CONCERNING CERTAIN
MINIMAL COVER REFINEABLE SPACES

A Dissertation

Presented to

the Faculty of the Department of Mathematics

University of Houston

In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

---

by

Upton J. Christian

May, 1970

CONCERNING CERTAIN
MINIMAL COVER REFINEABLE SPACES

An Abstract of a Dissertation

Presented to

the Faculty of the Department of Mathematics

University of Houston

In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

by

Upton J. Christian

May, 1970

ABSTRACT

The concept of minimal cover refineable spaces was first used by R. Arens and J. Dugundji. This paper extends their results by showing both additional properties which imply minimal cover refineability and additional properties which are implied by minimal cover refineability.

In the course of the research for this paper, some properties of dense subspaces of certain minimal cover refineable spaces were noted. In particular, it was noted that every Nagata space contains a dense metric subspace.

TABLE OF CONTENTS

CHAPTER I

INTRODUCTION

This paper presents an investigation of minimal cover refineable spaces. Arguments used to establish properties of minimal cover refineable spaces are employed to determine metric properties of dense subspaces of a variety of topological spaces. Minimal cover refineability will be defined in this chapter along with other properties that are less frequently used. In addition, two unconventional definitions are provided.

Chapter II will identify spaces that were or are now known to possess the property of minimal cover refineability. A chart at the end of chapter II gives an overall view of the properties that imply minimal cover refineability.

Chapter III discusses the relation between compactness and minimal cover refineability. An uncommon definition of compactness is given below as well as an explanation for the substitution. Chapter III also notes the strength of $\aleph_1$-compactness.

Chapter IV is primarily concerned with dense subspaces of certain minimal cover refineable spaces, and in particular with dense metric subspaces. Of particular interest is the result that each Nagata space contains a dense metric subspace. This chapter also discusses the density and the weight of spaces.

Chapter V gives properties that cause subsets of minimal cover refineable spaces to be minimal cover refineable.

Chapter VI covers material that was not readily incorporated in one of the other chapters.

In this paper, "space" means "topological space," the word "cover" means "open cover" unless otherwise stated, and a refinement is assumed to be composed of the same kind of elements (i.e., open or closed) and cover the same set as the collection which is refined unless otherwise stated.

The following definition was obtained from [39]:

A point p of a space S is an <u>M-limit point</u> of $A \subset S$ if , for every open set R containing p, the cardinality of $R \cap A$ is not less than M.

The statement that K is a <u>minimal cover</u> of the point set M means that K is an open cover of M and if g is an element of K, then $\{h\colon h \in K \text{ and } h \neq g\}$ is not a cover of M.

The statement that a point set M is <u>minimal cover refineable</u> means that if K is an open cover of M, then there is a refinement of K which is a minimal cover of M.

If B is a basis for a space S, then the statement that the space S is <u>basically minimal cover refineable with respect to B</u> means that every open cover of S has a refinement composed of elements of B which is a minimal cover of S.

The statement that a space S is <u>basically minimal cover refineable</u> means that the space is minimal cover refineable with respect to any basis.

The following definition was obtained from [41]:

A space S is collectionwise Hausdorff if every discrete subset can be covered by a collection of disjoint open sets such that no two points of the discrete subset are contained in an element of the collection.

A space S is strongly collectionwise Hausdorff if every discrete subset can be covered by a discrete collection of open sets such that no two points of the discrete subset are contained in an element of the discrete collection.

The following definition is useful because the property defined is shared by T_1 spaces, regular spaces, and spaces in which closed sets are G_δ:

The statement that a space is T_1-like means that if p and q are points of the space and there is an open set that contains p but not q, then there is an open set that contains q but not p.

The statement that S' is a T_1 representation of a T_1-like space S means that there is a function f on the points of S such that (1) $p \in S$ if and only if $f(p) \in S'$; (2) for $p, q \in S$, then $f(p) = f(q)$ if and only if $p \in \overline{q}$; (3) R is an open set in S' if and only if $f^{-1}(R)$ is an open set in S.

It will be noted that a theorem that is true for a T_1 space is true for a T_1-like space if the properties in the theorem do not require a distinction between individual points and groups of points. An example of a property that does not require such a distinction is paracompactness; a property that does require such a distinction is compactness as usually defined. These examples will be discussed further below.

The statement that a point set M is <u>σ-discrete</u> means that M is the union of countably many discrete point sets.

The following definition is from [7]:

The statement that a space S is <u>collectionwise normal</u> means that if K is a discrete collection of point sets, then there is a collection D of disjoint open sets covering K^* such that no element of D intersects two elements of K.

The statement that a space S is <u>completely normal</u> means that if H and K are subsets of S and both $\overline{H} \cap K = \emptyset$ and $H \cap \overline{K} = \emptyset$, then there are disjoint open subsets C and D such that $H \subset C$ and $K \subset D$.

The following definitions were obtained from [11]:

The statement that a collection K is <u>σ-closure preserving</u> means that there is a sequence, say $\{K_i\}$, of subcollections of K such that (1) $K = \{g\colon \text{for some integer } i > 0,\ g \in K_i\}$ and (2) for each integer $i > 0$, the union of the closures of any subcollection of K_i is closed.

The statement that a space S is $\underline{M_1}$ means that S is a T_3 space with a σ-closure preserving basis.

The statement that a collection B is a <u>quasi-basis</u> for a space S means that if R is an open subset of S and $p \in R$, then there is an element of B, say b, such that $p \in b^o \subset b \subset R$ where b^o denotes the interior of b.

The statement that a space is $\underline{M}_2$ means that S is a T_3 space with a σ-closure preserving quasi-basis.

The following definition was obtained from [8] except that the requirement that the space be T_3 is dropped since regular spaces are T_1-like.

The statement that a space is <u>stratifiable</u> (also called $\underline{M}_3$) means that there is a function G from the product of the collection of closed subsets of S with the natural numbers into the collection of open sets in S such that (i) $A \subset G(A, n)$ for each closed set A; (ii) $\bigcap_{i=1}^{\infty} \overline{G(A, n)} = A$ for each closed set A; and (iii) $G(A_1, n) \subset G(A_2, n)$ whenever $A_1 \subset A_2$.

The following definition may be found in [13]:

The statement that a space S is <u>semi-stratifiable</u> means that there is a function G from the product of the collection of closed sets of S with the natural numbers into the collection of open sets in S such that (i) $\bigcap_{i=1}^{\infty} G(A, n) = A$ for each closed set A and (ii) $G(A_1, n) \subset G(A_2, n)$ whenever $A_1 \subset A_2$.

A number of papers have dealt with stratifiable spaces. Material on stratifiable spaces may be found in [8, 11, 13, 14, 20, 24, 35]. Arkhangel′skii calls these spaces <u>laced</u> and discusses them in [5].

The following definitions were obtained from [10]:

The statement that a space S is <u>semi-pseudometric</u> means that there is a distance function d defined on S X S such that if x, y ε S, then (1) $d(x, y) \geq 0$; (2) $d(x, y) = d(y, x)$; and (3) if M is a subset of

S and $p \in S$, then $\inf\{d(p, x): x \in M\} = 0$ if and only if $x \in \overline{M}$.

The statement that a space S is <u>semi-metric</u> means that S is a semi-pseudometric space and if d is a semi-pseudometric on $S \times S$ and x, y are points of S, then $d(x, y) = 0$ if and only if $x = y$.

The statement that a space S is <u>pseudometric</u> means that S is a semi-pseudometric space and there is a semi-pseudometric d on $S \times S$ such that if x, y, and z are points of S, then $d(x, z) \leq d(x, y) + d(y, z)$.

The statement that a space S is <u>metric</u> means that S is both a semi-metric and a pseudometric space.

Note that semi-pseudometric spaces are T_1-like because closed sets are G_δ in a semi-pseudometric space.

In [11], Ceder proves that a space is a <u>Nagata</u> space if and only if it is first countable and stratifiable. This will serve as a definition for a Nagata space in this paper. For another definition, see [11].

The following definition was obtained from [24]:

The statement that a collection C of sets in a space S is <u>point-countable</u> means that if $p \in S$, then p belongs to only countably many elements of C.

The following definition was obtained from [1]:

The statement that B is a <u>point-regular basis</u> for a space S means that B is a basis for S and if $p \in S$, then any infinite set of elements of B containing p is a basis for p.

The following definition was obtained from [15]:

The statement that $\mathcal{M}$ is the <u>weight</u> of a space S means that $\mathcal{M}$ is the least cardinal which is the cardinality of a basis for the space.

The following definition was obtained from [10]:

The statement that $\mathcal{M}$ is the <u>density</u> (also called <u>character density</u>) of a space S means that $\mathcal{M}$ is the least cardinal which is the cardinality of a dense subset of the space.

The following definition is taken from [3[illegible]]:

A cardinal $\mathcal{M}$ is called <u>regular</u> if it is not the sum of less than $\mathcal{M}$ cardinals each less than $\mathcal{M}$.

The statement that a cardinal $\mathcal{M}$ is <u>$\aleph_0$-regular</u> means that the cardinal is either regular or it is not the sum of countably many cardinals each less than $\mathcal{M}$.

The usual definition of a compact space states that every infinite subset has a limit point, but this definition is not suitable for this paper because there is a T_0 space in which this definition is not equivalent to the statement that no infinite subset is discrete. Therefore, in this paper, the definition of compact shall be:

The statement that a space S is <u>compact</u> means that no infinite subset is discrete.

The statement that a space S is <u>$\mathcal{M}$-compact</u> means that no subset of cardinality $\mathcal{M}$ is discrete.

The following definitions were obtained from [18]:

A topological space S is <u>(M, N)-compact</u> if for every open covering of S whose cardinality is at most N, one can select a subcovering whose cardinality is at most M.

A topological space S is <u>(M, ∞)-compact</u> if for every open covering of S, one can select a subcovering whose cardinality is at most M.

The following definitions were taken from [27]:

If M is a cardinal, the point set N is said to be <u>strongly M-separable</u> provided that there exists an H such that (1) $H \subset N \subset \overline{H}$ and (2) either H is countable or its cardinality is less than M.

If M is the cardinality of a point set N and N is strongly M-separable, then N is said to be <u>semi-separable</u>.

To the definition of paracompact which follows, some papers add the stipulation that the space is Hausdorff and others require that the space be regular. This is done because it is convenient for a paracompact space to be fully normal. There is a paracompact T_1 space which is not fully normal. Stone [ ] established that a paracompact Hausdorff space s fully normal; it need only be noted that a regular space is T_1-like to establish that a paracompact regular space is fully normal. This paper drops both Hausdorff and regular from the definition of paracompact in order to make the definition more like the definitions for metacompact and hypocompact. The definitions which follow were obtained from [41]:

A collection K of sets in a space is point-finite (point-countable) if each point of the space is in at most finitely (countably) many members of K.

