MATHEMATICS APPLIED TO SOME ASPECTS OF DYNAMIC METEOROLOGY

A Thesis

Presented to

the Faculty of the Department of Mathematics

The University of Houston

In Partial Fulfillment

of the Requirements for the Degree

Master of Science

by

Marvin R. Rogers

June 1954

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It is the purpose of this thesis to review and collect fundamental mathematics that bears on the motion of air particles in dynamic meteorology. Many of the derivations have been supplied by the author. The section on the curl of the relative acceleration and the development of the vector equation concerning the center of curvature are the author's own work.

TABLE OF CONTENTS

CHAPTER I

FORCES ON A ROTATING EARTH AND ATMOSPHERE

The aim of this chapter is to demonstrate mathematically the acceleration that a moving particle experiences from the forces that are present in a rotating system, and integrate these forces into a consideration of an air particle's movement with respect to the earth. From a meteorological view point, the total acceleration of the air particle combines the effect of the rotating system and of the forces that act upon the air particle. In particular, these include gravity, pressure, and friction.[1]

I. OUTLINE OF MATHEMATICS IN A ROTATING SYSTEM[2]

If a system of coordinates whose fundamental vectors are $\bar{i}$, $\bar{j}$, and $\bar{k}$, changes in position relative to a second system which is regarded as fixed, then the fundamental vectors themselves become functions of the time with respect to the fixed system.

To analyze this statement, consider the dot product

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1 Sverre Petterson, Weather Analysis and Forecasting (first edition; New York: McGraw-Hill Book Company, Inc., 1940), p. 206.

2 Arthur Haas, Introduction to Theoretical Physics (Vol. 1; second edition; London: Constable and Company, Ltd., 1928), pp. 38-51.

of unit vector $\bar{i}$.

$$\bar{i} \cdot \bar{i} = 1.$$

Differentiating with respect to time

$$\bar{i} \cdot \frac{d\bar{i}}{dt} + \frac{d\bar{i}}{dt} \cdot \bar{i} = 0$$

$$\bar{i} \cdot \frac{d\bar{i}}{dt} = 0.$$

Now $\frac{d\bar{i}}{dt} \neq 0$, thus $\frac{d\bar{i}}{dt}$ is perpendicular to $\bar{i}$.

From this it follows:

$$\bar{i} \cdot \frac{d\bar{i}}{dt} = 0, \ \bar{j} \cdot \frac{d\bar{j}}{dt} = 0, \ \bar{k} \cdot \frac{d\bar{k}}{dt} = 0$$

are all at right angles to corresponding fundamental vectors.

These three vectors $\frac{d\bar{i}}{dt}$, $\frac{d\bar{j}}{dt}$, and $\frac{d\bar{k}}{dt}$ are co-planar. The proof follows from the following relationship:

$$\bar{i} \times \bar{j} = \bar{k}.$$

Differentiating with respect to time

$$\frac{d\bar{k}}{dt} = \frac{d\bar{i}}{dt} \times \bar{j} + \bar{i} \times \frac{d\bar{j}}{dt}.$$

Performing a cross product with $\frac{d\bar{j}}{dt}$,

$$\frac{d\bar{j}}{dt} \times \frac{d\bar{k}}{dt} = \frac{d\bar{i}}{dt} \times \frac{d\bar{j}}{dt} \times \bar{j} + \frac{d\bar{j}}{dt} \times \bar{i} \times \frac{d\bar{j}}{dt}.$$

Making use of the following formula from vector analysis:

$$\bar{a} \times \bar{b} \times \bar{c} = \bar{b}(\bar{c} \cdot \bar{a}) - \bar{c}(\bar{a} \cdot \bar{b}),$$

the right side of the equation becomes

$$\frac{d\bar{i}}{dt}\left(\frac{d\bar{j}}{dt} \cdot \bar{j}\right) - \bar{j}\left(\frac{d\bar{j}}{dt} \cdot \frac{d\bar{i}}{dt}\right) + \bar{i}\left(\frac{d\bar{j}}{dt} \cdot \frac{d\bar{j}}{dt}\right) - \frac{d\bar{j}}{dt}\left(\frac{d\bar{j}}{dt} \cdot \bar{i}\right).$$

The term $\qquad \frac{d\bar{i}}{dt}\left(\frac{d\bar{j}}{dt} \cdot \bar{j}\right) = 0$

since $\qquad \frac{d\bar{j}}{dt}$ and $\bar{j}$

are perpendicular to each other.

Dotting both sides of the remaining equation with $\frac{d\bar{i}}{dt}$,

$$\frac{d\bar{i}}{dt} \cdot \frac{d\bar{j}}{dt} \times \frac{d\bar{k}}{dt} = -\left(\frac{d\bar{i}}{dt} \cdot \bar{j}\right)\left(\frac{d\bar{j}}{dt} \cdot \frac{d\bar{i}}{dt}\right)$$

$$+\left(\frac{d\bar{i}}{dt} \cdot \bar{i}\right)\left(\frac{d\bar{j}}{dt} \cdot \frac{d\bar{j}}{dt}\right)$$

$$-\left(\frac{d\bar{i}}{dt} \cdot \frac{d\bar{j}}{dt}\right)\left(\frac{d\bar{j}}{dt} \cdot \bar{i}\right).$$

The second term on the right drops out since

$$\frac{d\bar{i}}{dt} \cdot \bar{i} = 0,$$

and regrouping the terms

$$\frac{d\bar{i}}{dt} \cdot \frac{d\bar{j}}{dt} \times \frac{d\bar{k}}{dt} = -\left(\frac{d\bar{i}}{dt} \cdot \frac{d\bar{j}}{dt}\right)\left(\frac{d\bar{i}}{dt} \cdot \bar{j} \quad \frac{d\bar{j}}{dt} \cdot \bar{i}\right).$$

The term $\qquad \left(\frac{d\bar{i}}{dt} \cdot \bar{j} \quad \frac{d\bar{j}}{dt} \cdot \bar{i}\right) = \frac{d}{dt}(\bar{i} \cdot \bar{j})$

and $\qquad (\bar{i} \cdot \bar{j}) = 0,$

thus $\qquad \frac{d\bar{i}}{dt} \cdot \frac{d\bar{j}}{dt} \times \frac{d\bar{k}}{dt} = 0.$

The scalar triple product represents the volume of a parallelepiped formed by coterminous sides, $\bar{a}$, $\bar{b}$, and $\bar{c}$.

$$\bar{a} \cdot \bar{b} \times \bar{c} = |\bar{a}|\ |\bar{b}|\ |\bar{c}| \sin \theta \cos a$$

$$\bar{a} \cdot \bar{b} \times \bar{c} = hA = \text{volume}.$$ [3]

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3 Harry Lass, Vector and Tensor Analysis (New York: McGraw-Hill Company, Inc., 1950), p. 23.

If $$\bar{a} \cdot \bar{b} \times \bar{c} = 0,$$

then the vectors are co-planar.[4]

Suppose a unit vector $\bar{w}_o$ is perpendicular to the plane common to the three vectors

$$\frac{d\bar{i}}{dt}, \frac{d\bar{j}}{dt}, \text{ and } \frac{d\bar{k}}{dt},$$

then $\frac{d\bar{i}}{dt}$ is perpendicular to both $\bar{w}_o$ and also to $\bar{i}$; similar relations hold for

$$\frac{d\bar{j}}{dt} \text{ and } \bar{j}, \text{ and } \frac{d\bar{k}}{dt} \text{ and } \bar{k}.$$

Then the following is true:

$$(1) \quad a(\bar{w}_o \times \bar{i}) = \frac{d\bar{i}}{dt}, \; b(\bar{w}_o \times \bar{j}) = \frac{d\bar{j}}{dt}, \text{ and}$$
$$c(\bar{w}_o \times \bar{k}) = \frac{d\bar{k}}{dt}.$$

Now from the time derivative of $\bar{i} \cdot \bar{j} = 0$,

$$(2) \quad \frac{d\bar{i}}{dt} \cdot \bar{j} + \bar{i} \cdot \frac{d\bar{j}}{dt} = 0.$$

Then substituting values above for $\frac{d\bar{i}}{dt}$ and $\frac{d\bar{j}}{dt}$

$$(3) \quad \bar{j} \cdot a(\bar{w}_o \times \bar{i}) + \bar{i} \cdot b(\bar{w}_o \times \bar{j}) = 0,$$

interchanging dot and cross, and carrying out cyclic process, and letting our scalar be associated with $\bar{w}_o$

$$(4) \quad a\bar{w}_o \cdot (\bar{i} \times \bar{j}) + b\bar{w}_o \cdot (\bar{j} \times \bar{i}) = 0,$$

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4 *Ibid.*, p. 24.

(5) $a\overline{w}_e \cdot \overline{k} - b\overline{w}_e \cdot \overline{k} = 0,$

(6) $(\overline{w}_e \cdot \overline{k})\,(a - b) = 0.$

Similar statements could be made for the x and y axes, and hence both $\overline{w}_e \cdot \overline{i}$ and $\overline{w}_e \cdot \overline{j}$ would also be equal to zero. This, of course, is not true since $\overline{w}$ would be perpendicular to all three co-ordinate axes. Consequently, $a = b$. Similarly, by cyclic interchange, $b = c$ and $c = a$. Thus, the three fundamental vectors $\frac{d\overline{i}}{dt}$, $\frac{d\overline{j}}{dt}$, and $\frac{d\overline{k}}{dt}$ can be represented as vector products of one vector $\overline{w}$:

(7) $\frac{d\overline{i}}{dt} = \overline{w} \times \overline{i},\ \frac{d\overline{j}}{dt} = \overline{w} \times \overline{j},$ and $\frac{d\overline{k}}{dt} = \overline{w} \times \overline{k}.$

Carrying the investigation further, consider an arbitrary vector $\overline{a}$. Let $\overline{a}$ be associated with a system whose fundamental vectors are $\overline{i}$, $\overline{j}$, and $\overline{k}$. The projection of $\overline{a}$ along each axis of the co-ordinate system yields

(8) $\overline{a} = \overline{i}\,a_x + \overline{j}\,a_y + \overline{k}\,a_z.$

Differentiating with respect to time:

(9) $\frac{d\overline{a}}{dt} = \overline{i}\,\frac{da_x}{dt} + \overline{j}\,\frac{da_y}{dt} + \overline{k}\,\frac{da_z}{dt} + a_x\,\frac{d\overline{i}}{dt} + a_y\,\frac{d\overline{j}}{dt} + a_z\,\frac{d\overline{k}}{dt}.$

Letting the time rate of change of $\overline{a}$ with respect to the co-ordinate system $\overline{i}$, $\overline{j}$, and $\overline{k}$ be denoted by

(10) $\frac{d^*\overline{a}}{dt} = \overline{i}\,\frac{da_x}{dt} + \overline{j}\,\frac{da_y}{dt} + \overline{k}\,\frac{da_z}{dt}$

and

$$(11)\quad a_x \frac{d\bar{i}}{dt} + a_y \frac{d\bar{j}}{dt} + a_z \frac{d\bar{k}}{dt} = (\bar{w} \times \bar{i})a_x + (\bar{w} \times \bar{j})a_y + (\bar{w} \times \bar{k})a_z$$

$$= \bar{w} \times a_x \bar{i} + \bar{w} \times a_y \bar{j} + \bar{w} \times a_z \bar{k}$$

which is simply $\bar{w} \times \bar{a}$.

