CONTINUOUS CHARACTERS
OF COMMUTATIVE SEMIGROUPS

A Thesis

Presented to

the Faculty of the Department of Mathematics

University of Houston

In Partial Fulfillment

of the Requirements for the Degree

Master of Science

---

by

Susan Price Shrader

January 1968

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ABSTRACT

Let S be a compact commutative topological semigroup and H a closed subsemigroup of S. If χ is a continuous unit-character of H, it is possible to obtain the following necessary and sufficient conditions for χ to be extendable to S. First, $(x,y,a) \varepsilon H \times H \times S$ and $xa = ya$ then $\chi(x) = \chi(y)$. Also, $(x,y) \varepsilon H \times H$ and $xe = ye$ then $\chi(x) = \chi(y)$ where e is the least idempotent of S. Using these results, if χ is a continuous character of S, not necessarily a unit-character, further necessary and sufficient conditions for the extendability of χ are found. It is shown that χ can be extended to S if and only if there exists an open and closed prime ideal P such that $H \cap (S \setminus P)$ is the support of χ, and if x and y are elements of the support and a an element of the complement of P with $xa = ya$ then $\chi(x) = \chi(y)$. From these conditions, other criteria for extendability can be derived with the additional hypothesis that S is a pseudo-invertible semigroup. Finally, results are obtained which show that, to some extent, it suffices to consider continuous characters defined on closed subsemigroups of S which are unions of components of S.

The results in this paper parallel those of R. O. Fulp in his recent paper bearing the same title.

TABLE OF CONTENTS

CHAPTER		PAGE
I.	INTRODUCTION, DEFINITIONS OF TERMS USED, AND PROPERTIES OF CHARACTERS	1
	Introduction	1
	Definition of Terms Used	2
	Properties of Characters	5
II.	BACKGROUND PRELIMINARIES	7
III.	CONDITIONS FOR EXTENDABILITY	17
IV.	PROBLEMS FOR FURTHER RESEARCH	39
BIBLIOGRAPHY		41

CHAPTER I

INTRODUCTION, DEFINITIONS OF TERMS USED, AND PROPERTIES OF CHARACTERS

INTRODUCTION

There are two general areas of research on characters of semigroups. For characters of groups, the Pontryagin-van Kampen duality theorem asserts that a locally compact Abelian group is, in a natural way, iseomorphic to its second character group. The first area of research is concerned with whether an analogue of this theorem exists for commutative semigroups and with determining the structure of the character semigroup. The second area is concerned with determining necessary and sufficient conditions for a character defined on a subsemigroup of a commutative semigroup S to be extendable to a character on S.

The majority of papers published on characters have dealt with semigroups endowed with the discrete topology. The study of characters was initiated by Schwarz [17]* and Hewitt and Zuckerman [8,9]. Contributors to the further development and expansion of the theory include Clifford

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*Throughout this paper a bracketed number refers to the corresponding reference in the bibliography.

and Preston [2], Comfort [3,4], Ross [15,16], Hill [4,6,10, 11], and Fulp [6]. The pioneer in the study of characters of compact commutative Hausdorff topological semigroups was, again, Schwarz [18]. His contribution has been followed recently by Austin [1] and Fulp [5].

In his paper [5], Fulp determines certain necessary and sufficient conditions for a character defined on a subsemigroup of a compact commutative Hausdorff topological semigroup to admit an extension to the semigroup. He considers the particular case in which the range of the character is a subset of the boundary of the complex disc plus zero. It is the purpose of this paper to show many of Fulp's results hold for the entire complex disc and that if, in addition, the domain of the character is a pseudo-invertible semigroup, that the remainder of his results also apply. In [14], Y.-F. Lin defines a generalized character and describes the generalized character semigroup. The topic of generalized characters suggests an area of further research, that of determining necessary and sufficient conditions for a generalized character to admit an extension. In conclusion, some of the problems arising in this area are defined.

DEFINITIONS OF TERMS USED

Definition 1.1. A commutative semigroup is a nonempty set S together with a mapping $(x,y) \to xy$ on $S \times S$

to S such that $x(yz) = (xy)z$ and $xy = yx$ whenever x, y, $z \in S$. If S has an identity element, it is denoted by 1. An element z of S which has the property that for each s in S, $zs = sz = z$, is called a <u>zero</u> of S and is denoted by 0. S is said to be cancellable if for any non-zero elements x, y, z of S, $xy = xz$ implies $y = z$.

<u>Definition 1.2</u>. If S is a commutative semigroup and is also a compact Hausdorff topological space such that the mapping $(x,y) \to xy$ is continuous on $S \times S$, then S is a <u>compact commutative topological semigroup</u>. Since the term compact implies a topology exists on the space S, to say S is a <u>compact commutative semigroup</u> implies S is a compact commutative topological semigroup.

<u>Definition 1.3</u>. A <u>character</u> of a compact commutative semigroup S is a bounded, continuous, complex-valued function χ on S such that $\chi(x) \neq 0$ for some x in S and $\chi(xy) = \chi(x)\chi(y)$ for all x, y in S. The set of all characters of S is denoted by $S^\wedge$. The subset of $S^\wedge$ of characters of S which do not assume the value zero anywhere on S is denoted by S^*, and members of S^* are called <u>unit-characters</u>.

<u>Definition 1.4</u>. The set of all complex numbers z such that $|z| \leq 1$ is denoted by C. A <u>basis element for the topology on C</u> is of the form $\{z \in C: |z-a| < \varepsilon\}$ for each a in C and each positive real number ε.

Definition 1.5. For each compact subset K of S and each open subset U of C, let $X(K,U)$ denote the collection of all characters χ of S such that $\chi(K)$ is contained in U. The family of all such collections is a subbasis for the compact-open topology of $S^\wedge$. Thus a basis element in the compact-open topology of $S^\wedge$ is of the form $\bigcap_{i=1}^{n} X(K_i,U_i)$, where each K_i is compact in S and each U_i is open in C.

Definition 1.6. If χ is a character of a compact commutative semigroup S, the support of χ, denoted by S_χ, is defined to be $S_\chi = \{x\varepsilon S: \chi(x) \neq 0\}$.

Definition 1.7. If an element e in a semigroup S has the property that $e^2 = e$, then e is said to be an idempotent of S. The collection of all idempotents of S is denoted by E(S). There is a natural partial ordering of the set E(S) defined by $e \leq f$ if and only if $ef = fe = e$ for e, f in E(S).

Definition 1.8. An ideal of a commutative semigroup S is a subset I of S such that $IS \subset I$. A prime ideal is an ideal P such that $S\setminus P$ is a semigroup.

Definition 1.9. A semigroup S is said to be pseudo-invertible if for each x in S there is a positive integer n such that x^n is in some subgroup of S.

Definition 1.10. If S is a semigroup, H a sub-semigroup of S, and χ a character of H, then a character $\bar{\chi}$ of S is an extension of χ if $\bar{\chi}(x) = \chi(x)$ for all points x in H. If a character $\bar{\chi}$ with this property exists, then χ is said to be _extendable_ or to _admit an extension_ to all of S.

Definition 1.11. If S and T are topological semigroups, and f a function from S into T such that f is algebraically an isomorphism and topologically a homeomorphism, then f is said to be an _iseomorphism_.

PROPERTIES OF CHARACTERS

The following properties of a character of a compact commutative semigroup are immediate consequences of the definition of a character. These properties are used in the remainder of this study without specific reference being made to them.

Property 1. Let S be a compact commutative semigroup and χ a character of S. Then for each s in S, $|\chi(s)|\leq 1$.

Proof: Assume there exists an s in S such that $|\chi(s)|>1$. Then for each real number r, there is a positive integer n such that $|\chi(s^n)| = |\chi(s)|^n > r$. But this implies that χ is unbounded which contradicts the definition of a character. Hence the original assumption was incorrect, and for every s in S, $|\chi(s)| \leq 1$.

Property 2. Let S be a compact commutative semigroup and χ a character of S. If e is an idempotent of S, then either $\chi(e) = 0$ or $\chi(e) = 1$.

Proof: Suppose $\chi(e) \neq 0$. Since $e^2 = e$, $\chi(e)\chi(e) = \chi(e^2) = \chi(e) = \chi(e)\cdot 1$. Then C cancellable implies $\chi(e) = 1$. Hence $\chi(e) = 0$ or $\chi(e) = 1$.

Property 3. Let S be a compact commutative semigroup with a zero element and χ a character of S that is not identically 1. Then $\chi(0) = 0$.

Proof: Let $s \varepsilon S$ such that $\chi(s) \neq 1$. Since 0 is an idempotent, Property 2 implies that either $\chi(0) = 0$ or $\chi(0) = 1$. Assume $\chi(0) = 1$. Then $1 = \chi(0) = \chi(0\cdot s) = \chi(0)\chi(s) = \chi(s)$, a contradiction since $\chi(s) \neq 1$. Hence $\chi(0) = 0$ if χ is not identically 1 on S.

Property 4. Let G be a compact commutative group, e the identity of G, and χ a character of G. Then $\chi(e) = 1$.

Proof: Since e is an idempotent of G, either $\chi(e) = 0$ or 1. Assume $\chi(e) = 0$. Then for each g in G, $\chi(g) = \chi(eg) = \chi(e)\chi(g) = 0\cdot\chi(g) = 0$. But this contradicts the definition of a character since there does not exist a g in G for which $\chi(g) \neq 0$. Hence $\chi(e) = 1$.

Property 5. If G is a compact commutative group and χ a character of G, then for each g in G, $\chi(g) \neq 0$.

Proof: Assume for some g in G, $\chi(g) = 0$. Then $\chi(e) = \chi(gg^{-1}) = \chi(g)\chi(g^{-1}) = 0$, a contradiction. Hence for each g in G, $\chi(g) \neq 0$.

CHAPTER II

BACKGROUND PRELIMINARIES

The results in this chapter are included as preliminaries to be used as substantiation for the statements made in obtaining the primary results in Chapter III. If only the statement of a theorem is given, a complete proof appears in the reference cited. In all that follows, unless a specific designation is given, S denotes a compact commutative semigroup.

Lemma 2.1. Let S be a compact semigroup with a partial order $\leq$ such that for each s in S, $\{x \in S: x \leq s\}$ is closed. Then S contains a minimal element.

