{(1-\nu)} \mathcal{C}_{\rho\nu}(\gamma_j r)/H_1^{(2)}(\gamma_j \rho) \tag{4.20}$$

where

$$\mathcal{C}_{\rho\nu}(\gamma_j r) = J_\nu(\gamma_j r) Y_1(\gamma_j \rho) - Y_\nu(\gamma_j r) J_1(\gamma_j \rho) \tag{4.21}$$

and in all these expressions $\nu = 0, 1$. Function (4.21) satisfies the boundary conditions $\mathcal{C}_{\rho 1}(\gamma_j \rho) \equiv 0$

identically for $\varrho = a, b$, and $\mathcal{C}_{\varrho o}(\gamma_j \varrho) = -2/\pi\gamma_j\varrho$.

Recalling the definitions introduced in Para. 3.4, we put

$$\begin{aligned}
p_{ja} &\overset{d}{=} 1/a\ \mathcal{C}_{b1}(\gamma_j a) = P_{ja} / H_1^{(2)}(\gamma_j b) \\
q_{ja} &\overset{d}{=} \mathcal{C}_{bo}(\gamma_j a)/\mathcal{C}_{b1}^2(\gamma_j a) = i\ Q_{ja} / H_1^{(2)}(\gamma_j b) \\
r_{ja} &\overset{d}{=} \tau / \gamma_j a\ \mathcal{C}_{b1}(\gamma_j a) = i\ R_{ja} / H_1^{(2)}(\gamma_j b) \qquad (4.22) \\
s_{ja} &\overset{d}{=} b\ \mathcal{C}_{bo}(\gamma_j b) / a\ \mathcal{C}_{a1}(\gamma_j b)\ \mathcal{C}_{b1}(\gamma_j a) \\
&= i\ S_{ja} / H_1^{(2)}(\gamma_j a)
\end{aligned}$$

For convenience it is also advisable to define

$$ {}_a\underset{\sim}{w}_j = \begin{bmatrix} 1 & & & \\ & -i & & \\ & & 1 & \\ & & & -i \end{bmatrix} {}_a\underset{\sim}{\mathcal{W}}_j \qquad (4.23)$$

so that

$$ {}_a\underset{\sim}{w}_j = \begin{bmatrix} -(q_{ja}-r_{ja}) & -r\,p_{ja} & -s_{ja} & 0 \\ -r\,p_{ja} & (q_{ja}+r_{ja}) & 0 & s_{ja} \\ \gamma_j r\,p_{ja} & -\gamma_j q_{ja} & 0 & -\gamma_j s_{ja} \\ \gamma_j(q_{ja}-\frac{2}{\tau} r_{ja}) & \gamma_j r\,p_{ja} & \gamma_j s_{ja} & 0 \end{bmatrix} \begin{bmatrix} \mathcal{C}_{bo}(\gamma_j r) \\ \mathcal{C}_{b1}(\gamma_j r) \\ \mathcal{C}_{ao}(\gamma_j r) \\ \mathcal{C}_{a1}(\gamma_j r) \end{bmatrix} \qquad (4.24)$$

4.2 THE MIXED-MIXED PROBLEMS PM1 AND PM2

In terms of the notations put forward in the preceeding paragraph, we may recollect the results of Chapter III in the forms:

PM1:

$$\chi = (1-\ni_{ab})\sum_j \frac{b\,r\ \mathcal{M}(\gamma_j a|\gamma_j b)\,\hat{c}_j(a)}{\hat{\mathcal{V}}_j}\, e^{-\gamma_j z}$$

$$\hat{c}_j(a) = \frac{1}{\tau}\int_a^b \left[e(r)\,{}_a\mathcal{W}_{1j} + f(r)\,{}_a\mathcal{W}_{3j}\right] r\,dr \tag{4.25}$$

PM 2:

$$\chi = (1-\ni_{ab})\sum_j \frac{b\,r\ \mathcal{M}(\gamma_j a|\gamma_j b)\,\hat{d}_j(a)}{\hat{\mathcal{V}}_j}\, e^{-\gamma_j z}$$

$$\hat{d}_j(a) = \frac{1}{\tau}\int_a^b \left[g(r)\,{}_a\mathcal{W}_{2j} + h(r)\,{}_a\mathcal{W}_{4j}\right] r\,dr \tag{4.26}$$

4.3 THE GENERAL SOLUTION

A general representation of Love's function for circular hollow semi-inifinite cylindrical regions can now be constructed based on solutions (4.25) and (4.26). To this end we take:

$$\chi = (1-\ni_{ab})\sum_j \frac{b\,r\ \mathcal{M}(\gamma_j a|\gamma_j b)\,\hat{a}_j(a)}{\hat{\mathcal{V}}_j}\, e^{-\gamma_j z} \tag{4.27}$$

$$\hat{a}_j(a) \overset{d}{=} \hat{c}_j(a) + \hat{d}_j(a)$$

$$= \frac{1}{\tau}\int_a^b \big[e(r)\, {}_a w_{1j} + f(r)\, {}_a w_{3j} + g(r)\, {}_a w_{2j} + h(r)\, {}_a w_{4j} \big]\, r\, dr \tag{4.28}$$

Using (4.27) in the evaluation of formulae (2.17) and (2.19) we obtain:

$$t_{zz} = (1-\backepsilon_{ab}) \sum_k \frac{\hat{a}_k(a)\, {}_a\mathcal{R}_{1k}}{\hat{\mathcal{V}}}\, e^{-\gamma_k z} \tag{4.29}$$

$$u_r = \frac{1}{2\mu}(1-\backepsilon_{ab}) \sum_k \frac{\hat{a}_k(a)\, {}_a\mathcal{R}_{3k}}{\hat{\mathcal{V}}}\, e^{-\gamma_k z} \tag{4.30}$$

$$t_{rz} = (1-\backepsilon_{ab}) \sum_k \frac{\hat{a}_k(a)\, {}_a\mathcal{R}_{2k}}{\hat{\mathcal{V}}}\, e^{-\gamma_k z} \tag{4.31}$$

$$u_z = \frac{1}{2\mu}(1-\backepsilon_{ab}) \sum_k \frac{\hat{a}_k(a)\, {}_a\mathcal{R}_{4k}}{\hat{\mathcal{V}}}\, e^{-\gamma_k z} \tag{4.32}$$

The values $\bar{t}_{zz}, \bar{u}_r, \bar{t}_{rz}, \bar{u}_z$ of these components at the plane boundary Π_o are derived from the above by putting $z = 0$.

For shortness we have defined for use in the above expressions:

$$a\,\underset{\sim}{\mathcal{Y}}_k \triangleq \gamma_k\, b \begin{bmatrix} \gamma_k[c_k-(2+\tau)d_k] & \gamma_k^2 r\, d_k & \gamma_k[e_k-(2+\tau)f_k] & \gamma_k^2 r\, f_k \\ -\gamma_k^2 r\, d_k & \gamma_k[c_k-\tau d_k] & -\gamma_k^2 r\, f_k & \gamma_k[e_k-\tau f_k] \\ \gamma_k r\, d_k & -c_k & \gamma_k r\, f_k & -e_k \\ -[c_k-2\tau d_k] & -\gamma_k r\, d_k & -[e_k-2\tau f_k] & -\gamma_k r\, f_k \end{bmatrix} \begin{bmatrix} J_0(\gamma_k r) \\ J_1(\gamma_k r) \\ H_0^{(2)}(\gamma_k r) \\ H_1^{(2)}(\gamma_k r) \end{bmatrix} \tag{4.33}$$

where c_k, d_k, e_k, f_k are defined like in (4.13).

4.4 SOLUTION OF SPECIFIC BOUNDARY CONDITIONS

We return to equation (4.28) which can be expressed as

$$\hat{a}_j(\rho) = \frac{1}{\tau}\int_a^b [\, \bar{E}_{zz}\ {}_\rho w_{1j} + 2\mu\ \bar{U}_r\ {}_\rho w_{3j}]\, r\, dr$$
$$+ \frac{1}{\tau}\int_a^b [\, \bar{E}_{rz}\ {}_\rho w_{2j} + 2\mu\ \bar{U}_z\ {}_\rho w_{4j}]\, r\, dr \tag{4.34}$$

i.e., the coefficients involved on the general representation are expressed as an integral equation in the non-zero components of the stress tensor and displacement vector at the plane boundary Π_o. However, only a pair of these four components may be chosen to be arbitrary self-equilibrating stresses or displacements.

The form (4.34) is appropriate to discuss the particular problems PM1 and PM2. In the first problem we denote the first integral in (4.34), the one containing the specified boundary conditions, by ${}_\rho\mathcal{G}_j$, while the same symbol will represent the second integral when discussing PM2. We also define the hybrid integrals, ($\rho_1, \rho_2 = a, b$):

$$\mathcal{J}_{jk}^{\langle\rho_1,\rho_2\rangle} \triangleq \frac{1}{\tau\,\hat{\mathcal{V}}_k}\int_a^b [\,{}_{\rho_1}w_{2j}\ {}_{\rho_2}\mathcal{X}_{2k} + {}_{\rho_1}w_{4j}\ {}_{\rho_2}\mathcal{X}_{4k}]\, r\, dr \tag{4.35}$$

$$\mathcal{J}_{jk}^{<\rho_1,\rho_2>} \triangleq \frac{1}{\tau \hat{\mathcal{V}}_k} \int_a^b [{}_{\rho_1}w_{1j}\, {}_{\rho_2}\mathcal{r}_{1k} + {}_{\rho_1}w_{3j}\, {}_{\rho_2}\mathcal{r}_{3k}]\, r\, dr \tag{4.36}$$

The determination of $\hat{a}_j(\rho)$ is thus reduced in each case to solving a doubly infinite set of linear equations, namely

PM1

$$\hat{a}_j(\rho) = {}_\rho g_j + \sum_k \hat{a}_k(a)\, \mathcal{J}_{jk}^{<\rho,a>} - \sum_k \hat{a}_k(b)\, \mathcal{J}_{jk}^{<\rho,b>} \tag{4.37}$$

PM2

$$\hat{a}_j(\rho) = {}_\rho g_j + \sum_k \hat{a}_k(a)\, \mathcal{J}_{jk}^{<\rho,a>} - \sum_k \hat{a}_k(b)\, \mathcal{J}_{jk}^{<\rho,b>} \tag{4.38}$$

where as always $\rho =$ a,b.

