WIGNER'S DISTRIBUTION
AND THE QUANTIZED VORTEX

A Thesis

Presented to

the Faculty of the Department of Physics

University of Houston

In Partial Fulfillment

of the Requirements for the Degree

Master of Science

By

Diana Lund Power

May, 1978

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ABSTRACT

A quantum model for the vorticity of a single fluctuating rectilinear vortex is obtained by assuming Wigner's distribution for the vorticity. The starting point for this calculation is Fetter's quantum theory of the superfluid vortex, in which the vortex waves are induced by a quantum rather than a classical random noise source. Thermal averaging of the quantum model does, however, give the classical result for the distribution. From the vorticity, a quantum model for the velocity around the vortex is obtained. A thermal averaging of the velocity allows an explicit verification of McCauley's proposal that the effect of quantum fluctuations is to smear out the vortex singularity and give the appearance of a classical core. A quantum model for the kinetic energy density is also obtained. From this it is found that the zero-point velocities are uncorrelated.

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I. DISCUSSION AND PRESENTATION OF RESULTS

A. Classical Treatment of the Vortex Problem

1. The Circulation and Hamiltonian Formalism

The first study of rotational flow was done by Helmholtz, in Crelle's Journal, in 1858. The investigation and application of vortex motion has continued from that day to this, necessarily including an eventual integration with quantum theory. Many aspects of the vortex problem have been treated classically, but let me here recapitulate a few points of classical theory which will prove pertinent to the discussion later in this work. The circulation (or strength) of a vortex, denoted by κ, is defined by

$$\oint_c \vec{v} \cdot d\vec{\ell} = \kappa . \qquad (1)$$

By Stokes's Theorem, this becomes

$$\int_S (\nabla \times \vec{v}) \cdot d\vec{S} = \kappa , \qquad (2)$$

where S is bounded by c. So we see that the curl of the velocity is the circulation per unit area, or the vorticity.

In 1880, W. Thomson (Lord Kelvin)[1] solved the hydrodynamic equations of motion for the vortex. The exact nature of the solutions depended on certain boundary conditions as well as the distribution of vorticity. One particular case assumed a vortex in an unbounded, incompressible, non-viscous fluid, with a uniform vorticity over a certain core radius, a, and irrotational flow outside this region with no slip between. That is,

$$\nabla \times \vec{v} = 2\,\vec{\omega} = 2\omega\hat{z} , \qquad r < a , \qquad (3)$$

$$\nabla \times \vec{v} = 0, \qquad r > a. \tag{4}$$

This distribution predicts that the core area rotates like a solid body with angular velocity $\vec{\omega}$. The significance of Thomson's calculation is that if this vortex is given a transverse perturbation, simple harmonic vibrations result. This was the first prediction of self-induced vortex waves.

The next major step in the development of classical theory of the vortex problem was a Hamiltonian formalism[2,3]. The Hamilitonian for an array of rectilinear vortices is proportional to the interaction energy of the array which is also the kinetic energy of the circulating fluid to within a constant. The x,y coordinates of the i^{th} vortex form the i^{th} pair of conjugate variables. This dynamical approach to the formalism is for a two dimensional system. It cannot be extended to three dimensions except for the special case of small oscillations of a rectilinear vortex.

2. The Vorticity

One approach to the vorticity for a rectilinear vortex is to define it by the circulation times a delta function

$$\nabla \times \vec{v} = K\, \delta(\vec{r} - \vec{r}')\, \hat{z}. \tag{5}$$

Thus the delta function locates the vortex singularity. If the vortex is subjected to a classical noise source, then x',y' may be taken as Gaussian random variables. Using an appropriately normalized Gaussian distribution function, the expectation of the delta function becomes

$$\langle \delta(\vec{r}-\vec{r}')\rangle = \iint_{-\infty}^{+\infty} dx'dy'\; \delta(\vec{r}-\vec{r}')\, \frac{e^{-r'^2/\sigma^2}}{\pi\sigma^2}. \tag{6}$$

$$= \frac{e^{-r^2/\sigma^2}}{\pi\sigma^2} \quad . \tag{7}$$

where σ^2 is the spread of the distribution. In this case then, we see that the expectation of the vorticity distribution is just the probability density for locating the vortex.

The classical foundation for this thesis is now laid. The quantum theory of a vortex is discussed in Part I-B and the self-energy (in a quantum description) is discussed in Part I-C. The results of the calculations of this thesis are presented and discussed in Part I-D,E,and F. Part II presents the details of the calculations, and Part III is a summary.

B. Quantum Theory of the Vortex

1. Circulation Quantization and Consequences

The first to suggest the existence of quantized vortices was Onsager[4]. He proposed that such quantized vortices would allow a superfluid to rotate, as well as providing a possible explanation for the λ-point transition (that when the vortices become concentrated to the point of a connected tangle, the superfluid becomes normal). The rotation problem for a superfluid is as follows (here I condense Feynman[5]): suppose a "can" of solid helium (under pressure $>$ 25atm at 0^{o}K) is rotating. If the pressure is reduced, the solid melts and the liquid rotates with the angular momentum given to the solid. The problem is to find the state which minimizes the energy for a given angular momentum. The conclusion reached by Feynman is that a discontinuous velocity distribution is necessary, and that the maximum number of minimum strength vortices produces the lowest energy state. The minimum vortex strength is defined by

$$\oint_C \vec{v} \cdot d\vec{\ell} = K = n\left(\frac{h}{m}\right), \qquad n = 1, \tag{8}$$

where h is Planck's constant and m is the mass per (helium) atom. The maximum number of vortices possible is angular velocity dependent since the curl of the velocity (2ω) is the circulation per unit area, i.e.,

$$\frac{2\omega}{K} = \frac{2\omega m}{h} = 2.1 \times 10^{3}\, \omega\ \frac{\text{vortices}}{\text{cm}^2} . \tag{9}$$

The idea of quantized vortex motion has found a home with the theory of another phenomenon, as well. Particularly, one sees the idea arising for another "super" state, as the Abrikosov array[6] for bulk type-II superconductors.

This method of quantizing the vortex by assuming the strength to be integer multiples of h/m is a Bohr-Sommerfeld type of quantization. For the most part, then, any treatment of a vortex problem could be handled by classical theories, with Planck's constant entering only through the circulation. But given the type of Hamiltonian a vortex possesses, if the axis of the vortex is localized, zero-point motion follows[7]. And zero-point motion cannot be encompassed within any classical theory.

