ON THE CONGRUENCE LATTICE OF A SEMILATTICE

A Dissertation

Presented to

The Faculty of the Department of Mathematics

University of Houston

In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

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Elliott L. Evans

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ABSTRACT

Here is attempted an examination of three aspects of the lattice $\Theta(S)$ of congruence relations of a semilattice S (usually a join semilattice). The impetus for two of the three investigations is provided by the recent result of Fajtlowicz and Schmidt [5] that $\Theta(S)$ is dual to the lattice $\Theta^*(S^*)$ of algebraic closure subfamilies of S^* (the ideal lattice of S, $\emptyset$ included). So in Chapter I we generalize the notion of algebraic closure family and examine the lattice $\Theta^*(X)$ of algebraic closure families in a complete lattice X. There we obtain a second order characterization theorem for $\Theta^*(X)$ by axiomatizing the occurence in $\Theta^*(X)$ of a copy of X^{op} (the dual of X). This is called the upper spot in $\Theta^*(X)$. Of course a characterization of $\Theta^*(S^*)$ (and so of $\Theta(S)$) follows in the case that X is algebraic. We also find that the posession by a complete atomic lattice L of such a spot is tantamount to its being decomposable in a certain nice way ("atomwise") into disjoint complete join semilattices. Chapter II plays off the fact that $\Theta^*(S^*)$ is atomic and investigates the possibility that it is the presence (in sufficient quantities) of certain special types of atoms (especially primes) which results in $\Theta^*(S^*)$'s various properties. The notion of a prime atom is presented in $\Theta^*(S^*)$ as a correlate to the notion of finitely meet irreducible element of S^* and it is shown that under certain conditions a complete atomic lattice with enough prime atoms

will be dually-semi-Brouwerian, M^*-symmetric (so lower semimodular) and, if algebraic, fully dually quasi-decomposable. So $\Theta^*(S^*)$ gets these properties, hence $\Theta(S)$ their duals. In Chapter III we take a quite different tack. Here only, we work with meet semilattices S and $\Theta^*(S^*)$ does not come into play. Our work is based on the fact that for a meet semilattice M, $\Theta(M)$ is distributive iff M is a tree. So a concrete examination of semilattice trees and their congruences is attempted in the first part of this chapter. But gradually, we find that the compact congruences on M form a Boolean ring, and a special one at that. We then find ourselves studying the Boolean ring B[M] universal over a meet semilattice M. In the end we find that M is a semilattice tree iff $\Theta(M) \cong \mathcal{I}(E_M)$ the ideal lattice of the evenly generated ideal E_M of B[M]. So with E_M a Boolean ring we see that the compact congruences of a tree T form a Boolean ring E_T. Better yet if T has a zero, E_T = B[T] so $\Theta(T) \cong \mathcal{I}(B[T])$ so compact$(\Theta(T)) \cong$ B[T].

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CHAPTER 0

PRELIMINARIES

1. Review of some general facts

Pseudocomplemented Semilattices. Let S be a meet semilattice with 0. S is called pseudocomplemented if for each $x \varepsilon S$ there is a largest element of S disjoint from x, that is $MAX\{y | x \wedge y = 0\}$ exists. Call this latter element the pseudocomplement of x and denote it $\bar{x}$. Now in such a semilattice: $x \leq \bar{y}$ iff $x \wedge y = 0$. So $x \leq \bar{y}$ iff $y \leq \bar{x}$ and so the pair $(\overline{(\ )}, \overline{(\ )})$, where by $\overline{(\ )}$ we mean the map $x \longmapsto \bar{x}$, establishes an Ore type Galois connection between S and itself. The usual consequences of this then follow. For instance: $im\overline{(\ )} = im\overline{\overline{(\ )}}$ is a complete lattice which is self dual, $im\overline{(\ )}$ is meet closed in S, $\overline{\overline{(\ )}}$ is a closure operator in S and if $s = \sup_S s_t$ then $\bar{s} = \inf_S \bar{s}_t$. But in fact more holds. First $im\overline{(\ )} = B$ turns out to be a Boolean lattice whose complementation is $b \longmapsto \bar{b}$ (and whose meet is the same as S's). We call B the Boolean algebra of closed elements. Also, and the first result depends on this, the equation of Glivenko: $\bar{\bar{x}} \wedge \bar{\bar{y}} = \overline{\overline{x \wedge y}}$, holds. We call $x \varepsilon S$ dense iff $\bar{\bar{x}} = 1$ (S, it turns out has to have a 1) or $\bar{x} = 0$. The dense elements form a filter called the filter of dense elements. S is called quasi-decomposable if each x in S can be written $x = \bar{\bar{x}} \wedge d$ for some dense d. (See Schmidt [16].) It is known that if the pseudocomplemented semilattice S is distributive then S is quasi-decomposable.

Some words are in order for the dual notion. Now S is a join semilattice with 1 and the dual pseudocomplement of x will be denoted $\neg x = \mathrm{MIN}\{y \mid x \vee y = 1\}$, with S being dually pseudocomplemented if all $\neg x$'s exist. Now $B = \mathrm{im}(\neg) = \mathrm{im}(\neg\neg)$ is a Boolean lattice, the Boolean algebra of open elements with $\neg$ as its complement and the same join as S. The map $\neg\neg : S \longrightarrow S$ is a kernel operator, Glivenko's euqation now reads $\neg\neg x \vee \neg\neg y = \neg\neg(x \vee y)$. An element x is called meager if $\neg\neg x = 0$ or $\neg x = 1$; the meager elements form an ideal, the ideal of meager elements. Quasi-decomposability (dual ...) now means $x = \neg\neg x \vee m_x$ for some meager m_x (for any $x \in S$).

<u>Semi-Brouwerian Semilattices</u>. A meet semilattice S is called <u>semi-Brouwerian (SB)</u> if for each $x \in S$ the semilattice $[x) = \{y \in S \mid y \geq x\}$ is pseudocomplemented. If $y \geq x$ we will write $y \to x$ for the pseudocomplement of y in the lattice $[x)$. Now if $z \geq y \geq x$ one would like to establish a relation between $z \to y$, $z \to x$ and $y \to x$ but in general no such relation is known. We write B_x for the Boolean algebra of x-closed elements, $B_x = \{z \mid (z \to x) \to x = z\}$. We call a SB semilattice <u>fully quasi-decomposable</u> (FQD) if each of $[x)$ is quasi-decomposable.

The dual notion, dually semi-Brouwerian (DSB) would require a join semilattice S and would mean each $(x] = \{y \mid y \leq x\}$ is dually pseudocomplemented. If $y \geq x$ we write $y \dot{-} x$ for the dual pseudocomplement of x in $(y]$. Then we would have the notion dually fully quasi-decomposable which the reader can define appropriately.

<u>Brouwerian</u> <u>Semilattices</u>. A meet semilattice S is called <u>Brouwerian</u>

if for each $x, y \in S$, $\mathrm{MAX}\{z \mid z \wedge x \leq y\}$ (also denoted $x \to y$) exists. If S is Brouwerian it is semi-Brouwerian. If L is a complete lattice then L is Brouwerian iff L satisfies $x \wedge (\bigvee_t y_t) = \bigvee_t (x \wedge y_t)$. A semi-Brouwerian lattice is Brouwerian iff it is distributive iff it is modular. The reader can formula the dual Brouwerian notion.

<u>Complete Atomic Lattices</u>. In a lattice L with 0 an element is called an <u>atom</u> if it covers 0. L is <u>atomic</u> if each element x of L is the supremum of the set of atoms below it. Here we usually deal with complete atomic lattices. Generally atoms will be denoted with small Greek letters $\alpha, \beta, \gamma, \ldots$, etc. and the set of atoms of L will be denoted by Gothic $\mathfrak{A}$ or by the symbol $at(L)$.

<u>Algebraic Lattices</u>. An element x of a lattice L is called <u>compact</u> if $x \leq \sup_L x_t$ implies $x \leq x_{t_1} \vee \cdots \vee x_{t_n}$ for certain $t_1, \ldots, t_n$. If L is complete then x is compact iff $x \leq \sup_L d_t$, d_t updirected implies the existence of a t_0 so that $x \leq d_{t_0}$. The compact elements of L form a join subsemilattice $c(L)$ of L. A complete lattice is called <u>algebraic</u> if each element is the supremum of the set of compact elements below it. For example, a complete atomic lattice L is algebraic iff each atom α of L is compact. Now for any join semilattice S let $\mathcal{J}(S)$ denote the collection of nonempty ideals of S. (Our definition of ideal, $I \subseteq S$, $x \leq y$, $y \in I$ force $x \in I$ and $x, y \in I$ force $x \vee y \in I$; hence $\emptyset$ is an ideal. S^* denotes the collection of all ideals of S.) Then naturally $\mathcal{J}(c(L)) \cong L$ for any algebraic lattice L. For any join semilattice S <u>with zero</u>, $c(\mathcal{J}(S)) \cong S$ (naturally) with $\mathcal{J}(S)$ an algebraic lattice.

<u>Closure Families on a Set</u>. Let X be any set. P(X) denotes the power set of X. A collection $\mathcal{E} \subseteq P(X)$ is called a <u>closure family in X</u> iff $\mathcal{D} \subseteq \mathcal{E}$ implies $\bigcap \mathcal{D} \in \mathcal{E}$. $\mathcal{E}$ is called an <u>algebraic closure family</u> if besides being a closure family in X it satisfies: $\mathcal{D} \subseteq \mathcal{E}$, $\mathcal{D}$ updirected implies $\bigcup \mathcal{D} \in \mathcal{E}$. (Sometimes we also demand $\emptyset \in \mathcal{E}$). Each algebraic closure family on X is an algebraic lattice. The notions involved here can all be expressed in terms of closure operators on X, an activity we leave to the reader.

2. Some facts about $\Theta(S)$

$\Theta(S)$ is of course an algebraic lattice whose compact elements are its finitely generated congruences (finite joins of principal congruences). Actually $\Theta(S)$ is a complete sublattice of Eq(S), the lattice of equivalences on the set S. Papert [14] established that $\Theta(S)$ is SB. She also demonstrated a one-one correspondence between proper ideals I of a join semilattice S (use filters in the meet semilattice case) and dual atoms of $\Theta(S)$ given by the map

$$I \longmapsto \rho_I = \{(x,y) \in S \times S \mid x \in I \text{ iff } y \in I\};$$

and then she proved for any $\sigma \in \Theta(S)$, $\sigma = \bigcap_{\sigma \subseteq \rho_I} \rho_I$. Thus $\Theta(S)$ is dually atomic. Papert [14] and simultaneously Dean and Oehmke [4] established that $\Theta(S)$ is distributive iff the join semilattice S satisfies the condition: for each x, $[x) = \{y \mid y \geq x\}$ is totally ordered (in the meet semilattice case each $(x] = \{y \mid y \leq x\}$ is totally ordered). We call such a join semilattice a <u>dual</u> <u>tree</u> (in the meet semilattice case call it a tree). Varlet [17] carried these activities one step further showing that $\Theta(S)$ is Boolean iff in the join semilattice dual tree S all intervals are finite. Hall [8] showed that $\Theta(S)$ is upper semimodular and Freese and Nation [6] showed that the $\Theta(S)$'s satisfy no nontrivial lattice identities. More recently Fajtlowicz and Schmidt [5] established that $\Theta(S)$ is dually isomorphic to the lattice $\Theta^*(S^*)$ of algebraic closure subfamilies of S^* containing $\emptyset$. They proved many of the previously mentioned results in the latter context, greatly simplifying the proofs. Now in this work we will see that (i) if $\Theta(S)$ is dually algebraic then it is FQD and (ii) any $\Theta(S)$ is M-symmetric (see Chapter II) a result somewhat stronger than Hall's.

It has been wondered whether a characterization theorem for $\Theta(S)$ is possible. Źitomirskiĭ [18] advanced a second order characterization theorem; he, in a rather nice straightforward way, axiomatized an occurence of S in $\Theta(S)$. In a sense we do a similar thing here (Chapter I) axiomatizing in the dual case an occurence of S^* in $\Theta^*(S^*)$. This writer is not terribly happy about second order characterization theorems, but he must express the realistic sentiment that any better result (first order theorem) is highly unlikely. It is felt here though that in special cases better results are possible. (This is what led me to look closer at the distributive case, where characterization now boils down to finding which Boolean rings are universal over trees, itself a tough question.)

3. The duality of $\Theta(S)$ and $\Theta^*(S^*)$

Closure Family Viewpoint. For the join semilattice S, S^* denotes the collection of ideals of S and $\Theta^*(S^*)$ denotes the lattice of algebraic closure families on the set S which contain $\emptyset$ and are contained in S^*. If $\sigma \in \Theta(S)$ write $\mathcal{E}(\sigma) = \{I | \sigma \subseteq \rho_I\}$ (note $\rho_S = \rho_\emptyset = S \times S$) and for $\mathcal{E} \in \Theta^*(S^*)$ let $\vartheta(\mathcal{E}) = \bigcap\{\rho_I | I \in \mathcal{E}\}$. $\mathcal{E}(\sigma) \in \Theta^*(S^*)$, $\vartheta(\mathcal{E}) \in \Theta(S)$. The maps $\mathcal{E} : \Theta(S) \longrightarrow \Theta^*(S^*)$ and $\vartheta : \Theta^*(S^*) \longrightarrow \Theta(S)$ are order isomorphisms inverse to one another.

It should be emphasized that the updirected join in the lattice S^* is just set theoretic union and so we can say: For $\mathcal{E} \subseteq S^*$, $\mathcal{E} \in \Theta^*(S^*)$ iff $\emptyset$, $S \in \mathcal{E}$ (the zero and one of the lattice S^*) and $\mathcal{E}$ is closed under arbitrary meets and updirected joins all taken in the lattice S^*. This fact underlies our activities in Chapter I where we define in any abstract lattice X algebraic closure families (by demanding those last mentioned properties in X) and study the lattice $\Theta^*(X)$ of these. $\Theta^*(X)$ is really a generalization of $\Theta^*(S^*)$.

Topological Viewpoint. With S still a join semilattice $S^* \subseteq P(S)$; $P(S)$ is a Boolean topological space under set theoretic order convergence and S^* is topologically closed in $P(S)$. So it is that we consider the space S^*, with the relative topology from $P(S)$, as a compact T_2 space. Treating intersection as a binary operation and $\emptyset$, S as nullary operations we form the algebra $(S^*, \bigcap, \emptyset, S)$ of type (2, 0, 0) which is a compact topological algebra in the just mentioned topology. It turns out that for $\mathcal{E} \subseteq S^*$: $\mathcal{E} \in \Theta^*(S^*)$ iff $\mathcal{E}$ is a closed subalgebra of $(S^*, \bigcap, \emptyset, S)$. So the above indicated results actually establish a duality between $\Theta(S)$ and the lattice of closed subalgebras of $(S^*, \bigcap, \emptyset, S)$.

Compact Congruences and Clopen Subalgebras. Continue to use the notation of the last two paragraphs. This writer has noticed that if $a < b$, $a, b \in S$ and if $\Theta_{(a,b)}$ denotes the least congruence identifying a and b then $\mathcal{C}(\Theta_{(a,b)})$ is a clopen subalgebra of $(S^*, \bigcap, \emptyset, S)$. (So $\Theta_{(a,b)}$ is complemented in $\Theta(S)$.) Hence if σ is compact in $\Theta(S)$, $\mathcal{C}(\sigma)$ is clopen in $\Theta^*(S^*)$. But the converse is true too. Hence in the order anti-isomorphism between $\Theta(S)$ and $\Theta^*(S^*)$ the compact congruences of S correspond to the clopen subalgebras of $(S^*, \bigcap, \emptyset, S)$. So restricting the maps $\mathcal{C}$, ϑ we get an order anti-isomorphism between the join semilattice of compact congruences and the meet semilattice of clopen subalgebras of $(S^*, \bigcap, \emptyset, S)$.

Other Notations. L^{op} denote the dual of L. If $x, y \in L$ and $x \leq y$ then $[x,y] = \{z \mid x \leq z \leq y\}$ and is called the interval from x to y.

CHAPTER I

CLOSURE FAMILIES, UPPER SPOTS AND $\Theta^*(X)$

In this chapter we examine some properties of closure families in complete lattices, looking especially at the lattice $\Theta^*(X)$ of all algebraic closure families in a complete lattice X. The latter is of particular relevance since we find that if X is algebraic, $\Theta^*(X) \cong \Theta^*(S^*)$ for some join semilattice S. We axiomatize a special kind of closure family which may occur in a complete atomic lattice, the so-called upper spot. We find that the possession by such a lattice L of an upper spot is tantamount to L being a $\Theta^*(X)$. So we get a not-too-impressive characterization theorem for $\Theta^*(X)$. The special conditions on the upper spot required to make X (in $\Theta^*(X)$) algebraic are examined (bringing us back to $\Theta(S)$) and some alternative ways of describing the spot (decompositions into semilattices) are given.

1. Algebraic closure families in complete lattices

We begin by making some general remarks about closure families in complete lattices. Let X be a complete lattice. A subset $\mathcal{C}$ of X is called a <u>closure family in X</u> if $\mathcal{D} \subseteq \mathcal{C}$ implies that $\inf_X \mathcal{D} \in \mathcal{C}$. A function $o: X \longrightarrow X$ is called a <u>closure operator on X</u> if it is extensive ($x \leq o(x)$), order preserving and idempotent ($oo(x) = ox$). The closure families in X form a complete lattice under (set) inclusion while the closure operators on X do so also under the ordering $o_1 \leq o_2$ iff $o_2 \circ o_1 = o_1$. The lattice of closure families in X is order isomorphic to the lattice of closure operators on X according to the following scheme. For a closure family $\mathcal{C}$ in X define a map $o_{\mathcal{C}} : X \longrightarrow X$ by $o_{\mathcal{C}}(x) = \inf_X\{c \in \mathcal{C} \mid x \leq c\}$ (the least element of $\mathcal{C}$ which is $\geq$ to x). The map $o_{\mathcal{C}}$ is then a closure operator on X. Given a closure operator o on X define $\mathcal{C}_o = \{x \in X \mid o(x) = x\}$; the latter is a closure family in X. These assignments $\mathcal{C} \longmapsto o_{\mathcal{C}}$ and $o \longmapsto \mathcal{C}_o$ are order preserving and $\mathcal{C}_{(o_{\mathcal{C}})} = \mathcal{C}$ and $o_{(\mathcal{C}_o)} = o$ for any $\mathcal{C}$ (closure family in X) and any o (closure operator on X). So it is that we often use the notions closure family in X and closure operator on X interchangeably. (But we have a slight preference for closure families since the order on these seems more natural.)

A family $\mathcal{C} \subseteq X$ is called an <u>algebraic closure family in X</u> if $\mathcal{C}$ is a closure family in X and $0 = 0_X$ is in $\mathcal{C}$ and $\mathcal{C}$ is closed under updirected joins (if $d_t \in \mathcal{C}$, d_t updirected then $\sup_X d_t \in \mathcal{C}$). For closure operators we have: $o : X \longrightarrow X$ is an <u>algebraic closure operator on X</u> if o is a closure operator on X, $o(0_X) = 0_X$ and o preserves updirected joins. We expect what turns out: $\mathcal{C}$ is an algebraic closure family in X iff $o_{\mathcal{C}}$ is an algebraic closure operator on X. We write $\mathcal{C} \leq a$ X to denote the fact that $\mathcal{C}$ is an algebraic closure family in X.

We now make some warnings to the reader. First, our definition of algebraic closure operator is NOT the same as that given by others. For instance E. T. Schmidt [15] gives a quite different definition and restricts it to the case where X is an algebraic lattice. But notice that if X is an algebraic lattice the E. T. Schmidt notion and ours coincide. It seems that his objective is to make the image of an algebraic closure operator itself an algebraic lattice. This brings us to our second warning. If X is (just) a complete lattice, o an algebraic closure operator on X, then im o = $o[X]$ is NOT necessarily an algebraic lattice. But as we will soon see, X is algebraic iff im(o) is an algebraic lattice for each algebraic closure operator o on X.

<u>Proposition 1.1</u>. Let X be a complete lattice and let $\mathcal{C} \leq a$ X. If x is compact in X then $o_{\mathcal{C}}(x)$ is compact in the lattice $\mathcal{C}$. Thus, (a) if X is algebraic then a closure family $\mathcal{C}$ in X containing 0 is is an algebraic closure family in X iff $o_{\mathcal{C}}$ preserves compact elements. Also (b) (due to E. T. Schmidt [15]) the lattice X is algebraic iff for each algebraic closure operator o on X the lattice im o is algebraic.

<u>Proof</u>. Suppose $\mathcal{C} \leq a$ X with X complete. Let x be compact in X. Working in the complete lattice $\mathcal{C}$ we show $o_{\mathcal{C}}(x)$ is compact (in $\mathcal{C}$). Suppose $(c_d)_{d \varepsilon D}$ is an updirected collection of elements of $\mathcal{C}$ such that $o_{\mathcal{C}}(x) \leq \sup_{\mathcal{C}}\{c_d | d \varepsilon D\}$. Because $\mathcal{C} \leq a$ X we have $\sup_{\mathcal{C}}\{c_d | d \varepsilon D\} = \sup_X\{c_d | d \varepsilon D\}$ and so with $x \leq o_{\mathcal{C}}(x)$ we get: $x \leq \sup_X\{c_d | d \varepsilon D\}$. But x is X compact so for some $d_0 \varepsilon D$, $x \leq c_{d_0}$. Applying $o_{\mathcal{C}}$ gives $o_{\mathcal{C}}(x) \leq c_{d_0}$ the desired result.

<u>Proof of (a)</u>. Assume X is algebraic, $\mathcal{C}$ is a closure family in X and $0 \varepsilon \mathcal{C}$. One implication of (a) is clear now. Suppose for the

converse that $o_{\mathcal{E}}(c)$ is compact in the lattice $\mathcal{E}$ for each c compact in X. Let $\mathcal{D}$ be an updirected subset of $\mathcal{E}$. To show that $\sup_X \mathcal{D} \in \mathcal{E}$ it is sufficient to show $o_{\mathcal{E}}(\sup_X \mathcal{D}) \leq \sup_X \mathcal{D}$. Since X is algebraic we can show this as follows. Take x to be X compact with $x \leq o_{\mathcal{E}}(\sup_X \mathcal{D})$. Then we get $o_{\mathcal{E}}(x) \leq o_{\mathcal{E}}(\sup_X \mathcal{D})$. But $\sup_{\mathcal{E}} \mathcal{D} = o_{\mathcal{E}}(\sup_X \mathcal{D})$ so we have $o_{\mathcal{E}}(x) \leq \sup_{\mathcal{E}} \mathcal{D}$, $\mathcal{D}$ updirected in $\mathcal{E}$. Then because $o_{\mathcal{E}}(x)$ is $\mathcal{E}$ compact (because x is X compact), there is a $d_0 \in \mathcal{D}$ so that $o_{\mathcal{E}}(x) \leq d_0$. Hence $x \leq d_0$ and so $x \leq \sup_X \mathcal{D}$. Thus $o_{\mathcal{E}}(\sup_X \mathcal{D}) = \sup_X\{x \mid x$ is X compact and $x \leq o_{\mathcal{E}}(\sup_X \mathcal{D})\} \leq \sup_X \mathcal{D}$.

<u>Proof of (b)</u>. Because $1_X : X \longrightarrow X$ is an algebraic closure operator on X the "if" part of (b) is clear. Suppose X is an algebraic lattice and o is an algebraic closure operator on X. Let $x \in X$. Now $x = \sup_X\{c \mid c \leq x$ and c is X compact$\}$ an updirected join. Applying o to this gives: $o(x) = \sup_X\{o(c) \mid c \leq x,$ c is X compact$\} = \sup_{\mathcal{C}_o}\{o(c) \mid c \leq x,$ c is X compact$\}$. But c being X compact forces $o(c)$ to be $\mathcal{C}_o$ compact. Hence $o(x) = \sup_{\mathcal{C}_o}\{c \mid c$ is compact in o and $c \leq o(x)\}$. So the complete lattice $\mathcal{C}_o = \mathrm{im}\, o$ is compactly generated.

<u>Note</u>. Hence if X is an algebraic lattice and o is a closure operator on X preserving zero then o is an algebraic closure operator on X iff o preserves compact elements (x is X compact implies $o(x)$ is $\mathcal{C}_o$ compact).

We have been careful to say algebraic closure family <u>in X</u> (emphasis on "in X") in order to make clear the relative nature of things;

that our definitions depend on infima and suprema taken in the complete lattice X. The next result tells us how much leeway we have in these matters.

