CONSUMPTION AND LABOR SUPPLY UNDER DYNAMIC OPTIMIZATION

A Dissertation

Presented to

the Faculty of the Department of Economics

University of Houston

In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

By

Tony H. Elavia

August, 1984

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Acquiring a doctoral degree takes time and I have accumulated many intellectual debts in the process. My greatest debt in completing the dissertation is to Jeff Parker and Peter Zadrozny. They stimulated my interest in macroeconomics and helped me far beyond the call of duty.

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ABSTRACT

The goal of this dissertation is to test the hypothesis that economic agents jointly choose current consumption and labor supply, so as to maximize the present discounted value of current and future utility. We then investigate the response of such maximizing agents to transitory and permanent shocks from the exogenous environment consisting of real wages, real interest rates and taxes. The response of consumption is in broad conformity with the predictions of the premanent income hypothesis. Labor supply exhibits an inelastic response with respect to both shocks - a result at odds with an important postulate of the new classical macroeconomics.

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CHAPTER I

INTRODUCTION

The goal of this dissertation is to test the hypothesis that economic agents jointly choose current consumption and labor supply, so as to maximize the present discounted value of current and future utility. We then investigate the response of such maximizing agents to transitory and permanent shocks from the exogenous environment consisting of real wages, real interest rates and taxes.

Previous studies have shown no agreement on the nature of these responses. Friedman (1957) and Hall (1978) show that consumption responds to changes in permanent rather than transitory income, while Flavin (1981) shows transitory income to be important as well. The results of Lucas and Rapping (1969) indicate that labor supply is highly elastic with respect to transitory shocks in the real wage, but completely inelastic with respect to permanent shocks. As per reasons given in Chapter II, these results are also disputed. In our view, most previous studies are subject to at least one of the following two shortcomings. First, the failure to model consumption and labor supply jointly rather than separately. Second, the absence of an explicit utility maximization hypothesis as a premise for explaining behavior.

In this dissertation we attempt to remove both these shortcomings.[1]

In our model, agents' utility is a function of consumption, labor supply and financial assets. They derive income from supplying their labor (wage income), and earning interest on their financial assets (interest income). Income earned can be spent on consumption or saved, and saving is added to augment last period's stock of financial assets. Our agents are assumed to live in an environment characterized by exogenous real wages, real interest rates and taxes. Their objective is to pick an optimal time path for consumption and labor supply so as to maximize the present discounted value of current and future utility. Their expectations regarding the future are rational in the sense of Muth (1961).

This, together with the structure of the maximum problem imposes numerous cross-equation parameter restrictions. The traditional, elasticity coefficients in the consumption and labor supply functions, emerge as non-linear combinations of parameters belonging to the utility function and the exogenous stochastic processes characterizing the economic environment. This is in stark contrast to the usual representation of the elasticity coefficients as a set of

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[1]That such an analysis is desirable is stated in Lucas (1977).

free parameters unrestricted by economic theory. That such a representation (i.e., with free parameters) can lead to very misleading policy evaluation was first shown by Lucas (1976). He argued that according to virtually any form of dynamic economic theory (of which the present model is an example), the traditional elasticity coefficients are functions of parameters of the maximum problem (as discussed). Under alternative policy regimes which alter the form of the exogenous stochastic processes, the elasticity coefficients will shift. Hence, a behavioral relation whose elasticity coefficients have been estimated from past data as free parameters, cannot be used for policy evaluation purposes. Our model allows us to track the shift in the elasticity coefficients, given a change in the exogenous stochastic processes, and the assumption that the utility function parameters are policy invariant. In this way, the imposition of our theoretically motivated cross-equation parameter restrictions allows us to perform policy evaluation not subject to the Lucas critique. Also, ignoring them can lead to inefficient parameter estimates, as shown by Nerlove (1972) and Sims (1974).

Another feature of our model is the presence of non-linear adjustment costs, represented as quadratic terms in the utility function.[1] These imply diminishing marginal utility of consumption and assets, and increasing marginal disutility of labor supply. It is the presence of such non-linear adjustment costs which allow us to interpret a seemingly inefficient lagged response as being optimal, or a disequilibrium system to be in equilibrium.[2]

The chapters in this dissertation are organized in the following way. Chapter II presents a selective review of the literature. In Chapter III, we state the maximum problem and solve it to obtain optimal decision rules for consumption and labor supply. We also discuss model simulation over alternative policy regimes. Chapter IV contains the approach to model estimation, parameter estimates and the simulation results. Finally, Chapter V contains our concluding remarks and suggestions for further study.

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[1]Second order effects are recognized in the utility function. The optimal decision rules (derived later) recognize only first order effects.

[2]See Zadrozny (1980) for a critique of a typical "disequilibrium" viewpoint in the labor demand literature.

CHAPTER II

REVIEW OF THE LITERATURE

As indicated earlier, past studies have dealt separately with consumption and labor supply behavior. There is a very extensive literature on consumption behavior; both theoretical and empirical.[1] Empirical research on aggregate labor supply behavior is limited, though considerable theoretical research has been done.[2] In this chapter we will limit ourselves to reviewing research which bears some methodological resemblance to the present study.

Recently much research has concentrated on examining the validity of the permanent income hypothesis (PIH), first propounded by Friedman (1957). Basically, the PIH implies that agents base current consumption on their expected lifetime (permanent) income and not current income. Recently several efforts were made to study the PIH when expectations of income were formed rationally in the sense of Muth (1961). Hall (1978) showed that utility maximizing agents with rational expectations will tend to generate a consumption series which would follow a random walk. This striking result was not rejected by his empirical analysis and has generated much heated discussion in the literature. Sargent

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[1]Ott, Ott, and Yoo (1975) present an exhaustive listing of the pre-1960's research. More recent literature can be found in Gordon (1981).

[2]For a good listing of the empirical microeconomic literature on labor supply see Lucas and Rapping (1969).

(1978) statistically shows that income Granger-causes[1] consumption and hence the latter does not follow a random walk. He sought to test the PIH, where consumption is a function of permanent income and permanent income is generated by a high order autoregressive process. The resulting cross-equation parameter restrictions (imposed by the rational expectations assumption) were soundly rejected by the post-war United States data. Flavin (1981) showed that Sargent's definition of permanent income was flawed, and if corrected, would yield the same model as Hall's. She proposed a structural model of consumption, of which Hall (1978) is a reduced form. This allows her to recover a parameter for the "excess sensitivity" of consumption to current income. This parameter tracks the impact of current income on consumption after the change in permanent income due to an innovation in the current income has been accounted for. Since her empirical estimates revealed a large and significant "excess sensitivity" she rejected the PIH. Using a more general framework, Hayashi (1982) shows how the definition of the consumption variable plays a critical role in testing. The PIH is rejected for total consumption expenditures but not rejected for a series which substitutes service flows from durables instead of expenditures on them.

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[1]Causation is in the sense of Granger (1969).

Our interest in this literature stems from the forward-looking aspect of the PIH, a characteristic shared by our model. Also policy evaluation research based on PIH type models like Blinder (1981) reach conclusions which need to be checked. Blinder concludes (after estimating a version of the consumption function based on the PIH), that a transitory tax cut is roughly half as effective as a permanent tax cut, in stimulating consumption over a 1 year horizon. Though his policy evaluation is not subject to the Lucas critique, his model is not derived from a choice-theoretic basis. Another example of a policy related conclusion is work by Boskin (1978) which showed highly interest elastic saving (and therefore consumption) behavior, and Howrey and Hymans (1978) who found through more careful testing that no such relationship exists. The results of Blinder (1975, 1981) are in agreement with Howrey and Hymans (1978).

From our discussion of the modern literature, no consensus seems to be emerging regarding the validity of the PIH (i.e., the forward-looking hypothesis), or the specific influences of income and interest rate variables on consumption. The need for a more detailed study of the problem is evident. A similar need is also evident for the subject of aggregate labor supply, as the review below indicates.

A well-known empirical estimate of an aggregate labor supply function is in Lucas and Rapping (1969). Their influential paper showed the 1929-65 data for the United States to be consistent with an equilibrium interpretation of the labor market. Also, agents are assumed to choose current consumption and labor supply so as to maximize the present discounted value of current and future utility. However, in their empirical estimation, only the supply and demand curves for labor were estimated. Forward-looking suppliers of labor responded to current and expected future wages. Labor supply was shown to be highly elastic (elasticity = 1.6), with respect to transitory and completely inelastic with respect to permanent wage shocks respectively. In this way the authors explained short run cyclical fluctuations in employment and the long run insensitivity with respect to wages. Altonji and Ashenfelter (1980) showed that another (and equivalent) way of formulating Lucas and Rapping's model is to express the deviation of current from normal employment as a function of the deviation of the current from the expected future wage. They infer that in order to explain cyclical employment fluctuations, the difference between the current and expected future wage must exhibit cyclical fluctuations. Their empirical analysis failed to show the necessary cyclical fluctuations, because real wages seemed to follow a random walk.