Proof: Assume that S does not have a minimal element. Let C' be a chain in S. By the Maximal Principle [12], there is a chain C in S which contains C' and is not contained in any other chain in S. For each t in C, let $C_t = \{x \in S: x \leq t\}$. By hypothesis, C_t is closed for each t. Consider the intersection of a finite number of the sets C_t, $\bigcap_{i=1}^{n} C_{t_i}$. Let t' be the least element of $\{t_1, \ldots t_n\}$ with respect to the simple ordering in C. Then $\bigcap_{i=1}^{n} C_{t_i} = C_{t'}$ is nonempty. Since S is compact, S has the finite intersection property, and hence $\bigcap_{t \in C} C_t$ is nonempty. Let $y \in \bigcap_{t \in C} C_t$. Since S does not have a minimal element, there exists a z in S such that $z < y$. Then

$z \varepsilon C$ and $y \notin C_z$. But this implies that $y \notin \bigcap_{t \varepsilon C} C_t$, which is a contradiction. Therefore, S contains a minimal element.

Theorem 2.1. Every compact semigroup contains at least one idempotent. [7]

Lemma 2.2. Let S be a compact commutative semigroup. Then

1. E(S) is a closed subsemigroup of S,

2. for each e in E(S), the set $\{f \varepsilon E(S): f \leq e\}$ is closed in E(S), and

3. E(S) contains a least element with respect to the natural partial order.

Proof: From Theorem 2.1, E(S) is nonempty. If E(S) contains only one element, the result is obvious. Suppose then that E(S) has more than one element. For the proof of 1., let e, $f \varepsilon E(S)$. Then $e^2 = e$, $f^2 = f$, and since S is commutative, it follows that $(ef)^2 = e^2 f^2 = ef$. Thus $ef \varepsilon E(S)$, and E(S) is a subsemigroup of S. Assume that E(S) is not closed. Then there exists an element p in $S \setminus E(S)$ such that p is a limit point of E(S). Since $p \notin E(S)$, $p^2 \neq p$. Let U and V be disjoint open sets containing p and p^2, respectively. Since multiplication is continuous, there exists an open set $\bar{U}$ containing p such that $\bar{U} \subset U$ and for every $u \varepsilon \bar{U}$, $u^2 \varepsilon V$. Now $\bar{U}$ is an open set containing the limit point p of E(S), and hence $\bar{U}$ contains an element e of E(S). Now $e^2 = e$ implies $e^2 \varepsilon \bar{U}$ and $e^2 \varepsilon V$. But this is clearly impossible since $\bar{U}$ and V are disjoint. Thus the assumption that $E(S)$ is not closed is incorrect, and the proof of 1. is complete.

For the proof of 2. let $e \varepsilon E(S)$ and $K = \{f \varepsilon E(S): f \leq e\}$. Assume that K is not closed in $E(S)$. Then there exists a p in $E(S) \setminus K$ such that p is a limit point of K, and $p^2 = p$, $pe \neq p$. Let U and V be disjoint open sets containing p and pe, respectively. Since multiplication is continuous, there exist open sets U_p containing p and U_e containing e such that for all $f \varepsilon U_p$, $g \varepsilon U_e$, $fg \varepsilon V$. Let $\bar{U} = U_p \cap U$. Then $\bar{U}$ is open, contains the limit point p, and thus contains a point f of K. Now $f \varepsilon K$ implies $fe = f$. Thus $f \varepsilon \bar{U}$ and $fe = f \varepsilon V$, a contradiction, and statement 2. follows.

From Lemma 2.1, $E(S)$ contains a minimal element e. For the proof of 3., it suffices to show that e is unique and is related to all other elements of $E(S)$. Suppose, then, that e is not unique and that f is another minimal element. Then $e \leq ef$, $f \leq ef$ and $e = e(ef) = ef = (ef)f = f$ implies $e = f$. Now let $g \varepsilon E(S)$. Then $e(eg) = eg$ implies $eg \leq e$. But since e is minimal, $e \leq eg$ and hence $e = eg$. Now $(eg)g = eg$ implies $eg \leq g$ and, therefore, $e \leq g$. Thus e and g are related, and the proof of the Lemma is complete.

<u>Lemma 2.3</u>. If G is a compact commutative group, then $G^\wedge$ is a group.

<u>Proof</u>: The product in $G^\wedge$ defined by $(\chi_1\chi_2)(g) = \chi_1(g)\chi_2(g)$ is clearly associative. Let χ_i denote the character identically one on G. For each χ in $G^\wedge$ and g in G,

$(\chi\chi_i)(g) = \chi(g)\chi_i(g) = \chi(g)$, and hence $\chi\chi_i = \chi$ which implies χ_i is the identity for $G^\wedge$. For each χ in $G^\wedge$, define χ^{-1} by $\chi^{-1}(g) = \chi(g^{-1})$. Then
$(\chi\chi^{-1})(g) = \chi(g)\chi^{-1}(g) = \chi(g)\chi(g^{-1}) = \chi(gg^{-1}) = \chi(e) = 1 = \chi_i(g)$.
Thus χ^{-1} is the inverse of χ. It must be shown that χ^{-1} is continuous. Since G is a topological group, the mapping I defined by $I(g) = g^{-1}$ is continuous. Thus the composition χI is continuous and is clearly χ^{-1}.

Theorem 2.2. The continuous characters of a compact commutative group separate elements in the group. [7]

Theorem 2.3. Let G be a compact Abelian group and H a subgroup of G that is either open or closed. If χ is a continuous character of H, then χ can be extended to a continuous character of G. [7]

Theorem 2.4. (Pontryagin-van Kampen duality theorem) Let G be a compact commutative group, $G^\wedge$ the character group of G, and $G^{\wedge\wedge}$ the character group of $G^\wedge$. For each $x \varepsilon G$, let $\Psi_x \varepsilon G^{\wedge\wedge}$ be the character on $G^\wedge$ defined by $\Psi_x(\chi) = \chi(x)$ for all $\chi \varepsilon G^\wedge$. Let τ be the mapping from G into $G^{\wedge\wedge}$ given by $\tau(x) = \Psi_x$. Then τ is an iseomorphism. [7]

Theorem 2.5. If S is a compact commutative semigroup, then the idempotent characters of S form a discrete subspace of $S^\wedge$. [1]

Theorem 2.6. Let $\Gamma(x)$ denote the closure of the set $\{x, x^2, \ldots\}$. Then $\Gamma(x)$ contains a unique idempotent called the idempotent belonging to x and denoted by e_x. [18]

Let ρ be the relation on S defined by $(x,y)\varepsilon\rho$ if $e_x = e_y$. Then ρ is an equivalence relation, and the equivalence classes modulo ρ are of the form $P_e = \{x\varepsilon S\colon e_x = e\}$. Then $S = \bigcup_{e\varepsilon E(S)} P_e$ and $P_e \cap P_f = \phi$ if $e \neq f$.

<u>Lemma 2.4</u>. Let P be an open prime ideal in the compact semigroup S. Then if $P_e \cap P \neq \phi$, $P_e \subset P$.

<u>Proof</u>: Let $x\varepsilon P_e \cap P$. Since P is an ideal $xS \subset P$ and, in particular, $xe\varepsilon P$. Also $(xe)S \subset P$. Since $S = \bigcup_{e\varepsilon E(S)} P_e$ and $eP_e \subset P_e$, $(xe)(eP_e) \subset P$. But since $x\varepsilon P_e$, $xe\varepsilon eP$ and $(xe)(eP_e) = eP_e$. Thus $eP_e \subset P$ and $e\varepsilon P$. Assume there exists a $y\varepsilon P_e$ such that $y \not\varepsilon P$, that is, $y\varepsilon S \setminus P$. Then P open implies $S \setminus P$ closed and $y\varepsilon S \setminus P$ implies $\Gamma(y) \subset S \setminus P$. Then $e_y \varepsilon S \setminus P$ which is a contradiction since $y\varepsilon P_e$ implies $e_y = e\varepsilon P$. Hence $y\varepsilon P_e$ implies $y\varepsilon P$, and $P_e \subset P$.

<u>Lemma.2.5</u>. Let S be a compact commutative semigroup, P an open prime ideal of S, and e the least element of $E(S \setminus P)$. Then $P = \bigcup\{P_f\colon ef \neq e\}$ and $S \setminus P = \bigcup\{P_f\colon ef = e\}$.

<u>Proof</u>: Since e is the least element of $E(S \setminus P)$, for every element f in $S \setminus P$, $e \leq f$ and hence $ef = e$. Suppose now that $f\varepsilon P$. Then since P is an ideal, $ef\varepsilon P$. If $ef = e$, then $ef\varepsilon S \setminus P$ which is impossible. Thus $f\varepsilon P$ implies $ef \neq e$. From Lemma 2.4, $f\varepsilon P$ implies $P_f \subset P$. Hence $P = \bigcup\{P_f\colon ef \neq e\}$ and $S \setminus P = \bigcup\{P_f\colon ef = e\}$.

<u>Theorem 2.7</u>. A compact cancellative semigroup is a topological group. [7]

<u>Theorem 2.8</u>. Let S be a compact commutative semigroup. Then Se is the maximal subgroup of S containing the idempotent e. [13]

<u>Lemma 2.6</u>. Let S be a compact commutative semigroup. Then $Se \subset P_e$ and $Se = eP_e$ for each e in E(S).

<u>Proof</u>: Let $xe \varepsilon Se$. Then $\Gamma(xe) \varepsilon Se$ since Se is closed, and $e_{xe} \varepsilon \Gamma(xe)$ implies $e_{xe} \varepsilon Se$. But since Se has a unique idempotent e, $e_{xe} = e$ which implies $xe \varepsilon P_e$. Thus $Se \subset P_e$, and it follows that $eSe = Se \subset eP_e$. Obviously $eP_e \subset Se$, and hence $Se = eP_e$.

<u>Lemma 2.7</u>. Let S be a compact commutative semigroup and χ a character of S. Then S_χ is an open subsemigroup of S, and $S \setminus S_\chi$ is a closed prime ideal.