Proposition 1.2. Suppose X is a complete lattice and $\mathcal{C} \leq a\ X$ and $\mathcal{D} \subseteq \mathcal{C}$. Then $\mathcal{D} \leq a\ X$ iff $\mathcal{D} \leq a\ \mathcal{C}$.

The proof is straightforward.

Application 1.3. Let S be a join semilattice and let $\mathcal{C} \subseteq S^* = \{I | I \text{ ideal in } S\}$. Then $\mathcal{C}$ is a set theoretic algebraic closure family (algebraic closure family in P(S), the power set) iff $\mathcal{C}$ is an algebraic closure family in S^*.

Proof. This follows from 1.2 because S^* is an algebraic closure family in the complete lattice P(S).

Comment. In Fajtlowicz and Schmidt [5] the symbol $\Theta^*(S^*)$ is used to denote the lattice of all set theoretic algebraic closure families contained S^*. In view of 1.3, $\Theta^*(S^*)$ can be defined inside S^* (relative to S^*) without appealing outside to P(S); it is the lattice of all algebraic closure families in S^* (according to our above definition). We now extend the notation. For any complete lattice X the symbol $\Theta^*(X)$ will hereafter denote the complete lattice of all algebraic closure families in X. Set-wise, $\Theta^*(X) = \{\mathcal{C} | \mathcal{C} \leq a\ X\}$. This is a generalization of the notation $\Theta^*(S^*)$ for in the case that $X = S^*$ (for a join semilattice S) the Fajtlowicz and Schmidt meaning of $\Theta^*(S^*)$ and our meaning of $\Theta^*(X)$ coincide. But we also have:

Proposition 1.4. Let L be a complete lattice. Then $L \cong \Theta^*(S^*)$ for some join semilattice S iff $L \cong \Theta^*(X)$ for some algebraic lattice X.

Proof. Since S^* is algebraic one implication is trivial. Suppose $L \cong \Theta^*(X)$ for some algebraic X. Let $S = \{x \in X | x \neq 0, x \text{ is } X \text{ compact}\}$. S is a join semilattice (use the join in X) and easily $S^* \cong X$. From this it follows that $L \cong \Theta^*(X) \cong \Theta^*(S^*)$.

2. A topological view of $\Theta^*(X)$

Let X be a complete lattice. Suppose $(x_\lambda)_{\lambda \varepsilon \Lambda}$ is a net of elements of X. Recall that by definition the upper limit in X of $(x_\lambda)_{\lambda \varepsilon \Lambda}$ is:

$$\overline{\mathrm{Lim}}_X\, x_\lambda = \inf_X\{\sup_X\{x_{\lambda'} | \lambda' \geq \lambda\} | \lambda \varepsilon \Lambda\}$$

while the lower limit in X of (x_λ) is

$$\underline{\mathrm{Lim}}_X\, x_\lambda = \sup_X\{\inf_X\{x_{\lambda'} | \lambda' \geq \lambda\} | \lambda \varepsilon \Lambda\}.$$

Always $\underline{\mathrm{Lim}}_X\, x_\lambda \leq \overline{\mathrm{Lim}}_X\, x_\lambda$. Recall also that the net (x_λ) is said to be order convergent to $x \in X$ (written $x_\lambda \xrightarrow[o]{} x$) iff $\underline{\mathrm{Lim}}_X\, x_\lambda = x = \overline{\mathrm{Lim}}_X\, x_\lambda$. A subset $D \subseteq X$ is called order closed iff for any net $x_\lambda \in D$, $x_\lambda \xrightarrow[o]{} x$ forces $x \in D$. It is well known (Birkhoff [1], Lawson [9], etc.) that the collection of all order closed sets forms a topology $\mathcal{T}$. That is, there is a topology $\mathcal{T}$ on X whose closed sets are precisely the order closed sets. This $\mathcal{T}$ is called the order topology of X or the topology of X order convergence. A set is closed in the order topology iff it is order closed. Now X is T_1 in this topology; points are closed. Also if X_1 and X_2 are complete lattices and if $F:X_1 \longrightarrow X_2$ preserves order convergence then F is continuous relative to the respective order topologies (though continuous maps do NOT necessarily preserve order convergence).

Now we must make more warnings. Now that X is a topological space with the order topology $\mathcal{T}$ we have the notion of general topological convergence of a net. Write $x_\lambda \xrightarrow[\mathcal{T}]{} x$ for this. ($x_\lambda \xrightarrow[\mathcal{T}]{} x$ means for each open set 0 containing x, x_λ is eventually in 0.) We warn: $x_\lambda \xrightarrow[\mathcal{T}]{} x$ does not imply $x_\lambda \xrightarrow[o]{} x$ (although $x_\lambda \xrightarrow[o]{} x$ implies $x_\lambda \xrightarrow[\mathcal{T}]{} x$). It may be that $x_\lambda \xrightarrow[\mathcal{T}]{} x$ and $\underline{\mathrm{Lim}}_X\, x_\lambda < \overline{\mathrm{Lim}}_X\, x_\lambda$. All we can say is: if $x_\lambda \xrightarrow[\mathcal{T}]{} x$ then $\underline{\mathrm{Lim}}_X\, x_\lambda \le x \le \overline{\mathrm{Lim}}_X\, x_\lambda$. So if $x_\lambda \xrightarrow[\mathcal{T}]{} x$ and x_λ is order convergent then $x_\lambda \xrightarrow[o]{} x$. Our second warning, it is not necessarily the case that the lattice operations of X will be continuous in the order topology. It is well known that this is tantamount to X satisfying certain infinite distributive laws (see Birkhoff [1]). We point out here that the meet operation will be continuous in the order topology iff X satisfies $x \wedge (\bigvee_t y_t) = \bigvee_t x \wedge y_t$ for updirected y_t's (Maeda and Maeda [10] call such lattices upper continuous). Notice that these conditions hold in an algebraic lattice; hence the meet operation is jointly continuous in the order topology. We now prove a lemma which moves us closer to the activities of the last section.

Lemma 2.1. Suppose X is a complete lattice and $\mathcal{C} \le a\ X$ and (x_λ) is a net in $\mathcal{C}$. Then

$$\underline{\mathrm{Lim}}_{\mathcal{C}}\, x_\lambda = \underline{\mathrm{Lim}}_X\, x_\lambda \le \overline{\mathrm{Lim}}_X\, x_\lambda \le \overline{\mathrm{Lim}}_{\mathcal{C}}\, x_\lambda.$$

Proof. Certainly for each λ we have

$$\inf_{\mathcal{C}}\{x_{\lambda'} \mid \lambda' \ge \lambda\} = \inf_X\{x_{\lambda'} \mid \lambda' \ge \lambda\},$$

since $\mathcal{C}$ is closed under updirected joins we get the left equality above. Now for each λ, $\sup_X\{x_{\lambda'}|\lambda' \geq \lambda\} \leq \sup_{\mathcal{C}}\{x_{\lambda'}|\lambda' \geq \lambda\}$ and hence $\overline{\mathrm{Lim}}_X\, x_\lambda \leq \inf_X\{\sup_{\mathcal{C}}\{x_{\lambda'}|\lambda \geq \lambda\}|\lambda\}$ but because each $\sup_{\mathcal{C}}\{x_{\lambda'}|\lambda' \geq \lambda\}$ is in $\mathcal{C}$ we get $\inf_X \sup_{\mathcal{C}}\{x_{\lambda'}|\lambda' \geq \lambda\}|\lambda\} = \inf_{\mathcal{C}}\{\sup_{\mathcal{C}}\{x_{\lambda'}|\lambda' \geq \lambda\}|\lambda\}$ $= \overline{\mathrm{Lim}}_{\mathcal{C}}\, x_\lambda$ and so $\overline{\mathrm{Lim}}_X\, x_\lambda \leq \overline{\mathrm{Lim}}_{\mathcal{C}}\, x_\lambda$.

Now we can relate our topological activities with our activities of the last section. For a complete lattice X consider the algebra of type (2, 0, 0), namely $(X, \wedge, 0, 1)$ ($\wedge$ is a binary operation, 0, 1, nullary operations).

Proposition 2.2. Let X be a complete lattice, $\mathcal{C} \subseteq X$. The following statements are equivalent:

(i) $\mathcal{C} \leq a\ X$

(ii) $\mathcal{C}$ is a subalgebra of $(X, \wedge, 0, 1)$ which is closed in the X order topology

(iii) $\mathcal{C}$ is a closure family in X, $0 \in \mathcal{C}$ and the inclusion mapping $\mathcal{C} \hookrightarrow X$ is order convergence preserving (i.e., if in the complete lattice $\mathcal{C}$, $c_\lambda \xrightarrow[o]{} c$ then in X, $c_\lambda \xrightarrow[o]{} c$) and hence is continuous in the respective order topologies.

Proof (i) $\Longrightarrow$(ii) is trivial. For (ii) $\Longrightarrow$(i) assume that is a topologically closed subalgebra of $(X, \wedge, 0, 1)$. Then $\mathcal{C}$ has 0 in it and it is closed under updirected joins (if $x = \sup_X x_t$, x_t updirected with $x_t \in \mathcal{C}$ then in X, $x_t \xrightarrow[o]{} x$ and so $x \in \mathcal{C}$). So to complete (i) we must show that $\mathcal{C}$ contains the meet of each of its nonempty subsets (because already $1 \in \mathcal{C}$). Let $\mathcal{D} \subseteq \mathcal{C}$, $\mathcal{D} \neq \emptyset$.

Let $\mathcal{F} = \{F | F$ finite subset of $\mathcal{D}\}$. Then $\mathcal{F}$ is updirected. For each $F \in \mathcal{F}$ let $x_F = \inf_X F$. Then $(x_F)_{F\in\mathcal{F}}$ is an anti-isotone net, that is, $F_1 \subseteq F_2$ implies $x_{F_1} \geq x_{F_2}$. It is also clear that each $x_F \in \mathcal{C}$ (since $\mathcal{C}$ is a subalgebra of $(X, \wedge, 0, 1)$) and that $x_F \xrightarrow[o]{} x$ where by x we mean $x = \inf_X\{x_F | F \in \mathcal{F}\} = \inf_X \mathcal{D}$. But $\mathcal{C}$ is order closed so we get $x \in \mathcal{C}$. Hence $\inf_X \mathcal{D} \in \mathcal{C}$. We show (ii)$\Longrightarrow$(iii) requiring now only that $\mathcal{C} \hookrightarrow X$ preserve order convergence. But this is obvious in light of lemma 2.1. Now finally (iii) $\Longrightarrow$ (i). Take d_λ updirected, each $d_\lambda \in \mathcal{C}$. We show $\sup_X d_\lambda \in \mathcal{C}$. In the lattice $\mathcal{C}$ $d_\lambda \xrightarrow[o]{} \sup_{\mathcal{C}} d_\lambda$ so in the lattice X it must be that $d_\lambda \xrightarrow[o]{} \sup_{\mathcal{C}} d_\lambda$. But certainly in X, $d_\lambda \xrightarrow[o]{} \sup_X d_\lambda$. Hence $\sup_{\mathcal{C}} d_\lambda = \sup_X d_\lambda$ and so $\sup_X d_\lambda \in \mathcal{C}$.

It is now apparent that $\Theta^*(X)$ is the lattice of all topologically closed subalgebras of $(X, \wedge, 0, 1)$. We then have topological versions of Propositions 1.2 and 1.3.

<u>Proposition 2.3</u>. Let $\mathcal{C} \leq a$ X and let $\mathcal{D}$ be a subalgebra of $(\mathcal{C}, \wedge, 0, 1)$. Then $\mathcal{D}$ is closed in the order topology of $\mathcal{C}$ iff $\mathcal{D}$ is closed in the order topology of X. So $\Theta^*(\mathcal{C}) = \{\mathcal{D} | \mathcal{D} \in \Theta^*(X)$ and $\mathcal{D} \subseteq \mathcal{C}\}$.

<u>Proposition 2.4</u>. Let S be any join semilattice. Let $\mathcal{C} \subseteq S^*$. Then the following are equivalent:

(i) $\mathcal{C}$ is a subalgebra of $(S^*, \cap, \emptyset, S)$ which is closed in the order topology of P(S) (i.e., $\mathcal{C}$ is closed under set theoretic order convergence)

(ii) $\mathcal{C}$ is a subalgebra of $(S^*, \cap, \emptyset, S)$ which is closed in the order topology of S^* (i.e., $\mathcal{C}$ is closed under ideal theoretic order convergence).

3. A closure family in $\Theta^*(X)$

Let X be a complete lattice. Let x be any proper element of X ($x \neq 0$, $x \neq 1$) and write $\alpha_x = \{0,x,1\}$, $u(x) = \{0\} \cup [x,1]$ and $\ell(x) = [0,x] \cup \{1\}$. Then each of α_x, $u(x)$, $\ell(x)$ is in $\Theta^*(X)$. In fact the atoms of the lattice $\Theta^*(X)$ are precisely the elements of the form α_x ($x \in X \setminus \{0,1\}$). From this it is easy to see that $\Theta^*(X)$, beyond being complete, is also atomic.

We might extend our notation and write $u(0) = [0,1] = X$ (the "one" of the lattice $\Theta^*(x)$) and $u(1) = \{0,1\}$ (the "zero" of the lattice $\Theta^*(X)$). Now in this section we concentrate on the properties of the family $\{u(x) | x \in X\}$ in $\Theta^*(X)$, examining how it lies in $\Theta^*(X)$. We aim to axiomatize this family of elements and show that it is the presence of this family, situated as it is, which characterizes the lattice $\Theta^*(X)$. Certainly the elements $\{\ell(x) | x \in X\}$ can be similarly employed, an activity we forego here for two reasons. First what <u>is</u> true about this latter family can be obtained by careful dualization of what we do here. (This writer spent too much time checking that out and decided that it was too much to put here). Second, using the family $\{u(x) | x \in X\}$ there is a nice decomposition theorem which expresses $\Theta^*(X)$ as almost the union of intervals. The decomposition simply fails to be that nice using the $\ell(x)$'s. So we just axiomatize $\mathcal{U} = \{u(x) | x \in X\}$ and call this family, or its axiomatized version, the (an) upper spot in $\Theta^*(X)$ ($\{\ell_x\}$ would have been a lower spot). We will actually work with the associated closure operator.

For any $\mathcal{C} \in \Theta^*(X)$ let $i(\mathcal{C})$ denote $\inf\{x | x \in \mathcal{C}, x \neq 0\}$.

Now define $F:\Theta^*(X) \longrightarrow \Theta^*(X)$ by $F(\mathcal{C}) = \{0\} \cup [i(\mathcal{C}),1] = u(i(\mathcal{C}))$. Then F is a closure operator on $L = \Theta^*(X)$. (For the remainder of this section write L for $\Theta^*(X)$.) Clearly F is the closure operator associated with the closure family $\mathcal{U}$ and $F(0) = 0$ (meaning $F\{0,1\} = \{0,1\}$). Now for any x proper in X we see that $F(\alpha_x) = u(x)$ and so apparently F is one-one on the set $\mathcal{O}\!\mathcal{l}$ of atoms of L and $F[\mathcal{O}\!\mathcal{l}]$ is almost all of $\mathcal{U}$. So it is natural to build a map which traces back the atom from whence sprang a closed element (element of $\mathcal{U}$) i.e., that would map $u(x) \longmapsto \alpha_x$ (x proper). Such a map is given as follows:
$\Delta: L \longrightarrow \mathcal{O}\!\mathcal{l} \cup \{0,1\}$ and for $\mathcal{C} \in \Theta^*(X) = L$

$$\Delta(\mathcal{C}) = \begin{cases} 1 = X & \text{if } i(\mathcal{C}) = 0 \\ 0 = \{0,1\} & \text{if } i(\mathcal{C}) = 1 \\ \alpha_{i(\mathcal{C})} & \text{if } i(\mathcal{C}) \text{ is proper.} \end{cases}$$

Then $\Delta(\alpha_x) = \alpha_x$, $\Delta(u_x) = \alpha_x$ and so forth. We begin examining the properties of how F and Δ work with each other. We have first

US1 $$F(\Delta(\mathcal{C})) = F(\mathcal{C}) \quad \text{for any } \mathcal{C} \in L.$$

For if $i(\mathcal{C}) = 0$ then $\Delta(\mathcal{C}) = X$ and so the left side is X while the right is $\{0\} \cup [i(\mathcal{C}),1] = [0,1] = X$. Now if $i(\mathcal{C}) = 1$, necessarily $\mathcal{C} = \{0,1\}$ so $\Delta(\mathcal{C}) = \{0,1\}$ and $F\Delta\mathcal{C} = F0 = 0$ and $F(\mathcal{C}) = \{0\} \cup [i(\mathcal{C}),1] = \{0,1\} = 0$. Now if $i(\mathcal{C})$ is proper then $\Delta(\mathcal{C}) = \alpha_{i(\mathcal{C})}$ so $F\Delta\mathcal{C} = u(i(\mathcal{C}))$ and we know $F(\mathcal{C}) = u(i(\mathcal{C}))$. So US1 holds true. We also have

US2 $$\Delta(F(\mathcal{C})) = \Delta(\mathcal{C}) \text{ for any } \mathcal{C} \in L.$$

Again cases, first $i(\mathcal{C}) = 0$. Then $i(X) = 0$ also. So then $F(\mathcal{C}) = X$ and $\Delta F\mathcal{C} = X$; meanwhile $\Delta(\mathcal{C}) = X$ because $i(\mathcal{C}) = 0$. Suppose now $i(\mathcal{C}) = 1$. Then $\mathcal{C} = 0(\{0,1\})$ and $\Delta(F(\mathcal{C})) = \Delta(0) = 0$ while $\Delta(\mathcal{C}) = \Delta(0) = 0$. So finally take $i(\mathcal{C})$ proper. Then $\Delta(F(\mathcal{C})) = \Delta(\{0\} \cup [i(\mathcal{C}),1]) = \alpha_{i(\mathcal{C})}$ but $\Delta(\mathcal{C}) = \alpha_{i(\mathcal{C})}$ by definition. So we get US2.

Now comes

US3 $$\Delta(F(\beta)) = \beta \text{ for all } \beta \in \mathfrak{A} \cup \{0\}$$

a statement which is by now apparent.

Also

US4 $$\Delta(F(\alpha) \vee F(\beta)) \leq \alpha \vee \beta \text{ for all } \alpha,\ \beta \in \mathfrak{A},\ \Delta(\alpha \vee \beta) \neq 1.$$

To prove this first observe that $\Delta(\alpha \vee \beta) = \Delta(F(\alpha) \vee F(\beta))$. Write $\alpha = \alpha_x$, $\beta = \alpha_y$ for x, y proper in X and WLOG $x \neq y$. Then $\alpha \vee \beta = \{0,1,x,y,x \wedge y\}$ so that $i(\alpha \vee \beta) = x \wedge y$. Since we have assumed $\Delta(\alpha \vee \beta) \neq 1$ then $x \wedge y \neq 0$. So $x \wedge y$ is proper hence $\Delta(\alpha \vee \beta) = \alpha_{x \wedge y} = \{0,1,x \wedge y\}$ and so apparently $\Delta(F(\alpha) \vee F(\beta))$ $= \alpha_{x \wedge y} \subseteq \alpha_x \vee \alpha_y$.

Now notice that if we choose proper elements x_i in $X(i \in I)$ then (recalling that $\mathcal{U} = \text{im}F$)

$$\inf_{\mathcal{U}}\{F(\alpha_{x_i})|i \in I\} = \inf_L\{F(\alpha_{x_i})|i \in I\}$$

$$= \bigcap_{i \in I} (\{0\} \cup [x_i,1]) = \{0\} \cup [\sup_X x_i,1$$

while

$$\sup_{\mathcal{U}}\{F(\alpha_{x_i})|i \in I\} = \sup_{\mathcal{U}}(\{0\} \cup [x_i,1])$$

$$= \{0\} \cup [\inf_X x_i,1].$$

From these we can now establish

US5 If $F(\alpha_d)$ is a net order convergent in the F-order topology and if $\Delta(\lim F(\alpha_d)) \neq 1$ then

$$\Delta(\lim F(\alpha_d)) \leq \sup_L \alpha_d$$

(for any net (α_d) of atoms of L).

We point out that by the F-order topology we mean the order topology of the complete lattice imF = $\mathcal{U}$. For each d choose x_d proper in X so that $\alpha_{x_d} = \alpha_d$. Since $\Delta(\mathrm{Lim}\ F(\alpha_d)) \neq 1$, $\Delta(\mathrm{Lim}\ F(\alpha_d)) \in \mathcal{Q} \cup \{0\}$. If it is zero our result follows trivially. So suppose $\Delta(\mathrm{Lim}\ F\ (\alpha_d)) = \alpha \in \mathcal{Q}$. Choose x proper in X so that $\alpha = \alpha_x$. From what we have assumed we can say $F(\alpha) = \inf_{d}{}_{\mathcal{U}} \sup_{d' \geq d}{}_{\mathcal{U}} F(\alpha_{d'})$ which, reinterpreted says that

$$\{0\} \cup [x,1] = \inf_{d}{}_{\mathcal{U}} \{0\} \cup [\inf_X\{x_{d'}|d' \geq d\},1]$$

implying

$$\{0\} \cup [x,1] = \bigcap_d \{0\} \cup [\inf_X\{x_{d'} | d' \geq d\},1]$$

$$= \{0\} \cup [\sup_X\{\inf_X\{x_{d'} | d' \geq d\} | d\},1]$$

$$= \{0\} \cup [\underline{\mathrm{Lim}}_X\, x_d,\ 1].$$

Now if for each d, $\{\inf_X\{x_{d'} | d' \geq d\} = 0$ then the right side of the above equation is X and then we can conclude that $\inf_X\{z \in X | z \text{ proper}\}$ is 0; but then $\{0\} \cup [x,1]$ would have to be X and so x would have to be a least proper element of X, yielding $\inf_X\{z \in X | z \text{ proper}\} = x$, a contradiction. Thus for some d, $\inf_X\{x_{d'} | d' \geq d\}$ is not zero; hence $\underline{\mathrm{Lim}}_X\, x_d \neq 0$. Thus the equality $\{0\} \cup [x,1] = \{0\} \cup [\underline{\mathrm{Lim}}_X\, x_d,\ 1]$ forces $x = \underline{\mathrm{Lim}}_X\, x_d$. Now $\sup_L\{\alpha_{x_d} | d\}$ is the algebraic closure family in X generated by $\{x_d\}_d$ so it is clear that $x = \underline{\mathrm{Lim}}_X\, x_d \in \sup_L\{\alpha_{x_d} | d\}$. Going back to our original notation we now have $\alpha \leq \sup_L\{\alpha_d | d\}$ which is $\Delta(\mathrm{Lim}\ F\ \alpha_d) \leq \sup_L\{\alpha_d | d\}$. So US5 is proven.

We point out now that if X has no least positive element the map $x \longmapsto \{0\} \cup [x,1]$ of X into $\mathcal{U}$ establishes an order anti-isomorphism of X into $\mathcal{U}$. If however X has a least proper element x_0 we have to throw out the zero of X and consider only the interval $[x_0,1]$ in X and the map $[x_0,1] \longrightarrow \mathcal{U}$ given by $x \longmapsto \{0\} \cup [x,1]$ is an order anti-isomorphism.

Working in the complete lattice $\mathcal{U} = \mathrm{im}F$, for any subset D of let $\overline{D}^F$ denote the order closed join subsemilattice of $\mathcal{U}$ generated by D (we use F-order convergence!). Notice that $\overline{D}^F \cup \{0,1\}$ is the dual algebraic closure family in $\mathcal{U}$ generated by D (of course here 0 means $\{0,1\}$ and 1 means X).

We now state the last property of (F,Δ).

