MODELLING CONTROL SYSTEM WITH INDUSTRIAL SPECIFICATIONS

A Thesis

Presented To

the Faculty of the Department of Electrical

Engineering

University of Houston

In Partial Fulfillment

of the Requirements for the Degree

Master of Science in Electrical Engineering

---

by

Hsi-Zen Chow

April, 1976

Acknowledgement

The author wishes to express his sincere gratitude to his advisor, Dr. L. S. Shieh , for his advice and encouragement, and Dr. W. P. Schneider and Dr. R. D. Sinkhorn for serving on the committee.

Specific thanks are due to Mr. and Mrs. Pine, who reviewed the manuscripts, and Mrs. Mary Avant, who typed the whole thesis.

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ABSTRACT

In the synthesis of control systems, first the design goals are often specified. Then appropriate compensators are chosen to achieve the design goals. In order to design a control system more effectively, the practicing engineers are quite often required to construct a standard mathematical model by using industrial specifications which are assigned. In this thesis, a new method is presented to construct the standard mathematical model for single variable and multivariable systems.

The steps of research are listed as follows: First, a second-order model with a phase advance factor is established to investigate the relationship between the time domain specifications and frequency domain specifications. Next, an original synthesis method is established to construct a high order standard model by using industrial specifications. Two elegant methods are derived to improve the convergence of the Newton-Raphson multidimensional method. Finally, a method is presented to formulate a multivariable control system in the frequency domain and in the time domain by using industrial specifications.

TABLE OF CONTENTS

CHAPTER I

INTRODUCTION

In the synthesis of control systems, the design goals are often specified first. Then, appropriate compensators are chosen to achieve the goals. A set of industrial specifications suggested by Gibson and Rekasius[1] has been used for the design goals of the single-input-single-output systems. These specifications can be classified into two major groups, one is the specifications defined in the frequency domain; the other is the specifications defined in the time domain. The systematic study of the relationships between these two kinds of specifications has been done by several researchers. Seshadri[2] has found empirical relations between them by using analog simulation of a large number of systems. The simplest summary and clearest presentation of these rules were contributed by Axelby[3] which are shown as follows

1. $$M_t = M_p = \frac{1}{\sin \phi_m} \qquad (1.1)$$

   M_t: maximum value of unit step response.

   M_p: maximum value of the closed loop frequency response

   ϕ_m: phase margin

2. $$M_e = \frac{1}{\omega_c} \qquad (1.2)$$

   M_e: maximum value of the error of the unit ramp function

ω_c: cross over frequency

3. $\omega_p = \omega_c$ (1.3)

ω_p: the frequency when the maximum frequency response occurs

4. $\dot{M}_t = \omega_c$ (1.4)

$\dot{M}_t$: maximum value of the unit impulse response

5. $t_p = \frac{3}{\omega_c}$ (1.5)

t_p: the time when the maximum value of the unit step response occurs

6. $t_v = \frac{1.8}{\omega_c}$ (1.6)

t_v: the time when the maximum error of the ramp function with respect to it's input occurs

7. $t_c = \frac{1}{\omega_c}$ (1.7)

t_c: the time when the maximum value $\dot{M}_t$ occurs

Concerning Axelby's rules, Park[4] did a theoretical study by using the well-known second order model. The transfer function is

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} \qquad (1.8)$$

where

ξ: Damping Ratio in the complex domain

ω_n: System natural angular frequency

ξ and ω are the specifications used to specify transient response. Obviously the steady-state characteristic is ignored in this second order model. So Shieh and Huang[5] established a second order model with a phase-advance factor. The transfer function is

$$\frac{C(s)}{R(s)} = \frac{\tau\omega_n s + \omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} \tag{1.9}$$

In order to synthesize a control system which meets the design goals, Chen and Shieh[6] developed an original synthesis technique to determine the transfer function. However their methods deal only with a low order single-input-single-output system. The extended research should be done for the high order systems and the multivariable systems.

This thesis involves the following steps:

Chapter II deals mainly with Axelby's rules by using the new second order model with a phase-advance factor. An example will be shown to compare the specifications of a high order system with its simplified second order model.

Chapter III gives a method to construct a standard multivariable model in the frequency domain. The corresponding state equations can be obtained by an algebraic method which used matrix continued fraction. The processes of constructing a standard model are described as follows.

First, a multivariable system with various numbers of input and output is viewed as a composite system of single-input-single-output subsystems in the s plane, and the least common denominator of a multivariable system is constructed from the eigenvalues assigned in the s plane.

Second, the numerator polynomial of each subsystem is determined by using the basic definitions of the industrial specifications, and the coefficients of the common denominator assigned. Then, the properties of the composite system, which are formed from each subsystem, are examined.

Finally, the corresponding state-space equations which are the minimal realizations of the standard multivariable model are constructed by means of the matrix continued fraction.[8] An example will be used to demonstrate these methods.

The last chapter will be the summary of this research.

CHAPTER II

THE RELATIONSHIP BETWEEN TIME DOMAIN AND FREQUENCY DOMAIN SPECIFICATIONS

2.1 Introduction

The primary purpose of this chapter is to derive a number of the relationships between the time-domain specifications and the frequency-domain specifications by using the new second order model presented in Chapter I and Chapter III, which is written as follows:

$$\frac{C(s)}{R(s)} = \frac{\tau\omega_n s + \omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} \tag{2.1}$$

where ξ = damping ratio.

ω_n = natural angular frequency

By using the new model shown in Eq.(2.1) and following the basic definition of each control specification, we derive the mathematical expression for each specification. A set of curves will be plotted in the figures to show the relationship among the specifications. Based on the curves plotted, we will discuss and verify Axelby's rules.

2.2 Maximum Value of Unit Step Response, Frequency Response, and Phase Margin

(1) Derivation of the relationship among M_t, ξ, and τ. Based on the feedback system shown in Fig. 2.1.

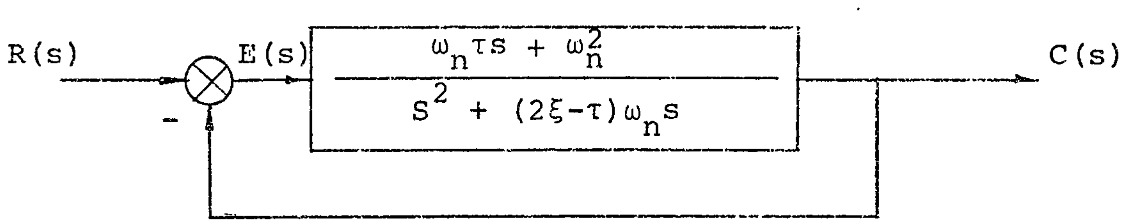


Fig. 2.1

we derive the relationship among M_t, ξ, and τ. By applying a unit step signal to the system, or letting $r(t) = 1$, we can calculate the output of the system $c_1(s)$, in the frequency domain which is written as follows:

$$c(s) = \frac{\tau\omega_n s + \omega_n^2}{s(s^2 + 2\xi\omega_n s + \omega_n^2)} \tag{2.2}$$

Using the Heaviside expansion we obtain the following equation

$$c(s) = \frac{k_1}{s} + \frac{k_2}{s + \xi\omega_n - j\omega_n\sqrt{1-\xi^2}} + \frac{k_3}{s + \xi\omega_n + j\omega_n\sqrt{1-\xi^2}} \tag{2.3}$$

where

$$k_1 = c(s) \times s\big|_{s=0} = 1$$

$$k_2 = c(s) \times (s + \xi\omega_n - j\omega_n\sqrt{1-\xi^2})\big|_{s = -\xi\omega_n + j\omega_n\sqrt{1-\xi^2}}$$

$$= \frac{\sqrt{1-\xi^2} + j(\tau-\xi)}{-2\sqrt{1-\xi^2}}$$

$$k_3 = k_2^* = \frac{\sqrt{1-\xi^2} - j(\tau-\xi)}{-2\sqrt{1-\xi^2}}$$

The inverse Laplace transformation of $c(s)$, Eq.(2.3), is as follows:

$$c(t) = 1 - e^{-\xi\omega_n t}\left[\cos \omega_n\sqrt{1-\xi^2}\, t + \frac{\xi - \tau}{\sqrt{1-\xi^2}}\sin \omega_n\sqrt{1-\xi^2}\, t\right] \tag{2.4}$$

In order to find t_p, we differentiate C(t), Eq.(2.4), obtaining Eq.(2.5)

$$\frac{dC(t)}{dt} = \tau\omega_n\ e^{-\xi\omega_n t}\cos\omega_n\sqrt{1-\xi^2}\,t + \frac{\omega_n(1-\tau\xi)}{\sqrt{1-\xi^2}}\ e^{-\xi\omega_n t}\ \sin\omega_n\sqrt{1-\xi^2}\,t \tag{2.5}$$

Setting Eq.(2.5) equal to zero and simplifying, we have

$$t_p = \frac{\Pi + \tan^{-1}\dfrac{\tau\sqrt{1-\xi^2}}{\tau\xi - 1}}{\omega_n\sqrt{1-\xi^2}} \tag{2.6}$$

Substituting Eq.(2.6) into Eq.(2.4), we finally obtain the maximum value of unit step response

$$M_t = 1-e^{-\xi\omega_n t_p}[\cos\omega_n\sqrt{1-\xi^2}\,t_p - \frac{\tau - \xi}{\sqrt{1-\xi^2}}\sin\omega_n\sqrt{1-\xi^2}\,t_p] \tag{2.7}$$

Eq.(2.7) shows the relationship of M_t, ξ, and τ.

(2) Derivation of the relationship among M_p, ξ , and τ. By definition, M_p means the maximum value of the closed loop frequency response. In order to find the relationship among M_p, ξ, and τ, we must follow Higgins and Siegel's[18] technique which uses complex variable differentiation. The closed-loop frequency response is expressed as follows:

$$M(j\omega) = |M(j\omega)|e^{j\phi(\omega)} = M(\omega)e^{j\phi(\omega)} \tag{2.8}$$

where $M(\omega)$and $\phi(\omega)$ denote the magnitude and the phase of the frequency response respectively. The derivative of $M(j\omega)$ with respect to ω is:

$$\frac{dM(j\omega)}{d\omega} = \frac{dM(\omega)}{\omega} e^{j\phi(\omega)} + jM(\omega)e^{j\phi(\omega)}\frac{d\phi(\omega)}{d\omega} \tag{2.9}$$

Dividing each of the terms of Eq.(2.9) by $M(j\omega) = M(\omega)\ e^{j\phi(\omega)}$ yields

$$\frac{1}{M(j\omega)} \frac{dM(j\omega)}{d\omega} = \frac{1}{M(\omega)} \frac{dM(\omega)}{\omega} + j\frac{d\phi(\omega)}{d\omega} . \tag{2.10}$$

From Eq.(2.10), we obtain

$$Re[\frac{1}{M(j\omega)} \frac{dM(j\omega)}{d\omega}] = \frac{1}{M(\omega)} \frac{dM(\omega)}{d\omega} \tag{2.11}$$

and

$$Im[\frac{1}{M(j\omega)} \frac{dM(j\omega)}{d\omega}] = \frac{d\phi(\omega)}{d\omega} . \tag{2.12}$$

From Eq.(2.11) we observe that if the left hand side of the equation vanishes for a particular value of $\omega = \omega_p$, then so does part of the right hand side. According to the definition of M_p and ω_p, we have the equation as follows:

$$|\frac{dM(\omega)}{d\omega}|_{\omega=\omega_p} = 0 \tag{2.13}$$

Comparing Eqs.(2.13) and (2.11) yields

$$R_e[j\frac{1}{M(s)} \cdot \frac{dM(s)}{ds}]_{s=j\omega} = 0 \tag{2.14}$$

The closed-loop transfer function M(s) is expressed by the ratio of two polynomials, which is shown as follows:

$$M(s) = \frac{A(s)}{B(s)} \tag{2.15}$$

Taking the derivative of Eq.(2.15) we have

$$\frac{dM(s)}{ds} = \frac{1}{B(s)} \frac{dA(s)}{ds} - \frac{A(s)}{B^2(s)} \frac{dB(s)}{ds} \tag{2.16}$$

Substituting Eq.(2.15) and Eq.(2.16) into Eq.(2.14), we have

$$R_e\{j[\frac{1}{A(s)}\frac{dA(s)}{ds} - \frac{1}{B(s)}\frac{dB(s)}{ds}]\}_{s=j\omega} = 0 \qquad (2.17)$$