Then

$$(12)\quad \frac{d\bar{a}}{dt} = \frac{d^*\bar{a}}{dt} + \bar{w} \times \bar{a}.$$

Suppose vector $\bar{a}$ is of constant magnitude and is directed from an origin of a co-ordinate system to a fixed point P. It is obvious, then that

$$\frac{d^*\bar{a}}{dt} = 0,$$

and $$\frac{d\bar{a}}{dt} = \bar{w} \times \bar{a}.$$

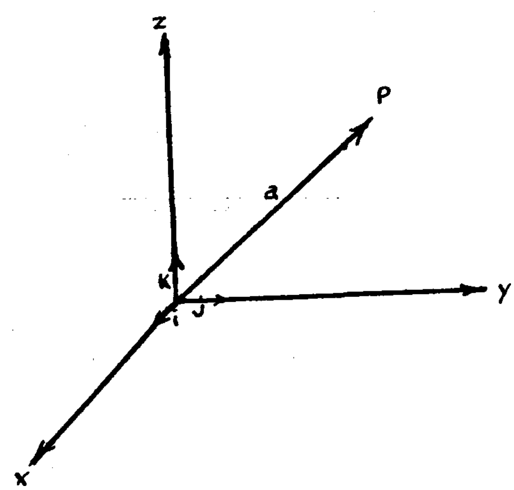

Figure 1

$\frac{d\bar{a}}{dt}$ is a vector perpendicular to the plane of $\bar{w} \times \bar{a}$ and its direction is the same as that of a right-hand screw.

Further consideration shows that $\overline{w} \times \overline{a} = wa \sin (w,a)\, \overline{e}$ where $\overline{e}$ is a unit vector.

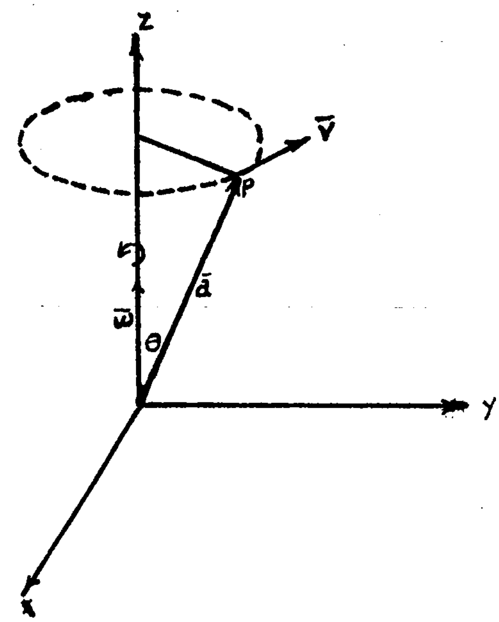

Figure 2

In Figure 2, the point P moves in a circle of radius $|\overline{a}|$ with $\overline{w}$ constant. If $\overline{w}$ is not constant, then it is considered as the instantaneous motion. It is now apparent that $\overline{w}$ is the angular velocity of the particle P, its direction, in the future, shall be directed along the axis of earth, and its magnitude is $\frac{d\overline{e}}{dt}$.

II. RELATIVE MOTION

Motion that is described as relative must be relative to some particular thing. In this case, movement with respect to a system of co-ordinates or a frame O, whose fundamental vectors $\overline{i}$, $\overline{j}$, and $\overline{k}$ vary with the time in regard to a fixed system or frame O', is denoted as relative motion.

Consider a moving point P, and let its position be described from both reference frames O and O'. Point P may be represented by drawing a directed vector length from O to P and O' to P. This may describe the motion relative to a particular frame, but since O may move with respect to O', a fixed system, a directed vector length from O' to O completes the description of motion of point P. Note Figure 3.

In vector form, let $\bar{r}'$ be a position vector from O' to P, $\bar{r}$ be a position vector from O to P, and $\bar{a}$ be directed from O' to O, thus

(1) $\bar{r}' = \bar{a} + \bar{r}.$

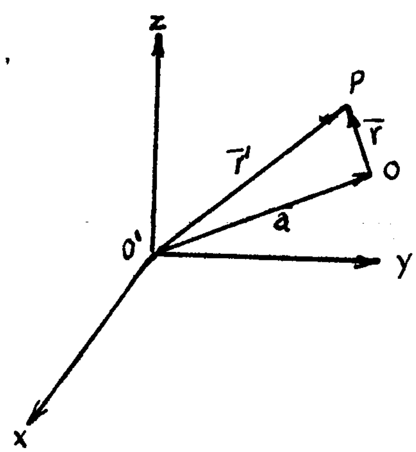

Figure 3

If $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ is a position vector of a moving particle P(x,y,z) in three dimension, then the change in $\bar{r}$ is $\quad d\bar{r} = dx\bar{i} + dy\bar{j} + dz\bar{k},$

and the velocity is

$$\bar{v} = \frac{d\bar{r}}{dt} = \frac{dx}{dt}\,\bar{i} + \frac{dy}{dt}\,\bar{j} + \frac{dz}{dt}\,\bar{k}.$$

The acceleration, being time rate of change of the velocity, is of form

$$\bar{a} = \frac{d\bar{v}}{dt} = \frac{d^2x}{dt}\,\bar{i} + \frac{d^2y}{dt}\,\bar{j} + \frac{d^2z}{dt}\,\bar{k}.$$ [5]

Thus, in describing the motion of point P, equation (1) is differentiated with respect to time

(2) $$\frac{d\bar{r}'}{dt} = \frac{d\bar{a}}{dt} + \frac{d\bar{r}}{dt}.$$

Examining each term of (2) separately:

$$\bar{v} = \frac{d\bar{r}'}{dt},$$

represents the velocity of moving particle with respect to first system of co-ordinates O'.

$$\bar{v}_t = \frac{d\bar{a}}{dt}$$

is the motion carried out by the origin of second system O with respect to first O'. Now

$$\frac{d\bar{r}}{dt} = \frac{d^*\bar{r}}{dt} + \bar{w} \times \bar{r}.$$

The $\frac{d^*\bar{r}}{dt}$ term is the time rate of change with respect to system O, that is

$$\bar{v}_r = \frac{d^*\bar{r}}{dt} = \bar{i}\,\frac{dx}{dt} + \bar{j}\,\frac{dy}{dt} + \bar{k}\,\frac{dz}{dt}.$$

The $\bar{w} \times \bar{r}$ term, as presented in the last section, is the velocity of a particle attached rigidly to system O, which in turn rotates around O'.

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5 *Ibid.*, p. 30.

Thus equation (2) takes the form

$$(3)\quad \bar{v} = \bar{v}_t + \bar{v}_r + (\bar{w} \times \bar{r}).$$

It follows, that the total acceleration may be accomplished by differentiating (3) with respect to time

$$(4)\quad \frac{d\bar{v}}{dt} = \frac{d\bar{v}_t}{dt} + \frac{d(\bar{v}_r)}{dt} + \frac{d\bar{w}}{dt} \times \bar{r} + \bar{w} \times \frac{d\bar{r}}{dt}.$$

Examining each term on the right separately, $\frac{d\bar{v}_t}{dt}$ is the acceleration of the origin of the second system with respect to the first. The term $\frac{d(\bar{v}_r)}{dt}$ breaks up into two parts, since $\bar{v}_r$ is the velocity of a particle relative to the second system, and thus it takes the following form

$$\frac{d(\bar{v}_r)}{dt} = \frac{d^*\bar{v}_r}{dt} + \bar{w} \times \bar{v}_r.$$

The last term becomes

$$\bar{w} \times \left(\frac{d^*\bar{r}}{dt} + (\bar{w} \times \bar{r})\right) = \bar{w} \times \frac{d^*\bar{r}}{dt} + \bar{w} \times (\bar{w} \times \bar{r})$$

$$= \bar{w} \times \bar{v}_r + (\bar{w} \times \bar{w} \times \bar{r}).$$

Finally,

$$(5)\quad \frac{d\bar{v}}{dt} = \frac{d\bar{v}_t}{dt} + \frac{d^*\bar{v}_r}{dt} + 2(\bar{w} \times \bar{v}_r) + \frac{d\bar{w}}{dt} \times \bar{r} + \bar{w} \times (\bar{w} \times \bar{r})$$

From equation (5) the significance of the term $\bar{w} \times \bar{w} \times \bar{r}$ can be better understood if examined closely.

Let $\bar{w}_o$ be a unit vector along $\bar{w}$ and r_w projection of $\bar{r}$ on $\bar{w}$. Note Figure 4.

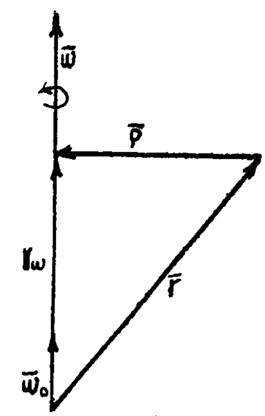

Figure 4

Then $$\bar{w} \times \bar{w} \times \bar{r} = (\bar{w} \cdot \bar{r})\,\bar{w} - (\bar{w} \cdot \bar{w})\,\bar{r},$$

$$\bar{w} \times \bar{w} \times \bar{r} = \bar{w}\,(wr\cos(w,r) - \bar{r}\,(w^2),$$

$$\bar{w} \times \bar{w} \times \bar{r} = \bar{w}_o\,(rw^2\cos(w,r) - \bar{r}\,(w^2),$$

$$\bar{w} \times \bar{w} \times \bar{r} = w^2\,(r_w\bar{w}_o - \bar{r}).$$

Now $$r_w\bar{w}_o = \bar{r} + \bar{P}$$

so that $$\bar{w} \times \bar{w} \times \bar{r} = w^2\,(\bar{r} + \bar{P} - \bar{r}) = w^2\bar{P}.$$

Then it is evident, if a particle is rigidly connected with system O, and the vector of angular velocity is constant both in magnitude and direction, and there is no translational acceleration, equation (5) becomes

$$\frac{d\bar{v}}{dt} = w^2\bar{P}.$$

$w^2\bar{P}$ is directed towards the center and reveals itself as a center seeking acceleration and is commonly called centripetal acceleration.

If $\overline{w}$ vanishes, the axes of the second system remains constantly parallel to those of the first, then

$$\frac{d\overline{v}}{dt} = \frac{d\overline{v}_t}{dt} + \frac{d\overline{v}_r}{dt}.$$

Limiting conditions to the case in which the motion of the origin of the second with respect to the first is uniform, then

$$\frac{d\overline{v}}{dt} = \frac{d\overline{v}_r}{dt},$$

and the acceleration is identical for both systems.

In conclusion to this analysis, two co-ordinate systems in a state of uniform translatory motion with respect to each other are known as an inertial system and are equivalent for the description of mechanical processes provided speeds are $v \ll c$ where c equal speed of light.[6] This conclusion is known as the mechanical principle of relativity.

III. FORCES PRODUCING MOTION RELATIVE TO EARTH

In the preceding section, mathematics was applied to show accelerations and forces on a rotating system. Now it

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6 Robert Lindsey, General Physics (New York: John Wiley and Sons, Inc., 1940), p. 516.

remains to show what are the contributors to forces and accelerations in regard to the earth and its atmosphere. According to Newton

$$\frac{d(m\bar{v})}{dt} = \frac{md^2\bar{r}}{dt^2} = \bar{F}$$

if the mass is constant. The force resulting is the net unbalanced force acting on a particle, air in our study. Thus it becomes apparent that in the atmosphere, the acting forces are the pressure force, the force of gravity, and the frictional force. These forces coupled with forces experienced by a rotating system tell the story of a particle's acceleration. In this section, however, frictional force will not be considered. A word definition of Newton's second law, here, will portray the following work most appropriately: "The change of motion is proportional to the force and takes place in the direction in which the force acts."

Reference system. The systems of co-ordinates or frames used are commonly called the relative frame and the absolute frame. The former is attached rigidly to the surface of the earth with origin O, and quantities referred to the relative or "local" system will carry subscripts of r. The absolute system O' will be attached to some point on the axis of the earth and be oriented so that the "fixed stars" appear fixed. References of quantities to the absolute frame will carry no subscript.

It should be noted here that Newton's second law, for astronomical calculations, should refer motion to a system located at the center of gravity of the solar system and fixed with respect to the stars, but for dynamic meteorology the system to be used is sufficient.[7]

Pressure force. The pressure force arises from interaction of the air elements and is independent of the reference system from which it is observed. In general, the atmosphere can be handled as a fluid medium, and mathematical equations expressing its motions follow hydrodynamic equations.[8] Rigorous derivations can be studied in textbooks on hydrodynamics, but for the purposes of this study, the equations and statements will be very compact and brief.