<u>Proof</u>: Let $s, t \varepsilon S_\chi$. Then $\chi(s) \neq 0$, $\chi(t) \neq 0$, and $\chi(st) = \chi(s)\chi(t) \neq 0$ which implies $st \varepsilon S_\chi$. Thus S_χ is a subsemigroup of S. Now let $p \varepsilon S \setminus S_\chi$ and $s \varepsilon S$. Then $\chi(p) = 0$ and $\chi(ps) = \chi(p)\chi(s) = 0$ which implies $S \setminus S_\chi$ is an ideal of S, and it is prime since S_χ is a semigroup. The continuity of χ implies $S \setminus S_\chi$, the inverse image under χ of the closed set $\{0\}$, is closed. Thus S_χ is open.

<u>Lemma 2.8</u>. Let S be a compact commutative semigroup, H a closed subsemigroup of S, and e and f the least elements of E(S) and E(H), respectively. Then if H and He have a nonempty intersection, e = f.

<u>Proof</u>: Let $h \varepsilon H \cap He$. Then there exists an h_1 in H such that $h_1 e = h$. By definition of e and f, $e \leq f$ and

$ef = e$. Then $h_1ef = h_1e = hf$ which is an element of the subgroup Hf with identity f. Let $\hat{h}f$ denote the inverse of h_1e in Hf. Then $(h_1e)(\hat{h}f) = f$ which implies $h_1e\hat{h} = f$. But also $(h_1e\hat{h})e = fe$ implies $h_1e\hat{h} = e$. Thus $e = h_1e\hat{h} = f$.

<u>Lemma 2.9</u>. Let S be a compact commutative pseudo-invertible semigroup. Then P_e is a pseudo-invertible semigroup for each e in $E(S)$.

<u>Proof</u>: Let $y, z\varepsilon P_e$. Then $e_y = e$, $e_z = e$ and $e_{yz} = e_ye_z = e^2 = e$ implies $yz\varepsilon P_e$. Thus P_e is a semigroup. Now suppose $x\varepsilon P_e$. Since S is a pseudo-invertible semigroup, there exists a positive integer n such that x^n is in some subgroup G of S. From Theorem 2.8, the maximal subgroups of S are of the form Sf for f in $E(S)$. Thus for some $f\varepsilon E(S)$, $x^n\varepsilon G \subset Sf \subset P_f$. But $x^n\varepsilon P_e$ since x is an element of the semigroup P_e. Hence $x^n\varepsilon P_e \cap P_f$ and since $P_e \cap P_f = \phi$ if $e \neq f$, it follows that $e = f$, and $x\varepsilon Se \subset P_e$. It is now evident that for $x\varepsilon P_e$ there is a positive integer n such that x^n is in some subgroup of P_e, namely Se. Thus P_e is pseudo-invertible for each $e\varepsilon E(S)$.

<u>Lemma 2.10</u>. Let S be a compact commutative pseudo-invertible semigroup and χ a character of S. Then for each s in S, $|\chi(s)| = 1$ or $|\chi(s)| = 0$.

<u>Proof</u>: $S = \bigcup_{e\varepsilon E(S)} P_e$. Let $x\varepsilon S$. Then $x\varepsilon P_e$ for some $e\varepsilon E(S)$. Suppose first that $\chi(e) = 0$. Then for each $y\varepsilon Se \subset P_e$, $\chi(y) = 0$. If $x\varepsilon P \setminus Se$, since Lemma 2.9 implies P_e

is pseudo-invertible, there exists a positive integer n such that x^n is in some subgroup of P, and that this subgroup must be Se. This implies $\chi(x^n) = 0$. Then if $|\chi(x)| > 0$, $|\chi(x^n)| = |\chi(x)|^n > 0$ which is a contradiction. Therefore, if $\chi(e) = 0$, $\chi(x) = 0$ for each x in P_e. Now suppose that $|\chi(e)| = 1$. Then since χ is a homomorphism and Se a group, $\chi(Se)$ must be a group containing 1. $\{z\varepsilon C\colon |z| = 1\}$ is the maximal subgroup of C containing 1 which implies for each $y\varepsilon Se$, $|\chi(y)| = 1$. If $x\varepsilon P_e \setminus Se$, as above, there is a positive integer n such that $x^n\varepsilon Se$, and for each positive integer j, $(x^n)^j\varepsilon Se$. If $|\chi(x)| < 1$, then as $j \to \infty$, $|\chi(x^{nj})| = |\chi(x^n)|^j \to 0$. But this is a contradiction since $(x^n)\varepsilon Se$ implies $|\chi(x^n)|^j = 1$ for every j. Thus $|\chi(x)| = 1$. Therefore, if $|\chi(e)| = 1$, $|\chi(x)| = 1$ for each x in P_e. Since x is an arbitrary element of S, $|\chi(x)| = 0$ or 1 for each x in S.

Lemma 2.11. Let S be a compact commutative pseudo-invertible semigroup and χ a character of S. Then the open subsemigroup S_χ is also closed, and the closed prime ideal $S \setminus S_\chi$ is also open.

Proof: Since S is pseudo-invertible, Lemma 2.10 implies if $x\varepsilon S_\chi$, $|\chi(x)| = 1$, and if $x\varepsilon S \setminus S_\chi$, $\chi(x) = 0$. $U = \{z\varepsilon C\colon |z| < \tfrac{1}{2}\}$ is an open set in C, and the continuity of χ implies $\chi^{-1}(U)$ is open. Clearly $\chi^{-1}(U) = S \setminus S_\chi$. Thus $S \setminus S_\chi$ is open, and S_χ is closed.

Lemma 2.12. Let S be a compact commutative semigroup. Then the following statements are equivalent:

(1) ρ is the maximal cancellative congruence of S,

(2) ρ is the maximal group congruence of S,

(3) $(x,y)\varepsilon\rho$ if and only if $xa = ya$ for some $a\varepsilon S$, and

(4) $(x,y)\varepsilon\rho$ if and only if $xe = ye$ where e is the least element of E(S). [5]

Theorem 2.9. Let S be a compact commutative semigroup, e the least element of E(S), and G the maximal cancellative homomorphic image of S. Then

1. G is a topological group and is the maximal group homomorphic image of S,

2. G is iseomorphic to the maximal subgroup of S which contains e, and

3. the function Ψ defined by $x \to xe$ is a continuous homomorphism from S onto the maximal subgroup of S which contains e. [5]

Lemma 2.13. Let S be a compact commutative semigroup, $e_x\varepsilon\Gamma(x)$, $e_y\varepsilon\Gamma(y)$, $e_{xy}\varepsilon\Gamma(xy)$. Then $e_{xy} = e_xe_y$.

Proof: $e_{xy}\varepsilon\Gamma(xy)$ implies there exists a net $\{(xy)^{n_\alpha}\} \to e_{xy}$ and since S is commutative $\{(x^{n_\alpha}y^{n_\alpha})\} \to e_{xy}$. Now $\{x^{n_\alpha}\}$ contains a subnet $\{x^{n_\beta}\}$ such that $\{x^{n_\beta}\} \to e_x$ and $\{y^{n_\alpha}\}$ contains a subnet $\{y^{n_\gamma}\}$ such that $\{y^{n_\gamma}\} \to e_y$. Then $\{x^{n_\beta}y^{n_\gamma}\}$ contains a subnet $\{x^{n_\delta}y^{n_\delta}\}$ such that $\{x^{n_\delta}y^{n_\delta}\} \to e_xe_y$. But $\{x^{n_\delta}y^{n_\delta}\}$ is a subnet of $\{x^{n_\alpha}y^{n_\alpha}\}$ and hence $\{x^{n_\delta}y^{n_\delta}\} \to e_{xy}$. Thus $e_{xy} = e_xe_y$.

Theorem 2.10. Suppose that L is a connected subset of a space S and that $\{L_\alpha\}$ is a collection of connected subsets of S, each of which intersect L. Then $L \cup (\cup L_\alpha)$ is connected. [12]

Lemma 2.14. Let S be a topological semigroup and L_1 and L_2 components of S. Then L_1L_2 is contained in a component of S.

Proof: Let $s \varepsilon L_1$. Then sL_2, the image under continuous multiplication of the connected set L_2 is connected. Similarly, if $t_0 \varepsilon L_2$, L_1t_0 is connected. Now $L_1L_2 = \cup\{sL_2\colon s\varepsilon L_1\} = L_1t_0 \cup (\cup\{sL_2\colon s\varepsilon L_1\})$. For each set sL_2, $st_0 \varepsilon sL_2$ and $st_0 \varepsilon L_1t_0$. Hence Theorem 2.10 implies $L_1t_0 \cup (\cup\{sL_2\colon s\varepsilon L_1\})$ is connected. Thus L_1L_2 is contained in a component of S.

Theorem 2.11. If $\{X_n\}$ is a sequence of connected sets in a compact Hausdorff space S, and if lim inf X_n is not empty, then lim sup X_n is connected. [12]

It can be shown that this theorem also applies to a net $\{X_\alpha\}$ of connected sets in a compact Hausdorff space in which lim inf X_α is not empty.

CHAPTER III

CONDITIONS FOR EXTENDABILITY

<u>Theorem 3.1</u>. Let e be the least element of $E(S)$ and H a closed subsemigroup of S. If $\chi \varepsilon H^*$, then the following statements are equivalent:

1. there is an extension of χ to a continuous unit-character of S,

2. $(x,y,a) \varepsilon H \times H \times S$ and $xa = ya$ imply $\chi(x) = \chi(y)$, and

3. $(x,y) \varepsilon H \times H$ and $xe = ye$ imply $\chi(x) = \chi(y)$.