2. Fetter's Quantum Theory

Fetter[7] proposed a fully quantum-mechanical system for an incompressible, non-viscous fluid with mass density ρ, by quantizing the position of a fluctuating vortex. He assumed small displacements, $\vec{u}$, of the vortex line position in the x,y plane. By parameterizing the deformation variable by the z coordinate, $\vec{u}(z)$, an extension to three dimensions was allowed. From this point, the energy calculation is analagous to the self-inductance calculation for a current-carrying wire in classical electromagnetic theory. Fetter then made a harmonic approximation for the Hamiltonian, taking it to second order in $\vec{u}(z)$. He identified the cannonical field variables for the i^{th} vortex as proportional to the deformation,

$$q_i(z) = \sqrt{\rho\kappa}\; u_{x_i}(z)\,, \tag{10}$$

$$p_i(z) = \sqrt{\rho\kappa}\; u_{y_i}(z)\,, \tag{11}$$

where each vortex is assumed to have the same minimum strength, κ. This produces an exact set of Hamiltonian equations

$$\delta H/\delta q_i(z) = -\dot{p}_i(z)\,, \tag{12}$$

$$\delta H/\delta p_i(z) = \dot{q}_i(z)\,. \tag{13}$$

These canonical variables Fetter interpreted as quantum-mechanical field operators obeying Heisenberg equations of motion:

$$i\hbar\, \dot{q}_i(z) = [\, q_i(z), H]\,, \tag{14}$$

$$i\hbar\, \dot{p}_i(z) = [\, p_i(z), H]\,. \tag{15}$$

And in order to correctly reproduce the equations of motion for the vortex array, these operators are subject to the usual canonical commutation relations

$$[\, q_i(z)\,,\; p_j(z')\,] = i\hbar\, \delta_{ij}\, \delta(z-z')\,. \tag{16}$$

The translational invariance of this system is continuous along the z axis, but discrete in the x,y plane[7]. This anisotropy is reflected in the commutation relations by the appearance of the two different delta functions; the Dirac delta function of the z coordinate and the Kronecker delta function of the lattice position variables in the x,y plane.

C. Self-energy of a Single Rectilinear Vortex in an Unbounded Fluid

The self-energy of the vortex core can be formulated as an interaction of two parallel vortex filaments, with a single quantum-mechanical operator sufficient to describe each elementary length of filament. Again, this is analogous to the self-inductance calculation of classical electromagnetic theory. So after Fetter Fourier-analyzed his operators (Part II-A.1), the self-energy was found by setting i=j for the operators and averaging the interaction energy over the core of radius a, thus obtaining

$$H = \frac{1}{2} \sum_{\ell} \omega(\ell) \left(q_\ell q_{-\ell} + p_\ell p_{-\ell} \right). \tag{17}$$

Here the dispersion relation is given by

$$\omega(\ell) = \frac{\kappa}{4\pi} \ell^2 \left\langle K_0(\ell |\vec{r} - \vec{r}'|) \right\rangle, \tag{18}$$

where K_0 is the zero-order modified Bessel function of the second kind and $\langle\ \rangle$(for Eq.(18) only) means

$$\frac{1}{(\pi a^2)^2} \int_{r \le a} d^2 r \int_{r' \le a} d^2 r' .$$

Now Fetter only approximated this dispersion relation for the case $a\ell \ll 1$. McCauley[8] has made an exact calculation, yielding

$$\omega(\ell) = \frac{\kappa}{\pi a^2} \left(\frac{1}{2} - I_1(a\ell) K_1(a\ell) \right), \tag{19}$$

where I_1 and K_1 are first-order modified Bessel functions of the first and second kind, respectively. Eq.(19) must be taken with a grain of salt at high frequencies because of its semi-classical origins. However, this problem is ameliorated by the fact that it becomes necessary to take an upper cut-off for the wavenumber, ℓ_{max} (Part II-A.2).

After several more transformations (Part II-A.1), the following Hamiltonian is obtained

$$H = \sum_{\ell \geq 0} \hbar \omega(\ell) \left(N_+(\ell) + N_-(\ell) + 1 \right), \tag{20}$$

where N_+ and N_- can be identified as the number operators for quanta of vibrations in either of two polarizations. These polarizations correspond to vortex waves travelling in either the +z or -z direction. An illustration of this helical arrangement is shown in Figure 1.

D. The Vorticity

1. The Distribution Function

Fetter's model assumed[8] a sectionally continuous distribution[1] for the classical vorticity, namely,

$$\nabla \times \vec{v} = \frac{K}{\pi a^2} \left(1 - H(r, a) \right), \tag{21}$$

$$H(r,a) = \begin{cases} 0, & r < a, \\ 1, & r > a. \end{cases} \tag{22}$$

Classically, this is equivalent to a core in solid-body rotation. It is at this point that I extend Fetter's work by assuming a quantum-mechanical discrete vorticity, as first suggested by Onsager[4]:

$$\nabla \times \vec{v} = K\, \delta(\vec{r} - \vec{u}(z))\, \hat{n}. \tag{23}$$

The location of the vortex is given by Fetter's canonically conjugate deformation operators and $\hat{n}$ is the local direction (see Fig. 1).

