Homotopy-Systems, H-Spaces and Sheaf Cohomology

A Dissertation
Presented to
The Faculty of the Department of Mathematics
University of Houston

In Partial Fulfillment
of the Requirements for the Degree
Doctor of Philosophy

---

by
James Michael Parks
May 1971

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ABSTRACT

The concepts of sheaves and sheaf cohomology are central throughout the work. Certain natural generalizations of these concepts are investigated in the latter part of the dissertation.

The induced sheaf of a locally constant sheaf under a homotopy of a map of base spaces is shown to behave similar to the induced bundle of a locally constant bundle space, with respect to a homotopy of a map of the base spaces. The question: Are all sheaves limits of locally constant sheaves? is answered in the negative by demonstrating that such sheaves inherit certain homotopy properties of locally constant sheaves. Several related sheaf cohomology mapping theorems are proved, using sheaf cohomology with coefficients in locally constant sheaves or restrictions on the mapping or both, thus giving results concerning sheaf cohomology and homotopy type. A continuity theorem for a system of locally constant sheaves over a homotopy-inverse system of spaces is proved. (Homotopy-systems of spaces are introduced and investigated in the beginning of the work and numerous applications are found throughout the dissertation.)

By relaxing the algebraic structure on the stalks of a sheaf to admit H-structures, the concept of a sheaf of H-spaces is introduced. A cohomology theory with coefficients in a sheaf of H-spaces is defined using the Cech technique. This cohomology theory is shown to satisfy Cartan's axioms for a sheaf cohomology theory. Other properties are explored and the theory is shown to contain the sheaf cohomology theory.

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CHAPTER 1 HOMOTOPY-SYSTEMS

In most cases the basic definitions and properties given here hold for both inverse and direct homotopy-systems. However, our interests lie mainly towards the inverse situation (see the applications which follow below). Therefore, various additional properties of the homotopy-inverse system are investigated.

Definition 1.1 The collection of spaces and maps, $\{X_\alpha, \varphi_\alpha^{\ \beta}\}_\Lambda$, $(\{X_\alpha, \varphi^\alpha_{\ \beta}\}_\Lambda)$, indexed by the directed set Λ, is called a homotopy-inverse (direct) system, or h-inverse (direct) system, whenever:

(1.1a) if $\alpha,\beta \in \Lambda$ and $\alpha < \beta$ then there exists a map $\varphi_\alpha^{\ \beta}: X_\beta \to X_\alpha$, $(\varphi^\alpha_{\ \beta}: X_\alpha \to X_\beta)$,

(1.1b) if $\alpha,\beta,\gamma \in \Lambda$ and $\alpha < \beta < \gamma$, then $\varphi_\alpha^{\ \beta}\varphi_\beta^{\ \gamma} \simeq \varphi_\alpha^{\ \gamma}$, $(\varphi^\beta_{\ \gamma}\varphi^\alpha_{\ \beta} \simeq \varphi^\alpha_{\ \gamma})$.

Note that in contrast to the usual situation for inverse systems only the condition $\varphi_\alpha^{\ \alpha}(x) \simeq x$ for some $x \in X_\alpha$ and all $\alpha \in \Lambda$ is necessary in order to obtain nontrivial limits.

Definition 1.2 If $\{X_\alpha, \varphi_\alpha^{\ \beta}\}_\Lambda$ is an h-inverse system of spaces, then define the h-inverse limit, $\underleftarrow{L}\{X_\alpha, \varphi_\alpha^{\ \beta}\}_\Lambda$, or $\underleftarrow{L}\, X_\alpha$ when the system is understood, as the subspace of $\prod_\Lambda X_\alpha$ given by the set:

(1.2a) $\{x \in \Pi X_\alpha |$ if $\alpha < \beta$ then $p_\alpha(x) \simeq \varphi_\alpha^{\ \beta} p_\beta(x)\}$,

where $p_\alpha: \Pi X_\alpha \to X_\alpha$ is the projection map. Denote the map $p_\alpha | \underleftarrow{L}\, X_\alpha$ by φ_α.

Dually, if $\{X_\alpha, \varphi^\alpha_{\ \beta}\}_\Lambda$ is an h-direct system of spaces, then define the h-direct limit, $\underrightarrow{L}\{X_\alpha, \varphi^\alpha_{\ \beta}\}_\Lambda$, or $\underrightarrow{L}\, X_\alpha$ when the system is understood, as the quotient space $\sum\limits_{\Lambda} X_\alpha / \sim$, where $\Sigma\, X_\alpha$ is the free union of the spaces X_α and $\sim$ is the equivalence relation determined by the equivalence:

(1.2b) $x_\alpha \sim x_\beta$ iff there exists a $\gamma > \alpha, \beta$ such that $\varphi^{\alpha}{}_{\gamma}(x_\alpha) \simeq \varphi^{\beta}{}_{\gamma}(x_\beta)$, where x_α is the α-th coordinate of a point $x = \{x_\alpha\}$ in $\Sigma\, X_\alpha$. Let $p\colon \Sigma\, X_\alpha \to \underset{\rightsquigarrow}{L}\, X_\alpha$ be the natural map and denote $p|X_\alpha$ by φ_α.

As an immediate result one has the following lemma.

Lemma 1.3

(1.3a) If $\{X_\alpha, \varphi_\alpha{}^\beta\}_\Lambda$ is an h-inverse system and $\alpha < \beta$, then the diagram commutes up to homotopy.

$$\begin{tikzcd} & X_\beta \arrow[dd, "\varphi_\alpha{}^\beta"] \\ \underset{\leftsquigarrow}{L}\, X_\alpha \arrow[ur, "\varphi_\beta"] \arrow[dr, "\varphi_\alpha"'] & \\ & X_\alpha \end{tikzcd}$$

(1.3b) If $\{X_\alpha, \varphi^\alpha{}_\beta\}_\Lambda$ is an h-direct system and $\alpha < \beta$, then the diagram commutes up to homotopy.

$$\begin{tikzcd} X_\alpha \arrow[dr, "\varphi_\alpha"] \arrow[dd, "\varphi^\alpha{}_\beta"'] & \\ & \underset{\rightsquigarrow}{L}\, X_\alpha \\ X_\beta \arrow[ur, "\varphi_\beta"'] & \end{tikzcd}$$

Proof Part a is immediate from Definition 1.2 and part b follows from the observation: $x_\alpha \sim \varphi^\alpha{}_\beta(x_\alpha)$ whenever $\alpha < \beta$.

We list some examples and observations on h-inverse and h-direct systems.

Example 1.4 Let $\{X_\alpha, \varphi_\alpha{}^\beta\}_\Lambda$ be an (h-) inverse system and $\{P_\alpha\}_\Lambda$ a collection of spaces such that P_α dominates X_α for each $\alpha \in \Lambda$.

For instance one might have one of the following situations:

a.) X_α is an ANR with respect to metrizable spaces and P_α is a polyhedron [2].

b.) X_α is an ANR with respect to metrizable spaces and compact (separable) and P_α is a (locally) finite polyhedron [2].
c.) X_α is a compact space with the homotopy type of a CW-complex and P_α is a finite CW-complex [15], [11].
d.) X_α is a metric space dominated by a polytope and P_α is the nerve of a grating (a collection of mutually disjoint open sets the union of whose closures covers the space) on X_α [6].
e.) X_α and P_α have the same homotopy type.

The following system is determined by each of the above possibilities.

$$\begin{tikzcd}
\cdots & P_\alpha \arrow[l, dashed] \arrow[d, shift left, "g_\alpha"] & P_\beta \arrow[l, dashed, "\psi_\alpha{}^\beta"'] \arrow[d, shift left, "g_\beta"] & P_\gamma \arrow[l, dashed, "\psi_\beta{}^\gamma"'] \arrow[d, shift left, "g_\gamma"] & \cdots \arrow[l, dashed] \\
\cdots & X_\alpha \arrow[l] \arrow[u, shift left, "f_\alpha"] & X_\beta \arrow[l, "\varphi_\alpha{}^\beta"'] \arrow[u, shift left, "f_\beta"] & X_\gamma \arrow[l, "\varphi_\beta{}^\gamma"'] \arrow[u, shift left, "f_\gamma"] & \cdots \arrow[l]
\end{tikzcd}$$

The maps f_α, g_α satisfy $g_\alpha f_\alpha \simeq 1_{X_\alpha}$ for all $\alpha \in \Lambda$. The maps $\psi_\alpha{}^\beta : P_\beta \to P_\alpha$ are defined by $\psi_\alpha{}^\beta = f_\alpha\, \varphi_\alpha{}^\beta\, g_\beta$ whenever $\alpha < \beta$.

Thus $\{P_\alpha, \psi_\alpha{}^\beta\}_\Lambda$ is an h-inverse system. If $\{X_\alpha, \varphi_\alpha{}^\beta\}_\Lambda$ is assumed to be an h-direct system in the above, then the system $\{P_\alpha, \psi_\alpha{}^\beta\}_\Lambda$ is an h-direct system.

<u>Example 1.5</u> Let X_n be a contractible space for all integers $n \in J$, and suppose $X_n \subset X_m$ whenever $m < n$. Define connecting maps by the following formula:

$$\varphi_m{}^n = \begin{cases} i : X_n \to X_m, \text{ an inclusion map whenever } n \text{ and } m \\ \qquad\qquad \text{are both odd or both even,} \\ \text{trivial otherwise, i.e. a constant map.} \end{cases}$$

Thus, one has:
$$\varphi_m^{m+k} = \begin{cases} i, & \text{for } k \text{ even, and} \\ \text{trivial}, & \text{for } k \text{ odd.} \end{cases}$$

Since all spaces are contractible this map determines an h-inverse system. Clearly, $p_m(x) \simeq \varphi_m^{\,n}\, p_n(x)$ for all $x \in \Pi X_n$, whenever $m < n$. Thus $\underset{\leftarrow}{L}\, X_n = \Pi X_n$.

Example 1.6 Let $\{X_\alpha, \varphi_\alpha^{\,\beta}\}_\Lambda$ be an inverse system directed by inverse inclusion, that is if $\alpha < \beta$, then $X_\beta \subset X_\alpha$ and $\varphi_\alpha^{\,\beta}$ is an inclusion map. Let $\{P_\alpha\}_\Lambda$ be a collection of spaces of the same homotopy type as $\{X_\alpha\}_\Lambda$, that is $X_\alpha \simeq P_\alpha$ for all $\alpha \in \Lambda$, (see Example 1.4 above).

Then, if $\alpha < \beta$, $P_\beta \simeq X_\beta \subset X_\alpha \simeq P_\alpha$, and P_β is homotopic to a subspace of P_α through the induced map $f_\alpha\, \varphi_\alpha^{\,\beta}\, g_\beta = \psi_\alpha^{\,\beta}$ as in Example 1.4 above (see also Example 1.5).

If one requires that P_α dominates X_α instead of $P_\alpha \simeq X_\alpha$ in the above, then P_β dominates P_α, whenever $\alpha < \beta$, through the induced map $\psi_\alpha^{\,\beta}$.

On the other hand, let $\{X_n, \varphi^m_{\ n}\}_J$ be an h-direct system and let $Z_{\varphi^m_{\ n}}$ denote the mapping cylinder of $\varphi^m_{\ n}: X_m \to X_n$. Then $Z_{\varphi^m_{\ n}} \simeq X_n$ and $Z_{\varphi^m_{\ n}} \simeq Z_{\varphi^k_{\ n}}$ for all $k,m < n$. Let $Z_n = \bigcup_{m<n} Z_{\varphi^m_{\ m+1}}$. Then $Z_n \simeq Z_{\varphi^m_{\ n}}$ and the induced map $\psi^m_{\ n}: Z_m \to Z_n$ is an inclusion map. Also $\underset{\rightarrow}{L}\, Z_n = \cap Z_n \simeq \underset{\rightarrow}{L}\, X_n$, (see Proposition 1.13).

Example 1.7 Graphically, $\underset{\leftarrow}{L}\, X_\alpha = \bigcap_\Lambda (\Gamma_\alpha^{\,\beta} \times \Pi X_\gamma)$, $\alpha,\beta \neq \gamma$, where $\Gamma_\alpha^{\,\beta} = \{\ (x_\alpha, x_\beta) \mid x_\alpha \simeq \varphi_\alpha^{\,\beta}(x_\beta)\ \}$, whenever $\alpha < \beta$, is a collection of path components in $X_\alpha \times X_\beta$. Thus $\underleftarrow{L}\, X_\alpha$ is not necessarily closed in $\Pi\, X_\alpha$.

If each X_α is path connected, then obviously $\underleftarrow{L}\, X_\alpha = \Pi\, X_\alpha$, (see Example 1.5 above).

On the other hand if each X_α is totally disconnected, the h-inverse limit becomes an ordinary inverse limit.

<u>Example 1.8</u> Note that if a cohomology functor (or cohomotopy functor in the proper setting) is applied to an h-inverse system, $\{X_\alpha, \varphi_\alpha{}^\beta\}_\Lambda$, then a direct system, $\{H^*(X_\alpha), \Phi^\alpha{}_\beta\}_\Lambda$ (or $\{\pi^*(X_\alpha), \Phi^\alpha{}_\beta\}_\Lambda$), results, where $\Phi^\alpha{}_\beta = (\varphi_\alpha{}^\beta)^*$.

Similarly, if a homology (or homotopy) functor is applied to an h-inverse system, an inverse system results.

If $\{X_\alpha, \psi^\alpha{}_\beta\}_\Lambda$ is an h-direct system, then $\{H_*(X_\alpha), \Psi^\alpha{}_\beta\}_\Lambda$ is a direct system, (and $\{\pi_*(X_\alpha), \Psi^\alpha{}_\beta\}_\Lambda$ is a direct system), where $\Psi^\alpha{}_\beta = (\psi^\alpha{}_\beta)_*$.

In Example 1.7 it was noted that the h-inverse limit of an h-inverse system was not necessarily closed in the product space. This is remedied by the following lemma.

<u>Lemma 1.9</u> If $\{X_\alpha, \varphi_\alpha{}^\beta\}_\Lambda$ is an h-inverse system of locally path connected spaces, then $\underleftarrow{L}\, X_\alpha$ is closed in $\Pi\, X_\alpha$.

<u>Proof</u> Local path connectedness is equivalent to path components being closed (and open) [7].

<u>Corollary 1.10</u> If $\{X_\alpha, \varphi_\alpha{}^\beta\}_\Lambda$ is an h-inverse system of locally path connected compact spaces, then $\underleftarrow{L}\, X_\alpha$ is compact.

<u>Definition 1.11</u> If $\{X_\alpha, \varphi_\alpha^{\ \beta}\}_\Lambda$ and $\{Y_\alpha, \psi_\alpha^{\ \beta}\}_\Lambda$ are h-inverse systems and for each $\alpha \in \Lambda$ there exists a map $f_\alpha : X_\alpha \to Y_\alpha$ such that the diagram commutes up to homotopy, then the family $F = \{f_\alpha\}$ is called a <u>map of the systems</u>. (Similarly for h-direct systems.)

$$\begin{tikzcd} X_\alpha \arrow[d, "f_\alpha"'] & X_\beta \arrow[l, "\varphi_\alpha^{\ \beta}"'] \arrow[d, "f_\beta"] \\ Y_\alpha & Y_\beta \arrow[l, "\psi_\alpha^{\ \beta}"] \end{tikzcd}$$

<u>Proposition 1.12</u>

(1.12a) If $F = \{f_\alpha\}$ is a map of the h-inverse system $\{X_\alpha, \varphi_\alpha^{\ \beta}\}_\Lambda$ to the h-inverse system $\{Y_\alpha, \psi_\alpha^{\ \beta}\}_\Lambda$, then F induces a map $F' : \varprojlim X_\alpha \to \varprojlim Y_\alpha$.

(1.12b) If f_α is a homotopy equivalence for each $\alpha \in \Lambda$ in part a, then F' is a homotopy equivalence.

<u>Proof</u> Let x be a point in $\varprojlim X_\alpha$. Then, if $\alpha < \beta$, one has

$$\varphi_\alpha^{\ \beta}(x_\beta) = \varphi_\alpha^{\ \beta}\varphi_\beta(x) \simeq \varphi_\alpha(x) = x_\alpha ,$$

by Lemma 1.3a. By Definition 1.11, F satisfies:

$$\psi_\alpha^{\ \beta} f_\beta(x_\beta) \simeq f_\alpha \varphi_\alpha^{\ \beta}(x_\beta) \simeq f_\alpha(x_\alpha) .$$

Thus $(\Pi\ f_\alpha)(x)$ is a point in $\varprojlim Y_\alpha$ or

$$(\Pi\ f_\alpha)\big| \varprojlim X_\alpha = F' : \varprojlim X_\alpha \to \varprojlim Y_\alpha .$$

If f_α is a homotopy equivalence, let g_α be the homotopy inverse of f_α. Then, since

$$\varphi_\alpha^{\ \beta} g_\beta \simeq g_\alpha f_\alpha \varphi_\alpha^{\ \beta} g_\beta \simeq g_\alpha \psi_\alpha^{\ \beta} f_\beta g_\beta \simeq g_\alpha \psi_\alpha^{\ \beta},$$

whenever $\alpha < \beta$, the map $G = \{g_\alpha\}$ is a map of the systems.