A collection K of sets in a space is locally finite (locally countable) if each point is contained in an open set which intersects at most finitely (countably) many members of K.

A collection K of sets in a space is star-finite (star-countable) if each member of K intersects at most finitely (countably) many members of K.

A space is metacompact (paracompact) (hypocompact) if every open cover has a point-finite (locally finite) (star-finite) refinement.

A space is metalindelöf (paralindelöf) (hypolindelöf) if every open cover has a point-countable (locally countable) (star-countable) refinement.

If B is a basis for a space S, then the statement that the space S is basically metacompact with respect to B means that every open cover of S has a refinement consisting of elements of B which is point-finite.

The statement that a space is basically metacompact means that the space is basically metacompact with respect to any basis.

The following definition is taken from [6]:

The statement that a space S is quasi-developable means that there is a sequence $\{G_i\}$ of collections of open sets such that if $p \in S$

and R is an open set containing p, then there is an integer $I > 0$ such that some element of G_I contains p and if $p \in g \in G_I$, then $g \subset R$.

The following definitions were obtained from [7]:

The statement that a space S is screenable means that for each open cover H of the space, there is a sequence $\{H_i\}$ of collections of disjoint open sets such that $\{g: i$ is a natural number and $g \in H_i\}$ is a cover of S which refines H.

The statement that a space S is strongly screenable means that for each open cover H of the space, there is a sequence $\{H_i\}$ of discrete collections of open sets such that $\{g: i$ is a natural number and $g \in H_i\}$ is a cover of S which refines H.

The following definition is from [28]:

The statement that a space S if F_σ-screenable means that for each open cover H of the space, there is a sequence $\{H_i\}$ of discrete collections of closed point sets such that $\{h: i$ is a natural number and $h \in H_i\}$ is a closed cover of S which refines H.

The following definition was obtained from [32]:

The statement that a collection B of closed subsets of a space S dominates S means that B covers S and if $A \subset S$ and A has a closed intersection with every element of some subcollection of B which covers A, then A is closed.

The statement that a well-ordered collection K of closed sets weakly dominates a space S means that $K^* = S$ and if H is an initial segment of K, then H^* is closed.

The following definitions were obtained from [9]:

A space S is strong cover compact if an only if for each open cover G of S, there exists an open refinement H such that if $\{h_i\}$ is a countably infinite subcollection of distinct elements of H, $p_i, q_i \in h_i$ for each i, $p_i \neq p_j$ and $q_i \neq q_j$ for $i \neq j$, and the point set $\{p_i\}$ has a limit point in S, then the point set $\{q_i\}$ does.

A space S is weak cover compact if and only if for each open cover G of S, there exists an open refinement H such that if $\{h_\alpha\}_{\alpha \epsilon A}$ is an uncountable subcollection of distinct elements of H, $p_\alpha, q_\alpha \epsilon h_\alpha$ for each $\alpha \epsilon A$, $p_\alpha \neq p_\beta$, $q_\alpha \neq q_\beta$ for $\alpha \neq \beta$, and the point set $\{p_\alpha\}_{\alpha \epsilon A}$ has a limit point in S, then the point set $\{q_\alpha\}_{\alpha \epsilon A}$ does.

The following definition was obtained from [38].

A Souslin space is a linearly ordered connected space which has the property that every disjoint collection of open sets is countable but the space is not separable.

No example of a Souslin space is known.

The following note is appended for clarity. The statement that a definition was taken from a particular source is not meant to imply that the author felt constrained to copy the definition, it merely provides an opportunity to verify that the definitions are equivalent.

CHAPTER II

MINIMAL COVER REFINEABILITY

The concept of minimal cover refineability is due to R. Arens and J. Dugundji. They showed that every metacompact space is minimal cover refineable, and thus, every metric space is minimal cover refineable. In this chapter, other properties that imply that a space is minimal cover refineable are presented. It is shown that not only are all metric spaces minimal cover refineable, but also semi-metric spaces and still weaker spaces.

The following theorem is proved in [4] (see also [15, pp. 160-161]):

<u>Theorem 2.1.</u> (Arens, Dugundji) Every metacompact space is minimal cover refineable.

In fact, they proved a somewhat stronger result. Namely,

<u>Theorem 2.2.</u> (Arens, Dugundji) Every open point-finite cover of a space contains a minimal subcover.

It was suggested to the author that he determine whether minimal cover refineability could be substituted for metacompact in well-known theorems concerning metacompact spaces. Before undertaking this, the author wished to be certain that minimal cover refineability was not implied by another part of the hypothesis of such a theorem. For instance, would a hypothesis that stated that a space was both Moore and minimal cover refineable be stronger than just the hypothesis that the space was Moore? The author was able to prove, successively, that a Moore space, a developable space, a semi-metric space, and a semi-stratifiable space is minimal cover refineable. Finally, a property shared by these spaces was discovered to imply minimal cover refineability resulting in the following theorem:

Theorem 2.3. Every F_σ-screenable space is minimal cover refineable.

Proof. Let S be an F_σ-screenable space and C any open cover of S. Let $\{F_i\}$ be a countable collection of discrete collections of closed refinements of C such that $\bigcup\{F_i\} = S$. For each integer $i > 0$, associate with each element f of F_i an open set $c = g\setminus(F_i^*\setminus f)$ where $f \subset g \in C$. Let C_i' be the collection of open sets associated with elements of F_i. Let $C_1 = C_1'$. For each integer $i > 1$, let $C_i = \{c\colon c' \in C_i',\ c = c'\setminus\bigcup_{j=1}^{i-1} F_j^*, \text{ and } c \cap F_i^* \neq \emptyset\}$. This forms a minimal cover of S and establishes the theorem.

In [13], G. D. D. Creede notes that every semi-stratifiable space is F_σ-screenable (for a proof, see [14]). Thus,

Corollary 2.4. Every semi-stratifiable space is minimal cover refineable.

It is well-known that a regular strongly screenable space is paracompact and, thus, minimal cover refineable. The following theorem is readily proved and presented here without proof.

Theorem 2.5. Every strongly screenable T_1-like space is minimal cover refineable.

Recall (from chapter 1) that if a space is T_1, regular, or has the property that closed sets are G_δ, then it is T_1-like.

Stone proves [40] without stating that fully normal implies both paracompact and F_σ-screenable, so by theorem 2.1 or 2.3:

Corollary 2.6. Every fully normal space is minimal cover refineable.

It will now be useful to look at two spaces which are not minimal cover refineable.

Example 2.7. Let the space S be the countable ordinals with the order topology. This space is denoted by $[0, \Omega)$ where Ω is the first uncountable ordinal. This example is well-known.

The following theorem uses the previous example to indicate some properties that do not imply that a space is minimal cover refineable.

Theorem 2.8. There exists a collectionwise normal completely normal first countable Hausdorff space which is not minimal cover refineable.

Proof. Let S be $[0, \Omega)$ as described in example 2.7.

First, it will be shown that the space S is not minimal cover refineable. Let C be a collection of open sets covering S such that if $g \in C$, then there is a point p in S such that if $q \in g$, $q < p$. Suppose C' is an open refinement of C such that C' is a minimal cover of S. Let P be a collection of one and only one point from each element of C'

which is in no other element of C'. Then P contains uncountably many points since no countable subcollection of C covers S. Further, P is discrete. But S is compact and this is a contradiction.

It is well-known that S is first countable and Hausdorff. If the reader has seen arguments that S is collectionwise normal and completely normal, he may wish to proceed directly to the next example.

It will now be shown that S is collectionwise normal. Let H be a discrete collection of closed subsets of S. For each point $p \in H^*$, let α be a point such that $\alpha < p$ and $O_p = (\alpha, p+1)$ intersects only one element of **H** (for 0, use $[0, 1)$). Then the collection $\{O_h : h \in H, O_h = \bigcup\{O_p : p \in h\}\}$ verifies that S is collectionwise normal.

Finally, it will be shown that S is completely normal. Let M and N be two separated subsets of S. For each point p of M or N form an open set O_p as in the previous argument. The open sets $\bigcup\{O_p : p \in M\}$ and $\bigcup\{O_p : p \in N\}$ establishes that S is completely normal.

The following is an example of a non T_1-like space which is not minimal cover refineable.

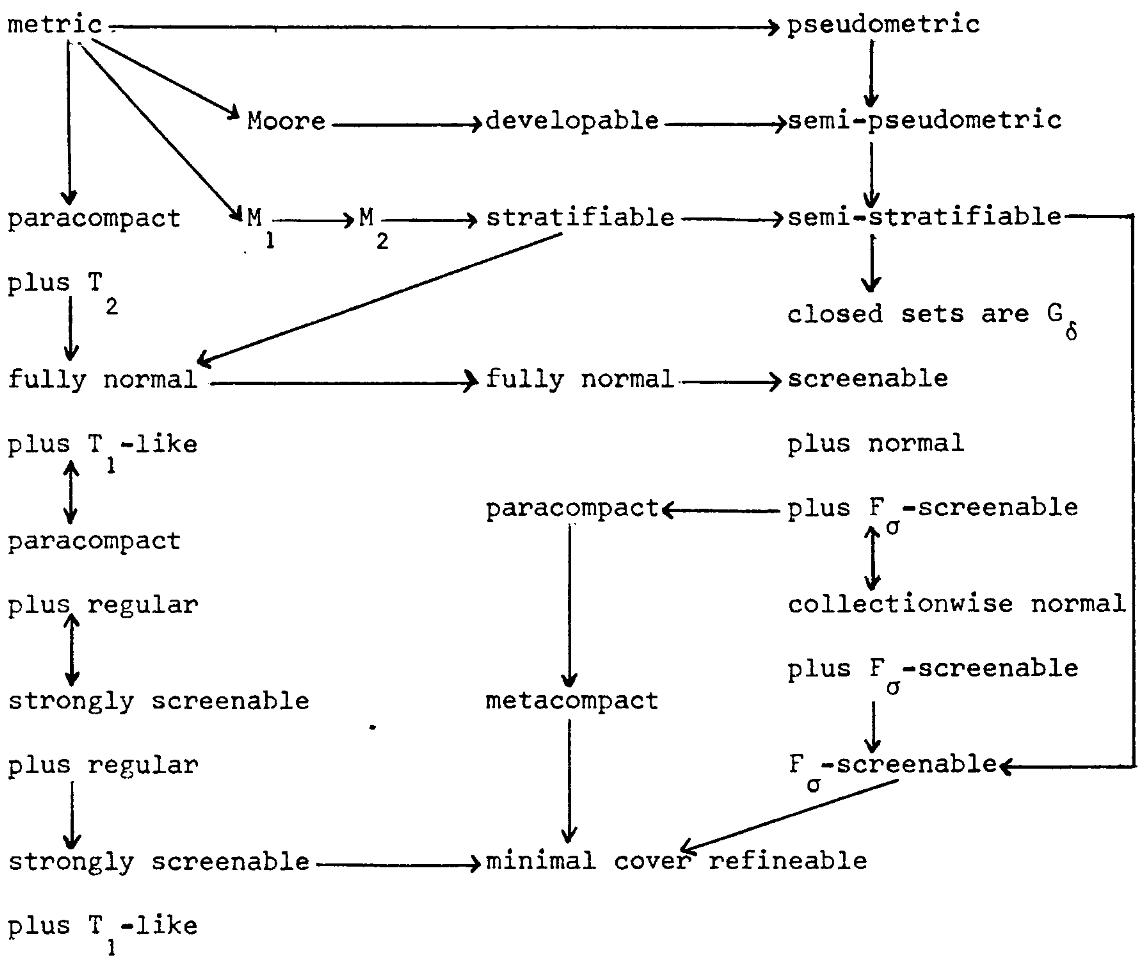
Example 2.9. Let S be the points of an infinite sequence $\{p_i\}$. For each integer $i > 0$, let $\bigcup_{j=1}^{i} p_j$ be a basis element for S.