Though the results of Lucas and Rapping (1969) are intuitively appealing, the above mentioned paper casts some doubt on their validity. Their model is not subject to the Lucas (1976) critique of policy evaluation because they clearly distinguish between the structural (policy invariant) and forecasting (policy variant) parameters. The fault could lie in their assumption of adaptive expectations and neglect of the consumption function in their estimation.[1]

In conclusion it can be said that no effort has been made to estimate jointly consumption and labor supply functions which have been derived from an explicit maximum problem. Since this is the direction in which both theoretical microeconomics and macroeconomics are going, it is only natural that empirical efforts be made in the same direction.

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[1]Even if Lucas and Rapping (1969) would have estimated the consumption function jointly with labor supply, they could not have incorporated the full set of cross-equation parameter restrictions, in the absence of an explicit maximum problem. This omission would lead to an efficiency loss for their parameter estimates.

CHAPTER III

THE DYNAMIC CONSUMPTION AND LABOR SUPPLY MODEL

This chapter presents a model where economic agents choose optimal levels of consumption and labor supply. In Section 1 we state the maximum problem, and solve for the optimal decision rules. Section 2 discusses the properties of the model and Section 3 concludes the chapter with a note on model-simulation.

3.1 The Maximum Problem

Our notation will use upper case letters to denote the original variables of the model, and lower case for their logarithms except where noted. Denoting time by the subscript t, the original variables of the model are

C_t = total real consumption in period t.

N_t = total employment in period t.

P_t = nominal price level in period t.

A_t = stock of real financial assets at the end of period t.

$\tilde{r}^*_t$ = nominal rate of interest (not in log form) in period t.

$\tilde{W}^*_t$ = nominal wage rate in period t.

TAX_t = total nominal taxes on income in period t.

A superscript * indicates a nominal variable corresponding to the real variables above. We will begin by discussing the utility function and the transition equation for financial assets, followed by a statement of the maximum problem.

The aggregate utility function is assumed to be (with all parameters positive)

$$(3.1) \qquad J_t = b_1c_t - b_2n_t + b_3a_t - b_4c_t^2 - b_5n_t^2 - b_6a_t^2,$$

hence utility is increasing in consumption and assets, and decreasing in labor supply. The function is concave in its arguments implying a diminishing marginal utility of consumption and assets, and an increasing marginal disutility of labor. For tractability we have ignored cross-product terms and the temporal dependence of J_t on lagged values of its arguments.

Usually financial assets (a_t) are excluded from a utility function. The argument, that it is the level of consumption, and not assets, which the agent cares about is used to justify this exclusion. This argument has merit if capital markets are perfect but not if agents have liquidity constraints. With liquidity constrained agents, the stock of financial assets becomes a variable which they care for (see Hayashi 1982). Though it is not very satisfactory, we offer this explanation for including assets in the utility function. In this framework, it is plausible to model agents as choosing optimal levels of consumption and assets, or equivalently consumption and saving levels, given past period's assets. This choice is possible if agents can vary their income by supplying desired quantities of labor at a given point in time, i.e., labor markets are in equilibrium. This view of labor markets requires us to view observed unemployment as "search unemployment." These features of our model will now be presented in a mathematical form.

The equation governing the motion of financial assets is

$$(3.2) \qquad \overset{*}{A}_t = \overset{*}{A}_{t-1} + \tilde{\overset{*}{r}}_t \overset{*}{A}_{t-1} + \tilde{\overset{*}{W}}_t N_t - P_t C_t - TAX_t,$$

where current period stock of assets equals lagged assets augmented with interest income ($\overset{*}{r}_t \overset{*}{A}_{t-1}$) and wage income ($\overset{*}{W}_t N_t$), less expenditures on consumption ($P_t C_t$) and taxes (TAX_t). It is assumed that income can be earned by supplying labor (wage income) or capital (interest income). Earned income can be spent on consumption or saved to augment financial assets.

To obtain a log-linear approximation of (3.2) we decompose TAX_t into proportional and non-proportional (to income) components by estimating the aggregate tax function

$$(3.3) \qquad TAX_t = B + s\,(\tilde{\overset{*}{W}}_t N_t + \tilde{\overset{*}{r}}_t \overset{*}{A}_{t-1}) + e_t$$

where e_t is a regression error term.[1]

After estimating (3.3), define the expression

$$M_t^* = \hat{B} + \hat{e}_t, \tag{3.4}$$

where ^ denotes a regression estimate. Substituting (3.4) into (3.3) we obtain

$$A_t^* = A_{t-1}^* + \tilde{r}_t^* A_{t-1}^* + \tilde{W}_t^* N_t - P_t C_t - M_t^* - \hat{s}\,(\tilde{W}_t^* N_t + \tilde{r}_t^* A_t^*) \tag{3.5.1}$$

$$A_t^* = (1+r_t^*) A_{t-1}^* + W_t^* N_t - P_t C_t - M_t^* \tag{3.5.2}$$

where $r_t^* = (1-\hat{s})\tilde{r}_t^*$ and $W_t^* = (1-\hat{s})\,\tilde{W}_t^*$. Dividing both sides of (3.5.2) by P_t, and multiplying and dividing A_{t-1}^* by P_{t-1}, we get

$$A_t = (1+r_t^*)\ (P_{t-1} / P_t)\ (A_{t-1}^* / P_{t-1}) + W_t N_t - C_t - M_t. \tag{3.5.3}$$

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[1] Since N is endogenous, a better way would be to estimate the regression by two stage least squares. Our estimated value for s was 0.15.

After some manipulation and approximation[1], this can be written out as the State Difference Equation (SDE)

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[1]Define $R_t = \dfrac{1+r_t^*}{P_t/P_{t-1}} = \dfrac{1+r_t^*}{1+\dfrac{P_t-P_{t-1}}{P_{t-1}}}$, hence (3.5.3)

can be written as $A_t = R_t A_{t-1} + W_t N_t - C_t - M_t$ using the above definition of R_t. Taking first differences, this expression can be written as

$$\Delta A_t = R_t \Delta A_{t-1} + A_{t-1} \Delta R_t + N_t \Delta W_t + W_t \Delta N_t - \Delta C_t - \Delta M_t$$

As explained in Appendix A, the variables in log terms were purged of their constant and trend components. Hence the mean of these logged variables was zero and consequently, the mean for the original variables was 1 (because $e^0 = 1$). Hence the above expression can be written in percentage change terms

$$\% \Delta A_t = \% \Delta A_{t-1} + \% \Delta R_t + \% \Delta W_t + \% \Delta N_t - \% \Delta C_t - \% \Delta M_t$$

which is approximately equal to

$$\ln A_t = \ln A_{t-1} + \ln Rt + \ln W_t + \ln N_t - \ln C_t - \ln M_t$$

i.e.,

$$a_t = a_{t-1} - c_t + n_t + w_t + r_t - m_t$$

where lower case letters denote natural logarithms and $r_t = \ln R_t = \ln(1 + r_t^*) - \ln(P_t / P_{t-1}) = r_t^* - \ln(P_t / P_{t-1})$, a real rate of interest using the approximation $\ln(1 + r_t^*) = r_t^*$ for small r_t^*.

$$a_t = a_{t-1} - c_t + n_t + w_t + r_t - m_t \tag{3.6}$$

where $r_t = \ln [(1 + r_t^*) (P_{t-1} / P_t)]$; i.e., a real rate of interest. The solution of the control problem would be considerably simplified if we substitute (3.6) into (3.1). We can ignore linear terms in the resulting objective function because they only contribute constant terms to our decision rules. If our data is purged of all linearly deterministic components prior to estimation (as it will be), the constant terms become redundant and hence are dropped from the decision rules. The quadratic exogenous terms are also dropped from the objective function because they do not affect the decision rules at all. For references regarding these simplifications, see Sargent (1979) and Zadrozny (1980).