Also, $G' = (\Pi\ g_\alpha)\big| \varprojlim Y_\alpha : \varprojlim Y_\alpha \to \varprojlim X_\alpha$ by part a. Thus

$$G'F'(x) = G'(< f_\alpha x_\alpha >) = < g_\alpha f_\alpha x_\alpha > = < x_\alpha > ,$$

since $g_\alpha f_\alpha \simeq 1_{X_\alpha}$. Thus $G'F' \simeq 1_{\varprojlim X_\alpha}$.

Similarly, $F'G' \simeq 1_{\varprojlim Y_\alpha}$ and F' (and G') is a homotopy equivalence.

Note that the proposition does not hold if inverse limits are involved, since in that case the map of the systems does not commute up to homotopy, necessarily.

Corollary 1.13 If $\{X_\alpha\}$ dominates $\{Y_\alpha\}$ by F above, then $\varprojlim X_\alpha$ dominates $\varprojlim Y_\alpha$ by F' (and $\{Y_\alpha\}$ need only be an inverse system).

Corollary 1.14 If $F = \{f_\alpha\}$ and $G = \{g_\alpha\}$ are maps of the system $\{X_\alpha, \varphi_\alpha^{\ \beta}\}_\Lambda$ to the system $\{Y_\beta, \psi_\alpha^{\ \beta}\}_\Lambda$ and $f_\alpha \simeq g_\alpha$ for each $\alpha \in \Lambda$, then $F' = G'$.

Proof $F'< x_\alpha > = < f_\alpha(x_\alpha)> = < g_\alpha(x_\alpha)> = G'< x_\alpha >$.

Definition 1.15 If $\{X_\alpha, \varphi_\alpha^{\ \beta}\}_\Lambda$ is an h-inverse system, then it is h-ordered iff for each $\alpha \in \Lambda$, X_α has the homotopy type of some space Y_α, $X_\alpha \overset{h_\alpha}{\simeq} Y_\alpha$, such that $\{Y_\alpha, \psi_\alpha^{\ \beta}\}_\Lambda$ is an inverse system where $\psi_\alpha^{\ \beta}$ is induced by $\varphi_\alpha^{\ \beta}$ and $H = \{h_\alpha\}$ is a map of the systems.

The system $\{X_\alpha, \varphi_\alpha^{\ \beta}\}_\Lambda$ is strongly h-ordered iff it is h-ordered and $\varprojlim X_\alpha \simeq \varprojlim Y_\alpha$.

Proposition 1.16 If $\{X_\alpha, \varphi_\alpha^{\ \beta}\}_J$ is strongly h-ordered, then there exists an inverse system $\{Y_\alpha, \psi_\alpha^{\ \beta}\}_J$ ordered by inverse inclusion, such that $\varprojlim X_\alpha \simeq \bigcap Y_\alpha$, all spaces compact.

Proof Since $\{X_\alpha\}$ is h-ordered, there exists an inverse system $\{\hat{Y}_\alpha, \hat{\varphi}_\alpha^{\ \beta}\}_J$ such that $X_\alpha \simeq \hat{Y}_\alpha$ for each $\alpha \in J$.

It is well known that an inverse system may be imbedded in a space in such a way that it is ordered by inverse inclusion and the limit space is invariant. Denote the imbedded system by $\{Y_\alpha, \psi_\alpha{}^\beta\}_J$ and apply the definition of strongly h-ordered systems.

That an h-ordered system will not suffice in Proposition 1.16 is evident from the remark in the proof of Proposition 1.13.

An application of h-inverse systems is the following generalization of Eilenberg's Theorem [8].

<u>Theorem 1.17</u> If X is a compact space, then there exists an h-inverse system of spaces having the homotopy type of triangulable spaces, $\{X_\alpha, \varphi_\alpha{}^\beta\}_\Lambda$, such that $X \simeq \underleftarrow{L}\, X_\alpha$.

<u>Proof</u> Imbed X in a cube I^ξ and let $p_\zeta : I^\xi \to I^\zeta$ denote the projection map, where $\zeta \subset \xi$.

Define an index set $\Lambda = \{(\zeta, M)\}$, where $\zeta \subset \xi$ is finite, $M \subset I^\xi$ is closed with the homotopy type of a triangulable subset of I^ζ, and $p_\zeta(X) \simeq Y \subset \mathrm{Int}M$.

$$\begin{tikzcd} I^\xi \arrow[rr, "p_\zeta"] & & I^\zeta \\ \cup & & \cup \\ X \arrow[rr] \arrow[drr, "p_\zeta|X"'] & & M \\ & & Y \arrow[u, "h"'] \end{tikzcd}$$

Order Λ by: $(\lambda, L) < (\mu, M)$ iff $\lambda \subset \mu$ and $p_\lambda(M) \simeq M_\lambda \subset L$. Then Λ is directed, for if $(\lambda,L), (\mu,M) \in \Lambda$, let $\nu = \lambda \cup \mu$ and let $U = I^\lambda \cap p_\lambda{}^{-1}(L) \cap p_\mu{}^{-1}(M)$. Then $p_\nu(X) \simeq V \subset \mathrm{Int}U \subset I^\nu$ and there exists a space $N \subset \mathrm{Int}U$ closed, such that $p_\nu(X) \simeq V \subset \mathrm{Int}N$ and N is triangulable (see [8]). Thus, $(\nu,N) \in \Lambda$ and (ν,N) follows (λ,L) and (μ,M).

Define an h-inverse system as follows. Let $X_m = M$ and $X_l = L$, where $m = (\mu,M)$ and $l = (\lambda,L)$. Let $\varphi_l{}^m : X_m \to X_l$ be the map defined by $h(p_\lambda|M)$, where $h : p_\lambda(M) \to M_\lambda \subset L$ is a homotopy equivalence given in the definition of the order on Λ, and $m = (\mu,M) > l = (\lambda,L)$.

If $l = (\lambda,L) < m = (\mu,M) < n = (\nu,N)$, then

$$\begin{tikzcd} I^{\mu} \arrow[r, "p_\lambda | I^{\mu}"] & I^{\lambda} \\ \cup & \cup \\ M \arrow[r, "\varphi_1^{\,m}"] \arrow[dr, "p_\lambda | M"'] & L \\ & M_\lambda \arrow[u, "h"'] \end{tikzcd}$$

$$\varphi_1^{\,m}\varphi_m^{\,n} = h_1(p_\lambda | M)\ h_2(p_\mu | N),$$

and $\quad \varphi_1^{\,n} = h_3(p_\lambda | N)$.

Thus $h_3^{-1}\varphi_1^{\,n}(N) \simeq p_\lambda(N) = p_\lambda p_\mu(N) \simeq p_\lambda(h_2 p_\mu(N)) \simeq h_1 p_\lambda(h_2 p_\mu(N))$, but $\varphi_1^{\,m}\varphi_m^{\,n}(N) = h_1 p_\lambda(h_2 p_\mu(N))$, therefore $\varphi_1^{\,n} \simeq \varphi_1^{\,m}\varphi_m^{\,n}$.

Let $X' = \varprojlim\{X_k, \varphi_j^{\,k}\}$ and define a map $f_k : X \to X_k$ by $f_k = p_\zeta | X$. Then $\{f_k\}$ defines a map of the space X to the system $\{X_k, \varphi_j^{\,k}\}$. Let $f = \varprojlim f_k : X \to X'$.

$$\begin{tikzcd} & & X' \arrow[dl] \arrow[dr, "\varphi_k"'] & & X \arrow[ll, "f"'] \arrow[dl, "f_k"] \arrow[dr] & & \\ \cdots & X_j \arrow[l] & & X_k \arrow[ll] & & X_1 \arrow[ll] & \cdots \arrow[l] \end{tikzcd}$$

If x is a point in X' and $\varphi_k(x) = x_k \in X_k$, then x_k is in the path component of some point in $p_\zeta(X)$ by definition of Λ. Thus, if $x \in X_k$, then there exists a point $x' \in I^{\xi}$ such that $p_\zeta(x') \simeq x_k = \varphi_k(x)$, or $x' \in X$.

Thus $f_k(x') = p_\zeta | X\ (x') \simeq x_k$, and $f(x') \simeq x$, or every point $x \in X'$ is in a path component of some point in X.

If $x_1, x_2 \in X$ such that $C(x_1) \neq C(x_2)$, where $C(x)$ denotes the path component of x in X, then there exists a finite set $\alpha \subset \xi$ such that $C(p_\alpha(x_1)) \neq C(p_\alpha(x_2))$, or $C(f_\alpha(x_1)) \neq C(f_\alpha(x_2))$ by definition of f_α. Since $\varphi_\alpha f \simeq f_\alpha$, $C(f(x_1)) \neq C(f(x_2))$ and f is one-to-one on the path components of X to the path components of X'.

Since the corresponding path components of X and X' have the same homotopy type, X and X' have the same homotopy type.

<u>Corollary 1.18</u> If X is a compact space, then there exists an inverse system of triangulable spaces $\{X_\alpha, \varphi_\alpha^{\,\beta}\}$ such that $\varprojlim \{X_\alpha\}$ is homeomorphic to X.

<u>Proof</u> If the system in the proof of Theorem 1.17 is constructed using triangulable spaces, an inverse system results. By removing the homotopies, f is an open one-to-one map of X to $X' = \underleftarrow{L}\, X_\alpha$, (see the proof by Eilenberg [8]).

It should be noted that applications of Theorem 1.17 to iterated inverse systems as in [10] are not possible, since introducing the variant of homotopy type removes all controls on dimension. However, several results in this direction are given in the latter part of Chapter 2. The main observation to bear in mind is that Theorem 1.17 may be applied to the terms of an (h-)inverse system which is either given or obtained from some previous application of Theorem 1.17.

CHAPTER 2 MAPPING THEOREMS, LOCALLY CONSTANT SHEAVES AND CONTINUITY

We shall be interested in obtaining results concerning the invariance of sheaf cohomology on homotopy type, mapping theorems for sheaf cohomology, and continuity theorems for systems which involve h-inverse systems of spaces as base spaces.

Let $\mathfrak{a}_X$ denote the category of sheaves of abelian groups over X, (see [4], [3] and [9] for the basic definitions and properties of sheaf theory). (One could use sheaves of R-modules.)

<u>Definition 2.1</u> Let $f:X \to Y$, $\mathcal{A} \in \mathfrak{a}_X$ and $\mathcal{B} \in \mathfrak{a}_Y$. Then <u>$\mathcal{A}$ is f-isomorphic to $\mathcal{B}$</u> , and we denote $\mathcal{A} \overset{f}{\approx} \mathcal{B}$, iff $\mathcal{A} \approx f^*\mathcal{B}$ and $\mathcal{B} \approx f_*\mathcal{A}$.

Thus $\mathcal{A} \overset{f}{\approx} \mathcal{B}$ implies $\mathcal{A} \approx f^*f_*\mathcal{A}$ and $\mathcal{B} \approx f_*f^*\mathcal{B}$.

<u>Proposition 2.2</u> If $f:X \to Y$ is closed and onto, $\mathcal{B} \in \mathfrak{a}_Y$ and $f^{-1}(y)$ is connected and taut ([4]) in X for all y in Y, then there exists a sheaf $\mathcal{A} \in \mathfrak{a}_X$ such that $\mathcal{A} \overset{f}{\approx} \mathcal{B}$.

<u>Proof</u> Let $\psi: \mathcal{B} \to f_*f^*\mathcal{B}$ be the homomorphism induced by the f-cohomomorphism $\mathcal{B} \to f^*\mathcal{B}$ and the definition of the direct image sheaf.

Then one has the canonical homomorphism

$$f^*: S(U,\mathcal{B}) \to S(f^{-1}(U), f^*\mathcal{B}) ,$$

$$\begin{tikzcd} f^*\mathcal{B} & \mathcal{B} \arrow[l, "f^*"'] \\ f^{-1}(U) \arrow[u, "s'"] \arrow[r, "f"] & U \arrow[u, "s"'] \end{tikzcd}$$

where $S(f^{-1}(U), f^*\mathcal{B}) = S(U, f_*f^*\mathcal{B})$ by definition.

Stalkwise, for each y in Y, ψ is the map

$$\mathcal{B}_y \to S(f^{-1}(y), f^*\mathcal{B}) \overset{\approx}{\to} (f_*f^*\mathcal{B})_y ,$$

where the isomorphism follows from the assumption that $f^{-1}(y)$ is taut.

By definition, $S(f^{-1}(y), f^*\mathcal{B})$ is constant with $\mathcal{B}_y$; since $f^{-1}(y)$ is connected one has $\mathcal{B}_y \approx S(f^{-1}(y), f^*\mathcal{B}) \approx (f_*f^*\mathcal{B})_y$, or ψ is an isomorphism.

Let $\mathcal{A} = f^*\mathcal{B}$, then $\mathcal{A} \overset{f}{\approx} \mathcal{B}$.

<u>Corollary 2.3</u> If X is compact and Y dominates X by $f: X \to Y$ and f is onto, then there exists a sheaf $\mathcal{A} \in \mathfrak{A}_X$ such that $\mathcal{A} \overset{f}{\approx} \mathcal{B}$, where $\mathcal{B} \in \mathfrak{A}_Y$.

<u>Proof</u> If X is compact then f is a closed map and $f^{-1}(y)$ is taut in X,[4].

Since Y dominates X, $f^{-1}(y)$ is connected for all y in Y, for if not, let $x_1, x_2 \in f^{-1}(y)$ such that $C(x_1) \neq C(x_2)$ and let g be the homotopy inverse of f. Then $gf \simeq 1_X$ implies $gf(x_1)$ and $gf(x_2)$ are in the same path component, contradicting the assumption on $f^{-1}(y)$.

Apply Proposition 2.2 to get the desired result.

<u>Definition 2.4</u> A map $f: X \to Y$ is a <u>relative map</u> [12] iff for each open set $U \subset X$ there exists an open set $V \subset Y$ such that $U = f^{-1}(V)$.

<u>Proposition 2.5</u> If $f: X \to Y$ is a relative map and $\mathcal{A} \in \mathfrak{A}_X$ then there exists a sheaf $\mathcal{B} \in \mathfrak{A}_Y$ such that $\mathcal{A} \overset{f}{\approx} \mathcal{B}$.

<u>Proof</u> If $\psi: \mathcal{B} \to \mathcal{A}$ is an f-cohomomorphism of some sheaf $\mathcal{B} \in \mathfrak{A}_Y$ to the sheaf $\mathcal{A} \in \mathfrak{A}_X$, then ψ factors through $f^*\mathcal{B}$ (or $f_*\mathcal{A}$) by some homomorphism $f^*\mathcal{B} \to \mathcal{A}$, (or $\mathcal{B} \to f_*\mathcal{A}$).

This reflects the following natural isomorphisms of functors, (see [4]).

$$\begin{tikzcd}
f^*\mathcal{B} \arrow[d] & f_*\mathcal{A} \arrow[d] \arrow[dl] \\
\mathcal{A} \arrow[d] & \mathcal{B} \arrow[l, "\psi"'] \arrow[ul] \arrow[d] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

$$\mathrm{Hom}(f^*\mathcal{B}, \mathcal{A}) \approx f\text{-cohom}(\mathcal{B}, \mathcal{A}) \approx \mathrm{Hom}(\mathcal{B}, f_*\mathcal{A}).$$

Let $\Phi: \mathrm{Hom}(f^*\mathcal{B}, \mathcal{A}) \overset{\approx}{\to} \mathrm{Hom}(\mathcal{B}, f_*\mathcal{A})$ denote the above natural isomorphism.

If $\mathcal{A} = f^*\mathcal{B}$ one obtains the isomorphism:

$$\Phi': \mathrm{Hom}(f^*\mathcal{B}, f^*\mathcal{B}) \overset{\approx}{\to} \mathrm{Hom}(\mathcal{B}, f_*f^*\mathcal{B}),$$

where we denote $\Phi'(1) = \beta$.