If we apply Eq.(2.17) to the second-order model by making the following substitutions

$$A(s) = \tau\omega_n s + \omega_n^2$$

$$B(s) = s^2 + 2\xi\omega_n s + \omega_n^2$$

$$\frac{dA(s)}{ds} = \tau\omega_n$$

$$\frac{dB(s)}{ds} = 2s + 2\xi\omega_n$$

then M_p and ω_p are found as follows:

$$\left.\begin{aligned}\omega_p &= \omega_n\sqrt{1-2\xi^2}\\ M_p &= \frac{1}{2\xi\sqrt{1-\xi^2}}\end{aligned}\right\} \text{ when } \tau = 0 \qquad (2.18)$$

$$\omega_p = \frac{\omega_n}{\tau}(-1 + \sqrt{(\tau^2+1)^2 - 4\tau^2\xi^2})^{1/2}$$

when $\tau \neq 0$ (2.19)

$$M_p = \tau^2/\sqrt{2}\left(\frac{1}{\sqrt{(\tau^2+1)^2+4\xi^2\tau^2} - [(\tau^2+1)-2\xi^2\tau^2]}\right)^{1/2}$$

(3) Derivation of the relationship among ϕ_m, ξ and τ. Since the closed-loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{\tau\omega_n s + \omega_n^2}{s^2 + 2\xi\omega_n\tau + \omega_n^2} \qquad (2.20)$$

By definition, we can express the open-loop transfer function as follows:

$$G(s) = \frac{\omega_n \tau s + \omega_n^2}{s^2 + (2\xi-\tau)\,\omega_n s} \tag{2.21}$$

The phase margin ϕ_m is defined as follows

$$\phi_m = \underline{/G(s)}\,|_{s=j\omega_c} + \Pi$$

where ω_c is called the gain crossover frequency, or

$$|G(s)|_{s=j\omega_c} = \left|\frac{\tau\omega_n s + \omega_n^2}{s^2+(2\xi-\tau)\omega_n s}\right|_{s=j\omega_c} = 1 \tag{2.22}$$

By squaring Eq.(2.22) and performing the multiplication, we have

$$\omega_c^4 + (4\xi^2-4\xi\tau)\omega_n^2\,\omega_c^2 - \omega_n^4 = 0 \tag{2.23}$$

Solving Eq.(2.23) yields

$$\left(\frac{\omega_c}{\omega_n}\right)^2 = -(2\xi^2-2\xi\tau)+\sqrt{(2\xi^2-2\xi\tau)^2+1} \tag{2.24}$$

Therefore the phase margin is

$$\phi_m = \tan^{-1}\left\{\frac{1}{1-2\xi\tau+\tau^2}\left[(2\xi-\tau)\left(\frac{1}{\sqrt{(2\xi^2-2\xi\tau)^2+1}-(2\xi^2-2\xi\tau)}\right)^{1/2}\right.\right.$$

$$\left.\left. + \tau\left(\sqrt{(2\xi^2-2\xi\tau)^2+1} - (2\xi^2-2\xi\tau)\right)^{1/2}\right]\right\} \tag{2.25}$$

(4) Summary of the results which were obtained in sections 2.2-(1), 2.2-(2), 2.2-(3). Based on the foregoing derivation, several plots can be presented shown in Fig. 2.2, Fig. 2.3, Fig. 2.4, Fig. 2.5, and Fig. 2.6. The coordinates of these plots are M_t, M_p, ϕ_m versus the damping ratio, ξ, and the parameter τ.