For any atoms α_i and α:

US6 $$\alpha \leq \sup_L\{\alpha_i\} \text{ implies } F(\alpha) \in \overline{\{F(\alpha_i)\}}^F$$

To prove US6 choose proper elements x, x_i of X so that $\alpha = \alpha_x$, $\alpha_i = \alpha_{x_i}$. Then the assumption $\alpha \leq \sup_L\{\alpha_i\}$ tells us that x is in the algebraic closure family in X generated by $\{x_i\}$. Case 1. X has no least proper element. Then X is anti-isomorphic to $\mathcal{U}$ = imF via $x \longrightarrow u(x)$. But if z is proper in X, $F(\alpha_z) = u(z)$ so we conclude that $F(\alpha_x)$ is in the dual algebraic closure family in $\mathcal{U}$ generated by the $F(\alpha_{x_i}) = F(\alpha_i)$. Hence $F(\alpha_x) \in \overline{\{F(\alpha_i)\}}^F \cup \{0,1\}$. But x is proper and X has no least proper element so $F(\alpha_x) \notin \{0,1\}$ so $F(\alpha) = F(\alpha_x) \in \overline{\{F(\alpha_i)\}}^F$. Case 2. X has a least positive element x_0. Then $\mathcal{U}$ is order anti-isomorphic to $[x_0,1]$. Write $Z = [x_0,1]$. Z is a complete lattice, closed in X's order topology and in fact the order topology of Z is that of X relativized. Now each $x_i \in Z$ so let G denote the topologically closed meet subsemilattice of Z generated by $\{x_i\}$. It is not difficult to see that $G \cup \{0_X, 1_X\}$ is the algebraic closure family in X generated by $\{x_i\}$ from which it then follows that $x \in G$. The mapping $z \longmapsto u(z)$ (for proper elements $z \longmapsto F(\alpha_z)$) gives an order anti-morphism of Z into $\mathcal{U}$ and under this mapping G necessarily passes into $\overline{\{F(\alpha_{x_i})\}}^F$. Thus $x \in G$ implies $F(\alpha_x) \in \overline{\{F(\alpha_{x_i})\}}^F$ and going back to our original notation we now have $F(\alpha) \in \overline{\{F(\alpha_i)\}}^F$.

So we have found a special closure family $\mathcal{U}$ in $L = \Theta^*(X)$, its associated closure operator F and an associated retraction type mapping Δ and have established for these mappings the properties US1 through US6. Soon we will see that the presence in a complete atomic lattice L of maps F, Δ satisfying these properties will be sufficient to make L a $\Theta^*(X)$.

4. Upper Spots

Let L be any complete atomic lattice with $\mathcal{O}\!\mathcal{L}$ its set of atoms. Suppose F is a closure operator on L satisfying $F(0) = 0$ and suppose $\Delta : L \longrightarrow \mathcal{O}\!\mathcal{L} \cup \{0,1\}$. We call the pair (F,Δ) an upper spot of the lattice L if it satisfies all of the following conditions:

US1	$F(\Delta(x)) = F(x)$	for all $x \in L$
US2	$\Delta(F(x)) = \Delta(x)$	for all $x \in L$
US3	$\Delta(F(\beta)) = \beta$	for all $\beta \in \mathcal{O}\!\mathcal{L} \cup \{0\}$
US4	$\Delta(F(\alpha) \vee F(\beta)) \leq \alpha \vee \beta$	for all $\alpha, \beta \in$ with $\Delta(\alpha \vee \beta) \neq 1$
US5	$\Delta(\lim F(\alpha_d)) \leq \sup_L \alpha_d$	for any net of atoms so that $F(\alpha_d)$ is order convergence in the F order topology and $\Delta(\lim F(\alpha_d)) \neq 1$

US6 $\quad F(\alpha) \in \overline{\{F(\alpha_i)\}}^F \quad$ for any $\alpha \leq \sup_L \alpha_i$.

Of course in US5 by the F order topology we mean the order topology of the complete lattice imF; and in US6 $\overline{\{F(\alpha_i)\}}^F$ denotes the topologically closed (F-order topology) join subsemilattice of imF generated by $\{F(\alpha_i)\}$.

The activities of the last section lead us to our first result.

Proposition 4.1. If X is any complete lattice then the lattice $L = \Theta^*(X)$ has an upper spot. Specifically if $F(\mathcal{C}) = \{0\} \cup [i(\mathcal{C}),1]$ for each $\mathcal{C} \in \Theta^*(X)$ and

$$\Delta(\mathcal{C}) = \begin{cases} X & \text{if } i(\mathcal{C}) = 0 \\ \{0,1\} & \text{if } i(\mathcal{C}) = 1 \\ \alpha_{i(\mathcal{C})} & \text{if } i(\mathcal{C}) \text{ is proper in } X \end{cases}$$

where $i(\mathcal{C}) = \inf_X\{c \in \mathcal{C} \mid c \neq 0\}$ then (F,Δ) is a spot and imF is order anti-isomorphic to X if X has no least proper element while, if X has a least proper element then imF is order anti-isomorphic with the nonzero elements of X.

We aim for a converse to 4.1. So let L be a complete atomic lattice with an upper spot (F,Δ). We will consider first the special case where there is an atom α_o so that $F(\alpha_0) = 1$. Then $\Delta(1) = \alpha_0$ ($\Delta(1) = \Delta(F(\alpha_0)) = \alpha_0$ by US3) and for all ℓ, $\Delta(\ell) \in \mathcal{A} \cup \{0\}$ (if $\Delta(\ell) = 1$ then $\Delta(1) = \Delta F1 = \Delta F\Delta\ell = \Delta F\ell = \Delta(\ell) = 1$ so $\Delta(1) = 1$ contradicting $\Delta(1) = \alpha_o$). Now imF is a complete lattice and let $Y = \text{imF} \cup \{I\}$ where we define $I \geq f$ for all $f \in$ imF. Since imF is already a complete lattice, it is closed in the order topology of Y, its order topology being the restriction of that of Y (im$F$ is a complete sublattice of Y).

Now let $\ell \in L$ and consider $\psi(\ell) = \{F(\alpha)\,|\,\alpha \in \mathcal{O}\!\mathcal{L} \cup \{0\}, \alpha \leq \ell\} \cup \{I\}$. It is apparent that $\psi(\ell)$ is finitely join closed in Y (by US4) and that $\psi(\ell)$ is closed in the order topology of Y(US5). Hence it's clear that $\Psi(\ell)$ is a dual algebraic closure family in Y. So if we let $X = Y^{op}$ then the set $\psi(\ell) \in \Theta^*(X)$. This then gives us a mapping $\psi\colon L \longrightarrow \Theta^*(X)$ defined $\psi(\ell) = \{F(\alpha)\,|\,\alpha \leq \ell, \alpha \in \mathcal{O}\!\mathcal{L} \cup \{0\}\} \cup \{I\}$. ψ is apparently order preserving. With X still Y^{op} we can build a map $\phi\colon\Theta^*(X) \longrightarrow L$ by the scheme $\phi(\mathcal{C}) = \sup_L\{\Delta(y)\,|\,y \in \mathcal{C}, y \neq I\}$. Then ϕ is order preserving.

Now for $\ell \in L$, $\phi\psi(\ell) = \sup_L\{\Delta y\,|\,y \neq I \text{ and } y \in \psi\ell\}$. But note that $y \in \psi(\ell)$ with $y \neq I$ is equivalent to: $y = F(\alpha)$ for $\alpha \leq \ell$ ($\alpha \in \mathcal{O}\!\mathcal{L} \cup \{0\}$). So for such a y, $\Delta(y) = \Delta F\alpha = \alpha$ so that $\{\Delta(y)\,|\,y \neq I, y \in \psi\ell\} = \{\alpha\,|\,\alpha \leq \ell, \alpha \in \mathcal{O}\!\mathcal{L} \cup \{0\}$ and hence taking suprema in L we get $\phi\psi(\ell) = \ell$ (the sup of the above right side is ℓ because L is atomic).

On the other hand for $\mathcal{C} \subseteq Y$ and $\mathcal{C} \in \Theta^*(X)$

$$\psi(\phi(\mathcal{C})) = \left\{F(\alpha)\,\middle|\,\begin{array}{l}\alpha \in \mathcal{O}\!\mathcal{L} \cup \{0\}\\ \alpha \leq \phi(\mathcal{C})\end{array}\right\} \cup \{I\} \text{ and so clearly } \mathcal{C} \subseteq \psi\phi\mathcal{C}.$$

So let $F(\alpha) \in \psi\phi\mathcal{C}$. Then $\alpha \in \mathcal{O}\!\mathcal{L} \cup \{0\}$ and $\alpha \leq \sup_L\{\Delta(y)\,|\,y \in \mathcal{C}, y \neq I\}$. But by US6 we then have

$$F(\alpha) \in \overline{\{F\Delta y\,|\,y \in \mathcal{C}, y \neq I\}}^F$$

which is:

$$F(\alpha) \in \overline{\{y \mid y \in \mathcal{C},\ y \neq I\}}^{F} \qquad \begin{pmatrix} \text{since } y \in \mathcal{C} \\ y \neq I \text{ forces} \\ y \in \lim F \end{pmatrix}$$

but the latter is certainly contained in $\mathcal{C}$. So $F(\alpha) \in \mathcal{C}$. Thus $\psi\phi(\mathcal{C}) = \mathcal{C}$ for each $\mathcal{C} \in \Theta^*(X)$. So finally we see that ϕ, ψ are order isomorphisms between $\Theta^*(X)$ and L inverse to one another. So we get our converse in the case where there is an atom α_0 whose F value is 1.

Now consider the other possibility. Hereafter suppose that L has a spot (F,Δ) and for no atom α is $F(\alpha) = 1$. Then $\Delta(1) = 1$ and $\{x \mid \Delta(x) = 1\} = \{x \mid Fx = 1\}$ and this set has no atom in it. This time let $Y = \mathrm{im}F$ and then $X = Y^{op}$.

Let $\ell \in L$. Much like the first case let $\psi(\ell) = \{F(\alpha) \mid \alpha \in \mathcal{Q} \cup \{0\},\ \alpha \leq \ell\} \cup \{1_L\}$. Thanks to US4, $\psi(\ell)$ is closed under finite joins (join taken in imF); and due to US5 $\psi(\ell)$ is closed in the F-order topology. So then from the dual viewpoint $\psi(\ell) \in \Theta^*(X)$. So we have the order preserving mapping $\chi : L \longrightarrow \Theta^*(X)$ defined by

$$\psi(\ell) = \{F(\alpha) \mid \alpha \leq \ell,\ \alpha \in \mathcal{Q} \cup \{0\}\} \cup \{1\}.$$

Going the other way define $\phi : \Theta^*(X) \longrightarrow L$ by $\phi(\mathcal{C}) = \sup_L \{\Delta(y) \mid y \in \mathcal{C},\ y \neq 1\}$. Then ϕ is order preserving and in a manner quite analogous to the work above one can show: for any $\mathcal{C} \in \Theta^*(X)$ $\psi(\phi(\mathcal{C})) = \mathcal{C}$.

Let $\ell \in L$. $\phi\psi\ell = \sup_L \left\{ \Delta y \,\middle|\, \begin{matrix} y \in \psi\ell \\ y \neq 1 \end{matrix} \right\}$. But $\left\{ \Delta y \,\middle|\, \begin{matrix} y \in \psi\ell \\ y \neq 1 \end{matrix} \right\} = \left\{ \alpha \,\middle|\, \begin{matrix} \alpha \leq \ell \\ \alpha \in \mathcal{Q} \cup \{0\} \end{matrix} \right\}$

(if α is in the right set, $y = F(\alpha) \in \psi(\ell)$, $y = F(\alpha) \neq 1$ and so $\alpha = \Delta(y) = \Delta F\alpha$ is in the left set. If Δy is in the left set then $y \in \psi\ell$, $y \neq 1$ forces $y = F(\alpha)$ for some $\alpha \leq \ell$, $\alpha \in \mathfrak{A} \cup \{0\}$. Then $\Delta y = \Delta F\alpha = \alpha$ is in the right side set). So taking suprema in L we get $\phi\psi\ell = \ell$.

So with the slight modification in the definition of X we get the maps $\psi: L \longrightarrow \Theta^*(X)$, $\phi: \Theta^*(X) \longrightarrow L$ to be order isomorphisms inverse to one another. Summing up we have:

Proposition 4.2. Suppose the complete atomic lattice L has an upper spot (F,Δ). Then if there is an atom α_0 so that $F(\alpha_0) = 1$ and if we let $X = (\mathrm{im}F \,\dot{\cup}\, \{I\})^{op}$ (where $I \notin \mathrm{im}F$ and is defined so that $I \geq \mathrm{im}F$) then $\Theta^*(X) \cong L$. If for each atom α, $F(\alpha) \neq 1$ then letting $X = (\mathrm{im}F)^{op}$, $\Theta^*(X) \cong L$.

So our main result of this section is

Theorem 4.3. A complete atomic lattice L is isomorphic to $\Theta^*(X)$ for some complete lattice X if and only if L has an upper spot (F,Δ).

We conclude this section with some comments on the algebraic case. Let L be a complete atomic lattice with upper spot (F,Δ). In a rather straightforward way we can call an atom α of L F-compact if the element $F(\alpha)$ is dually compact in the complete lattice imF. Then we will say that the upper spot (F,Δ) is algebraic if for each $\alpha \in \mathfrak{A}$ we have $\alpha \leq \sup_L\{\beta \mid \alpha \leq F(\beta)$ and β is F-compact$\}$. We then get

Proposition 4.4. Let L be a complete atomic lattice. Then the following statements are equivalent:

(i) L has an algebraic upper spot

(ii) $L \cong \Theta^*(X)$ for some algebraic lattice X

(iii) $L^{op} \cong \Theta(S)$ for some join semilattice S

Proof. (ii)$\Longleftrightarrow$(iii) already. Assume (i). From proposition 4.2 it is clear that all we need to get (ii) is to show imF dually algebraic. To show imF dually algebraic it is sufficient to show that each proper element of imF is the meet of dually compact elements. So take $F(\alpha)$ proper in imF ($\alpha \in \mathfrak{A}$). Since the spot is algebraic we know $\alpha \leq \sup_L\{\beta | \alpha \leq F(\beta),\ \beta$ F-compact$\}$. US6 then forces $F(\alpha) \in \overline{\{F(\beta) | \alpha \leq F(\beta),\ \beta \text{ F-compact}\}}^F$. If $f = \inf\{F(\beta) | \alpha \leq F(\beta),$ β F-compact$\}$ then the interval $[f,1]$ in $F[L]$ is join closed and closed in the F-order topology and each $F(\beta) \in [f,1]$ (for all F-compact $\beta \ni \alpha \leq F(\beta)$). Hence

$$\overline{\left\{F(\beta) \,\middle|\, \begin{matrix}\beta \text{ F-compact} \\ \alpha \leq F(\beta)\end{matrix}\right\}}^F \subseteq [f,1]$$

So $F(\alpha) \in [f,1]$. But already $F(\alpha) \leq f$ so finally $F(\alpha) = \inf\{F(\beta) | \alpha \leq F(\beta),\ \beta$ F-compact$\}$ so $F(\alpha)$ is the infinum of a set of dually compact elements. (ii) $\Longrightarrow$(i): Suppose $L \cong \Theta^*(X)$ for X algebraic. Construct the natural upper spot (F,Δ) described in Proposition 4.1. It is then clear that imF is a dually algebraic lattice. Let $\alpha \in \mathfrak{A}$. Then

$$F(\alpha) = \inf\{F(\beta) | F(\alpha) \leq F(\beta) \text{ and } F(\beta) \text{ is dually compact}\}$$

$$= \inf\{F(\beta) | \alpha \leq F(\beta) \text{ and } \beta \text{ is F-compact}\}.$$

Notice that this is the infimum of a downdirected collection and the Δ value of this infimum is α. US5 then tells us $\alpha \leq \sup_L\{\beta | \alpha \leq F(\beta)$ and β is F-compact$\}$ our desired result. So (F,Δ) is an algebraic spot.

5. Alternative axioms for the upper spot

In this section we prove the following result.

Proposition 5.1. Let L be a complete atomic lattice, F a closure operator on L so that $F(0) = 0$ and Δ a map of L into $\mathcal{O}\!\mathcal{L} \cup \{0,1\}$. The following statements are equivalent:

(i) (F,Δ) is an upper spot of L

(ii) (F,Δ) satisfies US6 and the following conditions

A1	$\Delta\Delta x = \Delta x$	for all $x \in L$
A2	$\Delta x \leq x$	for all x such that $\Delta(x) \neq 1$
A3	$\mathcal{O}\!\mathcal{L} \subseteq \operatorname{im} \Delta$	
A4	$\Delta x = \Delta y$ iff $Fx = Fy$	for all x, y
A5	$\Delta(\inf F(\alpha_i)) \leq \sup_L \alpha_i$	for downdirected $\{F(\alpha_i)\}$.

Proof. Assume (ii) holds true. US1, US2 follow easily because of A4 and the fact that $F^2 = F$, $\Delta^2 = \Delta$. US3 is easy because of A3, A1 (and the newly established US1 & US2). To prove US4: if $\Delta(\alpha \vee \beta) \neq 1$ we get $\Delta(\alpha \vee \beta) \leq \alpha \vee \beta$ from A2, but easily $\Delta(\alpha \vee \beta) = \Delta(F\alpha \vee F\beta)$ so US4 follows. For US5: Take a net $(F(\alpha_d))_{d\varepsilon D}$ F order convergent and suppose $\Delta(\operatorname{Lim} F\alpha_d) \neq 1$. Write $s_d = \sup_{\operatorname{im}F}\{F(\alpha_{d'}) | d' \geq d\}$. Then $\operatorname{Lim} F(\alpha_d) = \inf_d s_d$ and $(s_d)_{d\varepsilon D}$ is downdirected and in imF. Since $\Delta(s_d) = 1$ forces $s_d = 1$ (and since $\Delta(x) = 1$ for some x forces $\Delta(1) = 1$) apparently $\{d | \Delta(s_d) = 1\}$ cannot be cofinal in D, hence there is a d_0 so that for all $d \geq d_0$, $\Delta(s_d) \neq 1$. Now we can apply A5 to the net $(F(\Delta s_d))_{d \geq d_0}$ which is downdirected and get $\Delta(\inf_{d \geq d_0} F\Delta s_d) \leq \sup_{d \geq d_0} \Delta s_d$. But clearly $\inf_{d \geq d_0} F\Delta s_d = \inf_{d \geq d_0} s_d = \inf_d s_d = \lim F(\alpha_d)$ so we have

$$(1) \qquad \Delta(\lim F\,\alpha_d) \leq \sup_L\{\Delta(s_d) \mid d \geq d_0\}.$$

Looking at a given s_d $(d \geq d_0)$ we see that $s_d = F(\sup_L\{\alpha_{d'} \mid d' \geq d\})$ so $\Delta(s_d) = \Delta F(\sup_L\{\alpha_{d'} \mid d' \geq d\}) = \Delta(\sup_L\{\alpha_{d'} \mid d' \geq d\}) \leq \sup_L\{\alpha_{d'} \mid d' \geq d\}$ (we have just used A2 and the fact that $\Delta s_d \neq 1$). So for each $d \geq d_0$ we have

$$(2) \qquad \Delta(s_d) \leq \sup_L\{\alpha_{d'} \mid d' \geq d\}.$$

Putting together (1) and (2) above and using general associativity we have

$$\Delta(\lim F(\alpha_d)) \leq \sup_L\{\alpha_d \mid d \in D\}$$

giving US5. We now have (ii)$\Longrightarrow$(i).

Assume (i) holds true. $\Delta(x) = \Delta y$ implies $F\Delta x = F\Delta y$ giving $Fx = Fy$. Conversely $Fx = Fy$ implies $\Delta Fx = \Delta Fy$ hence $\Delta x = \Delta y$. We have A4. A1 is the same as $F\Delta x = Fx$, the latter is known true. Suppose $\Delta(x) \neq 1$. We show $\Delta(x) \leq x$. We claim that $\mathcal{D} = \{F(\alpha) \mid \alpha \leq x, \alpha \in \mathcal{A} \cup \{0\}\}$ is updirected. Take $F(\alpha_1), F(\alpha_2) \in \mathcal{D}$ (α_1, α_2 distinct atoms below x). If $\Delta(\alpha_1 \vee \alpha_2) = 1$ then $F(\alpha_1 \vee \alpha_2) = 1$ and then $F(x) = 1$ (since $\alpha_1 \vee \alpha_2 \leq x$) and then $\Delta x = \Delta Fx = \Delta 1 = \Delta\Delta(\alpha_1 \vee \alpha_2) = \Delta(\alpha_1 \vee \alpha_2) = 1$ so $\Delta x = 1$.$\otimes$ Thus $\Delta(\alpha_1 \vee \alpha_2) \neq 1$ and $\Delta(\alpha_1 \vee \alpha_2) \in \mathcal{A} \cup \{0\}$ and

$\Delta(\alpha_1 \vee \alpha_2) = \Delta(F\alpha_1 \vee F\alpha_2) \leq \alpha_1 \vee \alpha_2$ (by US4) $\leq x$. Hence $F(\alpha_1 \vee \alpha_2)$ is of the form $F(\alpha)$ for $\alpha \leq x$, $\alpha \in \mathcal{A} \cup \{0\}$ (namely $\alpha = \Delta(\alpha_1 \vee \alpha_2)$) and $F(\alpha_1 \vee \alpha_2)$ is then an upper bound in $\mathcal{D}$ for $F(\alpha_1)$ and $F(\alpha_2)$. So $\mathcal{D}$ is updirected. Now $\sup_{\mathrm{im}F}\left\{F(\alpha) \middle| \begin{array}{l}\alpha \in \mathcal{A} \cup \{0\} \\ \alpha \leq x\end{array}\right\} = F(\sup_L\{\alpha | \alpha \leq x,$ $\alpha \in \mathcal{A} \cup \{0\}\}) = F(x)$. So $F(x) = \lim_F \mathcal{D}$ ($\mathcal{D}$ is an order convergent net in imF) and so by US5

$$\Delta(\lim \mathcal{D}) \leq \sup_L\{\alpha | \alpha \leq x, \alpha \in \mathcal{A} \cup \{0\}\}$$

(we have used $\Delta(x) \neq 1$ to tell us that $\Delta(\lim \mathcal{D}) = \Delta Fx = \Delta x \neq 1$ which is needed to apply US5). Rewriting the last result we get $\Delta x \leq x$. We now have A2.

A3 is given by US3 and A5 comes easily from US5. So we get all of (ii).

6. The upper spot decomposition

In this section we show that L's having an upper spot is equivalent to L's being decomposable in a certain nice way into join complete join semilattices (here this always means that any nonempty subset has a supremum).

Proposition 6.1. There is a one-one correspondence between closure operators F on a complete lattice L and decompositions $\mathcal{S}$ of L into disjoint join complete join semilattices with the property that for any $S_1, S_2 \in \mathcal{S}$ there exists $S \in \mathcal{S}$ so that $S_1 \vee S_2 \subseteq S$. (By $S_1 \vee S_2$ we mean $\{x \vee y | x \in S_1$ and $y \in S_2\}$).

<u>Proof</u>. Given such an F let $\mathcal{S}_F = \{S_f | f \in \text{im}F\}$ where $S_f = \{x | Fx = f\}$. Each S_f is a join complete join semilattice and given S_f and S_g certainly $S_f \vee S_g \subseteq S_{F(f \vee g)}$. So $\mathcal{S}_F$ is a decomposition of the required type.

Conversely for such a decomposition $\mathcal{S}$ we define a map $F_\mathcal{S} : L \longrightarrow L$ as follows: for any $x \in L$ take $S \in \mathcal{S}$ so that $x \in S$. Then let $F_\mathcal{S}(x) = \sup_L S \in S$. Then $F_\mathcal{S}$ is a closure operator on L.

Now $F_{(\mathcal{S}_F)}(x) = \sup S_f$ where $x \in S_f$. But $x \in S_f$ iff $F(x) = f$ so $F_{(\mathcal{S}_F)}(x) = F(x)$. $F_{\mathcal{S}_F} = F$. Also we consider $\mathcal{S}_F$. For any x, $F_\mathcal{S}(x) = \sup S$ where $x \in S \in \mathcal{S}$. If $S \in \mathcal{S}$ then $x \in S$ iff $F_\mathcal{S}(x) = \sup S$ so that $S = S_{F_\mathcal{S}(x)} \in \mathcal{S}_{(F_\mathcal{S})}$. If we take a typical $S_{F_\mathcal{S}(x)} \in \mathcal{S}_{F_\mathcal{S}}$ we have $S_{F_\mathcal{S}(x)} = \{x | x \in S \in \mathcal{S} , \sup S = F_\mathcal{S}(x)\}$ $= S \in$
So $\mathcal{S}_{F_\mathcal{S}} = \mathcal{S}$.

The join complete join semilattices mentioned in proposition 6.1 are necessarily convex and we have also for any collection (S_λ) in there is $S \in \mathcal{S}$ so that

$$\bigvee S_\lambda = \{ \bigvee x_\lambda | x_\lambda \in S_\lambda \} \subseteq S.$$

We know our upper spot is a kind of closure family and so can be expressed as some type of decomposition of L. We expect (because of Δ) the elements of the decomposition to correspond to the atoms of L (and also 0 and 1). So we index the elements of the decomposition with the set $\mathcal{A} \cup \{0,1\}$ and attempt to rephrase in the decomposition language the upper spot properties.