If $\mathcal{B} = f_*\mathcal{A}$ one obtains the isomorphism:

$$\Phi'': \mathrm{Hom}(f^*f_*\mathcal{A}, \mathcal{A}) \overset{\approx}{\to} \mathrm{Hom}(f_*\mathcal{A}, f_*\mathcal{A}),$$

where we denote $\Phi''^{-1}(1) = \alpha$.

If $\chi \in \mathrm{Hom}(f^*\mathcal{B}, \mathcal{A})$, then by the naturality of Φ the following diagram commutes:

$$\begin{tikzcd} \mathrm{Hom}(f^*\mathcal{B}, f^*\mathcal{B}) \arrow[r, "\Phi'"] \arrow[d, "\mathrm{Hom}(f^*\mathcal{B},\ \chi)"] & \mathrm{Hom}(\mathcal{B}, f_*f^*\mathcal{B}) \arrow[d, "\mathrm{Hom}(\mathcal{B}, f_*(\chi))"] \\ \mathrm{Hom}(f^*\mathcal{B}, \mathcal{A}) \arrow[r, "\Phi"] & \mathrm{Hom}(\mathcal{B}, f_*\mathcal{A}) \end{tikzcd}$$

Thus $\Phi(\chi) = f_*(\chi)\ \Phi'(1) = f_*(\chi)\ \beta$.

Let $\alpha = \psi \in \mathrm{Hom}(f^*f_*\mathcal{A}, \mathcal{A})$, where $\mathcal{B} = f_*\mathcal{A}$. Then

$1 = \Phi''(\alpha) = f_*(\alpha)\ \beta \in \mathrm{Hom}(f_*\mathcal{A}, f_*\mathcal{A})$, and $f_*(\alpha)$ is surjective.

Recall, $S(U, f_*\mathcal{A}) = S(f^{-1}(U), \mathcal{A})$, thus

$S(U, f_*f^*f_*\mathcal{A}) = S(f^{-1}(U), f^*f_*\mathcal{A}) \overset{\alpha}{\to} S(f^{-1}(U), \mathcal{A}) = S(U, f_*\mathcal{A})$, and α is onto, that is $f^*f_*\mathcal{A} \overset{\alpha}{\to} \mathcal{A}$ is onto.

Looking at stalks one has $(f^*f_*\mathcal{A})_x \overset{\alpha}{\to} \mathcal{A}_x$, but by the relativeness of f,

$$\begin{tikzcd} f^*f_*\mathcal{A} \arrow[d, "\alpha"'] & f_*f^*f_*\mathcal{A} \arrow[l, "f_*"'] \arrow[d, shift left, "f_*(\alpha)"] \\ \mathcal{A} & f_*\mathcal{A} \arrow[l, "f_*"] \arrow[ul, "f^*"'] \arrow[u, shift left, "\beta"] \end{tikzcd}$$

$$\mathcal{A}_x = \varinjlim_{x\in V} S(V, \mathcal{A}) = \varinjlim_{x\in f^{-1}(U)} S(f^{-1}(U), \mathcal{A}) = \varinjlim_{f(x)\in U} S(U, f_*\mathcal{A}) = (f_*\mathcal{A})_{f(x)} \approx (f^*f_*\mathcal{A})_x .$$

Thus α is an isomorphism (recall sheaf homomorphisms are open), and if $\mathcal{B} = f_*\mathcal{A}$, the proposition follows.

The above ideas are now applied to obtain a mapping theorem relating homotopy type and sheaf cohomology. The following lemma will be needed in the proof of the main theorem. Let $\mathcal{T}^*$ denote the serration functor, (see [3], [4] and [9]).

<u>Lemma 2.6</u> If $f: X \to Y$ is a relative map and $\mathcal{A} \in \mathcal{Q}_X$ then $\mathcal{T}^*f_*\mathcal{A} \approx f_*\mathcal{T}^*\mathcal{A}$.

Proof By the definition of $\mathcal{T}^*$ and $f_* \mathcal{A}$,

$$(\mathcal{T}^0 f_* \mathcal{A})_y = \varinjlim_{y \in V} S(V, \mathcal{T} f_* \mathcal{A}) = \varinjlim_{y \in V} \prod_{y' \in V} (f_* \mathcal{A})_{y'} = \varinjlim_{y \in V} \prod_{y' \in V} \varinjlim_{y' \in U} S(f^{-1}(U), \mathcal{A}),$$

for all y in Y.

Since f is relative,

$$(f_* \mathcal{T}^0 \mathcal{A})_y = \varinjlim_{y \in V} S(f^{-1}(V), \mathcal{T}\mathcal{A}) = \varinjlim_{y \in V} \prod_{\substack{y' \in V \\ y' = f(x)}} \mathcal{A}_x \approx \varinjlim_{y \in V} \prod_{\substack{y' \in V \\ y' = f(x)}} \varinjlim_{y' \in U} S(f^{-1}(U), \mathcal{A}),$$

for all y in Y.

That is, $(\mathcal{T}^0(f_* \mathcal{A}))_y \approx (f_* \mathcal{T}^0 \mathcal{A})_y$ for all y in Y, since

$$\mathcal{A}_x \approx \varinjlim_{f(x) \in U} S(f^{-1}(U), \mathcal{A}) = (f_* \mathcal{A})_{f(x)} .$$

Repeating the argument for $\mathcal{T}^1$, etc. gives the desired result.

Theorem 2.7 If $f: X \to Y$ is a relative map and $\mathcal{A} \in \mathcal{Q}_X$, then there exists a sheaf $\mathcal{B} \in \mathcal{Q}_Y$ such that $\mathcal{A} \overset{f}{\approx} \mathcal{B}$ and $H_c^*(X, \mathcal{A}) \approx H_c^*(Y, \mathcal{B})$.

Proof In view of (2.5), choose $\mathcal{B} = f_* \mathcal{A}$. Then, by definition of the Grothendieck cohomology, (cf. [3], [4] and [9]), $H^n(X, \mathcal{A}) = H^n(S(X, \mathcal{T}^* \mathcal{A}))$, where $\mathcal{T}^* \mathcal{A}$ is the canonical resolution of $\mathcal{A}$ determined by the serration functor [3].

If $\mathcal{B} = f_* \mathcal{A}$, then $S(X, \mathcal{A}) = S(f^{-1}(Y), \mathcal{A}) = S(Y, f_* \mathcal{A}) = S(Y, \mathcal{B})$, and in view of (2.6),

$S(X, \mathcal{T}^n \mathcal{A}) = S(f^{-1}(Y), \mathcal{T}^n \mathcal{A}) = S(Y, f_* \mathcal{T}^n \mathcal{A}) \approx S(Y, \mathcal{T}^n f_* \mathcal{A})$, thus $S(X, \mathcal{T}^* \mathcal{A}) \approx S(Y, \mathcal{T}^* f_* \mathcal{A})$, and $H^*(X, \mathcal{A}) = H^*(Y, f_* \mathcal{A}) = H^*(Y, \mathcal{B})$.

Definition 2.8 A sheaf $\mathcal{A} \in \mathcal{Q}_X$ is locally constant iff for each x in X there exists a neighborhood, N, of x such that $p^{-1}(N) = \mathcal{A} | N$ is constant (trivial), that is $p^{-1}(N)$ has the form $N \times G$, where G is an abelian group.

Let $\underline{\mathcal{Q}}_X$ denote the subcategory of $\mathcal{Q}_X$ of locally constant sheaves on X. Note that the inverse image sheaf of a locally constant sheaf is locally constant.

<u>Theorem 2.9</u> If $f,g:X \to Y$, $f \overset{H}{\simeq} g$, X is compact, and $\mathscr{A} \in \underline{\mathcal{Q}}_Y$, then $f^*\mathscr{A} \approx g^*\mathscr{A}$.

<u>Proof</u> Let $H = \{h_t\}:X \times I \to Y$ be the given homotopy and let $\pi:X \times I \to X$ be the projection map. Then the following diagram is determined.

$$\begin{tikzcd}
h_t^*\mathscr{A} \arrow[dr] \arrow[dddd] & & & & \mathscr{A} \arrow[llll] \arrow[dl] \arrow[dddd] \\
& \pi^*h_t^*\mathscr{A} \arrow[dr, "p_1"'] & & H^*\mathscr{A} \arrow[dl, "p_2"] & \\
& & X \times I \arrow[dll, "\pi"'] \arrow[drr, "H"] & & \\
& & & & \\
X \arrow[rrrr, "h_t"] & & & & Y
\end{tikzcd}$$

Since $H|X \times I = h_t\pi$, $H^*\mathscr{A}|X \times t \approx (\pi^*h_t^*\mathscr{A} = (h_t\pi)^*\mathscr{A})|X \times t$. Since $H^*\mathscr{A}$ and $\pi^*h_t^*\mathscr{A}$ are locally constant sheaves, cover $X \times t$ by neighborhoods determined by the neighborhoods which express the local constantness of each sheaf. By compactness of X reduce the cover to a finite subcover. Thus there exists an $\epsilon > 0$ such that if $X \times (t - \epsilon, t + \epsilon) = M$, then $H^*\mathscr{A}|M = (h_t\pi)^*\mathscr{A}|M$.

Thus $(h_t\pi)^*\mathscr{A}$ is locally constant as a function of t, and since I is connected, $(h_t\pi)^*\mathscr{A}$ is constant as a function of t. Therefore $h_t^*\mathscr{A}$ is constant as a function of t, or $f^*\mathscr{A} = h_0^*\mathscr{A} \approx h_1^*\mathscr{A} = g^*\mathscr{A}$.

The following relations are immediate from Theorem 2.9 and the above results concerning relative maps.

Corollary 2.10 Assuming the hypothesis of Theorem 2.9:

(2.10a) If $X = Y$ and f is a homotopy equivalence, then $f^*\mathcal{A} \approx \mathcal{A}$.

(2.10b) If X is contractible to a point x_o, then every locally constant sheaf on X is constant.

(2.10c) If f and g are relative maps, then $f^*f_*g^*\mathcal{A} \approx g^*g_*f^*\mathcal{A}$.

(2.10d) If f in part a is a relative map, then $\mathcal{A} \approx f^*\mathcal{A} \approx f^*f_*\mathcal{A} \approx \ldots$.

(2.10e) If $\mathcal{A} \in \mathcal{Q}_Y$ and $f_*\mathcal{A}$ and $g_*\mathcal{A}$ are locally constant, then $\mathcal{A} \in \underline{\mathcal{Q}}_Y$ and $g^*f_*\mathcal{A} \approx f^*g_*\mathcal{A} \approx \mathcal{A}$.

Proof In general $\mathcal{A} \not\approx f^*f_*\mathcal{A}$, $\mathcal{A} \not\approx f_*f^*\mathcal{A}$, $\mathcal{A} \not\approx f^*g^*\mathcal{A}$, and $\mathcal{A} \not\approx f_*g_*\mathcal{A}$, where g is the homotopy inverse of f.

Part a follows immediately from Theorem 2.9 with $g = 1_X$.

Part b is immediate from part a, where the trivial sheaf is determined by $\mathcal{A}_{x_o}$.

For part c, by Proposition 2.5 we know $f^*f_*f^*\mathcal{A} \approx f^*\mathcal{A}$ and $g^*g_*g^*\mathcal{A} \approx g^*\mathcal{A}$, thus $g^*g_*f^*\mathcal{A} \approx g^*g_*g^*\mathcal{A} \approx g^*\mathcal{A} \approx f^*\mathcal{A} \approx f^*f_*f^*\mathcal{A} \approx f^*f_*g^*\mathcal{A}$.

Applying parts a and c with $g = 1_X$ one obtains $f^*f_*\mathcal{A} \approx f^*\mathcal{A} \approx f^*f_*f^*\mathcal{A}$. Continuing in this manner one obtains the sheaf isomorphisms for part d.

For e use the fact that by the relativeness of the maps, $g^*g_*\mathcal{A} \approx \mathcal{A} \approx f^*f_*\mathcal{A}$ on the one hand, but by Theorem 2.9, $f^*g_*\mathcal{A} \approx g^*g_*\mathcal{A}$ and $g^*f_*\mathcal{A} \approx f^*f_*\mathcal{A}$.

Definition 2.11 If $\{\mathcal{A}_\alpha, \Phi^\alpha{}_\beta\}_\Lambda$ is a direct system of sheaves on X, then $\varinjlim \mathcal{A}_\alpha$ is the sheaf generated by the presheaf $U \to \varinjlim S(U, \mathcal{A}_\alpha)$, (see [3], [4] or [9]).

When referring to sheaves which are limits of locally constant sheaves, we denote the category by $\underset{\to}{\mathcal{Q}}_X$. Note that $(\varinjlim \mathcal{A}_\alpha)_x \approx \varinjlim(\mathcal{A}_\alpha)_x$ for direct limits of sheaves, by the properties of direct limits.

<u>Theorem 2.12</u> If $\mathcal{A} \in \underrightarrow{\mathcal{Q}}_Y$, $f,g:X \to Y$, $f \overset{H}{\simeq} g$ and X is compact, then $f^* \mathcal{A} \approx g^* \mathcal{A}$.

<u>Proof</u> By Theorem 2.9, $f^*\mathcal{A}_\alpha \approx g^*\mathcal{A}_\alpha$ for all α. By the properties of direct limits of sheaves (noted above),

$$(\underrightarrow{L}\, f^*\mathcal{A}_\alpha)_x \approx \underrightarrow{L}\, (f^*\mathcal{A}_\alpha)_x = \underrightarrow{L}\, (\mathcal{A}_\alpha)_{f(x)} = (\underrightarrow{L}\,\mathcal{A}_\alpha)_{f(x)} \approx \mathcal{A}_{f(x)} = (f^*\mathcal{A})_x,$$

or $f^* \mathcal{A} \approx \underrightarrow{L}\, f^*\mathcal{A}_\alpha$. Similarly for $g^* \mathcal{A}$.

By Theorem 2.9 and the functor properties of f^* and g^*, $\underrightarrow{L}\, f^*\mathcal{A}_\alpha \approx \underrightarrow{L}\, g^*\mathcal{A}_\alpha$, or $f^* \mathcal{A} \approx g^* \mathcal{A}$.

$$\begin{tikzcd}
f^*\mathcal{A}_\alpha \arrow[dr, "\approx"] \arrow[dd] & & \mathcal{A}_\alpha \arrow[ll, "f^*"'] \arrow[dl] \arrow[dd] \\
 & g^*\mathcal{A}_\alpha \arrow[dd] & \\
f^*\mathcal{A}_\beta \arrow[dr, "\approx"'] & & \mathcal{A}_\beta \arrow[ll] \arrow[dl, "g^*"] \\
 & g^*\mathcal{A}_\beta &
\end{tikzcd}$$

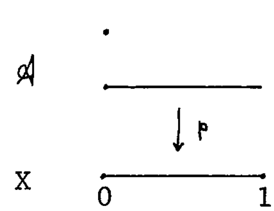
<u>Corollary 2.13</u> Assuming the hypothesis of Theorem 2.12:
(2.13a) If $X = Y$ and f is a homotopy equivalence, then $\mathcal{A} \approx f^* \mathcal{A}$.
(2.13b) If X is contractible to a point x_0, then $\mathcal{A}$ is trivial.

<u>Proof</u> Let $g = 1_X$ in Theorem 2.12 to obtain part a.

Part b follows from part a with the trivial sheaf determined by $(\mathcal{A})_{x_0}$, (recall $\underrightarrow{L}\, \mathfrak{B} = \mathfrak{B}$).

<u>Example 2.14</u> That not all sheaves are limits of locally constant sheaves follows from the example of a nontrivial sheaf on a contractible space, in view of Corollary 2.13.

Let $X = I$, and $\mathcal{A}$ be the sheaf which is trivial, that is zero, on $(0,1]$ and $\mathcal{A}_0 = J_2$.



The important fact to observe is that sheaves which are limits of locally constant sheaves behave similar to locally constant sheaves under homotopies of maps of the base spaces.

<u>Definition 2.15</u> Let $H:X \times I \to Y$ be a homotopy and G an open cover of Y. Then H is a <u>G-homotopy</u> iff for each x in X there exists a $U \in G$ such that $H(x,I) \subset U$.

If $f:X \to Y$ is a homotopy equivalence with homotopy inverse g, $\mathcal{A} \in \underline{\mathfrak{A}}_Y$ with respect to some open cover G of Y, and $F = f^{-1}(G)$, then f is an <u>(F,G)-homotopy equivalence</u> (relative to $\mathcal{A}$) iff $fg \simeq 1_Y$ by a G-homotopy and $gf \simeq 1_X$ by an F-homotopy.

<u>Theorem 2.16</u> If $\mathcal{A} \in \underline{\mathfrak{A}}_Y$ (with respect to a cover G of Y), $f:X \to Y$ is an (F,G)-homotopy equivalence, X and Y are compact, then there exists a sheaf $\mathcal{B} \in \underline{\mathfrak{A}}_X$ such that $H^*_c(X,\mathcal{B}) \approx H^*_c(Y,\mathcal{A})$.