Our objective function (3.1) becomes

$$J_t = - \beta_o c_t^2 - \beta_1 n_t^2 - \gamma a_{t-1}^2 \tag{3.7}$$

where $\beta_o = b_4 + b_6$

$\beta_1 = b_5 + b_6$

$\gamma = b_6$

Expressing (3.7) and (3.6) in matrix form, we respectively obtain (' denotes matrix transposition)

(3.8) $$J_t = u_t^{'}Ru_t + x_t^{'}Qx_t$$

(3.9) $$x_t = Kx_{t-1} + Lu_t + Hf_t$$

where

(3.10.1) $$u_t^{'} = [c_t \quad n_t]$$

(3.10.2) $$x_t = [a_t]$$

(3.10.3) $$f_t^{'} = [w_t \quad r_t \quad m_t]$$

(3.10.4) $$R = \begin{bmatrix} \beta_o & 0 \\ 0 & -\beta_1 \end{bmatrix}$$

(3.10.5) $$Q = [\,-\gamma\,]$$

(3.10.6) $$K = [1]$$

(3.10.7) $$L = [-1 \quad 1]$$

(3.10.8) $$H = [1 \quad 1 \quad -1]$$

As mentioned earlier, agents maximize the present discounted value of current and expected future utility. Denoting the discount rate by δ, such that $0 < \delta < 1$, the maximum problem can be stated as

$$\max_{\{u_{t+k}\}_{k=0}^{\infty}} E_t \sum_{k=0}^{\infty} \delta^k J_{t+k}$$

$$\text{(3.11)} \qquad \text{s.t.} \quad x_t = Kx_{t-1} + Lu_t + Hf_t.$$

were E_t denotes the expectation operator conditional on information at time t. The maximization is done by choosing optimal current values for u_t and a contingency plan for u_{t+1}, u_{t+2}, ... The structure of the problem, and its solution is closely patterned after the Linear Optimal Regulator Problem in the engineering literature. This literature is well presented in Anderson and Moore (1971) and Kwakernaak and Sivan (1972). However, we will follow the exposition in Zadrozny (1980), for it extends the usual formulation to include the presence of exogenous variables in the SDE and incorporates discounting over time. We will often refer to this work for proofs and details.

The maximization in (3.11) is done under the assumptions listed below.

<u>Assumption (3.1).</u> The utility function J_t is strictly concave in controls u_t and state x_t.

This assumption holds if and only if matrix

$$\begin{bmatrix} R & 0 \\ 0 & Q \end{bmatrix}$$

is negative definite. This implies that R and Q are negative definite.[1] This assumption is satisfied because R and Q are negative definite as given by (3.10.4) and (3.10.5).

<u>Assumption (3.2).</u> The state vector x_t is completely controllable - i.e., ignoring the effects of exogenous variables in the SDE, - the state vector can be transferred from an arbitrary initial state to the origin in a finite number of periods by an appropriate control sequence.

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[1]This corresponds to the second order conditions in static maximization.

Complete controllability occurs if and only if the controllability matrix

$$[K \quad KL \quad K^2L \;\ldots\ldots\; K^{n-1}L]$$

has rank n, where n is the dimension of the state vector. This condition is satisfied for our problem because the rank of the controllability matrix $[1 \quad -1 \quad 1]$ is one, which is also the dimension of x_t.

Assumption 3.3. The exogenous vector f_t in (3.10.3) is generated by an AR1 process

$$f_t = \theta f_{t-1} + v_t$$

of mean exponential order less than $1/\sqrt{\delta}$ (where θ is the parameter matrix).

In general, one could specify a stationary invertible ARMA representation like Zadrozny (1980). Since no significant moving average component or higher order autoregressive lags were detected, the simpler alternative is justified.

Assumption 3.4. Agents have rational expectations in the sense of Muth (1961).

If the information set of our agents is

(3.12) $I_t = (u_t, u_{t-1}, \ldots\ldots, x_t, x_{t-1}, \ldots\ldots f_t, f_{t-1}, \ldots\ldots)$

then agents can use linear regression i.e., regress f_t on I_t to generate forecasts for t+1, t+2, ... at time t. These forecasts are optimal in the sense of minimizing the mean squared forecast error, if the forecast error vector $e_{f,t}$, is independent of all past data, i.e.,

$$E\,(e_{f,t+k} / I_t) = 0$$

for k = 1,2, ...

We can now solve the maximum problem (3.11) by the method of dynamic programming.[1] The procedure involves formulating and solving the Bellman and Riccati equations. We will assume that at any time t, agents have a finite time horizon t+N-1 periods long. At any time t, they make current decisions and contingency plans for t+1, t+2, ... t+N-1, for their controls u_t as a function of I_t and the remaining number of periods, so

(3.13) $$u_{t+k} = D\,(I_{t+k}, N-K)$$

for k = 0, 1, 2, ... N-1. The above decision rule D, once known, would allow us to compute

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[1]The original reference for dynamic programming is Bellman (1957).

$$(3.14) \qquad V(D,\ I_t,\ N) = \sum_{k=0}^{N-1} \delta^k E_t\ J_{t+k},$$

where V (∘) denotes the expected present value of a utility stream conditional on policy D. The expectation operator, conditional on I_t is denoted by E_t. Our objective is to find an optimal decision rule D^* (which is known to be linear), that maximizes V (∘) in (3.14) above, and thus maximize the present discounted value of current and expected future utility. But first we must introduce some more notation. Let W (I_t, N) be the maximum value of V(∘), so that

$$(3.15) \qquad W\ (I_t,\ N) = \underset{\{D_{N-k}\}_{k=0}^{N}}{\text{Max}} \qquad V(D,\ I_t,\ N)$$

is the Optimal Performance Function. Since the maximization of V (∘) is done over a sequence of D, it (D) does not appear in W(∘). Notice that W(∘) gives the maximum value of discounted utility if the optimal value of D is selected. We now apply Bellman's Principle of Optimality[1] to obtain the Fundamental Recurrence Relation of Dynamic Programming

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[1]Bellman's (1957) Principle of Optimality states that "An Optimal policy has the property that, whatever the initial state and decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.

$$W(I_t, N) = \underset{u_t}{\text{Max}} \ [J_t + \delta E_t W(I_{t+1}, N-1)]. \tag{3.16}$$

The Principle of Optimality allows us to transform a complicated problem of maxmizing $V(\cdot)$ in (3.15) over a control sequence $\{\dot{P}_{N-k}\}_{k=0}^{N}$ into a simple maximization over the current control vector u_t. Typically, the problem for the last period t+N-1 is solved first, given the initial condition I_{t+N}. Having found the optimal policy u_{t+N-1}, conditional on information in the earlier period, we now find U_{t+N-2} conditional on U_{t+N-1}, and so on. Notice that (a) the problem is solved recursively backward in time, and (b) at each step only one unknown vector u_t is determined. The Bellman Principle of Optimality guarantees that each solution from this procedure is optimal, regardless of the initial condition.

Because utility is quadratic and the constraint linear in variables, the Optimal Performance Function in (3.16) is known to have the following form,

$$\begin{aligned} W(I_t, N) &= d_o + x'_{t-1} W_x(N) x_{t-1} \\ &\quad + \sum_{k=0}^{N-1} x'_{t-1} W_k(N) E_t f_{t+k} \end{aligned} \tag{3.17}$$

where W_x (N) and W_o (N), W_1 (N) ..... W_{N-1}(N) are matrices and the constant d_o is a quadratic function in the exogenous variables. Similarly, current expectations regarding the one-step-ahead Optimal Performance Function is

$$(3.18) \qquad E_t W\ (I_{t+1},\ N-1) = d_1 + x_t'\ W_x\ (N-1) x_t + \sum_{k=0}^{N-2} x_t'\ W_k\ (N-1)\ E_t\ f_{t+k+1}$$

where all subscripts are advanced by one period and d_1 has an interpretation similar to d_o earlier.

To maximize (3.16), differentiate it with respect to u_t and set the result to zero,[1] i.e.,

$$(3.19) \qquad \frac{\partial\,[W(I_t,\ N)]}{\partial u_t} = Ru_t + \delta\, L'W_x\ (N-1)x_t + \sum_{k=0}^{T-2} \delta\, L'W_k\ (N-1)E_t f_{t+k+1} = 0$$

Substituting for x_t from the SDE in (3.9), yields

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[1]From (3.17) W(·) is continuously differentiable.

$$(3.20)\qquad [R + \delta L'W_x(N-1)L]u_t + \delta L'W_x(N-1)F\,x_{t-1} + [\delta L'W_x(N-1)H]f_t + \sum_{k=0}^{N-2} \delta L'W_k(N-1)E_t f_{t+k+1} = 0.$$

Define

$$(3.21)\qquad S(N-1) = -[R + \delta L'W_x(N-1)L]^{-1}$$

and premultiply (3.20) by (3.21)[1] to obtain the Optimal Decision Rule (ODR)

$$(3.22)\qquad u_t = \delta S(N-1)L'W_x(N-1)K\,x_{t-1} + S(N-1)[\delta L'W_x(N-1)H]f_t + \sum_{k=0}^{N-2} \delta S(N-1)L'W_k(N-1)E_t f_{t+k+1}$$

In order to express $W_x(N-1)$, $W_o(N-1)$, $W_1(N-1)$, ..., in terms of the structural parameters of the maximum problem (3.11), we first set controls optimally for current period t, i.e.,

$$(3.23)\qquad W(I_t,N) = J_t^o + \delta E_t W(I_{t+1}, N-1),$$

[1]By induction, it can be shown that the strict concavity of utility implies that $W_x(N-1)$ is negative and therefore $S(N-1)$ is negative definite and has an inverse.