Let $\mathcal{S}$ be a collection of pairwise disjoint join complete join semilattices S_α, $\alpha \in \mathcal{A} \cup \{0,1\}$ such that

(1) $$\dot{\bigcup_{\alpha \in \mathfrak{A} \cup \{0\}}} S_\alpha = L$$

(2) For each $\alpha \in \mathfrak{A}$, α is in S_α and is minimum there

(3) $S_0 = \{0\}$, $S_1 = \emptyset$ if $1 \notin S_1$

(4) For each pair of atoms α, β there exists $\gamma \in \mathfrak{A} \cup \{0,1\}$ so that $S_\alpha \vee S_\beta \subseteq S_\gamma$. (We can show $\gamma \leq \alpha \vee \beta$ if $\gamma \neq 1$.)

Call such a collection $\mathcal{S}$ an <u>atomwise semilattice decomposition of L</u>. We can see how it will give rise to a closure operator and a function Δ. But there still is not enough to form an upper spot.

Note that if L has an atomwise semilattice decomposition $\mathcal{S}$, it is almost a decomposition into intervals. For each $\alpha \in \mathfrak{A} \cup \{0\}$ let $u_\alpha = \sup_L S_\alpha \in S_\alpha$. Then $S_\alpha = [\alpha, u_\alpha]$ ($\alpha \in \mathfrak{A} \cup \{0\}$) so that

$$L = \dot{\bigcup_{\alpha \in \mathfrak{A} \cup \{0\}}} [\alpha, u_\alpha] \;\dot{\cup}\; S_1$$

where $S_1 = \emptyset$ if $1 \notin S_1$.

The elements $\{u_\alpha\}_{\alpha \in \mathfrak{A} \cup \{0\}} \cup \{1\}$ then form a closure family with its own order convergence. Now we state versions of A5 and US6.

(I) If $\{u_{\alpha_i}\}$ is downdirected and ρ is chosen so that $\bigwedge_i u_{\alpha_i} \in S_\rho$ then $\rho \leq \bigvee_i \alpha_i$.

(II) $\alpha \leq \bigvee_i \alpha_i$ implies $u_\alpha \in \overline{\{u_{\alpha_i} | i\}}$

where in both I and II all α_i, α are atoms and $\overline{\{u_{\alpha_i} | i\}}$ denotes the join closed order closed subset of the complete lattice $\{u_\beta | \beta \in \mathfrak{A} \cup \{0\}\} \cup \{1\}$ generated by $\{u_{\alpha_i} | i\}$. So our result is

<u>Proposition 6.2</u>. Let L be a complete atomic lattice. Then L has an upper spot iff L has an atomwise semilattice decomposition satisfying (I) and (II).

<u>Proof</u>. Suppose L has an upper spot (F,Δ). For each α in $\mathcal{O}\!\mathcal{L} \cup \{0,1\}$ let $S_\alpha = \{x \mid \Delta x = \alpha\} = \{x \mid \Delta x = \Delta\alpha\}$ and if $\alpha \neq 1$, $S_\alpha = \{x \mid Fx = F\alpha\}$. Clearly each S_α is a join complete join semilattice and $\alpha \in \mathcal{O}\!\mathcal{L} \cup \{0\}$ implies α is the least element of S_α. Also $S_1 = \emptyset$ if $1 \notin S_1$. Take $\alpha, \beta \in \mathcal{O}\!\mathcal{L} \cup \{0,1\}$. If one is zero, say α, then $S_\alpha \vee S_\beta \subseteq S_\beta$. So suppose $\alpha, \beta \in \mathcal{O}\!\mathcal{L} \cup \{1\}$. If $\Delta(\alpha \vee \beta) = 1$ (then $S_1 \neq 0$, $1 \in S_1$) and $S_\alpha \vee S_\beta \subseteq S_1$. Now if $\Delta(\alpha \vee \beta) \leq \alpha \vee \beta$ (which is the only other possibility by A2) then $S_\alpha \vee S_\beta \subseteq S_{(\alpha \vee \beta)}$. For suppose $\Delta(x) = \Delta\alpha = \alpha$, $\Delta(y) = \Delta(\beta) = \beta$. Then $F(x \vee y) = F(Fx \vee Fy)$ $= F(F\Delta x \vee F\Delta y) = F(F\alpha \vee F\beta) = F(\alpha \vee \beta)$ so $F(x \vee y) = F(\alpha \vee \beta)$ so A4 gives $\Delta(x \vee y) = \Delta(\alpha \vee \beta)$ hence $x \vee y \in S_{\Delta(\alpha \vee \beta)}$. So now we have an atomwise semilattice decomposition. To get (I): Take u_{α_i} downdirected, $\alpha_i \in \mathcal{O}\!\mathcal{L}$ and choose ρ so that $\bigwedge_i u_{\alpha_i} \in S_\rho$. Now $u_{\alpha_i} = \sup\{x \mid \Delta x = \alpha_i\}$ $= \sup\{x \mid F(x) = F(\alpha_i)\} = F(\alpha_i)$. Now $\bigwedge_i F(\alpha_i) \in S_\rho$ means $\Delta(\bigwedge_i F\alpha_i) = \rho$ so we easily get $\rho \leq \bigvee_i \alpha_i$ from A5. It is clear to see that II follows now from US6.

Conversely assume L has an atomwise semilattice decomposition satisfying (I) and (II). Build F as in Proposition 6.1, that is let $F(x) = \sup S_\alpha$ where $x \in S_\alpha$. F becomes a closure operator so that $F(\alpha) = u_\alpha$ for each $\alpha \in \mathcal{O}\!\mathcal{L} \cup \{0\}$. Define $\Delta(x) = \alpha$ iff $x \in S_\alpha$. Then $\Delta\colon L \longrightarrow \mathcal{O}\!\mathcal{L} \cup \{0,1\}$ and A1, A2, A3 easily follow. Apparently A4 holds and A5 is obtained from I. Finally we get US6 from II. So L has an upper spot.

Theorem 6.3. For a complete atomic lattice L the following are equivalent:

(i) $L \cong \Theta^*(X)$ for some complete lattice X

(ii) L has an atomwise semilattice decomposition satisfying (I) and (II).

We leave to the reader the following easily obtained special cases.

Proposition 6.4. For a complete atomic L the statements below are equivalent:

(i) $L \cong \Theta^*(X)$ for some complete lattice X with a least proper element.

(ii) L has an atomwise semilattice decomposition into intervals $[\alpha, u_\alpha]$ $(\alpha \in \mathfrak{A} \cup \{0\})$ satisfying I and II

and in this case we can choose X so that $\{z \in X | z > 0\}$ is anti-isomorphic to $\{u_\alpha | \alpha \in \mathfrak{A} \cup \{0\}\}$.

And the finite version:

Proposition 6.5. Let L be a finite atomic lattice. The following are equivalent:

(i) $L \cong \Theta^*(S^*)$ for some finite join semilattice S

(ii) L has an atomwise semilattice decomposition satisfying $\alpha \leq \sup_L\{\alpha_i | i\}$ implies $u_\alpha = \bigvee_j u_{\alpha_j}$ for certain $\alpha_j \in \{\alpha_i\}$.

CHAPTER II

PRIME ATOMS IN COMPLETE ATOMIC LATTICES

It is the objective of this chapter to show that certain of the properties which have been demonstrated for $\Theta(S)$ are actually consequences of certain relations holding among its dual atoms. We will assume the duality of Fajtlowicz and Schmidt [5] and work in $\Theta^*(S^*)$. So we actually will look at complete atomic lattices whose atoms satisfy certain conditions and in such lattices we will demonstrate the duals of the properties semi-Brouwerian, upper semimodularity, etc. Our theorems will then apply to a wider class of lattices than just the $\Theta^*(S^*)$'s (often including for instance the lattice of subsemilattices of a semilattice). In the process we obtain properties of $\Theta(S)$ not already known, such as M-symmetry and, when $\Theta(S)$ is dually algebraic, quasi-decomposability. This work owes much to the paper of Fajtlowicz and Schmidt [5] which was the chief motivation for this work. The techniques of section 3 are essentially an abstraction of methods used by Fajtlowicz and Schmidt [5].

1. Some definitions and examples

In this section we set forth some basic definitions and discuss our two motivating examples. Let L be a complete atomic lattice. We say L is <u>neatly atomic</u> if for each pair x, y of nonzero elements of L and for each atom α, $\alpha \leq x \vee y$ implies that $\alpha \leq \beta \vee \gamma$ for certain atoms

$\beta \leq x$, $\gamma \leq y$. L is said to satisfy the <u>2-3 condition</u> if for each pair α, β of atoms of L there are either two or three atoms below $\alpha \vee \beta$. We define a binary relation T_0 among the atoms of L as follows: for distinct atoms α, β of L we write $\alpha T_0 \beta$ iff there is an atom $\gamma \notin \{\alpha, \beta\}$ so that $\alpha \leq \beta \vee \gamma$. Notice that if L is neatly atomic, $\alpha T_0 \beta$ iff α is <u>subperspective</u> to β, as that notion is defined in Maeda and Maeda [10] (x subperspective to y if there is a $z \in L$ so that $x \wedge z = 0$, $x \leq y \vee z$). We turn now to our examples.

<u>Example 1</u>. This is of course $L = \Theta^*(S^*)$ for a join semilattice S. $\Theta^*(S^*)$ is complete and atomic, its atoms being elements of the form $\mathcal{C}_I = \{\emptyset, S, I\}$ where I is any proper ideal ($I \neq \emptyset$, $I \neq S$) of S. To say $\mathcal{C}_I \leq \mathcal{A} \in \Theta^*(S^*)$ simply means $I \in \mathcal{A}$ and so $\mathcal{A} = \sup_L \{\mathcal{C}_I | I \in \mathcal{A}$, I proper$\}$. The supremum in L of a set of atoms $\{\mathcal{C}_{I_j} | j\}$ is just the algebraic closure family (containing $\emptyset$) in S^* generated by $\{I_j | j\}$. For $\mathcal{A}$, $\mathcal{B}$ in $\Theta^*(S^*)$ their join is $\mathcal{A} \vee \mathcal{B} = \{I \cap J | I \in \mathcal{A}, J \in \mathcal{B}\}$. Hence for I, J proper ideals $\mathcal{C}_I \vee \mathcal{C}_J = \{\emptyset, S, I, J, I \cap J\}$. Clearly L satisfies the 2-3 condition. Also L is neatly atomic. Notice also that the T_0 relation is asymmetric. Finally, because of the theorem of Birkhoff and Frink, for each $\mathcal{A}$ in $\Theta^*(S)$, $\mathcal{A}$ is generated by the set $\{J | J \in \mathcal{A}$, J completely meet irreducible in $\mathcal{A}\}$. Thus

$$\mathcal{A} = \sup_L \left\{ \mathcal{C}_J \,\middle|\, \begin{array}{l} J \text{ completely meet} \\ \text{irreducible in } \mathcal{A} \end{array} \right\} = \sup_L \left\{ \mathcal{C}_J \,\middle|\, \begin{array}{l} J \text{ finitely meet} \\ \text{irreducible in } \mathcal{A} \end{array} \right\}.$$

So for instance $S^* = \sup_L \{\mathcal{C}_J | J$ finitely meet irreducible$\}$. Notice that $I \in \mathcal{A}$ is finitely meet irreducible in $\mathcal{A}$ iff for each finite set $\mathcal{C}_{J_1}, \ldots, \mathcal{C}_{J_n}$ of atoms of $\mathcal{A}$ (below $\mathcal{A}$) $\mathcal{C}_I \leq \mathcal{C}_{J_1} \vee \cdots \mathcal{C}_{J_n}$ implies $\mathcal{C}_I = \mathcal{C}_{I_K}$ for some K.

Example 2. Let M be any meet semilattice. Let L be Sub(M) = $\{N \subseteq M | N \wedge N \subseteq N\}$, the lattice of subsemilattices of M. L is a complete lattice. The atoms of L are the singletons $\{m\}$ ($m \varepsilon M$), the zero is $\emptyset$. Each element N of L is $N = \sup_L\{\{n\}|n \varepsilon N\}$, so that L is atomic. If $\mathcal{D}$ is any set of atoms of L then $\sup_L\{\{m\}|\{m\} \varepsilon \mathcal{D}\}$ is the subsemilattice of M generated by $\{m|\{m\} \varepsilon \mathcal{D}\}$ and so $\sup \mathcal{D} =$ $\{x \varepsilon M | x = m_1 \wedge \cdots \wedge m_t$ for certain m_i with $\{m_i\} \varepsilon \mathcal{D}\}$. Then for m, $n \varepsilon M$, $\{m\} \vee \{n\} = \{m,n,m \wedge n\}$ and so L satisfies the 2-3 condition. Now for K, $N \varepsilon L$

$$K \vee N = K \cup N \cup \{k \wedge n | k \varepsilon K \text{ and } n \varepsilon N\}.$$

From this it follows that L is neatly atomic. Also $\{m\}T_0\{n\}$ implies $m < n$ (in M) and so T_0 is asymmetric; in fact, T_0 has no cycles and so its transitive closure is an irreflexive partial order (which recovers some of the order of M). Finally L = Sub(M) is an algebraic lattice, its compact elements being those which are the join of finitely many atoms.

Notational Conventions. We will use the abbreviations: PC to stand for "pseudo-complemented" (or "pseudo-complement" as the reader can tell from the context), DPC for "dually pseudo-complemented" (or "dual pseudo-complement"), SB for "semi-Brouwerian" and DSB for dually semi-Brouwerian.

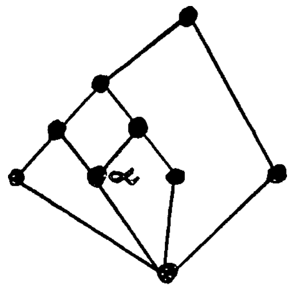
2. Various types of atoms

Let L be a complete atomic lattice. We begin with a description of various types of atoms which might occur in L. An atom α is called <u>prime</u> iff the condition $\alpha \leq \beta_1 \vee \dots \vee \beta_n$ implies $\alpha = \beta_i$ for some i (for any atoms $\beta_1, \dots, \beta_n$). Π will denote the set of prime atoms. An atom α below a given element x of L will be called <u>x-prime</u> if α is a prime atom of the lattice $[0,x]$. Π_x denotes the set of these. $\mathcal{C}$ will denote the collection of complemented atoms, that is atoms α for which there is an element z in L with $\alpha \vee z = 1$ and $\alpha \wedge z = 0$ ($\alpha \nleq z$). An atom α is called (join) <u>primitive</u> if for any x, y ε L $\alpha \leq x \vee y$ implies $\alpha \leq x$ or $\alpha \leq y$. $\mathcal{P}$ denotes the set of these. Finally an atom α is called <u>1-primitive</u> if for any coprime pair x, y of elements of L (i.e., $x \vee y = 1$) either $\alpha \leq x$ or $\alpha \leq y$. $\mathcal{P}_1$ denotes the set of these.

Now we always have $\mathcal{P} \subseteq \mathcal{P}_1$, $\mathcal{P} \subseteq \Pi$. If L is neatly atomic then $\mathcal{P} = \Pi$, while if L is a finite support lattice (each element is the join of finitely many atoms) then $\mathcal{P} = \Pi \subseteq \mathcal{C}$. So in each finite atomic lattice L we have

$$\begin{array}{c} \mathcal{P}_1 \\ | \\ \mathcal{P} = \Pi \subseteq \mathcal{C}. \end{array}$$

The containment $\Pi \subseteq \mathcal{C}$ can, even in the finite atomic case, be proper. Consider



here $\alpha \in \mathcal{C}$ but $\alpha \notin \mathcal{T}$. Notice that if a complete atomic lattice L is algebraic (which is equivalent to saying that all atoms are compact) then $\mathcal{T} \subseteq \mathcal{C}$.

Our first interest is the effect of L being DPC on the relations between the various types of atoms.

<u>Lemma 2.1</u>. Suppose L is complete atomic and DPC. Then an atom α is open iff $\alpha \not\leq \neg\alpha$ iff $\alpha \in \mathcal{C}$.

<u>Proof</u>. Recall that α open means $\alpha = \neg\neg\alpha$. If $\alpha \leq \neg\alpha$ then with α open the meet of $\neg\neg\alpha$ and $\neg\alpha$ in the Boolean lattice of open elements is $\geq \alpha$ and so not zero (a contradiction). If α is an atom so that $\alpha \not\leq \neg\alpha$ then $\alpha \wedge \neg\alpha = 0$ so with $\alpha \vee \neg\alpha$ already 1 we get $\alpha \in \mathcal{C}$. Finally if $\alpha \in \mathcal{C}$ there is an x in L so that $\alpha \not\leq x$ but $\alpha \vee x = 1$. Hence $\neg\alpha \leq x$, so $\alpha \not\leq \neg\alpha$. But $\neg\neg\alpha \leq \alpha$ so $\neg\neg\alpha = 0$ or α. If $\neg\neg\alpha = 0$ then $\neg\alpha = 1$ giving $\alpha \leq \neg\alpha$. So $\neg\neg\alpha = \alpha$, α is open.

<u>Lemma 2.2</u>. Suppose L is complete atomic and DPC. Suppose $x \in L$ and for distinct atoms α, β: $x \vee \alpha = x \vee \beta = 1$. Then $x = 1$.

<u>Proof</u>. $x \vee \alpha = 1$ gives $\neg x \leq \alpha$; $x \vee \beta = 1$ gives $\neg x \leq \beta$. Hence $\neg x \leq \alpha \wedge \beta = 0$. So $\neg x = 0$ so $x = 1$.

Corollary 2.3. If L is complete atomic DPC then $\mathcal{C} \subseteq \Pi$. If L is also neatly atomic then $\mathcal{C} \subseteq \mathcal{P} = \Pi \subseteq \mathcal{P}_1$.

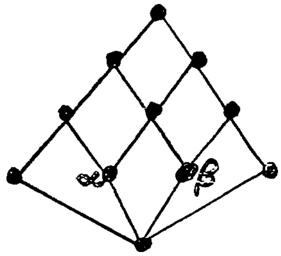
Proof. Let $\alpha \in \mathcal{C}$. Suppose $\alpha \leq \beta_1 \vee \cdots \vee \beta_n$ where $\beta_1, \ldots, \beta_n$ are atoms of L but $\alpha \neq \beta_i$ for each $i = 1, \ldots, n$. Then $(\neg\alpha \vee \beta_1 \vee \cdots \vee \beta_{n-1}) \vee \beta_n = (\neg\alpha \vee \cdots \vee \beta_{n-1}) \vee \alpha = 1$. The above lemma gives $\neg\alpha \vee \beta_1 \vee \cdots \vee \beta_{n-1} = 1$. But then $(\neg\alpha \vee \beta_1 \vee \cdots \vee \beta_{n-2}) \vee \beta_{n-1} = (\neg\alpha \vee \beta_1 \vee \cdots \vee \beta_{n-2}) \vee \alpha = 1$ so again by the lemma, $\neg\alpha \vee \beta_1 \vee \cdots \vee \beta_{n-2} = 1$. Repeating this activity eventually gives $\neg\alpha = 1$, so $\alpha \leq \neg\alpha$ forcing $\alpha \notin \mathcal{C}$.

While it is so accesible we state a result about the relation T_0 in DSB lattices. (Recall L is DSB if for each x the lattice $[0,x]$ is DPC.)

Corollary 2.5. Let L be complete, atomic, satisfying the 2-3 condition. If L is DSB then the relation T_0 is asymmetric.

Proof. Suppose by way of contradiction that for atoms α and β, $\alpha T_0 \beta$ with $\alpha \leq \beta \vee \gamma$ where $\gamma \notin \{\alpha,\beta\}$ and $\beta T_0 \alpha$ with $\beta \leq \alpha \vee \rho$ for $\rho \notin \{\alpha,\beta\}$ (note that by definition T_0 is irreflexive so $\alpha \neq \beta$). Let $x = \gamma \vee \rho$ and $y = \gamma \vee \rho \vee \alpha \vee \beta$. The lattice $[0,y]$ is complete, atomic and DPC (since L is DSB) and $x \vee \alpha = x \vee \alpha = y$, the latter being the 1 of the lattice $[0,y]$. So lemma 2.2 gives us $x = y$ and so $\alpha, \beta \leq \gamma \vee \rho$. But then $\gamma \vee \rho$ has too many atoms below it. (If $\gamma \neq \rho$ then $\gamma \vee \rho$ has four atoms below it, violating the 2-3 condition, if $\gamma = \rho$ then we have $\alpha, \beta \leq \gamma \vee \rho = \gamma$ which is impossible.)

Note. That 2.5 fails without the 2-3 condition is seen by considering



This is a complete atomic DSB lattice, but $\alpha T_0 \beta$ and $\beta T_0 \alpha$.

3. Expressible dual pseudocomplements

Let L be a complete, atomic DPC lattice. Recall that for $x \in L$, $\neg x$ denotes the DPC of x, namely $\neg x = \mathrm{MIN}\{y \mid x \vee y = 1\}$. Let $\mathcal{E} \subseteq at(L) = \{\alpha \mid \alpha \text{ is an atom of } L\}$. We say that dual pseudocomplementation is <u>$\mathcal{E}$-expressible</u> (or just: <u>$\neg$ is $\mathcal{E}$-expressible</u>) iff for each $x \in L$, $\neg x = \sup_L\{\alpha \in \mathcal{E} \mid \alpha \nleq x\}$. We say <u>$\neg$ is expressible</u> if for some $\mathcal{E} \subseteq \mathcal{A}$, $\neg$ is $\mathcal{E}$-expressible. Our main result about expressible DPC's is:

<u>Proposition 3.1</u>. Let $\mathcal{E} \subseteq at(L)$. The following are equivalent:

(i) L is DPC with $\neg$ $\mathcal{E}$-expressible

(ii) $\mathcal{E} \subseteq \mathcal{P}_1$ and $\sup_L \mathcal{E} = 1$.

<u>Proof</u>. <u>(i) $\Longrightarrow$ (ii)</u>. Let $\alpha \in \mathcal{E}$. Suppose x, y are coprime, $x \vee y = 1$. Suppose $\alpha \nleq x$. We know $\neg x = \sup_L\{\beta \in \mathcal{E} \mid \beta \nleq x\}$ and so $\alpha \leq \neg x$. But $x \vee y = 1$ forces $\neg x \leq y$ so we get $\alpha \leq y$. Thus $\alpha \in \mathcal{P}_1$ and we get $\mathcal{E} \subseteq \mathcal{P}_1$. Note that $\neg 0 = \sup_L\{\beta \in \mathcal{E} \mid \beta \nleq 0\} = \sup_L \mathcal{E}$, and since $\neg 0 = 1$ we get $\sup_L \mathcal{E} = 1$. So we have (ii).
<u>(ii) $\Longrightarrow$ (i)</u>. Take $x \in L$ and let $y = \sup_L\{\alpha \in \mathcal{E} \mid \alpha \nleq x\}$. Because

$\sup_L \mathcal{E} = 1$ we know $x \vee y = 1$. Now suppose $x \vee z = 1$. Since $\mathcal{E} \subseteq \mathcal{P}_1$ then for any $\alpha \in \mathcal{E}$ if $\alpha \not\leq x$ then $\alpha \leq z$; thus, $y \leq z$. So y is the DPC of x and $\neg x = y = \sup_L\{\alpha \in \mathcal{E} \mid \alpha \leq x\}$. So (i) holds.

Trivially we get:

<u>Corollary 3.2</u>. If L is complete, atomic and DPC then $\neg$ is expressible iff $\sup_L \mathcal{P}_1 = 1$.

Also:

<u>Corollary 3.3</u>. If L is complete, atomic and DPC. Suppose $\neg$ is $\mathcal{E}$-expressible and $\mathcal{D} \subseteq \mathcal{E}$ with $\sup_L \mathcal{D} = 1$. Then $\neg$ is $\mathcal{D}$-expressible.

Another easy result is then:

<u>Corollary 3.4</u>. Let L be complete and atomic. L has an expressible DPC iff $\sup_L \mathcal{P}_1 = 1$.

Now with $\mathcal{E} \subseteq at(L)$ we say L has <u>enough $\mathcal{E}$-atoms</u> iff $\sup_L \mathcal{E} = 1$.