<u>Proof</u> Let $\mathcal{B} = f^*\mathcal{A}$ and $(f,f^*):S(Y,\mathcal{A}) \to S(X,f^*\mathcal{A})$ be the homomorphism defined by $(f,f^*)(s)(x) = f^*_x\ s\ f(x)$. If g is the homotopy inverse of f, let $(g,g^*):S(X,f^*\mathcal{A}) \to S(Y,\mathcal{A})$ be the homomorphism defined by $(g,g^*)(s')(y) = g^*_y\ s'\ g(y)$.

Recall $\mathcal{A} \approx g^*f^*\mathcal{A}$ and $f^*\mathcal{A} \approx f^*g^*f^*\mathcal{A}$ by Corollary 2.10a.

$$\begin{tikzcd}
f^*\mathcal{A} \arrow[d] & \mathcal{A} \arrow[l, "f^*"'] \arrow[d, shift right] & f^*\mathcal{A} \arrow[l, "g^*"'] \arrow[d, shift right] \\
X \arrow[r, "f"] & Y \arrow[r, "g"] \arrow[u, shift right, "s"'] & X \arrow[u, shift right, "s'"']
\end{tikzcd}$$

Note that (f,f^*) and (g,g^*) are onto, since $\operatorname{Im} f \cap U \neq \emptyset$ for all U in G by the (F,G)-homotopy equivalence property on f. Similarly for g.

Combining these homomorphisms one has:

$$\begin{aligned}
(f,f^*)(g,g^*)(s')(x) &= (f,f^*)(g^*_{f(x)}s'g)(x) \\
&= f^*_x\ g^*_{f(x)}s'(gf(x)) \\
&= (gf,f^*g^*)(s')(x),
\end{aligned}$$

and

$$\begin{aligned}
(g,g^*)(f,f^*)(s)(y) &= (g,g^*)(f^*_{g(y)}s\ f)(y) \\
&= g^*_y\ f^*_{g(y)}s\ (fg(y)) \\
&= (fg,\ g^*f^*)(s)(y).
\end{aligned}$$

Stalkwise these homomorphisms are isomorphisms, by the definition of the inverse image sheaf. The image of each of these homomorphisms may be extended in a natural way to the whole group, since f is an (F,G)-homotopy equivalence (and g is a (G,F)-homotopy equivalence). Thus $S(X,f^*\mathcal{A}) \approx S(Y,\mathcal{A})$.

Similarly, $S(X,\mathcal{J}^o f^*\mathcal{A}) \approx S(Y,\mathcal{J}^o\mathcal{A})$, thus $S(X,T(\mathcal{J}^o f^*\mathcal{A}/f^*\mathcal{A})) \approx S(Y,T(\mathcal{J}^o\mathcal{A}/\mathcal{A}))$, or $S(X,\mathcal{J}^1 f^*\mathcal{A}) \approx S(Y,\mathcal{J}^1\mathcal{A})$, and by iteration one has $S(X,\mathcal{J}^* f^*\mathcal{A}) \approx S(Y,\mathcal{J}^*\mathcal{A})$, or $H^*(X,f^*\mathcal{A}) \approx H^*(Y,\mathcal{A})$.

<u>Corollary 2.17</u> Suppose $\{Y_\alpha, \psi_\alpha{}^\beta\}_\Lambda$ is an inverse system of compact spaces, $\{\mathcal{A}_\alpha, \Psi^\alpha{}_\beta\}_\Lambda$ is a direct system of locally constant sheaves on $\{Y_\alpha\}$ with respect to covers $\{G_\alpha\}$ on $\{Y_\alpha\}$, $\{X_\alpha, \varphi_\alpha{}^\beta\}_\Lambda$ is an inverse system of compact spaces with $F = \{f_\alpha : X_\alpha \to Y_\alpha \mid f_\alpha$ is an (F_α,G_α)-homotopy equivalence$\}$ a map of the systems. Let $Y = \underleftarrow{L}\, Y_\alpha$, $X = \underleftarrow{L}\, X_\alpha$, $\mathcal{A} = \underrightarrow{L}\, \Psi_\alpha^* \mathcal{A}_\alpha$ and $f^*\mathcal{A} = \underrightarrow{L}\, \varphi_\alpha^* f_\alpha^* \mathcal{A}_\alpha$, where $f = \underleftarrow{L}\, f_\alpha$. Then $H^*(X,f^*\mathcal{A}) \approx H^*(Y,\mathcal{A})$.

<u>Proof</u> By Theorem 2.16, $H^*(X_\alpha,f_\alpha^*\mathcal{A}_\alpha) \approx H^*(Y_\alpha,\mathcal{A}_\alpha)$, and by continuity $\underrightarrow{L}\, H^*(Y_\alpha,\mathcal{A}_\alpha) \approx H^*(Y,\mathcal{A})$, and $\underrightarrow{L}\, H^*(X_\alpha,f_\alpha^*\mathcal{A}_\alpha) \approx H^*(X,f^*\mathcal{A})$, where $f^*\mathcal{A} = \underrightarrow{L}\, \varphi_\alpha^* f_\alpha^* \mathcal{A}_\alpha \approx \underrightarrow{L}\, f^*\psi_\alpha^*\mathcal{A}_\alpha$. Thus $H^*(Y,\mathcal{A}) \approx H^*(X,f^*\mathcal{A})$.

In order to obtain a theorem similar to Theorem 2.16 for the case $\mathcal{A} \in \underrightarrow{\mathcal{Q}}_X$ we prove an existence theorem in which the conditions of the hypothesis of Corollary 2.17 are satisfied.

<u>Theorem 2.18</u> Let $\mathcal{A} \in \underrightarrow{\mathcal{Q}}_Y$, $g : X \to Y$ an (F_α,G_α)-homotopy equivalence for all $\alpha \in \Lambda$, where $\mathcal{A} = \underrightarrow{L}\, \mathcal{A}_\alpha$ and $\Lambda = I^Y$, X and Y are compact spaces. Then there exists a sheaf $\mathcal{B} \in \mathcal{Q}_X$ such that $H^*(Y,\mathcal{A}) \approx H^*(X,\mathcal{B})$.

Proof Imbed Y in a cube $\prod_{\alpha\in\Lambda} I_\alpha$, and construct an inverse system of finite polyhedra, $\{Y_\alpha, \varphi_\alpha{}^\beta\}_\Lambda$, such that $Y = \varprojlim Y_\alpha$, (see Theorem 1.17 above,[8]).

Let Z_{φ_α} be the mapping cylinder of the projection map $\varphi_\alpha : X \to Y_\alpha$, (recall $Z_{\varphi_\alpha} \simeq Y_\alpha$). Then $\{Z_{\varphi_\alpha}, \widetilde{\varphi}_\alpha{}^\beta\}_\Lambda$ is an inverse system, where $\widetilde{\varphi}_\alpha{}^\beta$ is induced by $\varphi_\alpha{}^\beta$, and $\varprojlim Z_{\varphi_\alpha} = Y \times I \simeq Y$.

Let $\mathcal{A}_\alpha \times [0,1)$ be the sheaf on $Y \times [0,1)$ which satisfies $(\mathcal{A}_\alpha \times [0,1))_{(x,t)} = (\mathcal{A}_\alpha)_x$ for all $(x,t) \in Y \times [0,1)$. Note that $Y \times [0,1)$ is open in Z_{φ_α} and thus locally closed. Extend $\mathcal{A}_\alpha \times [0,1)$ by zero to Z_{φ_α} and denote this sheaf by $\widetilde{\mathcal{A}}_\alpha$.

Clearly, $\mathcal{A}_\alpha \in \underline{\mathfrak{a}}_Y$ implies $\mathcal{A}_\alpha \times [0,1)$ is locally constant over $Y \times [0,1)$, while $\varinjlim \widetilde{\mathcal{A}}_\alpha = \widetilde{\mathcal{A}} = (\mathcal{A} \times [0,1)) \cup \theta$. Carry out a similar construction on X and the system $\{g^*\mathcal{A}_\alpha\}$.

Clearly $H^*(Y,\mathcal{A}) \approx H^*(Y \times [0,1), \mathcal{A} \times [0,1))$, and, assuming compact supports, $H^*(Y \times [0,1), \mathcal{A} \times [0,1)) \approx H^*(Y \times I, \widetilde{\mathcal{A}})$. Similarly for X and $\{g^*\mathcal{A}_\alpha\}$.

By continuity,

$H^*(Y,\mathcal{A}) \approx H^*(Y \times I, \widetilde{\mathcal{A}}) = H^*(\varprojlim Z_{\varphi_\alpha}, \varinjlim \widetilde{\mathcal{A}}_\alpha) \approx \varinjlim H^*(Z_{\varphi_\alpha}, \widetilde{\mathcal{A}}_\alpha)$, and

$H^*(X,g^*\mathcal{A}) \approx H^*(X \times I, \widetilde{g^*\mathcal{A}}) = H^*(\varprojlim Z_{\psi_\alpha}, \varinjlim \widetilde{g}^*\widetilde{\mathcal{A}}_\alpha) \approx \varinjlim H^*(Z_{\psi_\alpha}, \widetilde{g}^*\widetilde{\mathcal{A}}_\alpha)$,

where $\widetilde{g}$ is the extension of g to $Z_{\psi_\alpha} \to Z_{\psi_\alpha}$, and $\widetilde{g^*\mathcal{A}} = \varinjlim \widetilde{g}^*\widetilde{\mathcal{A}}_\alpha$.

By Theorem 2.16, $H^*(Z_{\varphi_\alpha}, \widetilde{\mathcal{A}}_\alpha) \approx H^*(Z_{\psi_\alpha}, \widetilde{g}^*\widetilde{\mathcal{A}}_\alpha)$ for all $\alpha \in \Lambda$, since the condition that g is an (F_α, G_α)-homotopy equivalence implies $\widetilde{g}$ is an $(\widetilde{F}_\alpha, \widetilde{G}_\alpha)$-homotopy equivalence, where $\widetilde{G}_\alpha$ is the cover $G_\alpha \times [0,1) \cup Y_\alpha$. Thus $H^*(Y,\mathcal{A}) \approx H^*(X, g^*\mathcal{A})$ and $\mathcal{B} = g^*\mathcal{A}$.

The material above on homotopy systems, locally constant sheaves and mapping theorems for locally constant sheaves and limits of locally constant sheaves is unified in the following material on continuity theorems.

Definition 2.19 A homotopy-inverse system of spaces, $\{X_\alpha, \varphi_\alpha{}^\beta\}_\Lambda$, is called an (α,β)-homotopy-inverse system of spaces iff whenever $\alpha,\beta,\gamma \in \Lambda$ and $\alpha < \beta < \gamma$, then $\varphi_\alpha{}^\beta\varphi_\beta{}^\gamma \simeq \varphi_\alpha{}^\gamma$ by an F_α-homotopy for some cover F_α of X_α.

Theorem 2.20 If $\{X_\alpha, \varphi_\alpha{}^\beta\}_\Lambda$ is an (α,β)-homotopy-inverse system of locally path connected compact spaces with respect to covers $\{F_\alpha\}$ determined by some system of locally constant sheaves $\{\mathcal{A}_\alpha, \Phi^\alpha{}_\beta\}_\Lambda$ on $\{X_\alpha, \varphi_\alpha{}^\beta\}_\Lambda$, where the $\Phi^\alpha{}_\beta$ are $\varphi_\alpha{}^\beta$-cohomomorphisms, then $H_c^*(X,\mathcal{A}) = H_c^*(\varprojlim X_\alpha, \varinjlim \varphi_\alpha^*\mathcal{A}_\alpha) \approx \varinjlim H_c^*(X_\alpha,\mathcal{A}_\alpha)$.

Proof The theorem is immediate from the observation that $\varphi_\alpha{}^\beta\varphi_\beta{}^\gamma \simeq \varphi_\alpha{}^\gamma$ by an F_α-homotopy, where $\mathcal{A}_\alpha$ is locally constant with respect to F_α, implies that the systems $\{X_\alpha\}$ and $\{\mathcal{A}_\alpha\}$ behave identically to the usual situation (see [3], [4] or [9]).

Thus if $X = \varprojlim X_\alpha$ and $\varphi_\alpha : X \to X_\alpha$ is the projection map, then $\varphi_\alpha^*\mathcal{A}_\alpha$ is a locally constant sheaf on X with respect to the cover $\varphi_\alpha^{-1}(F_\alpha)$ and if $\mathcal{A} = \varinjlim \varphi_\alpha^*\mathcal{A}_\alpha$, it is known that $\varinjlim H^*(X,\varphi_\alpha^*\mathcal{A}_\alpha) \approx H^*(X,\mathcal{A})$, (see [4]).

Corollary 2.21 If X is a compact space and $\mathcal{A}$ is the limit of sheaves which are members of $\underset{\to}{\mathfrak{A}}_X$, then $H^*(X,\mathcal{A})$ may be expressed as a doubly iterated limit of cohomologies of spaces of the homotopy type of polyhedra with coefficients in locally constant sheaves.

Proof Consider the diagram:

$$\begin{array}{ccccccc}
 & & \downarrow & & \downarrow & & \\
 & & H^*(X_i', \mathcal{A}_{\alpha_i}) & & H^*(X_i', \mathcal{A}_{\beta i}) & & \\
 & & \downarrow & & \downarrow & & \\
 & & H^*(X_j', \mathcal{A}_{\alpha_j}) & & H^*(X_j', \mathcal{A}_{\beta_j}) & & \\
 & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \\
 & & \Downarrow & & \Downarrow & & \\
\cdots \longrightarrow & & H^*(X, \mathcal{A}_\alpha) & \longrightarrow & H^*(X, \mathcal{A}_\beta) & \longrightarrow \cdots & \Longrightarrow H^*(X, \mathcal{A})
\end{array}$$

The lower horizontal system follows from the usual continuity theorem, where $\mathcal{A} = \underrightarrow{L}\, \mathcal{A}_\alpha$, and $\mathcal{A}_\alpha \in \underrightarrow{\mathcal{Q}}_X$ for all α. The vertical systems are obtained by applying Theorem 1.17.

An h-inverse system $\{X_i, \varphi_j{}^i\}$ is thus obtained, and by passing to the mapping cylinder Z_{φ_i}, as in the proof of Theorem 2.18 above, one has the system $\{H^*(X_i', \mathcal{A}_{\alpha_i})\}$, where $\underrightarrow{L}\, H^*(X_i', \mathcal{A}_{\alpha_i}) \approx H^*(X, \mathcal{A}_\alpha)$ by Theorem 2.20, where $\underleftarrow{L}\, X_i' = \underleftarrow{L}\, Z_{\varphi_i} = \underleftarrow{L}\, X_i \times I \simeq X$.

If constant (trivial) sheaves are present, the (α,β)-homotopy condition in Theorem 2.20 may be dropped and the following Corollary is immediate.

<u>Corollary 2.22</u> If $\{X_\alpha, \varphi_\alpha{}^\beta\}_\Lambda$ is an h-inverse system of locally path connected compact spaces, then $H_c^*(\underleftarrow{L}\, X_\alpha, R) \approx \underrightarrow{L}\, H_c^*(X_\alpha, R)$, (Alexander-Čech cohomology).

It should be noted that an analogous result on continuity may be obtained with h-direct systems and Čech homology.

CHAPTER 3 SHEAVES OF H-SPACES

Definition 3.1 Let $\{H_x, \mu_x\}_X$ be a collection of H-spaces with corresponding multiplications indexed by a given space X.

Let $\mathcal{H} = \bigcup_X H_x$ and $p: \mathcal{H} \to X$ be defined by $p(H_x) = x$. Given a point $a \in \mathcal{H}$, a set which contains a, N, is open iff p(N) is open in X, where $N \cap H_x$ is open and path connected for all x. Such sets form a basis for a topology on $\mathcal{H}$.

If for each point in $\mathcal{H}$ a path connected neighborhood in the basis above exists, and the operations μ_x are continuous in $\mathcal{H}$, then we call the structure ($\mathcal{H}$,p,X), (or $\mathcal{H}$ when X and p are understood), a sheaf of H-spaces. Note that $\mathcal{H}$ itself need not be an H-space, (see Example 3.5 below).

At this point the H-space structures are assumed to satisfy $\mu_x(a,e_x) \simeq \mu_x(e_x,a) \simeq a$ for all $a \in H_x$ and some point $e_x \in H_x$, that is a two-sided identity up to homotopy must exist. It is not assumed that the spaces satisfy the homotopy extension property.