The following theorem uses the previous example to show some more properties that do not imply that a space is minimal cover refineable. In particular, it is shown that T_1-like can not be dropped from the hypothesis of theorem 2.5.

Theorem 2.10. There exists a quasi-developable strongly screenable second countable collectionwise normal T_0 space which is not minimal cover refineable.

Proof. Clearly, S is quasi-developable, strongly screenable, second countable, and T_0. Since S does not contain two disjoint closed sets, it is collectionwise normal. It is readily seen that a cover of basis elements has no refinement which is a minimal cover of S.

Chart 2.11, which follows, will help the reader to visualize the relation between various familiar properties and minimal cover refineability. The relationships shown in the chart are discussed further in [7, 11, 13, 14, 21, 28, 33, and 40]. Note that Stone's argument in [40] that every fully normal T_1 space is paracompact is sufficient to show that every fully normal space is strongly screenable and F_σ-screenable.



metric $\longleftrightarrow$ pseudometric plus T_1

Moore $\longleftrightarrow$ developable plus T_3

semi-metric $\longleftrightarrow$ semi-pseudometric plus T_1

Nagata $\longleftrightarrow$ stratifiable plus first countable

semi-pseudometric $\longleftrightarrow$ semi-stratifiable plus first countable

Chart 2.11

CHAPTER III

COMPACTNESS AND RELATED PROPERTIES

The concepts of $\mathcal{X}_{\alpha+1}$-compactness and $(\mathcal{X}_{\alpha},\infty)$-compactness are similar; indeed, so similar that one is surprised on first learning that they are not equivalent in a T_4 space. In this chapter, it is shown that an $\mathcal{X}_{\alpha+1}$-compact minimal cover refineable space is $(\mathcal{X}_{\alpha},\infty)$-compact, but an $(\mathcal{X}_{\alpha},\infty)$-compact space may not be minimal cover refineable.

Arens and Dugundji [4] proved part (ii) of the following theorem. It need only be noted that the collection of the closed sets unique to the elements of a minimal cover is discrete to establish the theorem.

Theorem 3.1. (i) If a space is minimal cover refineable and $\mathcal{X}_{\alpha+1}$-compact, then the space is $(\mathcal{X}_{\alpha},\infty)$-compact; (ii) (Arens, Dugundji) if a space is minimal cover refineable and compact, then the space is bicompact.

The following two theorems are also due to Arens and Dugundji [4].

Theorem 3.2. (Arens, Dugundji) In order for a space to be bicompact, it is necessary and sufficient that the space be compact and minimal cover refineable.

Theorem 3.3. (Arens, Dugundji) In order for a space to be compact, it is necessary and sufficient that no infinite open cover be minimal.

In chapter XI, section 1, problem 9, Dugundji [15, p. 251] gives a sufficient hint to solve the following interesting theorem:

Theorem 3.4. (Dugundji) In order for a space to be bicompact, it is necessary and sufficient that every open cover have a minimal subcover.

The following theorem can be easily verified and is presented without proof.

Theorem 3.5. In order for a T_1-like space to be Lindelöf, it is necessary and sufficient that the space be $\aleph_1$-compact and minimal cover refineable.

The following corollary to theorem 3.5 was proved by Creede [14] (Jones did the same for Moore spaces [25]):

Corollary 3.6. (Creede) In order for a semi-stratifiable space to be Lindelöf, it is necessary and sufficient that it be $\aleph_1$-compact.

It can be easily verified that if a space is $(\aleph_\alpha,\aleph_{\alpha+1})$-compact, then it is $\aleph_{\alpha+1}$-compact in order to establish theorem 3.7. Part (ii) of theorem 3.7 was proved in [4] (for a more explicit argument, see [15, pp. 229-230]).

Theorem 3.7. In a minimal cover refineable space, (i) the following are equivalent:

(a) $\aleph_{\alpha+1}$-compact,

(b) $(\aleph_\alpha,\infty)$-compact,

(c) $(\aleph_\alpha,\aleph_{\alpha+1})$-compact;

and (ii) (Arens, Dugundji) the following are also equivalent:

(a′) compact,

(b′) bicompact,

(c′) countably compact.

Theorem 3.8. (i) If a space is minimal cover refineable, $\aleph_{\alpha+1}$-compact, and has the property that closed sets are G_δ, then the space is hereditarily $(\aleph_\alpha,\infty)$-compact and, thus, hereditarily $\aleph_{\alpha+1}$-compact; (ii) if a space is minimal cover refineable, compact, and has the property that closed sets are G_δ, then the space is hereditarily Lindelöf and, thus, hereditarily $\aleph_1$-compact.

Proof. Let S be a space satisfying the hypothesis of (i). Let M be any subset of S and C any open cover of M. If $C^* = S$, then, by theorem 3.1, there is a subcollection of C which covers S containing at most $\aleph_\alpha$ many elements. Otherwise, consider $S\backslash C^*$. Since closed sets are G_δ, let $S\backslash C^* = \bigcap_{i=1}^{\infty} G_i$. For each integer $i > 0$, $S\backslash G_i$ is a closed set covered by C. Since $\{g\colon g \in C \text{ or } g = G_i\}$ covers S, there is a subcollection of C, say C_i, covering $S\backslash G_i$ containing at most $\aleph_\alpha$ many elements. Since $\bigcup_{i=1}^{\infty} C_i$ covers M, M is $(\aleph_\alpha,\infty)$-compact.

The argument for part (ii) is now obvious.

The following corollary to theorem 3.7 was proved by Creede [14]:

Corollary 3.9. (Creede) In order for a semi-stratifiable space to be bicompact, it is necessary and sufficient that the space be countably compact.

The following lemma is stated without proof:

Lemma 3.10. If C is an open cover of a space S, then the collection $\{g_p\colon p \in S,\ g_p = \bigcup\{h\colon h \in C \text{ and } h \cap p \neq \emptyset\}\}$ has a minimal subcover.

The following three theorems indicate the strength of $\aleph_1$-compactness.

<u>Theorem 3.11</u>. In an $\aleph_1$-compact T_1-like space, the following are equivalent:

(a) Lindelöf,

(b) metalindelöf,

(c) paralindelöf,

(d) hypolindelöf,

(e) screenable,

(f) strongly screenable,

(g) σ-(any of the above),

(h) minimal cover refineable.

Proof. By their definitions, (a) implies (b) through (g). By theorem 3.5, (a) and (h) are equivalent. By their definitions, (c) through (g) imply (b). G. Aquaro has shown that (b) implies (a) [3]. His argument seems unduly tedious. Lemma 3.10 can be applied to establish the same conclusion.

<u>Theorem 3.12</u>. In an $\aleph_1$-compact regular space, the following are equivalent:

(a) Lindelöf,

(b) metalindelöf,

(c) paralindelöf,

(d) hypolindelöf,

(e) screenable,

(f) strongly screenable,

(g) metacompact,

(h) F_σ-screenable,

(i) paracompact,

(j) hypocompact,

(k) σ-(any of the above),

(l) fully normal,

(m) minimal ccver refineable.

Proof. By theorem 3.11, (a) through (f) and (m) are equivalent. By theorems 2.1 and 2.3 and corollary 2.6, (g) through (j) and (l) imply (m). By Morita [34], in a regular space (a) implies (j). By their definitions, (j) implies (i) and (g). Clearly, in a regular space, (a) implies (h). By Stone [40], in a regular space, (i) implies (l). Noting that σ-(F_σ-screenable) is not distinct from F_σ-screenable, the balance of the argument is similar to the argument for theorem 3.11.

Theorem 3.13. In an $\aleph_1$-compact space which has the property that closed sets are G_δ, the following are equivalent:

(a) Lindelöf,

(b) metalindelöf,

(c) paralindelöf,

(d) hypolindelöf,

(e) screenable,

(f) strongly screenable,

(g) metacompact,

(h) F_σ-screenable,

(i) σ-(any of the above),

(j) minimal cover refineable,

(k) hereditarily (any of the above).

Proof. By theorem 3.11, (a) through (f) and (j) are equivalent. By theorems 2.1 and 2.3, (g) and (h) imply (j). Since each open set is F_σ, clearly, (a) implies (h).

It will now be shown that (a) implies (g). Let S be an $\aleph_1$-compact space which has the property that closed sets are G_δ. Let $\{g_i\}$ be a countable open cover of S. Let $\{M_{ij}\}$ be a countable collection of closed sets such that $\bigcup_{j=1}^{\infty} M_{ij} = g_i$ for each integer i > 0. Let $C_1 = g_1$. For each integer i > 0, let $C_i = g_i \setminus \bigcup_{j,k=1}^{i-1} M_{jk}$. Then $\{C_i\}$ is the point-finite cover of S required.

The equivalence of (a) and (i) is argued as in theorem 3.12. By theorem 3.8, (j) implies hereditarily Lindelöf and hereditarily $\aleph_1$-compact. Then, since closed sets are G_δ is hereditary, the other properties in the conclusion are hereditary.

Theorem 3.14. If M is a regular cardinal and a space is hereditarily M-compact and hereditarily minimal cover refineable, then if N is a subset of the space, the cardinality of the non M-limit points of N in N is less than M.

Proof. This argument is similar to one given by Gál [18] (also see Stone [39]). Let H be the collection of all non M-limit points in N. Cover each point of H with an open set containing less than M points of H. Let G be a minimal cover refinement of these open sets. Then G contains less than M open sets.

The following two corollaries to theorem 3.14 are stated without proof:

Corollary 3.15. If a space is hereditarily minimal cover refineable and M is a regular cardinal, then hereditarily M-compact is equivalent to the statement that if N is a subset of S, the non M-limit points of N in N has cardinality less than M.