It can be stated that the pressure force per unit volume is a potential vector, and its potential is the pressure. Since the potential vector is directed toward decreasing pressure, "the pressure force per unit volume is the gradient of the pressure, or simply the pressure gradient."[9]

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7 Jorgen Holmboe, Dynamic Meteorology (New York: John Wiley and Sons, Inc., 1945), p. 152.

8 Bernard Haurwitz, Dynamic Meteorology (New York: McGraw-Hill Book Company, Inc., 1941), p. 127.

9 Holmboe, op. cit., p. 99.

If p represents pressure and p(x,y,z) is a continuous differentiable space function, the calculus gives

$$(1) \quad dp = \frac{\partial p}{\partial x} dx + \frac{\partial p}{\partial y} dy + \frac{\partial p}{\partial z} dz .$$

Now, let $\bar{r}$ be a position vector to the point of pressure (x,y,z).

$$\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$$

and $$dr = dx\bar{i} + dy\bar{j} + dz\bar{k}.$$

Now let $$\text{del } p \equiv \nabla p$$

be denoted as $$\nabla p = \frac{\partial p}{\partial x}\bar{i} + \frac{\partial p}{\partial y}\bar{j} + \frac{\partial p}{\partial z}\bar{k} .$$

It is obvious then, that if

$$dp = d\bar{r} \cdot \nabla p = 0,$$

then the equation points out that ∇p is perpendicular to $d\bar{r}$ as long as $d\bar{r}$ represents a change from a point (x,y,z) to a point (x_0, y_0, z_0) on the surface p constant. ∇p then is normal to all the tangents to the surface at (x,y,z) and is normal to surface p(x,y,z) = constant. Since ∇p is fixed at any point (x,y,z) the change in p will depend on $d\bar{r}$. The term dp will be at a maximum when $d\bar{r}$ is parallel to p. Then p is in the direction of maximum increase. Now since the definition calls for the vector to point to decreasing pressure, it can be expressed

$$\bar{b} = -\nabla p \quad \text{pressure per unit volume}$$

and the pressure force for a volume δV is $-\delta V \nabla p$.

Dividing by δM

$$\bar{b} = -\frac{\delta v}{\delta M}\nabla P = -\alpha \nabla P \text{ for per unit mass.}$$

<u>Gravitation</u>. "Every particle in the universe attracts every other particle with a force which is directed along the line, joining the particles and varies directly as the product of the masses and inversely as the square of the distance between them" is a fundamental assumption proposed by Newton.[10] The coefficient of proportionality is called the constant of gravitation and is denoted by G. Thus,

$$F = \frac{Gm_1m_2}{r^2},$$

where $G = 6.658 \times 10^{-8}\ (M^{-1}L^3T^{-2})$, (Holmboe, P 153)

and $M = 5.988 \times 10^{21}$ Metric Tons.

Since the particle is of unit mass

$M = m_1m_2$ (m_1 = mass of earth, m_2 = unit mass)

and then the equation can take the form

(1) $\bar{g} = \frac{GM}{r^2}$, where $\bar{g} = F$.

The particle of air is in the gravitational field of force produced by M(earth). The relation holds only if the earth is considered as a perfect homogeneous sphere, but actually the earth is an oblate spheroid with the polar radius about

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10 Robert Lindsay, <u>General Physics</u> (New York: John Wiley and Sons, Inc., 1940), p. 96.

6357 kilometers and the equatorial radius 6378 kilometers. Using $r = 6371$, the equation (1) becomes

$$g = \frac{GM}{r^2} = 9.822 \text{ mps}^{-2}.[11]$$

The force of gravitation is directed along a line from the center of the earth to a point in question, thus, to get a vector representation of this force, it will be necessary to determine if it has a potential. In this case, potential energy is a function of the position of a particle and is independent of its velocity.[12] Furthermore, the total energy of a particle, the sum of its kinetic energy and its potential energy, remain constant. From these statements, equipotential surfaces can be considered as infinitesimal spherical shells and the distance between two consecutive shells is dr. Letting φ be the gravitational potential, the following relation must hold:

$$g\,dr = -d(\varphi)_r.[13]$$

Substituting from $g = \dfrac{GM}{r^2}$,

$$-d(\varphi)_r = \frac{GM}{r^2}\,dr .$$

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11 Holmboe, *op. cit.*, p. 153.

12 Lindsey, *op. cit.*, p. 93.

13 *Ibid.*, p. 92.

Integrating $$-\int d\phi = GM\int \frac{dr}{r^2}.$$

$$\phi \equiv \frac{GM}{r}$$ which is the gravitational potential.[14]

The directional derivative of ϕ in any direction s is
$$\frac{d\phi}{ds} = \frac{\partial\phi}{\partial x}\frac{dx}{ds} + \frac{\partial\phi}{\partial y}\frac{dy}{ds} + \frac{\partial\phi}{\partial z}\frac{dz}{ds}.$$[15]

or
$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$$

Treating this is the same manner applied in the section on pressure $$d\phi = \nabla\phi \cdot d\bar{r} = 0.$$

Obviously $d\phi$ will be at a maximum when $d\bar{r}$ is parallel to $\nabla\phi$. Then $\nabla\phi$ is in the direction of maximum increase, but letting $\nabla\phi$ point in the direction of maximum decrease in gravitational potential, the vector force of gravitation
$$\bar{g} = -\nabla\phi.$$

Absolute Motion. The forces observed from the absolute frame or system O' acting upon a particle of unit mass are the gravitational force $\bar{g}$ and the pressure force $\bar{b}$ which is the pressure force per unit mass.

Newton's second law equates the absolute acceleration

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14 Ivan S. Sokolnikoff, Higher Mathematics for Engineers and Physicists (New York: McGraw-Hill Book Company, Inc., 1941), p. 219.

15 Loc. cit.

to the resultant of the forces applied. Then obviously

$$\frac{d\overline{v}}{dt} = \overline{b} + \overline{g} = -\alpha \nabla p - \nabla Q.$$

By letting

$$-\frac{d\overline{v}}{dt} = \overline{F}';$$

the equation for absolute motion can be expressed as an equilibrum of forces

$$0 = \overline{b} + \overline{g} + \overline{F}'.$$

$\overline{F}'$ is called the inertial force of reaction, it arises from the inertia of a particle moving relative to the absolute frame.[16] An observer attached to the moving particle is unable to distinguish between real forces and the inertial force of reaction. Thus when forces are measured relative to a moving particle which is accelerating relative to the absolute frame, inertial forces appear. Whenever a particle is moving with respect to some reference system with constant velocity, the particle is said to be attached to an inertial system.

<u>Velocity and acceleration of a point of earth</u>. The earth rotates from west to east at a constant speed w. Since w is considered with respect to "fixed stars", it is necessary to determine

$$w = \frac{d\theta}{dt}$$

in that relationship.

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16 Jorgen Holmboe, *Dynamic Meteorology* (New York: John Wiley and Sons, Inc., 1945), p. 155.

In one year or approximately $365\frac{1}{4}$ solar days, earth has rotated $365\frac{1}{4}$ times with respect to the sun. Also, it has made one complete turn in absolute space around the sun from west to east. Thus in one year it has rotated $366\frac{1}{4}$ times with respect to the stars. The ratio

$$\frac{366\frac{1}{4}}{365\frac{1}{4}} = \text{a sidereal day, and}$$

$$w = \frac{2\pi \text{ radians}}{1 \text{ sidereal day}} = \frac{366\frac{1}{4} \cdot 2\pi \text{ radians}}{365\frac{1}{4} \text{ solar days}} = 7.292 \times 10^{-5} \text{ radians sec}^{-1}.$$

Let the earth and all points that appear at rest when observed from a point of the earth constitute a space. Call this space relative space. It is evident that every point of relative space rotates at a constant angular speed w around the axis of the earth in a fixed circle of curvature centered on the axis.

Now, consider a point P of earth, fixed in relative space. Let the system O' be located at the center of the earth with the x, y plane in the equatorial plane and z axis pointed towards north pole along the axis of the earth, Direct position vector $\bar{r}$ to a point P from O'. Note Figure 4. Let $|\bar{r}|$ be the radius of the earth in this discussion. Now, $\bar{w}$ is defined as the angular velocity of the earth; since the rotation is described by the numerical value of the angular speed, the orientation of the axis, and the sense of rotation. Specifically, the vector of magnitude w directed along the axis of rotation according to the right-hand screw rule portrays the necessary information about the rotation. The velocity of a point of the earth is the time derivative of the position vector $\bar{r}$ of constant length.

$$\bar{v}_e = \frac{d\bar{a}}{dt} = \bar{w} \times \bar{r}. \quad \text{ref: p. 6, if } \bar{r} = \bar{a}$$

The acceleration of a point of the earth is determined by differentiating $\bar{v}_e$ with respect to time. (Note $\bar{w}$ is constant in direction and magnitude.)

$$\frac{d\bar{v}_e}{dt} = \frac{d}{dt}(\bar{w} \times \bar{r}) = \frac{d\bar{w}}{dt} \times \bar{r} + \bar{w} \times \frac{d\bar{r}}{dt}.$$

$$\frac{d\bar{v}}{dt} = \bar{w} \times (\bar{w} \times \bar{r})$$

which is recognizable as $\frac{d\bar{v}_e}{dt} = w^2\bar{p}$ from the section on relative motion, as the centripetal force.

An examination of the movement of a point at rest in relative space demonstrates clearly that there is an unbalanced force exerted against the point which can be considered as a particle of unit mass. This force causes the particle to have an acceleration towards the center of curvature. Otherwise, according to Newton's first law of motion, the moving particle would travel in a straight line. Now, taking into consideration Newton's third law of motion which states: "For every action, there is an equal and opposite reaction, and the two are along the same straight line", it is obvious there exists an equal and opposite reaction directed radially outward from the center of curvature. This force is known as the centrifugal reaction. It should be stated that these forces do not balance each other because they are not acting upon the same object.

From these considerations, the equation of absolute motion for a particle at rest in relative space is

$$0 = \overline{b} + \overline{g} + \overline{k}. \quad (k = w^2\overline{P})$$

This is the equation of relative or hydrostatic equilibrium, expressed from the absolute frame or system O'. To an observer in space, the pressure force is balanced by the force $\overline{g}$ $\overline{k}$, but to an observer at rest in relative space, the pressure force $\overline{b}$ appears to be balanced by a single force, $\overline{g}$. Thus

$$\overline{g}_r = \overline{g} + \overline{k},$$

and the moving observer is unable to distinguish between real and inertial forces.[17]

<u>Velocity Equation</u>. In the last section, motion of a particle that was fixed to the earth was considered. The question arises as to what is the nature of arbitrary motion of a particle moving with respect to system O' and system O. From Figure 5 it is easily seen that a moving particle P can be described by the following vector relationship

$$r^1 = \overline{a} + \overline{r}.$$

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17 Holmboe, <u>op</u>. <u>cit</u>., p. 156.

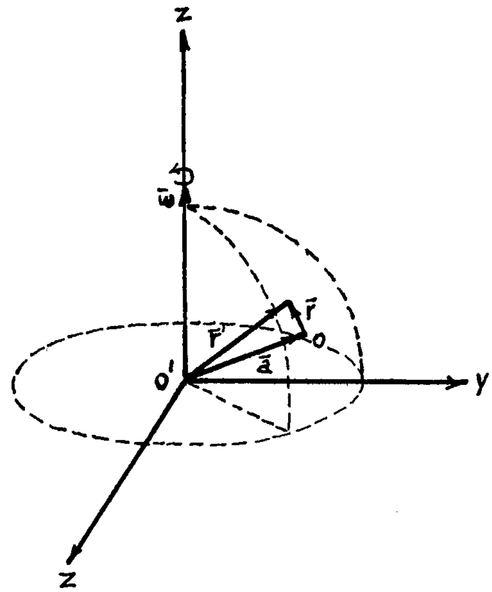

Figure 5

The time derivative

$$\frac{d\bar{r}^1}{dt} = \frac{d\bar{a}}{dt} + \frac{d\bar{r}}{dt}$$

gives an equation recognizable from the section on Relative Motion. Furthermore,

$$\frac{d\bar{r}^1}{dt} = \bar{v} = \bar{v}_t + \bar{v}_r + (\bar{w} \times \bar{r})$$

where $\bar{v}$ is the velocity of particle relative to system O', absolute frame. $\bar{v}_t$ is the motion of second system O, relative frame, with respect to first O'. $\bar{v}_r$ is the relative velocity of point P with respect to system O and $\bar{w} \times \bar{r}$ is the velocity of a point rigidly attached to earth.