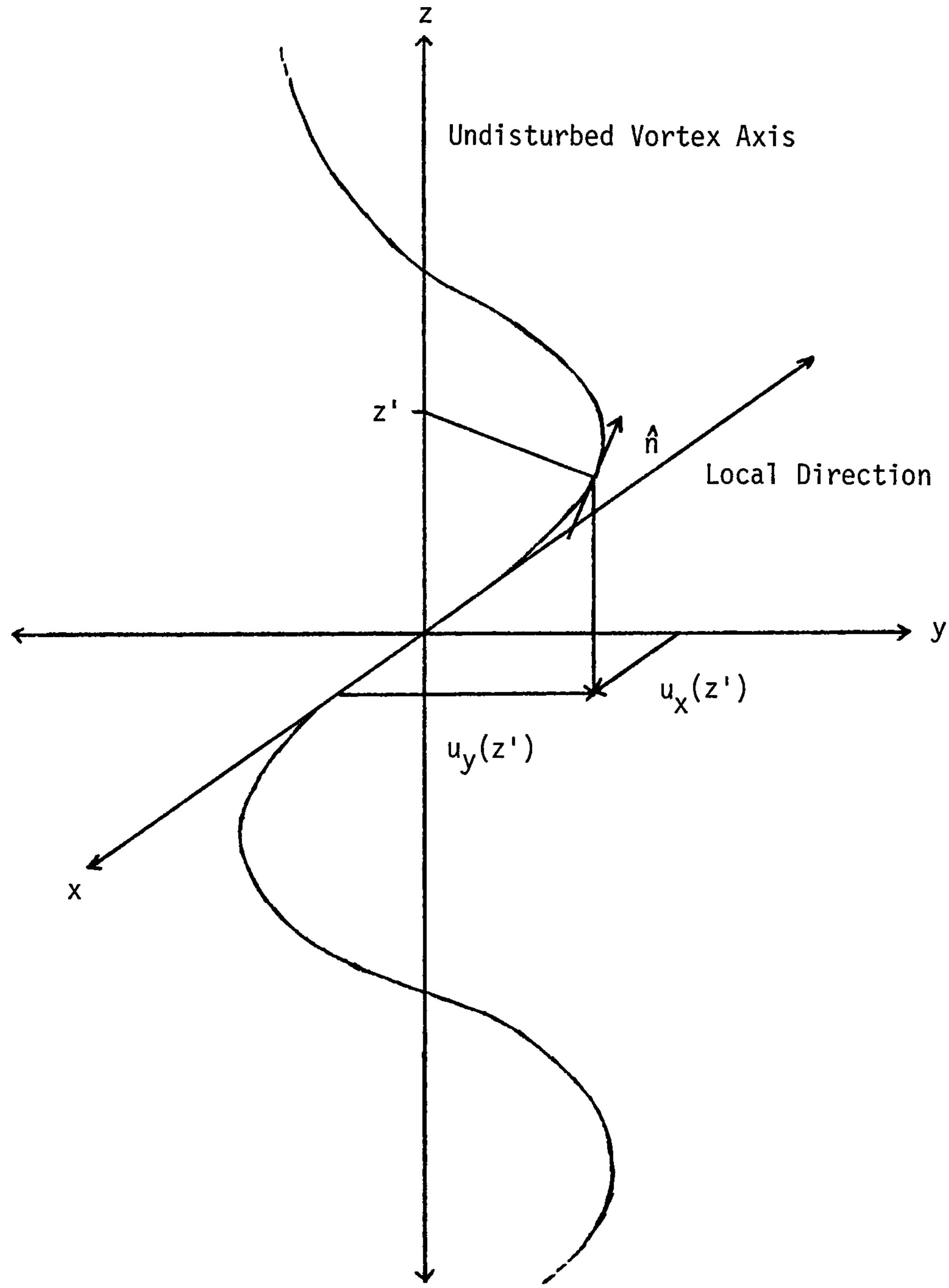

Figure 1: The Helical Configuration of the Fluctuating Vortex

$$\hat{n} = \left(\hat{z} + \frac{d\vec{u}}{dz}\right) \Big/ \sqrt{1 + \left|\frac{d\vec{u}}{dz}\right|^2} \, . \tag{24}$$

Thus we are subjecting the vortex to a quantum noise. The average of this directed delta function is interpreted as a vector probability density for locating the vortex. Throughout this work, the local direction is approximated to lowest order in $\vec{u}$, since the harmonic approximation requires the magnitude of bending to be small:

$$\hat{n} \simeq \hat{z} + \frac{d\vec{u}}{dz} \, . \tag{25}$$

A Fourier-expansion of this distribution function yields the two-dimensional characteristic function:

$$\vec{W} = \sum_{\vec{k}} e^{i\vec{k}\cdot\vec{r}} \vec{W}_k = \sum_{\vec{k}} e^{i\vec{k}\cdot\vec{r}} \left\langle e^{-i\vec{k}\cdot\vec{u}(z)} \left(\hat{z} + \frac{d\vec{u}}{dz}\right)\right\rangle . \tag{26}$$

The motivation for choosing this particular distribution function is two-fold. First, it is the same form (sans direction) as one of the pure state joint probabilities for non-commuting operators proposed by Margenau and Hill[9]. Second, it can be shown (Part II-B), at least for a single mode of vibration, that this is Wigner's distribution.

2. Wigner's Distribution and Joint Probabilities

Wigner's distribution for canonically conjugate variables p and q is defined[10] by

$$f_W(p,q) = \frac{1}{2\pi\hbar} \int_{-\infty}^{+\infty} d\eta \, e^{-i\eta p/\hbar} \rho(q + \eta/2, q - \eta/2) \, , \tag{27}$$

where $\rho(x,x')$ is the appropriate density matrix for the system. It should be pointed out that the definition of Wigner's distribution as a probability density is not unique. Wigner[11] has proven that, in general, one cannot define a non-negative joint probability distribution for the eigenvalues q and p of quantum-mechanical operators $\hat{q}$ and $\hat{p}$. Margenau and Hill[9] have shown that for some systems in a pure state, this distribution takes on negative values. However, for oscillator type systems, it can be shown that the thermal Wigner distribution is a non-negative Gaussian in both p and q. Fetter's model with the harmonic Hamiltonian falls into this category. Thus Wigner's distribution can be suitably chosen as a joint probability density for our canonical non-commuting operators.

3. The Thermal Average Distribution

The thermal averaging of the delta function distribution involves an application of the Baker-Hausdorff formula (Part II-A.3), which yields

$$\vec{W} = \frac{e^{-r^2/\overline{\langle u^2\rangle}}}{\pi\overline{\langle u^2\rangle}}\,\hat{z}\,. \tag{28}$$

As expected, this result is Gaussian, the spread being the expectation of the square displacement averaged over the z direction. We see that this is the same result as is obtained by subjecting the vortex to classical random noise (Eq.(7)). Unless a partial trace is taken, *i.e.*, a maximum wavenumber cut-off used in the calculation of the mean square displacement (Part II-A.2), $\overline{\langle u^2\rangle}$ diverges and the distribution vanishes. In fact, at a non-zero temperature, a lower cut-off on the order of $2\pi/L$ must also be taken in order that the distribution not vanish. These partial traces must be taken despite the fact that a strict definition of quantum field operators requires no limits on the wavenumbers (total traces). Also, an explicit calculation is made (Part II-A.4) to show that the components of the distribution in the x,y plane do not, on the average, contribute.

E. The Velocity Distribution

The velocity is solved for from the quantum delta function distribution (Part II-C.1), subject to the condition that the divergence of the velocity is zero. The result obtained for the thermal average velocity (Part II-C.2) is,

$$\langle \vec{v} \rangle = \frac{K}{2\pi} \frac{\left(1 - e^{-r^2/\langle u^2 \rangle}\right)}{r} \hat{\varphi}, \tag{29}$$

where $\hat{\varphi}$ is the azimuthal direction. For $r \gg \sqrt{\langle u^2 \rangle}$, the average velocity is irrotational,

$$\langle \vec{v} \rangle \sim \frac{K}{2\pi r} \hat{\varphi}. \tag{30}$$

For $r \ll \sqrt{\langle u^2 \rangle}$, the average velocity goes as the radius, indicating rotation as a solid body:

$$\langle \vec{v} \rangle \sim \frac{K r}{2\pi \langle u^2 \rangle} \hat{\varphi}. \tag{31}$$

The curl of the zero-point average velocity for $r \ll \sqrt{\langle u^2 \rangle}$ is

$$\nabla \times \langle \vec{v} \rangle_0 \sim \frac{K}{\pi \langle u^2 \rangle_0} \hat{z}. \tag{32}$$

This calculation verifies explicitly the proposal of McCauley[8] that the effect of purely quantum fluctuations is to smear out the vortex singularity and give the appearance of a classical core. Thus $\sqrt{\langle u^2 \rangle_0}$ is identified as the core radius, a.

F. Kinetic Energy Density

The kinetic energy density, assuming a near-uniform mass density, is

$$\epsilon = \rho v^2/2. \tag{33}$$

From the solution for the velocity Fourier coefficient (Eq.(105)), a calculation is made for the average of v^2. The averaging process here only involves a redefinition of the constants in the Baker-Hausdorff formula (Part II-D.1). Since the maximum wavenumber, ℓ_{max} must be left finite to some extent, otherwise everything vanishes, I used ℓ_{max} finite only insofar as it enters into the definition of a^2. Due to the complex nature of the correlation, the energy density is found only for zero temperature in order to simplify calculations. The result, in the limit of large wavenumbers, is that the velocities decouple (Part II-D.2):

$$\langle v^2 \rangle_0 = \langle \vec{v} \rangle_0 \cdot \langle \vec{v} \rangle_0 \, . \tag{34}$$

Thus, the zero-point average kinetic energy density distribution becomes:

$$\epsilon_0 = \frac{\rho}{2} \left(\frac{K}{2\pi}\right)^2 \frac{\left(1 - e^{-r^2/a^2}\right)^2}{r^2} , \tag{35}$$

where a^2 is the mean square fluctuation due to zero-point motion.