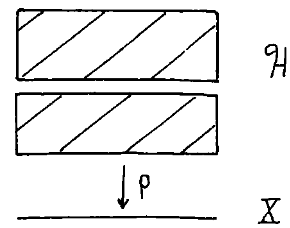
Continuity of μ_x in $\mathcal{H}$ is satisfied if given a path connected neighborhood N of $\mu_x(a_1,a_2)$, where $a_1,a_2 \in H_x$, there exist path connected neighborhoods N_1 of a_1 and N_2 of a_2 such that if $a_1' \in N_1$ and $a_2' \in N_2$ and $p(a_1') = p(a_2') = x'$, then $\mu_{x'}(a_1',a_2') \in N$.

If $a_1' \in N_1(a_1)$ and $a_2 \in C(e_x)$, the path component in H_x of e_x, and $a_2' \in N(a_2)$ such that $p(a_1') = p(a_2') = x'$, then $\mu_{x'}(a_1',a_2') \in N$ by continuity. But $\mu_x(a_1,a_2) \simeq \mu_x(a_1,e_x) \simeq a_1$, thus $\mu_{x'}(a_1',a_2') \simeq a_1 \simeq a_1'$ for all $a_2' \in N_2$. As a result, if the path component in $\mathcal{H}$ of e_x intersects $H_{x'}$, it does so on the path component of $e_{x'}$.

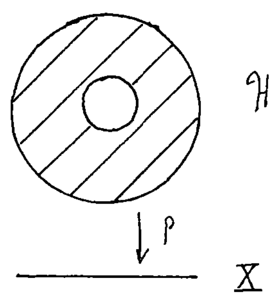
Clearly a sheaf of algebraic structures is a sheaf of H-spaces.

We list some examples of sheaves of H-spaces.

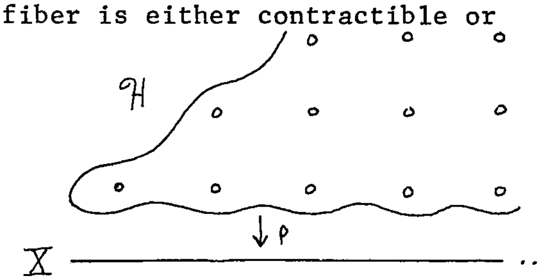
<u>Example 3.2</u> Let X be a given space and $\mathcal{H} = X \times ([-1,-\frac{1}{2}]\cup[\frac{1}{2},1])$. Define μ_x on $H_x = \{x\} \times ([-1,-\frac{1}{2}]\cup[\frac{1}{2},1])$ by $\mu_x(a,b) = 2ab$. Then if e_x is any point in $[\frac{1}{2},1]$, H_x is an H-space and $\mathcal{H}$ is a sheaf of H-spaces which is constant (trivial) in the sense of sheaf theory.



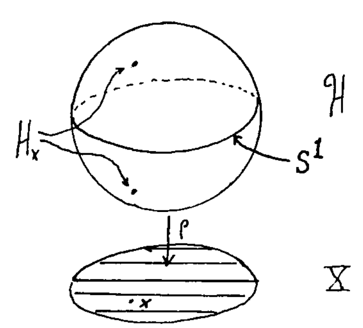
<u>Example 3.3</u> Let $\mathcal{H}$ be a solid annulus in R^2 and let X = I. Then $H_x = p^{-1}(x)$ is either contractible or has two path components. In the first case H_x is trivially an H-space and in the second case H_x has the homotopy type of the H-space in Example 3.2 above.



<u>Example 3.4</u> Example 3.3 may be generalized to an "annulus" with an infinite number of holes. Each fiber is either contractible or a finite union of contractible sets and is therefore an H-space trivially.



<u>Example 3.5</u> For an example of a sheaf of H-spaces which is not an H-space, let $\mathcal{H} = S^2$ and X = D, the projection of S^2 in R^2. Then $p^{-1}(x) \approx J_2$ if x is an interior point of X and $p^{-1}(x) = s \in S^1$ if $x \in \partial X$.



Both possibilities of the fibers are H-spaces, however the only spheres which are H-spaces are S^o, S^1, S^3 and S^7,([1]), so $\mathcal{H}$ is not an H-space.

<u>Definition 3.6</u> Let $\theta: X \to \mathcal{H}$ be the section ($p\theta = 1_X$) which satisfies: $\theta(x) \in C(e_x) \subset H_x$. That θ exists and is continuous follows from the observation on the behavior in $\mathcal{H}$ of the path components of e_x above, (3.1). (θ is not unique.)

<u>Definition 3.7</u> Let $S(X,\mathcal{H})$ denote the collections of global sections of $\mathcal{H}$, with the compact-open topology. Define a multiplication on $S(X,\mathcal{H})$ as follows: if $s,t \in S(X,\mathcal{H})$ then

$$\mu(s,t)(x) = \mu_x(s(x),t(x)),\ x \in X.$$

The multiplication μ is continuous by the following argument: Let U be an open set about $\mu(f,g)$. Then there exists a finite collection of sub-basic sets $\{M(C_i,O_i)\}_{i\in\pi}$ such that

$$\mu(f,g) \in \bigcap_{i\in\pi} M(C_i,O_i) \subset U ,$$

(we may assume that the sets O_i are path connected).

Since the multiplications μ_x are continuous in $\mathcal{H}$, choose neighborhoods U_i of $f(x)$ and V_i of $g(x)$ such that if $f_i(x') \in U_i$ and $g_i(x') \in V_i$ then $\mu(f_i,g_i)(x') = \mu_{x'}(f_i(x'),g_i(x')) \in O_i$.

Thus $f \in \bigcap M(C_i,U_i)$ and $g \in \bigcap M(C_i,V_i)$, and if $f' \in \bigcap M(C_i,U_i)$ and $g' \in \bigcap M(C_i,V_i)$, then $\mu(f',g') \in \bigcap M(C_i,O_i)$, since $\mu(f',g')(x) = \mu_x(f'(x),g'(x)) \in \bigcap O_i$ for all $x \in \bigcap C_i$, (for sections, $\bigcap M(C_i,O_i) = M(\bigcap C_i,\bigcap O_i)$).

We shall use the notation $s\odot t$ for $\mu(s,t)$. The identity of $S(X,\mathcal{H})$ is the section θ in view of the equation:

$$\mu(s,\theta)(x) = \mu_x(s(x),\theta(x)) \simeq \mu_x(s(x),e_x) \simeq s(x),\ x \in X.$$

Thus $S(X,\mathcal{H})$ is an H-space under the multiplication μ. If $U \subset X$, then $S(U,\mathcal{H}|U)$ is an H-space under the multiplication $\mu|S(U,\mathcal{H}|U)$, (we shall use the notation $S(U,\mathcal{H})$ for $S(U,\mathcal{H}|U)$).

Let F be a family of supports on X and $U \subset X$. Then $S_{F|U}(U,\mathcal{H})$ is the collection of sections in $S(U,\mathcal{H})$ which satisfies:

$$|s| \in F|U \text{ for all } s \in S_{F|U}(U,\mathcal{H}),$$

where $F|U = \{A \subset U | \ A \in F\}$. The collection $S_{F|U}(U,\mathcal{H})$ is closed under the multiplication $\mu_{F|U} = \mu|S_{F|U}(U,\mathcal{H})$, for if $s,t \in S_{F|U}(U,\mathcal{H})$, then $|s|$ and $|t| \in F|U$, and since

$$|\mu_{F|U}(s,t)|^{\sim} = \{x \in X | \ \mu_{F|U}(s,t) \in C(e_x)\} \supset |s|^{\sim} \cup |t|^{\sim},$$

it follows that $|\mu_{F|U}(s,t)| \subset |s| \cap |t|$ and thus $|\mu_{F|U}(s,t)| \in F|U$. Also $\theta \in S_{F|U}(U,\mathcal{H})$, since $|\theta| = \emptyset$, and it follows that $S_{F|U}(U,\mathcal{H})$ is an H-space.

Definition 3.8 Let $\mathcal{H}$ and $\mathcal{K}$ be sheaves of H-spaces on X. A map of sheaves of H-spaces is a map $\alpha: \mathcal{H} \to \mathcal{K}$ such that

$$\begin{tikzcd} \mathcal{H} \arrow[rr, "\alpha"] \arrow[dr, "p_1"'] & & \mathcal{K} \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

(3.8a) $p_2\alpha = p_1$,

(3.8b) α is an H-homomorphism on fibers.

If $\alpha: \mathcal{H} \to \mathcal{K}$ is a map of sheaves of H-spaces as above, then $\mathcal{H} \approx \mathcal{K}$ iff $\alpha_x: \mathcal{H}_x \to \mathcal{K}_x$ is a homotopy equivalence for all $x \in X$.

The map α induces a map $\alpha': S(X,\mathcal{H}) \to S(X,\mathcal{K})$ by the rule $\alpha'(s) = \alpha s$, $s \in S(X,\mathcal{H})$. The section functor S is thus a functor on the category of sheaves of H-spaces on some fixed base space to the category of H-spaces.

Definition 3.9 If $U \subset X$, then $S(U,\mathcal{H})$ is an H-space by Definition 3.7. Let $U,V \subset X$ be open sets with $V \subset U$ and define a map

$$r^U_V: S(U,\mathcal{H}) \to S(V,\mathcal{H})$$

by restriction. This map is an H-homomorphism by the following observation: $r^U_V(s \odot t) = (s\odot t)|V = s|V \odot t|V = r^U_V(s) \odot r^U_V(t)$.

Also, if $W \subset V \subset U$, then $r^U_W = r^V_W \, r^U_V$. Thus $\Sigma = \{S(U,\mathcal{H}), r^U_V\}$ is a direct system. Let $\tilde{\Sigma} = \{S(U,\mathcal{H}), r^U_V\}$ be the h-direct system determined by Σ as follows: if $W \subset V \subset U$, then $r^U_W \simeq r^V_W \, r^U_V$ in $S(W,\mathcal{H})$.

Take the limit over the collection of sets U about some fixed point x in X:

$$\mathcal{H}_x = \underset{x \in U}{\underrightarrow{L}} \{S(U,\mathcal{H}), r^U_V\} = \sum_{x \in U} S(U,\mathcal{H})/\sim .$$

The equivalence amounts to requiring that if $s \in S(U,\mathcal{H})$ and $t \in S(V,\mathcal{H})$, then $s \sim t$ iff there exists a neighborhood $W \subset U \cap V$ of x such that $r^U_W(s) \simeq r^V_W(t)$, (this need not be a vertical homotopy).

The induced multiplication on $\mathcal{H}_x$ is defined by :

$$\tilde{\mu}_x(\langle s\rangle,\langle t\rangle) = \langle s|W \odot t|W\rangle,$$

where $x \in W \subset U \cap V$, $s \in S(U,\mathcal{H})$ and $t \in S(V,\mathcal{H})$.

<u>Lemma 3.10</u> $\mathcal{H}_x$ is an H-space.

<u>Proof</u> The multiplication is well defined, since if $s' \in \langle s\rangle$, then $s' \sim s$ on some neighborhood U', so

$$\tilde{\mu}_x(\langle s'\rangle,\langle t\rangle) = \langle s'|W' \odot t|W'\rangle,$$

where $W' \subset U' \cap V$, but $s'|W' \odot t|W' \sim (s|W \odot t|W)|W''$, for some $W'' \subset W' \cap W$, thus $\langle s'|W' \odot t|W'\rangle = \langle s|W \odot t|W\rangle$.

Clearly $\tilde{\mu}_x(\langle s\rangle,\langle\theta\rangle) \simeq \langle s\rangle \simeq \tilde{\mu}_x(\langle\theta\rangle,\langle s\rangle)$, where $\langle\theta\rangle$ is the coset of θ.

<u>Theorem 3.11</u> $H_x \approx \mathcal{H}_x$ as H-spaces.

<u>Proof</u> Define maps $H_x \xrightarrow{f} \mathcal{H}_x \xrightarrow{g} H_x$ by: $f(a) = \langle s\rangle$, where $s(x) \simeq a$, $s \in S(U,\mathcal{H})$ and $a \in H_x$; $g\langle t\rangle = a_0$, where $t'(x) \simeq a_0$ for all $t' \in \langle t\rangle$. Then f and g are H-homomorphisms.

The compositions yield $gf(a) = g\langle s\rangle = a_o \simeq a$, so $gf \simeq 1_{H_x}$, and $fg\langle t\rangle = f(a_o) = \langle t'\rangle = \langle t\rangle$, so $fg = 1_{\mathcal{H}_x}$. Thus $\mathcal{H}_x \simeq H_x$.

The H-structure induced on $\mathcal{H}_x$ by the above homotopy equivalence, through f and g, is given by:

$$\hat{\mu}(\langle s\rangle,\langle t\rangle) = f\mu_x(g\langle s\rangle, g\langle t\rangle).$$

However,

$$f\,\mu_x(g\langle s\rangle, g\langle t\rangle) = f\,\mu_x(a_o, a_1) = f(a_o \odot a_1) = \langle u\rangle,$$

where $u(x) \simeq a_o \odot a_1$, and

$$\tilde{\mu}_x(\langle s\rangle,\langle t\rangle) = \langle s|W \odot t|W\rangle = \langle \mu_W(s\odot t)\rangle = \langle s\odot t\rangle.$$

Thus the H-structures, induced by the homotopy equivalence and given in Lemma 3.6, are identical and $\mathcal{H}_x \approx H_x$.

<u>Definition 3.12</u> A <u>presheaf</u> P (of H-spaces) on X is a contravariant functor on $\mathcal{T}_X$ and inclusions to the category of H-spaces and restriction (H-)homomorphisms such that $P(1_U) \simeq 1$ and if $U \subset V \subset W$, then $P(i_V{}^U)\, P(i_W{}^V) \simeq P(i_W{}^U)$.

Let $M = \bigcup_{U\in\mathcal{T}_X}(U \times P(U))$ and define $(x,a) \sim (y,b)$ iff $x = y$ and there exists a neighborhood of x, $W \subset U \cap V$, such that,

$$P(i_W{}^U)(a) \simeq P(i_W{}^V)(b).$$

Form the quotient $\tilde{\mathcal{H}} = M/\sim$, where the quotient topology is assumed. Let $\pi\colon \tilde{\mathcal{H}} \to X$ be the projection map induced by p which takes (x,a) to x, (π is open since p is open and $\tilde{p}$ is continuous).

$$\begin{tikzcd} M \arrow[rr, "\tilde{p}"] \arrow[dr, "p"'] & & \tilde{\mathcal{H}} \arrow[dl, "\pi"] \\ & X & \end{tikzcd}$$

Consider $\pi^{-1}(x) = \tilde{\mathcal{H}}_x = \{[x,a]\}$. This clearly the limit:

$$\underset{x\in U}{\underrightarrow{L}}\{P(U), r_U{}^V\} = \{ \langle a\rangle \mid (y,b) \in \langle a\rangle \text{ iff } r_W{}^U(a) \simeq r_W{}^V(b),\ x\in W\subset U\cap V\},$$

and has a natural H-structure. Thus $\tilde{\mathcal{H}}$ is essentially $\bigcup_{x\in X}\tilde{\mathcal{H}}_x$.

Since the multiplications in M are continuous, they are so in $\widetilde{\mathcal{H}}$, and $\widetilde{\mathcal{H}}$ is a <u>sheaf of H-spaces generated by P on X</u> .

Under certain conditions the sheaf $\widetilde{\mathcal{H}}$ generated by P is a sheaf of algebraic structures, (the main requirement being discreteness of the stalks $\mathcal{H}_x$).

If $\mathcal{H}$ is a sheaf of H-spaces and P is the presheaf of sections of $\mathcal{H}$, there is a map ψ: $\mathcal{H} \to \widetilde{\mathcal{H}}$ defined by $\psi(a_x) = \langle a_x \rangle = \{s \mid s(x) \simeq a\}$ which preserves the H-structure and is a homotopy equivalence stalkwise. Thus $\mathcal{H} \approx \widetilde{\mathcal{H}}$ as sheaves of H-spaces, (see (3.8)).

A Čech cohomology theory with values in a sheaf of H-spaces may be defined (cf. [3],[9],[13] and [14]). In the material below we shall assume that the fiber H-structures are associative, commutative and admit inversion (which is continuous in the sheaf space). Then $S(U,\mathcal{H})$ inherits an associative and commutative H-structure and admits inversion.

<u>Definition 3.13</u> Let $\omega = \{w_i\}$ and $\nu = \{v_j\}$ be open covers of X with corresponding nerves w and v. Denote the simplex $w_{i_o} \dots w_{i_q}$ by $i_o \dots i_q$, for convenience, where the nucleus $N = \bigcap_{m=o}^{q} w_{i_m} \neq \emptyset$.