<u>Acceleration Equation</u>. Acceleration, of course, is the time rate of change of the velocity, and in this case by performing a differentiation with respect to time of the

equation

$$\bar{v} = \bar{v}_t + \bar{v}_r + (\bar{w} \times \bar{r})$$

yields

$$\frac{d\bar{v}}{dt} = \frac{dv_t}{dt} + \frac{dv_r}{dt} + 2(\bar{w} \times \bar{v}_r) + \frac{d\bar{w}}{dt} \times \bar{r} + \bar{w} \times (\bar{w} \times \bar{r}).$$

(From section on Relative Motion, page 10)

The earth rotates at a constant velocity w and its rotation is described by vector $\bar{w}$ which is constant in direction and magnitude, thus the term

$$\frac{d\bar{w}}{dt} \times \bar{r} = 0.$$

Also, limiting the conditions to the instant that the motion of the origin of the second system O with respect to O' is uniform or rigidly attached,

$$\bar{v}_t = 0.$$

The resulting equation is

$$\frac{d\bar{v}}{dt} = \frac{d\bar{v}_r}{dt} + 2(\bar{w} \times \bar{v}_r) + \bar{w} \times (\bar{w} \times \bar{r}),$$

$$\frac{d\bar{v}}{dt} = \frac{d\bar{v}_r}{dt} + 2\bar{w} \times \bar{v}_r + \frac{d\bar{v}_e}{dt}.$$

This equation shows that the acceleration of a particle with respect to absolute frame O' is the sum of three vectors. The first term is the acceleration of a particle with respect to relative frame O. The last term is the centripetal acceleration of a coinciding point of the earth. The middle term is called the Coriolis acceleration.

Coriolis Acceleration. A clearer idea of Coriolis acceleration, named after its discoverer, can be determined by definition of the cross product. The two vectors $\bar{w}$ and $\bar{r}$ have been described previously. The cross product of these two vectors yields a third vector which is perpendicular to the plane of $\bar{w}$ and $\bar{v}$ and directed according to the right hand screw rule. Note Figure 6.

It should be added that Coriolis acceleration acts normal to the velocity $\bar{v}$, thus does not contribute to the tangential component of the motion. From the discussion of centripetal acceleration, it is apparent that if Newton's second law is to hold on a rotating earth, a fictitious force $-2\bar{w} \times \bar{v}$ must be added. This inertial force will be called the Coriolis force.

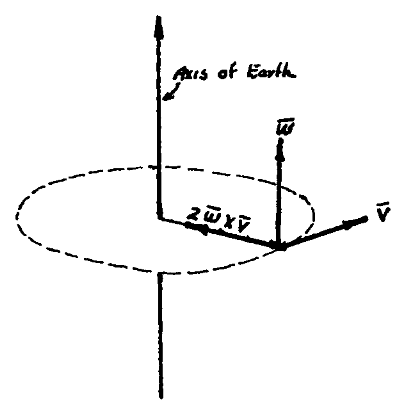

Figure 6.

Relative Motion, Earth. Eliminating the absolute velocity between the two equations

$$\frac{d\bar{v}}{dt} = \bar{b} + \bar{g}$$

and

$$\frac{d\bar{v}}{dt} = \frac{d\bar{v}_r}{dt} + 2\bar{w} \times \bar{v}_r + \frac{d\bar{v}_e}{dt},$$

$$\bar{b} + \bar{g} = \frac{d\bar{v}_r}{dt} + 2\bar{w} \times \bar{v}_r + \frac{d\bar{v}_e}{dt}.$$

Solving for $\frac{d\bar{v}_r}{dt}$,

$$\frac{d\overline{v}_r}{dt} = \overline{b} + \overline{g} - 2\overline{w} \times \overline{v}_r - \frac{d\overline{v}_e}{dt}.$$

This equation states that the acceleration relative to the earth is equal to the sum of all the forces, including the inertial forces arising from the absolute motion of the relative frame. The term $-\frac{d\overline{v}_e}{dt}$ which is equal and opposite to the centripetal acceleration is called the centripetal reaction. It has been shown previously that $\overline{g}_r = \overline{g} - \frac{d\overline{v}_e}{dt}$, thus the above equation becomes

$$\frac{d\overline{v}_r}{dt} = \overline{b} - 2\overline{w} \times \overline{v}_r + \overline{g}_r.$$

As stated before $-2\overline{w} \times \overline{v}_r$ is called the Coriolis force, the equal and opposite force of reaction. Letting

$$\overline{c} = -2\overline{w} \times \overline{v}_r$$

the final form of the equation is

$$\frac{d\overline{v}_r}{dt} = \overline{b} + \overline{c} + \overline{g}_r.$$

This is the equation of relative motion, because it gives Newton's second law of motion with respect to observations from a relative frame.

CHAPTER II

HORIZONTAL FLOW

Since a rigorous analysis of the study of motion of the air poses extremely complicated mathematical equations, solutions can be attained only by certain simplifying assumptions. Thus, an assumption that motion of the air is strictly horizontal in nature will be considered in this chapter. It might be added, that observations indicate that most large scale movements of the atmosphere are horizontal. Friction and some consideration of vertical motion will be considered later. The equations of motion developed in the last chapter, are valid for arbitrary motion of air particles on the earth, and thus, they are certainly valid for horizontal motion.

In introducing horizontal motion, the use of the standard co-ordinate system $\overline{i}$, $\overline{j}$, $\overline{k}$ will be supplemented by three fundamental vectors $\overline{t}$, $\overline{n}$, $\overline{k}$. The latter vectors will be oriented such that $\overline{t}$ is tangent to the flow, $\overline{n}$ is normal to the flow and $\overline{k}$ is perpendicular to the plane of $\overline{t}$ and $\overline{n}$; these are all unit vectors. The right-hand screw system prevails in both systems. Now any vector projected into the system $\overline{t}$, $\overline{n}$, $\overline{k}$, is equal to the sum of its projections along each perpendicular axis of the system. A vector $\overline{a}$ then would be

$$\bar{a} = a_s\bar{t} + a_n\bar{n} + a_z\bar{k}$$

in standard system

$$\bar{a} = a_x\bar{i} + a_y\bar{j} + a_z\bar{k}.$$

I. OUTLINE OF MATHEMATICS INVOLVED IN HORIZONTAL FLOW

Centripetal Acceleration. Consider a particle moving on a circle of radius r with a constant angular speed $w = \frac{d\theta}{dt}$ or instantaneous speed w. Note Figure 7.

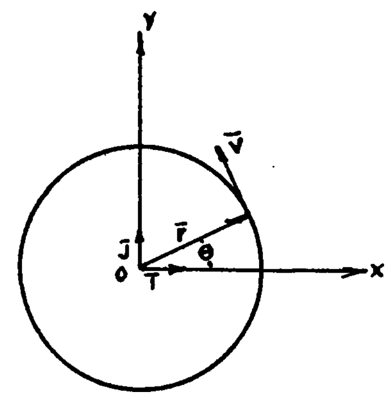

Figure 7

Now $\bar{r} = r\cos\theta\,\bar{i} + r\sin\theta\,\bar{j}$, and the time rate of change of $\bar{r}$ is the velocity

$$\bar{v} = \frac{d\bar{r}}{dt} = (-r\sin\theta\,\bar{i} - r\cos\theta\,\bar{j})\,\frac{d\theta}{dt}.$$

Obviously, the acceleration

$$\bar{a} = \frac{d\bar{v}}{dt} = \frac{d^2\bar{r}}{dt} = (-r\cos\theta\,\bar{i} - r\sin\theta\,\bar{j})\left(\frac{d\theta}{dt}\right)^2,$$

since $$\frac{d}{dt}\left(\frac{dr}{dt}\right) = 0.$$

Thus, the acceleration reveals itself to be

$$\bar{a} = -w^2\bar{r}$$

which is a center seeking acceleration. If a vector $\bar{P} = -\bar{r}$, and $\bar{n}$ is a unit vector directed towards center O from point P, the equation

$$\bar{a} = w^2\bar{n}\,P\ .$$

Arbitrary Motion. The point P is any point on the space curve

$$x = x(s),$$
$$y = y(s),$$
$$z = z(s),$$

where s is an arc length measured from some fixed point. Note Figure 8.

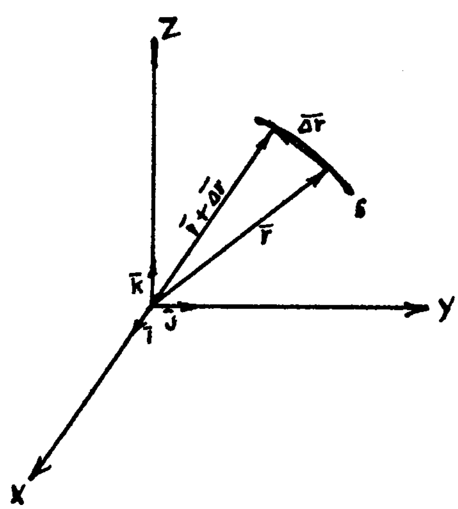

Figure 8

Then a position vector $\bar{r}$ from reference frame O is

(1) $\bar{r} = x(s)\bar{i} + y(s)\bar{j} + z(s)\bar{k}.$

The change in $\bar{r}$ along s

$$\frac{d\bar{r}}{ds} = \frac{dx}{ds}\,\bar{i} + \frac{dy}{ds}\,\bar{j} + \frac{dz}{ds}\,\bar{k}.$$

Now $\quad \frac{d\bar{r}}{ds} \cdot \frac{d\bar{r}}{ds} = \left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2 = \frac{dx^2 \quad dy^2 \quad dz^2}{ds^2} \equiv 1.$

This, of course, is the magnitude of the vector $\frac{d\bar{r}}{ds}$, and by definition it is a unit vector. Furthermore, it is tangent to the space curve under discussion.

Now consider the relation

(2) $\bar{v} = v\,\frac{d\bar{r}}{ds}$, where $\bar{v}$ = velocity.

If differentiated with respect to time

(3) $\frac{d\bar{v}}{dt} = \frac{dv}{dt}\,\frac{d\bar{r}}{ds} + v\,\frac{d}{dt}\left(\frac{d\bar{r}}{ds}\right).$

Now, $\frac{d}{dt}\left(\frac{d\bar{r}}{ds}\right)$, part of the last term on the right, is perpendicular to the unit vector $\frac{d\bar{r}}{ds}$ which is tangent to the curve. In other words, the acceleration has been divided into compements tangential and normal to the path of particle under consideration.

Investigating $\frac{d}{dt}\left(\frac{d\bar{r}}{ds}\right)$

(4) $\frac{d}{dt}\left(\frac{d\bar{r}}{ds}\right) = \frac{d\left(\frac{d\bar{r}}{ds}\right)}{ds}\,\frac{ds}{dt} = \frac{d^2\bar{r}}{ds^2}\,\frac{ds}{dt}.$

Since $\frac{d\bar{r}}{ds}$ is a unit vector it can be expressed as

$$\frac{d\bar{r}}{ds} = \cos\theta\,\bar{i} + \sin\theta\,\bar{j}.$$

Differentiating with respect to s

$$\frac{d}{ds}\left(\cos\theta\,\bar{i} + \sin\theta\,\bar{j}\right) = \left(-\sin\theta\,\bar{i} + \cos\theta\,\bar{j}\right)\frac{d\theta}{ds}$$

and $$\frac{d^2\bar{r}}{ds} \cdot \frac{d^2\bar{r}}{ds} = (\sin^2\theta + \cos^2\theta)\left(\frac{d\theta}{ds}\right)^2.$$

$$\left|\frac{d^2\bar{r}}{ds}\right| = \frac{d\theta}{ds} = \frac{d\theta}{rd\theta} = \frac{1}{r}.$$

This gives the magnitude, thus by denoting a unit vector $\bar{n}$ as being normal to the tangent,

$$\frac{d}{dt}\left(\frac{d\bar{r}}{ds}\right) = \bar{n} \cdot \frac{1}{r} \cdot \frac{ds}{dt}, \qquad \frac{ds}{dt} = v.$$

Substituting into equation (3)

$$(5)\quad \frac{d\bar{v}}{dt} = \frac{dv}{dt}\left(\frac{d\bar{r}}{ds}\right) + \frac{v^2}{r}\bar{n}.$$

When $w = \frac{d\theta}{dt}$,

$$v = \frac{ds}{dt} = \frac{ds}{d\theta}\frac{d\theta}{dt} = wr,$$

and $$\frac{d\bar{v}}{dt} = \frac{dv}{dt}\left(\frac{d\bar{r}}{ds}\right) + w^2P\bar{n}.$$

In previous work $\bar{r} = r\,\bar{R}$. $\bar{R}$ is a unit vector directed from origin O towards a point. The position vector $\bar{r}$ of magnitude r then becomes the vector radius of curvature. Thus, $\bar{r} = -r\bar{n}$ where $\bar{n}$ is the vector in the direction opposite to $\bar{R}$.