II. DETAILS OF THE CALCULATION

A. Calculation of the Average Vorticity by Assuming Wigner's Distribution

1. Transformation of Fetter's Hamiltonian

Fetter's Hamiltonian is

$$H = \frac{1}{2} \sum_{\ell} \omega(\ell) \left(q_\ell q_{-\ell} + p_\ell p_{-\ell} \right) , \tag{36}$$

$$\ell = n 2\pi / L , \qquad n = 0, \pm 1, \pm 2 \ldots \tag{37}$$

where L is the extent of the system in the z direction.

$$\omega(\ell) = \frac{K}{\pi a^2} \left(\frac{1}{2} - I_1(a\ell) K_1(a\ell) \right) . \tag{38}$$

$$q(z) = \frac{1}{\sqrt{L}} \sum_{\ell} e^{i\ell z} q_\ell \quad , \qquad p(z) = \frac{1}{\sqrt{L}} \sum_{\ell} e^{-i\ell z} p_\ell \; . \tag{39}$$

$$q_\ell = \frac{1}{\sqrt{L}} \int_{-\infty}^{+\infty} dz \, e^{-i\ell z} q(z) \; , \qquad p_\ell = \frac{1}{\sqrt{L}} \int_{-\infty}^{+\infty} dz \, e^{i\ell z} p(z) . \tag{40}$$

$$\left[q(z), p(z') \right] = i\hbar \, \delta(z - z') \quad \Rightarrow \quad \left[q_\ell , p_{\ell'} \right] = i\hbar \, \delta_{\ell\ell'} \; . \tag{41}$$

Since the integral representation for the modified Bessel function of the second kind that Fetter used in his development of the Hamiltonian is strictly defined for $\ell \geqslant 0$[12], let the sum in the Hamiltonian be restricted to non-negative wavenumbers:

$$H = \sum_{\ell \geqslant 0} \omega(\ell) \left(q_\ell q_{-\ell} + p_\ell p_{-\ell} \right) . \tag{42}$$

To transform to self-adjoint, canonical operators, let

$$q_\ell = \frac{q_{2\ell} + i p_{1\ell}}{\sqrt{2}} \; , \qquad p_\ell = \frac{p_{2\ell} + i q_{1\ell}}{\sqrt{2}} \; . \tag{43}$$

$$q_{1\ell} = \frac{p_\ell - p_{-\ell}}{\sqrt{2}\, i} \; , \qquad p_{1\ell} = \frac{q_\ell - q_{-\ell}}{\sqrt{2}\, i} \; . \tag{44}$$

$$q_{2\ell} = \frac{q_\ell + q_{-\ell}}{\sqrt{2}} \; , \qquad p_{2\ell} = \frac{p_\ell + p_{-\ell}}{\sqrt{2}} \; . \tag{45}$$

$$\left[q_{1\ell} , p_{1\ell} \right] = \left[q_{2\ell} , p_{2\ell} \right] = i\hbar . \tag{46}$$

$$\left[q_{1\ell} , q_{2\ell} \right] = \left[p_{1\ell} , p_{2\ell} \right] = \left[q_{1\ell} , p_{2\ell} \right] = \left[q_{2\ell} , p_{1\ell} \right] = 0 . \tag{47}$$

$q_{1\ell}$, $p_{1\ell}$ are odd functions of ℓ and,

$q_{2\ell}$, $p_{2\ell}$ are even functions of ℓ .

$$H = \frac{1}{2}\sum_{\ell\geq 0} \omega(\ell)\left(q_{1\ell}^{2} + p_{1\ell}^{2} + q_{2\ell}^{2} + p_{2\ell}^{2}\right). \tag{48}$$

Let $\quad a_-(\ell) = \dfrac{q_{1\ell} + i\,p_{1\ell}}{\sqrt{2\hbar}}, \qquad a_+(\ell) = \dfrac{q_{2\ell} + i\,p_{2\ell}}{\sqrt{2\hbar}}.$ (49)

$[a_-(\ell), a_-^\dagger(\ell)] = [a_+(\ell), a_+^\dagger(\ell)] = 1$: Bose creation and annihilation operators. (50)

$$a_-^\dagger a_- = N_-(\ell), \qquad a_+^\dagger a_+ = N_+(\ell). \tag{51}$$

$$H = \sum_{\ell\geq 0} \hbar\,\omega(\ell)\left(N_+(\ell) + N_-(\ell) + 1\right). \tag{52}$$

2. Mean Square Deformation

$$u_x(z) = \frac{q(z)}{\sqrt{\rho K}} = \frac{1}{\sqrt{\rho K L}}\sum_\ell e^{i\ell z} q_\ell = \frac{1}{\sqrt{2\rho K L}}\sum_\ell e^{i\ell z}(q_\ell + i p_\ell).$$

$$= \sqrt{\frac{2}{\rho K L}}\sum_{\ell\geq 0}\left(\cos\ell z\, q_{2\ell} - \sin\ell z\, p_{1\ell}\right). \tag{53}$$

$$u_y(z) = \frac{p(z)}{\sqrt{\rho K}} = \frac{1}{\sqrt{\rho K L}}\sum_\ell e^{-i\ell z} p_\ell = \frac{1}{\sqrt{2\rho K L}}\sum_\ell e^{-i\ell z}(p_{2\ell} + i q_{1\ell}).$$

$$= \sqrt{\frac{2}{\rho K L}}\sum_{\ell\geq 0}\left(\cos\ell z\, p_{2\ell} + \sin\ell z\, q_{1\ell}\right). \tag{54}$$

$$u^2(z) = \frac{1}{\rho K}\left(q^2(z) + p^2(z)\right). \tag{55}$$

$$\overline{u^2(z)} = \frac{1}{\rho K L}\int_{-\infty}^{+\infty} dz\left(q^2(z) + p^2(z)\right) = \frac{1}{\rho K L}\sum_\ell\left(q_\ell q_{-\ell} + p_\ell p_{-\ell}\right). \tag{56}$$

$$E = \langle H\rangle = \sum_{\ell\geq 0} E_\ell = \sum_{\ell\geq 0}\hbar\,\omega(\ell)\coth\left(\beta\hbar\omega(\ell)/2\right). \tag{57}$$

$$\overline{\langle u^2\rangle} = \frac{2}{\rho KL}\sum_{\ell\geq 0} E_\ell/\omega(\ell) = \frac{2\hbar}{\rho KL}\sum_{\ell\geq 0}\coth(\beta\hbar\,\omega(\ell)/2). \tag{58}$$

$\beta = 1/k_B T$; $T = {}^\circ K$, $k_B =$ Boltzmann's constant.

$$\overline{\langle u^2\rangle} = \frac{2\hbar}{\rho KL}\left(\frac{L}{2\pi}\right)\int_{\ell_{min}\sim 2\pi/L}^{\ell_{max}} d\ell\ \coth(\beta\hbar\,\omega(\ell)/2). \tag{59}$$

$$\overline{\langle u^2\rangle}_0 = \frac{\hbar}{\pi\rho K}\int_0^{\ell_{max}} d\ell = \frac{\hbar\ell_{max}}{\pi\rho K} = a^2. \tag{60}$$