In analogy with the solid cylinder solution discussed by CHILDS [1966], the w and $\mathcal{r}$ eigenfunctions are expected to constitute a biorthogonal countably infinite system such that

$$\mathcal{J}_{jk}^{<\rho_1,\rho_2>} = \frac{1}{2}\, \delta_{jk} = -\, \mathcal{J}_{jk}^{<\rho_1,\rho_2>} \tag{4.39}$$

in which case, proving the completeness of the representation for the mixed-mixed class of problems is automatic.

In PT (the traction problem) we are expected to solve

$$\hat{a}_j(\rho) = {}_\rho\mathcal{G}_j + \sum_k \hat{a}_k(a)\, \mathcal{R}_{jk}^{<\rho,a>} - \sum_k \hat{a}_k(b)\, \mathcal{R}_{jk}^{<\rho,b>} \qquad (4.40)$$

with

$$\mathcal{R}_{jk}^{<\rho_1,\rho_2>} \triangleq \frac{1}{\tau\,\hat{\mathcal{V}}} \int_a^b [{}_{\rho_1}w_{3j}\;{}_{\rho_2}\mathcal{R}_{3k} + {}_{\rho_1}w_{4j}\;{}_{\rho_2}\mathcal{R}_{4k}]\, r\, dr \qquad (4.41)$$

and ${}_\rho\mathcal{G}_j$ containing the prescribed boundary values. However, in this case no similar biorthogonality is expected.

The labor involved in verifying (4.39) is not trivial, the large number of integrals required and terms involved indicate that numerical evaluation is the only feasible method of verification.

The infinite systems (4.37), (4.38) and (4.40) should be soluble by truncation to obtain values of $\hat{a}_j(\rho)$ to any desired degree of accuracy as shown in the similar problem discussed by LITTLE and CHILDS [1967].

The less important mixed problem PM3 leads to a system of equations of the same form as (4.40). From another point of view this problem is discussed in

Appendix I (Para. A I.2).

To have only real values for the stress tensor and displacement vector, we have to proceed as in equation (A I.6) using the complex conjugates of the eigenvalues. It is useful to record here, in closed form, the integrals which appear in the evaluation of $\hat{a}_j(\rho)$. Some of these, for Bessel functions of order zero were given by PEAVY [1967]; however, by his method one is required to compute Struve functions of first kind, and use complicated polynomial expression.

Let $\mathscr{C}_{\rho\nu}(\gamma_j r)$ be the cylindrical function defined in (4.21) then:

$$I_{n\rho} \stackrel{d}{=} \int_a^b r^n \mathscr{C}_{\rho\nu}(\gamma_j r)\, dr$$

$$= (-1)^{n-1}\left\{\left[r^n \mathscr{C}_{\rho\mu}(\gamma_j r)\right]_a^b - (n-1)\, I_{(n-1)\rho}\right\} \tag{4.42}$$

$$\nu = \left[(-1)^n + 1\right]/2 \qquad , \qquad \mu = \left[(-1)^{n-1} + 1\right]/2$$

with $n = 1,2,3 \ldots$.

If $\mathscr{D}_\nu(\gamma_K r)$ is any of the Bessel or Hankel functions of order $\nu = 0,1$, we have:

$$
\begin{aligned}
\mathrm{I}_{\nu\rho} &\stackrel{d}{=} \int_a^b r\, \mathscr{C}_{\rho\nu}(\gamma_j r)\, \mathscr{D}_\nu(\gamma_k r)\, dr \\
&= \frac{(-1)^\nu}{\gamma_j^2-\gamma_k^2}\left[\gamma_j r\, \mathscr{C}_{\rho(1-\nu)} \mathscr{D}_\nu - \gamma_k r\, \mathscr{C}_{\rho\nu} \mathscr{D}_{(1-\nu)}\right]
\end{aligned}
\tag{4.43}
$$

also

$$
\begin{aligned}
\mathrm{II}_{\nu\rho} &\stackrel{d}{=} \int_a^b r^2\, \mathscr{C}_{\rho(1-\nu)}(\gamma_j r)\, \mathscr{D}_\nu(\gamma_k r)\, dr \\
&= \frac{(-1)^{(1-\nu)}}{\gamma_j^2-\gamma_k^2}\Big\{\left[\gamma_j r^2\, \mathscr{C}_{\rho\nu} \mathscr{D}_\nu + \gamma_k r^2\, \mathscr{C}_{\rho(1-\nu)} \mathscr{D}_{(1-\nu)}\right]_a^b \\
&\qquad - 2\,(\gamma_j)^{(1-\nu)}(\gamma_k)^\nu\, \mathrm{I}_{o\rho}\Big\}
\end{aligned}
\tag{4.44}
$$

and

$$
\begin{aligned}
\mathrm{III}_{\nu\rho} &\stackrel{d}{=} \int_a^b r^3\, \mathscr{C}_{\rho\nu}(\gamma_j r)\, \mathscr{D}_\nu(\gamma_k r)\, dr \\
&= \frac{(-1)^\nu}{\gamma_j^2-\gamma_k^2}\Big\{\left[\gamma_j r^3\, \mathscr{C}_{\rho(1-\nu)} \mathscr{D}_\nu - \gamma_k r^2\, \mathscr{C}_{\rho\nu} \mathscr{D}_{(1-\nu)}\right]_a^b \\
&\qquad - 2\left[\gamma_j\, \mathrm{II}_{\nu\rho} - \gamma_k\, \mathrm{II}_{(1-\nu)\rho}\right]\Big\}
\end{aligned}
\tag{4.45}
$$

which were derived by suitable integrations by parts. Of course, the formulae still apply when $\mathscr{C}_{\rho\nu}$ is any Bessel or Hankel function of order $\nu = 0,1$ and not just the one given in (4.21).

Formula (4.42) is used on the evaluation of ${}_{\rho}\mathscr{G}_{j}$ while the remaining ones appear in (4.35,36,41).

CHAPTER V

CONCLUDING REMARKS

5.1 SUMMARY

The problem of a semi-infinite circular hollow cylinder was considered, where the loading (assumed self-equilibrated) is applied at the finite end. The method uses Love's stress function with a suitable representation in the form of a Fourier-Bessel series and a coupled Fourier integral. We arrive at the general solution by a linear superposition of the solutions to two particular mixed-mixed problems. Eigenvalues and eigenfunctions are then obtained and the problem is formally reduced to the solution of a doubly infinite linear system of algebraic equations which can probably be solved by truncation. The eigenvalues are tabulated for certain values of the inner radius and Poisson's ratio in the case of a normalized outer radius. The labor involved in actually obtaining the stresses and displacements indicates that the problem can only be solved numerically, all the necessary tools for this being given in the text.

For the mixed-mixed problems at the plane end, a biorthogonality is expected which should considerably simplify the computations. However, this will not be the case in the displacement and traction problems.

5.2 CONCLUSIONS

The method employed can be extended to the case of an axisymmetric hollow body, albeit the solution will require a tremendous numerical effort.

For the axisymmetric case it is shown that no real or purely imaginary non-zero eigenvalues γ_j exist, and that if self-equilibrated loading is assumed that the solution is unique.

Write $\gamma_j = \mathcal{R}e[\gamma_j] + i\,\mathcal{I}m[\gamma_j]$ and let ω denote the least value of $\mathcal{R}e[\gamma_j]$ in the sequence of eigenvalues for a given section. Then $\omega^2 A(\Pi_o)$, where $A(\Pi_o)$ is the surface area of the end section, depends only on the shape of the section. For arbitrary sections $\omega^2 A(\Pi_o)$ forms a positive sequence. If Π_o is a circular annulus with fixed outer radius $\underline{b}$, $\omega^2 A(\Pi_o)$ is maximum for the solid cylinder and goes to zero steadily with $\underline{a} \to \underline{b}$. However, $|\omega|$ has a minimum in the open interval (0.4, 0.5) for b = 1.

Now suppose $A(\Pi_o)$ is fixed, it would be interesting to have an answer to the question: is the sequence $\omega^2 A(\Pi_o)$ bounded from below, and, if so, is the lower bound the one given by the solid cylinder or is there a hollow one with critical radii a*, b* with such attribute?

The last question has a definite importance from a practical viewpoint because $|\omega|$ presents the rate at which end effects decay as we pass along the cylinder ($|\omega|$ is inversely proportional to $\varsigma_c(\ell)$ in equation (1.1)). The greater $|\omega|$, the more rapid the decay. In engineering we are concerned with the end effects, because the Saint-Venant relaxed solutions (which have been proved to furnish an absolute minimum of the total strain energy by STERNBERG and KNOWLES [1966], except in the flexure case) give no information about them. The assignment of such lower bound might be more useful than the description of a complicated process for the evaluation of the eigenvalues.

Also, we conjecture that the energy characterization of the exact solution would coincide with the one for the relaxed solution.

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APPENDIX

APPENDIX I

SOME ASPECTS OF SAINT-VENANT PROBLEMS

A I.1 FORMULATION AND REDUCTION TO AN EIGENVALUE PROBLEM

Consider a semi-infinite cylindrical body of homogeneous isotropic elastic material. The cross section Π_ξ is arbitrary; it may be simply or multiply connected; the lateral bounding surface being denoted by Γ and the plane end section by Π_o. For convenience we deal with rectangular Cartesian coordinates and Cartesian tensors; the axis x_3 will be taken parallel to the generators of the cylinder. Latin suffixes have the range 1, 2, 3 while Greek ones are restricted to the values 1, 2.