Equation (5) which gives the vector curvature $\bar{P}$,

$|\overline{P}| = \frac{1}{r}$ and is directed toward the center of curvature, is composed of two components

$$\overline{P} = P_n\overline{n} + P_z\overline{k}.$$

$\overline{k}$ is perpendicular to the plane of $\overline{n}$ and $\frac{d\overline{r}}{ds}$, and thus the three unit vectors form an orthogonal system. If $\overline{t}$ and $\overline{n}$ are tangent to the earth's surface, a local system is realized. This system of $\frac{d\overline{r}}{ds} = \overline{t}$, $\overline{n}$, $\overline{k}$ will simplify some problems later. The components for horizontal flow are:

$$\frac{dv_s}{dt} = \frac{dv}{dt},$$

$$\frac{dv_n}{dt} = v^2P_n,$$

$$\frac{dv_k}{dt} = v^2P_z.$$

<u>Sense of Curvature</u>. At this point, the idea of what direction a curve is taking would clarify later discussions of circular motion. When a particle appears to be moving in a clockwise direction, viewed from the zenith, its cyclic movement will be negative, and anticlockwise will be positive. Movement that is along a great circle may be defined as positive or negative. Also, it is obvious that the cyclic sense of rotation of a particle fixed to the earth's surface is positive in the northern hemisphere and negative in the southern hemisphere.

<u>Angular Radius of Curvature.</u>[1]

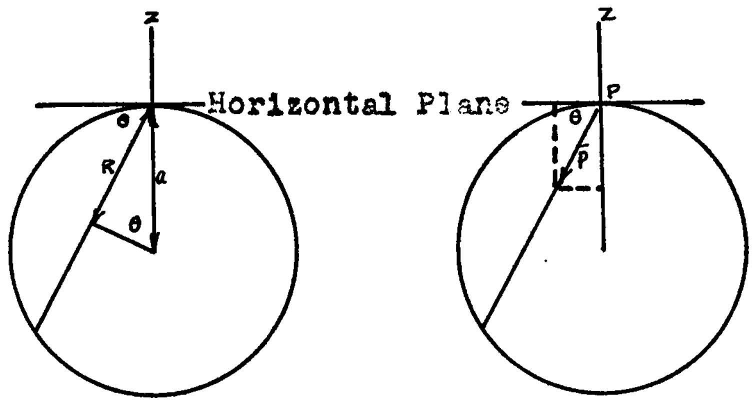

Figure 8a   Figure 8b

The angle θ will be called the angular radius of curvature. As shown in Figure 8a, it is the angle subtended at the center of the sphere (earth in later problems) by the radius of curvature r.

The radius of curvature subtending θ is $r = a \sin\theta$ and its reciprocal $\frac{1}{a \sin\theta}$ is the curvature P.

The vector curvature appears from a point on z to point to the left of flow in a positive circular sense. Since $\overline{n}$ points to the left of flow, then $\overline{P} = P\overline{n}$. Note Figure 8b.

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1 Jorgen Holmboe, <u>Dynamic Meteorology</u> (New York: John Wiley and Sons, 1945), pp. 177-178.

$$P_n = P \cos\Theta \quad (2).$$

$$P_z = -P \sin\Theta \quad (3).$$

Using

$$P = \frac{1}{a \sin\Theta},$$

$$P_n = P \cos\Theta = \frac{\cos\Theta}{\sin\Theta\, a} = \frac{1}{a \tan}.$$

$$P_z = -P \sin\Theta = -\frac{1}{a}\frac{\sin\Theta}{\sin\Theta} = -\frac{1}{a}$$

a = radius of sphere.

<u>Horizontal</u> <u>and</u> <u>Vertical</u> <u>Curvature</u>. The spherical path of a particle can be projected upon a horizontal plane, and the resulting curve is called the horizontal path. "The curvature of the horizontal projection of the path is equal to the horizontal component of the vector curvature."[2] Thus, if P_h represents the curvature of the horizontal projection,

$$P_h = P \cos\Theta = P_n.$$

From the previous section, the vertical component of the vector curvature

$$P_z = -\frac{1}{a}$$

or

$$R_z = -a$$

where R_z equals radius of curvature. a is the radius of a great circle and it can be stated that "the spherical path projects into the vertical plane as an arc of a great circle

2 <u>Ibid</u>., p. 179.

no matter how strongly curved the spherical path may be".[3] The vertical plane is normal to the horizontal plane and passes through the unit tangent. The mathematical proof of both statements concerning the curvature in the horizontal and vertical planes can be found in any good dynamic meteorology book.

It can be stated here that the three radii of curvature, R, R_h, and R_z emanate from points on the axis of rotation to a moving point P. In other words, the centers are collinear. Then vectors $\overline{P}_h$, $\overline{P}$, and $\overline{P}_z$ drawn from point P to respective centers of curvature can represent this condition quite clearly. Note Figure 9.

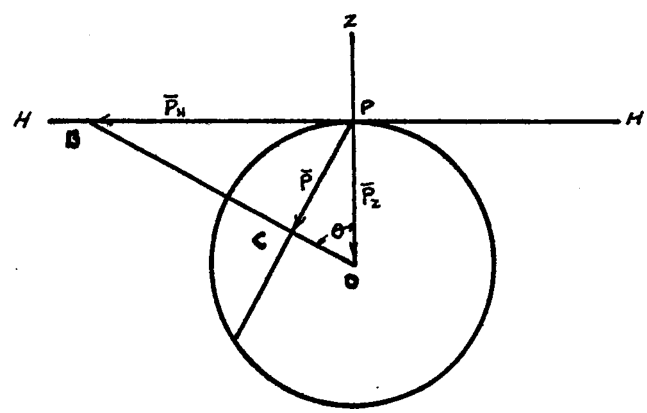

Figure 9

Further consideration of the relationship of these three vectors yields

$$\overline{P} = x\,\overline{P}_z + y\,\overline{P}_h$$

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3 <u>Ibid</u>., p. 182.

where $x = \sin^2\theta$

and $y = \cos^2\theta$.

The proof follows. Let C divide B O in the ratio x:y where $x + y = 1$. Now

$$\overline{P} = \overline{P}_h + \overline{BC},$$

$$\overline{P} = \overline{P}_h + x(\overline{P}_z - \overline{P}_h),$$

$$\overline{P} = x\overline{P}_z + (1-x)\overline{P}_h, \quad y = 1-x,$$

$$\overline{P} = x\overline{P}_z + y\overline{P}_h.$$

From Figure 9

$$\overline{P}_h \cdot \overline{P} = \overline{P}_h \cdot \overline{P}_z x + \overline{P}_h \cdot \overline{P}_h y.$$

Now, $\overline{P}_z \cdot \overline{P}_h = 0$

since the horizontal plane is tangent at P, therefore $\overline{P}_z$ is perpendicular to $\overline{P}_h$, and thus

$$P_h\, P \cos\,(P_h\, P) = P_h^2 \cos\,(P_h\, P_h) y,$$

$$y = \frac{P \cos\,(P_h\, P)}{P_h},$$

$$y = \frac{P \cos\theta}{P_h} = \frac{a \sin\theta \cos\theta}{a\,\frac{\sin\theta}{\cos\theta}} = \cos^2\theta.$$

$$P_h = a \tan\theta.$$

$$\overline{P}_z \cdot \overline{P} = (\overline{P}_z \cdot \overline{P}_z)x + (\overline{P}_z \cdot \overline{P}_h)y,$$

$$P_z\, P \cos\,(P_z\, P) = P_z^{\,2}\, x,$$

$$\frac{P \cos\,(P_z\, P)}{P_z} = x,$$

since $P = a \sin\theta,$

$$\frac{a \sin\theta \cos(90-\theta)}{a} = x,$$

thus $x = \sin^2\theta.$

From the $x + y = 1$

requirement $\sin^2\theta + \cos^2\theta = 1$

and then $\bar{F} = \sin^2\theta\, \bar{F}_z + \cos^2\theta\, \bar{F}_h.$

II. APPLICATIONS OF MATHEMATICS TO EARTH AND ATMOSPHERE

<u>The Angular Velocity of the Earth</u>. Since angular velocity is a vector in a meridional plane, its component $w_x = 0$, in the standard $\bar{I}$, $\bar{J}$, $\bar{K}$ system. Note Figure 10.

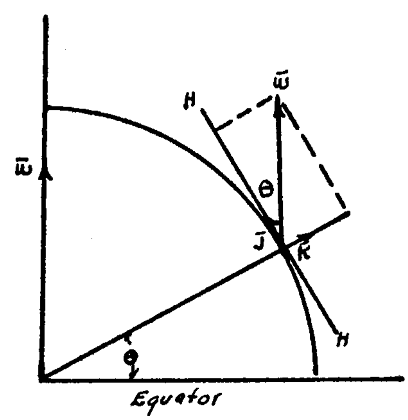

Figure 10

The components are then

$$\bar{w}_y = |\bar{w}| \cos\theta,$$

where y is directed towards the local north and θ is the degree of latitude, and

$$\bar{w}_z = |\bar{w}| \sin\theta$$

along the local zenith. It should be noted that w is positive in the northern hemisphere and negative in the southern hemisphere.

<u>Coriolis</u> <u>Force</u>. The vector equation of this force is

$$\bar{c} = -2\bar{w} \times \bar{v}_r,$$

and the equation can be expressed in the determinant form and expanded by the ordinary method of determinants.

$$\bar{c} = -2 \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 0 & w_y & w_z \\ v_x & v_y & 0 \end{vmatrix} = 2w_z\, v_y\, \bar{i} - 2w_z\, v_x\, \bar{j} + 2w_y\, v_x\, \bar{k}.$$

Its components in the standard system are

$$c_x = 2w_z\, v_y,$$

$$c_y = -2w_z\, v_x,$$

$$c_z = 2w_y\, v_x.$$

For the $\bar{t}$, $\bar{n}$, $\bar{k}$ system

$$\bar{c} = -2 \begin{vmatrix} \bar{t} & \bar{n} & \bar{k} \\ w_s & w_n & w_z \\ v & 0 & 0 \end{vmatrix} = -2w_z v\bar{n} + 2w_n v\bar{k},$$

and the components along each axis

$$c_s = 0,$$

$$c_n = -2w_z v = -2w \sin\theta\; v,$$

$$c_z = 2w_n v.$$

Obviously the only horizontal component of the Coriolis Force is normal to the flow, thus the horizontal vector component

$$\bar{c}_h = c_n \bar{n} = -2w_z v\bar{n} = -2\bar{w}_z \times \bar{v},$$

since $\bar{w}_z = w\bar{k}$ and $\bar{n} = \bar{k} \times \bar{t}$.

The Pressure Force and the Force of Gravity. Since the force of gravity has no horizontal components

$$g_z = \bar{g} \cdot \bar{k} = -g.$$

The horizontal force of pressure is

$$\bar{b} = -\alpha \nabla p_h = -\alpha \frac{\partial p}{\partial x} - \alpha \frac{\partial p}{\partial y},$$

and this force acts normal to the horizontal isobars (lines of constant pressure) towards lower pressure.