Axelby suggested that the relationship of M_t, M_p and ϕ_m

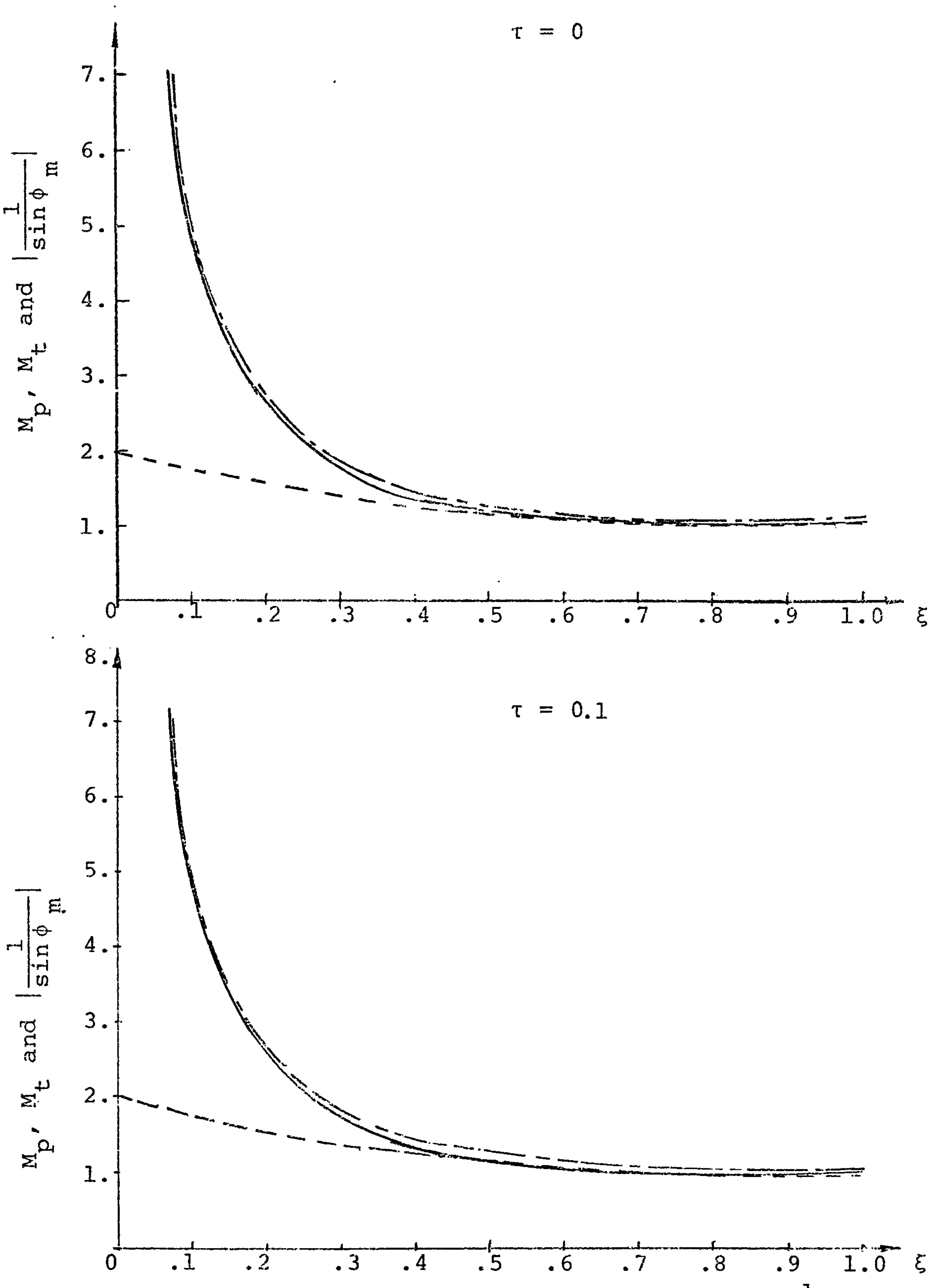


Fig. 2.2 Relationship among M_p, M_t and $\left|\frac{1}{\sin\phi_m}\right|$

——— M_p ······ M_t ——— · ——— $\left|\frac{1}{\sin\phi_m}\right|$

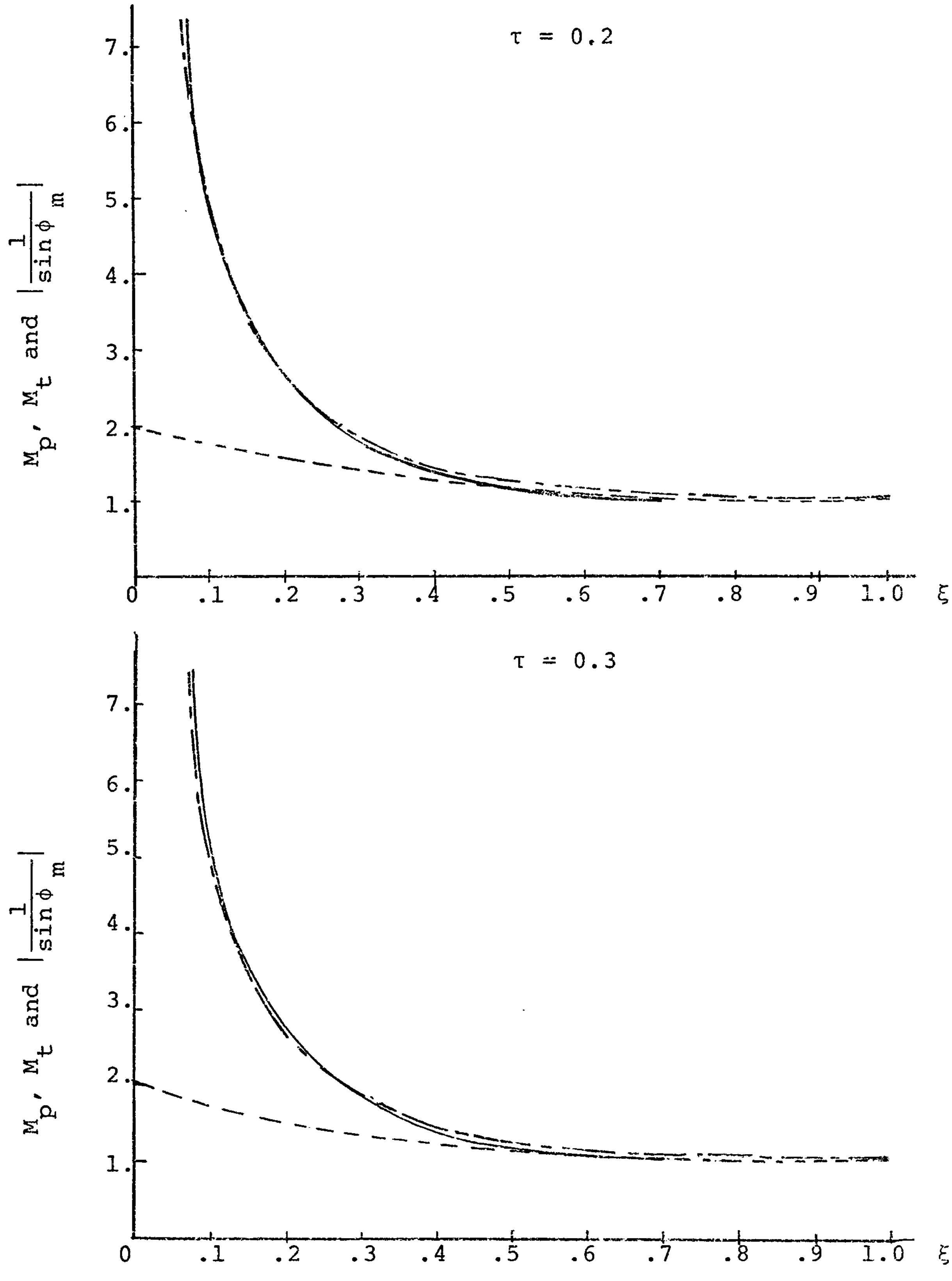


Fig. 2.3 Relationship among M_p, M_t and $\left|\frac{1}{\sin\phi_m}\right|$

———— M_p ...... M_t ——— · ——— $\left|\frac{1}{\sin\phi_m}\right|$

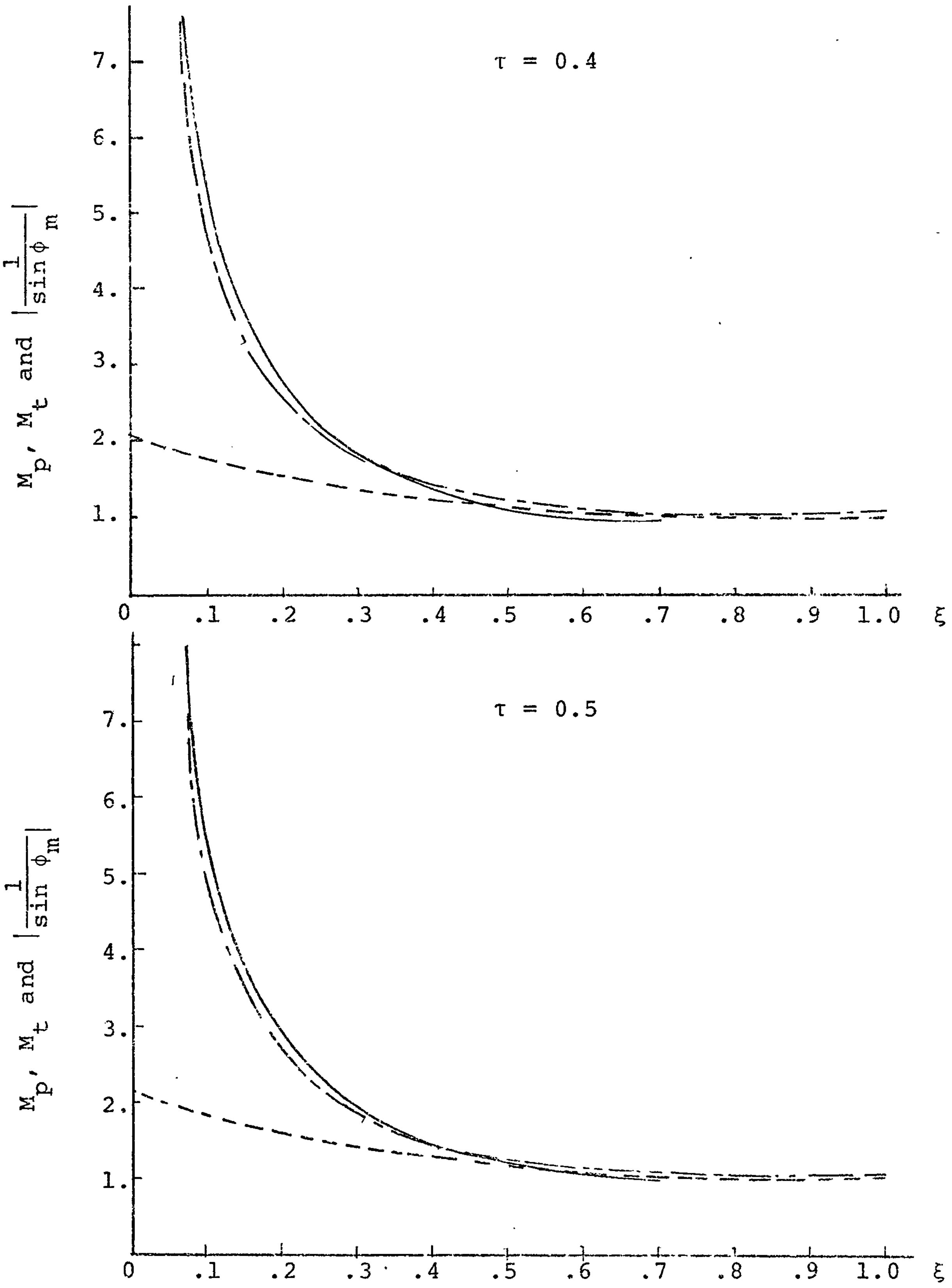


Fig. 2.4 Relationship among M_p, M_t and $|\frac{1}{\sin\phi_m}|$

——— M_p ...... M_t ——— - ———. $|\frac{1}{\sin\phi_m}|$

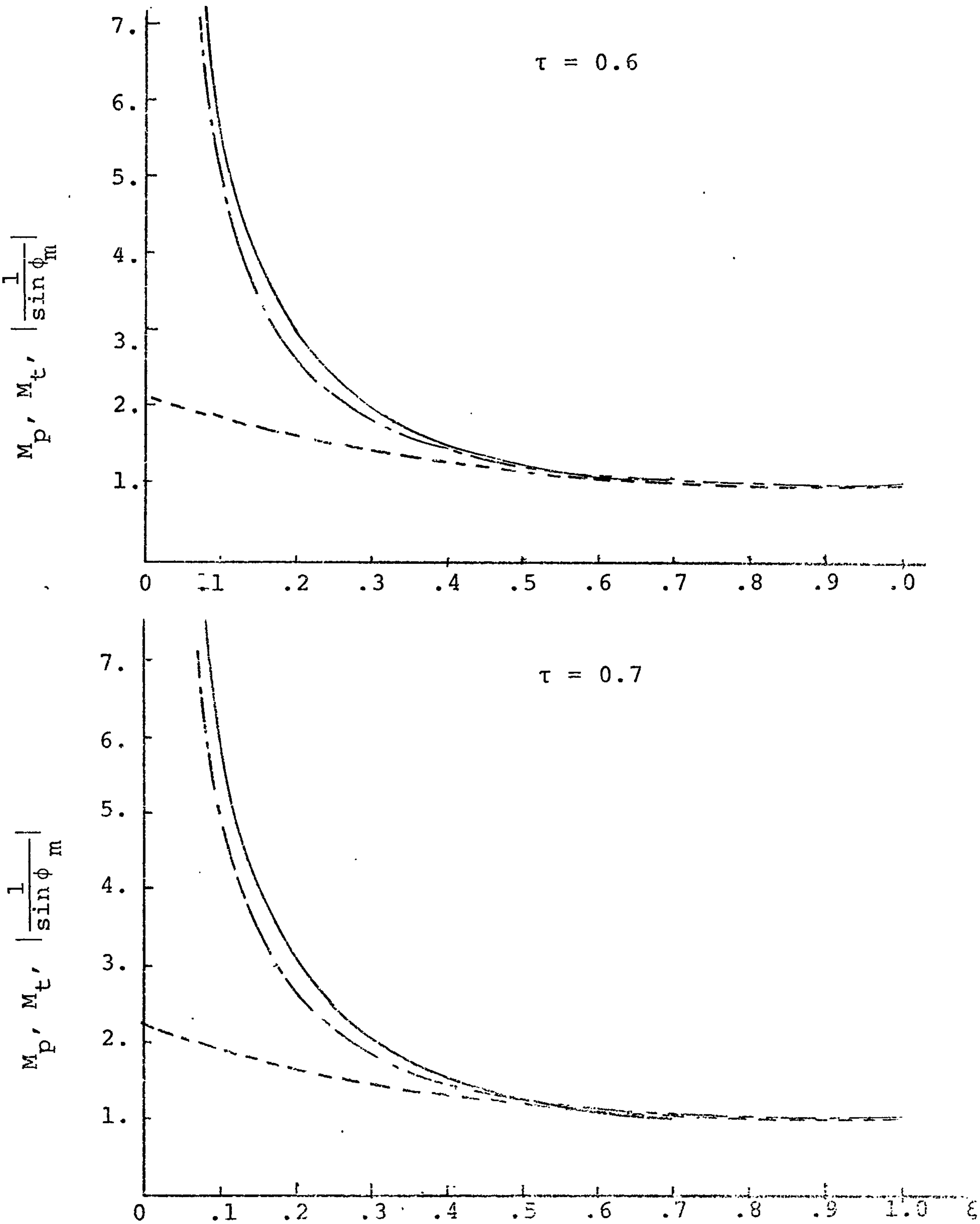


Fig. 2.5 Relationship among M_p, M_t and $\left|\frac{1}{\sin\phi_m}\right|$

——— M_p ...... M_t ——·——·——· $\left|\frac{1}{\sin\phi_m}\right|$

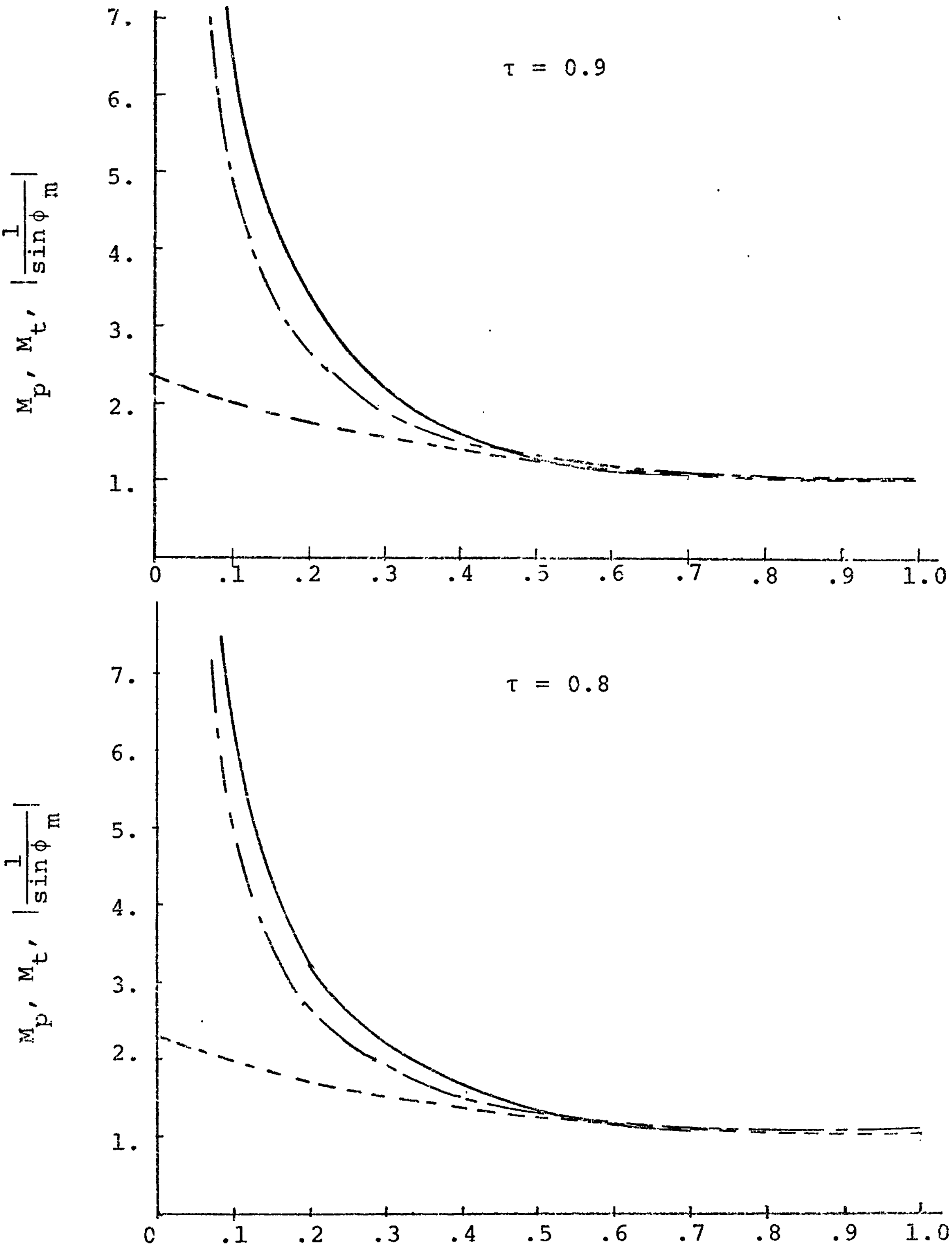


Fig. 2.6 Relationship among M_p, M_t and $\left|\frac{1}{\sin\phi_m}\right|$

———— M_p ······· M_t ——·——·—— $\left|\frac{1}{\sin\phi_m}\right|$

can be expressed by the following equation

$$M_t = M_p = \frac{1}{\sin \phi_m} \tag{2.26}$$

Inspecting the plots which we have, we found that Axelby's rule holds only if the damping ratio is larger than 0.4 and for any τ.

2.3 Crossover Frequency and Maximum Error Signal

(1) Derivation of the relationship among M_e, ξ and τ. If a ramp input is applied to the second-order system shown in Fig. 2.7, the Laplace transform of the output function C(s) can be written as follows:

$$C(s) = \frac{\omega_n \tau s + \omega_n^2}{s^2(S^2 + 2\xi\omega_n s + \omega_n^2)} \tag{2.27}$$

$$= \frac{A'}{s^2} + \frac{B'}{s} + \frac{C'}{s+\xi\omega_n - j\omega_n\sqrt{1-\xi^2}} + \frac{D'}{s+\xi\omega_n + j\omega_n\sqrt{1-\xi^2}}$$

where $A' = C(s) \times s^2|_{s=0} = 1$

$$B' = \frac{d}{ds}(C(s) \times s^2)|_{s=0} = \frac{\tau - 2\xi}{\omega_n}$$

$$C' = C(s) \times (s+\xi\omega_n - j\omega_n\sqrt{1-\xi^2})|_{s=-\xi\omega_n + j\omega_n\sqrt{1-\xi^2}}$$

$$= \frac{(2\xi-\tau)(1-\xi^2)+j(1-2\xi^2+\tau\xi)\sqrt{1-\xi^2}}{2\omega_n(1-\xi^2)} = \frac{B+jC}{2\omega_n A}$$

where $B = (2\xi-\tau)(1-\xi^2)$, $A=(1-\xi^2)$, $C=\sqrt{1-\xi^2}\,(1-2\xi^2+\tau\xi)$