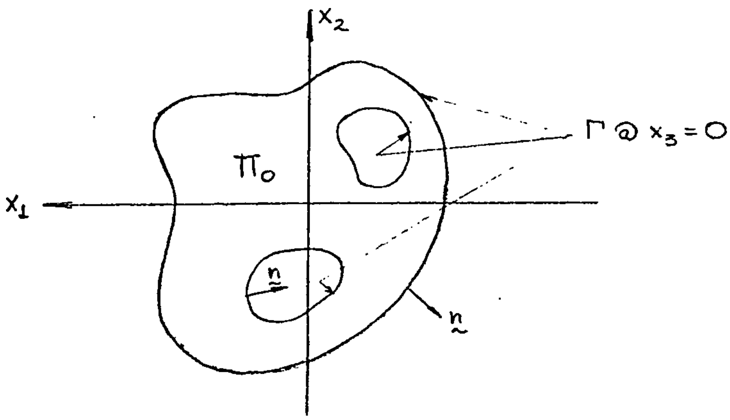

Fig. A I.1. Notations

Two formulations of the problem of SAINT-VENANT can be set out, one in terms of displacements, the other in terms of stresses. Accordingly, we have, SYNGE [1945]:

I. Stress formulation

(i) differential equations

$$t_{ij,j} = 0$$

$$(1+\sigma)\Delta_3 t_{ij} + t_{kk,ij} = 0$$

respectively the equilibrium and Beltrami-Michell compatibility equations. Here $\Delta_3 = \partial^2/\partial x_k \partial x_k$ is the 3-d Laplace operator.

(ii) boundary conditions

$$t_{\alpha\beta} n_\beta = 0 \ , \ t_{3\beta} n_\beta = 0 \qquad \text{on} \quad \Gamma$$

$$t_{33} = \overline{T}_3 \ , \ t_{3\alpha} = \overline{T}_\alpha \qquad \text{on} \quad \Pi_0$$

$\underset{\sim}{n}$ being the unit normal to Γ .

II. Displacement formulation

(i) differential equations

$$(1-2\sigma)\Delta_3 u_i + u_{j,ji} = 0$$

which are the Navier equations.

(ii) boundary conditions

$$\left\{\begin{array}{l} (u_{\beta,3} + u_{3,\beta}) n_\beta = 0 \qquad \text{on} \quad \Gamma \\ 2\sigma u_{k,k} n_\alpha + (1-2\sigma)(u_{\beta,\alpha} + u_{\alpha,\beta}) n_\beta = 0 \end{array}\right.$$

$$\left\{\begin{array}{l} \sigma u_{k,k} + (1-2\sigma) u_{3,3} = (1+\sigma)(1-2\sigma)\overline{T}_3 \\ u_{3,\alpha} + u_{\alpha,3} = 2(1+\sigma)\overline{T}_\alpha \end{array}\right. \quad \text{on } \Pi_0$$

The first formulation has the advantage of having simple boundary conditions while the second has simpler equations. It is natural from the geometry of the problem to introduce what is called the "exponential condition" (transitory free mode in DOUGALL [1912, 1914] terminology). We then assume that

$$t_{ij} = e^{\kappa x_3}\, \mathcal{T}(x_1, x_2) \tag{A I.1}$$

and since stress determines displacements to within a rigid body displacement, we may write the corresponding displacement in the form

$$u_i = e^{\kappa x_3}\, \mathcal{U}_i(x_1, x_2) \tag{A I.2}$$

The displacement formulation contains simpler partial differential equations, and by substituting (A I.2) into II(1) we obtain after algebraic manipulations:

$$\begin{aligned} (\Delta_2 + \kappa^2)\, \mathcal{U}_\alpha &= -\mathcal{V}_{,\alpha} \\ (\Delta_2 + \kappa^2)\, \mathcal{V} &= 0 \end{aligned} \tag{A I.3}$$

where $\Delta_2 = \partial^2/\partial x_\alpha \partial x_\alpha$, $\mathcal{V}(x_1, x_2)$ is an auxiliary function defined by

$$\mathcal{V} \stackrel{d}{=} \frac{1}{1-2\sigma}\left(\kappa\, \mathcal{U}_3 + \mathcal{U}_{\alpha,\alpha}\right) \tag{A I.4}$$

and to derive $(\mathrm{A\ I.3})_2$ we took advantage of the fact that the components of the displacement field in Cartesian coordinates are biharmonic functions in the absence of body forces.

The lateral boundary conditions II(ii) take the form

$$\begin{aligned} &2\sigma \mathcal{V} n_\alpha + (\mathcal{U}_{\beta,\alpha} + \mathcal{U}_{\alpha,\beta}) n_\beta = 0 \\ &(1-2\sigma)\mathcal{V}_{,\beta} n_\beta + \kappa^2 \mathcal{U}_\beta n_\beta - \mathcal{U}_{\beta,\beta\gamma} n_\gamma = 0 \end{aligned} \quad \text{on } \Pi_0 \qquad (\mathrm{A\ I.5})$$

We have thus a complex eigenvalue problem to solve, the system is consistent only for certain values of κ. There is no objection to complex eigenvalues, generating complex solutions for $\mathcal{V}, \mathcal{U}_\alpha$ and the corresponding stresses, for in such cases to have real values for $\underset{\sim}{t}$ and $\underset{\sim}{v}$ we should take

$$\begin{aligned} v_i &= \frac{1}{2}\left(e^{\kappa x_3}\mathcal{U}_i + e^{\bar{\kappa} x_3}\,\bar{\mathcal{U}}_i\right) \\ t_{ij} &= \frac{1}{2}\left(e^{\kappa x_3}\mathcal{T}_{ij} + e^{\bar{\kappa} x_3}\,\bar{\mathcal{T}}_{ij}\right) \end{aligned} \qquad (\mathrm{A\ I.6})$$

where superposed bars denote conjugates. If κ is an eigenvalue of the problem so also are $-\kappa$, and $\pm\bar{\kappa}$. In fact the eigenvalues occur in sets of two if they are real or purely imaginary and in sets of four if complex. That no purely imaginary eigenvalues should exist has been shown by DOUGALL [1912], his argument is so concise and elegant that we may well transcribe it: a purely imaginary κ implies a periodic distribution of displacement and stress; consider the energy stored in a length of the cylinder equal to this period; it is equal to the work done by the terminal stress in passing from the natural state to the strained configuration, but, from the periodicity, this is zero. Hence, the energy of a strained state is zero, and for $-1 < \sigma < 1/2$ this is contrary to a basic postulate in linear elasticity.

A I.2 MIXED PROBLEM PM 3

Consider a hollow cylinder with the geometry of Fig. 1.1 and let us investigate the mixed problem PM 3 of Par. 1.3, namely

(i) Traction problem on $\Gamma_1 \cup \Gamma_2$

$$\underset{\sim}{T}\Big|_{\Gamma_1 \cup \Gamma_2} = \underset{\sim}{0}$$

(ii) Displacement problem on Π_0

$$\underset{\sim}{u}\Big|_{\Pi_0} = \underset{\sim}{\bar{u}}$$

We note that as formulated the degree of indeterminacy of the problem does not lie within a class of rigid body displacements, and this will be shown in Par. A I.3.

Navier's equations in cylindrical coordinates in the axisymmetric case read (MARGUERRE [1955; p. 248]) after trivial manipulations

$$\tau \frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(r u_r)\right] + (\tau - 1)\frac{\partial^2 u_r}{\partial z^2} + \frac{\partial^2 u_z}{\partial r \partial z} = 0$$

(A I.7)

$$\frac{1}{\tau}\frac{\partial}{\partial z}\left[\frac{1}{r}\frac{\partial}{\partial r}(r u_r)\right] + \frac{\tau - 1}{\tau}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) + \frac{\partial^2 u_z}{\partial z^2} = 0$$

Using the fact that the dilatation

$$I_{\underset{\sim}{u}} = \frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{\partial u_z}{\partial z}$$

is a harmonic function, u_z can be eliminated from equations (A I.7) and we arrive at

$$\left[\Delta - \frac{1}{r^2}\right]^2 u_r = 0 \qquad \text{(A I.8)}$$

Let us assume a solution in the form (A I.2), for this purpose we take

$$u_r = e^{i\alpha z}\,\mathcal{R}(r) \qquad \text{(A I.9)}$$

and (A I.8) becomes

$$\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right) - \frac{1}{r^2} - \alpha^2\right]^2 \mathcal{R} = 0 \qquad \text{(A I.10)}$$

We remark that (A I.9) and (A I.10) are equivalent to invoke separation of variables, namely,

$$
\begin{aligned}
&u_r = \mathcal{R}(r)\,\mathcal{Z}(z) \\
&\frac{d^2\mathcal{Z}}{dz^2} = -\alpha^2\,\mathcal{Z} \qquad\qquad \text{(A I.11)}\\
&\left[\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}\right)r - \alpha^2\right]^2 \mathcal{R} = 0
\end{aligned}
$$

where (A I.11)$_3$ is just (A I.10) written in a more convenient fashion. We recall that here $i\alpha$ plays the role of κ in the previous paragraph.
(A I.11)$_3$ has for solution, YIH [1956]:

$$
\mathcal{R}(r) = A\,I_1(\alpha r) + B\,\alpha r\, I_0(\alpha r) + C\,K_1(\alpha r) + D\,\alpha r\, K_0(\alpha r) \qquad \text{(A I.12)}
$$

which substituted into (A I.9) gives $u_r(r,z)$.
Going back to equation (A I.7)$_1$ is an easy matter to solve for $u_z(r,z)$, then equation (A I.7)$_2$ gives the conditions on the undetermined functions obtained in the first integration, the final result is then

$$
\begin{aligned}
u_z = i\,e^{i\alpha z}\{ &A\,I_0(\alpha r) + B\,[2\tau I_0(\alpha r) + \alpha r\, I_1(\alpha r)] \qquad \text{(A I.13)}\\
&- C\,K_0(\alpha r) + D\,[2\tau\, K_0(\alpha r) - \alpha r\, K_1(\alpha r)]\}
\end{aligned}
$$