where J^{o} denotes the optimality of current controls. Differentiating (3.2.3) with respect to x_{t-1} (because it holds identically for all I_t) and using (3.17), (3.18), we obtain

$$(3.24)\quad \frac{\partial W(I_t,N)}{\partial x_{t-1}} = \frac{\partial J_t^{o}}{\partial x_{t-1}} + \frac{\partial x_t'}{\partial x_{t-1}'} \frac{\partial [E_t W(I_{t+1},N-1)]}{\partial x_t}$$

therefore

$$(3.25)\quad W_x(N)x_{t-1} + \sum_{k=0}^{N-1} W_k(N)E_t f_{t+k}$$

$$= Qx_{t-1} + \delta F' [W_x(N-1)x_t + \sum_{k=0}^{N-2} W_k(N-1)E_t f_{t+k+1}].$$

Substituting for x_t from the SDE in (3.9) into (3.25), and eliminating u_t by substituting (3.22) into the resulting equation, we get

(3.26)
$$W_x(N)x_{t-1} + \sum_{k=0}^{N-1} W_k(N) \quad E_t f_{t+k}$$

$$= [Q + \delta K' W_x(N-1) \quad \phi_x(N-1)] x_{t-1}$$

$$+ [\delta\phi_x(N-1)' \; W_x(N-1)H + \delta K' W_x(N-1) LS(N-1)] f_t$$

$$+ \sum_{k=0}^{N-2} \delta \phi_x(N-1)' \; W_k(N-1) E_t f_{t+k+1},$$

where

(3.27)
$$\phi_x (N-1) = [I + \delta LS (N-1) L' W_x (N-1)] K.$$

The relation (3.26) is known to hold for all subsets of I_t, see Zadrozny (1980). Hence when $x_{t-1} \neq 0$, and $f_t = E_t f_{t+k} = 0$ for $k = 0, 1, 2, \ldots.$ (i.e., in the absence of exogenous variables), we obtain

(3.28)
$$W_x(N) = Q + \delta K' W_x (N-1) \phi_x (N-1)$$

which is a version of the discrete time matrix Riccati equation. This is solved recursively starting with $W_x(0) = 0$, till convergence is achieved. The negative root of the resulting quadratic equation in the steady state (as $N \rightarrow \infty$) is the solution to (3.28). The solution of the Riccati equation is obviously in terms of the structural parameters of the maximum problem in (3.9).

We again exploit the property of (3.26) to hold for all subsets of I_t by setting $x_{t-1} = 0$ and $f_t \neq 0$, $E_t f_{t+k} \neq 0$ for $k = 1, 2, \ldots,$ obtaining

$$(3.29) \quad W_o(N) = \delta F' W_x(N-1)[H + LS(N-1)(\delta L' W_x(N-1)H)]$$

$$(3.30) \quad W_k(N) = [\delta \phi_x (N-1)']^k W_o(N-1).$$

With Riccati equation (3.28) solved, (3.29) and (3.30) can also be expressed in terms of the original structural parameters. We are now ready to state the ODR in terms of the original structural parameters:

$$(3.31) \quad u_t = D_x(N)x_{t-1} + D_o(N) f_t + \sum_{k=1}^{N-1} D_k(N) E_t f_{t+k}$$

where

$$(3.32.1) \quad D_x(N) = \delta S(N-1) L' W_x(N-1)K$$

$$(3.32.2) \quad D_o(N) = S(N-1) [\delta L' W_x(N-1)H]$$

$$(3.32.3) \quad D_k(N) = \delta S(N-1) L' W_k(N-1)$$

for k = 1, 2, ..., N-1.

The finite horizon ODR above, can be transformed into an infinite horizon ODR when $N \to \infty$,

$$(3.33) \qquad u_t = D_x x_{t-1} + D_o f_t + \sum_{k=1}^{\infty} D_k E_t f_{t+k}.$$

The coefficient matrices of (3.33) are the limits of (3.32) as $N \to \infty$. Notice the feedback relation between the control and state variables. This relation, as given by D_o is independent of the feedforward relation of the controls with respect to the exogenous vector. By independence, we mean that the same D_o will prevail in the absence of exogenous variables. When the control vector is set optimally according to (3.33), the state x_t moves according to the Optimal State Equation (OSE)

$$(3.34) \qquad x_t = \phi_x x_{t-1} + \phi_o f_t + \sum_{k=1}^{\infty} \phi_k E_t f_{t+k}$$

where

$$(3.35.1) \qquad \phi_x = K + LD_x$$

$$(3.35.2) \qquad \phi_o = H + LD_o$$

(3.35.3) $$\phi_k = LD_k$$

for k = 1,2 .... The OSE in (3.34) results from substituting the ODR in (3.33) into the original SDE in (3.9). In order to estimate the parameters of (3.33) we need to substitute observable expressions for the expectations term $E_t f_{t+k}$. These observable expressions are the optimal forecasting rules for f_{t+k}, k=1,2 ... at time t. Having assumed that the exogenous vector f_t is driven by a first order autoregressive process in Assumption (3.3), we specify

(3.36.1) $$f_{t+1} = \theta f_t + v_{t+1}$$

where

$$\theta = \begin{bmatrix} \theta_w & 0 & 0 \\ 0 & \theta_r & 0 \\ 0 & 0 & \theta_m \end{bmatrix}$$

and θ_w, θ_r, θ_m are the AR1 parameters of the stochastic processes for w_t, r_t and m_t, respectively. The 3x1 vector of independent error terms is denoted by v_{t+1}. The forecasting rule is given by

(3.36.2) $$E_t f_{t+k} = \theta^k f_t$$

for k = 1,2, ... The forecasts are optimal in the mean squared error sense if (3.36.1) has the following properties:

(3.37.1) $$E(v_{t+k}) = 0 \text{ for } k \geqslant 1$$

(3.37.2) $$E(v_{t+k}/I_t) = 0$$

(3.37.3) f_t exogenous with respect to $[x_t, u_t]$.

The estimated values of θ are all less than unity and therefore satisfy this condition, since by assumption $0 < \delta < 1$. Substituting (3.36.2) into (3.33) yields

(3.38) $$u_t = D_x x_{t-1} + D_o f_t + \sum_{k=1}^{\infty} D_k \theta^k f_t$$

which is the ODR in observable variables. We aim to estimate the parameters of (3.38) subject to restrictions given by (3.32). Recall that the coefficient matrices in (3.38) embody non-linear cross-equation restrictions in terms of the underlying structural parameters. Equation (3.38) expresses real consumption and labor supply as functions of lagged assets, the current real wage, real interest rate and real taxes, and their optimal forecasts. The free parameters to be estimated are β_o, β_1, γ , δ , the structural parameters of the maximum problem, and the forecasting parameters θ_w, θ_r, θ_m. The complete absence of parameters unrestricted by economic theory, and the clear separation into structural and forecasting parameters is a noteworthy feature of the ODR (3.38).

3.2 Properties of the Model

Our model has the certainly equivalence property due to its linear quadratic formulation. This implies that the same ODR in (3.38) would have resulted, had we chosen to model f_t as a deterministic rather than a stochastic process as in (3.36.1).

We are also interested in knowing the stability properties of our model. The system is considered to be stable if it has finite steady states in the long run for its control, state and exogenous variables, given information in the current period. Specifically

(3.39.1) $$\bar{u}_t = \lim_{j \to \infty} E_t u_{t+j}$$

(3.39.2) $$\bar{x}_t = \lim_{j \to \infty} E_t x_{t+j}$$

(3.39.3) $$\bar{f}_t = \lim_{j \to \infty} E_t f_{t+j}$$

Equations (3.33) and (3.34) can be written as

$$(3.40.1) \qquad E_t u_{t+j} = D_x E_t x_{t+j-1} + \sum_{k=0}^{\infty} D_k E_t f_{t+k+j}$$

$$(3.40.2) \qquad E_t x_{t+j} = \phi_x E_t x_{t+j-1} + \sum_{k=0}^{\infty} \phi_k E_t f_{t+k+j}$$

for $j = 1,2, \ldots$ This is done by applying the law of iterated projections as discussed in Sargent (1979). The steady state OSE in (3.40.2) is stable if $\bar{f}_t$ exists and ϕ_x has all eigen values less than unity in absolute value. The limit $\bar{f}_t$ exists if $-1 < | \theta_{ii} | \leq 1$ for $i = 1,2,3$ as is evident from (3.36.2). In the next chapter we show that this condition is satisfied by our parameter estimates. It can also be shown (using the ODR) that $\phi_x < 1$.[1] Also, define

---

[1]From 3.35.1, we can write

$$\phi_x = 1 + \frac{\delta(\beta_o + \beta_1)W_x}{\beta_o \beta_1 - \delta W_x(\beta_o + \beta_1)}$$

$$= \frac{\beta_o \beta_1}{\beta_o \beta_1 - \delta(\beta_o + \beta_1)W_x} < 1$$

because $W_x < 0$ (see footnote on page 24), and $0 < \delta < 1$, $\beta_o > 0$, $\beta_1 > 0$ by assumption.