$$D' = C'^*$$

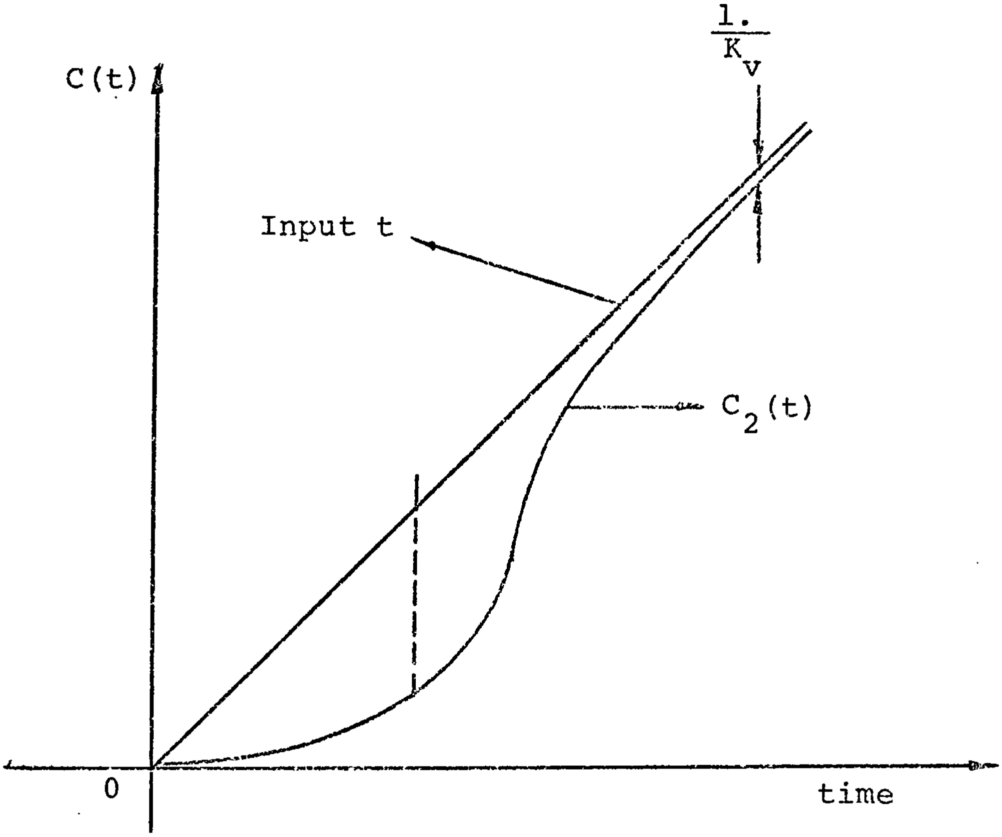


Fig. 2-7. Unit Ramp Response

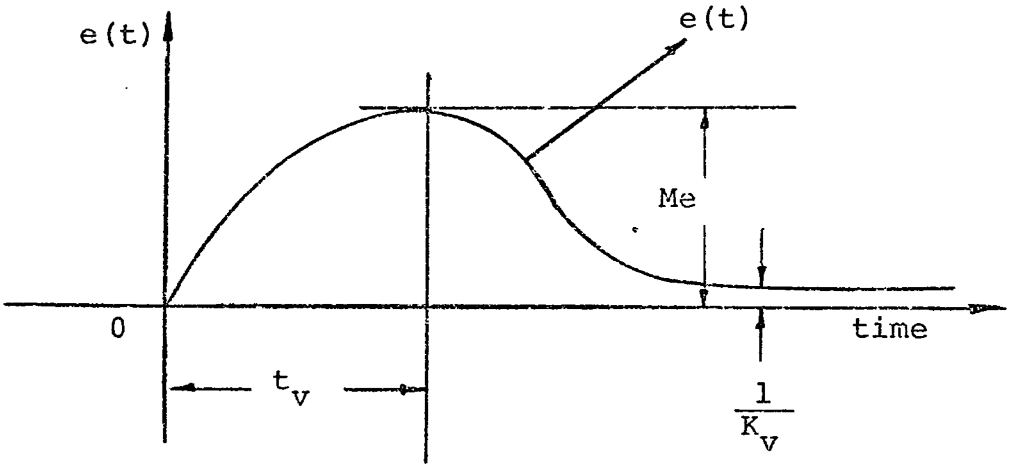


Fig. 2-8. Error Signal

Taking the inverse Laplace transformation of Eq.(2.27) yields

$$C(t) = t + (\frac{\tau-2\xi}{\omega_n}) + \frac{1}{\omega_n A} e^{-\xi\omega_n t}(B \cos\sqrt{1-\xi^2}\,\omega_n t$$

$$+ C \sin\sqrt{1-\xi^2}\,\omega_n t) \qquad (2.28)$$

The error signal e(t) is defined as follows:

$$e(t) = r(t) - C(t) \qquad (2.29)$$

Substituting Eq.(2.28) into Eq.(2.29), the result is:

$$e(t) = \frac{2\xi-\tau}{\omega_n} - \frac{1}{A\omega_n} e^{-\xi\omega_n t}(B \cos\sqrt{1-\xi^2}\,\omega_n t + C \sin\sqrt{1-\xi^2}\,\omega_n t) \qquad (2.30)$$

The error signal e(t) is shown in Fig. 2.8.

Differentiating the error signal with respect to time, we have

$$e'(t) = \frac{1}{A\omega_n} e^{-\xi\omega_n t}[\cos\sqrt{1-\xi^2}\,\omega_n t(\xi\omega_n B - C\sqrt{1-\xi^2}\,\omega_n)$$

$$+ \sin\sqrt{1-\xi^2}\,\omega_n t(C\xi\omega_n + B\sqrt{1-\xi^2}\,\omega_n)] \qquad (2.31)$$

Setting $e'(t) = 0$, we obtain

$$t_v = \frac{\tan^{-1}\dfrac{\xi B - C\sqrt{1-\xi^2}}{-B\sqrt{1-\xi^2} - \xi C}}{\omega_n\sqrt{1-\xi^2}} \qquad (2.32)$$

where t_v is the time at which the maximum value, M_e, occurs. Substituting Eq.(2.32) into Eq.(2.30), yields

$$M_e = \frac{2\xi-\tau}{\omega_n} - \frac{1}{A\omega_n} e^{-\xi\omega_n t_v}[B \cos\sqrt{1-\xi^2}\,\omega_n t_v + C \sin\sqrt{1-\xi^2}\,\omega_n t_v] \qquad (2.33)$$

(2) Derivation of the relationship among ω_c, ξ and τ. By definition, ω_c is the gain crossover frequency of the open-loop frequency response, on other words, the following relation holds when $\omega = \omega_c$.

$$\left|\frac{\omega_n \tau s + \omega_n^2}{s^2+(2\xi-\tau)\omega_n s}\right|_{s=j\omega_c} = 1 \tag{2.34}$$

From Eq.(2.34), ω_c is obtained as follows:

$$\omega_c = \omega_n\left[\sqrt{(2\xi^2 - 2\xi\tau)^2 + 1} - (2\xi^2 - 2\xi\tau)\right]^{1/2} \tag{2.35}$$

(3) Summary of the results which are obtained from sections 2.3-(1), 2.3-(2). Based on the previous derivations, we can present the plots shown in Fig. 2.9, Fig. 2.10, Fig. 2.11, Fig. 2.12, and Fig. 2.13. On the basis of the criterion #2 of Axelby's rule, we can obtain the following equation

$$M_e = \frac{1}{\omega_c} \tag{2.36}$$

Comparing the Eq.(2.36) with the plots, we found that Axelby's criterion #2 can hold only when τ is very small, in other words, there is a factor which is affected by τ which needs to be added when the zero of the second-order system occurs.

2.4 Peak Frequency ω_p and Gain Crossover Frequency ω_c

(1) Relation between ω_p and ω_c. Based on section 2.2

$$\omega_p = \omega_n\sqrt{1-\xi^2} \qquad \text{when } \tau = 0$$

$$\omega_p = \frac{\omega_n}{\tau}\left(-1+\sqrt{(\tau^2+1)^2-4\tau^2\xi^2}\right)^{1/2} \qquad \text{when } \tau \neq 0 \tag{2.37}$$