<u>Proposition 3.5</u>. Suppose L is complete and atomic with enough prime (enough Π) atoms. If L is neatly atomic <u>OR</u> of finite support <u>or</u> if 1 is compact (in L) then L is DPC with $\neg$ being Π-expressible. So each neatly atomic complete lattice with enough primes is Π-expressibly DPC.

<u>Proof</u>. Each of the conditions listed puts Π inside $\mathcal{P}_1$ which is all we need by 3.1.

<u>Corollary 3.6</u>. Suppose L is complete and neatly atomic. Suppose for each $x \in L$ the lattice $[0,x]$ has enough prime atoms (that is, $[0,x]$ has enough x-primes or $\sup_L \Pi_x = x$). Then L is DSB and for each x the DPC in the lattice $[0,x]$ is Π_x-expressible.

We apply these results to our examples.

Corollary 3.7. (i) Suppose M is a meet semilattice satisfying the ascending chain condition. Then the lattice L = sub(M) has enough x-primes in each interval [0,x]. Hence L is DSB with prime expressible local DPC's. In the lattice L an atom $\{m\}$ is prime iff m is finitely meet irreducible in M iff $\{m\}$ is complemented. (ii) If S is a join semilattice then $\Theta^*(S^*)$ is DSB with for each $x \in \Theta^*(S^*)$ the DPC (in the lattice [0,x] being Π_x-expressible. An atom $\mathcal{C}_I$ is prime iff the ideal I is finitely meet irreducible amongst the ideals of S. Hence $\Theta(S)$ is semi-Brouwerian for any join semilattice S.

Proof. (i) For any meet semilattice M, with $m \in M$, it is clear that the atom $\{m\}$ of L = sub(M) will be prime in L iff the element m is finitely meet irreducible (FMI) in M. Suppose M satisfies the ACC. Then each element is the finite meet of FMI elements. Hence the sub-semilattice of M generated by the FMI elements is all of M. Thus $\sup_L\{\{m\} \mid \{m\} \text{ prime}\} = M$ (the 1 of L). So there are enough primes in L. But this situation duplicates itself in each interval [0,N] of L. For in a subsemilattice N of M we again have the ACC so sub(N) = [0,N] has enough primes. So L is complete, neatly atomic and for each $x \in L$ the lattice [0,x] has enough prime atoms. So our claims follow from 3.6.

(ii). Let S be a join semilattice. Each $\mathcal{C}$ in $\Theta^*(S^*)$ is generated by its $\mathcal{C}$ completely meet irreducible elements and so $\mathcal{C}$ is generated by those ideals which are FMI in $\mathcal{C}$. Now $\mathcal{C}_I$ (for a proper ideal $I \in \mathcal{C}$) is $\mathcal{C}$-prime iff I is FMI in $\mathcal{C}$. So each interval $[0, \mathcal{C}]$ in $\Theta^*(S^*)$ has enough $\mathcal{C}$-primes. The claims follow from 3.6.

We move toward a partial converse of 3.6. But first:

<u>Lemma 3.8</u>. Suppose L is complete and atomic and suppose 1 is the join of finitely many atoms. Then L has enough complemented ($\mathcal{C}$-) atoms.

<u>Proof</u>. Since 1 is the join of finitely many atoms we can take s to be the least positive integer k so that 1 is the join of k atoms. Then $1 = \alpha_1 \vee \cdots \vee \alpha_s$, each $\alpha_i \in at(L)$. Then $\alpha_i \nleq \alpha_1 \vee \cdots \vee \alpha_{i-1} \vee \alpha_{i+1} \vee \cdots \vee \alpha_s$ for each $i = 1, \ldots, s$ (by the choice of s) and so $x_i = \alpha_1 \vee \cdots \vee \alpha_{i-1} \vee \alpha_{i+1} \vee \cdots \vee \alpha_s$ is a complement for α_i in L. Thus each $\alpha_i \in$ and so $\sup_L$ $= 1$. Notice that $\Pi \subseteq \{\alpha_1, \ldots, \alpha_s\}$ so $\Pi \subseteq \mathcal{C}$.

We conclude this section with a summary.

<u>Proposition 3.9</u>. Suppose L is complete, neatly atomic and 1 is the join of finitely many atoms. The following statements are equivalent:

(i) L has enough primes

(ii) L is DPC

(iii) L has enough $\mathcal{P}_1$ atoms

(iv) $\mathcal{C} \subseteq \mathcal{P}_1$

If any of these holds then $\neg$ is $\mathcal{C}$-expressible (and so Π-expressible).

<u>Proof</u>. (i)$\Rightarrow$(ii) is the content of 3.5. For (ii)$\Rightarrow$(iii): because of (ii) and neatly atomic we have by 2.3 that $\mathcal{C} \subseteq \mathcal{P} = \Pi \subseteq \mathcal{P}_1$. By 3.8, $\sup \mathcal{C} = 1$ so $\sup \mathcal{P}_1 = 1$. Now (iii) easily implies (ii) (Proposition 3.4). Also easy: (iv)$\Rightarrow$(iii) (our hypotheses

give $\sup_L \mathcal{C} = 1$ and so $\mathcal{C} \subseteq \mathcal{P}_1$ yields $\sup_L \mathcal{P}_1 = 1$). For (iv) $\Rightarrow$ (i): $\mathcal{C} \subseteq \mathcal{P}_1$ implies (with $\sup \mathcal{C} = 1$ from Lemma 3.8) that L has a $\mathcal{C}$-expressible $\neg$. Now because L is DPC we have $\mathcal{C} \subseteq \Pi$ so $\sup_L \Pi = 1$. At last (ii) $\Rightarrow$ (iv). Assume L is DPC. By the last part of corollary 2.3, $\mathcal{C} \subseteq \mathcal{P} = \Pi \subseteq \mathcal{P}_1$.

4. Quasi-decomposability results

Following J. Schmidt [16] we call a PC meet semilattice M quasi-decomposable (QD) if each element m of M can be written $m = \bar{\bar{m}} \wedge d$ where $\bar{\bar{m}}$ is the PC of m and where d is dense, $\bar{\bar{d}} = 1$. (See Schmidt [16] for what is almost the whole story on this.) It is the aim of this section to show that if S is a join semilattice so that $\Theta(S)$ is dually algebraic then $\Theta(S)$ is QD. (So for finite S, $\Theta(S)$ is QD.) As the reader might by now expect we will work and prove our results in the dual $\Theta^*(S^*)$ and as usual our assertion is that what happens is a result of certain relations holding among the atoms of $\Theta^*(S^*)$.

First we quickly recall the dual notions. A join semilattice with 1 which is DPC is called dually quasi-decomposable (DQD) if each element x can be written $x = \neg\neg x \vee m$ where $\neg\neg x$ is open (in the Boolean algebra of open elements) and m is meager ($\neg\neg m = 0$). A DSB join semilattice is called fully DQD if for each x the lattice $[0,x]$ is DQD.

Let S be a DSD join semilattice. Recall that for $w \leq x$, $\neg_x w$ denotes the DPC of w in $[0,x]$, that is $\neg_x w = x \dot{-} w$. So if $y \leq x$, $\neg_x\neg_x w$ is x-open ($z \leq x$ is x-open iff $\neg_x\neg_x z = z$). Certainly for any $y \leq x$ we have, in the semilattice $[0,x]$, the following expression for y

(1) $$y = (\neg_x\neg_x y) \vee (y \dot{-} \neg_x\neg_x y).$$

So certainly if for any $x, y \in S$, $y \le x$ implies that $y \dot{-} \neg_x\neg_x y$ is x-meager ($w \le x$ is x-meager iff $\neg_x w = x$ iff $\neg_x\neg_x w = 0$) then each $[0,x]$ will be DQD, i.e., S will be fully DQD. That the x-meagerness of $y \dot{-} \neg_x\neg_x y$ is characteristic of the full DQD condition is the easily established result that follows.

Lemma 4.1. Let S be a DSB join semilattice. Then S is fully DQD iff for each $x, y \in S$, $y \le x$ the element $y \dot{-} \neg_x\neg_x y$ is x-meager.

Proof. "only if": Take $y \le x$. Since S is fully DQD then $[0,x]$ is DQD so $y = \neg_x\neg_x y \vee m$ for some x meager $m \le x$. Now $y \dot{-} \neg_x\neg_x y = \mathrm{MIN}\{w \mid w \le x \text{ and } w \vee \neg_x\neg_x y = y\}$ so $y \dot{-} \neg_x\neg_x y \le m$. So applying gives $\neg_x m \le x\ [y \dot{-} \neg_x\neg_x y]$, but $\neg_x m = x$ so $\neg_x(y \dot{-} \neg_x\neg_x y) = x$ and so $y \dot{-} \neg_x\neg_x y$ is x meager.

For our next theorem we put in for the sake of completeness some already known results. But first we need to mention some notation. For any element x of a complete atomic lattice L,

$\mathcal{C}_x = \{\alpha \mid \alpha \le x,\ \alpha \text{ complemented in the lattice } [0,x]\}$,

$\Pi_x = \{\alpha \mid \alpha \le x,\ \alpha \text{ prime atom in the lattice } [0,x]\}$,

$\mathcal{P}_1(x) = \{\alpha \mid \alpha \le x \text{ and } \alpha \text{ is 1-primitive in the lattice } [0,x]\}$

(thus $\alpha \le x$ is in $\mathcal{P}_1(x)$ iff $\alpha \le y$ or $\alpha \le z$ for any y, z whose join is x). An element of $[0,x]$ is called x compact if it is compact in the lattice $[0,x]$.

Theorem 4.2. Suppose L is complete atomic and for each x in L the following statements are true:

(i) $\mathcal{C}_x \subseteq \mathfrak{M}_x \subseteq \mathcal{P}_1(x)$

(ii) $[0,x]$ has enough $\mathcal{C}_x$ atoms ($\sup_L \mathcal{C}_x = x$)

(iii) each $\mathcal{C}_x$ atom is x compact

Then L is DSB and for each $y \leq x$

$$(2) \qquad \overline{\chi}\, y = \sup_L \{\alpha \in \mathcal{C}_x \mid \alpha \nleq y\}$$

and $y \dot{-} \overline{\chi}\,\overline{\chi}\, y$ is x meager. Hence L is fully DQD.

<u>Proof</u>. The first result is clear from earlier sections. Let $y \leq x$. We need to show $y \dot{-} \overline{\chi}\,\overline{\chi}\, y$ is x meager, that is $\overline{\chi}[y \dot{-} (\overline{\chi}\,\overline{\chi}\, y)] = x$. Let α be any atom open in the lattice $[0,x]$. Notice that x-openness for α is equivalent to $\alpha \in \mathcal{C}_x$ (lemma 2.1 applied to $[0,x]$). We claim $\alpha \nleq y \dot{-} (\overline{\chi}\,\overline{\chi}\, y)$.

Suppose, by way of contradiction, $\alpha \leq y \dot{-} \overline{\chi}\,\overline{\chi}\, y$. Then $\alpha \leq \sup_L \{\beta \mid \beta \in \mathcal{C}_y,\ \beta \nleq \overline{\chi}\,\overline{\chi}\, y\} \leq x$. Since α is x compact and all the β's involved are below x (each $\beta \in \mathcal{C}_y$ so $\beta \leq y \leq x$) we can find $\beta_1, \ldots, \beta_n \in \mathcal{C}_y$ $\beta_i \nleq \overline{\chi}\,\overline{\chi}\, y$ $(i = 1, \ldots, n)$ so that $\alpha \leq \beta_1 \vee \ldots \vee \beta_n$. But $\alpha \in \mathcal{C}_x$ forces $\alpha \in \mathfrak{M}_x$ and all the β_i's are below x. So for some i, $\alpha = \beta_i$. Thus $\alpha \nleq \overline{\chi}\,\overline{\chi}\, y$. But $\alpha \leq y$ (because we have assumed $\alpha \leq y \dot{-} (\overline{\chi}\,\overline{\chi}\, y)$) and so $\overline{\chi}\,\overline{\chi}\,\alpha \leq \overline{\chi}\,\overline{\chi}\, y$. But α is x-open so $\alpha = \overline{\chi}\,\overline{\chi}\,\alpha \leq \overline{\chi}\,\overline{\chi}\, y$, a contradiction.

Thus $\alpha \nleq y \dot{-} \overline{\chi}\,\overline{\chi}\, y$. So each x open atom α is not below $y \dot{-} \overline{\chi}\,\overline{\chi}\, y$. Hence $\mathcal{C}_x = \{\alpha \in \mathcal{C}_x \mid \alpha \nleq y \dot{-} \overline{\chi}\,\overline{\chi}\, y\}$. So taking sups gives $x = \overline{\chi}(y \dot{-} \overline{\chi}\,\overline{\chi}\, y)$. The other results now follow.

We will use 4.2 mostly in the following form.

Corollary 4.3. Suppose L is complete, atomic, DSB and satisfies the conditions: for each x in L, $\sup_L \mathcal{C}_x = x$ and each $\mathcal{C}_x$ atom is x-compact. Then L is fully DQD.

Proof. The conditions listed suffice to put each ${}_x$ inside ${}_1(x)$. We can then obtain the result.

Proposition 4.4. If L is finite, atomic and if for each x ${}_x \subseteq {}_x$ then L is DSB and fully DQD.

Adding the DSB hypothesis

Corollary 4.5. If L is finite, atomic and DSB then L is fully DQD.

Improving somewhat on the situation we have

Corollary 4.6. Suppose L is complete, neatly atomic, algebraic with enough x-primes for each x. Then L is fully DQD.

Proof. L is neatly atomic and DSB. But L is algebraic implying that each atom of L is compact. Let $x \in L$ and let $\alpha \leq x$ with $\alpha \in \Pi_x$. Now $\overline{\chi}\alpha = \sup_L\{\beta \in \Pi_x \mid \beta \neq \alpha\}$. If $\alpha \leq \overline{\chi}\alpha$ then the compactness of α gives $\alpha \leq \beta_1 \vee \cdots \vee \beta_n$ for certain $\beta_1, \ldots, \beta_n$ in Π_x, $\beta_i \neq \alpha$. But $\alpha \leq \beta_1 \vee \ldots \vee \beta_n$ and $\alpha \in \Pi_x$ forces $\alpha = \beta_i$ for some i. $\otimes$ Hence $\alpha \nleq \overline{\chi}\alpha$. Thus $\alpha \in \mathcal{C}_x$. So $\Pi_x \subseteq \mathcal{C}_x$. But [0,x] is DPC so proposition 2.3 says $\mathcal{C}_x \subseteq \Pi_x$. So for each $x \in L$ we have shown $\mathcal{C}_x = \Pi_x$. Hence $\sup_L \mathcal{C}_x = x$ (because with enough x primes $\sup_L \Pi_x = x$). Also $\mathcal{C}_x \subseteq \{w \mid w \text{ x-compact}\}$ (all atoms are compact anyway so those below x are x compact). We now have all we need to apply 4.3 and that gives us our claimed result.

Corollary 4.6 works nicely with our ongoing examples.

Application 4.7. (i) Suppose the meet semilattice M satisfies the ACC. Then L = sub(M) is fully DQD. (ii) Let S be a join semilattice so that $\Theta^*(S^*)$ is algebraic. Then $\Theta^*(S^*)$ is fully DQD.

Proof. (ii) immediately follows 4.6 and (i) is clear once we observe that the ACC on M gives for each x in L enough prime atoms in the lattice [0,x]. So 4.6 applies to the (already algebraic) lattice sub(M).

We can make a slightly better statement for $\Theta^*(S^*)$'s full DQD. Following Fajtlowicz and Schmidt [5] we say a poset P satisfies the weakened ACC if each updirected subset of P which is bounded above in P has a maximum element. Now if S is a join semilattice so that $\Theta^*(S^*)$ is algebraic then the ideals of S other than S, namely $S^*\backslash\{S\}$ satisfy the weakened ACC (because each atom of $\Theta^*(S^*)$ is compact). However it is not necessarily the case that if $S^*\backslash\{S\}$ satisfies the weakened ACC then $\Theta^*(S^*)$ is algebraic (example easy to construct, use any semilattice S whose S^* contains an infinite decreasing sequence whose limit is a proper ideal). So the following result is somewhat stronger than 4.7 (ii).

Proposition 4.8. Let S be a join semilattice whose ideals other than S satisfy the weakened ACC. Then $\Theta^*(S^*)$ is fully DQD.

Proof. Let $\mathcal{A} \in \Theta^*(S^*)$ and let $I \in \mathcal{A}$, $I \neq \emptyset$, $I \neq S$. Because of the weakened ACC in $S^*\backslash\{S\}$ we have: $\mathcal{C}_I$ is $\mathcal{A}$-open iff $I \notin \mathcal{A} \doteq \mathcal{C}_I$ iff I is completely meet irreducible in $\mathcal{A}$. So there are enough $\mathcal{A}$-open atoms, $\mathcal{A} = \sup_{\Theta^*(S^*)} \{\alpha \leq \mathcal{A} \mid \alpha \text{ is } \mathcal{A}\text{-open}\}$. Notice also that each $\mathcal{A}$ open atom $\mathcal{C}_I$ is $\mathcal{A}$-compact (since I is is completely meet irreducible in $\mathcal{A}$). Now we have the hypotheses of 4.3, giving our result.

For completeness we state:

Proposition 4.9. Suppose S is a join semilattice so that $S^*\setminus\{S\}$ satisfies the weakened ACC. Then $\Theta(S)$ is fully QD; in particular $\Theta(S)$ is QD, so each finite $\Theta(S)$ is QD.

5. Some upper semimodularity results

In Hall [8] it is proven that for each join semilattice S, $\Theta(S)$ is upper semimodular and in fact satisfies a condition somewhat stronger. A brief review of the literature though indicates some diversity in the definition of the upper semimodularity condition. Wanting to avoid a full discussion of this complicated area (as one can find in Croisot [19]) we only state some of the notions involved and the relations between them. Our objective will be to show that the properties Hall demonstrated for $\Theta(S)$ follow from certain properties of its dual atoms (enough primes, for instance).

We list and label some notions. Let L be any lattice x, y, a, b etc. all elements of L.

A if x, y both cover $x \wedge y$ then $x \vee y$ covers both x, y.

B if x covers $x \wedge y$ then $x \vee y$ covers x.

B is condition (3) of Dubreil-Jacotin, Lesieur and Croisot [20, page 87].

Now write aMb if $w = (w \vee a) \wedge b$ for all w in the interval

$[a \wedge b, b]$. If aMb we call (a,b) a <u>modular</u> pair. Now write aM^*b if $w = (w \wedge a) \vee b$ for all w in $[b, a \vee b]$. ((a,b) is then a dual modular pair.) We now list two more conditions

C aMb implies bMa

and

D aM^*b implies bM^*a.

Hall [8], adapting this condition from Dubreil-Jacotin, Lesieur and Croisot [20], calls a lattice upper semimodular if it satisfies condition A (this should NOT be confused with the notion of semimodularity given by the latter three authors). Clearly B implies A and also, as proven in [20, page 88, theorem 1], B implies the following property: if a, b ε L, a < b and if there is a maximal chain from a to b which is finite then each maximal chain from a to b is finite. Hall shows that $\Theta(S)$ satisfies B and hence A and this just mentioned condition.

Birkhoff [1] defines a lattice <u>of finite length</u> to be upper semimodular if it satisfies A. He shows that if a finite length lattice satisfies A then it satisfies C. However, he defines an arbitrary lattice to be upper semimodular if it satisfies C. Maeda & Maeda [10] call C M-symmetry and D then is called M^*-symmetry.

We make the following convention. With Hall, we will call any lattice L <u>upper semimodular</u> if it satisfies A. L will be called <u>M symmetric</u> if it satisfies C. We will soon show $C \Longrightarrow B \Longrightarrow A$ and we will show each $\Theta(S)$ to be M-symmetric. Hence we will establish B, giving Hall's result. Now B $\Rightarrow$ A is trivial. C $\Rightarrow$ B is next.

Proposition 5.1. Each M symmetric lattice L satisfies condition B.

Proof. Suppose x covers $x \wedge y$. We show $x \vee y$ covers y. Let z be chosen so that $y \leq z \leq x \vee y$ and suppose that $z \neq x \vee y$. We claim $y = z$. Now $x \vee z = x \vee y$. Also $x \wedge y \leq x \wedge z \leq x$ and x covers $x \wedge y$. So $x = x \wedge z$ or $x \wedge z = x \wedge y$. If $x \wedge z = x$ then $x \leq z$ and so $x \vee y \leq z$ giving $z = x \vee y$. ⊗ Hence $x \wedge z = x \wedge y$. So x covers $x \wedge z$ and from this it easily follows that zMx (i.e., for all $w \in [z \wedge x, x]$, $w = (w \vee z) \wedge x$ since such a w must be either x or $z \wedge x$). But the M-symmetry of L now yields xMz. Since $y \in [x \wedge z, z]$ and since xMz we must get $y = (y \vee x) \wedge z$. But $z \leq y \vee x$ then forces $y = z$. Hence $y \vee x$ covers y.

Now M^* is dual to M so D is dual to C. We will work in complete neatly atomic lattices and show, under certain conditions, that D holds. For any element x of such a lattice at(x) denotes the set of atoms below x. An element z of such a lattice is called a line if it is the join of two atoms. (Warnings: (i) this does not mean # at(z) = 2 and (ii) the geometric language is quite deceptive since we aim for conditions which are quite ageometric; contrast with the conditions Maeda and Maeda [10] use in their work.) We will say that a complete neatly atomic lattice L satisfies the atom-line condition (ALC) if for each x in L which is either an atom or a line we have

$$x \vee \alpha = x \vee \beta,\ \alpha \neq \beta \text{ implies } \alpha,\ \beta \leq x.$$

Lemma 5.2. Let L be complete, neatly atomic and satisfying the ALC. Then for any a, b $\in$ L: aM^*b iff $at(a \vee b) = at(a) \cup at(b)$.

Proof. Suppose aM^*b. Now suppose $\gamma \in at(a \vee b)$. Assume $\gamma \nleq b$. Because $b \vee \gamma \in [b, a \vee b]$ and because aM^*b we have $b \vee \gamma = [(b \vee \gamma) \wedge a] \vee b$. Hence $\gamma \leq [(b \vee \gamma) \wedge a] \vee b$. If either of $(b \vee \gamma) \wedge a$ or b is zero then it is easy to see that $\gamma \leq a$ or $\gamma \leq b$. So suppose both are nonzero. By the neatly atomic condition there are atoms $\alpha \leq (b \vee \gamma) \wedge a$ and $\beta \leq b$ so that $\underline{\gamma \leq \alpha \vee \beta}$. Certainly $\alpha \neq \beta$ and $\gamma \neq \beta$; if $\gamma = \alpha$ then $\gamma \leq a$ (which is what we want). So assume $\#\{\alpha,\beta,\gamma\} = 3$. Now $\alpha \leq b \vee \gamma$ (and $b \neq 0$ can be assumed) so there is an atom $\delta \leq b$ so that $\underline{\alpha \leq \delta \vee \gamma}$. Now $x = \delta \vee \beta$ is either a line or an atom and $x \vee \alpha = x \vee \gamma$ (with $\alpha \neq \gamma$). Since ALC holds we get $\alpha, \gamma \leq x$. Mainly $\gamma \leq x$ and $x \leq b$ so $\gamma \leq b$. $\otimes$ So we get either $\gamma \leq a$ or $\gamma \leq b$. Thus $at(a \vee b) \subseteq at(a) \cup at(b)$.

Now suppose $a, b \in L$ and $at(a \vee b) = at(a) \cup at(b)$. To show aM^*b we take $c \in [b, a \vee b]$ and so $c = (c \wedge a) \vee b$ which requires only $c \leq (c \wedge a) \vee b$. Since L is atomic, this requires $at(c) \subseteq at((c \wedge a) \vee b)$. Let $\gamma \in at(c)$. Then $\gamma \in at(a \vee b)$. If $\gamma \in at(b)$ then $\underline{\gamma \in at((c \wedge a) \vee b)}$. Otherwise $\gamma \in at(a)$ so $\gamma \in at(a \wedge c)$ hence $\underline{\gamma \in at((c \wedge a) \vee b)}$. Thus $at(c) \subseteq at((c \wedge a) \vee b)$.

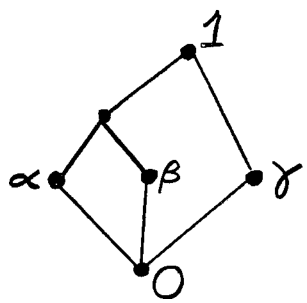
So in a complete neatly atomic ALC lattice it is easy to identify the dual modular pairs, they are the pairs of elements whose joins introduce no new atoms. It is easy to see

Theorem 5.3. Each complete neatly atomic lattice satisfying the ALC is M^* symmetric.