<u>Proof</u>: Suppose 1. is true. Let $\bar{\chi} \varepsilon S^*$ such that for each $x \varepsilon H$, $\bar{\chi}(x) = \chi(x)$. Then since $\bar{\chi}$ is a unit-character and C is cancellable, it is evident that each of 2. and 3. follow. Now let Ψ: $S \to Se$ be defined by $\Psi(x) = xe$ and consider the diagram:

$$\begin{tikzcd}
S \arrow[rr, "\Psi"] & & Se \\
H \arrow[u, phantom, sloped, "\subseteq"] \arrow[rr, "\Psi|H"] \arrow[dr] & & He \arrow[u, phantom, sloped, "\subseteq"] \arrow[dl] \\
& C &
\end{tikzcd}$$

where χ^*: $He \to C$ is defined by $\chi^*(he) = \chi(h)$. Then χ^* is well-defined, and since $\chi^*(\Psi(h)) = \chi^*(he) = \chi(h)$, χ^* is a homomorphism which makes the above diagram commutative. To see that χ^* is continuous, let F be a closed subset of C. Then since χ is continuous, $\chi^{-1}(F)$ is closed in the compact space H and, therefore, is compact. The continuity of Ψ implies $\Psi(\chi^{-1}(F))$ is compact and, hence, closed in He. Since $\chi^{*-1}(F) = \Psi(\chi^{-1}(F))$, $\chi^{*-1}(F)$ is closed. Thus the inverse image under χ^* of a closed set is closed, and χ^* is continuous. Now since S and H are compact and Ψ is continuous, Se and He are compact. From Theorem 2.8, Se and He are the maximal subgroups containing e of S and H respectively. Hence Se and He are compact commutative cancellative semigroups, and from Theorem 2.7, they are topological groups. By Theorem 2.9, χ^* can be extended to a continuous unit-character χ' of Se. Clearly, then, $\chi'\Psi$ is a continuous unit-character of S which extends χ.

The equivalence of 2. and 3. follows from Lemma 2.12.

<u>Corollary 3.1.1</u>. Let S be a compact commutative semigroup and H a closed subsemigroup of S. The following statements are equivalent:

1. each continuous unit-character of H can be extended to a continuous unit-character of S, and

2. if $(x,y,a)\varepsilon H\times H\times S$ and $xa = ya$, then $xe = ye$ where e is the least idempotent of H.

<u>Proof</u>: Suppose 2. is true. Let χ be a continuous unit-character of H and let $(x,y,a)\varepsilon H\times H\times S$ such that $xa = ya$. Then by hypothesis $xe = ye$ where e is the least idempotent of H, and $\chi(xe) = \chi(ye)$. Since $\chi\varepsilon H^*$, $\chi(e) \neq 0$ which implies that $\chi(e) = 1$, and

$\chi(x) = \chi(x)\chi(e) = \chi(xe) = \chi(ye) = \chi(y)\chi(e) = \chi(y)$.
Then by Theorem 3.1, χ can be extended to a continuous unit-character of S.

Suppose now that 1. is true. Assume there exists an element (x,y) in $H \times H$ such that $xe \neq ye$, but for some a in S, $xa = ya$. Now xe and ye are both elements of He, the maximal subgroup of H containing e. Let χ be any continuous unit-character of He such that $\chi(xe) \neq \chi(ye)$. The existence of χ is guaranteed by Theorem 2.2. If $\chi: H \to C$ is defined by $\chi(h) = \chi(he)$, clearly χ' is a continuous unit-character of H. By hypothesis, let χ^* be the extension of χ' to S. Then since C is cancellable, and $xa = ya$, $\chi^*(x) = \chi^*(y)$. But this implies $\chi(xe) = \chi'(x) = \chi^*(x) = \chi^*(y) = \chi'(y) = \chi(ye)$ which is a contradiction to the choice of χ. Hence the original assumption was false, and $xa = ya$ implies $xe = ye$ which is the desired conclusion.

<u>Corollary 3.1.2</u>. If S is a compact commutative semigroup, then the following statements are equivalent:

1. each continuous unit-character of each closed subsemigroup of S has an extension which is a continuous unit-character of S, and

2. if $(x,y,a) \in S \times S \times S$ and $xa = ya$, then $xe_xe_y = ye_xe_y$.

<u>Proof</u>: Let H be the closure of $\{x^iy^j : i = 1,2,\ldots; j = 1,2,\ldots\}$. Then from Lemma 2.13 H is a closed subsemigroup of S containing e_xe_y as the least idempotent. The proof

then follows from Corollary 3.1.1.

It is desirable to know not only when a continuous unit-character of a closed subsemigroup admits an extension, but under what conditions such an extension is unique. For compact commutative semigroups, the following theorem shows the question can be reduced to the corresponding question of the existence of a unique extension of a continuous character of a topological subgroup.

Theorem 3.2. Let S be a compact commutative semigroup, e the least idempotent of S, and χ a continuous unit-character of a closed subsemigroup H of S. Then any two extensions of χ to a continuous unit-character of S agree on He. For each χ which has an extension to S, let χ_e denote the restriction of any such extension to He. Then χ has a unique extension to S if and only if the continuous group character χ_e of the subgroup He of the topological group Se has a unique extension to Se.

Proof: Suppose χ_1 and χ_2 are continuous unit-characters of S which extend χ. Then $\chi_1(e) = \chi_2(e) = 1$, and if $he \in He$, $\chi_1(he) = \chi_1(h)\chi_1(e) = \chi_1(h)\cdot 1 = \chi_2(h)\chi_2(e) = \chi_2(he)$. Hence χ_1 and χ_2 agree on He. Since χ is a continuous unit-character, $\chi(e) = 1$, and for each s in S, $\chi(s) = \chi(s)\cdot 1 = \chi(s)\chi(e) = \chi(se)$. Hence any continuous unit-character of S is determined by its values on Se, and the remainder of the theorem follows.

If S and H are as is Theorem 3.2, let $\phi: S^* \to H^*$ be defined by $\phi(\chi) = \chi|H$. By the first Corollary to Theorem 3.1, ϕ is an onto mapping. Thus for each $\bar{\chi} \varepsilon H^*$ there is a χ in S^* such that $\phi(\chi) = \chi|H = \bar{\chi}$, if and only if $(x,y,a) \varepsilon H \times H \times S$ and $xa = ya$ imply $xe = ye$ where e is the least element of E(H).

<u>Lemma 3.1</u>. Let S be a compact commutative semigroup and H a closed subsemigroup of S. Let ϕ denote the function from S^* into H^* defined by $\phi(\chi) = \chi|H$. Then ϕ is an iseomorphism if and only if

1. $(x,y,a) \varepsilon H \times H \times S$ and $xa = ya$ imply $xe = ye$, where e is the least element of E(H) and

2. each two continuous group characters of the topological group Se which agree on the subgroup He of Se are identical.

<u>Proof</u>: Suppose 1. and 2. are true. From 1., the first Corollary to Theorem 3.1 implies ϕ is an onto mapping. Let $\chi_1, \chi_2 \varepsilon S^*$. Then
$\phi(\chi_1\chi_2) = (\chi_1\chi_2)|H = (\chi_1|H)(\chi_2|H) = \phi(\chi_1)\phi(\chi_2)$. Now if $\chi \varepsilon H^*$, from 1. and Corollary 3.1.1 again, χ has an extension to a continuous unit-character of S, and from 2. and Theorem 3.2, the extension is unique. Hence ϕ is a monomorphism. For the proof of the continuity of ϕ, let $\chi \varepsilon S^*$ and $V = \bigcap_{i=1}^{n} X(C_{H_i}, U_i)$ be a basis element in H^* containing $\phi(\chi)$. Then since C_{H_i} is compact in H and H is closed in S, C_{H_i} is compact in S for $i = 1,2,\ldots,n$. Then if $U = \bigcap_{i=1}^{n} X(C_{H_i}, U_i)$,

obviously $\phi(U) \subset V$ and hence ϕ is continuous. For the proof of the continuity of ϕ^{-1}, let $\chi \varepsilon H^*$ and $V = \bigcap_{i=1}^{n} X(C_{S_i}, U_i)$ be a basis element in S^* containing $\phi^{-1}(\chi)$. Since H is compact, $C_{S_i} \cap H$ is compact in S for i = 1, ..., n, and $U = \bigcap_{i=1}^{n} (C_{S_i} \cap H, U_i)$ is a basis element in H^*. It is clear that $\phi^{-1}(U) \subset V$, and hence ϕ^{-1} is continuous. Consequently, ϕ is an iseomorphism.

Suppose now that ϕ is an iseomorphism. Then the first Corollary to Theorem 3.1 immediately implies 1. Since ϕ is a monomorphism, each character χ of H has a unique extension to S. Hence from Theorem 3.2, 2. is satisfied.

Theorem 3.3. Let S be a compact commutative semigroup and H a closed subsemigroup of S. Let e and f be the least elements of E(S) and E(H) respectively, and let $\Gamma: He^* \to H^*$ be defined by $\Gamma(\chi)(h) = \chi(he)$ for each $\chi \varepsilon H^*$ and $h \varepsilon H$. Then

1. Γ is always a bicontinuous monomorphism and is an iseomorphism if and only if $(h_1, h_2) \varepsilon H \times H$ and $h_1 e = h_2 e$ imply $h_1 f = h_2 f$,

2. S^* and Se^* are iseomorphic, and

3. S^* and H^* are iseomorphic if and only if the topological groups Se and Hf are iseomorphic.

Proof: Let $\chi_1, \chi_2 \varepsilon He^*$. Then for each h in H, $\Gamma(\chi_1\chi_2)(h) = (\chi_1\chi_2)(he) = \chi_1(he)\chi_2(he) = (\Gamma(\chi_1)(h))(\Gamma(\chi_2)(h))$. Now suppose $\Gamma(\chi_1) \neq \Gamma(\chi_2)$. Then there is an h in H such that $\chi_1(he) = \chi_2(he)$ and hence $\chi_1 \neq \chi_2$. Thus Γ is a monomorphism. For the proof of the continuity of Γ, let $\chi \varepsilon H^*$ and $V = \bigcap_{i=1}^{n} X(C_{H_i}, U_i)$ be a basis element in H^* containing $\Gamma(\chi)$.