3. Expectation of the Distribution

$$(\vec{W})_z = \sum_{\vec{k}} e^{i\vec{k}\cdot\vec{r}}\left\langle e^{-i\vec{k}\cdot\vec{u}(z)}\right\rangle = \sum_{\vec{k}} e^{i\vec{k}\cdot\vec{r}}(\vec{W}_k)_z. \tag{61}$$

$$(\vec{W}_k)_z = \frac{\mathrm{Tr}\ e^{-\beta H}\,e^{-i\vec{k}\cdot\vec{u}(z)}}{Z}. \tag{62}$$

$$Z = \mathrm{Tr}\ e^{-\beta H}. \tag{63}$$

Because the operators for each mode commute, a product of thermal averages may be taken:

$$(\vec{W}_k)_z = \prod_{\ell\geq 0}\frac{\mathrm{Tr}\ \exp\left[-\frac{\beta\omega(\ell)}{2}\left(q_{1\ell}^2 + p_{1\ell}^2 + q_{2\ell}^2 + p_{2\ell}^2\right)\right]}{Z}\ \times$$
$$\exp\left[-ik_x\sqrt{\frac{2}{\rho KL}}\left(\cos\ell z\, q_{2\ell} - \sin\ell z\, p_{1\ell}\right) - ik_y\sqrt{\frac{2}{\rho KL}}\left(\cos\ell z\, p_{2\ell} + \sin\ell z\, q_{1\ell}\right)\right].$$

$$= \prod_{\ell\geq 0}\frac{\mathrm{Tr}\ \exp\left[\frac{-\beta\omega(\ell)}{2}\left(q_{1\ell}^2 + p_{1\ell}^2\right)\right]}{Z}\exp\left[i\sqrt{\frac{2}{\rho KL}}\sin\ell z\left(k_x p_{1\ell} - k_y q_{1\ell}\right)\right]\times$$
$$\exp\left[\frac{-\beta\omega(\ell)}{2}\left(q_{2\ell}^2 + p_{2\ell}^2\right)\right]\exp\left[-i\sqrt{\frac{2}{\rho KL}}\cos\ell z\left(k_y p_{2\ell} + k_x q_{2\ell}\right)\right]. \tag{65}$$

Eq.(65) is the same form as a particular application of the Baker-Hausdorff formula worked out by Weiss and Maradudin[13], namely

$$\left.\begin{aligned} &e^{\mathcal{P}^2+Q^2}\, e^{\alpha\mathcal{P}+\beta Q} = e^{\mathcal{P}'^2+Q'^2}\, e^{-\frac{1}{4}c\cot c\,(\alpha^2+\beta^2)} . \\ &c=[\mathcal{P},Q]. \\ &\mathcal{P}'=\mathcal{P}+\tfrac{1}{2}c(\beta+\alpha\cot c)\,,\quad Q'=Q+\tfrac{1}{2}c(\beta\cot c-\alpha)\,. \end{aligned}\right\} \quad (66)$$

Let the following identifications be made:

$$\left.\begin{aligned} &\mathcal{P}_1 = i\sqrt{\frac{\beta\omega(\ell)}{2}}\,p_{1\ell}\,, && Q_1 = i\sqrt{\frac{\beta\omega(\ell)}{2}}\,q_{1\ell}\,. \\ &\mathcal{P}_2 = i\sqrt{\frac{\beta\omega(\ell)}{2}}\,p_{2\ell}\,, && Q_2 = i\sqrt{\frac{\beta\omega(\ell)}{2}}\,q_{2\ell}\,. \\ &\alpha_1 = \frac{2k_x\sin\ell z}{\sqrt{\rho KL\beta\omega(\ell)}}\,, && \beta_1 = \frac{-2k_y\sin\ell z}{\sqrt{\rho KL\beta\omega(\ell)}}\,. \\ &\alpha_2 = \frac{-2k_y\cos\ell z}{\sqrt{\rho KL\beta\omega(\ell)}}\,, && \beta_2 = \frac{-2k_x\cos\ell z}{\sqrt{\rho KL\beta\omega(\ell)}}\,. \end{aligned}\right\} \quad (67)$$

$$c_1 = c_2 = [\mathcal{P}_{1,2}, Q_{1,2}] = \frac{-\beta\omega(\ell)}{2}[p_{1,2\ell}, q_{1,2\ell}] = \frac{i\beta\hbar\omega(\ell)}{2}. \quad (68)$$

$$(\vec{W}_k)_z = \prod_{\ell>0} \frac{\mathrm{Tr}\exp(Q_1'^2+\mathcal{P}_1'^2+Q_2'^2+\mathcal{P}_2'^2)}{Z} \times \exp\left[-\frac{i\beta\hbar\omega(\ell)}{4\cdot 2}\cot(i\beta\hbar\omega(\ell)/2)(\alpha_1^2+\beta_1^2+\alpha_2^2+\beta_2^2)\right]. \quad (69)$$

Since $(Q,P)\longrightarrow(Q',P')$ is a canonical transformation the remaining $\mathrm{Tr}\exp(Q_1'^2+P_1'^2+Q_2'^2+P_2'^2)$ cancels with Z.

$$(\vec{W}_k)_z = \prod_{\ell>0}\exp\left[-\frac{\beta\hbar\omega(\ell)}{8}\coth(\beta\hbar\omega(\ell)/2)\frac{4k^2}{\rho KL\beta\omega(\ell)}\right]. \quad (70)$$

$$(\vec{w}_k)_z = \prod_{\ell>0} \exp\left[\frac{-\hbar k^2}{2\rho KL} \coth\left(\frac{\beta\hbar\omega(\ell)}{2}\right)\right] = \exp\left[-\frac{k^2}{4}\left(\frac{2\hbar}{\rho KL}\right)\sum_{\ell>0} \coth\left(\frac{\beta\hbar\omega(\ell)}{2}\right)\right]. \quad (71)$$

$$(\vec{w}_k)_z = \exp(-k^2\overline{\langle u^2\rangle}/4). \quad (72)$$

$$(\vec{w})_z = \sum_{\vec{k}} e^{i\vec{k}\cdot\vec{r}} e^{-k^2\overline{\langle u^2\rangle}/4}. \quad (73)$$

$$(\vec{w})_z = \frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty} d^2k\, e^{i\vec{k}\cdot\vec{r}} e^{-k^2\overline{\langle u^2\rangle}/4} = \frac{e^{-r^2/\overline{\langle u^2\rangle}}}{\pi\overline{\langle u^2\rangle}}. \quad (74)$$

4. Components of the Vorticity in the Plane

$$\vec{w} = \sum_{\vec{k}} e^{i\vec{k}\cdot\vec{r}} \left\langle e^{-i\vec{k}\cdot\vec{u}}\left(\hat{z} + \frac{d\vec{u}}{dz}\right)\right\rangle. \quad (75)$$

$$(\vec{w}_k)_{x,y} = \left\langle e^{-i\vec{k}\cdot\vec{u}} \frac{d\vec{u}}{dz}\right\rangle. \quad (76)$$