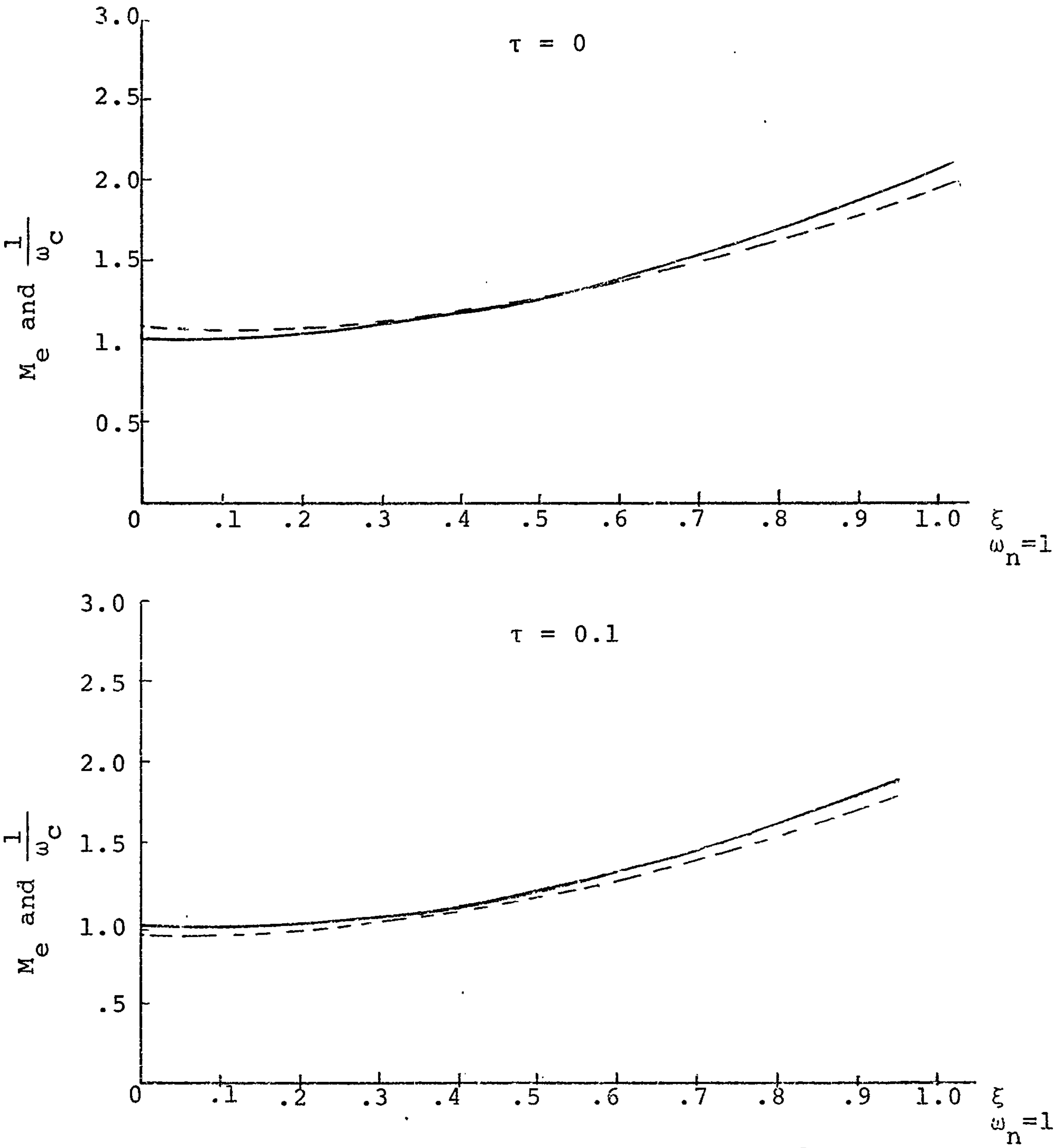


Fig. 2.9 Relationship between M_e and $\frac{1}{\omega_c}$

———— $\frac{1}{\omega_c}$ — — — — — . M_e

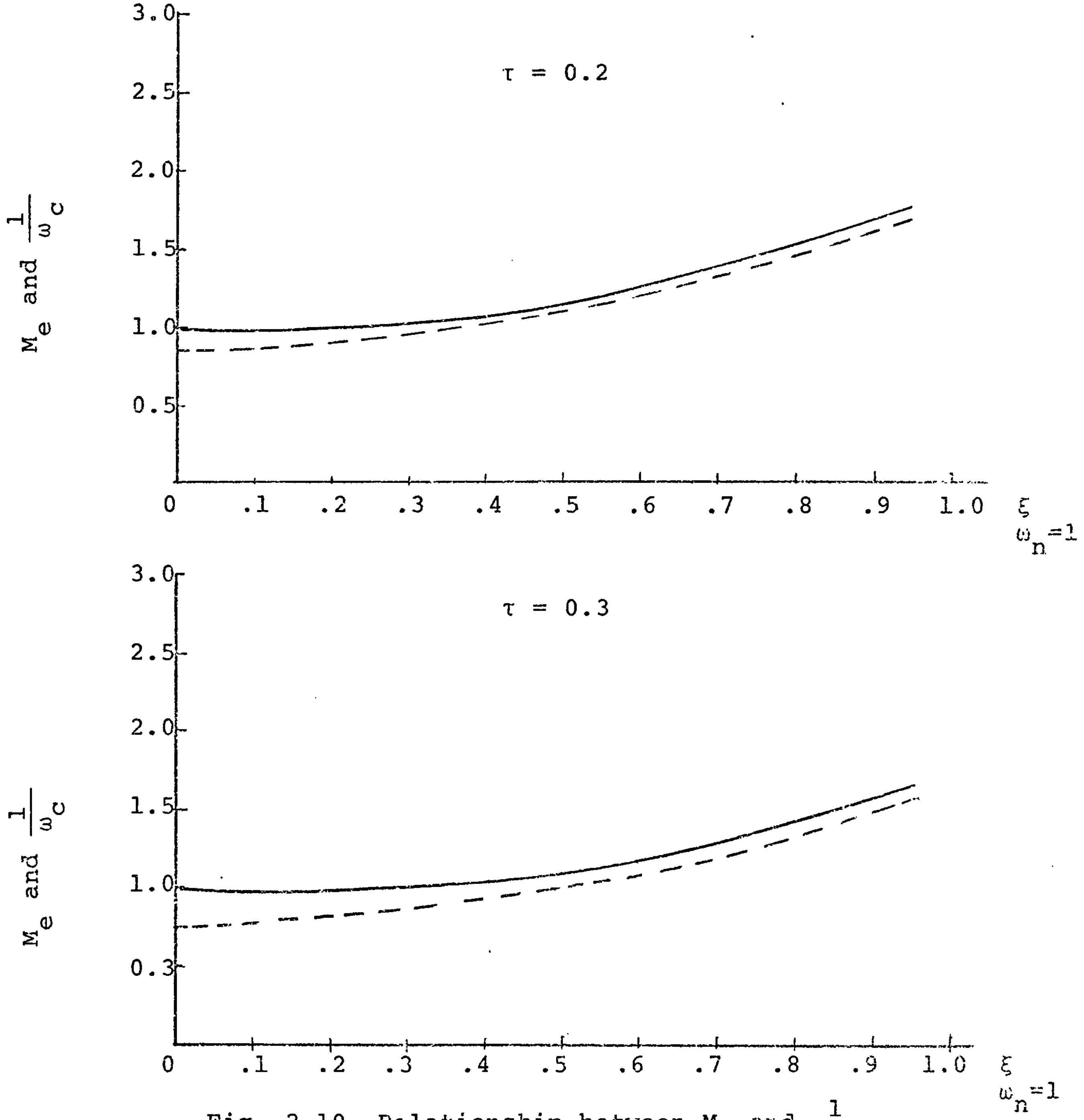


Fig. 2.10 Relationship between M_e and $\frac{1}{\omega_c}$

———— $\frac{1}{\omega_c}$ — — — — — — — — — M_e

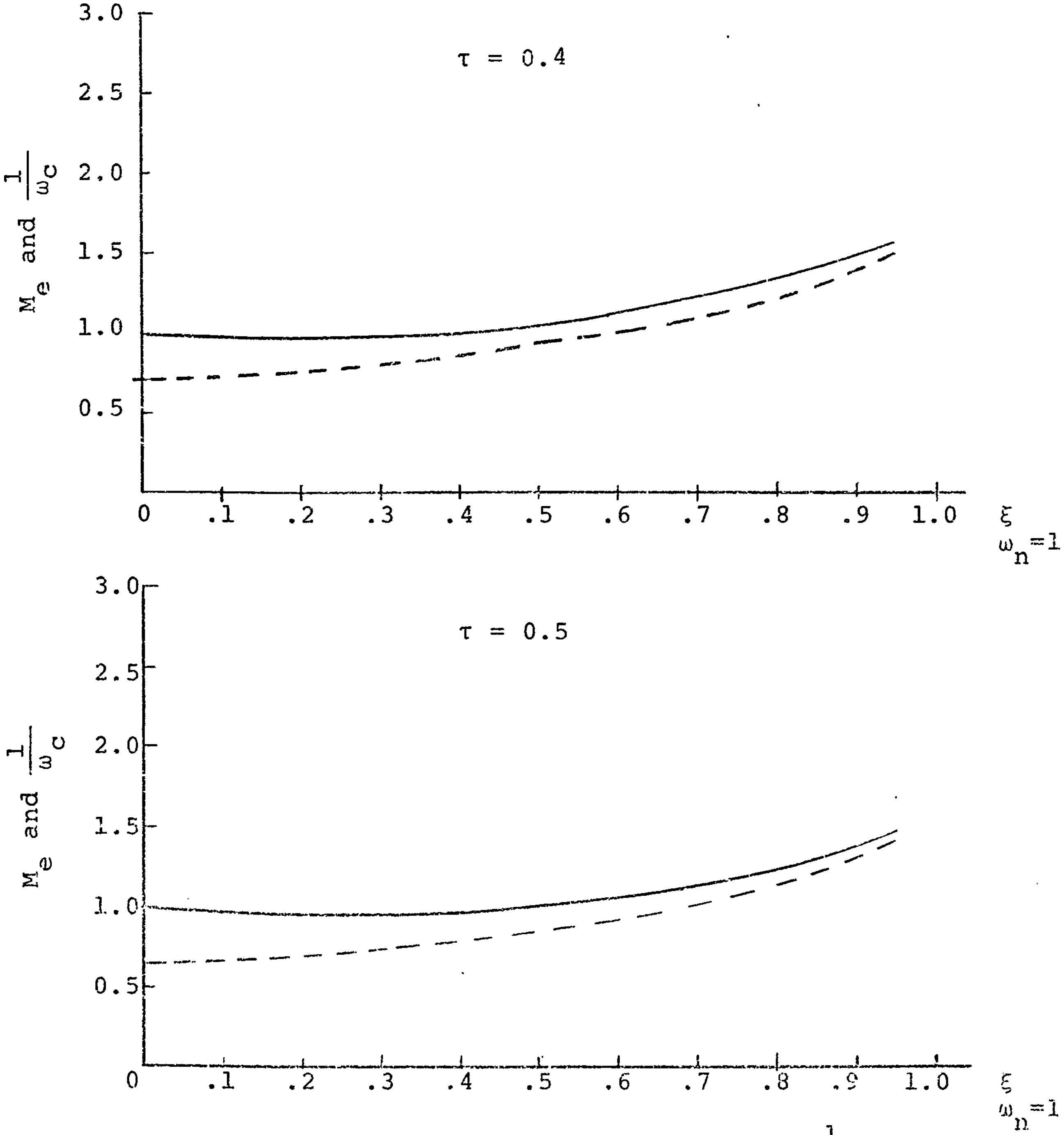
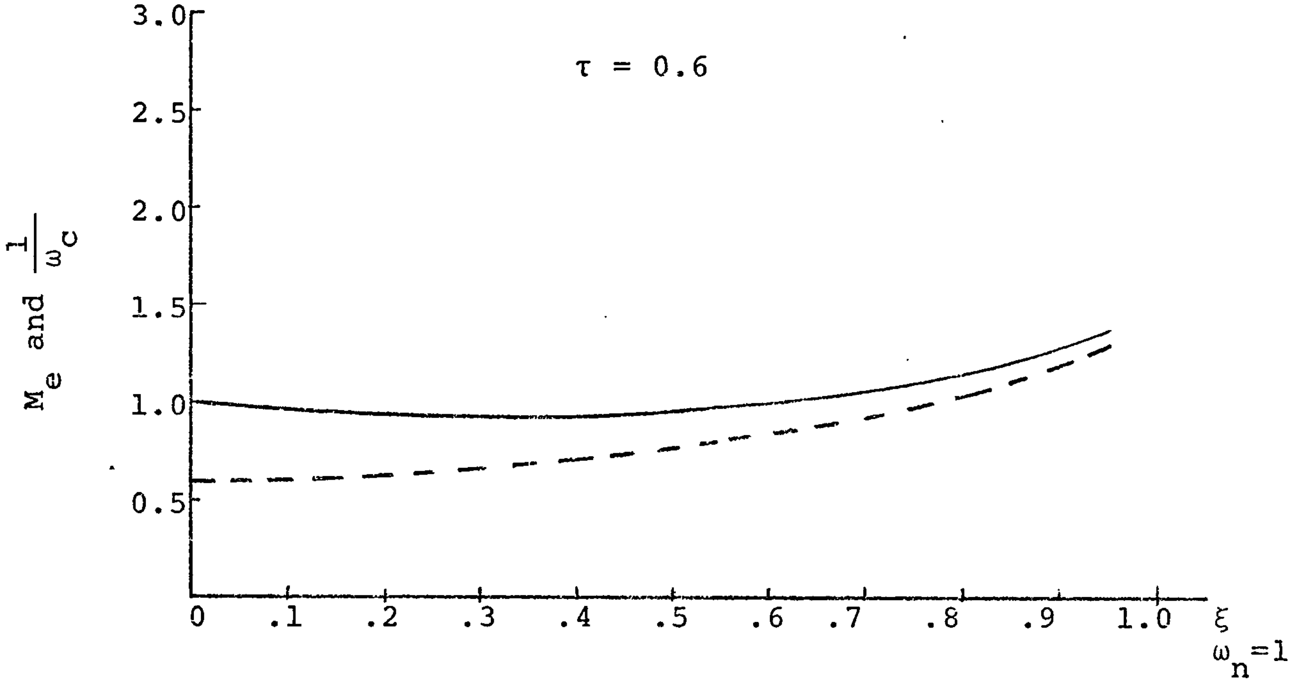