Since He is closed, $C_{H_i}e$ is a compact subset of He for $i = 1,\ldots,n$, and if $U = \bigcap_{i=1}^{n} X(C_{H_i}e, U_i)$, for each $\chi \varepsilon U$ and each h_ie in $C_{H_i}e$, $\Gamma(\chi)(h_ie) = \chi(h_ie\cdot e) = \chi(h_ie) = \chi(h_i)\chi(e) = \chi(h_i)$ for $1 = 1,\ldots,n$. Hence $\Gamma(U) \subset V$, and Γ is continuous. Now let $U = \bigcap_{i=1}^{n} X(K_i, U_i)$ be a basis element in He*. Then for $i = 1,\ldots,n$, $K_i = \hat{K}_ie$ for some $\hat{K}_i$, and since K_i is a compact subset in the compact space He, K_i is closed in He. Let p be a point of the closure of $\hat{K}_i$. Then p is the limit of a net $\{p_\alpha\}$ of points of $\hat{K}_i$. Since $\{p_\alpha\} \subset \hat{K}_i$, $\{p_\alpha e\}$ is a net from K_i, and by continuity of multiplication $\{p_\alpha e\} \to pe$. K_i closed in He implies $pe \varepsilon K_i = \hat{K}_ie$, and hence $p \varepsilon \hat{K}_i$. Consequently, since $\hat{K}_i$ contains each of its closure points, $\hat{K}_i$ is closed and hence compact in H. Thus $\bigcap_{i=1}^{n} X(\hat{K}_i, U_i)$ is a basis element in H*, and if $R(\Gamma)$ denotes the range of Γ, $V = \bigcap_{i=1}^{n} X(\hat{K}_i, U_i) \cap R(\Gamma)$ is a basis element in the range of Γ. For χ in U, since $\chi(K_i) \subset U_i$, $\chi(\hat{K}_ie) \subset U_i$. This implies $\Gamma(\chi)(k) = \chi(k_ie) \varepsilon U$ for each $k \varepsilon \hat{K}_i$, and hence $\Gamma(U) \subset V$. Now if $\chi \varepsilon V$, for each $k \varepsilon \hat{K}_i$, $\chi(k) \varepsilon U$ and $\Gamma^{-1}(\chi)(ke)$ or $\chi(k) \varepsilon U$. Thus $V \subset \Gamma(U)$. Clearly, then, $\Gamma(U) = V$, and Γ is an open mapping. Suppose now that Γ is an onto mapping. Then if $\bar{\chi} \varepsilon H^*$, there is a $\chi \varepsilon He^*$ such that for each h in H, $\chi(he) = \bar{\chi}(h)$. Let h_1 and h_2 be elements of H such that $h_1e = h_2e$. Then by definition of e and f, $e \leq f$ and hence $ef = e$. Now

$$\bar{\chi}(h_1f) = \chi(h_1fe) = \chi(h_1e) = \chi(h_2e) = \chi(h_2fe) = \bar{\chi}(h_2f).$$

Hence for every $\bar{\chi} \varepsilon H^*$, $\bar{\chi}(h_1f) = \bar{\chi}(h_2f)$. But $\bar{\chi}$ is a unit-character of the topological group Hf which contains h_1f and h_2f, and since unit-characters of a topological group separate elements of the group, $h_1f = h_2f$. Conversely, suppose $(h_1,h_2)\varepsilon H\times H$ and $h_1e = h_2e$ imply $h_1f = h_2f$. For the special case where $e = f$, e is an element of H, and He is a subgroup of H. Clearly $\bar{\chi}|H$ is such that $\Gamma(\bar{\chi}|H) = \bar{\chi}$. If $e \neq f$, from Lemma 2.8, $He \cap H$ is void. Thus if χ is the mapping from He into C defined by $\chi(he) = \bar{\chi}(hf)$, χ is well-defined, and χ is a continuous homomorphism. The first assertion is immediate from the condition $h_1e = h_2e$ implies $h_1f = h_2f$. Since Hf is a subgroup of H, and $\bar{\chi}|Hf$ is a continuous homomorphism, the second claim follows. Thus $\chi\varepsilon He^*$ and for each h in H, $\Gamma(\chi)(h) = \chi(he) = \bar{\chi}(hf) = \bar{\chi}(h)\bar{\chi}(f) = \bar{\chi}(h)$. Consequently, Γ is an onto mapping, and the proof of 1. is complete.

Statement 2. follows immediately from 1. since S is a compact subsemigroup of itself, and in this case $e = f$.

For the proof of 3., let $\cong$ denote the relation of iseomorphism, and suppose $Se \cong Hf$. Then $Se^* \cong Hf^*$, and from 2., $S^* \cong Se^* \cong Hf^* \cong H^*$ or $S^* \cong H^*$. Conversely, suppose S^* and H^* are iseomorphic. Then $Se^* \cong S^* \cong H^* \cong Hf^*$ implies $Se^* \cong Hf^*$. Thus $Se^{**} \cong Hf^{**}$, and since Se and Hf are compact topological groups, an application of the Pontryagin-Van-Kampen duality theorem gives $Se \cong Hf$, the desired conclusion.

<u>Lemma 3.2</u>. Let S be a compact commutative pseudo-invertible semigroup, H a closed subsemigroup of S, and χ a continuous character of H. Then χ can be extended to a continuous character of S if and only if there exists an open and closed prime ideal P of S satisfying

(1) $(S \setminus P) \cap H = S_\chi$, and

(2) $(x,y,a) \varepsilon S_\chi \times S_\chi \times (S \setminus P)$ and $xa = ya$ imply $\chi(x) = \chi(y)$.

<u>Proof</u>: Suppose there exists an open and closed prime ideal P of S satisfying (1) and (2). Then $S \setminus P$ is closed, and since H is closed, $S_\chi = (S \setminus P) \cap H$ is closed. Since P is a prime ideal, $S \setminus P$ is a compact commutative semigroup. Now $\chi | S_\chi$ is a unit-character of the closed subsemigroup S_χ of $S \setminus P$ for which condition (2) holds. Thus Theorem 3.1 implies $\chi | S_\chi$ can be extended to a continuous unit-character $\bar{\chi}$ of $S \setminus P$. If $\bar{\chi}(x) = 0$ for all x in P, clearly, $\bar{\chi}$ is a homomorphism which extends χ. For the proof of the continuity of $\bar{\chi}$, let U be an open set in C and s an element of $\bar{\chi}^{-1}(U)$. If $s \varepsilon P$, let $V_s = P$. Then V_s is an open set containing s such that $\bar{\chi}(V_s) \subset U$. If $s \varepsilon S \setminus P$, then $\bar{\chi}(s) \neq 0$. Since C is a Hausdorff topological space, there exist disjoint open sets 0_1 and 0_2 such that $0 \varepsilon 0_1$ and $\bar{\chi}(s) \varepsilon 0_2$. Then $0_2 \cap U$ is an open set in C that contains $\bar{\chi}(s)$ but does not contain 0. Since $\bar{\chi} | (S \setminus P)$ is a continuous unit-character of $S \setminus P$, there exists an open set V_s in $S \setminus P$ such that $\bar{\chi}(V_s) \subset 0_2 \cap U \subset U$.

If $V = \bigcup_{s \varepsilon \bar{\chi}^{-1}(U)} V_s$, V is an open set in S and $V = \bar{\chi}^{-1}(U)$. Thus the inverse image under $\bar{\chi}$ of an open set in C is open in S, and $\bar{\chi}$ is continuous. It is of interest to note here that pseudo-invertibility was not necessary for this part of the proof.

Suppose now that χ has an extension $\bar{\chi}$ to S. Then from Lemma 2.11 $S_{\bar{\chi}}$ is an open and closed subsemigroup of S, and $S \setminus S_{\bar{\chi}}$ is an open and closed prime ideal. Let $P = S \setminus S_{\bar{\chi}}$. Then, obviously, $(S \setminus P) \cap H = S_{\bar{\chi}}$. Also if $(x,y,a) \varepsilon S_{\chi} \times S_{\chi} \times (S \setminus P)$ and $xa = ya$, then $\chi(x) \neq 0$, $\chi(y) \neq 0$, $\bar{\chi}(a) \neq 0$ and since C is cancellable, $\chi(x)\bar{\chi}(a) = \bar{\chi}(xa) = \bar{\chi}(ya) = \chi(y)\bar{\chi}(a)$ implies $\chi(x) = \chi(y)$. Thus $P = S \setminus S_{\chi}$ satisfies conditions (1) and (2).

<u>Definition 3.1</u>. An element e in E(S) is said to be a <u>generating idempotent</u> of a compact semigroup S if the open prime ideal $\{P_f : ef \neq e\}$ is also closed. If F is a subset of some subsemigroup H of S, and if $F = H \cap [\bigcup_f \{P_f : e \leq f\}]$ for some e in E(S), then e <u>induces</u> F.

<u>Theorem 3.4</u>. Let S be a compact commutative pseudo-invertible semigroup and χ a character of a closed sub-semigroup H of S. Let e be the least element of $E(S_{\chi})$. In order that χ admit an extension which is a character of S, it is necessary and sufficient that

1. there exists a generating idempotent f of S which lies under e and which has the property that if x is a member

of the maximal subgroup of H containing e such that $xf = f$, then $\chi(x) = 1$, and

2. there exists a generating idempotent f' of S which induces S_χ.

<u>Proof</u>: Suppose that χ can be extended to a character of S. Then from Lemma 3.2 there exists an open and closed prime ideal P which satisfies conditions (1) and (2) of the Lemma. Let f be the least element of $E(S \setminus P)$. Since $S_\chi \subset S \setminus P$ and e is the least element of $E(S_\chi)$, $f \leq e$. From Lemma 2.5, P open and f the least element of $E(S \setminus P)$ imply $P = \bigcup\{P_g\colon fg \neq f\}$ and $S \setminus P = \bigcup\{P_g\colon fg = g\}$. Since P is closed, f is a generating idempotent. Let $x \varepsilon He$ such that $xf = f$. Then $e \varepsilon E(S_\chi)$ implies $\chi(e) = 1$, and since He is a subgroup containing x, $\chi(x) \neq 0$ and, therefore, $x \varepsilon S_\chi$. Then $(x,e,f) \varepsilon S_\chi \times S_\chi \times (S \setminus P)$ and $xf = f = ef$ implies $\chi(x) = \chi(e) = 1$ from condition (2) of Lemma 3.2. Thus 1. is satisfied. Statement 2. follows from the previous remark that $S \setminus P = \bigcup\{P_g\colon fg = f\}$ and the fact that $H \cap (S \setminus P) = S_\chi$.