$$(3.41.1) \qquad \overline{D}_f = \sum_{k=0}^{\infty} D_k$$

$$(3.41.2) \qquad \overline{\phi}_f = \sum_{k=0}^{\infty} \phi_k$$

From equation (3.35.3), it is evident that $\overline{\phi}_f$ exists if $\overline{D}_f$ exists. To verify whether $\overline{D}_f$ exists, substitute (3.30) into (3.32.3) obtaining

$$(3.42) \qquad D_k = \delta SG'(\delta\phi_x')^{k-1} W_o.$$

$\overline{D}_f$ exists if $\delta\phi_x' < 1$. Since we showed that the scalar $\phi_x < 1$ (see footnote on page 32) and assumed $0 < \delta < 1$, it follows that $\delta\phi_x < 1$. Using the above definitions and substituting (3.40.2) into (3.40.1), we obtain

$$(3.43.1) \qquad \overline{u}_t = [D_x (I - \phi_x)^{-1} \quad \overline{\phi}_f + \overline{D}_f] \overline{f}_t$$

$$(3.43.2) \qquad \overline{x}_t = (I - \phi_x)^{-1} \quad \overline{\phi}_f \overline{f}_t.$$

Since we have already shown that $\overline{f}_t$ exists, it follows that the steady-state system ODR and OSE exist and the system is asymptotically stable.

3.3 Dynamic Simulation

From the very beginning, we have emphasized the need to create models where effects of alternative policy regimes can be examined without being subject to the Lucas (1976) critique of econometric policy evaluation. This section outlines a method, following Zadrozny (1980), to determine the effect on controls and state variables of transitory and permanent shocks in the exogenous variables.

Since our data consists of deviations from constant and trend components (see Appendix A) the mean values of the variables are zero. Hence it is reasonable to assume that the model is in a steady-state equilibrium at the origin when $u_{t-1} = x_{t-1} = f_{t-1} = 0$. The system is subjected to a shock in the current period if $f_t \neq 0$. The shock is considered to be transitory if $f_{t+j} = 0$ for $j = 1, 2, \ldots\ldots$ and permanent if $f_t = f_{t+j} \neq 0$ for $j = 1, 2, \ldots$ It is also assumed that only one of the three components of f_t is responsible for the shock. When agents can distinguish between transitory and permanent shocks, it follows that the expectation sequence for transitory shocks is given by

$$E_{t+k} f_{t+k+j} = 0 \quad \text{for } k = 0, 1, 2, \ldots. \text{ and } j = 1, 2, \ldots\ldots. \tag{3.44.1}$$

and for permanent shocks by

(3.44.2) $$E_{t+k}f_{t+k+j} = f_t \quad \text{for } k = 0, 1, 2, \ldots. \text{ and } j = 1, 2, \ldots\ldots$$

To forecast the effect on control and state variables, substitute (3.44.1) into (3.33) and (3.34) to obtain the intertemporal sequence

$$u_{t-1} = 0 \qquad x_{t-1} = 0$$

$$u_t = D_x x_{t-1} + D_o f_t \qquad x_t = \phi_x x_{t-1} + \phi_o f_t$$

(3.45.1) $$u_{t+1} = D_x x_t \qquad x_{t+1} = \phi_x x_t$$

$$\ldots\ldots\ldots\ldots \qquad \ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots \qquad \ldots\ldots\ldots\ldots$$

$$u_{t+k} = D_x x_{t+k} \qquad x_{t+k} = \phi_x x_{t+k-1}$$

for transistory shocks. If instead (3.44.2) were to be substituted, then the intertemporal sequence

$$u_{t-1} = 0 \qquad x_{t-1} = 0$$

$$u_t = D_x x_{t-1} + \overline{D}_f f_t \qquad x_t = \phi_x x_{t-1} + \overline{\Phi}_f f_t$$

$$(3.45.2) \quad u_{t+1} = D_x x_t + \overline{D}_f f_t \qquad x_{t+1} = \phi_x x_t + \overline{\phi}_f f_t$$

$$\cdots\cdots\cdots\cdots \qquad \cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots \qquad \cdots\cdots\cdots\cdots$$

$$u_{t+k} = D_x x_{t+k-1} + \overline{D}_f f_t \qquad x_{t+k} = \phi_x x_{t+k-1} + \overline{\phi}_f f_t$$

for permanent shocks is generated. Generally, for an asymptotically stable system like ours, state and controls will return to the origin after a transitory shock, and attain new steady-states after a permanent shock. Numerical simulations in Chapter IV allow us to estimate the differential effect of transitory and permanent shocks in the exogenous variables on the state and controls.

CHAPTER IV

MODEL ESTIMATION AND SIMULATION RESULTS

Before estimating the parameters and simulating the model, it should be mentioned that the validity of assuming f_t to be exogenous was tested. Failure of this fundamental assumption to hold in our data would invalidate the basis of our maximum problem. The statistical exogeneity of f_t is a testable proposition, given the work of Granger (1969) and Sims (1972).

First we formulate a three equation system where each of the three elements of f_t is regressed on three lagged values of itself, of u_t and x_t (the data are described in Appendix A). This unrestricted system is estimated to give us the unrestricted log likelihood function value L_u. The restricted system is estimated by dropping the elements of u_t and x_t from the unrestricted system. We now obtain the restricted log likelihood function value L_r. The statistic

$$\lambda = -2\ (L_r - L_u)$$

is asymptotically distributed as $\chi^2(k)$, where k is the degrees of freedom parameter, given by the difference between the number of free parameters in the unrestricted and restricted cases. In our case k = 45-18 = 27, L_r = 1023, and

$L_u = 1030$, therefore $\lambda = 14$. The null hypothesis of restrictions is rejected if the calculated value of λ exceeds the table value of $\chi^2(k)$ for a given significance level. Since the table value $\chi^2(27) = 40.1$ at the 5% significance level is greater than the calculated value $\lambda = 14$, we do not reject the null hypothesis that f_t is not Granger-caused by u_t and x_t. Hence in a model where f_t is considered to be exogenous with respect to u_t and x_t, no misspecification error is involved (for proof, see Sims 1972). These results justify the formulation of our maximum problem (3.11) where f_t is assumed to be exogenous. We discuss the estimation strategy for the ODR in (3.38) in Section 1. The parameter estimates are discussed in Section 2 and the simulations are presented in Section 3.

4.1 Estimation Strategy

Following Sargent (1978) and Zadrozny (1980), the effect of unobservable components of f_t are modeled as additive error terms to the ODR. Let ε_t denote a two element column vector of error terms, hence (3.38) becomes

$$(4.1) \qquad u_t = D_x x_{t-1} + D_o f_t + \sum_{k=1}^{\infty} D_k \theta^k f_t + \varepsilon_t$$

where ε_t generally follows an ARMA process (see Zadrozny 1980). It is useful to write out the structural equation for c_t and n_t in greater detail so that the cross-equation restrictions can be clearly seen. Hence (4.1) can be written as:

$$
\begin{aligned}
c_t &= d_1 x_{t-1} + d_2 w_t + d_3 r_t + d_4 m_t + \hat{\varepsilon}_{1t} \\
n_t &= d_5 x_{t-1} + d_6 w_t + d_7 r_t + d_8 m_t + \varepsilon_{2t} \\
w_t &= d_9 w_{t-1} + v_{1t} \\
r_t &= d_{10} r_{t-1} + v_{2t} \\
m_t &+ d_{11} m_{t-1} + v_{3t}
\end{aligned}
\tag{4.2}
$$

where:

$$
\begin{aligned}
d_1 &= [\delta M(\beta_o - 1)] \;/\; P \\
d_2 &= [[\delta M(\beta_o - 1)] \;/\; P] + Z(\beta_o - 1)\theta_w \;\; [1 \;/\; (1 - Y\theta_w)] \\
d_3 &= [[\delta M(\beta_o - 1)] \;/\; P] + Z(\beta_o - 1)\theta_r \;\; [1 \;/\; (1 - Y\theta_r)] \\
d_4 &= -[[\delta M(\beta_o - 1)] \;/\; P] + Z(\beta_o - 1)\theta_m \;\; [1 \;/\; (1 - Y\theta_m)] \\
d_5 &= (\delta M\beta_o) \;/\; P \\
d_6 &= (\delta M\beta_o) \;/\; P + Z\beta_o\theta_w \;\; [1 \;/\; (1 - Y\theta_w)] \\
d_7 &= (\delta M\beta_o) \;/\; P + Z\beta_o\theta_r \;\; [1 \;/\; (1 - Y\theta_r)] \\
d_8 &= -(\delta M\beta_o) \;/\; P + Z\beta_o\theta_m \;\; [1 \;/\; (1 - Y\theta_m)] \\
d_9 &= \theta_w \\
d_{10} &= \theta_r \\
d_{11} &= \theta_m
\end{aligned}
\tag{4.3}
$$

and

$$M = -[[\delta\gamma - \beta_o(1-\beta_o)(1-\delta)] + [(\delta\gamma - \beta_o(1-\beta_o)(1-\delta))^2 + 4\delta\gamma\beta_o(1-\beta_o)]^{1/2}] / 2\delta$$

$$P = \beta_o(1-\beta_o) - \delta M$$

$$Z = [(\delta^2 m) / P] / [1 + (\delta M) / P]$$

$$Y = [\delta\beta_o(1-\beta_o)] / P.$$

The structural equations in (4.2) have been written with the normalization $\beta_o + \beta_1 = 1$. Ideally we would want to estimate (4.2) subject to the cross-equation restrictions (4.3). Such attempts using full information maximum likelihood methods proved to be unsuccessful because of severe identification problems. Therefore the following two-step limited information maximum likelihood method was used. The exogenous stochastic processes in (3.36.1) were first estimated. Then the (first two) structural equations in (4.2), subject to restrictions (4.3), were estimated. Therefore, the structural parameter estimates were conditioned on the estimated θ values from the first step. The parameter estimates of the structural equations will not be asymptotically efficient, though they will be consistent.