Because differentiation is independent of the thermal averaging process, the planar components of the distribution may be represented so:

$$(\vec{w}_k)_{x,y} = \Big|_{r'=0} i\,\nabla' \left\langle e^{-i\vec{k}\cdot\vec{u} - i\vec{r}'\cdot d\vec{u}/dz}\right\rangle. \quad (77)$$

Here the exponential operators may be combined because

$$\left[-i\vec{k}\cdot\vec{u}, -i\vec{r}'\cdot\frac{d\vec{u}}{dz}\right] = \frac{ik_x y'}{\rho KL}\sum_{\ell,\ell'} \ell' e^{i(\ell-\ell')z}[q_\ell, p_{\ell'}] + \frac{ik_y x'}{\rho KL}\sum_{\ell,\ell'} \ell\, e^{i(\ell-\ell')z}[q_\ell, p_{\ell'}].$$

$$= \frac{-\hbar}{\rho KL}(k_x y' + k_y x')\sum_{\ell} = 0. \quad (78)$$

The solution to Eq.(77) involves redefining the constants used in the Baker-Hausdorff formula, Eq.(67), in this way:

$$\left.\begin{aligned} \alpha_1' &= \frac{2(k_x \sin lz + x'l \cos lz)}{\sqrt{\rho K L \beta \omega(l)}}, & \beta_1' &= \frac{-2(k_y \sin lz + y'l \cos lz)}{\sqrt{\rho K L \beta \omega(l)}}. \\ \alpha_2' &= \frac{2(-k_y \cos lz + y'l \sin lz)}{\sqrt{\rho K L \beta \omega(l)}}, & \beta_2' &= \frac{2(-k_x \cos lz + x'l \sin lz)}{\sqrt{\rho K L \beta \omega(l)}}. \end{aligned}\right\} \tag{79}$$

$$(\vec{W}_k)_{x,y} = \Big|_{r'=0} i\,\nabla' \prod_{l \geq 0} \exp\left[\frac{-\hbar}{2\rho K L} \coth\left(\frac{\beta\hbar\omega(l)}{2}\right)(k^2 + r'^2 l^2)\right]. \tag{80}$$

$$(\vec{W}_k)_{x,y} = \Big|_{r'=0} i\,\nabla' \exp\left[\frac{-k^2\langle u^2\rangle}{4} \quad \frac{-\hbar r'^2}{2\rho K L}\sum_{l \geq 0} l^2 \coth\left(\frac{\beta\hbar\omega(l)}{2}\right)\right]. \tag{81}$$

$$(\vec{W}_k)_{x,y} = \Big|_{r'=0} \exp\left[\frac{-k^2\langle u^2\rangle}{4} \quad \frac{-\hbar r'^2}{2\rho K L}\sum_{l \geq 0} l^2 \coth\left(\frac{\beta\hbar\omega(l)}{2}\right)\right] \times$$

$$\left(\frac{-i r'}{\rho K L}\sum_{l \geq 0} l^2 \coth\left(\frac{\beta\hbar\omega(l)}{2}\right)\right)\hat{r} = 0. \tag{82}$$

B. Identification of Wigner's Distribution

For a single mode of vibration,

$$W_l = \sum_{\vec{k}} e^{i\vec{k}\cdot\vec{r}} \left\langle \exp\left(\frac{-ik_x}{\sqrt{\rho K L}} e^{ilz} q_l \quad \frac{-ik_y}{\sqrt{\rho K L}} e^{-ilz} p_l\right)\right\rangle. \tag{83}$$

$$\left.\begin{aligned} &\text{Let} \quad \xi = \frac{-\hbar k_x}{\sqrt{\rho K L}} e^{ilz}, & \eta &= \frac{-\hbar k_y}{\sqrt{\rho K L}} e^{-ilz}. \\ &q = \sqrt{\rho K L}\, x, & p &= \sqrt{\rho K L}\, y. \\ &\hat{q} = q_l, & \hat{p} &= p_l. \end{aligned}\right\} \tag{84}$$

$$[\hat{q}, \hat{p}] = i\hbar. \tag{85}$$

$$W_\ell = \frac{\rho K L}{\hbar^2} \sum_{\xi,\eta} e^{-i\xi q e^{-i\ell z}/\hbar} e^{-i\eta p e^{i\ell z}/\hbar} \left\langle e^{i\xi\hat{q}/\hbar + i\eta\hat{p}/\hbar} \right\rangle. \tag{86}$$

Now $e^{\hat{A}+\hat{B}} = e^{\hat{A}} e^{\hat{B}} e^{-\frac{1}{2}c}$ if $[\hat{A},\hat{B}] = c$, so

$$W_\ell = \frac{\rho K L}{\hbar^2} \sum_{\xi,\eta} e^{-i\xi q e^{-i\ell z}/\hbar} e^{-i\eta p e^{i\ell z}/\hbar} e^{+i\xi\eta/2\hbar} \times \left\langle e^{i\xi\hat{q}/\hbar} e^{i\eta\hat{p}/\hbar} \right\rangle. \tag{87}$$

For a pure state, the expectation of the operators is:

$$\langle\psi| e^{i\xi\hat{q}/\hbar} e^{i\eta\hat{p}/\hbar} |\psi\rangle.$$

By inserting the identity, this becomes

$$\langle\psi| e^{i\xi\hat{q}/\hbar} e^{i\eta\hat{p}/\hbar} |\psi\rangle = \sum_{q'} \langle\psi| e^{i\xi\hat{q}/\hbar} |q'\rangle\langle q'| e^{i\eta\hat{p}/\hbar} |\psi\rangle. \tag{88}$$

Since $\hat{p}/\hbar$ is the generator of infinitesimal translations in q-space (likewise $\hat{q}/\hbar$ for p-space),

$$= \sum_{q'} e^{i\xi q'/\hbar} \langle\psi|q'\rangle\langle q'+\eta|\psi\rangle. \tag{89}$$

$$W_{\ell,\psi} = \frac{\rho K L}{\hbar^2} \sum_{\xi,\eta} e^{-i\xi q e^{-i\ell z}/\hbar} e^{-i\eta p e^{i\ell z}/\hbar} e^{+i\xi\eta/2\hbar} \times \sum_{q'} e^{i\xi q'/\hbar} \langle\psi|q'\rangle\langle q'+\eta|\psi\rangle. \tag{90}$$

$$W_{\ell,\psi} = \frac{\rho K L}{\hbar^2} \sum_{\eta} e^{-i\eta p e^{i\ell z}/\hbar} \sum_{q'} \langle\psi|q'\rangle\langle q'+\eta|\psi\rangle \times \delta(q'/\hbar - q e^{-i\ell z}/\hbar + \eta/2\hbar). \tag{91}$$

$$W_{\ell,\psi} = \frac{\rho K L}{\hbar} \sum_{\eta} e^{-i\eta p e^{i\ell z}/\hbar} \langle\psi| q e^{-i\ell z} - \eta/2\rangle\langle q e^{-i\ell z} + \eta/2|\psi\rangle. \tag{92}$$