We now try to satisfy the stress free condition on $\Gamma_1 \cup \Gamma_2$; for this purpose we compute

$$\frac{1}{2\mu} t_{rr} = \frac{1}{2(\tau-1)} \left\{ \tau \frac{\partial u_r}{\partial r} + (2-\tau)\left[\frac{u_r}{r} + \frac{\partial u_z}{\partial r} \right] \right\}$$

$$= \alpha e^{i\alpha z} \left\{ A \left[I_0(\alpha r) - \frac{I_1(\alpha r)}{\alpha r} \right] + \right.$$

$$B[(\tau-1) I_0(\alpha r) + \alpha r\, I_1(\alpha r)] - C\left[K_0(\alpha r) + \right. \qquad \text{(A I.14)}$$

$$\left. \left. \frac{K_1(\alpha r)}{\alpha r} \right] + D[(\tau-1) K_0(\alpha r) - \alpha r\, K_1(\alpha r)] \right\}$$

and

$$\frac{1}{2\mu} t_{rz} = \frac{1}{2} \left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z} \right)$$

$$= i\alpha e^{i\alpha z} \left\{ A\, I_1(\alpha r) + B[\alpha r\, I_0(\alpha r) + \right.$$

$$\text{(A I.15)}$$

$$\tau\, I_1(\alpha r)] + C\, K_1(\alpha r) + D[\alpha r\, K_0(\alpha r) -$$

$$\left. \tau\, K_1(\alpha r)] \right\}$$

These stresses are required to vanish for all z at $r = a, b$. We have thus a homogeneous system of linear equations in A, B, C, D; the condition for the existence of a nontrivial solution requires that the determinant of the coefficients vanish. This is then the characteristic equation which gives the eigenvalues $i\alpha$. Explicitly written, the above mentioned determinant coincides with the determinant of the coefficients' matrix in (3.8), i.e., the eigenvalues satisfy

$$-\alpha^2 \det \mathcal{F}(\alpha) = 0 \qquad \text{(A I.16)}$$

We note thus, that the problems discussed in Chap. III and the one being discussed have, as expected, the same spectrum.

Problem PM 3 could be discussed further along the lines of the previous paragraph. For instance the homogeneity of the system implies that not all constants are independent, consequently the displacements can be expressed in the form (A I.6) with only two essential constants, the others being the corresponding complex conjugates. These constants would have, of course, to be determined by the boundary conditions along Π_o, but these would have to be expanded in terms of eigenfunctions. However, due to the nature of the problem these will be nonorthogonal and such representation will be difficult to obtain. The E. Schmidt's orthogonalization process (see e.g., SCHMEIDLER [1965; p. 14]) could be used to advantage, however the labor involved is considerable and satisfactory results are not warranted.

A I.3 THE ZERO EIGENVALUE

Let $\alpha = 0$ in (A I.11) the system to be solved is thus

$$u_r^o = \mathcal{R}_o(r)\, \mathcal{Z}_o(z)$$

$$\frac{d^2 \mathcal{Z}_o}{dz^2} = 0 \qquad \text{(A I.17)}$$

$$\left(\frac{d^4}{dr^4} + \frac{2}{r}\frac{d^3}{dr^3} - \frac{3}{r^2}\frac{d^2}{dr^2} + \frac{3}{r^2}\frac{d}{dr} - \frac{3}{r^4}\right)\mathcal{R}_o = 0$$

the solution to $(\text{A I.17})_2$ is trivial

$$\mathcal{Z}_o(z) = a_o z + b_o \qquad \text{(A I.18)}$$

To solve $(\text{A I.17})_3$ we note that it is a homogeneous linear differential equation which can be transformed in one with constant coefficients by introducing a new variable t such that $t = \ln r$, which is easily solvable. Return to the current variables gives

$$\mathcal{R}_o(r) = c_o r^3 + d_o r + e_o r \ln r + f_o r^{-1} \qquad \text{(A I.19)}$$

Substituting $(\text{A I.17})_1$ as given in terms of (A I.18) and (A I.19) into $(\text{A I.7})_1$ and solving for u_z, we obtain

$$u_z = -\tau\left(a_o \frac{z^2}{2} + b_o z\right)\left(4 c_o r^2 + 2 e_o \ln r + g_o\right) + h_o \qquad \text{(A I.20)}$$

where a_o through h_o are integration constants (we may remark that we are not interested in checking the compatibility of these solutions, the discussion is not affected by this). In order that these displacements do not increase beyond bounds with z we must choose; $c_o = e_o = g_o = 0$ and $a_o = 0$. Then:

$$u_r = k_o r + l_o r^{-1}$$
$$u_z = h_o \tag{A I.21}$$

that is, u_r is composed of a uniform expansion, associated with a uniformly distributed body force over the cylinder, and an "eversion" which has no meaning for the case of solid cylinders. The axial displacement is a rigid body translation in the direction of z. To remove such solutions we may either redefine the stresses and displacements or, regard the prescribed distributions of the displacements (or stresses) as arising from a self-equilibrated traction system at the plane end.

Since the last assumption was made in our derivations, this justifies the noninclusion of the contribution of the zero eigenvalue.

A I.4 THE NON-EXISTENCE OF REAL EIGENVALUES

Using $\alpha = i\gamma$, the determination of the eigenvalues is equivalent to finding the roots of the equation

$$\hat{m}_a \hat{q}_b + \hat{m}_b \hat{q}_a - 2\hat{s}_a \hat{s}_b + \frac{4}{\pi^2}\frac{1}{(\gamma ab)^2}\left[a^2 + b^2 - \frac{2\tau}{\gamma^2}\right] = 0 \tag{A I.22}$$

where

$$\begin{aligned} \hat{m}_\rho &\stackrel{d}{=} J_0^2(\gamma\rho) + \left(1 - \frac{\tau}{\gamma^2\rho^2}\right) J_1^2(\gamma\rho) \\ \hat{q}_\rho &\stackrel{d}{=} \left[H_0^{(2)}(\gamma\rho)\right]^2 + \left(1 - \frac{\tau}{\gamma^2\rho^2}\right)\left[H_1^{(2)}(\gamma\rho)\right]^2 \\ \hat{s}_\rho &\stackrel{d}{=} J_0(\gamma\rho)\, H_0^{(2)}(\gamma\rho) + \left(1 - \frac{\tau}{\gamma^2\rho^2}\right) J_1(\gamma\rho)\, H_1^{(2)}(\gamma\rho) \end{aligned} \tag{A I.23}$$

The corresponding equation for the solid cylinder is simply

$$\hat{m}_b = 0 \tag{A I.24}$$

as given e.g. by CHILDS [1966].

We show first that (A I.24) has no real solutions, and since DOUGALL (cf. A I.1) has shown that no purely imaginary eigenvalue exists, we conclude that only complex roots with non-zero real and imaginary parts have to be found.

Suppose γ is a real number, from WATSON [1944; p. 147] we have

$$J_\nu^2(\gamma\rho) = \sum_{k=0}^{\infty} \frac{(-1)\,(2k+2\nu)!\,\left(\frac{\gamma\rho}{2}\right)^{2(k+\nu)}}{k!\,(k+2\nu)!\,[(k+\nu)!]^2} \tag{A I.25}$$

Using this result for $\nu = 0, 1$ and suitable re-defining the dummy index we obtain after trivial manipulations:

$$\hat{m}_\rho = \sum_{k=0}^{\infty} \left[\frac{k^2 + (3-\tau)k + (2-\tau/2)}{(k+2)(k+1)} \right] \frac{(-1)^k \left(\frac{\gamma\rho}{2}\right)^{2k} (2k)!}{(k+1)! \ (k!)^3} \tag{A I.26}$$

Provided $\tau < 4, (\sigma > -1)$ the square bracketed quantity is a positive one for any k . Since the series is absolutely convergent we may rearrange its terms, grouping then the first term with the second, the third with the fourth and so on, we discover that each of these combinations is non-negative. Thus if γ is real $\hat{m}_\rho$ is a sum of non-zero positive terms.

In the case of the hollow cylinder, i.e., equation (AI.22) the same can be proved with a little more labor with the help of the expansions for the products $J_\nu Y_\nu$ as given in WATSON [1944; p.150]. We may also remark that equation (AI.22) can be re-written as:

$$\hat{m}_a \hat{v}_b + \hat{m}_b \hat{v}_a - 2\hat{t}_a \hat{t}_b - \frac{4}{\pi^2} \frac{1}{(\gamma a b)^2} \left[a^2 + b^2 - \frac{2\tau}{\gamma^2} \right] = 0 \tag{A I.27}$$

and an order of magnitude analysis shows that for real γ the l.h.s. of equation (AI.27) is positive definite; where we have defined

$$\hat{t}_\rho = Y_0^2(\gamma\rho) + \left(1 - \frac{\tau}{\gamma^2\rho^2}\right) Y_1^2(\gamma\rho) \tag{A I.28}$$

APPENDIX II

A II.1 THE COEFFICIENTS $C(\alpha a \mid \alpha b)$ ETC IN (3.13)

We define

$$[-\alpha^2 \det \mathcal{F}(\alpha)]\; C \overset{d}{=} -(1+\partial_{ab})\,[\alpha b\, C(\alpha a|\alpha b)\, \mathcal{B}(a)] \tag{A II.1}$$

identical expression being valid for D, E and F.