A first order autoregressive process was found to fit well the exogenous variables in (3.36.1). No significant moving average components were detected, and the Box-Pierce Q statistic indicated no significant serial correlation in the residual error series. The coefficients of θ, together with their t statistics and the Q statistics are presented in Table 1 (all tables are assembled in Appendix B). Our interest in the θ coefficients is restricted to their usefulness for estimating the structural equations, and hence we will not discuss their properties further.

We can now estimate the structural consumption and labor supply equations (4.2) taking the estimates of θ as given. This estimation was done by maximum likelihood methods. Since the structural estimates were still subject to identification problems, the value of the discount rate was fixed at $\delta = .92$.[1] Initial estimates of the structural equations indicated autocorrelated error terms. An analysis of the residuals did not indicate the presence of any moving average components. The error series was found to follow an AR1 process with coefficient 0.79. Hence all variables were multiplied by the linear filter (1-.79L), where L is a lag operator. The resulting maximum likelihood estimates of (4.2) did not indicate serial correlation in the error terms, as per the Q statistic. These results are presented in Table 2.

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[1]The resulting parameter estimates were not sensitive for higher values for δ up to 0.99.

4.2 Parameter Estimates

In the absence of any estimates of the structural parameters in the literature, we cannot directly evaluate β_o, β_1 and γ. The null hypothesis that the parameters are individually zero is not rejected for β_o and γ, but rejected for β_1, at the 5% significance level. The estimated values of these parameters, with their associated asymptotic t statistics are presented in Table 2.

Testing the validity of the model involves testing the reasonableness of the cross-equation restrictions (4.3) imposed by our theory on the ODR (4.2). This is done along the lines suggested for the exogeneity test earlier. The system (4.2) is estimated subject to restrictions (4.3) to give the value of the restricted log likelihood function L_r. Then (4.2) is estimated without the restrictions (4.3) to give the value of the unrestricted log likelihood function L_u. This involved estimating the d's of (4.2) as free parameters unrestricted by theory. The first order autoregressive filter (1-.79L) removed the serial correlation from the error terms, hence all variables were filtered in this way. Since all variables in the restricted and unrestricted cases were filtered identically, the likelihood ratio test can be performed.

The value of the test statistic was $\lambda = 74.8$ (see Table 2 for details). This value is greater than the $\chi^2(6)$ table value at the 5% significance level, which implies a rejection of our null hypothesis of restrictions (4.3).[1] This result implies that the aggregate post-war data for the United States rejects our hypothesis that agents maximize the discounted value of current and expected future utility as per the maximum problem (3.11).

In spite of the model's rejection, it is interesting to see the economic implications of our estimates. This is done in two ways. First we recognize that the d's of (4.2) are elasticity parameters and have meaningful interpretations. Second, it would be of interest to conduct simulations of the effects of transitory and permanent shocks in the exogenous variables, on state and controls.

Since the ODR (4.2) has been estimated both, subject to, and without imposing restrictions (4.3), we can obtain two sets of elasticity parameters. From the restricted estimates, we obtain estimates for β_0, β_1, and γ, which together with the previously estimated values for θ_w, θ_r, θ_m

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[1]Berndt and Savin (1977) proved a systematic bias amongst some popular tests for linear restrictions. Though no such results are available for non-linear restrictions (like in our model), further research to verify the robustness of tests is clearly required.

and δ , allow us to compute d_1, d_2, ... d_8 as per (4.3). These are the restricted elasticity estimates and are presented in Table 3.[1] We also have parameter estimates of (4.2) without imposing restrictions (4.3). This provides us with the unrestricted elasticity estimates, and are also presented in Table 3.

Examining the unrestricted elasticity coefficients first, we notice that real consumption (c_t) has an elasticity of .48 with respect to real wages (w_t). The closest proxy for w_t would be some measure of aggregate income, the elasticity for which ranges from 0.5 to 0.9 (see Gordon 1981). As wage income constitutes only a part of total income, it is obvious that the elasticity of c_t with respect to w_t should be smaller than that with respect to some measure of aggregate income--a result obtained by us.

On the other hand, our estimate of the unrestricted elasticity of c_t with respect to assets (.49) is higher than an estimate (.25) by Ando and Modigliani (1963). The elasticity with respect to the interest rate given by the coefficient $d_3 = -.09$ indicates a very small inverse response of c_t to r_t. As discussed in the literature review, there is much debate over the effect of interest rates on consumption (or saving). In a paper that received much attention Boskin

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[1]Notice the clear separation between "optimizing" parameters β_o, β_1, γ and "forecasting" parameters θ_w, θ_r, and θ_m. Under a different regime, the θ's would change giving us new elasticity values for the d's.

(1978) showed a significant inverse relation between the real rate of interest and consumption, while Howrey and Hyman's (1978) results reveal no such relation. Our conclusions are more in the line with the latter.

In summary, the unrestricted elasticities are not very different from the ones found in the literature. In fact it would be surprising if they were, given the similarity of the structures in the two cases.

When we turn our attention to the elasticity estimates generated by our hypothesis, i.e, the restricted coefficients--a totally different picture emerges. All elasticities turn out to be very small, though small values for d_3 and d_4 are not as different from the unrestricted case as the small values for d_1 and d_2. The elasticity with respect to real assets is .01 and real wages is .07. Also the elasticity with respect to r_t is positive, which is different from the unrestricted results.

The case of the labor supply function is similar in important respects. As pointed out in the literature review, well-known estimates of the aggregate labor supply function are restricted to the work of Lucas and Rapping (1969). They found an insignificant role for assets and the interest rate in the labor supply function. This insignificance is also obtained by us in the unrestricted estimates of d_5 and d_7, which are very small. Also, labor supply is inversely related to all variables, except taxes. We will prefer to discuss these results further in the context of simulations below.

4.3 Dynamic Simulation

The most important purpose in performing simulations is to examine the conditional forecasts generated by the model. In this section, the response of consumption, labor supply and financial assets to transitory and permanent shocks in the exogenous variables is analyzed. These simulations are based on equations (3.45.1.) and (3.45.2) as discussed in Chapter III. An examination of (3.33) reveals that identical simulations would result for real wage and real interest shocks, while those for taxes will differ only in their algebraic sign. Hence, one simulation will suffice, though we will make references to all three exogenous variables.

Following Zadrozny (1980), we assume that (the logarithms of) all variables, at time t-1 are zero i.e, $u_{t-1} = x_{t-1} = w_{t-1} = 0$, and (the log of) wages jumps to $w_t = .01$ in the current period. This shock is permanent if $w_{t+k} = .01$ and transitory if $w_{t+k} = 0$ for $k = 1,2, \ldots\ldots$

The magnitude of these shocks is irrelevant because the simulations are interpreted in elasticity terms. Denoting endogenous and exogenous variables of the model by uppercase letters M_t and Z_t respectively, and their natural logarithms by lower case m_t and z_t respectively, the elasticity

$$\eta_{m,z}(t) = [(M_t \quad M_{t-1})/M_{t-1}] \;/\; [(Z_o - Z_{t-1})/Z_{t-1}]$$

is the percentage change in M at time t (where t = 0,1,2, ...), relative to its initial position at t-1, with respect to a one percent shock in Z_o. In terms of natural logarithms, the elasticity formula is

$$\eta_{m,z}(t) = (e^{m_t} - 1) \;/\; (e^{z_t} - 1)$$

where $e^{m_{t-1}} = e^{z_{t-1}} = e^0 = 1$ by assumption.

Tables 4 and 5 present simulations for a one percent transitory (η^T) and permanent (η^P) shock respectively in w_t, for 20 quarters. Table 6 presents the steady-state elasticities i.e., $\eta_{m,z}(\infty)$, representing the steady-state response of endogenous variables to a permanent shock in the exogenous variable.