Suppose now that, respectively, f and f' satisfy 1. and 2. of the theorem. Let $P_1 = \bigcup\{P_g\colon f \nleq g\}$ and $P_2 = \bigcup\{P_g\colon f' \leq f\}$. Since f and f' are generating idempotents, P_1 and P_2 are open and closed subsemigroups of S. Thus $S \setminus (P_1 \cup P_2)$ is an open and closed subsemigroup and hence contains a least idempotent $\bar{\bar{f}}$. Since $(P_1 \cup P_2)$ is an open and closed prime ideal, $\bar{\bar{f}}$ is a generating idempotent.

Let $P = \bigcup\{P_g: \bar{f} \not\leq g\}$. Then $S \setminus P = \bigcup\{P_g: \bar{f} \leq g\}$, and if k is an idempotent in $S \setminus P$, $\bar{f} \leq k$ which implies $f \leq \bar{f} \leq k$ and $k \varepsilon S \setminus P_1$. Similarly $k \varepsilon S \setminus P$ implies $k \varepsilon S \setminus P_2$. Thus $k \varepsilon (S \setminus P_1) \cap (S \setminus P_2) = S \setminus (P_1 \cup P_2)$ and $S \setminus P \subset S \setminus (P_1 \cup P_2)$. Suppose now that $k \geq f$ and $k \geq f'$. Then $k \varepsilon S \setminus P_1$ and $k \varepsilon S \setminus P_2$ which implies that $k \varepsilon S \setminus (P_1 \cup P_2)$. Since $\bar{f}$ is the least idempotent of $S \setminus (P_1 \cup P_2)$, $k \geq \bar{f}$. For the proof that (1) of Lemma 3.2 holds, note that $S_\chi = (S \setminus P_2) \cap H$. Then if $x \varepsilon S_\chi$, $x \varepsilon H$ and $e_x \geq e \geq f$ since 1. is true. Also $x \varepsilon S \setminus P_2$ implies $e_x \geq f'$ and hence $e_x \geq \bar{f}$. Thus $x \varepsilon (S \setminus P) \cap H$ and $S_\chi \subset (S \setminus P) \cap H$. Suppose now that $x \varepsilon (S \setminus P) \cap H$. Then $e_x \geq \bar{f} \geq f'$ and $x \varepsilon (S \setminus P_2) \cap H = S_\chi$. Thus $(S \setminus P) \cap H \subset S_\chi$ and hence $S_\chi = (S \setminus P) \cap H$. Therefore (1) of Lemma 3.2 is satisfied. Suppose now that $(x,y,a) \varepsilon S_\chi \times S_\chi \times (S \setminus P)$ and $xa = ya$. Then for each positive integer n, $xa^n = ya^n$. By definition of e_a, there exists a subsequence $\{a^{n_i}\}$ of $\{a^n\}$ such that $\{a^{n_i}\} \to e_a$. Therefore, by continuity of multiplication, $xa^n = ya^n$ implies $xe_a = ye_a$. Now, $e_a \varepsilon S \setminus P$ implies $e_a \geq \bar{f}$ and $e_a \geq f$. Then $xe_a f = ye_a f$, $xf = yf$ and $xef = yef$. Now xe and ye are in He, the maximal subgroup of H containing e. Thus $(ye)^{-1}$ exists, $(ye)(ye)^{-1} = e$, and $(ye)^{-1}(xe)$ is also an element of He. Let $\bar{x} = (ye)^{-1}(xe)$. Then $\bar{x}f = (ye)^{-1}(xe)f = ef = f$, and from 1., $\chi(\bar{x}) = 1$. This implies $\chi((ye)^{-1}(xe)) = 1$ or $\chi(ye) = \chi(xe)$. Since $e \varepsilon S_\chi$, $\chi(e) = 1$ and $\chi(x) = \chi(xe) = \chi(ye) = \chi(y)$, and (2) of lemma

3.2 is satisfied. Thus χ can be extended to a continuous character of S.

Definition 3.2. Let A be a set that is partially ordered by $\leq$. A is said to be directed if given α,β in A, there exists γ in A such that $\alpha \leq \gamma$ and $\beta \leq \gamma$.

Definition 3.3. Let A be a directed set and $\{G_\alpha\colon \alpha\varepsilon A\}$ a family of groups indexed by A. For each pair of indices α,β satisfying $\alpha \leq \beta$, assume there is a homomorphism $\phi_{\alpha\beta}\colon G_\alpha \to G_\beta$, and assume further that these homomorphisms satisfy the condition: if $\alpha \leq \beta \leq \gamma$, then $\phi_{\alpha\gamma} = \phi_{\beta\gamma}\phi_{\alpha\beta}$. Then the family $[\{G_\alpha\}, \{\phi_{\alpha\beta}\}]$ is called a direct system of groups over A, with groups G_α and connecting homomorphisms $\phi_{\alpha\beta}$.

Definition 3.4. The image of $g_\alpha \varepsilon G_\alpha$ under any connecting homomorphism is called a successor of g_α. From the direct system of groups $[\{G_\alpha\}, \{\phi_{\alpha\beta}\}]$, a limit group is constructed in the following manner. Let $D = \Pi\{G_\alpha\colon \alpha\varepsilon A\}$ be the infinite direct product of the groups, and call two elements $g_\alpha\varepsilon G_\alpha$, $g_\beta\varepsilon G_\beta$ in D equivalent whenever they have a common successor in the direct system. This relation, ρ, is an equivalence relation. That ρ is reflexive and symmetric is obvious. For the verification of transitivity, suppose that g_α, g_β have a common successor in G_δ and that g_β, g_γ have one in G_σ. Then since A is directed, there is an index T such that $\rho \leq T$, $\sigma \leq T$ and the successor of g_β in G_T is evidently a successor of both g_α and g_γ.

Definition 3.5. Let $[\{G_\alpha\}, \{\phi_{\alpha\beta}\}]$ be a direct system of groups. The quotient group $\Pi G_\alpha/\rho$ is called the direct limit of the system, and is denoted by G^∞. It can be shown that direct limits exist in the category of Abelian groups.

In the following, unless otherwise specified, let S denote a compact commutative pseudo-invertible semigroup, H a closed subsemigroup of S, and F a subset of H whose complement in H is a proper open and closed prime ideal of H. Let e denote the least element of E(F) and E_e the set of all generating idempotents lying below e.

Lemma 3.3. If f, $f' \varepsilon E_e$ and $f \leq f'$, let $\phi_{ff'}$ denote the function from the group $(fHe)^*$ into the group $(f'He)^*$ defined by

$\phi_{ff'}(\chi)(xf') = \chi(xff') = \chi(xf)$. Then

$[\{(fHe)^*\}_{f\varepsilon E_e}, \{\phi_{ff'}\}_{f\leq f'}]$ is a direct system of groups.

Proof: To see the $\phi_{ff'}$ is well-defined, let $\chi\varepsilon(fHe)^*$ and suppose $\phi_{ff'}(\chi) = \chi_1$ and $\phi_{ff'}(\chi) = \chi_2$. Then for each xf' in $f'He$, $\chi_1(xf') = \phi_{ff'}(\chi)(xf) = \chi_2(xf')$. Thus $\chi_1 = \chi_2$. Now let $\phi_{ff'}\varepsilon\{\phi_{ff'}\}_{f\leq f'}$ and χ, $\Psi\varepsilon(fHe)^*$. Then $\phi_{ff'}(\chi\Psi)(xf') = (\chi\Psi)(xf) = \chi(xf)\Psi(xf) = (\phi_{ff'}(\chi)(xf'))(\phi_{ff'}(\Psi)(xf') = (\phi_{ff'}(\chi)\phi_{ff'}(\Psi))(xf')$ which implies $\phi_{ff'}$ is a homomorphism. Now for $f \leq f' \leq f''$, if $\chi\varepsilon(fHe)^*$, denote $\phi_{ff'}(\chi)$ by χ'. Then $(\phi_{f'f''}(\phi_{ff'}(\chi))(xf'') = (\phi_{ff''}(\chi'))(xf'') = \chi'(xf') = \phi_{ff'}(\chi)(xf') = \chi(xf)$. Then since $\phi_{ff''}(\chi)(xf'') = \chi(xf)$, $\phi_{f'f''}\phi_{ff'} = \phi_{ff''}$. Also $\phi_{ff}(\chi)(xf) = \chi(xf)$ implies ϕ_{ff} is the identity homomorphism on $(fHe)^*$. Thus $[\{(fHe)^*\}_{f\varepsilon E_e}, \{\phi_{ff'}\}_{f\leq f'}]$ is a direct system of groups. Let $H_e^{*\infty}$ denote the direct limit of this system.

Lemma 3.4. Let Ψ denote the function from $H_e^{*\infty}$ into He^* defined by $\Psi(t)(x) = \chi(xf)$ where $x\varepsilon He$, $f\varepsilon E_e$, $\chi\varepsilon(fHe)^*$ and $t\varepsilon H_e^{*\infty}$ is the equivalence class which contains the "string" $\langle\phi_{fg}(\chi)\rangle_{f\leq g\varepsilon E_e}$. Then Ψ is a monomorphism embedding $H_e^{*\infty}$ into He^*, and if $e\varepsilon E_e$, Ψ is an isomorphism. Ψ is called the <u>natural embedding</u>.

<u>Proof</u>: For the proof that Ψ is well-defined, suppose $t = [\langle\phi_{fg}(\chi)\rangle]_{f\leq g\varepsilon E_e}$ and $t = [\langle\phi_{f'g}(\chi')\rangle]_{f'\leq g\varepsilon E_e}$. Then there exists a $g\varepsilon E_e$ such that $\phi_{fg}(\chi) = \phi_{f'g}(\chi')$, and $\chi(xf) = \phi_{fg}(\chi)(gx) = \phi_{f'g}(\chi')(gx) = \chi'(xf')$. Hence Ψ is well-defined. Multiplication of two elements t and t' of $H_e^{*\infty}$ is defined in the following manner. Let $t = [\langle\phi_{fg}(\chi)\rangle_{f\leq g\varepsilon E_e}]$, $t' = [\langle\phi_{f'g}(\chi')\rangle_{f'\leq g\varepsilon E_e}]$ and let f" be any element of E_e lying above both f and f'. Then $tt' = [\langle\phi_{f''g}(\phi_{ff''}(\chi)\cdot\phi_{ff''}(\chi'))\rangle_{f''\leq g\varepsilon E_e}]$. Now

$$\Psi(tt')(x) = (\phi_{ff''}(\chi)\cdot\phi_{f'f''}(\chi'))(xf'') = \phi_{ff''}(\chi)(xf'')\phi_{f'f''}(\chi)(xf'')$$
$$= (\Psi(t)\Psi(t'))(xf'').$$

Thus Ψ is a homomorphism. Suppose now that $\Psi(t) = \Psi(t')$. Then for some $x\varepsilon H^*$, $\chi(xf) = \Psi(t)(x) = \Psi(t')(xf') = \chi'(xf')$. Assume that $t \neq t'$. Then $\chi(xf) = \phi_{fg}(\chi)(xf) \neq \phi_{f'g}(\chi')(xf') = \chi'(xf')$, a contradiction. Thus $\Psi(t) = \Psi(t')$ implies $t = t'$, and Ψ is a monomorphism. Suppose that $e\varepsilon E_e$. Let $\chi\varepsilon H^*$ and let t be the equivalence class containing the "string" $\langle\phi_{eg}(\chi)\rangle_{e\leq g\varepsilon E_e}$.