Fig. 2.11 Relationship between M_e and $\frac{1}{\omega_c}$

——— $\frac{1}{\omega_c}$ — — — — — — — M_e



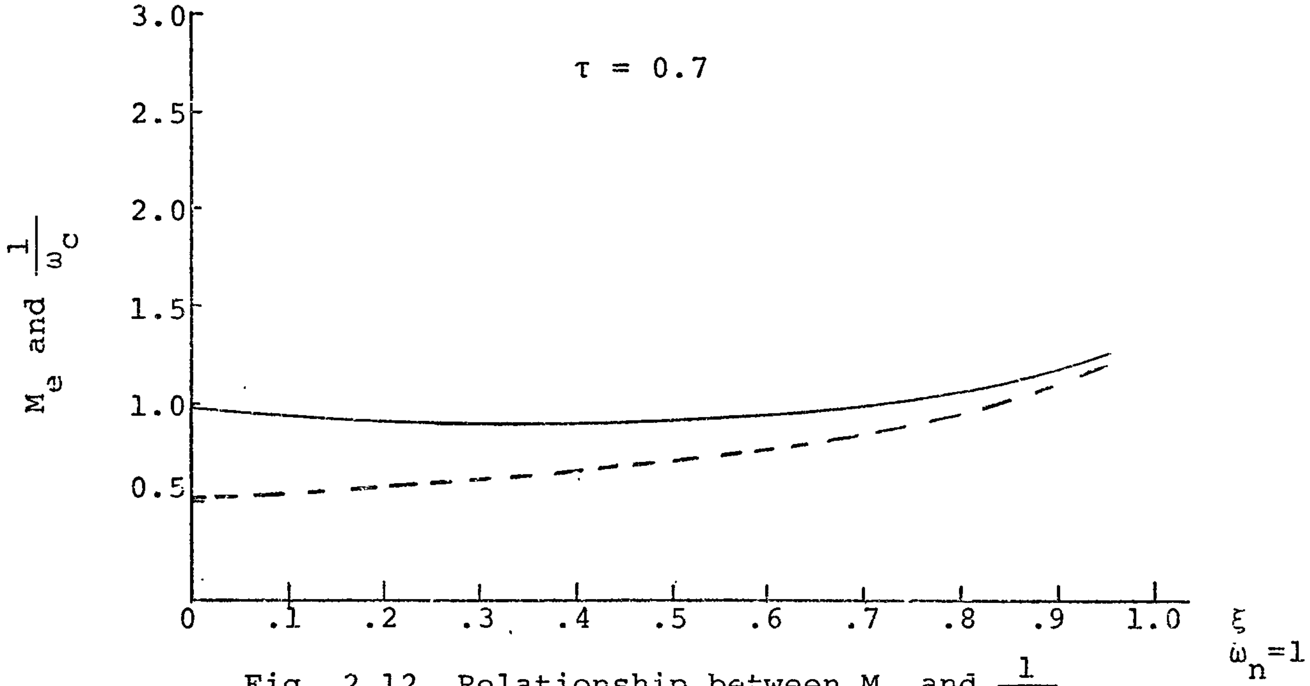


Fig. 2.12 Relationship between M_e and $\frac{1}{\omega_c}$

———— $\frac{1}{\omega_c}$ — — — — — M_e

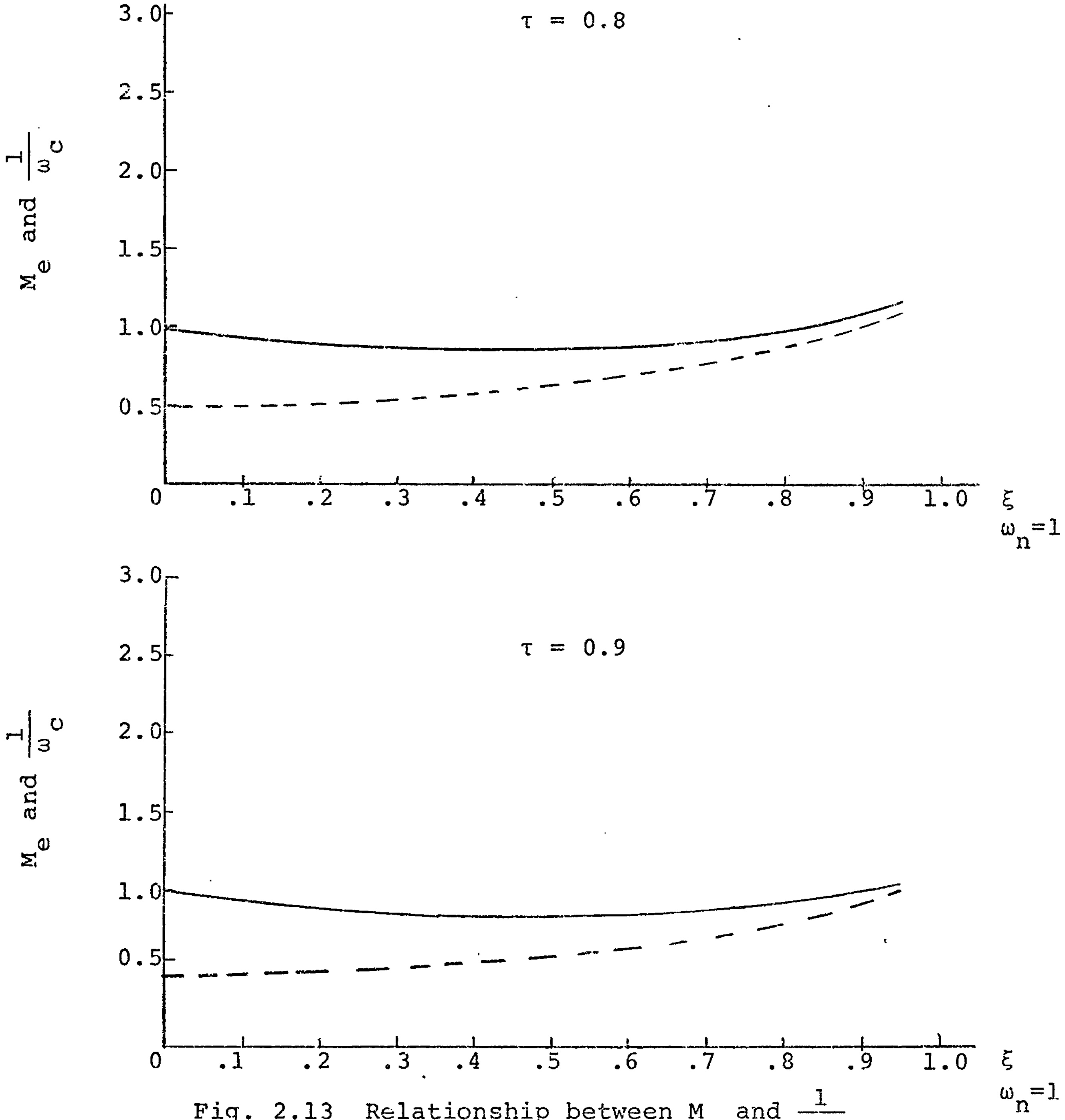


Fig. 2.13 Relationship between M_e and $\frac{1}{\omega_c}$

———— $\frac{1}{\omega_c}$ — — — — — M_e.

and also from section 2.3

$$\omega_c = \omega_n \left[\sqrt{(2\xi^2-2\xi\tau)^2 + 1} - (2\xi^2-2\xi\tau)\right]^{1/2} \tag{2.38}$$

Plotting these two equations versus the damping ratio and the parameter τ, we found the relationship between ω_p and ω_c. Axelby's criterion #3 gives us the following equation

$$\omega_p = \omega_c \tag{2.39}$$

Again, we inspect the plots according to Eq.(2.39) It is easy to find that these two quantities are almost the same in a certain region when τ is small. When a zero occurs in the transfer function, the rule in Eq.(2.19) suggested by Axelby should be modified. The plots of Eqs.(2.37) and (2.38) are shown in Fig. 2.14, Fig. 2.15, Fig. 2.16, Fig. 2.17, and Fig. 2.18.

2.5 Maximum Value of Impulse Response and Crossover Frequency

$\dot{M}_t$ is the maximum value of the unit impulse response of a system. To derive the relationship among $\dot{M}_t$, ξ, and τ is a straight forward process. We start with the closed-loop transfer function

$$M(s) = \frac{\tau\omega_n s + \omega_n^2}{s^2 + 2\xi\omega_n + \omega_n^2} \tag{2.40}$$

It can be considered as the Laplace transform of unit impulse response. In order to find the maximum value of the impulse response, we differentiate Eq.(2.40) obtaining the following equation

$$sM(s) = \frac{s(\tau\omega_n s + \omega_n^2)}{s^2+2\xi\omega_n s + \omega_n^2} \tag{2.41}$$

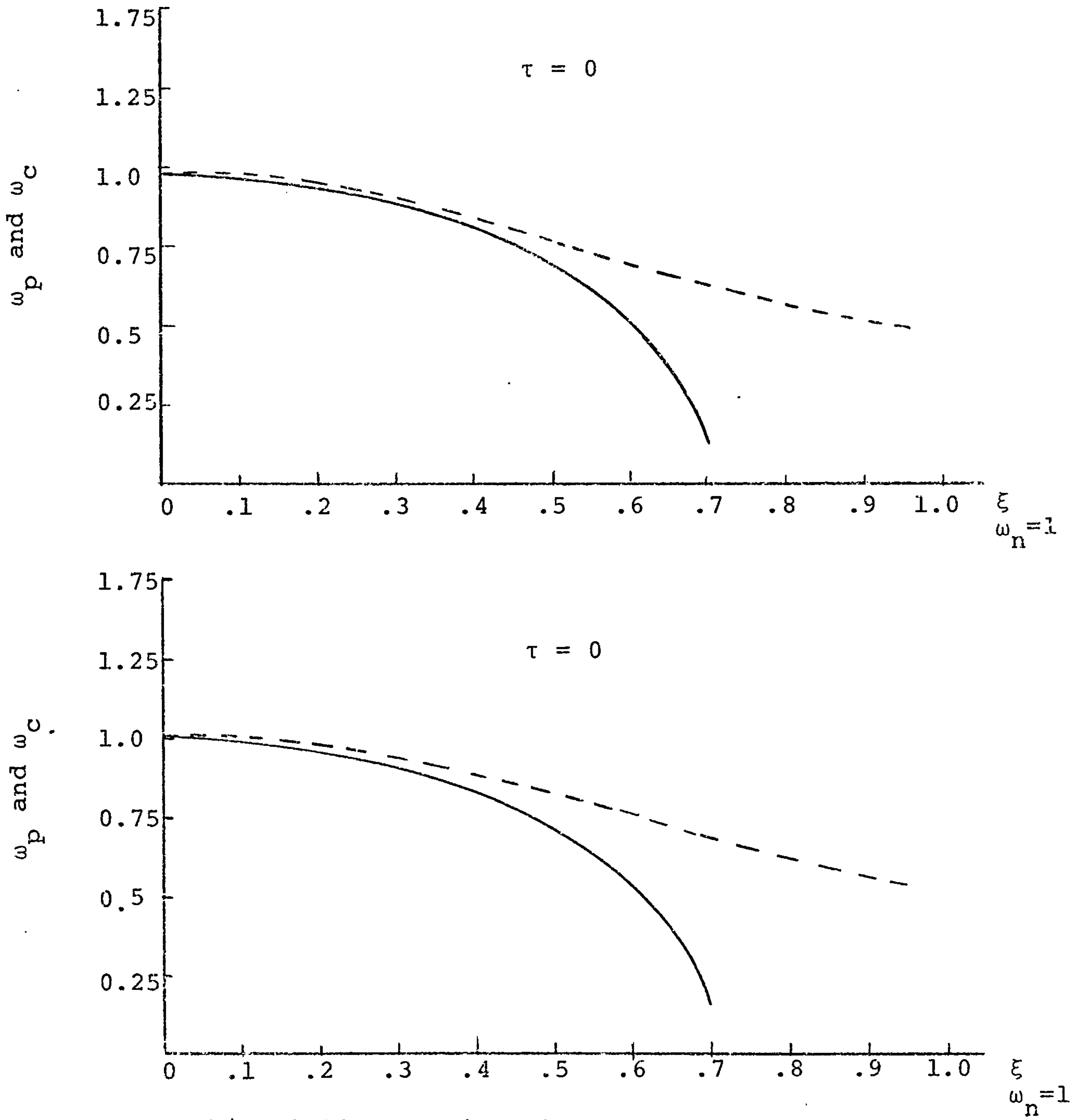


Fig. 2.14 Relationship between ω_p and ω_c

———— ω_p — — — — — ω_c.

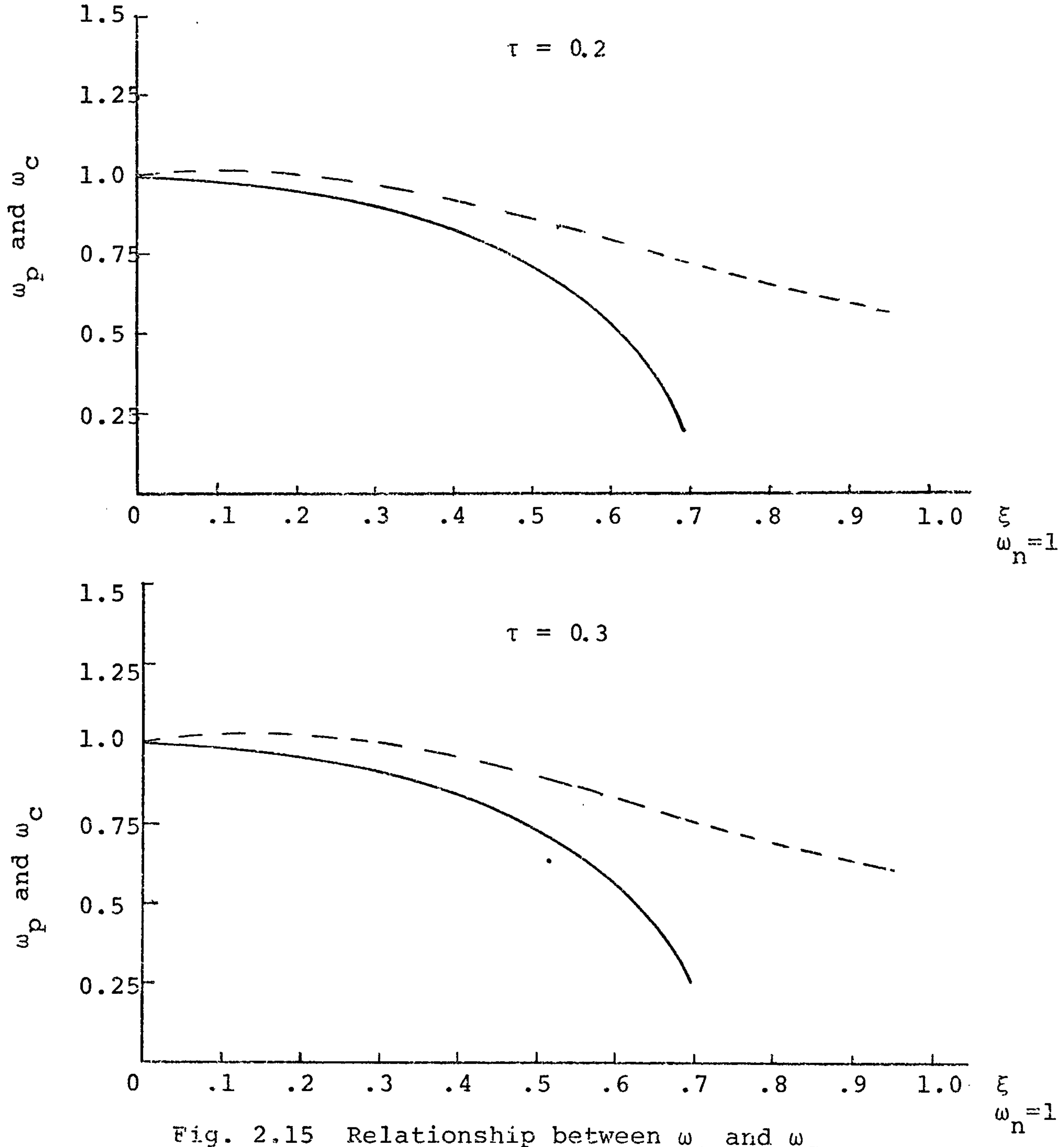


Fig. 2.15 Relationship between ω_p and ω_c

———— ω_p — — — — — ω_c.