The impact effect of a transitory shock in w_t (see table 4), is an increase in financial assets ($\eta^T_{x,w}$ = .98), an increase in consumption ($\eta^T_{c,w}$ = .01), and a decrease in labor supply ($\eta^{T}_{n,w}$ = .00069). Over time, both state and controls return to their initial steady-states at the origin.[1]

The impact effect of a permanent shock in w_t (see Table 5) is a decrease in financial assets ($\eta^{P}_{x,w}$ = .11), an increase in consumption ($\eta^{P}_{c,w}$ = .11) and a decrease in labor supply ($\eta^{P}_{n,w}$ = .0077). Over time, both state and controls approach new steady-state positions. The long run steady-state elasticities are presented in Table 6.

In evaluating our results, it is helpful to recall that they are not subject to the Lucas (1976) critique of econometric policy evaluation. Hence in principle they can be useful in simulating alternative policy regimes for exogenous variables. Below we attempt to analyze our results in the context of the relevant economic theory and other empirical results.

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[1]Theoretically, it was shown that this will be true if $\phi_x < 1$, a condition which our system satisfies (see footnote on page 32). Empirically the estimate of ϕ_x was .98 with a standard error of .0056. (Since ϕ_x is a non-linear function of parameters, its variance was computed by linearizing it around the estimated parameter values and then using standard formulas for the variance and covariance of linear functions of random variables - see Kmenta (1971)). A test of the null hypothesis $\phi_x = 1$ against the alternative $\phi_x < 1$ has a t statistic of -1.8 indicating a rejection of the null hypothesis at the 5% level.

The permanent income hypothesis (PIH), first propounded by Friedman (1957), occupies an important place in discussions regarding the consumption function. The PIH's basic idea is to relate current consumption to a measure of permanent income. In some rough sense, permanent income is a proxy for the current and discounted future earning power of the individual. The main results derived by Friedman were that consumption is (a) perfectly inelastic with respect to a transitory shock in income and (b) approximately unitary elastic with respect to a permanent shock in income.

Our results are in broad agreement with those of Friedman. A transitory shock in w_t is largely absorbed by a rise in financial assets, with a very small increase in consumption ($\eta^T_{c,w} = .01$). A permanent shock has a larger impact effect on consumption ($\eta^P_{c,w} = .11$), with a steady-state elasticity of .09. Recently the validity of the PIH has been challenged. Flavin (1981) rejects the parameter restrictions of the PIH under rational expectations, and concludes that transitory shocks in income do affect current consumption. Hayashi (1982) found that acceptance of the PIH was conditional on the data for the consumption variable.

The effect of transitory and permanent tax cuts can be similarly modeled. The PIH predicts that consumption is unitary elastic with respect to a permanent, but inelastic with respect to a transitory tax cut. As stated earlier, our simulations could be interpreted as shocks in m_t, verifying the PIH (the results are the same as for the shock in w_t). Other authors like Blinder (1981) have also concluded that permanent tax cuts are more influential than transitory cuts in increasing consumption. Blinder's results indicate that the ratio of the propensity to consume out of a transitory tax cut to that out of a permanent cut, is .48 for the current quarter, and rises steadily to .80 in the eighth quarter. Hence both kinds of tax cuts are effective though they differ in their speed with which they affect consumption.

We will now examine the response of labor supply to shocks in the real wage rate. In a simple static framework, an increase in the real wage rate leads to an increase in labor supply (as agents' substitute labor for leisure). It is also possible that agents now consume more leisure and hence supply less labor (because of a positive income effect on leisure). Therefore the net effect is ambiguous. Most "new classical macroeconomics" models postulate a high positive elasticity of labor supply with respect to transitory wage shocks, in order to explain short run employment fluctuations as movements along a supply curve. Empirical evidence at the aggregate level was obtained by

Lucas and Rapping (1969). They estimated the elasticity of labor supply with respect to the real wage rate to be 1.6 for transitory shocks, and zero with respect to permanent shocks. Our results indicate an almost perfectly inelastic response with respect to a transitory shock ($\eta^{T}_{n,w} = -.0077$) and a similar steady-state response ($\eta^{P}_{n,w}(\infty) = -.0062$) to a permanent shock. Hence the results of this study are in partial contrast with Lucas and Rapping (1969).

To summarize, our simulations agree with the basic prediction of the PIH, that consumption responds to permanent and not transitory fluctuations in real wages (interest rates and taxes). The direction of our response agrees with the general literature, though the magnitude is smaller. The case of labor supply is different in that it is not affected by transitory or permanent shocks in real wages (or interest rates and taxes). This result is in contrast to the usual belief in a highly elastic response to transitory shocks.

Before concluding this chapter we must mention an effort on our part to examine the variation in simulation results under a variety of settings for the structural parameter γ. The results are presented in Tables 7 and 8. Table 7 presents the simulations for a transitory shock in w_t. Basically, the direction of the response is similar to Table 4--c_t, x_t increase and n_t decreases, though the magnitude of the responses are different. The impact elasticity of c_t with respect to w_t ranges from .60 (for γ = .1) to .94 (for γ = 1000). The large consumption elasticities are consistent with Flavin's (1981) results indicating a significant response of consumption to transitory income. Hence a transitory increase in w_t is largely spent and not saved compared to the Table 4 results. Labor supply is negatively related to w_t, but the magnitude of the response is greater (in absolute value) than that in Table 4. The elasticity ranges from -.04 to -.065 and monotonically rises in absolute value, as γ increases (given the trend of estimates in Table 7). This rules out the possibility of our model exhibiting a large positive labor supply elasticity at some structural parameter setting. For the case of a permanent shock in w_t, the steady-state elasticity of c_t varies from .14E-02 to .81 and x_t from -.99 to 1.2. The possibility of high consumption elasticities (like .81) is consistent with the predictions of the permanent income hypothesis. The labor supply elasticity ranges from -0.5 E-07 to 0.022, a response which is not very different from that in Table 5.

To summarize, under alternative settings for the structural parameter γ, consumption and financial assets exhibit and labor supply does not exhibit, a wide range of responses to transitory and permanent shocks respectively in the exogenous variables.

CHAPTER V

SUMMARY

Our object in this dissertation has been to test the hypothesis, that economic agents jointly choose current consumption and labor supply, so as to maximize the present discounted value of current and expected future utility. This goal is met subject to the condition that, policy evaluation using the estimated decision rules for consumption and labor supply, should not be subject to the Lucas (1976) critique. Our hypothesis is embodied in the cross-equation parameter restrictions resulting from the solution of a maximum problem, and the assumption of rational expectations for forecasting future exogenous variables.

Empirical tests using quarterly postwar data for the United States, indicate a rejection of the cross-equation restrictions imposed by our hypothesis. However, the simulations for real consumption, labor supply and real financial assets, in response to transitory and permanent shocks in real wages, real interest rates and real taxes seem to be plausible. Our results are in basic agreement with Friedman's (1957) permanent income hypothesis. Hence consumption is affected by permanent and not transitory shocks in the real wage. Our result on labor supply is not in agreement with the seminal finding of Lucas and Rapping

(1969). They concluded that labor supply had a high positive elasticity (with respect to the real wage rate) for transitory shocks, and was highly inelastic with respect to permanent shocks. Our results indicate a highly inelastic response to both shocks.

We end this chapter with some general comments and suggestions for future research. Our objective has been to test the validity of the utility maximization assumption by testing the reasonableness of its implications in a consumption - labor supply framework. Technically, this involved formulating and solving an optimal control problem with a quadratic objective (utility) function and a linear constraint (equation of motion for the state). With some exceptions, closed form solutions are possible for certain linear - quadratic formulations only. Barring advances in mathematics which permit us to solve more general problems, we have to be constrained by such linear - quadratic formulations.

A richer specification of the utility function would involve the presence of interaction terms for consumption and labor supply. To the extent that utility is interdependent, such a specification would constitute an improvement over what we have. Another improvement would be the addition of a labor demand function to our model. Though we showed that real wages were exogenous with respect to labor supply, there may be some benefit in including the demand function.

The last suggestion we would like to make concerns the issue of aggregation. Scant regard is paid to issues of aggregation in the conventional econometric literature. The need for further research on such issues is acute for models like ours, where we try to motivate macrobehavior on the basis of microbehavior. In light of this, an attempt to estimate our model using panel data may turn out to be fruitful.

APPENDIX A

DESCRIPTION OF THE DATA

We used seasonally adjusted quarterly data for the U.S. from 1948.2 to 1980.4. All data was drawn from various issues of the Survey of Current Business and Business Statistics.

The variable list given in section 3.1 consists of variables on which (a) data were available, or (b) data was created by us. The following variables had data available on them:

C_tP_t	=	nominal value of total personal consumption.
TAX_t	=	total personal tax payments.
N_t	=	total civilian employment.
$\tilde{W}_t^* N_t$	=	compensation of employees.
P_t	=	consumption price deflator.
PI_t	=	personal income.