Also easy is the fact

Theorem 5.4. A complete neatly atomic lattice satisfying the ALC is modular iff it is distributive.

Note. Let L be complete and neatly atomic. We point out that $x \vee \alpha = x \vee \beta$, $\alpha \neq \beta$ $\alpha, \beta \leq x$ if assumed just for lines x is not sufficient to give the results of lemma 5.2 and theorem 5.3. Consider



This is complete, neatly atomic and for lines x satisfies the above condition. But $at(\alpha \vee \gamma) \neq at(\alpha) \cup at(\gamma)$. Also this lattice is NOT M^*-symmetric ($\alpha M^* \gamma$ but not $\gamma M^* \alpha$).

We now observe some conditions which make the ALC come true.

Proposition 5.4. Suppose L is a complete and neatly atomic lattice wherein the T_0 relation is asymmetric. Then L satisfies the ALC and so is M^*-symmetric.

Proof. Let $x \in L$, $x \neq 0$ and suppose $x \vee \alpha = x \vee \beta$ for atoms α, β $\alpha \neq \beta$. Then $\alpha \leq x \vee \beta$ so there is an atom $\chi \leq x$ so that $\alpha \leq \chi \vee \beta$. If $\alpha = \chi$ then $\alpha \leq x$ so both $\alpha, \beta \leq x$. So suppose $\alpha \neq \chi$. Then $\alpha T_0 \beta$. Also with $\beta \leq x \vee \alpha$ we can find an atom $\nu \leq x$ so that $\beta \leq \nu \vee \alpha$. If $\alpha = \nu$ then $\alpha \leq x$ and so $\alpha, \beta \leq x$. If $\beta = \nu$ then $\beta \leq x$ and so $\alpha, \beta \leq x$. So assume $\#\{\alpha,\beta,\nu\} = 3$. Hence $\beta T_0 \alpha$. But now we have violated the asymmetry of T_0.

Corollary 5.5. For each meet semilattice M the lattice sub(M) is M^*-symmetric and so satisfies the dual of condition B. (Implying among other things that sub(M) is lower semimodular.)

Note. Sub(M) is modular iff it is distributive iff M is a chain.

Proof of 5.5. We need only recall that in sub(M), for atoms $\{m\}$, $\{n\}$: $\{m\}T_0\{n\}$ implies that in M, $m < n$ so T_0 is asymmetric in sub(M).

Proposition 5.6. Suppose L is a complete, neatly atomic, DSB lattice. Then L satisfies the ALC and so is M^* symmetric. So each complete neatly atomic lattice with enough x-primes in each interval $[0,x]$ is M^*-symmetric.

Proof. Lemma 2.2 implies that the ALC will hold in DSB atomic lattices.

So for the record we state:

Corollary 5.7. For each join semilattice S, $\Theta^*(S^*)$ is M^* symmetric and so $\Theta(S)$ is M-symmetric. Hence $\Theta(S)$ satisfies condition B (Hall) and is upper semimodular and satisfies: if $x, y \in \Theta(S)$, $x < y$ and if there is a finite maximal chain from x to y in $\Theta(S)$ then each maximal chain of $\Theta(S)$ from x to y is finite.

6. Summary of this chapter

We have examined various types of atoms which may occur in complete atomic lattices and have seen that their presence in sufficient richness results in the lattice having certain properties. For our summary we emphasize the role of the prime atoms. We will say a complete atomic lattice has enough primes everywhere if each interval

[0,x] has enough prime atoms.

Let L be complete, neatly atomic with enough primes everywhere. Then (1) L is DSB with for each x the DPC operator in [0,x] being expressible in terms of x primes. Also (2) if L is algebraic then L is fully DQD with the natural formula

$$y = (\overline{x}\rceil\,\overline{x}\rceil\, y) \vee (y \dot{-} \overline{x}\rceil\,\overline{x}\rceil\, y)$$

with $y \leq x$ and $y \dot{-} \overline{x}\rceil\,\overline{x}\rceil\, y$ turning out to be meager in the lattice [0,x]. Finally (3) L is M^*-symmetric and so L is lower semimodular.

Our outstanding concrete examples of such a lattice were (i) sub(M) for a meet semilattice M with the ACC and (ii) $\Theta^*(S^*)$ for a join semilattice S.

CHAPTER III

CONGRUENCES OF SEMILATTICE TREES

In this chapter we study the special case of the distributive $\Theta(S)$. Here we work with meet rather than join semilattices. For such a semilattice M, $\Theta(M)$ is distributive iff each principal lower end $(m] = \{y \in M | y \leq m\}$ in M is a chain. If M satisfies this condition we call it a semilattice tree. Here T will always denote a semilattice tree. So we will examine $\Theta(T)$ paying special attention to its compact elements which we will examine both individually (how is a given compact congruence built?) and overall (what kind of lattice do they form?).

The chapter breaks up into two parts. In the first few sections we concretely examine congruences of T. We examine the role of the convex subsemilattices of T in determining the congruences of T. We examine to some extent a decomposition for a single compact congruence. We find that the compact congruences of T are complemented in $\Theta(T)$ and hence form a generalized Boolean lattice (Boolean ring) which is sublattice of $\Theta(T)$. If T has a zero we can even put T inside the Boolean ring $c(\Theta(T))$ of compact elements of $\Theta(T)$ and in fact T generates this ring. All these considerations result from straightforward tampering with congruences themselves.

But now we have a Boolean ring, and a special one at that, related to T. So we have another angle to approach things from. In the second part of the chapter we take up the general question of the Boolean ring

B[M] universal over a meet semilattice M. It is defined and characterized in section 5, further properties of it come up in section 6. In section 7 we return to our main preoccupation, congruences of semilattices. There we use the classical technique of extension and contraction of congruences (in exactly the same way as that technique is used by E. T. Schmidt [15] and Byrd, Mena and Troy [2] to compare $\Theta(M)$ with the ideal lattice $\mathcal{I}(B[M])$. We find the matchup best when M is a tree. Specifically if T is a tree, $\Theta(T) \cong \mathcal{I}(E_T)$ where E_T is a rather special ideal of B[T], the evenly generated ideal. Hence again the compact congruences of T form a Boolean ring, E_T. If T has a zero, $E_T = B[T]$ and so $\Theta(T) \cong \mathcal{I}(B[T])$ for any semilattice tree T with 0.

<u>Notation</u>. The symbol $\mathbb{Z}_2$ denotes the two element field.

1. Convex subsemilattices and congruences of trees

Let P be any poset $x, y \in P$. A <u>(lower) connecting bridge</u> of length n from x to y is a finite sequence $z_0, z_1, \ldots, z_n, z_1', \ldots, z_n'$ of elements of P so that $x = z_0$, $z_n = y$ and each $z_i' \leq z_{i-1}$ and $z_i' \leq z_i$. The elements z_j, z_i' might be called the <u>nodes</u> of the bridge. A subset D of P is said to be <u>(lower connected</u> if and only if for each pair x, y in D there is a connecting bridge from x to y (of some length $n \in N^+$) <u>all of whose nodes are in D</u>. Consider P as a topological space wherein a subset $U \subseteq P$ is open if and only if U is an upper end of P ($x \leq y$, $x \in U$ and $y \in P$ imply $y \in U$). Then it is easily seen that a subset D is (lower) connected if and only if D is topologically connected. Hence the usual topological theorems about connected sets apply to (lower) connected subsets of a poset.

Suppose now that the poset P is a tree (not necessarily a semilattice) and $D \subseteq P$. Suppose there is a bridge in D of length 2 from x to y; that is, there are elements z_0, z_1, z_2, z_1', z'_2 all in D with $x = z_0$, $z_2 = y$, $z_1' < z_0$, $z'_1 \leq z_1$, $z'_2 \leq z_1$ and $z'_2 \leq z_2$. Then z_1' and z_2' have a common upper bound and so are necessarily comparable. Let z' denote the smaller. Then we have $x \geq z'$, $y \geq z'$, $x, y, z' \varepsilon D$; namely a connecting bridge of length 1 in D from x to y. By induction we conclude: in a tree, a subset D is connected if and only if for each $x, y \varepsilon D$ there is a $z \varepsilon D$ so that $z \leq x$, $z \leq y$, i.e., D is downward directed.

Another notion which finds use here is that of convexity, with as usual a subset D being <u>convex</u> if the conditions $x \leq y \leq z$, $x, z \varepsilon D$ together imply $y \varepsilon D$. In a semilattice tree T the comments of the last paragraph allow a nice combination of our two notions: a subset D of T is convex and connected if and only if D is a convex subsemilattice. Such subsets of a semilattice tree are important because of the following result.

<u>Proposition 1.1</u>. Let S be a semilattice tree. For each equivalence relation E of S the following statements are equivalent:

(i) each equivalence class of E is convex and connected (a convex subsemilattice);

(ii) E is a meet congruence relation on S, $E \varepsilon \Theta(S)$.

In addition, if S is just a semilattice so that for each equivalence relation E on S the above statements (i) and (ii) are equivalent, then S is a semilattice tree.

Proof. (ii) $\Longrightarrow$ (i) is trivial. We show (i) $\Longrightarrow$ (ii). Take an equivalence E on a semilattice tree S for which the statement (i) holds. Assume xEy and show for arbitrary $a \in S$ that $(x \wedge a)E(y \wedge a)$. Consider first the special case where we assume x and y comparable, say $x \leq y$. Then x and $y \wedge a$ have y as a common upper bound and hence are comparable. If $y \wedge a \leq x$ then $y \wedge a \leq x \wedge a$ but $x \wedge a \leq y \wedge a$ anyway, so $x \wedge a = y \wedge a$ and so $(x \wedge a)E(y \wedge a)$. If, however, with the fact that the class $[x]_E = [y]_E$ is convex gives $(x \wedge a)E(y \wedge a)$. Thus xEy implies $(x \wedge a)E(y \wedge a)$ in the special case where x and y are comparable. Now consider the general case where x and y are unrelated. Now $x, y \in [x]_E$ and this class is connected, so there is a z in $[x]_E$ so that $z \leq x$ and $z \leq y$. Now zEx, $z \leq x$ and so by the first case, for any a, $(z \wedge a)E(x \wedge a)$. Similarly zEy, $z \leq y$ gives $(z \wedge a)E(y \wedge a)$. Transitivity of E finally gives: for any a, $(x \wedge a)E(y \wedge a)$. So E is a meet congruence.

We now prove the addition. So now assume S is just a meet semilattice so that, for each equivalence relation E on S, (i) $\Longleftrightarrow$ (ii) holds. Suppose a and b are elements of S with a common upper bound c; and suppose $b \nleq a$. We show $a < b$. The set $[b,c]$ is convex and connected (also it is a subsemilattice) as are each of the singletons $\{x\}$ (where $x \in S$, $x \notin [b,c]$). So if E is the equivalence on S corresponding to the decomposition $\{[b,c]\} \bigcup \{\{x\} | x \in S, x \notin [b,c]\}$, then E is a meet congruence relation on S. Since bEc and E is a meet congruence, $(a \wedge b)E(a \wedge c)$ and so $(a \wedge b)E\, a$. But $a \notin [b,c]$, so $[a]_E = \{a\}$ hence $a \wedge b = a$ so $a < b$. Thus S is a semilattice tree.

We then get the well known result:

Corollary 1.2. Suppose S is a chain, E an equivalence relation on S. Then the following statements are equivalent:

(i) each class of E is convex

(ii) E is a join congruence

(iii) E is a meet congruence

(iv) E is a lattice congruence of S.

It is well known (Papert [14], Varlet [17], etc.) that if S is an n element semilattice tree then $\Theta(S)$ is the Boolean algebra 2^{n-1}. Hence we get:

Corollary 1.3. If S is a semilattice tree with n elements, then there are exactly 2^{n-1} partitions of S into convex connected subsets (convex subsemilattices).

For all the following T is to be a semilattice tree and $\mathcal{C}$ denotes the collection of convex connected subsets of T. We conclude this section with an examination of the properties of $\mathcal{C}$.

Proposition 1.4. $\mathcal{C}$ is a closure family, $\emptyset \in \mathcal{C}$. If $\mathcal{D}$ is a family of elements of $\mathcal{C}$ one of which meets each of the others then $\bigcup \mathcal{D} \in \mathcal{C}$. Hence $\mathcal{C}$ is an algebraic closure family on T.

Proof. The first statement is clear, we show the second. Take $\mathcal{D}$ as described and let D_0 denote that element of $\mathcal{D}$ which meets each element of $\mathcal{D}$ and for each $D \in \mathcal{D}$ choose $z_D \in D \cap D_0$. Since our connectivity is topological it follows from the usual methods of topology that $\bigcup \mathcal{D}$ is connected. As to its convexity: suppose $x \in D_1$ and $y \in D_2$ and $x \leq a \leq y$. Now $y \wedge z_{D_2}$ and a have a common upper bound, namely y, and so a and $y \wedge z_{D_2}$ are comparable. If $y \wedge z_{D_2} \leq a$ then $y \wedge z_{D_2} \leq a \leq y$; and y, $y \wedge z_{D_2}$ are both in D_2 which is convex so we get $a \in D_2$. So suppose $a \leq y \wedge z_{D_2}$. Then a and $z_{D_1} \wedge z_{D_2}$ have a common upper bound (z_{D_2}) and so are comparable. If $z_{D_1} \wedge z_{D_2} \leq a$ then $z_{D_1} \wedge z_{D_2} \leq a \leq z_{D_2}$ and both $z_{D_1} \wedge z_{D_2}$ and z_{D_2} are in D_0 which is convex so $a \in D_0$. If however $a \leq Z_{D_1} \wedge Z_{D_2}$ then $x \leq a \leq Z_{D_1}$ and then with D_1 convex we get $a \in D_1$. At any rate we get $a \in \bigcup \mathcal{D}$. So $\bigcup \mathcal{D}$ is convex.

2. Formation of congruences generated by elements of $\mathcal{C}$

Let T be a semilattice tree. Let $D \in \mathcal{C}$. The decomposition $\{D\} \cup \{\{x\} | x \notin D\}$ is a decomposition of T into convex connected subsets and so yields a congruence $\Theta_D = \{(x,y) | x = y \text{ or } x, y \in D\}$. The latter is necessarily the least congruence identifying all elements of D (so our notation agrees with the standard notation used in these cases, see for instance Gratzer [7]). For example if $a < b$, $a, b \in T$, $\Theta_{[a,b]} = \Theta_{(a,b)}$ where the latter denotes the principal congruence generated by the pair (a,b) while the former is Θ_D for D being the interval $[a,b] \in \mathcal{C}$. In general if $a, b \in T$, possibly unrelated, $\Theta_{(a,b)} = \Theta_{[a \wedge b,a]} \vee \Theta_{[a \wedge b,b]}$.

The comments of section 1 have an impact on the formation of Θ_D for $D \in \mathcal{C}$. For $D_\lambda \in \mathcal{C}$ ($\lambda \in \Lambda$) we have

(2.1) $$\bigcap_{\lambda \in \Lambda} \Theta_{D_\lambda} = \Theta_{\bigcap_{\lambda \in \Lambda} D_\lambda} \text{ ; and}$$

if some D_{λ_0} meets each of the other D_λ then

$$(2.2) \qquad \bigvee_\lambda \Theta_{D_\lambda} = \Theta_{\bigcup_\lambda D_\lambda}.$$

Also if the D_λ's are mutually exclusive

$$(2.3) \qquad \bigvee_\lambda \Theta_{D_\lambda} = \bigcup_\lambda \Theta_{D_\lambda}.$$

We will say $D \in \mathcal{C}$ is nondegenerate if $\#D > 1$. If D_1 is nondegenerate then

$$(2.4) \qquad \Theta_{D_1} \subseteq \Theta_{D_2} \text{ iff } D_1 \subseteq D_2.$$

All the above can easily be established by the reader.

Consider the following application. For each $x \in T$ let $(x] = \{y \in T | y \leq x\}$. Then $(x] \in \mathcal{C}$. The formula (2.4) yields:

$$\Theta_{(x]} \subseteq \Theta_{(y]} \text{ iff } x \leq y \quad (\text{all } x, y \in T).$$

Hence the map $\phi: T \rightarrow \Theta(T)$ defined by $\phi(t) = \Theta_{(t]}$ is an order embedding. Better still (2.1) gives:

$$\Theta_{(x]} \cap \Theta_{(y]} = \Theta_{(x \wedge y]} \quad (\text{all } x, y \in T).$$

Hence ϕ is a meet homomorphism. Consider for each $x \in T$, $\sigma_x = \{(a,b) | a \wedge x = b \wedge x\}$. For each x, σ_x is a congruence. (Papert

[14] used this congruence in the join semilattice case to embed a join semilattice into its structure lattice.) The tree hypothesis is not even required to show

$$\sigma_x \vee \Theta_{(x]} = T \times T \quad \text{(the 1 of } \Theta(T)).$$

[Proof: For if $a \in T$ then $(a \wedge x, x) \in \theta_{(x]}$ and $(a, a \wedge x) \in \sigma_x$ so that $(a,x) \in \sigma_x \vee \theta_{(x]}$. Hence for any $a, b \in T$, $(a,x) \in \sigma_x \vee \theta_{(x]}$ and $(b,x) \in \sigma_x \vee \theta_{(x]}$ and so $(a,b) \in \sigma_x \vee \theta_{(x]}$.] Now using the tree hypothesis it is trivial that

$$\sigma_x \cap \Theta_{(x]} = \Delta_T \text{ (the zero of } \Theta(T)).$$

Hence σ_x is the complement of $\Theta_{(x]}$ in the lattice $\Theta(T)$. Let $cL(\Theta(T))$ denote the collection of complemented elements of $\Theta(T)$; $cL(\Theta(T))$ is a sublattice of $\Theta(T)$ (because the latter is distributive). Now $\text{im}\phi$ is actually in $cL(\Theta(T))$. Since the ordering and meet operation in $cL(\Theta(T))$ are the same as in $\Theta(T)$ we can conclude:

Proposition 2.1. If T is a semilattice tree the mapping $\phi: T \to cL(\Theta(T))$ given by $\phi(x) = \Theta_{(x]}$ is an order embedding meet homomorphism into the Boolean sublattice of complemented elements of $\Theta(T)$. The complement of $\phi(x)$ is $\sigma_x = \{(a,b) \mid a \wedge x = b \wedge x\}$.

We can actually say more. Let $a, b \in T$, $a < b$. From (2.1) and (2.2) we obtain:

$$\Theta_{(a]} \cap \Theta_{[a,b]} = \Delta_T \tag{2.5}$$

and

$$(2.6) \qquad \Theta_{(a]} \vee \Theta_{[a,b]} = \Theta_{(b]}$$

(for in a tree $(a] \cup [a,b] = (b]$, because $(b]$ is a chain). Hence

$$(\sigma_b \vee \Theta_{(a]}) \vee \Theta_{[a,b]} = T \times T$$

and

$$(\sigma_b \vee \Theta_{(a]}) \cap \Theta_{[a,b]} = \Delta_T$$

(use distributivity and $\sigma_b \cap \Theta_{[a,b]} = \Delta_T$ to get the latter). Thus $\Theta_{[a,b]} = \Theta_{(a,b)}$ is complemented in $\Theta(T)$ and its complement is:

$$\overline{\Theta}_{[a,b]} = \overline{\Theta}_{(b]} \vee \Theta_{(a]} = \sigma_b \vee \Theta_{(a]}.$$

We might use the Boolean ring notation in $cL(\Theta(T))$ and write

$$\Theta_{[a,b]} = \Theta_{(a]} + \Theta_{(b]}$$

which is given by (2.5) and (2.6) now that we know $\Theta_{[a,b]} \in cL(\Theta(T))$.

So each principal congruence of the form $\Theta_{(a,b)}$, with $a < b$, is complemented. Each compact congruence is the join in $\Theta(T)$ of finitely many congruences of this form. Since $cL(\Theta(T))$ is a sublattice of $\Theta(T)$ we can conclude:

Proposition 2.2. If T is a semilattice tree then each compact congruence of T is complemented,

$$c(\Theta(T)) \subseteq cL(\Theta(T)).$$

3. Compact and complemented elements in bounded distributive lattices

In the last section we left off our study of $\Theta(T)$ having demonstrated that its compact elements are complemented. In this section we aim to see the effect, in arbitrary bounded distributive lattices, of this condition (compact elements being complemented). So temporarily we leave the friendly confines of $\Theta(T)$ and work in the more general setting of bounded distributive lattices, returning at the end to applications in $\Theta(T)$.

Proposition 3.1. Let L be a distributive lattice, $a, b \in L$. If each of $a \wedge b$ and $a \vee b$ is compact then each of a and b is compact.

Proof. We show a compact (proof for b is similar). Suppose $a \leq \bigvee_t z_t$. Then $a \wedge b \leq \bigvee_t z_t$ and since $a \wedge b$ is compact there are finitely many t, say $t_1, \ldots, t_n$, so that $a \wedge b \leq z_{t_1} \vee z_{t_2} \vee \cdots \vee z_{t_n}$. Now $a \vee b \leq \bigvee_t (z_t \vee b)$ and so with $a \vee b$ compact we can find $t_1', \ldots, t_k'$ so that $a \vee b \leq \bigvee_{i=1}^{k} (z_{t_i'} \vee b)$. Hence $a \vee b \leq \left(\bigvee_{i=1}^{k} z_{t_i'} \right) \vee b$. Then

$$a = a \wedge (a \vee b) \leq a \wedge \left[\left(\bigvee_{i=1}^{k} z_{t_i'}\right) \vee b\right]$$

$$= \left[a \wedge \left(\bigvee_{i=1}^{k} z_{t_i'}\right)\right] \vee \left[a \wedge b\right]$$

$$\leq \left(\bigvee_{i=1}^{k} z_{t_i'}\right) \vee \left(\bigvee_{j=1}^{n} z_{t_j}\right) .$$

But now we see that a is covered by finitely many of the z_t's. So a is compact.

Our chief application of 3.1 will be, with L distributive, L having zero:

if $a \wedge b = 0$ and $a \vee b$ is compact then each of a and b is compact. It might be noted that the result 3.1 is something of a generalization of the standard result: if I, J are ideals of a distributive lattice (or just join semilattice) and $I \vee J$ and $I \cap J$ are principal then I and J are principal (see for instance Grätzer [7], page 71, lemma 5). We now prove a technical lemma of some use later.

Lemma 3.2. Let L be a distributive lattice with 0 and 1. Suppose each compact element of L is complemented in L, $c(L) \subseteq cL(L)$. Suppose $a \in c(L)$ and $b \in cL(L)$. Then $a \wedge b \in c(L)$.

Proof. Let $\bar{b}$ denote the complement of b in L. Then $(a \wedge b) \vee (a \wedge \bar{b}) = a$ and $(a \wedge b) \wedge (a \wedge \bar{b}) = 0$. So $a \wedge b$ and $a \wedge \bar{b}$ are elements with meet 0 and with compact join. So each of $a \wedge b$ and $a \wedge \bar{b}$ is compact.

Corollary 3.3. Let L be a bounded distributive lattice so that $c(L) \subseteq cL(L)$. Then c(L) is an ideal of the Boolean ring cL(L). Hence

$c(L)$ is a sublattice of L which is itself a generalized Boolean lattice. Finally $c(L) = cL(L)$ iff 1 is compact.

Proof. We need only show the first statement since the others follow from it (since each ideal of a Boolean ring is itself a Boolean ring, a subring of the original). That $c(L)$ is a lower end in $cL(L)$ is clear from lemma 3.2. But $c(L)$ is certainly join closed in $cl(L)$ and so it is a lattice ideal of $cL(L)$ $(0 \in c(L))$ and so a ring ideal.

Our main application is to $\Theta(T)$.

Corollary 3.4. If T is a semilattice tree then $c(\Theta(T))$ is an ideal of the lattice $cL(\Theta(T))$. So $c(\Theta(T))$ is a sublattice of $\Theta(T)$ which is a generalized Boolean lattice. So for any semilattice tree T, $\Theta(T)$ is an ideal lattice of a Boolean ring.

But before leaving the generalities of this section, we have another application.

Theorem 3.5. Let L be a distributive algebraic lattice. The following statements are equivalent:

(i) L is the ideal lattice of some Boolean ring B,

(ii) each compact element of L is complemented,

(iii) if $x \in c(L)$ then $(x \to 0) \vee x = 1$,

(iv) L is dually atomic and if $x \in c(L)$ and d is a dual atom $d \geq x$ then $d \ngeq x \to 0$.