Total Components of Relative Motion, Earth. Each vector in the equation

$$\frac{d\bar{v}_r}{dt} = \bar{b} - 2\bar{w} \times \bar{v}_r + \bar{g}_r$$

has been examined from the standpoint of their components. The component equations for the standard $\bar{i}$, $\bar{j}$, $\bar{k}$ system are

$$\frac{dv_x}{dt} = -\alpha\frac{\partial p}{\partial x} + 2w_z v_y,$$

$$\frac{dv_y}{dt} = -\alpha\frac{\partial p}{\partial y} - 2w_z v_x,$$

$$\frac{dv_z}{dt} = -\alpha\frac{\partial p}{\partial z} + 2w_y v_x - g.$$

The component equations for the $\bar{t}$, $\bar{n}$, $\bar{k}$, system are:

$$\frac{dv}{dt} = -\alpha\frac{\partial p}{\partial s},$$

$$P_h\, v^2 = -\alpha\frac{\partial p}{\partial n} - 2w_z\, v,$$

$$\frac{-v^2}{a} = -\alpha\frac{\partial p}{\partial z} + 2w_y\, v_x - g.$$

These equations have the advantage of being dependent on the direction of motion.

Vertical Equation.[4]

$$\frac{dv_z}{dt} = -\alpha\frac{\partial p}{\partial z} + 2w_y\, v_x - g.$$

This equation states that the vertical acceleration is equal to the sum of the vertical components of the forces acting.

The term $2w_y\, v_x$ clearly indicates that the vertical Coriolis force acts upward for motion towards the east and downward for motion towards the west. Also since $w_y = w\cos\theta$, the absolute value of the Coriolis force in the vertical is greatest at the equator and zero at the pole.

The $\frac{dv_z}{dt}$ is the centripetal term and is equal to $\frac{v^2}{a}$ since the vertical path is an arc of a great circle. The equal and opposite centrifugal reaction opposes the force of gravity.

The last term on the right $-g$ is the force of gravity

4 Ibid., pp. 187-188.

measured at a fixed point, but it is of interest to note that the vertical accelerations created by horizontal flow actually affects the so-called "pull" of gravity. Examination of an observer's horizontal movement with the flow yields the fact that measure of gravity would be

$$g' = g - \frac{v^2}{a} - 2w_y\, v_x.$$

If $g' > g$, a moving particle is "heavier" and if $g' < g$ it is "lighter" is a conclusion reached by consideration of the above equation. Experiments and computations have shown that correction terms are very small thus

$$g' \approx g.$$

A consideration of strictly vertical motion from our equation

$$\frac{d\bar{v}_r}{dt} = \bar{b} + \bar{c} + \bar{g}$$

is of some interest. The first term on the right would be

$$\bar{b} = -\alpha\frac{\partial p}{\partial z}$$

$$\bar{g} = \bar{g} \cdot \bar{k} = -g$$

and the Coriolis force

$$\bar{c}_h = -2\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 0 & w_y & w_z \\ 0 & 0 & v_z \end{vmatrix} = -2w_y\, v_z\, \bar{i}.$$

The result is (relative acceleration components)

$$\frac{dv_z}{dt} = -\alpha\frac{\partial p}{\partial z} - g,$$

$$\frac{dv_y}{dt} = 0,$$

$$\frac{dv_x}{dt} = -2w_y\ v_z = -2w\ \sin\theta\ v_z.$$

CHAPTER III

FRICTION

If the earth were perfectly smooth, the equation of motion

$$\frac{d\bar{v}}{dt} = \bar{b} + \bar{c} + \bar{g}$$

could be satisfactorily used in the surface layers. However, observations demonstrate clearly that friction is present. Friction created by the rough surface of the earth and an internal friction within the air mass itself are principally effective in slowing air movements. Then, the scope of this chapter is to present general theory on friction; the type prevalent in air motion, and the effect it has on air movement. It might be added that laws concerning friction experienced by air masses are still in the process of study.

I. FRICTIONAL THEORY

Analysis of a particle resting on a rough horizontal plane indicates that when a force is imparted to the particle, a greater force is necessary to accelerate the particle in a given direction than under the same conditions for a smooth plane. It is reasonable to assume that the irregularities between two surfaces which touch each other produce accelerations opposite in direction to the movement. Then it can be stated that the force of friction is proportional to the force exerted by the particle against the rough plane.

This thrust against the plane is denoted by vector $\bar{n}$ and the proportionality factor is called the coefficient of friction μ.[1] Note Figure 11. It is easily seen that the magnitude of $\bar{n}$ is $n = mg$, and then the magnitude of $\bar{F}_f$

$$F_f = \mu n = \mu mg.$$

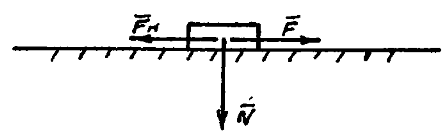

Figure 11

If a rough inclined plane is considered, it cannot be assumed that the total reaction of friction acts normal to the plane. In Figure 12, $\bar{n}$, the normal thrust, is the component of the weight of the particle normal to a flat plane. From Newton's third law, it is evident that an equal and opposite reaction takes place in the case of $\bar{n}$.

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1 Robert Bruce Lindsey, General Physics (New York: John Wiley and Sons, Inc., 1940), p. 69.

Thus, since $\overline{n}$ is the force which the particle pushes down perpendicular to the inclined plane, an opposite reaction is $\overline{R}$, the force which the plane pushes up on the particle. The total reaction of the surface on the particle is the resultant of two forces $\overline{R}$ and $\overline{F}_r$.

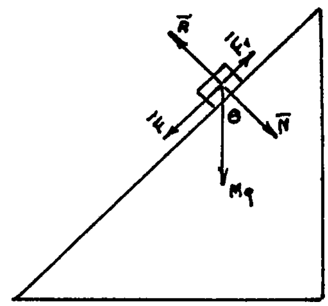

Figure 12

In Figure 12

$$F = mg \sin\theta,$$

$$F_f = \mu n = \mu mg \cos\theta,$$

and using these equations, the motion of the particle is

$$\frac{d\overline{v}}{dt} = \overline{F} + \overline{F}_f,$$

$$mg \sin\theta - \mu mg \cos\theta = ma,$$

$$a = g(\sin\theta - \mu \cos\theta).$$[2] $m = 1.$

II. RELATIONSHIP BETWEEN FRICTION AND CORIOLIS FORCE

It is now apparent, that a frictional force per unit mass $\overline{F}_f$ can be added to the equation of motion when motion takes place near the earth's surface. Thus the equation of

2 Ibid., p. 68.

motion is

$$\frac{d\overline{v}_r}{dt} = \overline{b} + \overline{c} + \overline{g} + \overline{F}_f.$$

According to various textbooks on Dynamic Meteorology, frictional force depends upon the motion, physical state of the atmosphere, and the underlying surface of the earth. The height to which it extends is roughly the half kilometer level where air movement is in good agreement with the general equation of motion.

An idea of the nature of the friction term $\overline{F}_f$ may be ascertained by assuming that constant rectilinear motion exists in a horizontal plane at the earth's surface. This flow is commonly called geostrophic or great circle flow. The equation of motion then takes the form

$$(1)\quad 0 = \overline{b}_h + \overline{c}_h + \overline{F}_h.$$

If the friction term did not exist, then the wind would blow along the isobar balanced by the pressure gradient and the Coriolis force, but the wind deviates from this type of flow and the deviation will be called $\overline{v}'$. Thus

$$(2)\quad \overline{v} = \overline{v}_g + \overline{v}'. \quad \text{Note Figure 13.}$$

Since

$$\overline{c}_h = -2\overline{w}_z \times \overline{v}$$

$$\overline{c}_h = -2\overline{w}_z \times (\overline{v}_g + \overline{v}')$$

$$= -2\overline{w}_z \times \overline{v}_g - 2\overline{w}_z \times \overline{v}'.$$

From the definition of geostrophic flow,

$$\overline{b} = 2\overline{w}_z \times \overline{v},$$

and substituting values in equation (1)

$$0 = 2\bar{w}_z \times \bar{v}_g - 2\bar{w}_z \times \bar{v}_g - 2\bar{w}_z \times \bar{v}' + \bar{F}_h,$$

$$2\bar{w}_z \times \bar{v}' = \bar{F}_h.$$

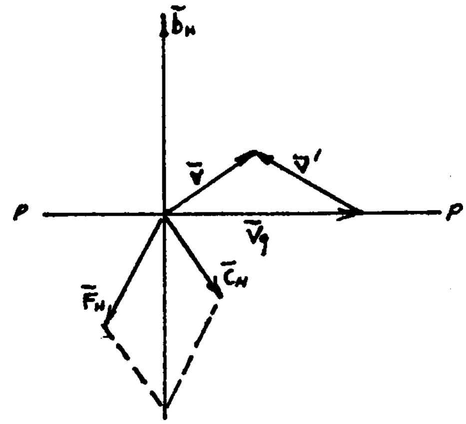

Figure 13

From the last equation, it is apparent that the Coriolis force arising from the deviation of the geostrophic wind must balance the frictional force.[3]

III. SURFACE FRICTION

An approach to this problem was presented by Guldberg and Mohn as early as 1875. They worked on assumptions similar to those presented in the section on Frictional Theory. They assumed that the frictional force is directed opposite to the velocity and its magnitude is proportional to the speed. Note Figure 14. Constant rectilinear motion is assumed

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3 Jorgen Holmboe, <u>Dynamic Meteorology</u> (New York: John Wiley and Sons, Inc., 1945), p. 234.

Once again $\quad (1)\ 0 = \bar{b}_h + \bar{c}_h + \bar{F}_f.$

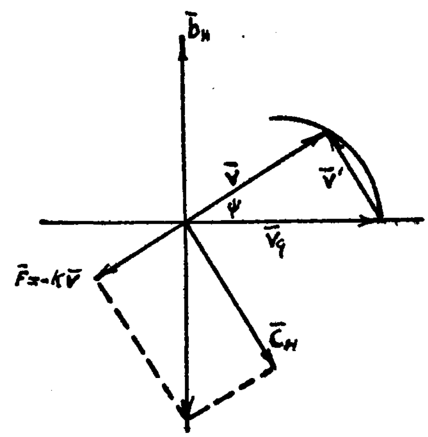

Figure 14

The force triangle is similar to the velocity triangle, as can be determined from the discussion in thelast section when this relation held with the factor of proportionality $2\bar{w}_z$. Thus the velocity triangle is a right triangle, and the horizontal component of the terms of equation (1) are

$$0 = -\alpha\frac{\partial p}{\partial n} - 2w_z\, v_g + kv_g \cos\psi - kv,$$

since $\quad c_h = -2w_z\, v_g + kv,$

$$kv = v_g \cos\psi. \quad \text{Note Figure 14.}$$

Finally,

$$\alpha\frac{\partial p}{\partial n} = -v_g\,(2w_z - k\cos\psi) - kv.$$

The proportionality factor k was assumed to decrease with height, and the angle ψ was largest at the ground and decreased with height. Calculations of values indicated that these assumptions were not accurate. Later, Sanstrom defined a residual force to be used with Guldberg and Mohns assumptions, and from this, Hesselburg and Sverdrup developed a method for determining the force of friction by adding in Sanstroms residual force.[4] This technique fit the observed facts but was empirical in nature and did not explain the physical characteristics of frictional resistance. Thus it became necessary to take into account the internal friction created by molecular activity and interaction of air masses of different velocity and direction. With this in mind, study was turned to fluid motion.

IV INTERNAL FRICTION

Considerable literature has been written concerning internal friction, and this writer does not propose to go into all the theoretical approaches, but rather to present some basic ideas. The essential parts of internal friction can be summed up into two parts: molecular friction and turbulence; eddies embodied in the general flow.

Viscous Stress and Viscosity. Consider two parallel plates which incase fluid at a distance z from each other. (Note Figure 15). Let the upper plate be moved at a horizontal velocity $\overline{v}$, while the lower plate remains stationary.