Next we recall the Wronskian result (WATSON [1944; p. 79])

$$W\{K_0(\alpha\rho), I_0(\alpha\rho)\} = \det \begin{bmatrix} I_0(\alpha\rho) & -K_0(\alpha\rho) \\ I_1(\alpha\rho) & K_1(\alpha\rho) \end{bmatrix} = \frac{1}{\alpha\rho} \tag{A II.2}$$

and also define

$$\begin{aligned}
\ell_\rho &\overset{d}{=} \alpha\rho\, I_0(\alpha\rho) + \tau\, I_1(\alpha\rho) \\
m_\rho &\overset{d}{=} I_0^2(\alpha\rho) - \left(1 + \frac{\tau}{\alpha^2\rho^2}\right) I_1^2(\alpha\rho) \\
p_\rho &\overset{d}{=} \alpha\rho\, K_0(\alpha\rho) - \tau\, K_1(\alpha\rho) \\
q_\rho &\overset{d}{=} K_0^2(\alpha\rho) - \left(1 + \frac{\tau}{\alpha^2\rho^2}\right) K_1^2(\alpha\rho) \\
s_\rho &\overset{d}{=} I_0(\alpha\rho) K_0(\alpha\rho) + \left(1 + \frac{\tau}{\alpha^2\rho^2}\right) I_1(\alpha\rho) K_1(\alpha\rho)
\end{aligned} \tag{A II.3}$$

We have then the following expressions for $C(\alpha a|\alpha b)$ etc:

$$C(\alpha a|\alpha b) = \rho_a q_b - p_a s_b - \frac{\tau}{\alpha^2 b^2}[\alpha a K_0(\alpha a) + K_1(\alpha a)] - K_1(\alpha a)$$

$$D(\alpha a|\alpha b) = I_1(\alpha a) q_b + K_1(\alpha a) s_b - \frac{a}{\alpha b^2} K_0(\alpha a)$$

$$E(\alpha a|\alpha b) = p_a m_b - \rho_a s_b + \frac{\tau}{\alpha^2 b^2}[\alpha a I_0(\alpha a) + I_1(\alpha a)] + I_1(\alpha a)$$

$$F(\alpha a|\alpha b) = K_1(\alpha a) m_b + I_1(\alpha a) s_b - \frac{a}{\alpha b^2} I_0(\alpha a) \tag{A II.4}$$

The linear terms in the modified Bessel functions appearing in these expressions are due to theorem (A II.2).

APPENDIX III

THE SERIES EXPANSIONS IN PM 1 AND PM 2

A III.1 A MODIFIED LOMMEL FORMULA

Let

$$\mathcal{C}_\nu(\beta_K r) = J_\nu(\beta_K r) + \lambda_K Y_\nu(\beta_K r) \qquad \text{(A III.1)}$$

and

$$\mathcal{Z}_\nu(\alpha_j r) = I_\nu(\alpha_j r) + \mu_j K_\nu(\alpha_j r) \qquad \text{(A III.2)}$$

then $\mathcal{C}_\nu(\beta_K r)$ and $\mathcal{Z}_\nu(\alpha_j r)$ satisfy

$$r^2 \mathcal{C}_\nu'' + r\,\mathcal{C}_\nu' + (\beta_K^2 r^2 - \nu^2)\,\mathcal{C}_\nu = 0 \qquad \text{(A III.3)}$$

$$r^2 \mathcal{Z}_\nu'' + r\,\mathcal{Z}_\nu' - (\alpha_j^2 r^2 - \nu^2)\,\mathcal{Z}_\nu = 0 \qquad \text{(A III.4)}$$

We multiply these equations by $\mathcal{Z}_\nu / r$ and $\mathcal{C}_\nu / r$ respectively and subtract; then

$$r\,[\mathcal{C}_\nu'' \mathcal{Z}_\nu - \mathcal{C}_\nu \mathcal{Z}_\nu''] + [\mathcal{C}_\nu' \mathcal{Z}_\nu - \mathcal{C}_\nu \mathcal{Z}_\nu'] + (\alpha_j^2 + \beta_K^2)\, r\, \mathcal{C}_\nu \mathcal{Z}_\nu = 0 \qquad \text{(A III.5)}$$

where $' \triangleq d/dr$. Integrating,

$$\int^r r\,\mathcal{C}_\nu(\beta_K r)\mathcal{Z}_\nu(\alpha_j r)\,dr = -\frac{1}{\alpha_j^2 + \beta_K^2}\,[r\,(\mathcal{C}_\nu' \mathcal{Z}_\nu - \mathcal{C}_\nu \mathcal{Z}_\nu')] \qquad \text{(A III.6)}$$

which we call a 'modified Lommel formula' and is not to be found in books known to the author.

Integration by parts in two different fashions yields, the useful formulae:

$$\int_a^b r^2 \mathscr{C}_1(\beta_K r)\mathscr{Z}_o(\alpha_j r)dr = -\frac{r^2}{\beta_K}\mathscr{Z}_o(\alpha_j r)\mathscr{C}_o(\beta_K r)\Big|_a^b +$$

$$\frac{2}{\beta_K}\int_a^b r\,\mathscr{C}_o(\beta_K r)\mathscr{Z}_o(\alpha_j r)dr + \frac{\alpha_j}{\beta_K}\int_a^b r^2\,\mathscr{C}_o(\beta_K r)\mathscr{Z}_1(\alpha_j r)dr \qquad \text{(A III.7)}$$

and

$$\int_a^b r^2 \mathscr{C}_1(\beta_K r)\mathscr{Z}_o(\alpha_j r)dr = -\frac{\beta_K}{\alpha_j}\int_a^b r^2\,\mathscr{C}_o(\beta_K r)\mathscr{Z}_1(\alpha_j r)dr \qquad \text{(A III.8)}$$

where β_K are such that

$$\mathscr{C}_1(\beta_K a) = 0 = \mathscr{C}_1(\beta_K b) \qquad \text{(A III.9)}$$

The latter equations allow to solve integrals with weight function r^2 .

A III.2 SUMMATION OF SERIES APPEARING IN PM 1 AND PM 2

(a) Orthogonality: Let $\mathscr{C}_\nu(\beta_K r)$ be a cylinder function; these functions are orthogonal on the interval [a,b] with respect to the weighting function r:

$$\int_a^b r\,\mathscr{C}_\nu(\beta_K r)\,\mathscr{C}_\nu(\beta_\ell r)\,dr = 0 \quad , \quad K \neq \ell \qquad \text{(A III.10)}$$

where boundary conditions of the form

$$K_1 \mathscr{C}_\nu(\beta_K r) + K_2 \mathscr{C}_\nu'(\beta_K r) = 0$$

apply at $r = a, b$ (Sturm-Liouville conditions).

An arbitrary function $f(r)$ can be expanded in a series of cylinder functions:

$$f(r) = \sum_{K=0}^{\infty} A_K \mathscr{C}_\nu(\beta_K r) \tag{A III.11}$$

where

$$A_K = \frac{1}{N_K} \int_a^b r f(r) \mathscr{C}_\nu(\beta_K r) dr \tag{A III.12}$$

and

$$N_K = \int_a^b r \left[\mathscr{C}_\nu(\beta_K r) \right]^2 dr = \tag{A III.13}$$

$$\frac{1}{2} r^2 \left\{ \left(1 - \frac{\nu^2}{\beta_K^2 r^2}\right) \mathscr{C}_\nu^2(\beta_K r) + \left[\mathscr{C}_\nu'(\beta_K r)\right]^2 \right\} \Bigg|_a^b$$

From now on we take:

$$\mathscr{C}_\nu(\beta_K r) \stackrel{d}{=} J_\nu(\beta_K r) + \lambda_K Y_\nu(\beta_K r) \ , \ \nu = 0, 1$$

$$\mathscr{C}_1(\beta_K a) = 0 = \mathscr{C}_1(\beta_K b) \tag{A III.14}$$

λ_K being given in (3.6).

We will also need

$$\mathscr{Z}_{\rho\nu}(\alpha_j r) \stackrel{d}{=} I_\nu(\alpha_j r) + \mu_j K_\nu(\alpha_j r)$$

$$\mu_j \stackrel{d}{=} (-1)^\nu I_1(\alpha_j \rho) / K_1(\alpha_j \rho) \tag{A III.15}$$

where $\rho = a, b$ and $\nu = 0, 1$.

(b) Let

$$\mathcal{Z}_{a0}(\alpha_j r) = \sum_K a_K \mathcal{C}_0(\beta_K r) \qquad \text{(A III.16)}$$

multiplying both members by $r\,\mathcal{C}_0(\beta_K r)$ using (3.18) on the r.h.s. of the equation and (A III.6) on the l.h.s. we obtain

$$a_K = \frac{1}{N_K} \frac{\alpha_j b}{(\alpha_j^2 + \beta_K^2)} \mathcal{C}_0(\beta_K b) \mathcal{Z}_{a1}(\alpha_j b) \qquad \text{(A III.17)}$$

which is used to express the series (S1a) in Table I. For $\mathcal{Z}_{b0}(\alpha_j r)$ the same method can be applied and the correspondent a_K will have the same form as (A III.17) except for a negative sign and the exchange of b by a in the r.h.s.