Now let $e^{-ilz} q \to q$ and $e^{ilz} p \to p$, then for a mixed state, and neglecting the normalization Tr $e^{-\beta H}$,

$$W_\ell = \frac{\rho K L}{2\pi\hbar} \int_{-\infty}^{+\infty} d\eta \, e^{-i\eta p/\hbar} \, \mathrm{Tr}\, e^{-\beta H} |q - \eta/2\rangle \langle q + \eta/2| . \tag{93}$$

$$W_\ell = \frac{\rho K L}{2\pi\hbar} \int_{-\infty}^{+\infty} d\eta \, e^{-i\eta p/\hbar} \int_{-\infty}^{+\infty} dq' \langle q' | \rho | q - \eta/2\rangle \underbrace{\langle q + \eta/2 | q' \rangle}_{\delta(q' - q - \eta/2)} . \tag{94}$$

$$W_\ell = \frac{\rho K L}{2\pi\hbar} \int_{-\infty}^{+\infty} d\eta \, e^{-i\eta p/\hbar} \langle q + \eta/2 | \rho | q - \eta/2 \rangle . \tag{95}$$

$$W_\ell = \rho K L \left(\frac{1}{2\pi\hbar}\right) \int_{-\infty}^{+\infty} d\eta \, e^{-i\eta p/\hbar} \rho(q + \eta/2, q - \eta/2) . \tag{96}$$

$$W_\ell = \rho K L \, f_W(q, p). \tag{97}$$ [10]

As a check, we see that

$$\iint_{-\infty}^{+\infty} dx\,dy\, W_\ell = \rho K L \iint_{-\infty}^{+\infty} dx\,dy\, f_w(p, q) = \iint_{-\infty}^{+\infty} dq\,dp\, f_w(q, p) = 1. \tag{98}$$

C. Calculation of the Average Velocity

1. Derivation of the Velocity Fourier Coefficient

$$\nabla \times \vec{v} = K \sum_{\vec{k}} e^{i\vec{k}\cdot\vec{r}} e^{-i\vec{k}\cdot\vec{u}(z)} \hat{z}. \tag{99}$$

$$\vec{v} = \sum_{\vec{k}} e^{i\vec{k}\cdot\vec{r}} \vec{V}(\vec{k}, \vec{u}(z)). \tag{100}$$

$$\nabla \times \vec{v} = \sum_{\vec{k}} e^{i\vec{k}\cdot\vec{r}} \, i\vec{k} \times \vec{V} . \tag{101}$$

$$i\vec{k}\times\vec{V} = K\, e^{-i\vec{k}\cdot\vec{u}(z)}\,\hat{z}. \tag{102}$$

$$i\vec{k}\times(\vec{k}\times\vec{V}) = i(\vec{k}\cdot\vec{V})\vec{k} - ik^2\vec{V} = K e^{-i\vec{k}\cdot\vec{u}(z)}\,\vec{k}\times\hat{z}. \tag{103}$$

$$\nabla\cdot\vec{v} = 0 \quad\Rightarrow\quad \vec{k}\cdot\vec{V} = 0. \tag{104}$$

$$\vec{V} = iKe^{-i\vec{k}\cdot\vec{u}(z)}\,\frac{\vec{k}\times\hat{z}}{k^2} = iKe^{-i\vec{k}\cdot\vec{u}(z)}\,\frac{(k_y\hat{x}-k_x\hat{y})}{k^2}. \tag{105}$$

2. The Averaging and Fourier Inversion

$$\langle\vec{V}\rangle = iK\,\frac{(k_y\hat{x}-k_x\hat{y})}{k^2}\left\langle e^{-i\vec{k}\cdot\vec{u}(z)}\right\rangle. \tag{106}$$

$$\langle\vec{V}\rangle = iK\,\frac{(k_y\hat{x}-k_x\hat{y})}{k^2}\; e^{-k^2\langle u^2\rangle/4}. \tag{107}$$

$$\langle\vec{v}\rangle = iK\sum_{\vec{k}} e^{i\vec{k}\cdot\vec{r}}\; e^{-k^2\langle u^2\rangle/4}\;\frac{(k_y\hat{x}-k_x\hat{y})}{k^2}. \tag{108}$$

$$\langle\vec{v}\rangle = \frac{iK}{(2\pi)^2}\int_0^\infty dk\, e^{-k^2\langle u^2\rangle/4}\underbrace{\underbrace{\int_0^{2\pi} d\varphi\, e^{ikr\cos(\varphi-\theta)}(\sin\varphi\,\hat{x}-\cos\varphi\,\hat{y})}_{\int_{-\theta}^{2\pi-\theta} d\alpha\, e^{ikr\cos\alpha}\left[(\sin\alpha\cos\theta+\cos\alpha\sin\theta)\hat{x}-(\cos\alpha\cos\theta-\sin\alpha\sin\theta)\hat{y}\right]}}_{2\pi i\, J_1(kr)(\sin\theta\,\hat{x}-\cos\theta\,\hat{y})}. \tag{109}$$

$$\langle\vec{v}\rangle = \frac{K}{2\pi}\,\hat{\varphi}\int_0^\infty dk\, e^{-k^2\langle u^2\rangle/4}\, J_1(kr). \tag{110}$$

$$\langle\vec{v}\rangle = \frac{K}{2\pi}\,\hat{\varphi}\,\frac{r}{(\langle u^2\rangle)^2}\, M(1,2,-r^2/\langle u^2\rangle).^{14} \tag{111}$$

J_1 is the first-order Bessel function, M is Kummer's function.

$$\langle \vec{v} \rangle = \frac{K}{2\pi}\ \hat{\varphi}\ \frac{r}{(\langle u^2\rangle)^2}\ e^{-r^2/2\langle u^2\rangle}\ \sinh(-r^2/2\langle u^2\rangle)/(-r^2/2\langle u^2\rangle).^{15} \tag{112}$$

$$\langle \vec{v} \rangle = \frac{K}{2\pi\langle u^2\rangle}\ \frac{(1-e^{-r^2/\langle u^2\rangle})}{r}\ \hat{\varphi}. \tag{113}$$

D. Calculation of the Average Kinetic Energy Density

1. The Averaging

$$\vec{V}(\vec{k},\vec{u}(z))\cdot\vec{V}(\vec{k}',\vec{u}(z')) = -K^2\ \frac{\vec{k}\cdot\vec{k}'}{k^2 k'^2}\ e^{-i\vec{k}\cdot\vec{u}(z)}\ e^{-i\vec{k}'\cdot\vec{u}(z')}. \tag{114}$$

$$\left[-i\vec{k}\cdot\vec{u}(z), -i\vec{k}'\cdot\vec{u}(z')\right] = \frac{-k_x k_y'}{\rho KL}\sum_{\ell,\ell'} e^{i\ell z - i\ell' z'}\left[q_\ell, p_{\ell'}\right]$$

$$-\frac{k_y k_x'}{\rho KL}\sum_{\ell,\ell'} e^{-i\ell z + i\ell' z'}\left[p_\ell, q_{\ell'}\right].$$

$$= \frac{i\hbar}{\rho KL}\left(-k_x k_y'\sum_{\ell,\ell'} e^{i\ell z - i\ell' z'}\delta_{\ell\ell'} + k_y k_x'\sum_{\ell,\ell'} e^{-i\ell z + i\ell' z'}\delta_{\ell\ell'}\right).$$

$$= -\frac{i\hbar}{\rho KL}\ kk'\sin(\varphi'-\varphi)\sum_{\ell} e^{i\ell(z'-z)}. \tag{115}$$

$$e^{-i\vec{k}\cdot\vec{u}(z)}\ e^{-i\vec{k}'\cdot\vec{u}(z')} = e^{-i\vec{k}\cdot\vec{u}(z) - i\vec{k}'\cdot\vec{u}(z')}\exp\left(\frac{-i\hbar kk'\sin(\varphi'-\varphi)}{2\rho KL}\sum e^{i\ell(z'-z)}\right) \tag{116}$$