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4 "Physics of the Earth", (Bulletin of National Research Council, Feb., 1931), Published by the National Research Council of the National Academy of Sciences, Washington, D. C., pp. 182-183.

Experiments have proved that when steady conditions exist, the velocity decreases linearly from the moving plate to the resting plate. The shear $\frac{\partial \bar{v}}{\partial z}$ is constant. The consensus is that the motion develops as a result of internal friction which arises from the disturbance of fluid molecules.

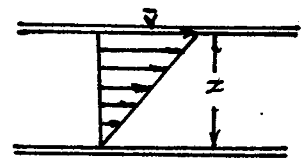

Figure 15

In order to keep the lower plate at rest, it is necessary to apply a force equal and opposite to the force applied to produce motion in the top plate. It might be added that no lost motion is assumed between the plates and the fluid in direct contact with them. It is reasonably assumed, from experiments, that a force $-\bar{t}$, which is proportional to velocity $\bar{v}$ of the upper plate and inversely proportional to the distance z, must be applied on a unit area of the resting plate.[5] Thus the viscous or shearing stress

$$-\bar{t} \sim -\frac{\partial \bar{v}}{\partial z} \text{ at the bottom plate}$$

and $\bar{t} \sim \frac{\partial \bar{v}}{\partial z}$ at the top plate.

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5 Bernhard Haurwitz, Dynamic Meteorology (New York: McGraw-Hill Book Company, Inc., 1941), p. 188.

Similarly, each horizontal infinitesimal fluid layer between the two plates can be shown to have the same shear. To arrive at an equation, a proportionality factor must be introduced. The symbol μ is used and

$$\overline{t} = \mu \frac{\partial \overline{v}}{\partial z}.$$

This equation is "Newton's formula" for the stress. μ is called the molecular viscosity which is variable from a physical and temperature standpoint, and it is a pure number.[6] The viscosity values for air in the meter, ton, second unit are: 1.7×10^{-8} at 0° centigrade and 2.2×10^{-8} at 100° centigrade.[7]

<u>Molecular Internal Friction Term</u>. Observations of wind direction and velocity with altitude have clearly demonstrated that a shear exist, thus in the case of horizontal motion with uniform velocity at each level the frictional force can be related to the shearing stress. Picture an infinitesimal cube of unit cross section and height dz. The stress exerted on the bottom face is $\overline{t}$ and the drag on the upper face is $\overline{t} + \left(\frac{\partial \overline{t}}{\partial z}\right) dz$. The difference between the two gives the force exerted on the element of

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6 Holmboe, <u>op</u>. <u>cit</u>., p. 236.

7 <u>Ibid</u>., p. 238.

volume $dv = dz$. Then $\left(\frac{\partial \overline{t}}{\partial z}\right)$ is the frictional force per unit volume. For unit mass,

$$\overline{f}_h = \alpha \frac{\partial \overline{t}}{\partial z} \text{ since } \alpha = \frac{1}{\rho}.$$

This equation is derived under the considerations that the horizontal variations of velocity components or horizontal shears are neglible.[8]

Application of this result to our horizontal equations of motion gives

$$\frac{dv_x}{dt} = -\alpha \frac{\partial p}{\partial x} + 2w_z\, v_y + \alpha \frac{\partial}{\partial z}\left(\mu \frac{\partial v_x}{\partial z}\right),$$

$$\frac{dv_y}{dt} = -\alpha \frac{\partial p}{\partial x} - 2w_z\, v_x + \alpha \frac{\partial}{\partial z}\left(\mu \frac{\partial v_y}{\partial z}\right).$$

It can be added that calculations of this source of friction yields results that are too small for what is actually observed.

Molecular and Eddy Viscosity. The equation that Newton developed for the stress

$$\overline{t} = \mu \frac{\partial \overline{v}}{\partial z}$$

where μ, the viscosity, was presented in a new form by Maxwell from a theoretical approach to molecular action of gas under similar considerations outlined in the section

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8 Ibid., p. 239.

on viscous stress. He found

$$\mu = \frac{1}{3}\rho cl$$

where ρ is the density of gas, l is the mean free path of a molecule, c is the mean heat speed due to internal heat energy.[9]

An analegous formula was developed when it was apparent that the above type of friction would create forces only a few meters in depth. The study of fluid motion subjected to mild disruption showed that small eddies appeared downstream for short distances beyond points of disturbance. From this transfer of momentum from layer to layer which is called eddy stress, a formula for stress was presented with the use of a new viscosity term μ'

$$\mu' = \rho wl.$$

To get the expression for eddy stress, it is assumed that the parcels of fluid which are affected by the eddies move an average distance l. It should be noted that this hypothesis has its limitations in that it assumes mixing to be a discontinuous process.[10] Also, as the parcels are displaced from z to z + dz, there arise components of the eddy velocity which are perpendicular to the constant flow $\bar{v}$. This is called w. Thus, the eddy stress can be written

$$\bar{\tau} = \rho wl \frac{\partial \bar{v}}{\partial z}.$$

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9 Ibid., p. 237.

10 Bernhard Haurwitz, Dynamic Meteorology (New York: McGraw-Hill Book Company, Inc., 1941), p. 195.

After due considerations, the above equation can be called semi-empirical.

V. WIND VARIATION FROM SURFACE TO GRADIENT LEVEL

W. F. Ekman solved the problem of the turning of ocean currents in the surface layers of the ocean in 1902.[11] He developed the "Ekman spiral" or logarithmic spiral of the currents, and analogous to this result, meteorologists developed the solution of the corresponding problem of wind deviation.

Essentially, the problem was attacked by assuming that horizontal pressure force has the same direction and magnitude for all levels. Thus the geostrophic wind is constant in magnitude and direction. As stated before, the level at which the wind is in fair agreement with our equations of motion is approximately at the five hundred meter level, thus the assumptions are not too great. The viscosity and specific volume are constant with height.

Using Newton's formula

$$(1) \quad \bar{t} = \mu \frac{\partial \bar{v}}{\partial z}$$

and the frictional force per unit mass

$$(2) \quad \bar{f}_r = \alpha \frac{\partial \bar{t}}{\partial z} = 2\bar{w}_z \times \bar{v}'.$$

Recalling $\bar{v} = \bar{v}_g + \bar{v}'$;

11 Ibid., p. 207.

$$\frac{\partial \bar{v}}{\partial z} = \frac{\partial \bar{v}'}{\partial z}, \text{ (since } \bar{v}_g \text{ is constant)}$$

$$(3) \quad \frac{\partial^2 \bar{v}}{\partial z^2} = \frac{\partial^2 \bar{v}'}{\partial z^2}.$$

Thus equation (2) can be written

$$(4) \quad \mu\alpha \frac{\partial^2 \bar{v}'}{\partial z^2} = 2\bar{w}_z \times \bar{v}'!$$

The right side of this equation is

$$2 \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 0 & w_y & w_z \\ v'_x & v'_y & 0 \end{vmatrix} = -v'_y\, w_z\, \bar{i} + w_z\, v'_x\, \bar{j} - w_y\, v_x\, \bar{k},$$

since $\bar{v}'$ is considered in the horizontal plane. The components in the x,y plane are then

$$(5) \quad \alpha\mu \frac{\partial^2 v'_x}{\partial z^2} = -2w_z\, v'_y,$$

$$(6) \quad \alpha\mu \frac{\partial^2 v'_y}{\partial z^2} = 2w_z\, v'_x.$$

(6) can be multiplied by i and added to (5) with the result that $\frac{\partial^2}{\partial z^2}(v'_x + iv'_y) = \frac{2w \sin\Theta}{\alpha\mu}(iv'_x - v'_y),$

$$-\frac{\partial^2}{\partial z^2}(v'_x + iv_y) = 2iB^2(-v'_x - iv'_y),$$

since $i^2 = -1$ and $B^2 = \frac{w \sin\Theta}{\alpha\mu}$,

and $$(7) \quad \frac{\partial^2}{\partial z^2}(v'_x + iv'_y) = 2iB^2(v'_x + iv'_y).$$

z is the only independent variable, thus (7) takes the form

$$\frac{d^2}{dz^2}(v'x + iv'_y) - 2iB^2(v'_x + iv'_y) = 0.$$

The final form of this linear differential equation is

$$\frac{d^2 v'}{dz^2} - (1 + i)^2 B^2 v' = 0$$

when $2i = (1 + i)^2$ and $v' = v'_x + iv'_y$.

The solution of this equation can be obtained by use of differential operator $D = \frac{d}{dz}$ procedure,

$$(D^2 - (1 + i)^2 B^2)\, v' = 0.$$

$$D = \pm (1 + i)\, B.$$

Thus $$v = c_1 e^{(1 + i)Bz} + c_2\, e^{-(1 + i)Bz}.$$

After due considerations of the assumptions and restrictions, the following definite solutions:

$$(1)\quad v' = v'_0 \quad e^{-Bz},$$

$$(2)\quad \theta = -Bz,$$

are acquired for the anemometer level.[12] This solution is known, as stated before, as the "Ekman spiral". Note Figure 16.

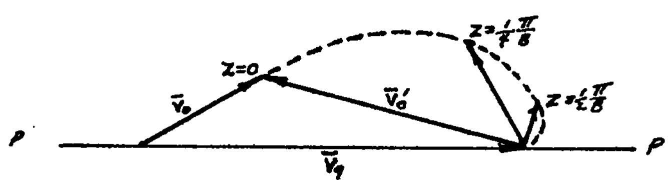

Figure 16

12 Ibid., p. 244.

CHAPTER IV

THE CURL OF THE VECTOR EQUATIONS OF MOTION

This chapter is concerned with a mathematical inspection of vector terms encountered in the development of the equations of motion from the standpoint of curl.

I. VELOCITY EQUATION

The velocity equation in vector form is

(1) $\bar{v} = \bar{v}_r + (\bar{w} \times \bar{r})$, from Chapter I.

<u>Curl</u>. The curl of the velocity equation is defined

(2) $\nabla \times \bar{v} = \nabla \times \bar{v}_r + \nabla \times (\bar{w} \times \bar{r})$.

Consideration of each term separately, yields for the first term on the left

$$\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\bar{i} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\bar{j} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\bar{k}.$$

Obviously the first term on the right is

$$\nabla \times \bar{v}_r = \left(\frac{\partial v_{rz}}{\partial y} - \frac{\partial v_{ry}}{\partial z}\right)\bar{i} + \left(\frac{\partial v_{rx}}{\partial z} - \frac{\partial v_{rz}}{\partial x}\right)\bar{j} + \left(\frac{\partial v_{ry}}{\partial x} - \frac{\partial v_{rx}}{\partial z}\right)\bar{k}.$$

The last term on the right of (2)

$$\nabla \times (\bar{w} \times \bar{r}) = (\bar{r} \cdot \nabla)\,\bar{w} - \bar{r}\,(\nabla \cdot \bar{w}) + \bar{w}\,(\nabla \cdot \bar{r}) - (\bar{w} \cdot \nabla)\bar{r}.$$

The first term on the right

$$(\bar{r} \cdot \nabla)\,\bar{w} = 0,$$

since $\bar{w}$ is constant in magnitude and direction.