The second fundamental expansion is obtained by letting

$$\mathcal{Z}_{a1}(\alpha_j r) = \sum_K b_K \mathcal{C}_1(\beta_K r) \qquad \text{(A III.18)}$$

Using (3.18) leads to

$$b_K = -\frac{1}{N_K} \frac{\beta_K b}{(\alpha_j^2 + \beta_K^2)} \mathcal{C}_0(\beta_K b) \mathcal{Z}_{a1}(\alpha_j b) \qquad \text{(A III.19)}$$

the same remarks, as before, apply to the expansion for $\mathcal{Z}_{b1}(\alpha_j r)$. By using (3.13) coupled with (A III.7) and (A III.8) to evaluate the expansions

$$r\,\mathcal{Z}_{a1}(\alpha_j r) = \sum_K c_K\, \mathcal{C}_0(\beta_K r) \qquad \text{(A III.20)}$$

and

$$r\,\mathcal{Z}_{a0}(\alpha_j r) = \sum_K d_K\, \mathcal{C}_1(\beta_K r) \qquad \text{(A III.21)}$$

and on this way generate (S3a) and (S4a). This, however, is not necessary since we find the following recursive relations:

$$(S3a) = \frac{d}{dr}(S1a)$$

$$(S4a) = \frac{d}{dr}(S2a) \qquad \text{(A III.22)}$$

$$(S5a) = \frac{1}{r}\frac{d}{dr}[r(S4a)]$$

The expansions (SNb) with, N = 1, ..., 5 in which $\mathcal{C}_0(\beta_K b)$ appear instead of $\mathcal{C}_0(\beta_K a)$ can be easily obtained through the rule

$$\partial_{ab}[\text{l.h.s. of }(SNa)] = -\partial_{ab}[\text{r.h.s. of }(SNa)] \qquad \text{(A III.23)}$$

Table I also assures us that the series which we were dealing with are uniformly and absolutely convergent; a requirement not stated explicitly in the body of Chapter III. These series lack a place in the only modern handbook of mathematical series known to the author, MANGULIS [1965; pp.106-123].

(c) Note: It is known that the MacRobert-Sneddon transform (usually called 'modified finite Hankel transform' but in fact due to MacROBERT [1931] and SNEDDON [1946]) plays an important role in potential problems associated with hollow circular regions. Such transforms are defined by

$$\mathcal{T}_\nu [f(r)] = \int_a^b r f(r)\, \mathcal{C}_\nu(\beta_k r)\, dr \; ; \quad b > a$$

where β_k are the positive roots of

$$J_\nu(\beta_k a)\, Y_\nu(\beta_k b) - J_\nu(\beta_k b)\, Y_\nu(\beta_k a) = 0$$

The determination of the coefficients a_k, b_k etc. in the series expansions above are closely related to such transforms of orders $\nu = 0, 1$ and thus are useful results in ways more than one. A comprehensive survey of the use of the MacRobert-Sneddon transforms mainly for heat conduction applications has been given by CINELLI [1965] .

TABLE I: SERIES SUMMATIONS

(S1a)	$\displaystyle\sum_K \frac{\mathcal{C}_0(\beta_K a)\,\mathcal{C}_0(\beta_K r)}{(\alpha_j^2+\beta_K^2)\,N_K} = -\frac{\mathcal{Z}_{b0}(\alpha_j r)}{\alpha_j a\;\mathcal{Z}_{b1}(\alpha_j a)}$
(S2a)	$\displaystyle\sum_K \frac{\mathcal{C}_0(\beta_K a)\,\mathcal{C}_0(\beta_K r)}{(\alpha_j^2+\beta_K^2)^2\,N_K} = \frac{r\,\mathcal{Z}_{b1}(\alpha_j r)}{2\alpha_j^2 a\;\mathcal{Z}_{b1}(\alpha_j a)} - \frac{b\,\mathcal{Z}_{b0}(\alpha_j b)\,\mathcal{Z}_{a0}(\alpha_j r)}{2\alpha_j^2 a\;\mathcal{Z}_{a1}(\alpha_j b)\,\mathcal{Z}_{b1}(\alpha_j a)} - \frac{\mathcal{Z}_{b0}(\alpha_j a)\,\mathcal{Z}_{b0}(\alpha_j r)}{2\alpha_j^2\;\mathcal{Z}_{b1}^2(\alpha_j a)}$
(S3a)	$\displaystyle\sum_K \frac{\beta_K\,\mathcal{C}_0(\beta_K a)\,\mathcal{C}_1(\beta_K r)}{(\alpha_j^2+\beta_K^2)\,N_K} = \frac{\mathcal{Z}_{b1}(\alpha_j r)}{a\;\mathcal{Z}_{b1}(\alpha_j a)}$
(S4a)	$\displaystyle\sum_K \frac{\beta_K\,\mathcal{C}_0(\beta_K a)\,\mathcal{C}_1(\beta_K r)}{(\alpha_j^2+\beta_K^2)^2\,N_K} = -\frac{r\;\mathcal{Z}_{b0}(\alpha_j r)}{2\alpha_j a\;\mathcal{Z}_{b1}(\alpha_j a)} + \frac{b\,\mathcal{Z}_{b0}(\alpha_j b)\,\mathcal{Z}_{a1}(\alpha_j r)}{2\alpha_j a\;\mathcal{Z}_{a1}(\alpha_j b)\,\mathcal{Z}_{b1}(\alpha_j a)} + \frac{\mathcal{Z}_{b0}(\alpha_j a)\,\mathcal{Z}_{b1}(\alpha_j r)}{2\alpha_j\;\mathcal{Z}_{b1}^2(\alpha_j a)}$
(S5a)	$\displaystyle\sum_K \frac{\beta_K^2\,\mathcal{C}_0(\beta_K a)\,\mathcal{C}_0(\beta_K r)}{(\alpha_j^2+\beta_K^2)\,N_K} = -\frac{r\;\mathcal{Z}_{b1}(\alpha_j r)}{2a\;\mathcal{Z}_{b1}(\alpha_j a)} + \frac{b\;\mathcal{Z}_{b0}(\alpha_j b)\,\mathcal{Z}_{a0}(\alpha_j r)}{2a\;\mathcal{Z}_{a1}(\alpha_j b)\,\mathcal{Z}_{b1}(\alpha_j a)} + \frac{1}{2\,\mathcal{Z}_{b1}^2(\alpha_j a)}\left[\mathcal{Z}_{b0}(\alpha_j a) - 2\,\frac{\mathcal{Z}_{b1}(\alpha_j a)}{\alpha_j a}\right]\mathcal{Z}_{b0}(\alpha_j r)$

APPENDIX IV

THE EIGENVALUES OF THE CIRCULAR HOLLOW CYLINDER

Written in full, the characteristic equation (4.17) for the determination of the eigenvalues reads when divided throughout by ab:

$$[J_0^2(\gamma_j a) + (1 - \frac{\tau}{\gamma_j^2 a^2}) J_1^2(\gamma_j a)] \{[H_0^{(2)}(\gamma_j b)]^2 + (1 - \frac{\tau}{\gamma_j^2 b^2}) [H_1^{(2)}(\gamma_j b)]^2\} +$$

$$[J_0^2(\gamma_j b) + (1 - \frac{\tau}{\gamma_j^2 b^2}) J_1^2(\gamma_j b)] \{[H_0^{(2)}(\gamma_j a)]^2 + (1 - \frac{\tau}{\gamma_j^2 a^2}) [H_1^{(2)}(\gamma_j a)]^2\}$$

$$- 2[J_0(\gamma_j a) H_0^{(2)}(\gamma_j a) + (1 - \frac{\tau}{\gamma_j^2 a^2}) J_1(\gamma_j a) H_1^{(2)}(\gamma_j a)] \quad \text{(A IV.1)}$$

$$[J_0(\gamma_j b) H_0^{(2)}(\gamma_j b) + (1 - \frac{\tau}{\gamma_j^2 b^2}) J_1(\gamma_j b) H_1^{(2)}(\gamma_j b)]$$

$$+ \frac{4}{\pi^2} \frac{1}{(\gamma_j ab)^2} [a^2 + b^2 - \frac{2\tau}{\gamma_j^2}] = 0$$

Of course, this equation can be expressed in terms of J_ν and Y_ν, if this is done by using (4.1), the equivalent form corresponds to replacing $H_\nu^{(2)}$ by Y_ν and reversing the sign of the first three product terms in (A.IV.1). Substitution of the first two terms of the asymptotic expansions for $J_\nu(\gamma_j \rho)$ and $H_\nu^{(2)}(\gamma_j \rho)$ with $|\gamma_j \rho| >> 1$ into the above equation and disregarding all terms of order

$(1/\gamma_j\rho)^2$ and higher, yields for the difference between two consecutive eigenvalues the asymptotic form:

$$\gamma_{j+1} - \gamma_j \simeq \frac{1}{b-a} \left[\pi + i \ln\left(\frac{2j-1}{2j-3}\right)\right] \qquad \text{(A IV.2)}$$

with $j = 1,2,3,\ldots$. The first root is expected to be in the neighborhood of the first eigenvalue for the solid cylinder, i.e., $\gamma_1 \sim 2.7 + i\,1.6$. With such initial guess, the Newton-Raphson algorithm

$$\gamma_N \Leftarrow \gamma_{N-1} - [\hat{A} / (d\hat{A}/d\gamma)]_{\gamma_{N-1}} \qquad \text{(A IV.3)}$$

can be used to advantage, and requires only a few number N of iterations for each root. For $a > .5$ the modulus of the individual terms in equation (A IV.1) are very large although the factors in square brackets are small, in consequence it would be necessary to have available a larger number of significant figures than that in use with the present computer facilities at the University of Houston.

In Fig. A IV.1 we picturize the eigenpaths $E_j = \gamma_j(a)$ with $j = 1,2, \ldots, 5$ for $\sigma = 0.3$, $b = 1$. For a approaching zero the eigenvalues reach those for the solid cylinder as expected. However, the thinner the cylinder the larger the imaginary part of the eigenvalues become which precludes the existence of an increasing number of vibrations modes. Increase of Poisson's ratio corresponds to a raising and a shift of the curve to the left.

In Tables II*, we present the first five eigenvalues in the range $a = 0.02\ (0.02) 0.08$ and the first twenty in $a = 0.1\ (0.1) 0.5$ for the useful values $\sigma = 0.25, 0.30$ and the academic values $\sigma = 0.0, 0.50$. The outer diameter b is taken as equal to one, without loss of generality since all the previous results can be normalized by b.

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*The complete programing for finding the roots of equation (4.17) $\equiv$ (A.IV.1) was done by Dr. Bart Childs. I am deeply grateful for his efforts, determination and consuming time in doing this so that these tables could be presented here. I also thank Mr. C. Hatfield for drawing Fig. A IV.1.