Let w(q) denote the collection of q-simplexes in w.

Define the q-cochains of w with coefficients in Σ by:

$$C^q(w,\Sigma) = \{ f^q : w(q) \to \Sigma \mid f^q(i_o \dots i_q) \in S(N,\mathcal{H})\},$$

where $N \neq \emptyset$ is the nucleus of $i_o \dots i_q$, a simplex in w(q). Topologize $C^q(w,\Sigma)$ with the compact-open topology.

Define a multiplication " $\odot$ " on $C^q(w,\Sigma)$ by the rule:

$$(f^q \odot g^q)(i_o \dots i_q) = \mu_N(f^q(i_o \dots i_q), g^q(i_o \dots i_q)),$$

where μ_N is the multiplication on $S(N,\mathcal{H})$ given in (3.7), and $f^q, g^q \in C^q(w,\Sigma)$.

The inversions $\{\rho_x\}$ on the stalks $\{H_x\}$ induce an inversion ρ on $S(N,\mathcal{H})$ which in turn induces an inversion ρ on $C^q(w,\Sigma)$. Denote the map $i_o\ldots i_q \to \theta_N \in S(N,\mathcal{H})$ by θ^q. By an argument similar to that given in (3.7), $C^q(w,\Sigma)$ is an H-space with an associative and commutative H-structure and admits inversion.

Let $C^q_F(w,\Sigma) = \{\ f^q \in C^q(w,\Sigma) \mid |f^q| \in F\ \}$, where $|f^q| = \overline{\bigcup_{i_o\ldots i_q} |f^q(i_o\ldots i_q)|}$. In view of (3.7) and the above definition of multiplication on $C^q(w,\Sigma)$, the space $C^q_F(w,\Sigma)$ is an H-space under the induced (inherited) multiplication from $C^q(w,\Sigma)$.

Definition 3.14 Define a map $d\colon C^q_F(w,\Sigma) \to C^{q+1}_F(w,\Sigma)$ by:

$$(3.14a)\qquad (df^q)(i_o\ldots i_{q+1}) = r^{N_o}{}_N f^q(i_1\ldots i_{q+1}) \odot \rho r^{N_1}{}_N f^q(i_o i_2\ldots i_{q+1})$$
$$\odot\ldots\odot\ (\rho)r^{N_k}{}_N f^q(i_o\ldots\hat{k}\ldots i_{q+1}) \odot\ldots$$
$$\odot\ (\rho)r^{N_{q+1}}{}_N f^q(i_o\ldots i_q),$$

where $N = \bigcap_o^{q+1} w_{i_m} \neq \emptyset$, $N_k = \bigcap_{m\neq k} w_{i_m} \neq \emptyset$, $k = 0,\ldots,q+1$, ρ is the inversion on $S(N,\mathcal{H})$ and (ρ) denotes ρ on the odd terms and the identity on the even terms. To shorten the notation we will denote the right-hand side of equation (3.14a) by:

$$(3.14b)\qquad \mathop{\odot}_{k=o}^{q+1} (\rho)r^{N_k}{}_N f^q(i_o\ldots\hat{k}\ldots i_{q+1}).$$

In view of (3.9) and (3.13),

$$|df^q| = \overline{\bigcup_{i_o\ldots i_{q+1}} |(df^q)(i_o\ldots i_{q+1})|}, \text{ where}$$

$$|(df^q)(i_o\ldots i_{q+1})| \subset \bigcap_{k=o}^{q+1} |(\rho)r^{N_k}{}_N f^q(i_o\ldots\hat{k}\ldots i_{q+1})|.$$

Lemma 3.15 The map $d\colon C^q_F(w,\Sigma) \to C^{q+1}_F(w,\Sigma)$ is an H-homomorphism, and d^2 is trivial.

<u>Proof</u> Let $f^q, g^q \in C_F^q(w,\Sigma)$ and $(i_o\ldots i_{q+1}) \in w(q+1)$. Then by (3.13) and (3.14),

$$(d(f^q \odot g^q))(i_o\ldots i_{q+1}) = \underset{k=o}{\overset{q+1}{\odot}}(\rho) r^{N_k}{}_N (f^q \odot g^q)(i_o\ldots\hat{k}\ldots i_{q+1})$$
$$\simeq \underset{k=o}{\overset{q+1}{\odot}}(\rho)(r^{N_k}{}_N f^q \odot r^{N_k}{}_N g^q)(i_o\ldots\hat{k}\ldots i_{q+1}),$$

and

$$(df^q \odot dg^q)(i_o\ldots i_{q+1}) = (df^q(i_o\ldots i_{q+1})) \odot (dg^q(i_o\ldots i_{q+1}))$$
$$= (\underset{k=o}{\overset{q+1}{\odot}}(\rho) r^{N_k}{}_N f^q(i_o\ldots\hat{k}\ldots i_{q+1})) \odot$$
$$(\underset{k=o}{\overset{q+1}{\odot}}(\rho) r^{N_k}{}_N g^q(i_o\ldots\hat{k}\ldots i_{q+1})).$$

Since the H-structures are assumed to be commutative, these two expressions are homotopic and d is an H-homomorphism.

Let $(i_o\ldots i_{q+2}) \in w(q+2)$, then

$$(d^2 f^q)(i_o\ldots i_{q+2}) = \underset{k=o}{\overset{q+2}{\odot}}(\rho) r^{N_k}{}_N (df^q)(i_o\ldots\hat{k}\ldots i_{q+2})$$
$$= \underset{k=o}{\overset{q+2}{\odot}}(\rho) r^{N_k}{}_N (\underset{\substack{j=o\\ j\neq k}}{\overset{q+2}{\odot}}(\rho) r^{N_{kj}}{}_{N_k} f^q(i_o\ldots\hat{j}\ldots\hat{k}\ldots i_{q+2})),$$

where $N_{kj} = \bigcap_{\substack{m\neq j\\ m\neq k}} w_{i_m}$.

By (3.9),

$$\underset{k=o}{\overset{q+2}{\odot}}(\rho) r^{N_k}{}_N (\underset{\substack{j=o\\ j\neq k}}{\overset{q+2}{\odot}}(\rho) r^{N_{kj}}{}_{N_k} f^q(i_o\ldots\hat{j}\ldots\hat{k}\ldots i_{q+2})) \simeq$$
$$\underset{k=o}{\overset{q+2}{\odot}}(\rho)(\underset{\substack{j=o\\ j\neq k}}{\overset{q+2}{\odot}}(\rho) r^{N_{kj}}{}_N f^q(i_o\ldots\hat{j}\ldots\hat{k}\ldots i_{q+2}))$$

$$= (r^{N_{o1}}{}_N f^q(i_2\ldots i_{q+2}) \odot \rho\, r^{N_{o2}}{}_N f^q(i_1 i_3\ldots i_{q+2}) \odot \ldots \odot (\rho) r^{N_{oq+2}}{}_N f^q(i_1\ldots i_{q+1}))$$
$$\odot \ldots$$
$$\odot ((\rho) r^{N_{q+2o}}{}_N f^q(i_1\ldots i_{q+1}) \odot \ldots \odot (\rho)(\rho) r^{N_{q+2q+1}}{}_N f^q(i_o\ldots i_q)).$$

Note that $N_{jk} = N_{kj}$ and $\rho\rho \simeq 1$. In the latter expanded form above, each term appears with an inversion and again without an inversion, and by the commutativity of the H-structures these terms may be rearranged to yield:

$$(d^2 f^q)(i_o \dots i_{q+2}) \simeq \theta^{q+2}(i_o \dots i_{q+2}),$$

where $\theta^{q+2}(i_o \dots i_{q+2}) = \theta_N \in S(N, \mathcal{H})$.

<u>Lemma 3.16</u> Im d^q and Ker d^q are H-spaces under the H-structure inherited from $C_F^q(w,\Sigma)$, $q \geq 0$.

<u>Proof</u> If $f^q, g^q \in$ Ker d^q, then

$$\begin{aligned} d(f^q \odot g^q)(i_o \dots i_{q+1}) &\simeq df^q(i_o \dots i_{q+1}) \odot dg^q(i_o \dots i_{q+1}) \\ &\simeq \theta^q(i_o \dots i_{q+1}) \odot \theta^q(i_o \dots i_{q+1}) \\ &\simeq \theta^q(i_o \dots i_{q+1}), \end{aligned}$$

and $f^q \odot g^q \in$ Ker d^q.

Also, $\theta^q \in$ Ker d^q, since $d^q\theta^q \simeq \theta^{q+1}$. Thus Ker d^q is an H-space under the induced multiplication of $C_F^q(w,\Sigma)$.

Let $f^{q+1}, g^{q+1} \in$ Im d^q. Then there exist f^q, $g^q \in C_F^q(w,\Sigma)$ such that $f^{q+1} = d^q f^q$ and $g^{q+1} = d^q g^q$. Thus

$$\begin{aligned} (f^{q+1} \odot g^{q+1})(i_o \dots i_{q+1}) &= (d^q f^q \odot d^q g^q)(i_o \dots i_{q+1}) \\ &\simeq (d^q(f^q \odot g^q))(i_o \dots i_{q+1}), \end{aligned}$$

or $f^{q+1} \odot g^{q+1} \in$ Im d^q.

Also, $\theta^{q+1} \in$ Im d^q, since $d^q\theta^q \simeq \theta^{q+1}$. Thus, Im d^q is an H-space.

<u>Definition 3.17</u> Since d^2 is trivial, Im $d^{q-1} \subset$ Ker d^q. Define $H_F^q(w,\Sigma) =$ Ker d^q/ Im d^{q-1}, where Ker d^q/Im d^{q-1} is the set

$\{ f^q \odot \mathrm{Im}\ d^{q-1} | \ f^q \in \mathrm{Ker}\ d^q \}$ and $f^q \odot \mathrm{Im}\ d^{q-1} = \{ f^q \odot g^q | g^q \in \mathrm{Im}\ d^{q-1} \}$. Under the quotient topology an H-structure is induced on $H^q_F(w,\Sigma)$ by the rule: $\hat{\mu}\ (f^q \odot \mathrm{Im}\ d^{q-1}, g^q \odot \mathrm{Im}\ d^{q-1}) = (f^q \odot g^q) \odot \mathrm{Im}\ d^{q-1}$. The neutral element is $\theta^q \odot \mathrm{Im}\ d^{q-1} = \mathrm{Im}\ d^{q-1}$. Let $\underline{f}^q$ denote $f^q \odot \mathrm{Im}\ d^{q-1}$.

Definition 3.18 If ν and ω are open covers of X and ω refines ν, $\nu < \omega$, then let $p^w{}_v$ denote the (nonunique) projection which maps simplexes $(i_o \ldots i_q) \in w$ into simplexes $(j_o \ldots j_r) \in v$, $r \le q$.

The projection $p^w{}_v$ induces a map $p^{w\ \#}_{\ v} : C^q_F(v,\Sigma) \to C^q_F(w,\Sigma)$, defined by: $p^{w\ \#}_{\ v} f^q(i_o \ldots i_q) = r^M{}_N f^q(pi_o \ldots pi_q)$, where $N = \cap\ w_{i_m}$, $M = \cap\ v_{j_k = pi_m}$, and $f^q \in C^q_F(v,\Sigma)$.

Note that $|p^{w\ \#}_{\ v} f^q| \subset |f^q|$ and that $p^{w\ \#}_{\ v}$ is an H-homomorphism since $r^M{}_N$ is an H-homomorphism.

Lemma 3.19 $d^q_w\, p^{w\ \#}_{\ v} \simeq p^{w\ \#}_{\ v}\, d^q_v$.

Proof Consider the diagram:

$$\begin{tikzcd}
 & \arrow[d] & \arrow[d] & \\
\cdots \arrow[r] & C^q_F(v,\Sigma) \arrow[r, "d^q_v"] \arrow[d, "p^{w\ \#}_{\ v}"] & C^{q+1}_F(v,\Sigma) \arrow[r] \arrow[d, "p^{w\ \#}_{\ v}"] & \cdots \\
\cdots \arrow[r] & C^q_F(w,\Sigma) \arrow[r, "d^q_w"] \arrow[d] & C^{q+1}_F(w,\Sigma) \arrow[r] \arrow[d] & \cdots \\
 & & &
\end{tikzcd}$$

Let $f^q \in C^q_F(v,\Sigma)$, then

$$\begin{aligned}
p^{w\ \#}_{\ v}(d^q_v f^q)(i_o \ldots i_{q+1}) &= r^M{}_N (d^q_v f^q)(pi_o \ldots pi_{q+1}) \\
&= r^M{}_N \Big(\bigodot_{k=o}^{q+1} (\rho) r^{M_k}{}_M f^q(pi_o \ldots \hat{k} \ldots pi_{q+1})\Big) \\
&\simeq \bigodot_{k=o}^{q+1} (\rho) r^{M_k}{}_N f^q(pi_o \ldots \hat{k} \ldots pi_{q+1}) \\
&= d^q_w (p^{w\ \#}_{\ v} f^q)(i_o \ldots i_{q+1}).
\end{aligned}$$

Lemma 3.20

(3.20a) $p^{w}_{v}{}^{\#}(\mathrm{Ker}\ d^q_v) \subset \mathrm{Ker}\ d^q_w$,

(3.20b) $p^{w}_{v}{}^{\#}(\mathrm{Im}\ d^{q-1}_v) \subset \mathrm{Im}\ d^{q-1}_w$.

Proof Let $f^q \in \mathrm{Ker}\ d^q_v$, then $p^{w}_{v}{}^{\#}f^q(i_o\ldots i_q) = r^M_N f^q(pi_o\ldots pi_q)$, by (3.18), and

$$\begin{aligned} d^q_w(p^{w}_{v}{}^{\#}f^q)(i_o\ldots i_{q+1}) &= \mathop{\odot}_{k=o}^{q+1}(\rho) r^{N_k}{}_N(p^{w}_{v}{}^{\#}f^q)(i_o\ldots\hat{k}\ldots i_{q+1}) \\ &= \mathop{\odot}_{k=o}^{q+1}(\rho) r^{N_k}{}_N r^{M_k}{}_{N_k} f^q(pi_o\ldots\hat{k}\ldots i_{q+1}) \\ &\simeq \mathop{\odot}_{k=o}^{q+1}(\rho) r^{M_k}{}_N f^q(pi_o\ldots\hat{k}\ldots pi_{q+1}), \end{aligned}$$

where $M_k = \bigcap_{m\neq k} w_{pi_m}$, $N_k = \bigcap_{m\neq k} v_{i_m}$, and $N = \bigcap v_{i_m}$.

But $d^q_v f^q \simeq \theta^{q+1}$, so the right-hand side of the equation is trivial in $C^{q+1}_F(w,\Sigma)$.

Let $f^q \in \mathrm{Im}\ d^{q-1}_v$, then $f^q(i_o\ldots i_q) = \mathop{\odot}_{k=o}^{q}(\rho) r^{M_k}{}_M g^{q-1}(i_o\ldots\hat{k}\ldots i_q)$, and

$$\begin{aligned} (p^{w}_{v}{}^{\#}f^q)(i_o\ldots i_q) &= r^M_N f^q(pi_o\ldots pi_q) \\ &= r^M_N\Big(\mathop{\odot}_{k=o}^{q}(\rho) r^{M_k}{}_M g^{q-1}(pi_o\ldots\hat{k}\ldots pi_q)\Big) \\ &\simeq \mathop{\odot}_{k=o}^{q}(\rho) r^{M_k}{}_N g^{q-1}(pi_o\ldots\hat{k}\ldots pi_q) \\ &= \mathop{\odot}_{k=o}^{q}(\rho) r^{N_k}{}_N r^{M_k}{}_{N_k} g^{q-1}(pi_o\ldots\hat{k}\ldots pi_q) \\ &= \mathop{\odot}_{k=o}^{q}(\rho) r^{N_k}{}_N(p^{w}_{v}{}^{\#}g^{q-1})(i_o\ldots\hat{k}\ldots i_q) \\ &= d^{q-1}_w(p^{w}_{v}{}^{\#}g^{q-1})(i_o\ldots i_q), \end{aligned}$$

thus $p^{w}_{v}{}^{\#}f^q \in \mathrm{Im}\ d^{q-1}_w$.

Definition 3.21 By Lemma 3.20, $p^{w}_{v}{}^{\#}$ induces an H-homomorphism $p^{w}_{v}{}^{*}: H^q_F(v,\Sigma) \to H^q_F(w,\Sigma)$ by the rule: $p^{w}_{v}{}^{*}(\underline{f}^q) = \underline{(p^{w}_{v}{}^{\#}f^q)}$, where $\underline{f}^q \in H^q_F(v,\Sigma)$.