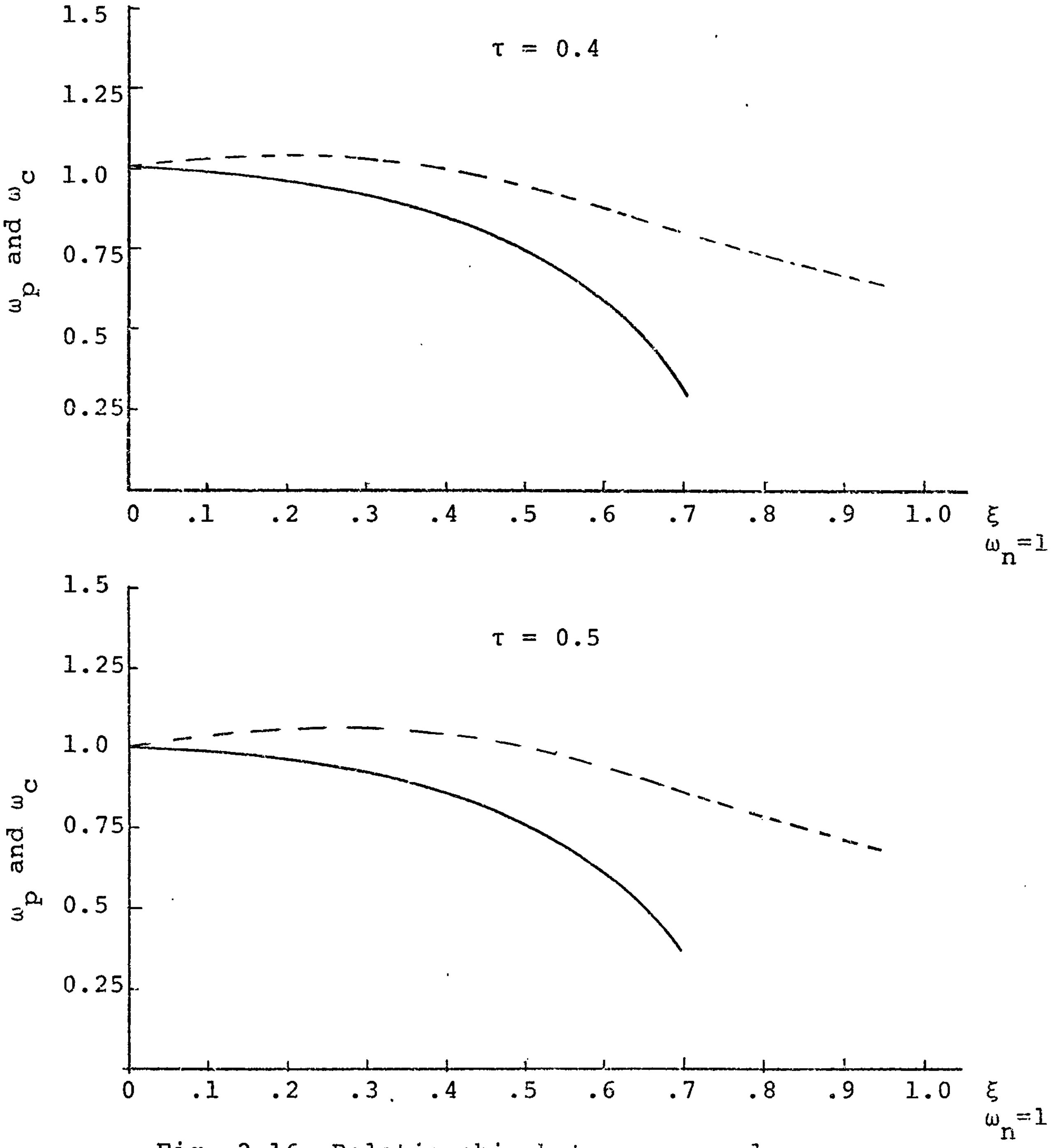


Fig. 2.16 Relationship between ω_p and ω_c

———— ω_p — — — — — ω_c.

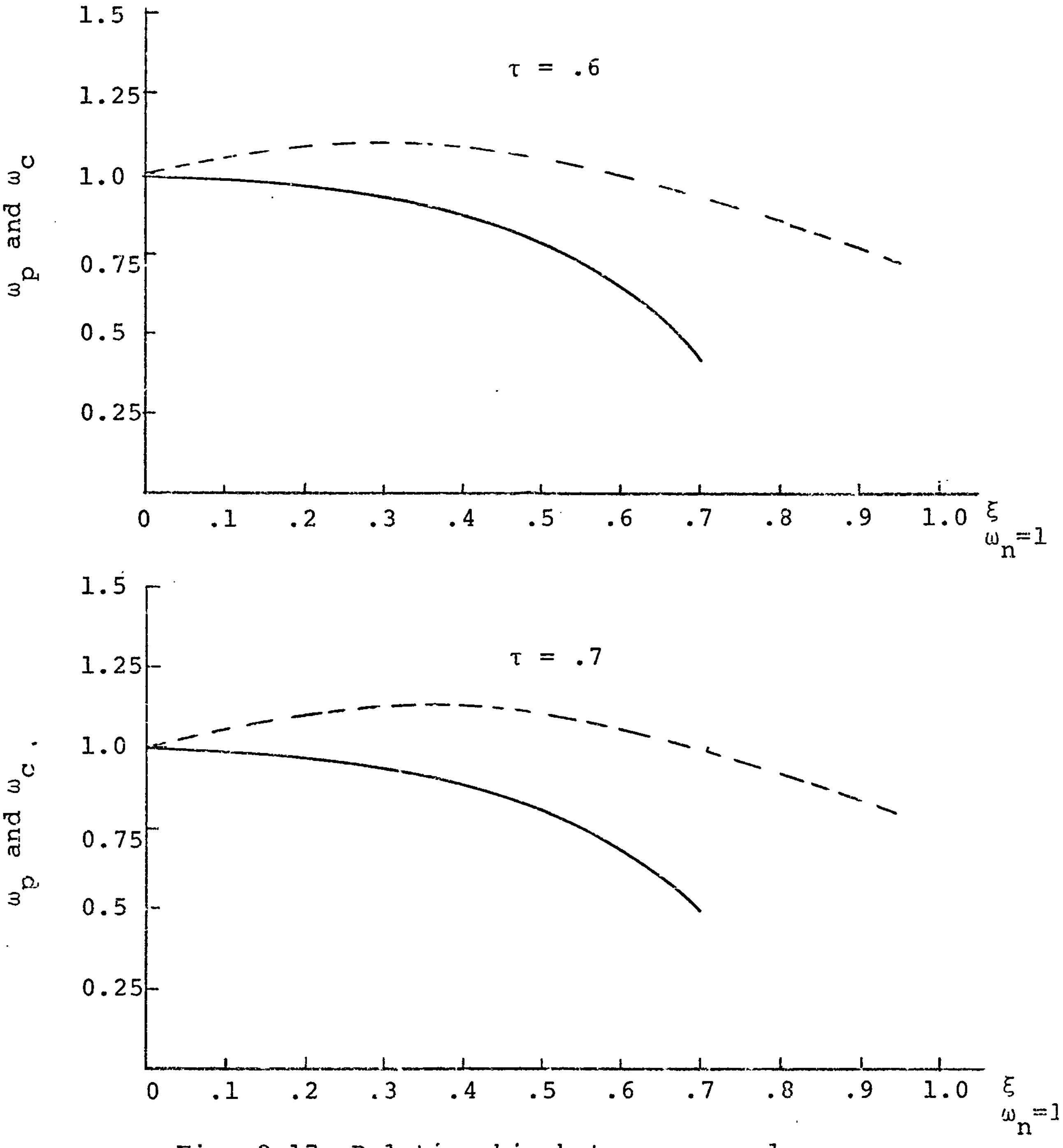


Fig. 2.17 Relationship between ω_p and ω_c

———— ω_p — — — — — ω_c.

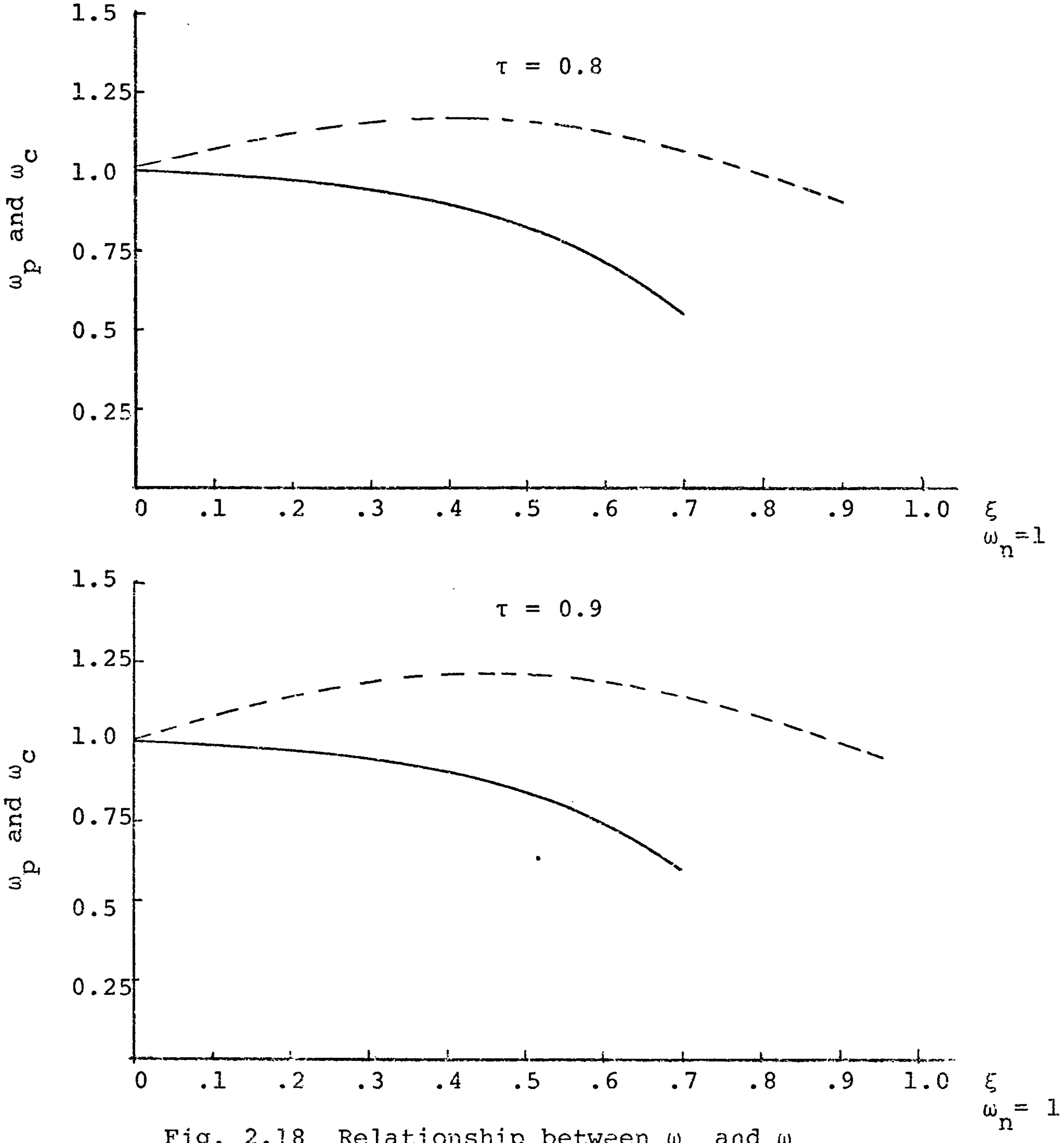


Fig. 2.18 Relationship between ω_p and ω_c

———————— ω_p – – – – – – – – ω_c.

The inverse Laplace transformation of Eq.(2.41) is

$$M_t'(t) = e^{-\xi\omega_n t}[(-\tau\xi\omega_n^2 + (1-\xi\tau)\omega_n^2)\cos\omega_n\sqrt{1-\xi^2}\,t - (\frac{\xi\omega_n^2(1-\xi\tau)}{\sqrt{1-\xi^2}} + \tau\omega_n^2\sqrt{1-\xi^2})\sin\omega_n\sqrt{1-\xi^2}\,t] \tag{2.42}$$

Setting $M_t' = 0$ and solving this equation, we find the time t_c at which the maximum value occurs

$$t_c = \frac{\tan^{-1}\frac{(1-2\xi\tau)\sqrt{1-\xi^2}}{\xi-2\tau\xi^2+\tau}}{\omega_n\sqrt{1-\xi^2}} \tag{2.43}$$

Substituting Eq.(2.43) into c'(t), we obtain the following equation.

$$\dot{M}_t = \omega_n e^{-\xi\omega_n t_c}[\tau\cos\omega_n\sqrt{1-\xi^2}\,t_c + \frac{(1-\xi\tau)}{\sqrt{1-\xi^2}} \sin\omega_n\sqrt{1-\xi^2}\,t_c] \tag{2.44}$$

In order to inspect the result which we got from the previous derivation, the plots of Eqs.(2.44) and (2.38) are shown in Fig. 2.19, Fig. 2.20, Fig. 2.21, Fig. 2.22 and Fig. 2.23. From these plots we found that $\dot{M}_t$ and ω_c are not exactly the same rule as suggested by Axelby which is $\dot{M}_t = \omega_c$ when ξ varies

2.6 Crossover Frequency and Time Domain Specifications

1. Time-domain specifications

The specifications used in the time-domain can be summarized by t_p and t_c shown in the graph, Fig. 2.24.