Corollary 3.16. Every hereditarily $\aleph_1$-compact hereditarily minimal cover refineable space has the property that every uncountable subset contains a condensation point.

The following theorem is proved by application of theorems 3.8, 3.13, and 3.14.

Theorem 3.17. Every $\aleph_1$-compact minimal cover refineable space which has the property that closed sets are G_δ has the property that every uncountable subset contains a condensation point.

The following corollary to theorem 3.17 was proved by Creede [14].

Theorem 3.18. (Creede) Every $\aleph_1$-compact semi-stratifiable space has the property that every uncountable subset contains a condensation point.

Creede [14] notes that a Souslin space is not semi-stratifiable. The following is a corollary to theorems 3.13 and 3.17:

Corollary 3.19. Every Souslin space is hereditarily fully normal and has the property that every uncountable subset contains a condensation point.

Proof. Since a Souslin space is regular, the property of hereditary strong screenability implies the property of hereditary full normality. It need only be shown that every Souslin space is $\aleph_1$-compact, minimal cover refineable, and has the property that closed sets are G_δ. Since the space is linearly ordered, connected, has the order topology, and every disjoint collection of segments is countable, it can not contain an uncountable discrete collection.

It will now be shown that a Souslin space is minimal cover refineable. Let S be a Souslin space and C an open cover of S. Let C′ be segments of S refining C and covering S. Let p be any point of S and consider $g_p = \bigcup\{g\colon p \in g \in C'\}$. Either there is a least point q_1 such that $p < q_1$ and $q_1 \notin g_p$ or g_p covers all points greater than p. In the latter case, there is a countable subcollection of $\{g\colon p \in g \in C'\}$ that covers these points which can be used to construct a refinement which is a minimal cover of these points. In the former case, the same procedure is repeated for q_1. Either the process stops because, for some integer $I > 0$, g_{q_I} covers all points greater than q_I or a countable sequence $\{q_i\}$ is found (Cf. [42]). In the latter case, there can be no point r of S such that $q_i < r$ for all integers $i > 0$ because of the construction. The same process can be used to traverse S in the opposite direction. It should now be apparent that this construction can now be used to form a refinement of C which is a minimal cover of S.

It will now be shown that closed sets are G_δ in S. If the reader is aware of an argument for this, he may wish to proceed to example 3.20. Let M be any closed subset of S. The complement consists of an at most countably infinite collection of maximal open connected sets. It is well-known that a Souslin space is first countable, so every open connected set is F_σ. This is sufficient to form a countable collection of open sets whose intersection is M.

Example 3.20. For any regular cardinal M, let S consist of all the ordinals with cardinality less than M. Let S have the order topology. Denote S by $[0, M)$. This example is well-known.

The following theorem is a well-known reference to example 3.20 and shows that the property of minimal cover refineability can not be dropped from theorem 3.1.

Theorem 3.21. For any cardinal M, a space may be constructed which is compact but not (M,∞)-compact.

Proof. Let S be the space $[0, N)$ described in example 3.20, where N is a regular cardinal greater than M. Because S is well-ordered and has the order topology, S is compact. Let C be a collection of open sets covering S such that if $g \in C$, then there is a point p in S such that if $q \in g$, $q < p$. Any refinement of C must contain at least N elements since N is a regular cardinal.

The following theorem shows that the property of T_1-likeness may not be dropped from theorem 3.5.

Theorem 3.22. There exists a T_0 Lindelöf space which is not minimal cover refineable.

Proof. Example 2.9 is such a space.

The following theorem shows that the property of minimal cover refineability can not be dropped in theorem 3.7.

Theorem 3.23. There exists a compact countably compact $(\aleph_1, \infty)$-compact Hausdorff space which is neither minimal cover refineable nor bicompact.

Proof. Example 2.7 is such a space. It is well-known that this space is countably compact, but not bicompact. The remaining properties have either been discussed or are obvious.

Example 3.24. This example was presented in [12]. Let S be the collection of all countable ordinals plus the first uncountable ordinal, Ω. Let each countable ordinal be a basis element. In addition, let $[\alpha, \Omega]$ be a basis element for each countable ordinal α.

Theorem 3.25 uses example 3.24 to show that the hypothesis of theorem 3.16 can not be weakened to $\aleph_1$-compact and hereditarily minimal cover refineable. The proof is obvious.

Theorem 3.25. There exists an $\aleph_1$-compact hereditarily minimal cover refineable space which does not have the property that every uncountable subset contains a condensation point.

Example 3.26. This example is a variation of an example suggested by D. R. Traylor. Let S be the points of the form (r_1, r_2) where r_1 and r_2 are rational in a simple sequence of copies of the plane, $\{P_i\}$. For each pair of integers i, j, $i \neq j$, let $P_i \cap P_j = (0,0)$. For each integer $i > 0$, let basis elements for P_i be any open set in the plane that does not intersect the point (0, 0). Let basis elements for the point (0, 0) be the union of any open set of the plane containing (0, 0) for each element of $\{P_i\}$.

The following theorem shows that theorems known for semi-metric spaces and proved for semi-stratifiable spaces are not redundant for even countable semi-stratifiable spaces, since such a space does not need to be semi-metric.

Theorem 3.27. There exists a countable semi-stratifiable T_4 space which is not semi-metric.

Proof. Let S be the space described in example 3.26. This space is countable. It is readily shown to be T_4 since the space minus an open set containing (0, 0) is a collection of disjoint metric spaces. It is not semi-metric because it has no countable basis for the point (0, 0). (Suppose $\{G_i\}$ is a countable basis for (0, 0). For each integer $i > 0$, let g_i be a proper open subset with respect to P_i of $P_i \cap \bigcap_{j=1}^{i} G_j$ containing (0, 0). No element of $\{G_i\}$ is a subset of $\bigcup\{g_i\}$.) The space is easily seen to be semi-stratifiable by letting G((0, 0), n) be the union of open disks of radius 1/n and center at (0, 0) intersected with each of the elements of $\{P_i\}$ for each integer $n > 0$.

Example 3.28. Let S be the collection of all countable ordinals plus the first uncountable ordinal, Ω. Let S have the order topology. Denote S by $[0, \Omega]$.

The following theorem uses example 3.28 to show that the property that closed sets are G_δ can not be dropped from theorem 3.8.

Theorem 3.29. There exists a minimal cover refineable compact Lindelöf space which is not hereditarily Lindelöf.

Proof. Let S be the space described in example 3.28. It is well-known that this space is compact, Lindelöf, and T_4. By theorem 3.5, S is minimal cover refineable. Because $[0, \Omega)$ is a subset of $[0, \Omega]$, S is not hereditarily Lindelöf.

CHAPTER IV

SEPARABILITY AND RELATED CONCEPTS

This chapter will be concerned with dense subsets of semi-stratifiable spaces and, in particular, with dense metric subspaces.

Several papers of the last decade have dealt with dense metric subspaces of Moore spaces. In [43], J. N. Younglove established that each complete Moore space contains a dense metrizable subspace. B. Fitzpatrick showed a Moore space that has no dense metric subspace in [16] and gave additional conditions for dense metrizable subspaces of Moore spaces. In [17], B. Fitzpatrick proved that a normal Moore space which is not a counterexample of type D has a dense metrizable subspace. In [36], C. W. Proctor showed that if a normal locally connected Moore space has a base with the property that the space is collectionwise normal with respect to each discrete subset that is contained in the boundary of some base element, then the space contains a dense metrizable subspace.

The author has inquired into the conditions in which a semi-stratifiable space has a dense metrizable subspace at the suggestion of D. R. Traylor.

Two papers that have theorems dealing with dense subspaces are in the bibliography [2, 12].

The following theorem was proved by the author after discovering that semi-stratifiable spaces are minimal cover refineable. The argument used illustrates how the author built a minimal cover refinement of an open cover for a semi-stratifiable space.

Theorem 4.1. Every semi-stratifiable space S contains a dense σ-discrete point set and if S is T_1, the complement of the σ-discrete point set is G_δ. Further, if M is the cardinality of S, then if either M is an $\aleph_0$-regular cardinal and S is M-compact, or $N < M$ and S is N-compact, then S is semi-separable and hereditarily strongly M-separable.

Proof. Patently, if S is T_1, the complement of the σ-discrete point set is G_δ. A dense σ-discrete point set will now be constructed.

Let G be a function from the product of the collection of closed subsets of S with the natural numbers into the collection of open sets in S such that (i) $\bigcap_{n=1}^{\infty} G(A, n) = A$ for each closed set A; (ii) $G(A_1, n) \subset G(A_2, n)$ whenever $A_1 \subset A_2$. Well-order the points of S. For each integer $m > 0$, do the following: for each point p of S, there is an integer $I > 0$ such that $p \notin G(S\setminus G(p, m), I)$. Associate I with p.

For each integer $i > 0$, let $W_i = \{p: i \text{ is associated with } p\}$.

Since S is well-ordered, there is a first element in the subcollection W_K, where $K > 0$ is the least integer such that W_K is not empty, say p_{K1}. Let $O_{K1} = G(p_{K1}, m)$.

Let p_{K2} be the first point of the well-ordered collection W_K which is not a point of O_{K1}. Let $O_{K2} = G(p_{K2}, m) \cap G(S\setminus O_{K1}, K)$

After each initial segment, let $p_{K\alpha}$ be the first point of the well-ordered collection W_K which is not a point of any of the $O_{K\beta}$

already constructed. Let $O_{K\alpha} = G(p_{K\alpha}, m) \cap G(S\setminus\bigcup_{\beta<\alpha} O_{K\beta}, K)$.

For each integer $i > K$, let p_{i1} be the first point of the well-ordered collection W_i which was not covered by any of the $O_{j\beta}$, $j < i$, already constructed. Let $O_{i1} = G(p_{i1}, m) \cap \bigcap_{j=K}^{i} G(S\setminus\bigcup_{\substack{h<i \\ \beta}} O_{h\beta}, j)$.

After each initial segment, let $p_{i\alpha}$ be the first point of the well-ordered collection W_i which is not a point of any of the $O_{j\beta}$, $j \leq i$, already constructed. Let

$O_{i\alpha} = G(p_{i\alpha}, m) \cap \bigcap_{j=K}^{i} G(S\setminus(\bigcup_{\substack{h<i \\ \beta}} O_{h\beta} \cup \bigcup_{\beta<\alpha} O_{i\beta}), j)$.

Let $Q_m = \{p_{K1}, p_{K2}, p_{K3}, \ldots, p_{K\Omega}, \ldots, p_{i1}, \ldots\}$. Note that Q_m is discrete since $\{O_{K1}, O_{K2}, O_{K3}, \ldots, O_{K\Omega}, \ldots, O_{i1}, \ldots\}$ covers S.