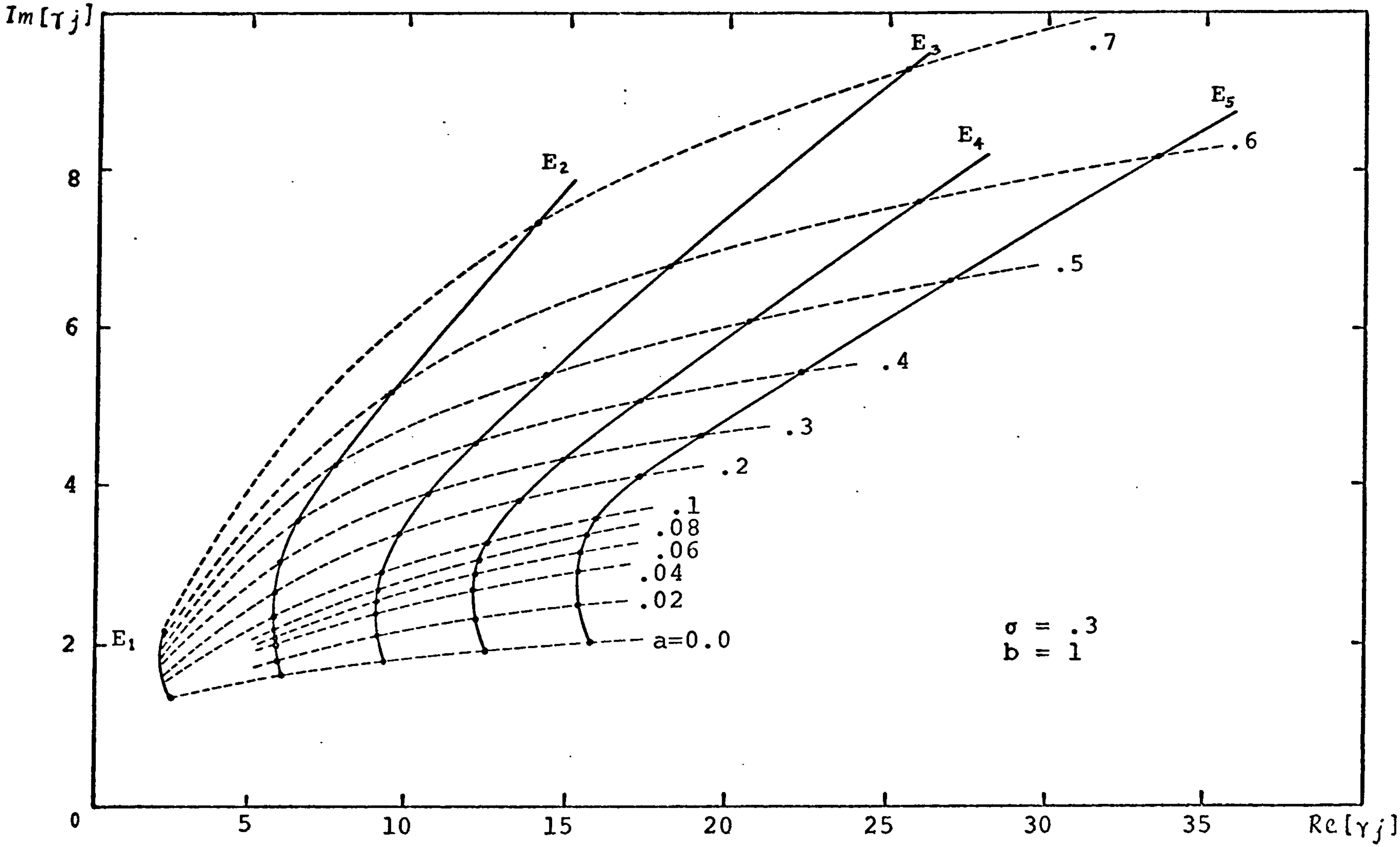

Fig. AIV.1. Eigenpaths $E_j = f(a, \gamma_j)$ for cylindrical regions

TABLE II A

EIGENVALUES OF THE CHARACTERISTIC EQUATION (4.17)

a =0.02

NO.	POISSON'S RATIO =0.50		POISSON'S RATIO =0.30	
1	2.803081	$+i$1.342661	2.717691	$+i$1.364237
2	6.051009	1.687871	6.031500	1.677019
3	9.196867	2.009703	9.203790	1.965626
4	12.33181	2.338611	12.35208	2.256190
5	15.47797	2.658850	15.50402	2.544134
NO.	POISSON'S RATIO =0.25		POISSON'S RATIO =0.00	
1	2.693665	$+i$1.369277	2.554509	$+i$1.390439
2	6.025215	1.675141	5.988993	1.667206
3	9.203357	1.957926	9.193983	1.929643
4	12.35474	2.241261	12.35965	2.185556
5	15.50810	2.522693	15.52011	2.440744

TABLE II B

EIGENVALUES OF THE CHARACTERISTIC EQUATION (4.17)

a =0.04

NO.	POISSON'S RATIO =0.50		POISSON'S RATIO =0.30	
1	2.782070	$+i$1.350701	2.704986	$+i$1.370288
2	5.960269	1.818185	5.970004	1.777041
3	9.085229	2.331905	9.124739	2.230349
4	12.25000	2.802906	12.30265	2.662932
5	15.43405	3.196086	15.50104	3.038061
NO.	POISSON'S RATIO =0.25		POISSON'S RATIO =0.00	
1	2.682352	$+i$1.374977	2.548068	$+i$1.394819
2	5.968969	1.769772	5.952102	1.742104
3	9.130511	2.211505	9.145347	2.139723
4	12.31116	2.635867	12.33849	2.529543
5	15.51194	3.006556	15.54829	2.879879

TABLE II C

EIGENVALUES OF THE CHARACTERISTIC EQUATION (4.17)

a =0.06

NO.	POISSON'S RATIO =0.50		POISSON'S RATIO =0.30	
1	2.750666	+i1.363512	2.685700	+i1.380105
2	5.869353	1.976460	5.907241	1.906337
3	9.021740	2.614008	9.088907	2.485062
4	12.23280	3.116439	12.32066	2.970840
5	15.46439	3.488264	15.57207	3.350755
NO.	POISSON'S RATIO =0.25		POISSON'S RATIO =0.00	
1	2.665139	+i1.384249	2.538194	+i1.402000
2	5.911487	1.893479	5.914837	1.844164
3	9.099728	2.459855	9.133870	2.360135
4	12.33533	2.940740	12.38490	2.816982
5	15.59067	3.320493	15.65529	3.191010

TABLE II D

EIGENVALUES OF THE CHARACTERISTIC EQUATION (4.17)

a =0.08

NO.	POISSON'S RATIO =0.50		POISSON'S RATIO =0.30	
1	2.712326	+i1.380114	2.661705	+i1.393232
2	5.795438	2.132760	5.857378	2.042730
3	9.000800	2.827938	9.091919	2.697292
4	12.27731	3.310828	12.38815	3.188658
5	15.58992	3.642896	15.70932	3.546085
NO.	POISSON'S RATIO =0.25		POISSON'S RATIO =0.00	
1	2.643656	+i1.396707	2.525753	+i1.411793
2	5.866247	2.025548	5.887682	1.958136
3	9.107265	2.670224	9.158613	2.558744
4	12.40785	3.161106	12.47742	3.041033
5	15.73197	3.522158	15.81587	3.410992

TABLE II E

EIGENVALUES OF THE CHARACTERISTIC EQUATION (4.17)

a = 0.10

NO.	POISSON'S RATIO = 0.50		POISSON'S RATIO = 0.30	
1	2.670084	+i1.399321	2.634744	+i1.409064
2	5.743233	2.274778	5.824673	2.174990
3	9.022223	2.985854	9.128905	2.868707
4	12.38757	3.438904	12.50367	3.346486
5	15.80014	3.745442	15.91176	3.681018
6	19.24385	3.973968	19.34456	3.930078
7	22.70632	4.157730	22.79528	4.127200
8	26.17973	4.312861	26.25821	4.290868
9	29.65957	4.447877	29.72926	4.431451
10	33.14332	4.567794	33.20574	4.555117
11	36.62951	4.675843	36.68591	4.665779
12	40.11726	4.774256	40.16865	4.766073
13	43.60605	4.864661	43.65321	4.857871
14	47.09555	4.948286	47.13910	4.942556
15	50.58552	5.026091	50.62597	5.021184
16	54.07582	5.098837	54.11359	5.094583
17	57.56636	5.167146	57.60177	5.163418
18	61.05706	5.231528	61.09039	5.228231
19	64.54789	5.292411	64.57936	5.289471
20	68.03879	5.350154	68.06861	5.347515

NO.	POISSON'S RATIO = 0.25		POISSON'S RATIO = 0.00	
1	2.619438	+i1.411830	2.511589	+i1.423937
2	5.837428	2.155125	5.874386	2.074994
3	9.147745	2.842830	9.213439	2.731342
4	12.52579	3.323756	12.60803	3.218071
5	15.93463	3.663530	16.02489	3.575600
6	19.36637	3.917196	19.45714	3.847766
7	22.81532	4.117710	22.90219	4.063618
8	26.27636	4.283742	26.35752	4.241346
9	29.74567	4.425966	29.82074	4.392260
10	33.22062	4.550788	33.28988	4.523537
11	36.69948	4.662284	36.76344	4.639879
12	40.18110	4.763196	40.24033	4.744488
13	43.66469	4.855462	43.71974	4.839621
14	47.14975	4.940508	47.20110	4.926927
15	50.63589	5.019420	50.68396	5.007650
16	54.12287	5.093047	54.16803	5.082747
17	57.61049	5.162068	57.65305	5.152977
18	61.09861	5.227035	61.13885	5.218949
19	64.58714	5.288403	64.62528	5.281163
20	68.07599	5.346554	68.11223	5.340033