From the above set, we created new variables as follows:

C_t	=	C_tP_t/P_t
$\tilde{r}_t^* A_{t-1}$	=	$PI_t - \tilde{W}_t^* N_t$ = all non-labor (capital) income.
A_t^*	=	the solution of equation (3.2) obtained recursively. The starting value for A^* was assumed to be roughly two and one-half times PI in 1948.3
$\tilde{r}_t^*$	=	the solution of equation (3.2) for $\tilde{r}_t^*$ obtained recursively, given A_t^* from above.

Prior to estimation, all variables of the model were transformed into a stationary, linearly indeterministic form by the following regression:

$$z_t = c_o + c_1 t + c_2 t^2 + u_t$$

where z_t is the variable and t is the trend term. The resulting regression residuals u_t were used as data instead of the original values. Since all variables were filtered in this way, no bias is introduced, see Maddala (1977). This procedure reduces multicollinearity among variables and reduces the possibility of "spurious regression" -- see Granger and Newbold (1974).

APPENDIX B

TABLES OF CHAPTER IV

This appendix contains all of the tables discussed in Chapter IV.

TABLE 1

ESTIMATES OF EXOGENOUS STOCHASTIC PROCESSES

	Dependent Variable		
	w_t	r_t	m_t
AR1 coefficient	0.95 (33.57)	0.41 (5.21)	0.87 (20.6)
R^2	0.91	0.85	0.81
Q	25.12	42.88	33.39
N	129	129	129

Notes:

1. The AR1 coefficients are parameter estimates of Eqns. (3.36.1)

2. The t statistics are in parentheses below the coefficient.

3. Q = Box-Pierce statistics of residuals for first twentyfive autocorrelations. The statistic is not significant at the 5% level for w_t and m_t and at the 1% level for r_t.

4. N = sample size.

TABLE 2

MAXIMUM LIKELIHOOD OPTIMAL DECISION RULE ESTIMATES

β_o	=	.064464 (.17)	L_r	=	901.5
β_1	=	.935537 (2.47)	L_u	=	938.9
γ	=	.0000773608 (.18)	λ	=	$-2(L_r-L_u) = 74.8$
Q_c	=	25.05	$\chi^2(6)$	=	12.6
Q_n	=	50.8	N	=	128

Notes:

1. β_o, β_1 and γ are the structural parameter estimates with asymptotic t statistics in parentheses.

2. Q_c and Q_n are the Box-Pierce statistics of residuals for the first thirty autocorrelations of the consumption and labor supply equations respectively. Q_c is not significant at the 5% level and Q_n is not significant at the 1% level.

3. L_r and L_u are the restricted and unrestricted values of the log likelihood functions.

4. $\chi^2(6)$ is the table value of the χ^2 statistic for 6 degrees of freedom.

5. N = sample size.

TABLE 3

ELASTICITY ESTIMATES

Coefficient	Restricted[1]	Unrestricted[2]
d_1	.01008	.486036
d_2	.06944	.459223
d_3	.0154	-.089881
d_4	-.05252	-.013105
d_5	-.000695	-.0037926
d_6	-.004785	.199374
d_7	-.001061	-.072814
d_8	.003619	.017588

Notes:

1. Estimates of the d coefficients in Eqn. (4.2) with restrictions (4.3) imposed.

2. Estimates of the d coefficients in Eqn. (4.2) without any restrictions.

TABLE 4

SIMULATION FOR A TRANSITORY SHOCK IN w_t

t	η_c^T	η_n^T	η_x^T
1	.100817E-01	-.694692E-03	.989224
2	.997306E-02	-.687206E-03	.978563
3	.986558E-02	-.679800E-03	.968018
4	.975927E-02	-.672474E-03	.957586
5	.965409E-02	-.665228E-03	.947267
6	.955006E-02	-.658059E-03	.937058
7	.944714E-02	-.650967E-03	.926961
8	.934534E-12	-.643952E-03	.916971
9	.924463E-02	-.637013E-03	.907090
10	.914500E-02	-.630148E-03	.897314
11	.904645E-02	-.623357E-03	.887645
12	.894897E-02	-.616640E-03	.878079
13	.885253E-02	-.609995E-03	.868616
14	.875713E-02	-.603421E-03	.859256
15	.866276E-02	-.596918E-03	.849996
16	.856941E-02	-.590486E-03	.840836
17	.847706E-02	-.584123E-03	.831775
18	.838571E-02	-.577828E-03	.822812
19	.829534E-02	-.571601E-03	.813945
20	.820595E-02	-.565441E-03	.805173

Note:

There will be identical simulations for r_t and m_t except for a change in sign for the latter.

TABLE 5

SIMULATION FOR A PERMANENT SHOCK IN w_t

t	η_c^P	η_n^P	η_x^P
1	.112126	-.772618E-02	-.119852
2	.110918	-.764291E-02	-.238413
3	.109722	-.756055E-02	-.355695
4	.108540	-.747908E-02	-.471714
5	.107370	-.739848E-02	-.586483
6	.106213	-.731875E-02	-.700015
7	.105069	-.723988E-02	-.812323
8	.103936	-.716186E-02	-.923421
9	.102816	-.708468E-02	-.103332E+01
10	.101708	-.700833E-02	-.114204E+01
11	.100612	-.693281E-02	-.124958E+01
12	.995280E-01	-.685810E-02	-.135597E+01
13	.984554E-01	-.678419E-02	-.146121E+01
14	.973944E-01	-.671108E-02	-.156531E+01
15	.963449E-01	-.663876E-02	-.166830E+01
16	.953066E-01	-.656733E-02	-.188017E+01
17	.942796E-01	-.649645E-02	-.187095E+01
18	.932636E-01	-.642644E-02	-.197064E+01
19	.922585E-01	-.635719E-02	-.206925E+01
20	.912643E-01	-.628868E-02	-.216681E+01

Note: There will be identical simulations for r_t and m_t except for a change in sign for the latter.

TABLE 6

STEADY-STATE ELASTICITIES

Endogenous Variable	Elasticity
c_t	0.09
n_t	-.0063
x_t	-2.1

Note: These elasticities are for t = 20.

TABLE 7

SIMULATION FOR A TRANSITORY SHOCK IN w_t

t	$\gamma = .1$	$\gamma = .5$	$\gamma = .75$	$\gamma = 50$	$\gamma = 250$	$\gamma = 1000$
			Dependent Variable: c_t			
1	.604	.821	.853	.934	.935	.935
2	.214	.101	.748	.144	.029 E-02	.077 E-02
3	.0758	.0124	.007	.222 E-07	.879 E-07	.006 E-06
4	.0269	.0015	.006 E-01	.344 E-08	.277 E-10	.004 E-10
			Dependent Variable: n_t			
1	.042	-.006 E-02	-.059	-.064	-.064	-.065
2	-.015	-.007	-.005	-.009 E-02	-.002 E-02	-.005 E-03
3	-.005	-.009 E-01	-.005 E-01	-.002 E-04	-.006 E-06	-.004 E-07
4	-.002	-.001 E-01	-.004 E-02	-.002 E-07	-.002 E-09	-.003 E-12
			Dependent Variable: x_t			
1	.354	.123	.088	.002	.003 E-01	.007 E-02
2	.125	.015	.007	.002 E-03	.009 E-05	.006 E-06
3	.045	.002	.007 E-01	.004 E-06	.003 E-08	.005 E-10
4	.016	.002 E-01	.006 E-02	.006 E-09	.009 E-12	.004 E-14

Note: Simulations over alternative values of γ.

TABLE 8

SIMULATION FOR A PERMANENT SHOCK IN w_t

t	$\gamma = .1$	$\gamma = .5$	$\gamma = .75$	$\gamma = 50$	$\gamma = 250$	$\gamma = 1000$
			Dependent Variable: c_t			
1	.896	.925	.928	.935	.935	.934
2	.317	.113	.814	.001	.289	.720
3	.113	.113	.814	.001	.289	.720
4	.039	.113	.814	.001	.289	.720
			Dependent Variable: n_t			
1	-.062	-.064	-.064	-.065	-.064	-.064
2	-.022	-.008	-.006	-.009 E-02	-.002 E-02	-.005 E-03
3	-.008	-.008	-.006	-.009 E-02	-.002 E-02	-.005 E-03
4	-.003	-.008	-.006	-.009 E-02	-.002 E-02	-.005 E-03
			Dependent Variable: x_t			
1	-0.957	-.988	-.922	.999	-.999	-.999
2	-1.29	-.988	-.922	.999	-.999	-.999
3	-1.42	-.988	-.922	.999	-.999	-.999
4	-1.46	-.988	-.922	.999	-.999	-.999

Note: Simulations over alternative values of γ.

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