Now $\chi\varepsilon(eHe)^* = (He)^*$, and $\chi(xe) = \chi(x)$. Thus $\Psi(t)(x) = \chi(xe) = \chi(x)$ implies that Ψ is an onto mapping and, hence, an isomorphism.

<u>Theorem 3.5</u>. Using all notation as previously established, each continuous character of H having support F can be extended to a continuous character of S if and only if $H_e^{*\infty}$ is isomorphic to H_e^* under the natural embedding and F is induced by some generating idempotent of S.

A complete proof of this theorem is omitted, but it can be shown that the theorem is a consequence of Lemma 3.4 and Theorem 3.4. The results obtained in the corollaries to this theorem are more readily applied than the theorem itself.

<u>Corollary 3.5.1</u>. If S is a compact commutative pseudo-invertible semigroup, H a closed subsemigroup of S, and χ a continuous character of H such that the least element of $E(S_\chi)$ is a generating idempotent of S, then χ has an extension which is a continuous character of S.

<u>Proof</u>: Let $F = S_\chi$. Then $e\varepsilon S_\chi$ and e a generating idempotent implies F is induced by a generating idempotent. Also since $e\varepsilon E_e$, from Lemma 3.4, $H_e^{*\infty}$ is isomorphic to H_e^*. Thus the hypothesis of Theorem 3.5 is satisfied, and χ can be extended to a continuous character of S.

For the next two corollaries, let S_χ/S denote the set $\{x\varepsilon S\colon xs\varepsilon S_\chi \text{ for some } s\varepsilon S\}$.

<u>Corollary 3.5.2</u>. If S is a compact commutative pseudo-invertible semigroup, H a closed subsemigroup of S, and χ a continuous character of H, then χ can be extended to a continuous character of S if S_χ/S is open.

<u>Proof</u>: Let e_o be the least element of $E(S_\chi)$, and let $x \in \bigcup\{P_f\colon e_o \leq f\}$. Then $e_o \leq e_x$ and $e_o e_x = e_o$ which implies $e_x \in S_\chi/S$. Thus if S_χ/S were closed, it would follow that $x \in S_\chi/S$ and that $\bigcup\{P_f\colon e_o \leq f\} \subset S_\chi/S$. For the proof that S_χ/S is closed, assume that this is not the case. Then there exists a limit point x of S_χ/S and $x \notin S_\chi/S$. Then there is a net $\{x_\alpha\}$ in S_χ/S such that $\{x_\alpha\} \to x$. For each x in S_χ/S, by definition, there exists a b_α such that $b_\alpha x_\alpha \in S_\chi$. The net $\{b_\alpha x_\alpha\}$ contains a subnet $\{b_\gamma x_\gamma\}$ such that $\{b_\gamma x_\gamma\} \to p$ which is a point of S_χ since S_χ is closed. Now $\{b_\gamma x_\gamma\} \to \hat{b}x$ which is not in S_χ since $x \notin S_\chi/S$. Then $\{b_\gamma x_\gamma\} \to p = \hat{b}x$ since $\{x_\gamma\} \to x$ and $\{b_\gamma x\} \to \hat{b}x$. But this implies that $p = \hat{b}x \in S_\chi$, a contradiction. Thus S_χ/S is closed and the desired inclusion $\bigcup\{P_f\colon e_o \leq f\} \subset S_\chi/S$ follows. Now let $x \in S_\chi/S$. Then there exists an s in S such that $xs \in S_\chi$ and $e_o \leq e_{xs}$. From Lemma 2.13, $e_{xs} = e_x e_s$. Also $(e_x e_s)e_x = e_x e_s$ implies $e_x e_s \leq e_x$. Thus $e_o \leq e_{xs} = e_s e_x \leq e_x$ implies $e_o \leq e_x$ and hence $x \in \bigcup\{P_f\colon e_o \leq f\}$. Then $S_\chi/S \subset \bigcup\{P_f\colon e_o \leq f\}$. Thus $S_\chi/S = \bigcup\{P_f\colon e_o \leq f\}$ and since S_χ/S is open by hypothesis, e_o is a generating idempotent of S. From Corollary 3.5.1, χ can be extended to a continuous character of S.

Corollary 3.5.3. Let S be a compact commutative semigroup and H a subsemigroup of S which is both open and closed. Then each continuous character of H can be extended to a continuous character of S.

Proof: Since H is open, S_χ is open in S. Let $x \varepsilon S_\chi/S$. Then there exists an $s \varepsilon S$ such that $xs \varepsilon S_\chi$. Since S_χ is open, there are open sets U and V such that $xs \varepsilon Us \subset UV \subset S_\chi$. Thus $x \varepsilon U \subset S_\chi/S$ and S_χ/S is open. Hence, from Corollary 3.5.2, χ can be extended to a continuous character of S.

Theorem 3.6. Let S be a compact commutative pseudo-invertible semigroup, H a closed subsemigroup of S and χ a continuous character of H. If K is the union of all components of S which intersect H, then K is a closed subsemigroup of S. If K_χ denotes the union of all those components which intersect S_χ, then in order that χ be extendable to K, it is necessary and sufficient that

1. for each component L of S either $(L \cap H) \cap S_\chi = \phi$, or $(L \cap H) \cap (S \setminus S_\chi) = \phi$, and

2. $(x,y,a) \varepsilon S_\chi \times S_\chi \times K_\chi$ and $xa = ya$ imply $\chi(x) = \chi(y)$.

Proof: For the proof that K is a semigroup, let $x, y \varepsilon K$. Then there exist components L_x and L_y of S which intersect H and contain x and y respectively. Now let $\hat{x} \varepsilon L_x \cap H$ and $\hat{y} \varepsilon L_y \cap H$. From Lemma 2.14, the product $L_x L_y$ of two components is contained in some component L of S. Since H is a semigroup and $\hat{x}$ and $\hat{y}$ are in H, $\hat{x}\hat{y} \varepsilon H$ and also

$\hat{x}\hat{y}\varepsilon L_x L_y \subset L$. Thus $\hat{x}\hat{y}\varepsilon L \cap H$ and hence $L \subset K$. Then $xy\varepsilon L \subset K$ and K is a semigroup. For the proof that K is closed, let x be a limit point of K and $\{x_\alpha\}$ a net in K such that $\{x_\alpha\} \to x$. Let L_α denote the component in K containing x_α for each x_α in the net. Since $L_\alpha \subset K$, each L_α intersects H. For each α, let $t_\alpha \varepsilon L_\alpha \cap H$. Then $\{t_\alpha\}$ contains a convergent subnet $\{t_\alpha'\} \to t$. Since each t_α' is in H, H closed implies $t\varepsilon H$. Let L denote the component containing t. Then $t\varepsilon L \cap H$ implies $L \subset K$. Now since $\{X_\alpha\} \to x$, for every open set 0 containing x, there is a γ such that for $\alpha' > \gamma$, $L_\alpha' \cap 0 \neq \phi$. Thus $x \,\varepsilon \lim\inf L_\alpha'$. From Theorem 2.11, $\lim\sup L_\alpha'$ is connected, and hence $\lim\sup L_\alpha' \subset L \subset K$. Thus $x \,\varepsilon \lim\inf L_\alpha' \subset \lim\sup L_\alpha' \subset L \subset K$ implies $x\varepsilon K$ and K is closed.

Suppose χ has an extension $\bar{\chi}$ to K. Let L be a component of S such that $L \cap H \neq \phi$. Assume there exist y, z such that $y\varepsilon (L \cap H) \cap S_\chi$ and $z\varepsilon (L \cap H) \cap (S \setminus S_\chi)$. Then $|\chi(y)| = 1$ and $\chi(z) = 0$. But L connected and χ continuous imply $\chi(L)$ is connected. Hence $|\chi(y)| = 1$ and $\chi(z) = 0$ is an obvious contradiction, and 1. follows. For each component $L \subset K_\chi$, it also follows that for each $a\varepsilon L$, $\chi(a) \neq 0$ by a similar argument. Thus if $(x,y,a)\varepsilon S_\chi \times S_\chi \times K_\chi$, then $\chi(x)\bar{\chi}(a) = \bar{\chi}(xa) = \bar{\chi}(ya) = \chi(y)\bar{\chi}(a)$ which implies $\chi(x) = \chi(y)$ since C is cancellable and none of these terms are zero. Thus 2. is true.

Suppose now that 1. and 2. are true. Since $S_\chi \subset K_\chi \subset K$, S_χ closed in S, and K closed in S, S_χ is closed in

K_χ and $\chi | K_\chi$ is a unit-character of K_χ. Then condition 2. together with Theorem 3.1 imply that χ can be extended to a unit-character χ' on K_χ. Define $\bar{\chi}(x) = 0$ if $x \varepsilon K \setminus K_\chi$ and $\bar{\chi}(K) = \chi'(K)$. Clearly, then, $\bar{\chi}$ is an extension of χ, and the proof of the theorem is complete.

<u>Definition 3.6</u>. Let S denote a compact commutative semigroup. Then there exists a totally disconnected compact commutative semigroup D and a continuous homomorphism ϕ from S onto D such that $\{\phi^{-1}(d): d \varepsilon D\}$ is precisely the set of components of S. [13] The semigroup D will be called the <u>canonical totally disconnected image</u> of S.