Let $p \in S$ and R any open set containing p. There is an integer $I > 0$ such that $p \notin G(S\setminus R, I)$. Therefore, $R \cap Q_I \neq \emptyset$. Hence, the collection $\{Q_i\}$ is the dense σ-discrete point set required.

The collection $\{Q_i\}$ demonstrates that if the hypothesis is met, S is strongly M-separable and, thus, semi-separable. It will now be shown that S is hereditarily strongly M-separable. Let N be any subset of S. Use the construction above to generate

$Q_m(N) = \{p_{K1}(N), p_{K2}(N), p_{K3}(N), \ldots, p_{K\Omega}(N), \ldots, p_{i1}(N), \ldots\}$ for each integer $m > 0$ except that the points of N are well-ordered instead of S and, so, the elements of $\{W_i\}$ contain only points of N. Note that for each integer $m > 0$, the collection $Q_m(N)$ can be decomposed into countably many discrete collections; i.e., $\{p_{K1}(N), p_{K2}(N), p_{K3}(N), \ldots, p_{K\Omega}(N), \ldots\}$, $\{p_{(K+1)1}(N), p_{(K+1)2}(N), \ldots\}$, $\{p_{(K+2)1}(N), p_{(K+2)2}(N), \ldots\}$, ... Thus, N is strongly M-separable.

The author discovered after proving the following corollary to theorem 4.1 that it had already been proved by Creede [14]. Creede did not use the minimal cover refineability of semi-stratifiable spaces in proving this corollary, though his argument can be modified by the author to establish the minimal cover refineability of semi-stratifiable spaces. Further, his argument can be modified to establish the previous theorem. The author preferred to show his own argument which is related to minimal cover refineability. A similar corollary for semi-metric spaces was proved by McAuley [29]. Grace and Heath noted that a metacompact Moore space has the property that every separable set is hereditarily separable and $\aleph_1$-compact [19].

Corollary 4.2. (Creede) Every $\aleph_1$-compact semi-stratifiable space is hereditarily separable.

Lemma 4.3. (Lubben [27]) Every hereditarily separable space is hereditarily $\aleph_1$-compact.

The following corollary now follows from lemma 4.3, corollaries 3.6, 3.15, and 4.2, and theorems 3.1, 3.8, 3.11, 3.12, and 3.13.

Corollary 4.4. (i) If a space is hereditarily separable and minimal cover refineable, then the space is Lindelöf;

(ii) (Creede [14]) in order for a semi-stratifiable space to be Lindelöf, it is necessary and sufficient that it be hereditarily separable;

(iii) if a space is minimal cover refineable, hereditarily separable, and has the property that closed sets are G_δ, then the space is hereditarily Lindelöf, and thus, hereditarily $\aleph_1$-compact;

(iv) theorem 3.11 with $\aleph_1$-compact replaced by hereditarily separable;

(v) theorem 3.12 with $\aleph_1$-compact replaced by hereditarily separable;

(vi) theorem 3.13 with $\aleph_1$-compact replaced by hereditarily separable;

(vii) if a space is hereditarily separable and hereditarily minimal cover refineable, then the cardinality of the non-condensation points of any subset of the space is less than $\aleph_1$.

Theorem 4.5. Every collectionwise Hausdorff semi-stratifiable space contains a dense screenable semi-stratifiable T_1 subspace.

Proof. Since closed sets are G_δ in a semi-stratifiable space, a semi-stratifiable space is T_1-like. Let S be a T_1 representation of a collectionwise Hausdorff semi-stratifiable space. Let G be a function from the product of the collection of closed subsets of S with the natural numbers into the collection of open sets in S such that (i) $\bigcap_{n=1}^{\infty} G(A, n) = A$ for each closed set A; (ii) $G(A_1, n) \subset G(A_2, n)$ whenever $A_1 \subset A_2$. Let $\{Q_i\}$ be a countable collection of discrete point sets such that $\overline{\{Q_i\}^*} = S$ whose existence was proved in theorem 4.1. For each integer $i > 0$, let O_i be a collection of disjoint open sets which covers Q_i and refines C, an arbitrary open cover of $\{Q_i\}^*$. This demonstrates that $\{Q_i\}^*$ is screenable. Creede [14] has noted that a semi-stratifiable space is hereditarily semi-stratifiable.

Theorem 4.6. Every collectionwise Hausdorff normal semi-stratifiable space contains a dense fully normal semi-stratifiable T_1 subspace.

Proof. Let S be a T_1 representation of a collectionwise Hausdorff normal semi-stratifiable space. Let $\{Q_i\}$ and $\{O_i\}$ be constructed for S in the same manner as the collections of the same name were constructed in the proof of theorem 4.5 (with C, an arbitrary cover of $\{Q_i\}^*$). Let H_i and K_i be disjoint open sets containing Q_i and $S\backslash O_i^*$, respectively. Let $O_i' = \{g\colon g' \in O_i \text{ and } g = g' \cap H_i\}$. The collection $\{O_i'\}$ contains countably many discrete collections of open sets. This demonstrates that $\{Q_i\}^*$ is strongly screenable. A normal T_1 space is regular. As noted in chart 2.11, a strongly screenable regular space is fully normal.

The hypothesis of the following theorem is almost strong enough to give a dense metric subspace.

Theorem 4.7. Every metacompact semi-pseudometric space contains a dense basically metacompact (and, thus, basically minimal cover refineable) developable T_1 subspace with a point-regular basis.

Proof. Let S be a T_1 representation of a metacompact semi-pseudometric space. A T_1 semi-pseudometric space is semi-metric and, thus, semi-stratifiable. Let $\{Q_i\}$ be a countable collection of discrete point sets such that $\overline{\{Q_i\}^*} = S$ whose existence was proved in theorem 4.1. For each integer $i > 0$, let O_i be a point-finite open cover of Q_i which refines the collection $\{g_p : p \in Q_i \text{ and } g_p = S\backslash(Q_i\backslash p)\}$. Let

$O'_1 = O_1$. For each integer $i > 1$, let $O'_i = \{g\colon g' \in O_i \text{ and } g = g' \setminus \bigcup_{j=1}^{i-1} Q_j\}$. Clearly, $\{g\colon i > 0,\ g \in O'_i\}$ is a point-finite cover of $\{Q_i\}^*$. Every subcollection of $\{g\colon i > 0,\ g' \in O'_i,\ p \in Q_i, \text{ and } p \in g \subset g'\}$ covering $\{Q_i\}^*$ contains a point-finite cover of $\{Q_i\}^*$. Then $\{Q_i\}^*$ is basically metacompact. By theorem 2.2, $\{Q_i\}^*$ is basically minimal cover refineable.

It will now be shown that $\{Q_i\}^*$ is developable and has a point-regular basis. Since S is first countable, let $\{h_{pi}\}$ be a countable basis for each point p of S. Let

$O'_{ij} = \{g\colon g' \in O'_i,\ p \in g' \cap Q_i,\ g = g' \cap \bigcap_{k=1}^{j} h_{pk}\}$. The collection $G = \{g\colon i, j > 0,\ g \in O'_{ij}\}$ is a point-countable basis for $\{Q_i\}^*$. Therefore, by Heath [24], $\{Q_i\}^*$ is developable. Further, for each point p of $\{Q_i\}^*$, there is an integer $J > 0$ such that if $j > J$, only one element of the collection $\{g\colon i > 0,\ g \in O'_{ij}\}$ contains p. Therefore, the collection G is a point-regular basis for $\{Q_i\}^*$.

The proof of the following theorem is a combination of the techniques developed to prove theorems 4.5 and 4.7.and is not shown since the argument is so similar to that used for those two theorems.

Theorem 4.8. Every collectionwise Hausdorff semi-pseudometric space contains a dense basically metacompact (and, thus, basically minimal cover refineable) developable T_1 subspace with a point-regular basis.

The following theorem is true simply because a T_1 representation of a pseudometric space is a dense metric subspace.

Theorem 4.9. Every pseudometric space contains a dense metric subspace.

The following theorem has been argued in the proof for theorems 4.7 and 4.8.

Theorem 4.10. If a semi-stratifiable space is either collectionwise Hausdorff or metacompact, it contains a dense basically metacompact (and, thus, basically minimal cover refineable) subspace.

Theorem 4.11. Every collectionwise Hausdorff normal semi-pseudometric space contains a dense metric subspace.

Proof. By the arguments used to establish theorems 4.6 and 4.8, the space contains a dense paracompact Hausdorff subspace with a point-regular basis. By Alexandroff [1], this dense subspace is metric.

Since a stratifiable T_1 space is a paracompact Hausdorff space [11] and, thus, collectionwise normal, the following theorem is true by theorem 4.6 since, by Ceder [11], stratifiability is hereditary.

Theorem 4.12. Every stratifiable space contains a dense fully normal stratifiable T_1 subspace.

Since a Nagata space is a first countable stratifiable space [11], the following theorem is true:

Theorem 4.13. Every Nagata space contains a dense metric subspace.

The following corollary is easily verified after noting only that a regular strongly screenable space is collectionwise normal [7]:

<u>Corollary 4.14</u>. (i) Theorem 4.5 with collectionwise Hausdorff replaced by either screenable or $\aleph_1$-compact;

(ii) theorem 4.6 with collectionwise Hausdorff plus normal replaced by any of the following: strongly collectionwise Hausdorff plus regular, strongly screenable plus regular, screenable plus normal, or $\aleph_1$-compact plus regular;

(iii) theorem 4.7 with metacompact replaced by screenable or $\aleph_1$-compact;

(iv) theorem 4.10 with collectionwise Hausdorff or metacompact replaced by screenable or $\aleph_1$-compact;

(v) theorem 4.11 with collectionwise Hausdorff plus normal replaced by any of the following: strongly collectionwise Hausdorff plus regular, strongly screenable plus regular, screenable plus normal, or $\aleph_1$-compact plus regular.

The following corollary is only stated to note that theorem 4.11 might be useful in investigating the existence of a dense metric subspace of a space which is not semi-pseudometric. The corollary follows from the well-known fact that if M is dense in N, then M is dense in $\overline{N}$.

<u>Corollary 4.15</u>. If a space S contains a dense subspace M with the properties of the hypothesis of one of the theorems 4.5 through 4.14, then M contains a subspace N dense in S which has the properties of the conclusion of the respective theorem.

The following theorem was effectively argued in proving theorem 4.1:

Theorem 4.16. Any dense subset of a semi-stratifiable space S contains a subset dense in the space which is the union of countably many discrete (in S) point sets.

The following theorem may prove useful in investigating the weight of a space. The theorem is an application of theorem 4.1.

Theorem 4.17. If M is an $\aleph_0$-regular cardinal greater than $\aleph_0$ and M is the density of a semi-stratifiable space S, then there is a discrete point set in S with cardinality M.