If any of these holds then B is a Boolean lattice if and only if the 1 of L is compact.

Proof. (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) are easy and well known. We prove (iii) $\Rightarrow$ (i). If (iii) holds then because L is Brouwerian,

$c(L) \subseteq cL(L)$. So by our previous results $c(L)$ is a generalized Boolean sublattice of L and so $\mathcal{J}(c(L)) \cong L$. Thus L is the ideal lattice of a Boolean ring.

Now it is well known that (i) $\Rightarrow$ (iv) (there can be no maximal proper ideal d above both a principal ideal x and its complement $x \to 0$). So at last we show (iv) $\Rightarrow$ (ii). Let $x \in c(L)$. Since L is dually atomic and no dual atom can be above both x and $x \to 0$ we have $x \vee (x \to 0) = 1$. But $x \wedge (x \to 0) = 0$ anyway and so x is complemented. Thus $x \in cL(L)$ and we have (ii), namely $c(L) \subseteq cL(L)$.

One might compare 3.5 and the next result to the work of Nachbin [13] and Monteiro [11] who first characterized the ideal lattice of a Boolean ring. We conclude with a restatement of 3.5.

<u>Corollary 3.6</u>. Let S be a distributive join semilattice with zero. Then S is a generalized Boolean lattice if and only if in its ideal lattice $\mathcal{J}(S)$ each principal ideal is complemented.

(Here distributivity means: if $x \leq a \vee b$ then there exist elements $a' \leq a$ and $b' \leq b$ so that $x = a' \vee b'$.)

4. Decomposition of a compact congruence

Let T be a semilattice tree. For $\Theta \in \Theta(T)$ write $(T/_\Theta)^*$ to denote the collection of nondegenerate congruence classes of Θ. Then trivially

$$(4.1) \qquad \Theta = \bigvee_{D \in (T/_\Theta)^*} \Theta_D = \bigcup_{D \in (T/_\Theta)^*} \Theta_D .$$

So Θ is the join of a lattice independent (in the lattice $\Theta(T)$) set of congruences generated by nondegenerate elements of $\mathcal{C}$. Because of the importance of the compact congruences we now interpret equation (4.1) in the case that Θ is compact.

Proposition 4.1. Let $\Theta \in \Theta(T)$ be expressed as in (4.1). Then Θ is compact in $\Theta(T)$ iff $(T/_{\Theta})^*$ is finite and each Θ_D is compact.

Proof. The "if" part is trivial. Then "only if" part uses proposition 3.1 and induction.

Realizing that $c(\Theta(T))$ is a generalized Boolean sublattice of $cL(\Theta(T))$ we can use its ring sum and write, for σ compact

$$\sigma = \sum_{D \in (T/_{\sigma})^*} \Theta_D, \tag{4.2}$$

because in any Boolean ring, ring sum and lattice join of an independent set of elements coincide. Next we have:

Lemma 4.2. Suppose $D \in \mathcal{C}$, D nondegenerate. Suppose $\Theta_D = \bigvee_{i=1}^{k} \Theta_{(c_i,d_i)}$ where each $c_i < d_i$ and the $d_1, \ldots, d_k$ are pairwise incomparable. Then $D = \bigcup_{i=1}^{k} [c,d_i]$ where $c = \bigwedge_{i=1}^{k} c_i$. Hence D has a minimum element, finitely many maximal elements, and each element of D dominated by some maximal element.

Proof. Assume the given hypothesis. Necessarily each of $c_1, \ldots, c_k$ and $d_1, \ldots, d_k$ is an element of D and so, letting $c = \bigwedge_{i=1}^{k} c_i$, we get $\bigvee_{i=1}^{k} \Theta_{(c,d_i)} \subseteq \Theta_D$. On the other hand, for each i, $c \leq c_i < d_i$ and this gives $\Theta_{(c_i,d_i)} \subseteq \Theta_{(c,d_i)}$ so that

$\Theta_D = \bigvee_{i=1}^{k} \Theta(c_i,d_i) \subseteq \bigvee_{i=1}^{k} \Theta(c,d_i)$. So finally $\Theta_D = \bigvee_{i=1}^{k} \Theta(c,d_i)$.

But formula 2.2 gives $\bigvee_{i=1}^{k} \Theta(c,d_i) = \bigvee_{i=1}^{k} \Theta[c,d_i] = \Theta_{\bigcup_{i=1}^{k}[c,d_i]}$. So we get $\Theta_D = \Theta_{\bigcup_{i=1}^{k}[c,d_i]}$. But D is nondegenerate so (2.4) allows us to conclude $D = \bigcup_{i=1}^{k} [c,d_i]$. The other comments follow easily.

We are now in a position to prove:

<u>Proposition 4.3</u>. Let $D \in \mathcal{C}$, $D \neq \emptyset$. The following statements are equivalent:

(i) Θ_D is compact

(ii) D has a minimum element, finitely many maximal elements and each element of D is dominated by some maximal element (i.e., D is the union of intervals with a common lower endpoint)

If either of these is true then

$$\Theta_D = \bigvee_{\substack{b \text{ MAXL} \\ \text{in } D}} \theta(a,b) \tag{4.3}$$

where a is the least element of D. Furthermore if D is nondegenerate and $\Theta_D = \bigvee_{i=1}^{k} \Theta(c_i,d_i)$ where each $c_i < d_i$ and the d_i's are pairwise incomparable then $\{d_1, \ldots, d_k\} = \{b \mid b \text{ maximal in } D\}$ and $\bigwedge_{i=1}^{k} c_i = a$, the least element of D.

<u>Proof</u>. For any nonempty $D \in \mathcal{C}$, (ii) $\Rightarrow$ (i) is trivial in view

of (2.2). Also (i) $\Rightarrow$ (ii) is clear if D is degenerate. Note that if (ii) holds true then formula (2.2) will yield the expression (4.3) above. So to complete the proof of the proposition we must show (i) $\Rightarrow$ (ii) for nondegenerate D ε $\mathcal{C}$. So suppose D is nondegenerated (in $\mathcal{C}$) and Θ_D is compact. Now $\Theta_D = \sup_{\Theta(T)}\{\Theta_{(x,y)} | x < y,\ x, y \varepsilon D\}$ and this (being a cover of the compact Θ_D) can be reduced to a finite subcover. Let n be the least positive integer s so that Θ_D is the join of S principal congruences of the form $\Theta_{(x,y)}$ where $x < y$ and x, y ε D. Then

$$(4.4) \qquad \Theta_D = \Theta_{(a_1,b_1)} \vee \Theta_{(a_2,b_2)} \vee \cdots \vee \Theta_{(a_n,b_n)}$$

for certain a_i, b_i in D with each $a_i < b_i$. The minimality of n forces the $b_1, \ldots, b_n$ to be pairwise incomparable. (If for $i \neq j$, $b_i \leq b_j$ we could replace $\Theta_{(a_i,b_i)} \vee \Theta_{(a_j,b_j)}$ in 4.4 with $\Theta_{(a_i \wedge a_j, b_j)}$.) So then lemma 4.2 applied to Θ_D gives $D = \bigcup_{i=1}^{n} [a,b_i]$ where $a = a_1 \wedge \cdots \wedge a_n$. So we get (ii). The "furthermore ..." part of the proposition is a consequence of lemma 4.2.

Recall from section 2 the map $\phi : T \longrightarrow cL(\Theta(T))$, with $\phi(x) = \Theta_{(x]}$. This map was an order embedding meet homomorphism. We have seen that if $x < y$ then

$$\Theta_{(x,y)} = \Theta_{(y]} + \Theta_{(x]} = \phi(x) + \phi(y) ,$$

the sum taken in the Boolean ring $cL(\Theta(T))$. (Actually one can show for any x, y in T, $\Theta_{(x,y)} = \phi x + \phi y$.)

Now if Θ_D is compact then working in $cL(\Theta(T))$ we can write

$$\Theta_D = \bigvee_{\substack{b \text{ MAXL} \\ \text{in } D}} (\Theta_{(b]} + \Theta_{(a]})$$

where a is the least element of D or

$$\Theta_D = \left(\bigvee_{\substack{b \text{ maxl} \\ \text{in } D}} \Theta_{(b]} \right) + \Theta_{(\text{MIN}(D)]}$$

which in the ϕ notation is

$$\Theta_D = \bigvee_{\substack{b \text{ maxl} \\ \text{in } D}} \phi(b) + \phi(\min D).$$

So apparently Θ_D is generated in the ring $cL(\Theta(T))$ by elements of $\text{im}\phi = \phi[T]$, our copy of T inside $cL(\Theta(T))$.

Putting things together we get a full decomposition of any compact congruence σ. For $\sigma \in c(\Theta(T))$ we have

$$\sigma = \bigvee_{D \in (T/_\sigma)^*} \left(\bigvee_{\substack{b \text{ maxl} \\ \text{in } D}} \Theta_{(b]} + \Theta_{(\min D]} \right).$$

Using properties holding in any Boolean ring we can rewrite this as

$$(4.4) \qquad \sigma = \sum_{D \in (T/_\sigma)^*} \left(\bigvee_{\substack{b \text{ maxl} \\ \text{in } D}} \Theta_{(b]} \right) + \sum_{D \in (T/_\sigma)^*} \Theta_{(\min D]}.$$

(Note that the sums and joins involved are all finite.)

Let $\langle T \rangle$ denote the Boolean subring of $cL(\Theta(T))$ generated by $\mathrm{im}\phi$. Then apparently

$$c(\Theta(T)) \subseteq \langle T \rangle \subseteq cL(\Theta(T)).$$

If the tree has a zero the situation improves somewhat. Then for each x, $\phi(x) = \Theta_{(0,x)} \in c(\Theta(T))$ and $\phi(0) = \Delta_T$, the zero of $\Theta(T)$. Hence because $c(\Theta(T))$ is an ideal in $cL(\Theta(T))$ we get $\langle T \rangle = c(\Theta(T))$. Summing up the situation we have:

<u>Proposition 4.4</u>. Suppose T is a semilattice tree with zero. The map $\phi: T \to c(\Theta(T))$, $\phi(t) = \Theta_{(t]}$ is an order embedding meet homomorphism (preserving zero) whose image generates the ring $c(\Theta(T))$. For any $\sigma \in c(\Theta(T))$ we have

$$\sigma = \sum_{D \in (T/_\sigma)^*} \ \bigvee_{\substack{b \ \max\ell \\ \text{in } D}} \Theta_{(0,b)} + \sum_{D \in (T/_\sigma)^*} \Theta_{(0,\min D)}.$$

We say more in later sections about how T relates to $c(\Theta(T))$.

5. The Boolean ring universal over a meet semilattice

We have seen that if T is a semilattice tree then $c(\Theta(T))$ is a Boolean ring and furthermore if T has a zero T can be thought of as a meet subsemilattice of $c(\Theta(T))$ (whose zero coincides with that of $c(\Theta(T))$) which ring generates $c(\Theta(T))$. In an attempt to gain insight into how T and $c(\Theta(T))$ relate we will, in the next few sections, abstract the situation and look at a special type of relation which might hold between a semilattice and a Boolean ring.

So for the next few sections we set aside the tree assumption; but in the end (section 7) we will return to it to see how nicely semilattice trees fit into the scheme of things. Our general context here will be a meet subsemilattice of a Boolean ring. If the meet subsemilattice has a least element, a zero, we will want it to coincide with that of the Boolean ring. Let $M \subseteq B$, B a Boolean ring. We call M an <u>admissible subsemilattice</u> of B iff $M \cdot M \subseteq M$ (subsemilattice) and M has a least element iff $0_B \in M$.

We now discuss how an admissible subsemilattice might generate as freely as possible a Boolean ring. Our motivation for the first few theorems is mainly the work of Mostowski and Tarski [12] on Boolean rings generated by chains and the more recent discussion in Grätzer [7, section 10]. We begin with a basic construction.

<u>Proposition 5.1</u>. Suppose M is a meet semilattice. Then there is a Boolean ring B so that (i) M is an admissible subsemilattice of B and (ii) $M \setminus \{0_B\}$ is a $\mathbb{Z}_2$ vector space basis of B.

<u>Proof</u>. This is almost the same as constructing the semigroup algebra $\mathbb{Z}_2[M]$ of the semigroup M over the field $\mathbb{Z}_2$ (see Clifford and Preston [3, page 159]). We let B be the $\mathbb{Z}_2$ vector space whose base is the set of nonzero elements of M (i.e., if M has a zero, throw it out, otherwise, leave M alone). If M has a zero identity it with the zero of B. Now we can view $M \subseteq B$. The meet operation on M extends to an associative, bilinear multiplication on B under which B becomes a Boolean ring. The described properties of B then follow easily. Note that if M has no zero then B is actually $\mathbb{Z}_2[M]$.

For any Boolean ring B and any meet homomorphism $\phi: M \to B$ we call ϕ admissible if whenever M has a least element 0_M then $\phi(0_M)$ is the zero of B. The ring constructed in 5.1 is significant because of the next result.

Proposition 5.2. Let M be a meet semilattice, B a Boolean ring, $\phi: M \to B$ admissible. The following statements are equivalent:

(i) ϕ is the universal admissible map into a Boolean ring (i.e., if R is a Boolean ring and $\psi: M \to R$ is admissible then there is exactly one ring homomorphism $\sigma: B \to R$ so that $\sigma \circ \phi = \psi$).

(ii) $(\phi(m) | m \in M, m \neq 0)$ is a basis for B as a $\mathbb{Z}_2$ vector space.

(iii) $(\phi(m) | m \in M, m \neq 0)$ is $\mathbb{Z}_2$ linearly independent in B and ring generates B.

Proof. Since $\phi[M]$ is a subsemilattice of B and B is a Boolean ring (ii) and (iii) are clearly equivalent. We show (ii)$\Rightarrow$(i). Assume (ii) holds. Note that because of (ii) the map ϕ is necessarily one-one so that B might be viewed as a vector space with basis $M \setminus \{0_B\}$. Actually ϕ is an order embedding. Let R be any Boolean ring, suppose $\psi: M \to R$ is admissible. Now there is exactly one $\mathbb{Z}_2$ linear map $\sigma: B \to R$ so that $\sigma \circ \phi = \psi$. But σ actually preserves the multiplication in B of elements of $\phi[M]$; that is if $m_1, m_2 \in M$, $\sigma(\phi m_1 \cdot \phi m_2) = \sigma(\phi(m_1 m_2))$ $= \psi(m_1 m_2) = \psi m_1 \psi m_2 = \sigma(\phi m_1)\sigma(\phi m_2)$. So σ preserves the multiplication on a generating subset of B. Hence σ preserves multiplication on a generating subset of B. Hence σ preserves multiplication and so is a ring homomorphism so that $\sigma \circ \phi = \psi$. Its uniqueness is clear. Finally by 5.1 and the usual techniques we get (i)$\Rightarrow$(ii).

Note. We will call the Boolean ring described in 5.2 the Boolean ring universal over M and write B[M] to denote it. The conditions of 5.2 imply that the map $\phi: M \to B[M]$ is an admissible order embedding. The universal Boolean ring over M can then be characterized as a Boolean ring B wherein M is an admissible subsemilattice whose nonzero elements form a $\mathbb{Z}_2$ vector space basis for B. Now every Boolean ring is a $\mathbb{Z}_2$ vector space and so has a $\mathbb{Z}_2$ vector space basis; our interest is in those with a multiplicatively closed basis. Our goal now is to translate $\mathbb{Z}_2$ linear independence of $M\setminus\{0\}$ into some order theoretic statement.

Proposition 5.3. Let M be an admissible subsemilattice of the Boolean ring B. Then $M\setminus\{0_B\}$ is $\mathbb{Z}_2$ linearly independent in B iff M satisfies the following condition in B:

(*) if $m_1, \ldots, m_k, m \in M$ and if each $m_i < m$ then the join in B, $\bigvee_{i=1}^{k} m_i < m$.

Actually the "if" part does not require the admissibility of M in B.

Proof. For the "if" part suppose M is just a meet subsemilattice of B satisfying (*). Assume by way of contradiction, that $M\setminus\{0_B\}$ is not $\mathbb{Z}_2$ linearly independent. Then there is a $\mathbb{Z}_2$ linear combination:

$$\sum_{\substack{m \in M \\ m \neq 0_B}} \lambda_m m = 0_B \qquad \text{where } \lambda_m \in \mathbb{Z}_2, \quad \text{a.a. } \lambda_m = 0$$

but not all λ_m'a are zero. Note that there are no repetitions of elements of M in this sum. Let $m_1, \ldots, m_k$ be the nonzero elements of M (i.e., $\neq 0_B$) whose coefficients in the above sum are not zero. Since our field of scalars is $\mathbb{Z}_2$ the equation now reads:

$$m_1 + \dots + m_k = 0_B.$$

Notice $m_i \neq m_j$ if $i \neq j$. Without loss of generality assume $k \geq 1$. Relabel these elements, if necessary, so that m_1 is maximal in the list. Multiplying the last equation through by m_1 and solving for m_1 we get

$$m_1 = m_1m_2 + m_1m_3 + \dots + m_1m_k.$$

Each of $m_1m_2, \dots, m_1m_k$ is in M. So if each of m_1m_j ($j = 2, \dots, k$) were less than m_1 then condition (*) would yield:

$$m_1 = m_1m_2 + \dots + m_1m_k \leq \bigvee_{j=2}^{k} m_1m_j < m_1,$$

a contradiction. So one of the m_jm_1 (for $j > 1$) is m_1 and hence $m_1 \leq m_j$ (for some $j > 1$). But m_1 was maximal so $m_1 = m_j$ (for some $j > 1$) which is a contradiction. So $M \setminus \{0_B\}$ is $\mathbb{Z}_2$ linearly independent.

Assume now M is an admissible subsemilattice of B. We show the "only if" part in the case M has a least element. The reader should make the appropriate adjustments in the other case. In $P(M) = \{D \mid D \subseteq M\}$ consider the interval $R = [\{0_M\}, M] = \{D \subseteq M \mid 0_M \in D\}$. R is a Boolean lattice whose join and meet are set theoretic union and intersection respectively. Consider the map $\rho : M \to R$ given by $\rho(m) = (m] = \{y \in M \mid y \leq m\}$. This ρ is an admissible map into a

Boolean ring. Assume now that $M\setminus\{0\}$ is $\mathbb{Z}_2$ linearly independent in B. The subring of B generated by M, $\langle M\rangle_B$, is by 1.2 isomorphic to B[M]. Hence with $\rho: M \to R$ admissible, there is exactly one ring homomorphism $\psi: \langle M\rangle_B \to R$ extending ρ. Notice that if $x, y \in \langle M\rangle_B$ and $x \vee y$ denotes their join in B we have: $\psi(x \vee y) = \psi(x + y + xy) = \psi x + \psi y + \psi x \psi y = \psi x \vee \psi y$ (join in R) $= \psi x \cup \psi y$. So for $x_1, \ldots, x_n$ in $\langle M\rangle_B$, if $\bigvee_{i=1}^{n} x_i$ denotes their join in B we get $\psi \bigvee_{i=1}^{n} x_i = \bigcup_{i=1}^{n} \psi(x_i)$. Now to show (*) holds for M in B, suppose (*) fails. Then we would have elements $m_1, \ldots, m_k$, m in M so that each $m_i < m$ but $\bigvee_{i=1}^{k} m_i = m$ (this sup taken in B). But then we could apply ψ to this and get $\bigcup_{i=1}^{k} \psi(m_i) = \psi(m)$. But ψ extends ρ so we would get $\bigcup_{i=1}^{k} (m_i] = (m]$, forcing $m \leq m_i$ for some i, a contradiction.

Comment. At this point we mention a simple consequence of independence which will be used often. Suppose $M \subseteq B$, B a Boolean ring. Suppose $M\setminus\{0_B\}$ is $\mathbb{Z}_2$ independent. Suppose $m, m_1, \ldots, m_k \in M$ and $m = m_1 + \ldots + m_k$ with $m \neq 0_B$. Then $m = m_i$ for some $i \in \{1,\ldots,k\}$. We leave the proof to the reader (try induction on k).

Among the consequences of 5.3 is the fact that any proper joins that already exist in M are lost in the transition to B[M]. To clarify: suppose m_1, m_2 are incomparable elements of M which have a join in M, say $m = \sup_M\{m_1,m_2\}$. Then because of condition (*) it is clear that $m \neq \sup_{B[M]}\{m_1,m_2\}$. Thus B[M] is a purely (meet) semilattice theoretic affair.

As another application of 5.3 notice that if T is any semilattice tree then T satisfies condition (*) in any Boolean ring where it is a meet subsemilattice. So

Corollary 5.4. The nonzero elements of a semilattice tree T are $\mathbb{Z}_2$ linearly independent in any Boolean ring wherein T is a meet subsemilattice (nonzero means not equal to the zero of the ring).

Thus B[T] is characterized by: it is a Boolean ring in which T is an admissible subsemilattice and which is ring generated by T.

We have seen that if T is a semilattice tree with 0 then the map $x \longmapsto \Theta_{(x]}$ is an admissible order embedding of T into $c(\Theta(T))$, the latter a Boolean ring. So the image of T under this map is an admissible subsemilattice tree in $c(\Theta(T))$ and this image ring-generates $c(\Theta(T))$. So we conclude: if T is a semilattice tree with zero then $c(\Theta(T)) \cong B[T]$ and so $\Theta(T) = \mathcal{L}(B[T])$ the lattice of ideals of B[T]. So in the case of a tree with zero we get a perfect matchup between congruences of the semilattice and congruences of its universal Boolean ring. But we will see later that these results come from the natural processes of extension and contraction of congruences and their proof depends only on the abstract properties of B[M]. This is the subject of section 7.

As a final application it can be shown (this writer proves it elsewhere) that B[M] will have a 1 (i.e., be a Boolean lattice) iff M has finitely many maximal elements with each element of M dominated by at least one of them.

Note. Below are listed some basic properties of Boolean rings used frequently here. Their proofs are easy and are left to the reader. Let B be any Boolean ring, x, y, $x_1, \ldots, x_n$ are all elements of B:

(1) If $x \cdot y = 0$ then $x + y = x \vee y$

(2) $x_1 + \ldots + x_n \leq x_1 \vee \ldots \vee x_n$

(3) If P is a prime ideal of B and if $x, y \in B$ then one of x or y or $x + y$ is in P.

6. Filters of M, primes of B[M]

Write B for B[M] when the meaning is clear. For any $x \in B$, x has a unique expression as a $\mathbb{Z}_2$ linear combination of nonzero elements of M,

$$x = \sum_{\substack{m \in M \\ m \neq 0}} \lambda_m(x) \cdot m$$

where $\lambda_m(x) \in \mathbb{Z}_2$ and almost all $\lambda_m(x) = 0$. Define $n(x) = \#\{m \mid \lambda_m(x) \neq 0\}$. Notice that $n(0) = 0$ and $n(x) = 1$ iff $x \in M \setminus \{0\}$.

Of some interest later will be the set $P_0 = \{x \in B \mid n(x) \text{ is even}\}$. We summarize for future use some properties of this set: (i) $0 \in P_0$ and if $m \in M$ and $m \neq 0$ then $m \notin P_0$. (ii) $P_0 + P_0 \subseteq P_0$. Also (iii) P_0 is an ideal of B iff the nonzero elements of M form a filter in M. So if M has no zero, P_0 is an ideal. (iv) If P_0 is an ideal, it is a prime ideal. Finally (v) if P_0 is an ideal then: $M \setminus P_0 = M$ if M has no zero, while $M \setminus P_0 = M \setminus \{0\}$ if M has a least element.

As we know by now, for any set X, $P(X) = \{D \mid D \subseteq X\}$ is a Boolean topological space under the topology of set theoretic order convergence (a net $(D_\lambda)_{\lambda \in \Lambda}$ converges to D iff for all $x \in X$, (i) $x \in D$ implies that eventually $x \in D_\lambda$ while (ii) $x \notin D$ implies that eventually $x \notin D_\lambda$).