<u>Theorem 3.7</u>. Let S be a compact commutative pseudo-invertible semigroup and H a closed subsemigroup of S. Using the notation of the previous paragraph, let $K = \phi^{-1}(\phi(H))$. Then K is a closed subsemigroup of S which is a union of components of S, and $E(K^\wedge)$ is iseomorphic to $E((\phi(H))^\wedge) = E((\phi(K))^\wedge)$.

<u>Proof</u>: Since H is closed in the compact space S, H is compact, and since ϕ is continuous, $\phi(H)$ is compact and hence closed in D. Then K, the inverse image under the continuous mapping ϕ of the closed set $\phi(H)$, is closed. Now $K = \phi^{-1}(\phi(H)) \subset H$ implies $\phi(H) \subset \phi(K)$. Also $\phi(K) = \phi(\phi^{-1}(\phi(H))) \subset \phi(H)$. Therefore, $\phi(H) = \phi(K)$. But this implies $(\phi(H))^\wedge = (\phi(K))^\wedge$ and $E(\phi(H))^\wedge = E(\phi(K))^\wedge$. To see that $\phi(K)$ is the canonical totally disconnected image of K, assume that this is not the case. Then some component L of S properly contains a subset of K and $\phi(K) = d_1$, $\phi(L \setminus K) = d_2$

for d_1, d_2 in D. But then $\phi(L)$ is the set $\{d_1, d_2\}$, and the continuous image of the connected set L is not connected, a contradiction. Therefore, $\phi(K)$ is the canonical totally disconnected image of K. Now if χ is an idempotent character of K, $\chi(k) = 0$ or $\chi(k) = 1$ for each $k\epsilon K$, and in particular, $\chi(L) = 0$ or $\chi(L) = 1$ for otherwise, a contradiction similar to the one just given is reached. Let Ψ denote the mapping from $E(K^\wedge)$ into $E((\phi(K))^\wedge)$ defined by $\Psi(\chi)(d) = \chi(k)$ where $\chi\epsilon E(K^\wedge)$ and $k\epsilon K$ such that $\phi(k) = d$. Now $\phi^{-1}(\phi(K)) = \phi^{-1}(d) = L$, a component of K, and since $\chi(L) = 0$ or $\chi(L) = 1$ for all $k\epsilon L$, the definition of $\Psi(\chi)$ is independent of the choice of $k\epsilon K$ such that $\phi(k) = d$, that is, Ψ is well-defined. For the proof that Ψ is a homomorphism, let $\chi_1, \chi_2 \epsilon E(K^\wedge)$. Then for each d in $\phi(K)$ and k such that $\phi(k) = d$,

$$\begin{aligned}\Psi(\chi_1\chi_2)(d) &= (\chi_1\chi_2)(k) = \chi_1(k)\chi_2(k) = (\Psi(\chi_1)(d))(\Psi(\chi_2)(d)) \\ &= (\Psi(\chi_1)\Psi(\chi_2))(d).\end{aligned}$$

Thus Ψ is a homomorphism. Suppose $\Psi(\chi_1) = \Psi(\chi_2)$ for $\chi_1, \chi_2 \epsilon E(K^\wedge)$. Then for all $d\epsilon\phi(K)$ $\Psi(\chi_1)(d) = \Psi(\chi_2)(d)$ and $\chi_1(k) = \Psi(\chi_1)(d) = \Psi(\chi_2)(d) = \chi_2(K)$ for each $k\epsilon K$ such that $\phi(k) = d$ implies $\chi_1 = \chi_2$. Thus Ψ is a monomorphism. From Theorem 2.5, $E(K^\wedge)$ and $E((\phi(K))^\wedge)$ are discrete. Thus Ψ is bicontinuous, and hence an iseomorphism.

<u>Corollary 3.7.1</u>. Let S be a compact commutative pseudo-invertible semigroup and ϕ the natural homomorphism from S onto its canonical totally disconnected image D. If H is a closed subsemigroup of S and $L \cap H$ is connected for

each component L of S, then $E(H^\wedge)$ is iseomorphic to $E((\phi(H))^\wedge)$.

Proof: Let $K = \phi^{-1}(\phi(H))$. Then from Theorem 3.6, $E(K^\wedge)$ is iseomorphic to $E((\phi(H))^\wedge)$. For the proof that $E(K^\wedge)$ is iseomorphic to $E(H^\wedge)$, let Ψ denote the mapping from $E(H^\wedge)$ into $E(K^\wedge)$ defined by

$$\Psi(\chi)(k) = \begin{cases} 1 & \text{if } k \in \phi^{-1}(\phi(\chi^{-1}(1))) \\ 0 & \text{if } k \in \phi^{-1}(\phi(\chi^{-1}(0))). \end{cases}$$

Since $L \cap H$ is connected for each component C of S, and χ is continuous for each $\chi \in E(H^\wedge)$, for each component $L \cap H$ of H, $\chi(L \cap H) = 0$ or $\chi(L \cap H) = 1$. Thus $\phi(\chi^{-1}(0)) \cap \phi(\chi^{-1}(1))$ is empty, and Ψ is well-defined. It can be shown, in a manner exactly analogous to the argument given in Theorem 3.7, that Ψ is an iseomorphism. Thus, letting $\cong$ denote the relation of iseomorphism, $E((\phi(H))^\wedge) \cong E(K^\wedge) \cong E(H^\wedge)$, and the Corollary follows.

Corollary 3.7.2. Let S be a compact commutative pseudo-invertible semigroup and ϕ the natural homomorphism from S onto its canonical totally disconnected image D. Then $E(S^\wedge)$ is iseomorphic to $E(D^\wedge)$.

Proof: Obviously S is a closed subsemigroup of itself such that $L \cap S$ is connected for each component L of S. An application of Corollary 3.7.1 gives $E(S^\wedge) \cong E((\phi(S))^\wedge) = E(D^\wedge)$.

CHAPTER IV

PROBLEMS FOR FURTHER RESEARCH

In [14], Lin defines a generalized character. He considers an arbitrary but fixed compact commutative cancellable semigroup T with zero z and unit u such that the complement of the maximal subgroup containing u, $T \setminus H(u)$, is a subsemigroup of T. A generalized character is, then, a continuous homomorphism from a compact commutative semigroup S into T. With the additional hypothesis that S is pseudo-invertible, Lin shows that the generalized character semigroup can be decomposed into the union of a disjoint family of groups.

One question to be considered is whether analogues of the theorems presented in the previous chapter exist for generalized characters. Basic to the development of the results of CHAPTER III is the fact that the characters of a compact commutative group separate elements of the group. It is not immediately obvious that this is also the case for generalized characters. For a character χ defined on a subgroup of a compact commutative group, Theorem 2.3 implies that χ can be extended to the group. The proof of this theorem depends strongly on the fact that the nonzero complex numbers under multiplication form a divisible group. It is well known that every homomorphism from a subgroup of a group into a divisible group can be extended to the group. [7] If the range T of a generalized character is required to be divisible, the problem of proving the continuity of the extension of a homomorphism remains. In the proofs of Theorems 3.6 and 3.7, a property relied upon is the fact that the range of a character of a compact commutative pseudo-invertible semigroup is a subset of the boundary of the complex disc plus zero. This seems to indicate that more restrictions must be placed on the range T in order to obtain analogous results for generalized characters. One possible solution is that T be required to be a topological group with an isolated zero.

The problem of determining conditions under which the second generalized character semigroup is isomorphic to the semigroup appears to be no less formidable than the extension problem. In attempting to determine such conditions for second character semigroups, the powerful Pontryagin-van Kampen duality theorem is available, at least for groups. But for generalized characters, even this tool is absent.

BIBLIOGRAPHY

1. C. W. Austin, Duality theorems for some commutative semigroups, Trans. Amer. Math. Soc. 109 (1963), 245-256.

2. A. H. Clifford and B. G. Preston, Algebraic theory of semigroups, Amer. Math. Soc. Mathematical Surveys (No. 7), 1961.

3. W. W. Comfort, The isolated points in the dual of a commutative semigroup, Proc. Amer. Math. Soc. 11 (1960), 227-233.

4. W. W. Comfort and Paul Hill, On extending nonvanishing semicharacters, Proc. Amer. Math. Soc. 4 (1966), 936-941.

5. R. O. Fulp, Continuous characters of compact semigroups, to appear.

6. R. O. Fulp and Paul Hill, Reflexive semigroups, Proc. Amer. Math. Soc. 16 (1965), 1096-1098.

7. E. Hewitt and K. A. Ross, Abstract harmonic analysis I, Springer-Verlag, Berlin, 1963.

8. E. Hewitt and H. S. Zuckerman, Finite dimensional convolution algebras, Acta Math. 93 (1955), 67-119.

9. E. Hewitt and H. S. Zuckerman, The ℓ_1-algebra of a commutative semigroup, Trans. Amer. Math. Soc, 83 (1956), 70-97.

10. Paul Hill, A solution to the nonvanishing semicharacter problem, Proc. Amer. Math. Soc. 5 (1966), 1178-1182.

11. Paul Hill, Characters of commutative semigroups, Journal of Algebra, Vol. 5, No. 1, Jan. 1967, 16-24.

12. J. G. Hocking and G. S. Young, Topology, Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1961.

13. K. H. Hofmann and P. S. Mostert, Elements of compact semigroups, Charles E. Merrill Books, Inc., Columbus, Ohio, 1966.

14. Y.-F. Lin, Generalized character semigroups: the Schwarz decomposition, Pacific Jour. Math., Vol. 15, No. 4, (1965), 1307-1312.

15. K. A. Ross, A note on extending semicharacters on semigroups, Proc. Amer. Math. Soc. 10 (1959), 579-583.

16. K. A. Ross, Extending characters on semigroups, Proc. Amer. Math. Soc. 12 (1961), 988-990.

17. S. Schwarz, The theory of characters of finite commutative semigroups (Russian), Czechoslovak Math. Jour. 4 (1954), 219-247.

18. S. Schwarz, The theory of characters of commutative Hausdorff bicompact semigroups, Czechoslovak Math. Jour. 6 (81) (1956), 330-361.