Theorem 4.18. Every hereditarily separable hereditarily minimal cover refineable regular space has the property that closed sets are G_δ.

Proof. Let M be any closed subset of S, a space with the properties of the hypothesis. Associate an open set C_p with each point $p \in S\backslash M$ such that $\overline{C_p} \subset S\backslash M$. Let C be a minimal cover which is a refinement of the collection $\{C_p : p \in S\backslash M\}$. Then C must be a countable collection and can be placed in a simple sequence, say $\{C_i\}$. Then $M = \bigcap_{i=1}^{\infty} S\backslash\overline{C_i}$.

Corollary 4.19. A necessary and sufficient condition that a hereditarily separable minimal cover refineable space be regular and hereditarily minimal cover refineable is that the space be perfectly normal.

Proof. By theorem 4.18, if a space is hereditarily separable, hereditarily minimal cover refineable, and regular, then it has the property that closed sets are G_δ; by corollary 4.4 (v), the space is fully normal and, therefore, normal. By corollary 4.4 (vi), if a space is hereditarily separable, minimal cover refineable, and has the property that closed sets are G_δ, then the space is hereditarily minimal cover refineable; finally, a normal T_1-like space is regular.

Grace and Heath [19] showed that a separable metacompact Moore space is hereditarily separable; McAuley [28] did the same for semi-stratifiable spaces.

Theorem 4.20. Every separable metacompact semi-stratifiable space is hereditarily separable.

Proof. A separable metacompact space is Lindelöf. By Creede [14], a Lindelöf semi-stratifiable space is hereditarily separable.

Theorem 4.21. There exists a bicompact fully normal Hausdorff space which is not separable.

Proof. Example 3.28 is such a space. This example and its properties are well-known.

Example 4.22. This example is due to McAuley [31]. The space S consists of the points of the plane. Let the function d be a semi-metric on $S \times S$ defined as follows: if p or q lies on the x-axis, let $d(p, q) = |p - q| + \alpha$ where $|p - q|$ is the ordinary Euclidean distance and α is the angle measured in radians between a line through p and q and the y-axis such that $0 \leq \alpha \leq \pi/2$. Otherwise, let $d(p, q)$ be the ordinary Euclidean distance.

The following theorem shows that the property of separability does not imply the properties of hereditary separability nor $\aleph_1$-compactness even in a Moore space. The theorem is well-known.

Theorem 4.23. There exists a separable Moore space which is not $\aleph_1$-compact, metacompact, nor normal.

Proof. Example 4.22 is such a space. McAuley showed that this example was a Moore space. It is well-known that this space possesses the other properties claimed.

The following theorem shows that minimal cover refineability is not necessary in order for a space to contain a dense metric subspace.

Theorem 4.24. There exists a first countable compact T_4 space which contains a dense metric subspace but is not minimal cover refineable.

Proof. Example 2.7, $[0, \Omega)$, is such a space. Since the other properties have already been dealt with, it will only be shown that the space contains a dense metric subspace. Let H consist of all the non-limit points of $[0, \Omega)$. This is a discrete subspace which is dense in S. Since H is discrete, it is metric.

Example 4.25. Let S consist of the points of example 2.7 plus the points x of the real line such that $0 < x \leq 1$. Let basis elements for the points of the real line in S be any open subset of the real line. Let a basis element for the points of $[0, \Omega)$ be the union of a segment of the form (0, x) of the real line where $0 < x \leq 1$ with a basis element of example 2.7. Thus, every open set in S contains a point of the real line.

The following theorem shows that the properties of separability and T_1-ness do not imply the property of minimal cover refineability.

Theorem 4.26. There exists a first countable compact T_1 space which contains a dense separable metric subspace but is not minimal cover refineable.

Proof. Clearly, example 4.25 is first countable, compact, and T_1. Further, the rationals on the real line between 0 and 1 are dense in the space. The space is not minimal cover refineable for the same reason that example 2.7 is not since the real line is hereditarily Lindelöf.

An example of a semi-stratifiable space which is not semi-metric was given in example 3.26. Two more follow.

Example 4.27. Let S consist of one copy of the plane for each countable ordinal. If P_α and P_β are two of these planes in S, let $P_\alpha \cap P_\beta = (0, 0)$. Let basis elements for a point p of S distinct from (0, 0) be any open set in the plane containing p which does not intersect (0, 0). For each countable ordinal α and each integer $i > 0$, let a basis element for the point (0, 0) consist of the point (0, 0) plus the points in open disks of radius $1/i$ with center at (0, 0) in each plane associated with an ordinal greater than α.

Example 4.28. This example was suggested by D. R. Traylor. A variation of the example was given in example 3.26. Let S be the points of a simple sequence of copies of the plane, $\{P_i\}$. For each pair of integers i, j > 0, i ≠ j, let $P_i \cap P_j = (0, 0)$. For each integer i > 0, let basis elements for P_i be any open set in the plane that does not intersect the point (0, 0). Let each basis element for the point (0, 0) be the union of an open disk of the plane P_i with radius $1/J_i$ and center at (0, 0) for each integer i > 0 where J_i is a positive integer.

The following theorem shows that the hypotheses for theorems 4.5, 4.6, 4.10, and 4.12 are not sufficient to imply semi-metricity.

Theorem 4.29. There exists an $\aleph_1$-compact fully normal connected M_1 T_4 space which is not semi-metric.

Example 4.28 is such a space. In theorem 3.27, an argument was given which shows that this space is T_4 and not semi-metric. Clearly, the space is $\aleph_1$-compact and connected. By theorem 3.12, it is fully normal.

It will now be shown that this space is M_1. Let $G_1 = \{g\colon g$ is a basis element for $(0, 0)\}$. Then G_1 is a closure preserving basis for (0, 0). For each pair of integers i, j > 0, let $G_{ij} = \{g\colon g'$ is an open disk in P_j with radius less than $1/i$, $g = g' \setminus (0, 0)\}$. Let O_{ij} be an open disk with radius less than 1/i and center at (0, 0). Then G_{ij} is a collection of open sets which covers $P_j \setminus O_{ij}$; there is a locally finite refinement, say G'_{ij}, which also covers $P_j \setminus O_{ij}$. Then G'_{ij} is closure preserving. For each integer i > 1, let $G_i = \{g\colon j > 0,\ g \in G'_{ij}\}$. Then $\{G_i\}$ is a σ-closure preserving basis for the space.

Example 4.30. This example is due to McAuley [31]. The space S consists of the points of the plane. Let the function d be a semi-metric on S X S defined as follows: if p or q lies on the x-axis, let $d(p, q) = |p - q| + \alpha$ where $|p - q|$ is the ordinary Euclidean distance and α is the angle measured in radians between the line through p and q and the x-axis such that $0 \le \alpha \le \pi/2$. Otherwise, let $d(p, q)$ be the ordinary Euclidean distance.

The following theorem shows that the hypotheses for theorems 4.7, 4.8, 4.11, and 4.13 are not sufficient to imply that a space be metric.

Theorem 4.31. There exists an $\aleph_1$-compact fully normal connected M_1 semi-metric T_4 space which is not developable.

Proof. McAuley [31] and Heath [20] noted that example 4.30 is hereditarily separable, fully normal, connected, M_1, and semi-metric, but not developable. Since the space is T_1, it is T_4. A hereditarily separable space is $\aleph_1$-compact.

CHAPTER V

MINIMAL COVER REFINEABLE SUBSETS

This chapter will be concerned with subsets of minimal cover refineable spaces. The chapter begins with the following obvious theorem:

Theorem 5.1. Every closed subset of the following spaces is minimal cover refineable:

(i) metacompact,

(ii) F_σ-screenable,

(iii) strongly screenable plus T_1-like.

The concept of a dominating collection is due to Michael [32]. Michael showed that a Hausdorff space is paracompact if and only if it is dominated by a collection of paracompact subsets. Borges [8] and Creede [14] have shown the same can be said for stratifiable and semi-stratifiable spaces. The following theorem uses a slightly weaker concept than a dominating collection.

Theorem 5.2. In order for every closed subset of a space to be minimal cover refineable, it is necessary and sufficient that the space be weakly dominated by a collection of subsets such that every proper closed subset of an element of the collection is minimal cover refineable.

Proof. For the reason noted by Michael [32], necessity is obvious. Let G be a collection of subsets which weakly dominates a space S such that every proper closed subset of an element of the collection is minimal cover refineable. Let C' be any open cover of S. Let g be an element of G and c an element of C' which covers a point of g. Then $g\backslash c$ is minimal cover refineable, so g is minimal cover refineable. Let M be any closed subset of S (not excluding S). Let $G' = \{g_\alpha: \alpha > 0, g'_\alpha \in G \text{ and } g_\alpha = g'_\alpha \cap M\}$. Let C be any open cover of M. Let C_1 be a refinement of C which is a minimal cover of g_1, the first element of G'. Let C_2 be a refinement of $\{c: c' \in C \text{ and } c = c'\backslash g_1\}$ which is a minimal cover of $g_2\backslash C_1^*$ where g_2 is the first element of G' not covered by C_1^*. After each initial segment, let g_α be the first element of G' not covered by $\bigcup_{\beta<\alpha} C_\beta^*$ already constructed. Let C_α be a refinement of $\{c: c' \in C \text{ and } c = c'\backslash\bigcup_{\beta<\alpha} g_\beta\}$ which is a minimal cover of $g_\alpha\backslash\bigcup_{\beta<\alpha} C_\beta^*$. Then $\{c: \alpha > 0 \text{ and } c \in C_\alpha\}$ is a minimal cover of M.

Corollary 5.3. Every open subset of a space which has the property that every closed subset is G_δ and minimal cover refineable is minimal cover refineable.

Proof. Let S be a space which has the property that every closed subset is G_δ and minimal cover refineable and let C be any cover of S. Clearly, S is minimal cover refineable. Let R be a proper open subset of S. Since $S\backslash R$ is closed, there is a sequence of open sets $\{G_i\}$ such that $S\backslash R = \bigcap_{i=1}^{\infty} G_i$. Then $\{S\backslash G_i\}$ weakly dominates R, though it may not dominate R.

Theorem 5.4. Every open subset of a perfectly normal minimal cover refineable space is minimal cover refineable.

Proof. The following argument seems unnecessarily complicated. However, the author has been unable to simplify the argument. Let S be a perfectly normal minimal cover refineable space. Let R be a proper open subset of S and C any cover of R. Since S is perfectly normal, let $\{G_i\}$ be a sequence of open sets such that $S\setminus R = \bigcap_{i=1}^{\infty} G_i = \bigcap_{i=1}^{\infty} \overline{G_i}$. Let $C' = \{g\colon g' \in C \text{ and } g = g' \cap R\}$.