<u>Lemma 3.22</u> If p^w_v and $\hat{p}^w_v$ are projection maps of w to v, then $p^{w\,*}_v \approx \hat{p}^{w\,*}_v$.

<u>Proof</u> Define a map $D: C^q_F(w,\Sigma) \to C^{q-1}_F(w,\Sigma)$ by:

$$Df^q(i_o \ldots i_{q-1}) = \underset{k=o}{\overset{q-1}{\odot}} (\rho) r^{M_k}{}_N f^q(pi_o \ldots pi_k \hat{p}i_k \ldots \hat{p}i_{q-1}) ,$$

where $M_k = \bigcap_{m=o}^{k} w_{pi_m} \cap \bigcap_{m=k}^{q-1} w_{\hat{p}i_m}$, $N = \bigcap v_i$.

Then

$$(Ddf^q \odot dDf^q)(i_o \ldots i_q) = (\underset{k=o}{\overset{q}{\odot}} (\rho) r^{M_k}{}_M df^q(pi_o \ldots pi_k \hat{p}i_k \ldots \hat{p}i_q))$$

$$\odot\ (\underset{k=o}{\overset{q}{\odot}} (\rho) r^{P_k}{}_N Df^q(i_o \ldots \hat{k} \ldots i_q))$$

$$= (\underset{k=o}{\overset{q}{\odot}} (\rho) r^{M_k}{}_N (\underset{\substack{h=o\\ h\neq k}}{\overset{q}{\odot}} (\rho) r^{M_{kh}}{}_{M_k} f^q(pi_o \ldots \hat{k} \ldots pi_q)))$$

$$\odot (\underset{k=o}{\overset{q}{\odot}} (\rho) r^{P_k}{}_N (\underset{\substack{h=o\\ h\neq k}}{\overset{q}{\odot}} (\rho) r^{M_{kh}}{}_{P_k} f^q(pi_o \ldots{}_k pi_k \hat{p}i_k \ldots \hat{p}i_q))).$$

Expanding and simplifying

$$(Ddf^q \odot dDf^q)(i_o \ldots i_q) \simeq (\hat{p}^{w\,\#}_v \odot \rho\ p^{w\,\#}_v)(i_o \ldots i_q).$$

Let $\underline{f}^q \in H^q_F(v,\Sigma)$, then $f^q \in \mathrm{Ker}\ d^q$, thus

$$(\hat{p}^{w\,*}_v \odot \rho\ p^{w\,*}_v)\ \underline{f}^q = \underline{(\hat{p}^{w\,\#}_v \odot \rho\ p^{w\,\#}_v)f^q} \simeq \underline{(Dd \odot dD)f^q} .$$

But $Ddf^q = \theta^q$ and $dDf^q \in \mathrm{Im}\ d^{q-1}$, thus $\underline{(Dd \odot dD)f^q} = \theta$ and $\hat{p}^{w\,*}_v \approx p^{w\,*}_v$.

<u>Definition 3.23</u> By the definition of $p^{w\,\#}_v$ and (3.9), the collection $\{H^*_F(w,\Sigma),\ p^{w\,*}_v\}$ forms an h-direct system. Define the cohomology of X with values in $\mathcal{H}$ and supports in F as the limit of this system:

$$H^p_F(X,\mathcal{H}) = \underset{\rightsquigarrow}{L} \{H^p_F(w,\Sigma),\ p^{w\,*}_v\}.$$

<u>Theorem 3.24</u> $H_F^*(X,\mathcal{H})$ is an H-space.

<u>Proof</u> If $\mu < \nu < \omega < \chi$, then

(3.24a) $\quad p^{\chi\ *}_{\ \omega}(\mu_\omega(p^{\omega\ *}_{\ \mu}(\underline{x}_\mu), p^{\omega\ *}_{\ \nu}(\underline{x}_\nu))) \simeq \mu_\chi(p^{\chi\ *}_{\ \mu}(\underline{x}_\mu), p^{\chi\ *}_{\ \nu}(\underline{x}_\nu))$,

and

(3.24b) $\quad \mu_\nu(p^{\nu\ *}_{\ \mu}(\underline{x}_\mu), p^{\nu\ *}_{\ \mu}(\underline{x}'_\mu)) \simeq p^{\nu\ *}_{\ \mu}\mu_\mu(\underline{x}_\mu, \underline{x}'_\mu)$,

since the connecting maps are H-homomorphisms (see the diagrams below).

$$\begin{tikzcd}
H_F^p(\mu,\Sigma) \times H_F^p(\nu,\Sigma) \arrow[rr, "p^{\chi\ *}_{\ \mu} \times p^{\chi\ *}_{\ \nu}"] \arrow[d, "p^{\omega\ *}_{\ \mu} \times p^{\omega\ *}_{\ \nu}"] & & H_F^p(\chi,\Sigma) \times H_F^p(\chi,\Sigma) \arrow[dd, "\mu_\chi"] \\
H_F^p(\omega,\Sigma) \times H_F^p(\omega,\Sigma) \arrow[d, "\mu_\omega"] & & \\
H_F^p(\omega,\Sigma) \arrow[rr, "p^{\chi\ *}_{\ \omega}"] & & H_F^p(\chi,\Sigma)
\end{tikzcd}$$

$$\begin{tikzcd}
H_F^p(\mu,\Sigma) \times H_F^p(\mu,\Sigma) \arrow[rr, "p^{\nu\ *}_{\ \mu} \times p^{\nu\ *}_{\ \mu}"] \arrow[d, "\mu_\mu"] & & H_F^p(\nu,\Sigma) \times H_F^p(\nu,\Sigma) \arrow[d, "\mu_\nu"] \\
H_F^p(\mu,\Sigma) \arrow[rr, "p^{\nu\ *}_{\ \mu}"] & & H_F^p(\nu,\Sigma)
\end{tikzcd}$$

The limit space $H_F^p(X,\mathcal{H})$, with the limit topology, has a continuous multiplication defined as follows:

$$\tilde{\mu}(\underline{x}_\mu, \underline{x}_\nu) = \underline{\mu_\omega(p^{\omega\ *}_{\ \mu}(\underline{x}_\mu), p^{\omega\ *}_{\ \nu}(\underline{x}_\nu))} ,$$

where $\underline{x}_\nu$, $\underline{x}_\mu \in H_F^p(X,\mathcal{H})$ and $\mu,\nu < \omega$.

The definition is independent of the choice of ω, for if $\omega' > \mu,\nu$ then there exists a cover $\omega'' > \omega',\omega$ such that

$$p^{\omega''\ *}_{\ \omega}(\mu_\omega(p^{\omega\ *}_{\ \mu}(\underline{x}_\mu), p^{\omega\ *}_{\ \nu}(\underline{x}_\nu))) \simeq p^{\omega''\ *}_{\ \omega'}(\mu_{\omega'}(p^{\omega'\ *}_{\ \mu}(\underline{x}_\mu), p^{\omega' *}_{\ \nu}(\underline{x}_\nu))),$$

by (3.24a) and (3.24b) above.

Thus $\underline{\mu_{\omega}(p^{\omega *}_{\mu}(\underline{x}_{\mu}), p^{\omega *}_{\nu}(\underline{x}_{\nu}))} = \underline{\mu_{\omega'}(p^{\omega' *}_{\mu}(\underline{x}_{\mu}), p^{\omega' *}_{\nu}(\underline{x}_{\nu}))}$.

Also, the definition of $\widetilde{\mu}$ is independent of the choice of representative of the elements of $H^p_F(X, \mathcal{H})$ by an argument similar to that above. Denote $\widetilde{\mu}(\underline{x}_{\mu}, \underline{x}_{\nu})$ by $\underline{x}_{\mu} \odot \underline{x}_{\nu}$.

Denote $\underline{\theta}_{\mu}$ by θ, then $\theta \odot \underline{x}_{\nu} = \underline{\theta}_{\mu} \odot \underline{x}_{\nu} = \underline{\mu_{\nu}(\theta_{\nu}, x_{\nu})} = \underline{x}_{\nu}$. Similarly, $\underline{x}_{\nu} \odot \theta = \underline{x}_{\nu}$.

The choice of representative of θ is independent of the choice of representative of $\underline{x}_{\nu}$, since the connecting maps are H-homomorphisms. Thus $\widetilde{\mu}$ determines an H-structure on $H^p_F(X, \mathcal{H})$.

If the spaces $H^p_F(w, \Sigma)$ have inversions ρ_w, then an inversion ρ on $H^p_F(X, \mathcal{H})$ is defined by $\rho(\underline{x}_{\nu}) = \underline{\rho_{\nu}(x_{\nu})}$. Denote $\rho(\underline{x}_{\nu})$ by $(\underline{x}_{\nu})^{-1}$, then $(\underline{x}_{\omega}) \odot (\underline{x}_{\omega})^{-1} = \underline{\mu_{\omega}(x_{\omega}, \rho_{\omega}(x_{\omega}))} = \underline{\theta}_{\omega} = \theta$.

If the H-structures on the spaces $H^p_F(w, \Sigma)$ are associative and/or commutative, then the H-structure on $H^p_F(X, \mathcal{H})$ is associative and/or commutative in view of the definition of the H-structure on $H^p_F(X, \mathcal{H})$.

The above cohomology theory may be shown to satisfy the axioms of a sheaf cohomology theory (for sheaves of H-spaces).

<u>Definition 3.25</u> By a cohomology theory with coefficients in a sheaf of H-spaces, we mean a covariant δ-functor ([14]) from the category of sheaves of H-spaces on a given space to the category of H-spaces which satisfies the following axioms (cf. [5], [9] and [13]):

I. $H^o_F(X, \mathcal{H}) \approx S_F(X, \mathcal{H})$,

II. If $0 \to \mathcal{H}' \xrightarrow{\alpha} \mathcal{H} \xrightarrow{\beta} \mathcal{H}'' \to 0$ is an exact sequence of sheaves of H-spaces on X, then the sequence

$$\cdots \to H_F^p(X,\mathcal{H}'') \xrightarrow{\delta} H_F^{p+1}(X,\mathcal{H}') \xrightarrow{\alpha^*} H_F^{p+1}(X,\mathcal{H}) \to \cdots$$

is exact,

III. $H_F^p(X,\mathcal{H}) = 0$ if $\mathcal{H}$ is fine and $p > 0$.

Additional properties of the above cohomology will be explored below. We begin by demonstrating the above axioms. The following definition is listed for later reference.

Definition 3.26 A sheaf of H-spaces $\mathcal{H}$ is fine iff for every fine covering of a locally compact space X ([3]) or locally finite cover $\{v_i\}$ of X ([14]) there exist sheaf maps $\alpha_i\colon \mathcal{H} \to \mathcal{H}$ such that:

(3.26a) $|\alpha_i| \subset \bar{v}_i$,

(3.26b) $\underset{i}{\odot}\, \alpha_i \simeq 1_{\mathcal{H}}$.

It is clear that if $1\colon \mathcal{H} \to \mathcal{H}$ is the identity map, then $1^*\colon H_F^q(X,\mathcal{H}) \xrightarrow{\approx} H_F^q(X,\mathcal{H})$. Also, if $\mathcal{H}' \xrightarrow{\alpha} \mathcal{H} \xrightarrow{\beta} \mathcal{H}''$, then $\beta^*\alpha^* = (\beta\alpha)^*\colon H_F^q(X,\mathcal{H}') \to H_F^q(X,\mathcal{H}'')$.

Theorem 3.27 $H_F^o(X,\mathcal{H}) \approx S_F(X,\mathcal{H})$.

Proof We are assuming $H_F^q(X,\mathcal{H})$ is trivial for $q < 0$, therefore $H_F^o(v,\Sigma) = \operatorname{Ker} d^o$, and $(d^o f^0)(i_o i_1) = r^{N_o}{}_N f^o(i_1) \odot r^{N_1}{}_N f^o(i_o) \simeq \theta^1(i_o i_1)$, where $N = v_{i_o} \cap v_{i_1}$, $N_j = v_{i_j}$, $j = 0,1$.

However,

$$\begin{aligned}(r^{N_o}{}_N f^o(i_1) \odot \rho r^{N_1}{}_N f^o(i_o)) \odot r^{N_1}{}_N f^o(i_o) &\simeq r^{N_o}{}_N f^o(i_1) \odot (\rho r^{N_1}{}_N f^o(i_o) \odot r^{N_1}{}_N f^o(i_o)) \\ &\simeq r^{N_o}{}_N f^o(i_1) \odot r^{N_1}{}_N \theta^o(i_o) \\ &\simeq r^{N_o}{}_N f^o(i_1),\end{aligned}$$

and $\theta^1(i_o i_1) \odot r^N 1_N f^o(i_o) \simeq r^N 1_N f^o(i_o)$, thus $r^N o_N f^o(i_1) \simeq r^N 1_N f^o(i_o)$.

Let $\{h_t\}$ be the latter homotopy of $f^o(i_1)$ and $f^o(i_o)$ on $v_{i_o} \cap v_{i_1}$. Then $\{h_t\}$ may be extended to the path components of $\mathrm{Im}\, f^o(i_1)|v_{i_o} \cap v_{i_1}$ and $\mathrm{Im}\, f^o(i_o)|v_{i_o} \cap v_{i_1}$ over $v_{i_o} \cup v_{i_1} \in v$ to obtain a section which is a homotopy of $f^o(i_o)$ on v_{i_o} and $f^o(i_1)$ on v_{i_1}. Thus $\bigcup_j f^o(i_j)$ determines a section of X.

Let $s \in S_F(X,\mathcal{H})$, then s determines a cocycle in $H^o_F(v,\Sigma)$.

The correspondence above determines a homomorphism Ker $d^o \to S_F(X,\mathcal{H})$ with trivial kernel, thus $H^o_F(v,\Sigma) \approx S_F(X,\mathcal{H})$, as an H-isomorphism, and $H^o_F(X,\mathcal{H}) \approx S_F(X,\mathcal{H})$.

Let $0 \to \mathcal{H}' \xrightarrow{\alpha} \mathcal{H} \xrightarrow{\beta} \mathcal{H}'' \to 0$ be an exact sequence of sheaves of H-spaces on X, that is $\mathrm{Im}\,\alpha \simeq \mathrm{Ker}\,\beta$, where $\mathrm{Ker}\,\beta = \{a \in \mathcal{H} \mid \beta(a) \simeq \theta\}$, α and β as defined in (3.8).

<u>Theorem 3.28</u> Let $0 \to \mathcal{H}' \xrightarrow{\alpha} \mathcal{H} \xrightarrow{\beta} \mathcal{H}'' \to 0$ be an exact sequence of sheaves of H-spaces on X, then there exists a map

$$\delta^q : H^q_F(X,\mathcal{H}'') \to H^{q+1}_F(X,\mathcal{H}'),$$

such that the sequence

$$\ldots \to H^q_F(X,\mathcal{H}'') \xrightarrow{\delta^*} H^{q+1}_F(X,\mathcal{H}') \xrightarrow{\alpha^*} H^{q+1}_F(X,\mathcal{H}) \to \ldots$$

is exact.

<u>Proof</u> Let $v = \{v_i\}$ be an open cover of X and $N = \bigcap_{m=o}^{q} v_{i_m} \neq \emptyset$. It is clear from (3.13) and (3.14) that α^* and β^* commute with the connecting maps required for the cohomologies involved here.

$$\begin{tikzcd}
0 \arrow[r] & C_F^q(v,\Sigma') \arrow[r, "\alpha^{\#}"] \arrow[d, "{'d^q}"] & C_F^q(v,\Sigma) \arrow[r, "\beta^{\#}"] \arrow[d, "d^q"] & C_F^q(v,\Sigma'') \arrow[r] \arrow[d, "{''d^q}"] & 0 \\
0 \arrow[r] & C_F^{q+1}(v,\Sigma') \arrow[r, "\alpha^{\#}"] & C_F^{q+1}(v,\Sigma) \arrow[r, "\beta^{\#}"] & C_F^{q+1}(v,\Sigma'') \arrow[r] & 0
\end{tikzcd}$$

If $\underline{f}^q \in H_F^q(v,\Sigma'')$, then define $\delta_v^q: H_F^q(v,\Sigma'') \to H_F^{q+1}(v,\Sigma')$ by

$$\delta(\underline{f}^q) = (\underline{\delta^q f^q}) = (\underline{\alpha\#^{-1} d^q \beta\#^{-1} f^q}) .$$

The map δ^q is well defined and commutes with the connecting maps.