For a Boolean ring B, $S(B) = \{P|P \text{ prime ideal } B\}$ (= $\{P|P$ maximal ideal of $B\}$) inherits the above topology from $P(B)$. There is another formally different topology on $S(B)$, the <u>spectral</u> (Zariski) topology, wherein a subset E of $S(B)$ is closed iff there is a subset D of B so that $E = \{P \in S(B)|D \subseteq P\}$. The lattice of ring ideals of B is order isomorphic to the lattice of open subsets of $S(B)$ in the spectral topology. The point to be made here is that for any Boolean ring B (even without 1) these topologies on $S(B)$ coincide. [The restriction of the power set's topology is generated by sets of the form $C_x = \{P \in S(B)|x \in P\}$ and $N_x = \{P \in S(B)|x \notin P\}$ for all $x \in B$. The spectral topology on $S(B)$ is generated by sets of the form N_x $(x \in B)$. But for any x, $C_x = \bigcup_{y > x} N_{x+y}$. So the topologies coincide.]

For any meet semilattice M a filter of M is a subset F so that: $x, y \in F \Longrightarrow x \wedge y \in F$ <u>and</u> $x \in F, y \geq x \Longrightarrow y \in F$. Let $\mathcal{F}(M) = \{F|F \text{ is a filter of } M\}$. Then $\mathcal{F}(M)$ becomes a topological space inheriting the topology of $P(M)$. Note that $\emptyset, M \in \mathcal{F}(M)$, while $S(B)$ has only proper ideals of B in it. For $P \in S(B[M])$ observe that $M\setminus P = \{x \in M|x \notin P\}$ is in $\mathcal{F}(M)$. We get a mapping $\Phi:S(B[M]) \to \mathcal{F}(M)$ whereby $P \longmapsto M\setminus P$. This map was first described by Mostowski and Tarski [12] in the case that M is a chain. The next theorem, which sums up the facts about Φ, is an attempted generalization of a result of Mostowski and Tarski.

<u>Theorem 6.1</u>. For any meet semilattice M the map $\Phi:S(B[M]) \to \mathcal{F}(M)$ given by $\Phi(P) = M\setminus P$ is a homeomorphism between $S(B[M])$ and in $\text{im}\Phi$. (The latter inherits its topology from $\mathcal{F}(M)$. Furthermore, each proper filter F of M (i.e., $F \neq \emptyset$, $F \neq M$) is in im Φ.

Proof. Write $B = B[M]$. We show first Φ is one-one. Let $P, Q \in S(B)$ and suppose $M \setminus P = M \setminus Q$. For any $x \in B$ let $A(x)$ denote the statement: $x \in P$ iff $x \in Q$. Now if $n(x)$ is 0 or 1 then $A(x)$ surely holds. Suppose x is chosen so that $A(x)$ fails and $n(x)$ is minimal making $A(x)$ fail. Write $k = n(x)$ and $x = m_1 + \ldots + m_k$ the unique linear combination of nonzero elements of M giving x. Since $A(x)$ fails we suppose, without loss of generality, that $x \in P$ but $x \notin Q$. One of $m_1, \ldots, m_k$ must fail to be in Q so (again without loss of generality) suppose $m_1 \notin Q$. Then $x + m_1 = m_2 + \ldots + m_k$ (note $k > 1$). Now $x \notin Q$, $m_1 \notin Q$ so since Q is a prime ideal, $x + m_1 = m_2 + \ldots + m_k$ is necessarily in Q. But $n(x + m_1) = k - 1$ and so $A(x + m_1)$ holds. Thus $x + m_1 \in P$. But remember that $x \in P$ so $m_1 = (x + m_1) + x$ is in P. But $m_1 \notin Q$ and $M \setminus P = M \setminus Q$ so $m_1 \notin P$. $\otimes$ Thus $A(x)$ holds for all x. Thus $P = Q$. So Φ is one-one.

The continuity of Φ is apparent so we now show $\Phi : S(B) \rightarrow \operatorname{im} \Phi$ is a homeomorphism. Suppose that for a net $(P_\lambda)_{\lambda \in \Lambda}$ in $S(B)$ the net $(M \setminus P_\lambda)_{\lambda \in \Lambda}$ converges in the topology of $\mathcal{F}(M)$ to $M \setminus P$ where $P \in S(B)$. We claim that $(P_\lambda)_{\lambda \in \Lambda}$ converges to P in the topology of $S(B)$. For $x \in B$ let $E(x)$ denote the statement:

$$x \in P \text{ implies that eventually } x \in P_\lambda$$

and

$$x \notin P \text{ implies that eventually } x \notin P_\lambda .$$

We show $E(x)$ holds for all $x \in B$. $E(x)$ is certainly true if $n(x) = 1$ (this is a consequence of $M \setminus P_\lambda \rightarrow M \setminus P$ in $\mathcal{F}(M)$). Suppose $E(x)$

fails for some $x \in B$. Take x in B with $n(x)$ minimal such that $E(x)$ fails. $(n(x) > 1)$ Write $k = n(x)$ and $x = m_1 + \ldots + m_k$ as above.

Case 1. $x \in P$. If each of $m_1, \ldots, m_k$ were in P then since $E(m_i)$ is true $(i = 1, \ldots, k)$ each of the m_i would eventually be in P_λ and so their sum x would eventually be in P_λ, and then $E(x)$ would be true. So one of $m_1, \ldots, m_k$ is not in P, say $m_1 \notin P$. Then $m_2 + \ldots + m_k \notin P$. So since $E(m_1)$ holds there is a λ_0 so that for all $\lambda \geq \lambda_0$, $m_1 \notin P_\lambda$. Since $E(m_2 + \ldots + m_k)$ holds there is a λ_1 so that for all $\lambda \geq \lambda_1$ $m_2 + \ldots + m_k \notin P_\lambda$. Choose $\lambda_2 \geq \lambda_0, \lambda_1$. Of necessity, for all $\lambda \geq \lambda_2$, $x \in P_\lambda$ ($m_1 \notin P_\lambda$, $m_2 + \ldots + m_k \notin P_\lambda$ but P_λ is prime so $x = m_1 + (m_2 + \ldots + m_k) \in P_\lambda$). So $E(x)$ holds, a contradiction.

Case 2. $x \notin P$. The one of m_1 or $m_2 + \ldots + m_k$ must be in P while the other is not. So we write x as $a + b$ where $a \in P$, $b \notin P$ and $n(a), n(b) < n(x)$. So $E(a)$ and $E(b)$ hold and so eventually $a \in P_\lambda$ and eventually $b \notin P_\lambda$. So eventually $x = a + b \notin P$. So again we get $E(x)$ true, a contradiction.

Thus for any x, $E(x)$ is true. So in $S(B)$, $P_\lambda \rightarrow P$.

Finally we show each proper filter of M is in im Φ. Let $F \in \mathcal{F}(M)$, $F \neq \emptyset$, $F \neq M$. We claim that the ideal I generated by $M \setminus F$ in B misses F. For otherwise there would be an $f_0 \in F$ and elements $m_1, \ldots, m_t$ of $M \setminus F$ so that $f_0 \leq m_1 \vee \ldots \vee m_t$. This would give $f_0 = \bigvee_{i=1}^{t} (m_i \cdot f_0)$. But condition (*) of proposition 5.3 holds for M in B so we would have to have $f_0 = m_i \cdot f_0$ for some i. This would force m_i into F, a contradiction. Now choose P to be an ideal of B maximal with respect to containing $M \setminus F$ and missing F. It is easy to show $P \varepsilon S(B)$ and $M \setminus P = F$. Hence $F = \Phi(P) \varepsilon \operatorname{im} \Phi$.

We now observe that M has a zero iff im Φ consists precisely of proper filters of M. If M has no zero, im Φ consists of the non-empty filters, and the prime ideal of B mapping to the improper filter M is $P_0 = \{x \varepsilon B \mid n(x) \text{ is even}\}$. We summarize:

Corollary 6.2. Let M be any meet semilattice. If M has a zero then $S(B[M])$ and $\mathcal{F}_p(M) = \{F \subseteq M \mid F \text{ proper filter of } M\}$ are homeomorphic. If M has no zero then $S(B[M])$ and $\mathcal{F}_p(M) \cup \{M\}$ are homeomorphic.

7. Comparison of congruences between M and B[M]

We now try to answer the natural question of how the congruences of a semilattice M compare to the congruences of its universal Boolean ring B[M]. To discuss this question we need some notation. Recall: $\Theta(M) = \{\sigma \mid \sigma \text{ is a meet congruence of } M\}$, for any Boolean ring B, $\Theta(B) = \{\rho \mid \rho \text{ is a lattice (ring) congruence of } B\}$, $\mathcal{J}(B) = \{J \mid J \text{ ideal of } B\}$. If $\rho \varepsilon \Theta(B)$ and $J \varepsilon \mathcal{J}(B)$ we say ρ and J are associated if the following condition holds: $(x,y) \varepsilon \rho$ iff $x + y \varepsilon J$ for all $x, y \varepsilon B$. The relation of being associated establishes an order isomorphism between $\Theta(B)$ and $\mathcal{J}(B)$.

For any meet congruence σ of M, σ has at least one extension to a (ring) congruence of B[M] namely to σ^e, the B[M] congruence generated by σ. Let $I(\sigma)$ denote the ideal of B[M] associated with the congruence σ^e. It is not difficult to see that $I(\sigma)$ is the ideal generated by the set $\{m + m' \mid m,m' \varepsilon M, m\sigma m'\}$. We denote this latter fact by writing

$I(\sigma) = (m + m' | m\sigma m']$. Let us call *$\sigma^e$ the extension of σ to $B[M]$* and *$I(\sigma)$ the ideal extension of σ*.

Now starting with an ideal I of $B[M]$ and its associated ring congruence ρ we let $\sigma(I)$ denote $\rho \bigcap (M \times M) = \rho^c$($\rho$ contracted to M), a meet congruence of M, and call *$\sigma(I)$ the contraction of I* (or ρ) to M. Certainly $\sigma(I) = \{(m,m') | m,m' \in M,\ m + m' \in I\}$.

We call a congruence σ of M *contracted* if for some $I \in \mathcal{I}(B[M])$, $\sigma = \sigma(I)$. A congruence of $B[M]$, ρ, is called *extended* if $\rho = \sigma^e$ for some $\sigma \in \Theta(M)$; while an ideal I of $B[M]$ is *extended* iff $I = I(\sigma)$ for some $\sigma \in \Theta(M)$. If ρ and I are associated then ρ is extended iff I is extended.

The two maps: $\Theta(M) \longrightarrow \mathcal{I}(B[M])$ where $\sigma \longmapsto I(\sigma)$ and $\mathcal{I}(B[M]) \longrightarrow \Theta(M)$ where $I \longmapsto \sigma(I)$ form a Galois connexion of mixed type. Namely for any $\sigma \in \Theta(M)$ and any $I \in \mathcal{I}(B[M])$ we have:

$$\sigma \leq \sigma(I) \text{ iff } I(\sigma) \leq I.$$

As a consequence of this fact we have several facts:

(i) the map $\Theta(M) \longrightarrow \mathcal{I}(B[M])$ is completely join preserving (and so order preserving). Extension preserves arbitrary joins;

(ii) the map $\mathcal{I}(B[M]) \longrightarrow \Theta(M)$ is completely meet preserving (and so order preserving). Contraction preserves arbitrary meets;

(iii) the map $\sigma \longmapsto \sigma(I(\sigma))$ is a closure operator in $\Theta(M)$ whose closure family of fixed elements is the collection of contracted congruences;

(iii') the map $I \longmapsto I(\sigma(I))$ is a kernel operator in $\mathcal{J}(B[M])$ whose kernel family of fixed elements is the collection of extended ideals;

(iv) the complete lattice of contracted congruences is order isomorphic to the complete lattice of extended ideals under the restriction of our mappings of extension and contractions. These restrictions are inverse to one another.

The previous results are general consequences of any Galois connexion. But in our specific case we can say more. Using the universal property of B[M] it is not difficult to show that for any congruence σ of M, $B[M/\sigma] = B[M]/_{\sigma^e}$, and hence $(\sigma^e)^c = \sigma$. Thus

Proposition 7.1. For each meet semilattice M each congruence of M is contracted. So for any $\sigma \in \Theta(M)$, $\sigma = \sigma(I(\sigma))$. Hence $\Theta(M)$ is order isomorphic to the lattice of extended ideals.

We turn now to the other side of the coin to examine the extended ideals of B[M]. Write B for B[M]. Let J be any ideal of B. Let $D(J) = \{m + \bar{m} | m, \bar{m} \in M \text{ and } m + \bar{m} \in J\}$. We have seen that $I(\sigma(J)) = (D(J)]$ (the ideal generated by the set D(J)). Certainly (D(J)] consists of all finite sums of elements of D(J). So J is extended iff $J = \{\sum_{i=1}^{n} e_i | n \in \mathbb{N}^+, e_i \in D(J)\}$.

There is a largest extended ideal, namely $I(M \times M) = I(\sigma(B[M]))$ $= (D(B[M])]$, so it is $\{\sum_{i=1}^{n} e_i | n \in \mathbb{N}^+, e_i = m + \bar{m} \text{ for } m, \bar{m} \in M\}$. Following Byrd, Mena and Troy [2], we will call this ideal the ideal evenly generated by M and denote it E_M. Though for the above authors M was a distributive sublattice of B generating B, their results are analogous to what we find here. For our next lemma, which summarizes the easily established facts about E_M, we remind the reader of the notation n(x) for x ε B. Recall that each element x of B can be uniquely expressed $x = \sum_{\substack{m\varepsilon M \\ m\neq 0}} \lambda_m m$. Then $n(x) = \#\{m|\lambda_m \neq 0\}$; remember a.a $\lambda_m = 0$.

Lemma 7.2. Let $P_0 = \{x \varepsilon B|n(x) \text{ is even}\}$ (see section 6 for a summary of facts about P_0). Then (a) $P_0 \subseteq E_M$, (b) $E_M = B$ iff $0_B \varepsilon M$. So (c) if M has no zero, $E_M = P_0$, hence a prime ideal. (d) The ideals of E_M are exactly the ideals of B contained in E_M. So extension-contraction is actually a Galois connexion between $\Theta(M)$ and $\mathcal{I}(E_M)$.

Proof. (a) is obvious and (c) holds once (b) is true. (d) holds for any ideal E_M of a Boolean ring B (i.e., viewing an ideal J as a Boolean ring itself $\mathcal{I}(J) = \{k \varepsilon \mathcal{I}(B)|k \subseteq J\}$). So we only show (b). If $0_B \varepsilon M$ then any m ε M can be written $m = 0_B + m$ which is in E_M. Hence $M \subseteq E_M$ so we get $E_M = B$ in the case that $0_B \varepsilon M$. So now suppose $0_B \notin M$ and that $E_M = B$. Since M has no least element it is clear that $D(B) \subseteq P_0$. But also P_0 is an ideal so $E_M = (D(B)] \subseteq P_0$ hence $P_0 = B[M]$ but this is impossible. (We have not explicitly said it but we assume $M \neq \emptyset$, so choosing m ε M and because M has no least element n(m) = 1 so $m \notin P_0$.)

Now all extended ideals of B are contained in E_M and so are ideals of the Boolean ring E_M. Also for each $\sigma \varepsilon \Theta(M)$, $\sigma = \sigma(I(\sigma))$ where $I(\sigma) \varepsilon \mathcal{I}(E_M)$ which means each M congruence is the contraction of an ideal of the ring E_M. We will hereafter treat our Galois connexion as between $\Theta(M)$ and $\mathcal{L}(E_M)$, with all elements of $\Theta(M)$ contracted. We are interested in what conditions on M will make all ideals of E_M extended. If this happens then $\Theta(M) \cong \mathcal{L}(E_M)$ which makes $\Theta(M)$ distributive. So in light of the result of Papert [14], we have: for each ideal of E_M to be extended it is necessary that M be a semilattice tree. We move toward a proof that M being a semilattice tree is sufficient to make all E_M ideals extended.

Lemma 7.3. Let B be any Boolean ring and let $x = b_1 + \ldots + b_k + y + z$ where $b_1, \ldots, b_k, y, z \in B$ and each $b_i \geq y \geq z$. Then if k is even, $x = (b_1 + \ldots b_k) \vee (y + z)$. If k is odd then $x = (b_1 + \ldots + b_k + y) \vee z$.

Proof. In a Boolean ring, if $a \bullet b = 0$ then $a \vee b = a + b$. If k is even then under the above hypotheses we have: $(b_1 + \ldots + b_k) \bullet (y + z) = (b_1y + \ldots + b_ky) + (b_1z + \ldots + b_kz) = (k \cdot y) + (k \cdot z) = 0 + 0 = 0$. Hence $x = (b_1 + \ldots + b_k) + (y + z) = (b_1 + \ldots + b_k) \vee (y + z)$. The proof of the k-odd case is similar.

The lemma enables us to prove:

Proposition 7.4. Let M be a semilattice tree with least element 0_M. Let $D = \{m + m' \mid m,m' \in M\}$ formed in B[M]. Then each element of B[M] is the finite join of elements of D.

Proof. Let x be an element of B = B[M] which is the sum of s nonzero elements of M and suppose that for each $y \in B[M]$: if y is the sum of fewer than s nonzero elements of M then y is the join of finitely many elements of D. Since $M \subseteq D$ we may as well assume $s \geq 2$. Write $x = m_1 + \ldots + m_s$. Let $i \in \{1, \ldots, s\}$. The set

$$\{m_1m_i, m_2m_i, \ldots, m_sm_i\}$$

is a subset of M which is bounded above by m_i. Since M is a tree this set must be totally ordered. Let $m_\ell m_i$ be the least element of this set and choose m_km_i to be least in $\{m_1m_i, \ldots, m_sm_i\} \setminus \{m_\ell m_i\}$. Then

$$m_ix = \left(\sum_{\substack{j=1 \\ j \neq \ell,k}}^{s} m_jm_i\right) + m_km_i + m_\ell m_i.$$

If s is even the above lemma says that

$$m_ix = \left(\sum_{\substack{j=1 \\ j \neq k,\ell}}^{s} m_jm_i\right) \vee (m_km_i + m_\ell m_i).$$

By the choice of s, $\sum_{j \neq k,\ell} m_jm_i$ is the finite join of elements of D and so apparently (with $m_km_i + m_\ell m_i \in D$) m_ix is the finite join of elements of D. If s is odd then the above lemma, plus the fact that

$$m_ix = \sum_{\substack{j=1 \\ j \neq \ell}}^{s} m_jm_i + m_\ell m_i$$

give us:

$$m_ix = \left(\sum_{\substack{j=1 \\ j \neq \ell}}^{s} m_jm_i\right) \vee (m_\ell m_i)$$

Again the choice of s gives: $\sum_{j \neq \ell} m_j m_i$ is the finite join of elements of D. But $m_\ell m_i$ is in D so we get $m_i x$ to be the finite join of elements of D.

Thus for each i = 1, ..., s, $(x \cdot m_i)$ is the finite join of elements of D. Then

$$x = m_1 + \ldots + m_s \leq m_1 \vee \ldots \vee m_s$$

and so

$$x = x \wedge \left(\bigvee_{i=1}^{s} m_i \right) = \bigvee_{i=1}^{s} (x \cdot m_i)$$

and is itself the finite join of elements of D.

As a corollary we have:

Corollary 7.5. Let M be a semilattice tree. Let $D = \{a + b \mid a, b \in B[M], a, b \in M \cup \{0\}\}$. Then each element of B[M] is the finite join of elements of D.

Proof. If M has a zero this corollary is identical to 7.4. Suppose M has no zero. Let $\hat{M} = M \cup \{0_B\}$. Then $\hat{M}$ is a tree with zero and $B[M] = B[\hat{M}]$. The claim of 7.5 then follows from 7.4 applied to $\hat{M}$ and $B[\hat{M}]$.

We come to our main result (c.f. [2, theorems 2.8, 2.9])

Proposition 7.6. Let M be any meet semilattice. The following statements are equivalent:

(i) M is a semilattice tree,

(ii) each ideal of E_M is extended.

Hence $\Theta(M) \cong \mathcal{A}(E_M)$ for any semilattice tree M. So the congruences of a semilattice tree are order isomorphic to the congruences of some Boolean ring. Also, M is a semilattice tree with 0 iff each ideal of B[M] is extended.

<u>Proof</u>. We already know (ii) $\Rightarrow$ (i). We show (i) $\Rightarrow$ (ii). Let $J \in \mathcal{A}(E_M)$. Then $J \in \mathcal{A}(B[M])$ and $J \subseteq E_M$. We must show $J \subseteq I(\sigma(J))$ (since already $I(\sigma(J)) \subseteq J$). Let $j \in J$. By corollary 7.5 we can write j as $j = \bigvee_{i=1}^{n} d_i$ where each $d_i = a_i + b_i$ for certain a_i, b_i in $M \cup \{0_B\}$. <u>If M has a least element</u> then each a_i, $b_i \in M$ and since $a_i + b_i \le j \in J$ then each $a_i + b_i \in J$; hence each $a_i + b_i \in \{m + \bar{m} | m, \bar{m} \in M$ and $m + \bar{m} \in J\} \subseteq I(\sigma(J))$. If on the other hand <u>M has no least element</u> then $E_M = P_0$ and each $a_i + b_i \le j \in J \subseteq E_M$ forces each $a_i + b_i$ into P_0. From this it follows that each $a_i + b_i$ is in $I(\sigma(J))$ (if both a_i, $b_i \in M$ then $a_i + b_i \in (m + \bar{m} | m + \bar{m} \in J] = I(\sigma(J))$; if one is zero and the other is not then $n(a_i + b_i)$ is 1 so $a_i + b_i \notin P_0$; if both are zero $a_i + b_i \in I(\sigma(J))$). So $j = \bigvee_{i=1}^{n} (a_i + b_i) \in I(\sigma(J))$. In any case, we get $j \in I(\sigma(J))$, hence $J \subseteq I(\sigma(J))$.

Since $E_M = B$ iff M has a zero, the other statements follow. Notice that if M is a tree with no least element, then $\Theta(M) \cong \mathcal{A}(P_0)$.

For trees with zero, we summarize our results.

<u>Corollary 7.7</u>. Let T be a semilattice tree with zero. Then $\Theta(T) = \mathcal{A}(B[T])$ under the mappings of extension and contraction. So $c(\Theta(T))$ (the compact congruences of T) is order isomorphic to B[T]. Also, $\Theta(T)$ is lattice isomorphic to the lattice of open subsets of the space $\mathcal{F}_P(T)$.

Proof. Only the last statement needs clarification. For any topological space X let $\mathcal{O}(X)$ denote the lattice of open subsets of X. We Have: for any Boolean ring B

$$\mathcal{I}(B) \cong \mathcal{O}(S(B))$$

so

$$\mathcal{I}(B[T]) \cong \mathcal{O}(S(B[T])).$$

But we know S(B[T]) is homeomorphic to $\mathcal{F}_P(T)$, so $\mathcal{O}(S(B[T])) \cong \mathcal{O}(\mathcal{F}_P(T))$ and thus $\mathcal{I}(B[T]) \cong \mathcal{O}(\mathcal{F}_P(T))$. But $\Theta(T) \cong \mathcal{I}(B[T])$ so the last statement of the corollary follows.

We have a restatement. We have mappings $\Theta(B[M]) \longrightarrow \Theta(M)$, $\rho \longmapsto \rho^c = \rho \cap (M \times M)$ and $\Theta(M) \to \Theta(B[M])$ whereby $\sigma \longmapsto \sigma^e$. These form a mixed type Galois connexion and so the usual consequences of this follow. Again the contracted M congruences are all M congruences. Because of 7.6, we get

Corollary 7.8. For any semilattice M, M is a semilattice tree with zero iff each lattice congruence of B[M] is extended (for each $\rho \in \Theta(B[M])$, $\rho = \sigma^e$ for some $\sigma \in \Theta(M)$). Hence $\Theta(M) \cong \Theta(B[M])$, if M is a tree with zero, under the extension and contraction mappings.

Proof. This is because whenever $\rho \in \Theta(B[M])$, $I \in \mathcal{I}(B[M])$ and ρ, I are associated then ρ is extended iff I is. So each ideal of B[M] is extended iff each congruence of B[M] is extended.

We close with a small application. Suppose T_1, T_2 are semilattice trees with 0 so that the lattices $\Theta(T_1)$ and $\Theta(T_2)$ are isomorphic. Then the lattices $c(\Theta T_1))$ and $c(\Theta(T_2))$ are isomorphic and hence

$B[T_1] \cong B[T_2]$. These latter are isomorphic as rings hence as $\mathbb{Z}_2$ vector spaces. So they have the same $\mathbb{Z}_2$ dimension, so $\# T_1 \setminus \{0\} = \# T_2 \setminus \{0\}$. We then have

Corollary 7.9. If each of T_1, T_2 is a semilattice tree with zero so that $\Theta(T_1) \cong \Theta(T_2)$ then $\# T_1 = \# T_2$.

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