The second term

$$-\bar{r}\,(\nabla\cdot\bar{w}) = -\bar{r}\left(\frac{\partial w_x}{\partial x}+\frac{\partial w_y}{\partial y}+\frac{\partial w_z}{\partial z}\right).$$

To simplify, let x be oriented along the easterly direction and consider only horizontal flow, then

$$= -\bar{r}\left(\frac{\partial w_y}{\partial y}\right)$$

$$= -\frac{\partial w_y}{\partial y}\,x\,\bar{i} - \frac{\partial w_y}{\partial y}\,\bar{j}.$$

The third term on the right

$$\bar{w}\,(\nabla\cdot\bar{r}) = \left(\frac{\partial x}{\partial x}+\frac{\partial y}{\partial y}+\frac{\partial z}{\partial z}\right)\bar{w} = 3\bar{w}$$

and $$-(\bar{w}\cdot\nabla)\bar{r} = w_x\frac{\partial\bar{r}}{\partial x}+w_y\frac{\partial\bar{r}}{\partial y}+w_z\frac{\partial\bar{r}}{\partial z}$$

$$= w_x\,\bar{i}+w_y\,\bar{j}+w_z\,\bar{k} = \bar{w},$$

thus $\bar{w}\,(\nabla\cdot\bar{r}) - (\bar{w}\cdot\nabla)\bar{r} = 2\bar{w}.$

Now adding the results after restricting x to the easterly direction and considering horizontal flow only,

$$\left(\frac{\partial v_y}{\partial x}-\frac{\partial v_x}{\partial y}\right)\bar{k} = \left(\frac{\partial v_{ry}}{\partial x}-\frac{\partial v_{rx}}{\partial z}\right)\bar{k}+2\bar{w}-\frac{\partial w_y}{\partial y}\,x\bar{i}-\frac{\partial w_y}{\partial y}\,y\bar{j}.$$

Dotting with $\bar{k}$

$$(3)\quad \left(\frac{\partial v_y}{\partial x}-\frac{\partial v_x}{\partial y}\right)=\left(\frac{\partial v_{ry}}{\partial x}-\frac{\partial v_{rx}}{\partial z}\right)+2w_z$$

since $2\bar{w}\cdot\bar{k} = 2w_z,\ \bar{k}\cdot\bar{i} = 0,\ \bar{k}\cdot\bar{j} = 0.$

This equation is a result gained from consideration of motion viewed from the absolute frame.

<u>Vorticity</u>. "The limit of the ratio of the circulation dc around an infinitesimal element to the area dA of that element is called vorticity."[1]

$$\zeta = \frac{dc}{dA} = \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right).$$

In the equation (3) of the last section the result would be

$$\zeta = \zeta_r + 2w \sin\Theta .$$

To acquire the vorticity term in the t, n, k system, consider two parallel curving streamlines at a distance dn apart. Let two normals extend from the outer streamline to the center of curvature of the outer streamline. Note Figure 17.

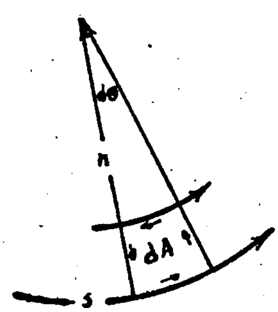

Figure 17

Circulation around this horizontal area in a counterclockwise direction yields

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1 Jorgen Holmboe, <u>Dynamic Meteorology</u> (New York: John Wiley and Sons, Inc., 1945), p. 320.

$$dc = vr_s de - \left(v + \frac{\partial v}{\partial n} dn\right)(r_s - dn)\, de,$$

$$(4)\quad dc = \left(\frac{v}{r_s} - \frac{\partial v}{\partial n} + \frac{1}{r_s}\frac{\partial v}{\partial n} dn\right) r_s d\, dn,$$

dividing by $dA = r_s dedn$

$$\zeta = \frac{dc}{dA} = \frac{v}{r_c} - \frac{\partial v}{\partial n},$$ [2]

after considering the third term in (4) which approaches zero.

<u>Vorticity and Horizontal Circular Motion</u>. In the chapter on horizontal motion the following component equations in the t, n, k system were presented:

$$(1)\quad \frac{dv}{dt} = -\alpha\frac{\partial p}{\partial s},$$

$$(2)\quad \frac{v^2}{r_c} = -\alpha\frac{\partial p}{\partial n} - 2w_z v,$$

$$(3)\quad -\frac{v^2}{a} = -\alpha\frac{\partial p}{\partial z} + 2w_y\, v_x - g.$$

In the following analysis, motion in a plane and tangential to the isobar will be considered. The streamlines and isobars are assumed to coincide. Thus equation (2) will be used in conjunction with the vorticity equation

$$\zeta = \frac{v}{r_c} - \frac{\partial v}{\partial n}.$$

Solving for r_c $$r_c = \frac{v}{\zeta + \frac{\partial v}{\partial n}}.$$

Substituting in (2) $$v\left(\zeta + \frac{\partial v}{\partial n}\right) = -\alpha\frac{\partial p}{\partial n} - 2w_z v.$$

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2 <u>Ibid</u>., p. 322.

Solving for ζ

$$\zeta + \frac{\partial v}{\partial n} = -\frac{\alpha}{v}\frac{\partial p}{\partial n} - 2w_z,$$

$$\zeta = -\frac{\partial v}{\partial n} - \frac{\alpha}{v}\frac{\partial p}{\partial n} - 2w_z.$$

The term $2w_z$ has a maximum value at the pole and its value is

$2 \times 7.292 \times 10^{-5} \times 1 = 1.4584 \times 10^{-4}$ radians per sec.

This term being so small, it can be deleted, thus

$$\zeta = -\frac{\partial v}{\partial n} - \frac{\alpha}{v}\frac{\partial p}{\partial n},$$

which gives a relationship for vorticity that is easily calculable from data existing on weather maps. The usefulness of this equation is not established yet, but its relationship to tornado or tornadic winds may bear fruit.

II. CURL OF ACCELERATION EQUATION

The acceleration equation in vector form is

$$\frac{d\bar{v}_r}{dt} = \bar{b} + \bar{c} + \bar{g},$$

but writing it in the form

$$\frac{d\bar{v}}{dt} = -\alpha\nabla p - 2\bar{w} \times \bar{v} - \nabla Q$$

it can be handled easier. The curl is then

$$(1) \quad \nabla \times \frac{d\bar{v}}{dt} = -\alpha\nabla \times \nabla p - 2 \times(\bar{w} \times \bar{v}) - \times \quad .$$

The right side of the equation may be analyzed by considering each term separately.

First term on the right

$$-\alpha\nabla \times \nabla p = -\alpha\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial p}{\partial x} & \frac{\partial p}{\partial y} & \frac{\partial p}{\partial z} \end{vmatrix} = -\alpha\Big[\Big(\frac{\partial^2 p}{\partial y \partial z} - \frac{\partial^2 p}{\partial y \partial z}\Big)\bar{i}$$

$$\Big(\frac{\partial^2 p}{\partial z \partial x} - \frac{\partial^2 p}{\partial x \partial z}\Big)\bar{j}$$

$$\Big(\frac{\partial^2 p}{\partial x \partial y} - \frac{\partial^2 p}{\partial x \partial y}\Big)\bar{k}\Big] = 0,$$

provided p has a continuous first partial derivative.

Similarly $\qquad \nabla \times \nabla q = 0.$

The second term

$$(2)\ -2\left[\nabla \times (\bar{w} \times \bar{v})\right] = -2\left[(\bar{v} \cdot \nabla)\bar{w} - \bar{v}(\nabla \cdot \bar{w}) + \bar{w}(\nabla \cdot \bar{v}) - (\bar{w} \cdot \nabla)\bar{v}\right].$$

Considering each term on the right separately

$$-2(\bar{v} \cdot \nabla)\bar{w} = v_x \frac{\partial \bar{w}}{\partial x} + v_y \frac{\partial \bar{w}}{\partial y} + v_z \frac{\partial \bar{w}}{\partial z} = 0, \quad (\bar{w} \text{ is constant})$$

$$2\bar{v}\ (\nabla \cdot \bar{w}) = 2\bar{v}\left(\frac{\partial w_x}{\partial x} \quad \frac{\partial w_y}{\partial y} \quad \frac{\partial w_z}{\partial z}\right),$$

since $w_y = w \cos\theta$, $w_x = 0$, $w_z = w \sin\theta$ when x points to east.

$$-2\bar{w}\ (\nabla \cdot \bar{v}) = -2\bar{w}\left(\frac{\partial v_x}{\partial x} \quad \frac{\partial v_y}{\partial y} \quad \frac{\partial v_z}{\partial z}\right)$$

and $\qquad 2(\bar{w} \cdot \nabla)\bar{v} = 2\left(w_x \frac{\partial \bar{v}}{\partial x} + w_y \frac{\partial \bar{v}}{\partial y} + w_z \frac{\partial \bar{v}}{\partial z}\right)$

$$= 2\left(w_x \frac{\partial v_x}{\partial x}\bar{i} + w_y \frac{\partial v_y}{\partial y}\bar{j} + w_z \frac{\partial v_z}{\partial z}\bar{k}\right).$$

Adding the terms of the last two

$$-2\overline{w}(\nabla\cdot\overline{v}) + 2(\overline{w}\cdot\nabla)\overline{v} = -2w_x \frac{\partial v_x}{\partial x}\overline{i} - 2w_x \frac{\partial v_y}{\partial y}\overline{i} - 2w_x \frac{\partial v_z}{\partial z}\overline{i}$$

$$-2w_y \frac{\partial v_x}{\partial x}\overline{j} - 2w_y \frac{\partial v_y}{\partial y}\overline{j} - 2w_y \frac{\partial v_z}{\partial z}\overline{j}$$

$$-2w_z \frac{\partial v_x}{\partial x}\overline{k} - 2w_z \frac{\partial v_y}{\partial y}\overline{k} - 2w_z \frac{\partial v_z}{\partial z}\overline{k}$$

$$+ 2w_x \frac{\partial v_x}{\partial x}\overline{i} + 2w_y \frac{\partial v_y}{\partial y}\overline{j} + 2w_z \frac{\partial v_z}{\partial z}\overline{k}$$

$$= -2w_x\left(\frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right)\overline{i}$$

$$-2w_y\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_z}{\partial z}\right)\overline{j}$$

$$-2w_z\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}\right)\overline{k}.$$

Considering the term on the left of equation (1)

$$\nabla \times \frac{d\overline{v}}{dt} = \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial v_x}{\partial t} & \frac{\partial v_y}{\partial t} & \frac{\partial v_z}{\partial t} \end{vmatrix} = \left(\frac{\partial}{\partial y}\frac{\partial v_z}{\partial t} - \frac{\partial}{\partial z}\frac{\partial v_y}{\partial t}\right)\overline{i}$$

$$+\left(\frac{\partial}{\partial z}\frac{\partial v_x}{\partial t} - \frac{\partial}{\partial x}\frac{\partial v_z}{\partial t}\right)\overline{j}$$

$$+\left(\frac{\partial}{\partial x}\frac{\partial v_y}{\partial t} - \frac{\partial}{\partial y}\frac{\partial v_x}{\partial t}\right)\overline{k}.$$

Restricting motion to a horizontal plane and the x axis pointed east, $\left(\frac{\partial}{\partial x}\frac{\partial v_y}{\partial t} - \frac{\partial}{\partial y}\frac{\partial v_x}{\partial t}\right)\overline{k} = -2w_y \frac{\partial v_x}{\partial x}\overline{j} - 2w_z\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}\right)\overline{k}$

$$2v_x \frac{\partial w_y}{\partial y}\overline{i} + \frac{\partial w_y}{\partial y} v_y \overline{j} + 2v_x \frac{\partial w_z}{\partial z}\overline{k}$$

$$+ 2v_y \frac{\partial w_z}{\partial z} \bar{k}.$$

Dotting with $\bar{k}$,

$$\frac{\partial}{\partial x}\frac{\partial v_y}{\partial t} - \frac{\partial}{\partial y}\frac{\partial v_x}{\partial t} = -2w_z\left(\frac{\partial v_y}{\partial y} + \frac{\partial v_x}{\partial x}\right) + 2v_x \frac{\partial w_z}{\partial z} + 2v_y \frac{\partial w_z}{\partial z},$$

$$\frac{\partial}{\partial t}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right) = -2w_z \left(\nabla_h \cdot \bar{v}\right) + 2\,w \cos\theta \frac{\partial \theta}{\partial z} \left(v_x + v_y\right),$$

$$\frac{\partial}{\partial t}(\zeta) = -2w_z \left(\nabla_h \cdot \bar{v}\right) \text{ since } 2w \cos\theta \frac{\partial \theta}{\partial z} \left(v_x + v_y\right) = 0.$$

The change in vorticity with time is equal to $-2w \sin\theta$ times the divergence of flow, and the right hand side of this equation is the time rate of change of the absolute vorticity as presented in Holmboe's Dynamic Meteorology, page 324. (This book is listed in the bibliography).

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