Let $g_1^q, g_2^q \in \beta\#^{-1}(f^q)$, then $d^q(g_1^q \odot g_2^q) \simeq d^q g_1^q \odot d^q g_2^q \simeq d^q \alpha^{\#} h^q \simeq \alpha^{\#} d^q h^q$ for some $h^q \in C_F^q(v,\Sigma')$, since $g_1^q \odot \rho g_2^q \in \operatorname{Ker} \beta^{\#}$. Thus

$$(d^q g_1^q \odot \rho d^q g_2^q) \odot (\rho\alpha^{\#}, d^q h^q) \simeq d^q g_1^q \odot \rho(d^q g_2^q \odot \alpha^{\#}, d^q h^q) \simeq \theta^{q+1},$$

or $(\underline{\alpha^{\#-1} d^q g_1^q}) = (\underline{\alpha^{\#-1} d^q g_2^q}) \in H_F^{q+1}(v,\Sigma)$.

The map δ^q is natural by the following argument. Let the diagram below have exact sequences and commuting squares.

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{H}' \arrow[r, "\alpha"] \arrow[d, "\lambda"] & \mathcal{H} \arrow[r, "\beta"] \arrow[d, "\mu"] & \mathcal{H}'' \arrow[r] \arrow[d, "\nu"] & 0 \\
0 \arrow[r] & \mathcal{K}' \arrow[r, "\hat{\alpha}"] & \mathcal{K} \arrow[r, "\hat{\beta}"] & \mathcal{K}'' \arrow[r] & 0
\end{tikzcd}$$

The following diagram results:

$$\begin{tikzcd}
H_F^q(X,\mathcal{H}'') \arrow[r, "\delta^q"] \arrow[d, "\nu^*"] & H_F^{q+1}(X,\mathcal{H}') \arrow[d, "\lambda^*"] \\
H_F^q(X,\mathcal{K}'') \arrow[r, "\hat{\delta}^q"] & H_F^{q+1}(X,\mathcal{K}')
\end{tikzcd}$$

Let $\underline{f}^q \in H_F^q(v,\Sigma'')$, then

$\lambda^*\delta^q(\underline{f}^q) = \lambda^*(\underline{\alpha\#^{-1} d^q \beta\#^{-1} f^q}) = (\underline{\lambda\#\alpha\#^{-1} d^q \beta\#^{-1} f^q})$,

and $\delta^*\nu^*(\underline{f}^q) = \delta^*(\underline{\nu\# f^q}) = (\underline{\hat{\alpha}\#^{-1} \hat{d}^q \beta\#^{-1} \nu\# f^q})$.

But $\hat{\beta}\#^{-1}\nu\# f^q \simeq \mu\#\beta\#^{-1} f^q$ and $\lambda\#\alpha\#^{-1} \simeq \hat{\alpha}\#^{-1}\mu\#$ on $\operatorname{Ker} \beta\#$.

Also, $\hat{d}^q\mu\# \simeq \mu\# d^q$, by definition of d. Thus,
$\underline{(\lambda\#\alpha\#^{-1}d^q\beta\#^{-1}f^q)} \simeq \underline{(\hat{\alpha}\#^{-1}\mu\# d^q\beta\#^{-1}f^q)} \simeq \underline{(\hat{\alpha}\#^{-1}\hat{d}^q\mu\#\beta\#^{-1}f^q)} \simeq$
$\underline{(\hat{\alpha}\#^{-1}\hat{d}^q\beta\#^{-1}\nu\# f^q)}$.

A long exact sequence results:

$$\cdots \to H^q_F(v,\Sigma'') \xrightarrow{\underline{\delta}^*} H^{q+1}_F(v,\Sigma') \xrightarrow{\underline{\alpha}^*} H^{q+1}_F(v,\Sigma) \to \cdots$$

Exactness at $H^q_F(v,\Sigma'')$: Let $\underline{f}^q \in \operatorname{Im}\beta^*$, thus $\underline{f}^q = \beta^*(\underline{g}^q) = \underline{\beta\# g}^q$. Then $\delta^*(\underline{f}^q) = \underline{\delta^q f}^q = \underline{\alpha\#^{-1}d^q\beta\#^{-1}f}^q = \underline{\alpha\#^{-1}d^q g}^q = \underline{\theta}^{q+1}$, and $\operatorname{Im}\beta^* \subset \operatorname{Ker}\delta^*$.

Let $\underline{f}^q \in \operatorname{Ker}\delta^*$, thus $\delta^*(\underline{f}^q) = \underline{\delta^q f}^q = \underline{\theta}^{q+1}$. But $\delta^q f^q = \alpha\#^{-1}d^q\beta\#^{-1}f^q$, so $g^q \in \beta\#^{-1}f^q$ has the property that $\beta^*(\underline{g}^q) = \underline{f}^q$.

Exactness at $H^{q+1}_F(v,\Sigma')$: Let $\underline{f}^{q+1} \in \operatorname{Im}\delta^*$, then $\underline{f}^{q+1} = \underline{\alpha\#^{-1}d^q\beta\#^{-1}g}^q$, where $\underline{g}^q \in H^q_F(v,\Sigma'')$, and $\alpha^*(\underline{f}^{q+1}) = \underline{d^q\beta\#^{-1}g}^q = \underline{\theta}^{q+1}$, or $\operatorname{Im}\delta^* \subset \operatorname{Ker}\alpha^*$.

Let $\underline{f}^{q+1} \in \operatorname{Ker}\alpha^*$, then $\alpha\# f^{q+1} \in \operatorname{Im} d^q$ and if $g^q \in \beta\# {d^q}^{-1}\alpha\# f^{q+1}$, then $\delta^*(\underline{g}^q) = \underline{f}^{q+1}$.

Exactness at $H^{q+1}_F(v,\Sigma)$: Let $\underline{f}^{q+1} \in \operatorname{Im}\alpha^*$, then $\underline{f}^{q+1} = \alpha^*(\underline{g}^{q+1}) = \underline{\alpha\# g}^{q+1}$ for some $\underline{g}^{q+1} \in H^{q+1}_F(v,\Sigma')$ and $\beta^*(\underline{f}^{q+1}) = \underline{\beta\#\alpha\# g}^{q+1} = \underline{\theta}^{q+1}$.

Let $\underline{f}^{q+1} \in \operatorname{Ker}\beta^*$, then $\beta^*(\underline{f}^{q+1}) = \underline{\theta}^{q+1}$ and there exists an element $\underline{g}^{q+1} \in H^{q+1}_F(v,\Sigma')$ such that $\alpha\# g^{q+1} \simeq f^{q+1}$, or $\alpha^*(\underline{g}^{q+1}) = \underline{f}^{q+1}$.

Passing to the limit of the sequence above, one obtains the desired exact sequence.

Theorem 3.29 $H^q_F(X,\mathcal{H})$ is trivial for $q > 0$, if $\mathcal{H}$ is fine.

Proof Let $v = \{v_i\}$ be a fine cover or locally finite cover, $\{\alpha_i\}$ a set of sheaf maps with supports in $\{\bar{v}_i\}$, and $\{\tilde{\alpha}_i\}$ the induced maps on $S(U,\mathcal{H})$.

Define a map $D: C_F^q(v,\Sigma) \to C_F^{q-1}(v,\Sigma)$, by

$$(Df^q)(i_o\ldots i_{q-1}) = \underset{j\in\pi}{\odot}\tilde{\alpha}_j(f^q(ji_o\ldots i_{q-1})).$$

Then if $x \in \bigcap_{m=o}^{q-1} v_{i_m} \setminus v_j$, $(Df^q)(i_o\ldots i_{q-1})(x) \simeq \theta_x$.

Note that $|\tilde{\alpha}_j s| \subset |s|$ for all $s \in S(X,\mathcal{H})$, thus $|Df^q| \subset |f^q|$.

Combining d and D,

$$d^{q-1}Df^q(i_o\ldots i_q) = \underset{k=o}{\overset{q}{\odot}}(\rho)r^{N_k}{}_N Df^q(i_o\ldots\hat{k}\ldots i_q)$$

$$= \underset{k=o}{\overset{q}{\odot}}(\rho)r^{N_k}{}_N(\underset{j\in\pi}{\odot}\tilde{\alpha}_j f^q(ji_o\ldots\hat{k}\ldots i_q)),$$

and $Dd^qf^q(i_o\ldots i_q) = \underset{h\in\pi,h}{\odot}\tilde{\alpha}_h d^qf^q(hi_o\ldots i_q)$

$$= \underset{h\in\pi,h}{\odot}\tilde{\alpha}_h(\underset{k=o}{\overset{q+1}{\odot}}(\rho)r^{M_k}{}_M f^q(i'_o\ldots\hat{k}\ldots i'_{q+1})),$$

where $i'_o\ldots i'_{q+1} = hi_o\ldots i_q$.

Note that $\tilde{\alpha}_j f^q(ji_o\ldots\hat{k}\ldots i_q) \simeq \theta$ if $v_j \cap N_k = \emptyset$, thus j must take on the values $i_o\ldots i_q$ in order to obtain nontrivial results.

Expanding and regrouping, one obtains

$$(d^{q-1}D \odot Dd^q)f^q(i_o\ldots i_q) \simeq \underset{n=o}{\overset{q}{\odot}}\tilde{\alpha}_{i_n} f^q(i_o\ldots i_q) \simeq f^q(i_o\ldots i_q),$$

or $d^{q-1}D \odot Dd^q \simeq 1_{C_F^q(v,\Sigma)}$.

Thus Ker $d^q \approxeq$ Im d^{q-1} and $H_F^q(v,\Sigma) \approxeq 0$. The result is immediate.

It is of interest to determine other properties which this cohomology theory enjoys. Mapping theorems present problems, for even though the inverse(and direct)image sheaf of a sheaf of H-spaces is well defined, the induced maps on the cochain spaces are not well defined in general. The excision property is satisfied, however. We begin the discussion with a definition of the relative cohomology spaces.

Definition 3.30 Let $A \subset X$, and $i:A \to X$ the inclusion map. We shall assume that A is locally closed in X if F is a paracompactifying family of supports.

The inclusion map induces an onto map

$$i_v^{\#}: C_F^q(v,\Sigma) \;\to\; C_{F|A}^q(v|A,\Sigma),$$

where $v|A = \{v_i \in v \mid v_i \cap A \neq \emptyset\}$, (every map in $C_{F|A}^q(v|A,\Sigma)$ may be extended trivially to $C_F^q(v,\Sigma)$), by the scheme

$$i_v^{\#}(f^q)(j_o,\ldots j_q) = f^q(ij_o\ldots ij_q) = f^q(j_o\ldots j_q),$$

where $(j_o\ldots j_q) \in v|A$, $f^q \in C_F^q(v,\Sigma)$, and $F|A = \{B \in F \mid B \subset A\}$.

Define $C_F^q(v,v|A,\Sigma) = \operatorname{Ker} i_v^{\#}$. Then, since $d^q(\operatorname{Ker} i_v^{\#}) \subset \operatorname{Ker} i_v^{\#}$, $H_F^q(v,v|A,\Sigma)$ is well defined and inherits a multiplication from $C_F^q(v,\Sigma)$.

In turn, $i_v^{\#}$ induces a map $i_v^*: H_F^q(v,\Sigma) \;\to\; H_{F|A}^q(v|A,\Sigma)$, since $i_v^{\#}d^q \simeq d^q i_v^{\#}$, defined by $i_v^*(\underline{f}^q) = \underline{i_v^{\#}f^q}$.

Note $\hat{p}^{w}{}_{v}^{*} = p^{w}{}_{v}^{*}|\operatorname{Ker} i_v^*: \operatorname{Ker} i_v^* \;\to\; \operatorname{Ker} i_w^*$, since $i_w^* p^{w}{}_{v}^{*} \simeq p^{w|A}{}_{v|A}^{*} i_v^*$, thus $\{H_F^q(v,v|A,\Sigma),\ \hat{p}^{w}{}_{v}^{*}\}$ forms an h-direct system.

Define the relative cohomology space as the limit

$$H_F^q(X,A,\mathcal{H}) = \underset{\rightsquigarrow}{L}\{\, H_F^q(v,v|A,\Sigma),\ \hat{p}^{w}{}_{v}^{*}\} .$$

Theorem 3.31 If $U \subset X$ is open, $\overline{U}$ is contained in the interior of $A \subset X$, and $j:X \setminus U, A \setminus U \;\to\; X,A$ is the inclusion map, then

$$j^*:\ H_F^*(X,A,\mathcal{H}) \overset{\approx}{\to} H_F^*(X \setminus U, A \setminus U, \mathcal{H}),$$

for any family of supports F.

<u>Proof</u> Consider the absolute spaces, that is $j: X \setminus U \to X$, and let ν be an open cover of X and $\omega = j^{-1}(\nu)$ an open cover of $X \setminus U$.

Then, by (3.30), the following short exact sequence is determined by j:

$$0 \to C_F^q(v,w,\Sigma) \to C_F^q(v,\Sigma) \xrightarrow{j^{\#}} C_{F|A}^q(w,\Sigma) \to 0 .$$

Let $f^q \in C_F^q(v,w,\Sigma)$, then $f^q(i_o \ldots i_q) \in S(N,\mathcal{H})$, where $N = \bigcap_{m=o}^{q} v_{i_m} \neq \emptyset$, and $j_v^{\#} f^q \simeq \theta^q \in C_{F|A}^q(w,\Sigma)$.

But $(j^{\#}f^q)(i_o \ldots i_q) = f^q(ji_o \ldots ji_q)$, and $j(i_o \ldots i_q) = (i_o \ldots i_q)$, so $j^{\#}f^q \simeq \theta^q$, or $C_F^q(v,w,\Sigma)$ is trivial and $j_v^{\#}$ is an isomorphism.

Thus j_v^* is an isomorphism and j^* is an isomorphism on the cohomology spaces as desired.

The relative case is obtained by the following argument. Restrict the covers of X to satisfy: $v_k \cap U \neq \emptyset$ implies $v_k \subset A$. Such a collection of covers is cofinal in the collection of all open covers of the pair X,A (see p. 243 [8]).

The following diagram is determined:

$$\begin{tikzcd}
0 \arrow[r] & C_F^q(v,v|A,\Sigma) \arrow[r, "\eta_v^{\#}"] \arrow[d, "\hat{j}_v^{\#}"] \arrow[dr, phantom, "\mathrm{I}"] & C_F^q(v,\Sigma) \arrow[r, "i_v^{\#}"] \arrow[d, "j_v^{\#}"] \arrow[dr, phantom, "\mathrm{II}"] & C_{F|A}^q(v|A,\Sigma) \arrow[r] \arrow[d, "j_{v|A}^{\#}"] & 0 \\
0 \arrow[r] & C_F^q(w,w|A,\Sigma) \arrow[r, "\eta_w^{\#}"] & C_F^q(w,\Sigma) \arrow[r, "i_w^{\#}"] & C_{F|A}^q(w|A,\Sigma) \arrow[r] & 0
\end{tikzcd}$$

The rows are exact and the maps $j_v^{\#}$ and $j_{v|A}^{\#}$ are isomorphisms by the above argument. Also, square I. commutes and square II. commutes up to homotopy.

Thus $\hat{j}_v^{\#}$ satisfies

$$C_F^q(v,v|A,\Sigma) \approx j_v^{\#}\eta_v^{\#}C_F^q(v,v|A,\Sigma) = \eta_w^{\#}\hat{j}_v^{\#}C_F^q(v,v|A,\Sigma) \approx \hat{j}_v^{\#}C_F^q(v,v|A,\Sigma),$$

and since $\eta_w^{\#}\hat{j}_v^{\#} = j_v^{\#}\eta_v^{\#}$, the map $\hat{j}_v^{\#}$ is onto and induces an isomorphism $\hat{j}_v^{*}: H_F^q(v,v|A,\Sigma) \to H_F^q(w,w|A,\Sigma)$.

Take limits to obtain the desired isomorphism on the cohomology spaces.

If $\mathcal{H}$ is a sheaf of H-spaces such that $\widetilde{\mathcal{H}} = \cup\mathcal{H}_x$ is a sheaf of algebraic structures (cf. (3.12)), then the following theorem demonstrates that the sheaf cohomology theory is contained in the cohomology theory defined above.

Theorem 3.32 $H_F^*(X,\mathcal{H}) \approx H_F^*(X,\widetilde{\mathcal{H}})$ as H-spaces.

Proof Consider the cochain spaces involved, $C_F^*(v,\Sigma)$ and $C_F^*(v,\widetilde{\Sigma})$. By (3.12), there is a map $\psi: \mathcal{H} \to \widetilde{\mathcal{H}}$ which is a homotopy equivalence on stalks. Thus, the map $\psi^*: C_F^*(v,\Sigma) \to C_F^*(v,\widetilde{\Sigma})$ is a homotopy equivalence and the cochain spaces are isomorphic as H-spaces. The desired isomrphism follows immediately.

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