TABLE II F

EIGENVALUES OF THE CHARACTERISTIC EQUATION (4.17)

a =0.20

NO.	POISSON'S RATIO =0.50		POISSON'S RATIO =0.30	
1	2.463435	+i1.504363	2.496081	+i1.509410
2	5.815449	2.788757	5.927013	2.716671
3	9.656808	3.475009	9.748708	3.433308
4	13.57250	3.895833	13.64421	3.873060
5	17.50605	4.203991	17.56313	4.190234
6	21.44363	4.449407	21.49056	4.440256
7	25.38141	4.654076	25.42111	4.647529
8	29.31832	4.829845	29.35267	4.824904
9	33.25415	4.983949	33.28440	4.980069
10	37.18894	5.121178	37.21597	5.118038
11	41.12283	5.244878	41.14725	5.242275
12	45.05594	5.357474	45.07821	5.355276
13	48.98837	5.460792	49.00884	5.458907
14	52.92023	5.556253	52.93917	5.554615
15	56.85160	5.644968	56.86923	5.643531
16	60.78256	5.727818	60.79904	5.726544
17	64.71316	5.805525	64.72863	5.804387
18	68.64342	5.878701	68.65801	5.877678
19	72.57343	5.947851	72.58722	5.946925
20	76.50320	6.013377	76.51628	6.012534
NO.	POISSON'S RATIO =0.25		POISSON'S RATIO =0.00	
1	2.493928	+i1.510108	2.436954	+i1.510044
2	5.948150	2.699113	6.025073	2.616382
3	9.768295	3.421576	9.848862	3.359447
4	13.66052	3.866065	13.73245	3.825765
5	17.57655	4.185804	17.63827	4.158949
6	21.50181	4.437235	21.55480	4.418372
7	25.43073	4.645339	25.47677	4.631429
8	29.36106	4.823240	29.40159	4.812565
9	33.29183	4.978758	33.32796	4.970303
10	37.22263	5.116977	37.25518	5.110107
11	41.15328	5.241397	41.18288	5.235698
12	45.08372	5.354535	45.11084	5.349726
13	49.01392	5.458273	49.03894	5.454157
14	52.94387	5.554066	52.96709	5.550501
15	56.87360	5.643049	56.89526	5.639929
16	60.80314	5.726119	60.82343	5.723363
17	64.73248	5.804008	64.75156	5.801555
18	68.66164	5.877338	68.67965	5.875141
19	72.59065	5.946618	72.60771	5.944636
20	76.51954	6.012255	76.53573	6.010459

TABLE II G

EIGENVALUES OF THE CHARACTERISTIC EQUATION (4.17)

a =0.30

NO.	POISSON'S RATIO =0.50		POISSON'S RATIO =0.30	
1	2.325267	+i 1.603825	2.399112	+i 1.621793
2	6.332077	3.225980	6.421647	3.188272
3	10.86315	3.967968	10.92237	3.950841
4	15.40105	4.441743	15.44391	4.432507
5	19.92832	4.794644	19.96165	4.788833
6	24.44664	5.076932	24.47384	5.072893
7	28.95850	5.312486	28.98148	5.309487
8	33.46572	5.514625	33.48560	5.512295
9	37.96948	5.691752	37.98701	5.689880
10	42.47073	5.849322	42.48639	5.847779
11	46.96991	5.991270	46.98407	5.989974
12	51.46762	6.120427	51.48055	6.119319
13	55.96399	6.238824	55.97588	6.237866
14	60.45941	6.348282	60.47042	6.347442
15	64.95398	6.449818	64.96423	6.449076
16	69.44782	6.544728	69.45741	6.544067
17	73.94109	6.633704	73.95009	6.633111
18	78.43385	6.717469	78.44234	6.716934
19	82.92619	6.796600	82.93421	6.796114
20	87.41814	6.871582	87.42576	6.871138

NO.	POISSON'S RATIO =0.25		POISSON'S RATIO =0.00	
1	2.406001	+i 1.623628	2.387276	+i 1.621804
2	6.440295	3.177767	6.514875	3.121749
3	10.93590	3.945540	10.99610	3.914384
4	15.45409	4.429529	15.50154	4.411199
5	19.96970	4.786933	20.00817	4.775004
6	24.48049	5.071569	24.51263	5.063185
7	28.98712	5.308506	29.01466	5.302275
8	33.49050	5.511535	33.51455	5.506710
9	37.99134	5.689272	38.01268	5.685418
10	42.49027	5.847280	42.50944	5.844123
11	46.98759	5.989557	47.00498	5.986922
12	51.48376	6.118963	51.49968	6.116725
13	55.97884	6.237560	55.99351	6.235636
14	60.47315	6.347175	60.48676	6.345500
15	64.96678	6.448841	64.97945	6.447369
16	69.45979	6.543859	69.47167	6.542555
17	73.95234	6.632924	73.96350	6.631760
18	78.44446	6.716766	78.45499	6.715720
19	82.93621	6.795961	82.94618	6.795016
20	87.42766	6.870999	87.43712	6.870140

TABLE II H

EIGENVALUES OF THE CHARACTERISTIC EQUATION (4.17)

a = 0.40

NO.	POISSON'S RATIO = 0.50		POISSON'S RATIO = 0.30	
1	2.265791	+i 1.711123	2.362228	+i 1.747135
2	7.215960	3.765203	7.280783	3.744879
3	12.58903	4.622496	12.62803	4.613691
4	17.91426	5.176402	17.94189	5.171468
5	23.21096	5.589701	23.23234	5.586480
6	28.49072	5.920195	28.50816	5.917901
7	33.76004	6.195603	33.77476	6.193868
8	39.02238	6.432144	39.03513	6.430772
9	44.27927	6.639216	44.29051	6.638111
10	49.53295	6.823187	49.54300	6.822266
11	54.78372	6.988988	54.79281	6.988210
12	60.03235	7.139793	60.04064	7.139125
13	65.27928	7.278090	65.28691	7.277510
14	70.52485	7.405797	70.53192	7.405288
15	75.76931	7.524421	75.77589	7.523970
16	81.01284	7.635168	81.01899	7.634765
17	86.25559	7.739019	86.26137	7.738657
18	91.49768	7.836783	91.50313	7.836455
19	96.73920	7.929133	96.74436	7.928835
20	101.9802	8.016637	101.9851	8.016365

NO.	POISSON'S RATIO = 0.25		POISSON'S RATIO = 0.00	
1	2.374591	+i 1.751621	2.380802	+i 1.754849
2	7.294959	3.738727	7.354781	3.703061
3	12.63719	4.610864	12.67939	4.593449
4	17.94856	5.169866	17.98032	5.159812
5	23.23756	5.585437	23.26284	5.578868
6	28.51244	5.917163	28.53340	5.912518
7	33.77840	6.193313	33.79626	6.189839
8	39.03828	6.430337	39.05385	6.427622
9	44.29330	6.637762	44.30710	6.635592
10	49.54550	6.821978	49.55787	6.820190
11	54.79507	6.987967	54.80629	6.986469
12	60.04271	7.138918	60.05297	7.137643
13	65.28882	7.277331	65.29827	7.276231
14	70.53368	7.405131	70.54244	7.404172
15	75.77753	7.523832	75.78569	7.522987
16	81.02053	7.634642	81.02817	7.633893
17	86.26281	7.738547	86.27000	7.737876
18	91.50449	7.836356	91.51127	7.835752
19	96.74565	7.928745	96.75206	7.928199
20	101.9864	8.016283	101.9924	8.015786

TABLE III

EIGENVALUES OF THE CHARACTERISTIC EQUATION (4.17)

a =0.50

NO.	POISSON'S RATIO =0.50		POISSON'S RATIO =0.30	
1	2.279673	+i1.846351	2.389953	+i1.900840
2	8.551486	4.513028	8.597045	4.501382
3	15.05541	5.542333	15.08162	5.537200
4	21.46392	6.208457	21.48234	6.205496
5	27.82835	6.705228	27.84258	6.703271
6	34.16870	7.102111	34.18029	7.100718
7	40.49631	7.433966	40.50611	7.432869
8	46.81323	7.717863	46.82172	7.716994
9	53.12357	7.966412	53.13105	7.965703
10	59.42912	8.187461	59.43581	8.186871
11	65.73109	8.386502	65.73714	8.386001
12	72.03030	8.567526	72.03583	8.567094
13	78.32733	8.733524	78.33242	8.733150
14	84.62262	8.886805	84.62732	8.886475
15	90.91647	9.029179	90.92086	9.028887
16	97.20915	9.162093	97.21326	9.161834
17	103.5008	9.286730	103.5047	9.286495
18	109.7917	9.404056	109.7953	9.403847
19	116.0818	9.514886	116.0853	9.514693
20	122.3714	9.619902	122.3746	9.619725

NO.	POISSON'S RATIO =0.25		POISSON'S RATIO =0.00	
1	2.405760	+i1.908397	2.428022	+i1.919320
2	8.607279	4.497665	8.651846	4.474887
3	15.08786	5.535527	15.11711	5.524993
4	21.48682	6.204537	21.50838	6.198481
5	27.84607	6.702643	27.86309	6.698690
6	34.18315	7.100275	34.19719	7.097492
7	40.50854	7.432524	40.52051	7.430390
8	46.82383	7.716723	46.83424	7.715055
9	53.13291	7.965484	53.14213	7.964141
10	59.43748	8.186690	59.44575	8.185583
11	65.73865	8.385849	65.74614	8.384919
12	72.03720	8.566964	72.04405	8.566172
13	78.33368	8.733036	78.33999	8.732353
14	84.62850	8.886376	84.63435	8.885779
15	90.92195	9.028797	90.92740	9.028273
16	97.21428	9.161755	97.21938	9.161287
17	103.5057	9.286423	103.5105	9.286006
18	109.7962	9.403783	109.8008	9.403403
19	116.0861	9.514640	116.0904	9.514292
20	122.3754	9.619670	122.3795	9.619364