Let $C_1 = \{g\colon g \in C' \text{ or } g = G_1\}$. Let C_1' be a refinement of C_1 which is a minimal cover of S. Let $C_1'' = \{g\colon g \in C_1' \text{ and } g \not\subset G_1\}$. Let M_1 be the collection of all points of $(C_1'')^*$ which are not in two elements of C_1'.

Let $C_2 = \{g\colon g' \in C' \text{ and } g = (g' \cap G_1)\setminus M_1,\ g = G_2, \text{ or } g = (C_1'')^*\}$. Let C_2' be a refinement of C_2 which is a minimal cover of S. Let $C_2'' = \{g\colon g \in C_2' \text{ and } g \not\subset (G_2 \cup (C_1'')^*)\}$. Let M_2 be the collection of all points of $(C_2'')^*$ which are not in two elements of C_2'.

For each integer $i > 2$, let $C_i = \{g\colon g' \in C'$ and $g = (g' \cap \bigcap_{j=1}^{i-1} G_j)\setminus \bigcup_{j=1}^{i-1} M_j,\ g = G_i, \text{ or } g = \bigcup_{j=1}^{i-1} (C_j'')^*\}$. Let C_i' be a refinement of C_i which is a minimal cover of S. Let $C_i'' = \{g\colon g \in C_i' \text{ and } g \not\subset (G_i \cup \bigcup_{j=1}^{i-1}(C_j'')^*\}$. Let M_i be the collection of all points of $(C_i'')^*$ which are not in two elements of C_i'.

Let $C_0 = \{g\colon i > 0,\ g' \in C_i'' \text{ and } g = g'\setminus \bigcup_{j=i+1}^{\infty} M_j\}$. By its construction C_0 is a minimal cover of R.

Theorem 5.5. In a space which has the property that closed sets are G_δ, any of the following conditions imply that the space is hereditarily minimal cover refineable:

(i) metacompact,

(ii) F_σ-screenable,

(iii) screenable.

Proof. Heath [23] notes that if a space has the property that closed sets are G_δ and is screenable, then the space is metacompact. Let S be a metacompact space in which closed sets are G_δ. Let M be any subset of S and C any cover of M. If $C^* = S$, there is a point-finite refinement of it, and thus, by the argument in [4], there is a minimal cover of M which refines C. Otherwise, let $S\backslash C^* = \bigcap_{i=1}^{\infty} G_i$ where $\{G_i\}$ is a sequence of open sets. Let C_1 be a point-finite refinement of C which refines $S\backslash G_1$. Once again, there must be a minimal cover, say C_1', of $M\backslash G_1$ which refines C. Let M_1 be the collection of all points of $M\backslash G_1$ not in two elements of C_1'. For each integer $i > 1$, let C_i' be a minimal cover of $M\backslash(G_i \cup \bigcup_{j=1}^{i-1}(C_j')^*)$ which refines C and let M_i be the collection of all points of $M\backslash(G_i \cup \bigcup_{j=1}^{i-1}(C_j')^*)$ not in two elements of C_i'. Then the collection $\{c\colon i > 1,\ c' \in C_i',\ c = c'\backslash\overline{\bigcup_{j=1}^{i-1} M_j},\ \text{or } c \in C_1'\}$ is a minimal cover of M which refines C.

Let S be an F_σ-screenable space which has the property that closed sets are G_δ. Let M be any subset of S and C any cover of M. If $C^* = S$, there is a countable collection, say $\{H_i\}$, of collections of discrete closed sets which refines C. Use the collection $\{\overline{h}\colon i > 0,\ h' \in H_i,\ \text{and } h = h' \cap M\}$ and the method used to prove theorem 2.3 to show a minimal cover of M which refines C. (Note the necessity of restricting ones attention to M since the cover must be a minimal cover of M and not just a minimal cover of the star of the cover.) Otherwise, proceed in a fashion analogous to that used above to show hereditary minimal cover refineability for the case that the space was metacompact.

Since closed sets are G_δ in a semi-stratifiable space and Creede [14] showed that a semi-stratifiable space is F_σ-screenable, the following is a corollary to theorem 5.5.

Corollary 5.6. Every semi-stratifiable space is hereditarily minimal cover refineable.

The following theorem shows that neither metacompact nor perfectly normal are necessary for a space to be hereditarily minimal cover refineable.

Theorem 5.7. There exists a hereditarily minimal cover refineable space which is neither metacompact nor normal.

Proof. Example 4.22 is such a space. The properties of the space have been adequately discussed in the proof of theorem 4.23.

The following theorem shows that full normality is not sufficient to guarantee hereditary minimal cover refineability.

Theorem 5.8. There exists a hypocompact fully normal space which does not have the property that the space is hereditarily minimal cover refineable.

Proof. Example 3.28, $[0, \Omega]$, is such a space since it contains example 2.7, $[0, \Omega)$. The space is easily shown to be hypocompact, T_2, and thus, fully normal.

Example 5.9. C. W. Proctor suggested inserting example 2.7 into another space to obtain an example of a minimal cover refineable space which contains a closed subset that is not minimal cover refineable. This led the author to construct the following space. Let T be the points of $[0, \Omega)$. For each point p of $[0, \Omega)$, let T_p be an independent copy of $[0, \Omega)$ and let T_p have the order topology. For each point p of T, let an open set containing p be the union of all the points between q and r where q, r ε T and $q < p < r$ (for 0, use $[0,r)$) plus the union of all the points greater than some point of T_s for each s ε T such that $q < s < r$. Let $S = \{p\colon p \ \varepsilon\ T \text{ or } q \ \varepsilon\ T \text{ and } p \ \varepsilon\ T_q\}$.

The following theorem shows that minimal cover refineability does not imply that every open or closed set is minimal cover refineable.

Theorem 5.10. There exists a minimal cover refineable space which contains a closed and an open set which are not minimal cover refineable.

Proof. Example 5.9 is such a space. First note that T of the example is not minimal cover refineable and is a closed subset of the space. Then note that for each point p of T, T_p is an open set which is not minimal cover refineable. It need only be shown that S is minimal cover refineable. Let C be any cover of S. Let g_0 be an open set in C containing the point 0 of T. Let p_0 be a point of $T_0 \wedge g_0$ and associate it with 0. Let α be the first point of T not covered by g_0. Let g_α contain α and be an open subset of some element of C such that $p_0 \notin g_\alpha$. Let p_α be a point of $T_\alpha \cap g_\alpha$. After each initial segment, let β be the first point of T not covered by the g_γ already constructed. Let g_β contain β and be an open subset of some element of C such that $p_\gamma \notin g_\beta$ for all p_γ already constructed. The points of S not covered by this construction may now be readily covered to form a minimal cover of S which refines C.

CHAPTER VI

MISCELLANEOUS

This chapter contains material that did not readily fit into one of the previous chapters.

<u>Theorem 6.1.</u> If M is an $\aleph_0$-regular cardinal greater than $\aleph_0$ and M is the weight of a collectionwise Hausdorff developable space, then the density of the space is also M.

Proof. Let S be a developable space, $\{G_i\}$ a sequence which is a development for S, and B a collection of M many open sets which is a basis for S. For each integer $i > 0$, let G_i' be a refinement of G_i which is a minimal cover of S and let P_i be a collection containing one and only one point from each element of G_i' which is in no other element of G_i'. Since $\{G_i'\}$ is also a development of S, the cardinality of $\{g: i > 0, g \in G_i'\}$ is not less than M. Then for some integer $I > 0$, the cardinality of G_I' is not less than M. Let H_I be a collection of disjoint open sets covering P_I such that no element of H_I contains two points of P_I. Then the density of S is not less than the cardinality of H_I nor more than the weight of S.

The following two obvious theorems are stated without proof:

<u>Theorem 6.2.</u> The cardinality of any minimal cover of a space is not more than the weight of the space.

<u>Theorem 6.3.</u> The density of any collectionwise Hausdorff space is not less than the cardinality of any minimal cover of the space.

Theorem 6.4. If the collection $\{G_i\}$ is a development for a space and has the property that for each integer $i > 0$, G_{i+1} is a refinement of G_i and G_i is a point-finite cover of the space, then the space is hereditarily basically metacompact (and, thus, hereditarily basically minimal cover refineable) with respect to the basis $\{g\colon \text{for some integer } i > 0,\ g \in G_i\}$.

Proof. Let $\{G_i\}$ be as described in the hypothesis for a space S. Let M be any subset of S (not excluding S) and C any cover of M. Let $C_1 = \{g\colon g \in G_1,\ g' \in C,\ g \subset g', \text{ and } g \cap M \neq \emptyset\}$. For each integer $i > 1$, let $C_i = \{g\colon g \in G_i,\ g' \in C,\ g \subset g', \text{ and } g \cap M \setminus \bigcup_{j=1}^{i-1} C_j^* \neq \emptyset\}$. Let p be any point of M. There is a least integer $I > 0$ such that $p \in C_I^*$. There is an integer $J > 0$ such that for $i > J$, if $p \in g \in G_i$, then $g \subset C_I^*$. Thus, $\{c\colon i > 0,\ c \in C_i\}$ is a point-finite cover of M.

The following three theorems are easily proved by arguments presented by Briggs [9].

Theorem 6.5. (Briggs) In a first countable T_3 space in which the set of isolated points is discrete, the following are equivalent:

(i) paracompact,

(ii) strong cover compact and minimal cover refineable,

(iii) weak cover compact and minimal cover refineable.

Proof. It need only be shown that (iii) implies (i) since the other implications needed are in Briggs' paper. It is useful to quote Briggs directly with the necessary changes placed in brackets:

"If S is a Lindelöf space, S is paracompact. If not, there is an open cover G of S such that no countable subcollection of G covers S [and such that each isolated point is itself the only open set in G containing it]. It is sufficient to show that such a cover has a locally finite refinement. Let H denote a weak cover compact refinement of G, and let K denote a [minimal cover refinement] of H. Let H be well-ordered by Ω and let D_α denote the collection to which an element k of K belongs if and only if h_α is the first element of H such that $k \subset h_\alpha$. Let O denote the collection to which an element O_α belongs if and only if $O_\alpha = D_\alpha^*$, $D_\alpha^* \neq \emptyset$, $\alpha \in \Omega$. O is a [minimal cover of S which refines G]. Moreover, if O_α and $O_\beta \in O$, $\alpha \neq \beta$, there exist elements h_α and $h_\beta \in H$, $h_\alpha \neq h_\beta$, such that $O_\alpha \subset h_\alpha$, $O_\beta \subset h_\beta$. Hence O is also a weak cover compact refinement of G. [Let $O' = O$.]

"Suppose O' is not locally finite. Then there is an infinite subcollection $\{O_i\}_{i=1}^\infty$ of O', a point p in S, and a sequence $\{p_i\}_{i=1}^\infty$ in S such that (1) $p_i \in O_i$, for each i; (2) $p_i \neq p_j$, $O_i \neq O_j$, for $i \neq j$; and (3) $\{p_i\}_{i=1}^\infty \to p$. [Note that if the isolated points were not discrete, the existence of p with these properties would not be certain.]