Here again, the only change necessary to solve Eq.(116) is a redefinition of the parameters in Eqs.(67):

$$\left.\begin{aligned}
\alpha_1'' &= \frac{2(k_x\sin\ell z + k_x'\sin\ell z')}{\sqrt{\rho KL\,\beta\omega(\ell)}}, &\qquad \beta_1'' &= \frac{-2(k_y\sin\ell z + k_y'\sin\ell z')}{\sqrt{\rho KL\,\beta\omega(\ell)}}. \\
\alpha_2'' &= \frac{-2(k_y\cos\ell z + k_y'\cos\ell z')}{\sqrt{\rho KL\,\beta\omega(\ell)}}, &\qquad \beta_2'' &= \frac{-2(k_x\cos\ell z + k_x'\cos\ell z')}{\sqrt{\rho KL\,\beta\omega(\ell)}}.
\end{aligned}\right\} \tag{117}$$

$$\left\langle e^{-i\vec{k}\cdot\vec{u}(z)-i\vec{k}'\cdot\vec{u}(z')}\right\rangle = \prod_{\ell\geq 0} \exp\left[\frac{-\hbar}{\rho K L}\coth\left(\frac{\beta\hbar\omega(\ell)}{2}\right)\times \left(k^2+k'^2+2\vec{k}\cdot\vec{k}'\cos\ell(z'-z)\right)\right]. \tag{118}$$

$$\left\langle e^{-i\vec{k}\cdot\vec{u}(z)-i\vec{k}'\cdot\vec{u}(z')}\right\rangle_0 = e^{-a^2(k^2+k'^2)/4} \times \exp\left[\frac{-\hbar\vec{k}\cdot\vec{k}'}{2\rho K L}\sum_\ell e^{i\ell(z'-z)}\right]. \tag{119}$$

$$\left\langle e^{-i\vec{k}\cdot\vec{u}(z)}\, e^{-i\vec{k}'\cdot\vec{u}(z')}\right\rangle_0 = e^{-a^2(k^2+k'^2)/4} \times \exp\left[\frac{-\hbar k k' Z_1}{2\rho K L}\sum_\ell e^{i\ell(z'-z)}\right]. \tag{120}$$

$$Z_1 = \cos(\varphi'-\varphi) + i\sin(\varphi'-\varphi). \tag{121}$$

$$\frac{-\hbar k k' Z_1}{2\rho K L}\sum_\ell e^{i\ell(z'-z)} = -\frac{\hbar k k' Z_1}{2\pi\rho K}\int_0^{\ell_{max}} d\ell\, \cos\ell(z'-z). \tag{122}$$

$$= -\frac{\hbar k k' Z_1}{2\pi\rho K}\;\frac{\sin\ell_{max}(z'-z)}{(z'-z)}.$$

$$= -\frac{a^2 k k' Z_1}{2}\;\frac{\sin\ell_{max}(z'-z)}{\ell_{max}(z'-z)}. \tag{123}$$

$$\lim_{\ell_{max}\to\infty}\left(\frac{-\hbar k k' Z_1}{2\rho K L}\sum_\ell e^{i\ell(z'-z)}\right) = -\frac{a^2 k k' Z_1}{2}\lim_{\ell_{max}\to\infty}\frac{\sin\ell_{max}(z'-z)}{\ell_{max}(z'-z)} = 0. \tag{124}$$

$$\left\langle \vec{V}(\vec{k},\vec{u}(z))\cdot\vec{V}(\vec{k}',\vec{u}(z'))\right\rangle_0 = -K^2\,\frac{\vec{k}\cdot\vec{k}'}{k^2 k'^2}\, e^{-a^2(k^2+k'^2)/4}. \tag{125}$$

2. Fourier Inversion

$$\langle v^2\rangle_0 = \lim_{z'\to z}\sum_{\vec{k},\vec{k}'} e^{i(\vec{k}+\vec{k}')\cdot\vec{r}}\left\langle \vec{V}(\vec{k},\vec{u}(z))\cdot\vec{V}(\vec{k}',\vec{u}(z'))\right\rangle_0. \tag{126}$$

$$\langle v^2\rangle_0 = -\frac{K^2}{(2\pi)^4}\int_0^\infty dk\, e^{-a^2k^2/4}\int_0^\infty dk'\, e^{-a^2k'^2/4} \;\times$$

$$\times \int_0^{2\pi} d\varphi\, e^{ikr\cos(\varphi-\theta)} \underbrace{\int_0^{2\pi} d\varphi' e^{ik'r\cos(\varphi'-\theta)} \cos(\varphi'-\varphi)}_{\underbrace{\int_{-\theta}^{2\pi-\theta} d\alpha\, e^{ik'r\cos\alpha} (\cos\alpha\cos(\varphi-\theta) + \sin\alpha\sin(\varphi-\theta))}_{2\pi i J_1(k'r)\cos(\varphi-\theta)}} . \tag{127}$$

$$\langle v^2\rangle_0 = \left(\frac{K}{2\pi}\right)^2 \int_0^\infty dk\, e^{-a^2k^2/4} J_1(kr) \int_0^\infty dk' e^{-a^2k'^2/4} J_1(k'r) . \tag{128}$$

$$\langle v^2\rangle_0 = \left[\frac{K}{2\pi}\frac{(1-e^{-r^2/a^2})}{r}\right]^2 = \langle \vec{v}\rangle_0 \cdot \langle \vec{v}\rangle_0 . \tag{129}$$

III. SUMMARY

As comprehensive as Fetter's quantum theory of the superfluid vortex is, he did not indicate how one would go about obtaining a quantum model for the vorticity, and from there the velocity and kinetic energy density. By a simple extension of Fetter's work, namely assuming Wigner's distribution for the vorticity with Fetter's deformation operators as the quantum noise source, I present here for the first time a quantum model for the vorticity. From this model, the velocity around the quantized vortex is calculated. Having obtained then a quantum model for the velocity, a thermal averaging provided an explicit verification of the proposal[8] that purely quantum fluctuations smear out the vortex singularity and give the appearance of a classical core. From the velocity, a quantum model of the kinetic energy density is obtained. It is also found that for the average zero-point energy density the velocities are uncorrelated.

The quantum model for the velocity may bear several potential applications. In particular, a classical model for the velocity has been used in the calculation of the mutual friction force (due to the scattering of phonons) between the normal and superfluid components of HeII. Pitaevskii[16] has made this calculation in the Born approximation, valid for distances from the vortex greater than the phonon wavelength. Fetter[17] has made the exact calculation, in the long wavelength limit. In both cases, the scattering of phonons does not probe the core. That this calculation may now be done taking into account the quantum fluctuations of the vortex, is a problem for future research. It is not clear whether the effect of quantum fluctuations will be experimentally measurable.

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