"Since O' is minimal, for each i there exists a point q_i in O_i such that $q_i \notin [\bigcup\{g\colon g \in O' \text{ and } g \neq O_i\}]$. Since O' refines G, O' is uncountable. Let $\{O_\beta\}_{\beta \varepsilon B}$ denote an uncountable subcollection of $O' - \{O_i\}_{i=1}^\infty$, and for each $\beta \in B$, let $q_\beta \in [O_\beta \backslash \bigcup\{g\colon g \in O' \text{ and } g \neq O_\beta\}]$. Then the subcollection $\{O_i\}_{i=1}^\infty + \{O_\beta\}_{\beta \varepsilon B}$ and the point sets $P = \{p_i\}_{i=1}^\infty + \{q_\beta\}_{\beta \varepsilon B}$ and $Q = \{q_i\}_{i=1}^\infty + \{q_\beta\}_{\beta \varepsilon B}$ contradict the fact that O is a weak cover compact refinement. Hence O' is locally finite."

Theorems 6.6 and 6.7 are proved by slight modifications to Briggs' arguments similar to the above modification.

Theorem 6.6. (Briggs) In a locally compact T_3 space in which the set of isolated points is discrete, the following are equivalent:

(i) paracompact,

(ii) strong cover compact and minimal cover refineable,

(iii) weak cover compact and minimal cover refineable.

Theorem 6.7. (Briggs) If a strong cover compact T_1 space in which the set of isolated points is discrete is either first countable or locally compact, then the space is paracompact.

The following is a corollary to theorem 6.5:

Corollary 6.8. In a semi-metric T_3 space in which the set of isolated points is discrete, the following are equivalent:

(i) paracompact,

(ii) strong cover compact,

(iii) weak cover compact.

McAuley [28] proved that in a semi-metric Hausdorff space, the following are equivalent: collectionwise normal, hereditarily collectionwise normal, paracompact, hereditarily paracompact. The following theorem extends these results to more general spaces:

Theorem 6.9. In an F_σ-screenable Hausdorff space which has the property that closed sets are G_δ, the following are equivalent:

(i) collectionwise normal,

(ii) hereditarily collectionwise normal,

(iii) paracompact,

(iv) hereditarily paracompact.

Proof. By McAuley [28], in an F_σ-screenable space, (i) implies (iii). By Stone [39], in a Hausdorff space , (iii) implies full normality and, by Bing [7], full normality implies (i). Since the Hausdorff property is hereditary, it need only be shown that (iii) implies (iv). Let S be a paracompact F_σ-screenable Hausdorff space which has the property that closed sets are G_δ. Let M be any subset of S and C any cover of M. Let $\{G_i\}$ be a sequence of open sets such that $S\backslash C^* = \bigcap_{i=1}^{\infty} G_i$. For each integer $I > 0$, let $\{F_{Ij}\}$ be a sequence of collections of discrete closed sets which refine $\{g\colon g \in C \text{ or } g = G_I\}$ such that $\bigcup_{j=1}^{\infty} F_{Ij} = S$. For each pair of integers $i, j > 0$, let $F'_{ij} = \{f\colon f' \in F_{ij} \text{ and } f = f' \cap S\backslash G_i\}$. Let $\{H_i\} = \{F'_{ij}\}$; i.e., $\{H_i\}$ is a resequencing of $\{F'_{ij}\}$. Since S is collectionwise normal, a collection $\{K_i\}$ can be constructed such that for each integer $i > 0$, K_i is a discrete collection of open sets, if $g \in K_i$, g contains one and only one element of H_i, and g is a subset of some element of C. This is sufficient to construct a locally-finite collection of open sets covering M.

Example 6.10. The following example is well-known. Let T be the space $[0, \Omega)$ of example 2.7. For each point α of T, let u_α be a copy of the unit segment (0, 1) of the real line such that if $x \in u_\alpha$, then $\alpha < x < \alpha+1$; if $x, y \in u_\alpha$, they have their natural order. Let S be the points of T plus the points of u_α for each point α of T. Let S have the order topology.

The following theorem uses the previous example to show that minimal cover refineability can not be removed from the theorems concerning weak and strong cover compactness.

Theorem 6.11. There exists a strong cover compact weak cover compact locally compact first countable collectionwise normal Hausdorff space which has no isolated points but is neither minimal cover refineable nor paracompact.

Proof. Example 6.10 is such a space. Clearly, the space has no isolated points. That S is first countable and Hausdorff are well-known. Collectionwise normality is shown in a manner similar to the argument given in theorem 2.8 for $[0, \Omega)$. It is also well-known that the space is compact, so the space is locally compact and both strong and weak cover compact. The argument that this space is not minimal cover refineable is similar to the argument given in proving theorem 2.8. Since the space is not minimal cover refineable, it can not be paracompact.

BIBLIOGRAPHY

1. Alexandroff, P., On some results concerning topological spaces and their continuous mappings, pp. 41-54 in General Topology and Its Relations to Modern Analysis and Algebra, Proceedings of the Prague Topological Symposium, 1961, Academic Press, New York, 1963.

2. Anderson, B. A., Topologies comparable to metric topologies, Topology Conference, Arizona State University, Arizona, 1967, pp. 15-21.

3. Aquaro, G., Point countable open coverings in countably compact spaces, pp. 39-41 in General Topology and Its Relations to Modern Analysis and Algebra II, Proceedings of the Second Prague Topological Symposium, 1966, Academic Press, New York, 1967.

4. Arens, R. and Dugundji, J., Remark on the concept of compactness, Portugaliae Math., 9 (1950), pp. 141-143.

5. Arkhangel'skii, A. V., Mappings and spaces, Russian Math. Surveys, 21 (1966), pp. 87-114.

6. Bennett, H. R., Quasi-developable spaces, Topology Conference, Arizona State University, Tempe, Arizona, 1967, pp. 314-317.

7. Bing, R. H., Metrization of topological spaces, Canad. J. Math., 3 (1951), pp. 175-186.

8. Borges, C. J. R., On stratifiable spaces, Pacific J. Math., 17 (1966), pp. 1-16.

9. Briggs, R. C., A comparison of covering properties in T_3 and T_4 spaces, University of Houston Dissertation (1968).

10. Čech, E., Topological Spaces, John Wiley and Sons, New York, 1966.

11. Čeder, J. G., Some generalizations of metric spaces, Pacific J. Math., 11 (1961), pp. 105-125.

12. Corson, H. H. and Michael, E., Metrizability of certain countable unions, Illinois J. Math., 8 (1964), pp. 351-360.

13. Creede, G. D. D., Semi-stratifiable spaces, Topology Conference, Arizona State University, Tempe, Arizona, 1967, pp. 318-323.

14. ______________, Semi-stratifiable spaces and a factorization of a metrization theorem due to Bing, Arizona State University Dissertation (1968)

15. Dugundji, J., Topology, Allyn and Bacon, Inc., Boston, 1966.

16. Fitzpatrick, B., On dense subspaces of Moore spaces, Proc. Amer. Math. Soc., 16 (1965), pp. 1324-1328.

17. ______________, On dense subspaces of Moore spaces II, Fund. Math., 61 (1967), pp. 91-92.

18. Gál, I. S., On a generalized notion of compactness I, II, Proc. Kon. Ned. Akad. v. Wetens., 60 (A) (1957), pp. 421-435. pp. 603-610.

19. Grace, E. E. and Heath, R. W., Separability and metrizability in pointwise paracompact Moore spaces, Duke Math. J., 31 (1964), pp. 603-610.

20. Heath, R. W., On certain first countable spaces, Topology Seminar, Wisconsin, 1965, Annals of Math. Studies, 60, Princeton University Press, Princeton, 1966, pp. 103-113.

21. ____________, On open mappings and certain spaces satisfying the first countability axiom, Fund. Math., 57 (1965), pp. 91-96.

22. ____________, A paracompact semi-metric space which is not an M_3-space, Proc. Amer. Math. Soc., 17 (1966), pp. 868-870.

23. ____________, Screenability, pointwise paracompactness, and metrization of Moore spaces, Canad. J. Math, 16 (1964), pp. 763-770.

24. ____________, Semi-metric spaces and related spaces, Topology Conference, Arizona State University, Tempe, Arizona, 1967, pp. 153-161.

25. Jones, F. B., Concerning normal and completely normal spaces, Bull. Amer. Math. Soc., 43 (1937), pp. 671-677.

26. Kelley, J. L., General Topology, Van Nostrand, Princeton, 1959.

27. Lubben, R. G., Separabilities of arbitrary orders and related properties, Bull. Amer. Math. Soc., 46 (1940), pp. 913-919.

28. McAuley, L. F., A note on complete collectionwise normality and paracompactness, Proc. Amer. Math. Soc., 9 (1958), pp. 796-799.

29. ______________, A note on naturally ordered sets in semi-metric spaces, Proc. Amer. Math. Soc., 8 (1957), pp. 384-386.

30. ______________, On semi-metric spaces, Summer Institute on Set-Theoretic Topology, Madison, Amer. Math. Soc., 1955, pp. 60-64.

31. ______________, A relation between perfect separability, completeness, and normality in semi-metric spaces, Pacific J. Math., 6 (1956), pp. 315-326.

32. Michael, E., Continuous selections I, Ann. of Math., 63 (1956), pp. 361-382.

33. ______________, A note on paracompact spaces, Proc. Amer. Math. Soc., 4 (1953), pp. 831-838.

34. Morita, K., Star-finite coverings and the star-finite property, Math. Japonicae, 1(1948), pp. 60-68.

35. Nagata, J., A contribution to the theory of metrization, J. Inst. Polytech., Osaka City University, 8 (1957), pp. 185-192.

36. Proctor, C. W., Metrizability in Moore spaces, University of Houston Dissertation (1969)

37. Rudin, M. E. E., Concerning abstract spaces, Duke Math. J., 17 (1950), pp. 317-327.

38. Souslin, M., Problem 3, Fund. Math., 1(1920), p. 223.

3 . Stone, A. H., Cardinals of closed sets, Mathematika, 6 (1959), pp. 99-107.

. ______________, Paracompactness and product spaces, Bull. Amer. Math. Soc., 54 (1948), pp. 977-982.

4 . Tall, F. D., Set-theoretic consistency results and topological theorems concerning the normal Moore space conjecture and related problems, University of Wisconsin Dissertation (1969).

42. Traylor, D. R., Concerning the Souslin problem, Auburn University Master's Thesis (1960).

4 . Younglove, J. N., Concerning dense metric subspaces of certain non-metric spaces, Fund. Math., 48 (1959), pp. 15-25.