2. The relationship between t_p and ω_c.

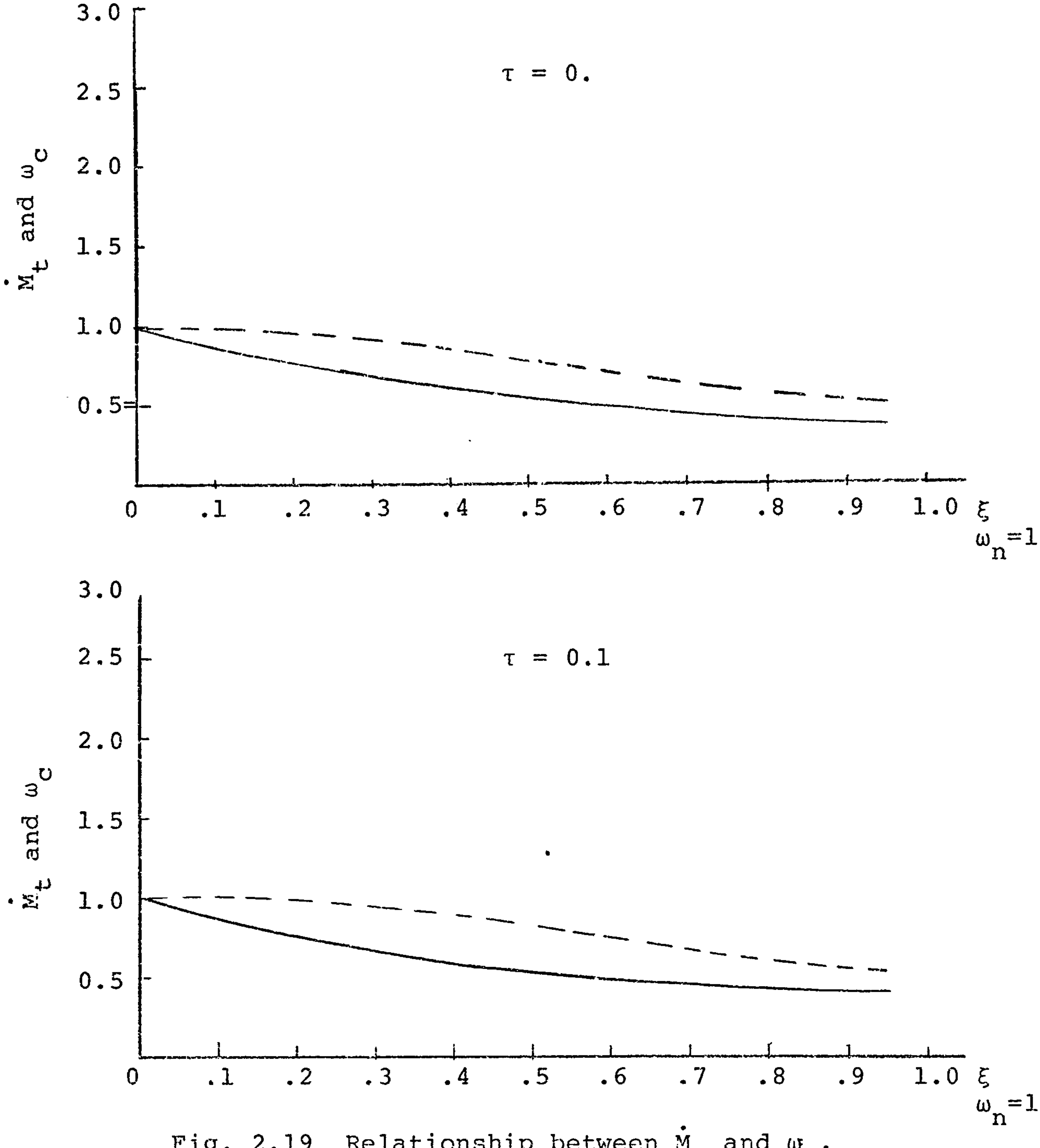


Fig. 2.19 Relationship between $\dot{M}_t$ and ω_c.

———— $\dot{M}_t$ — — — — — ω_c.

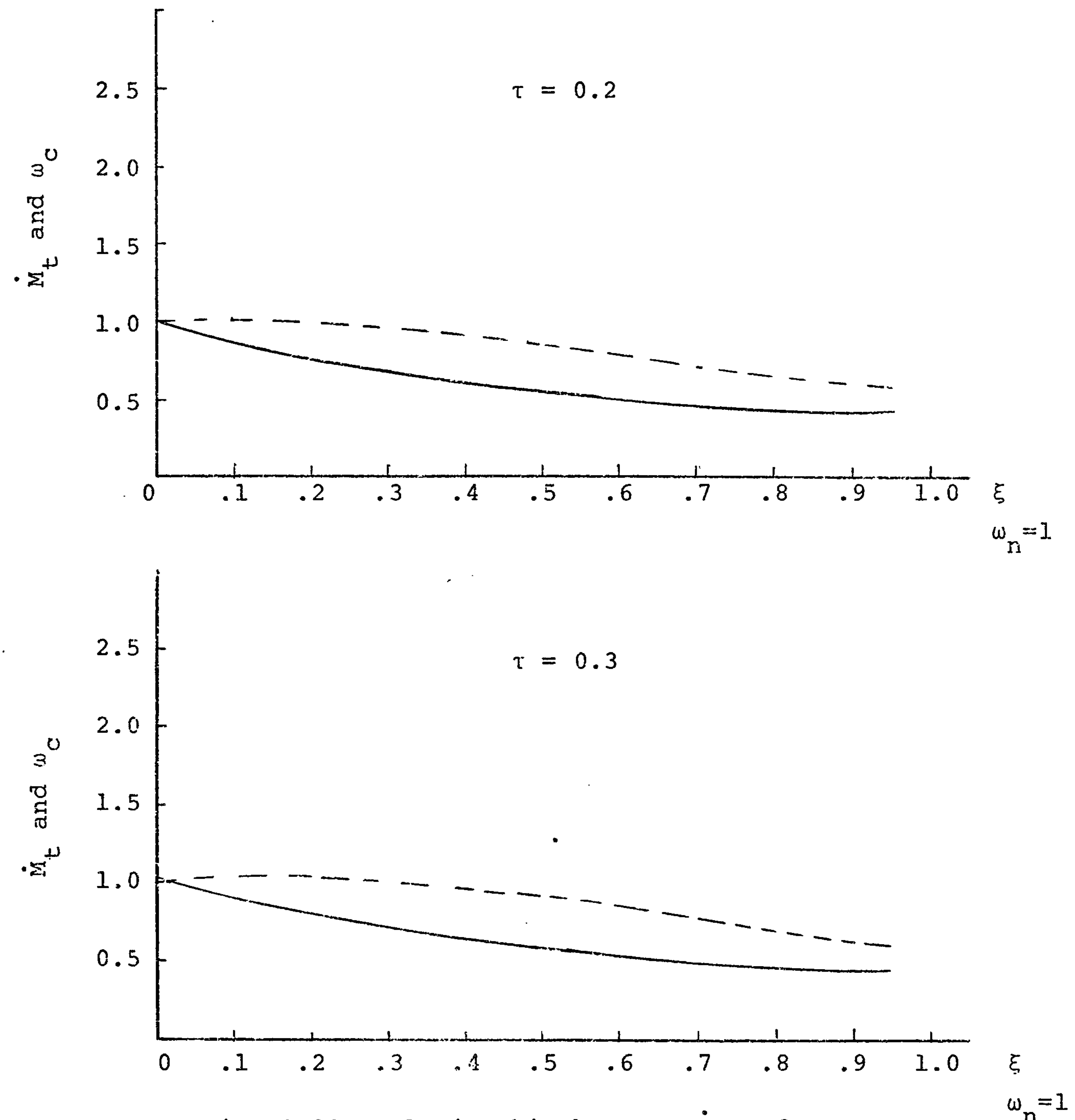


Fig. 2.20 Relationship between $\dot{M}_t$ and ω_c

———— $\dot{M}_t$ — — — — — ω_c.

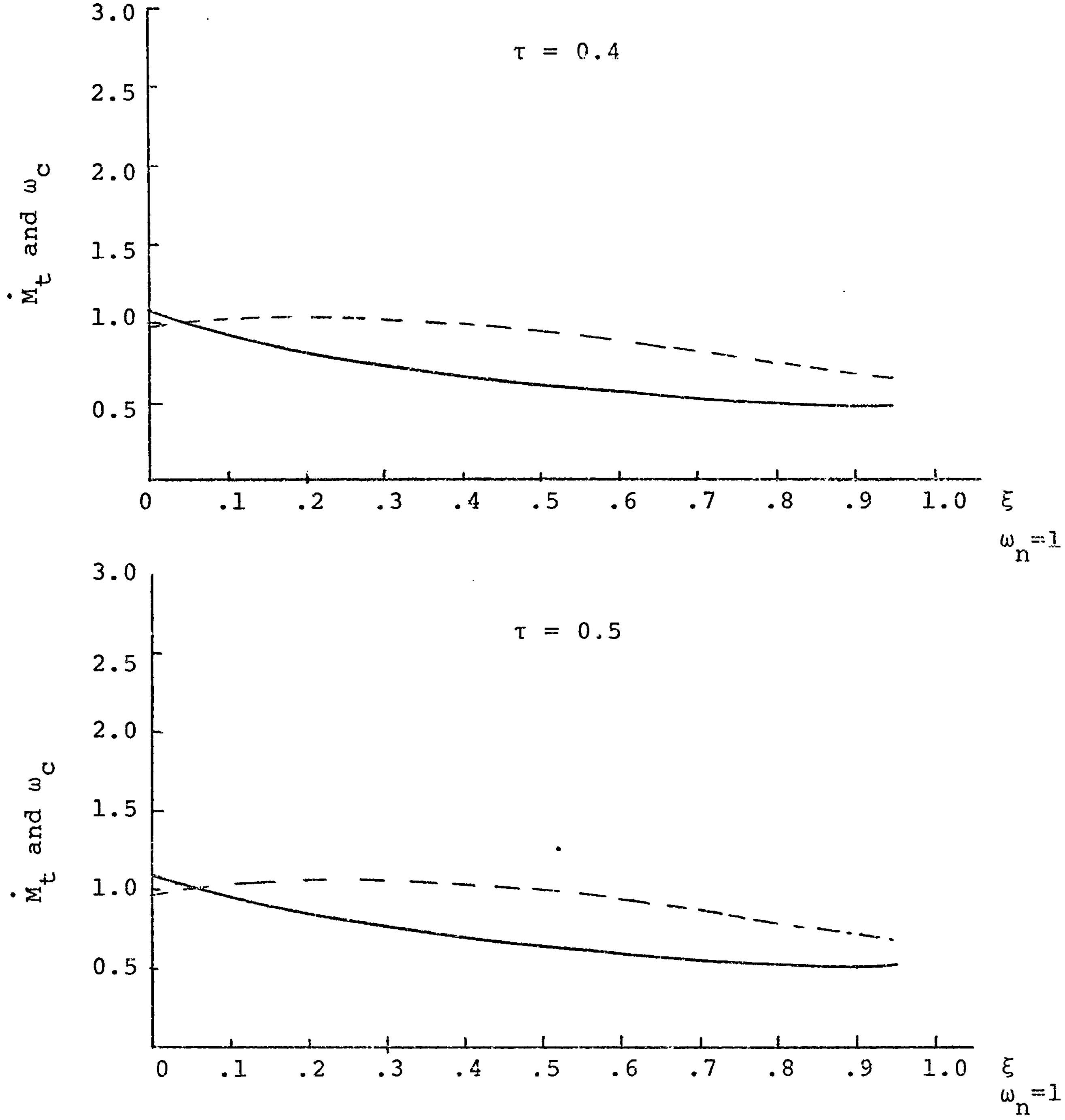


Fig. 2.21 Relationship between $\dot{M}_t$ and ω_c.

———— $\dot{M}_t$ — — — — — ω_c.

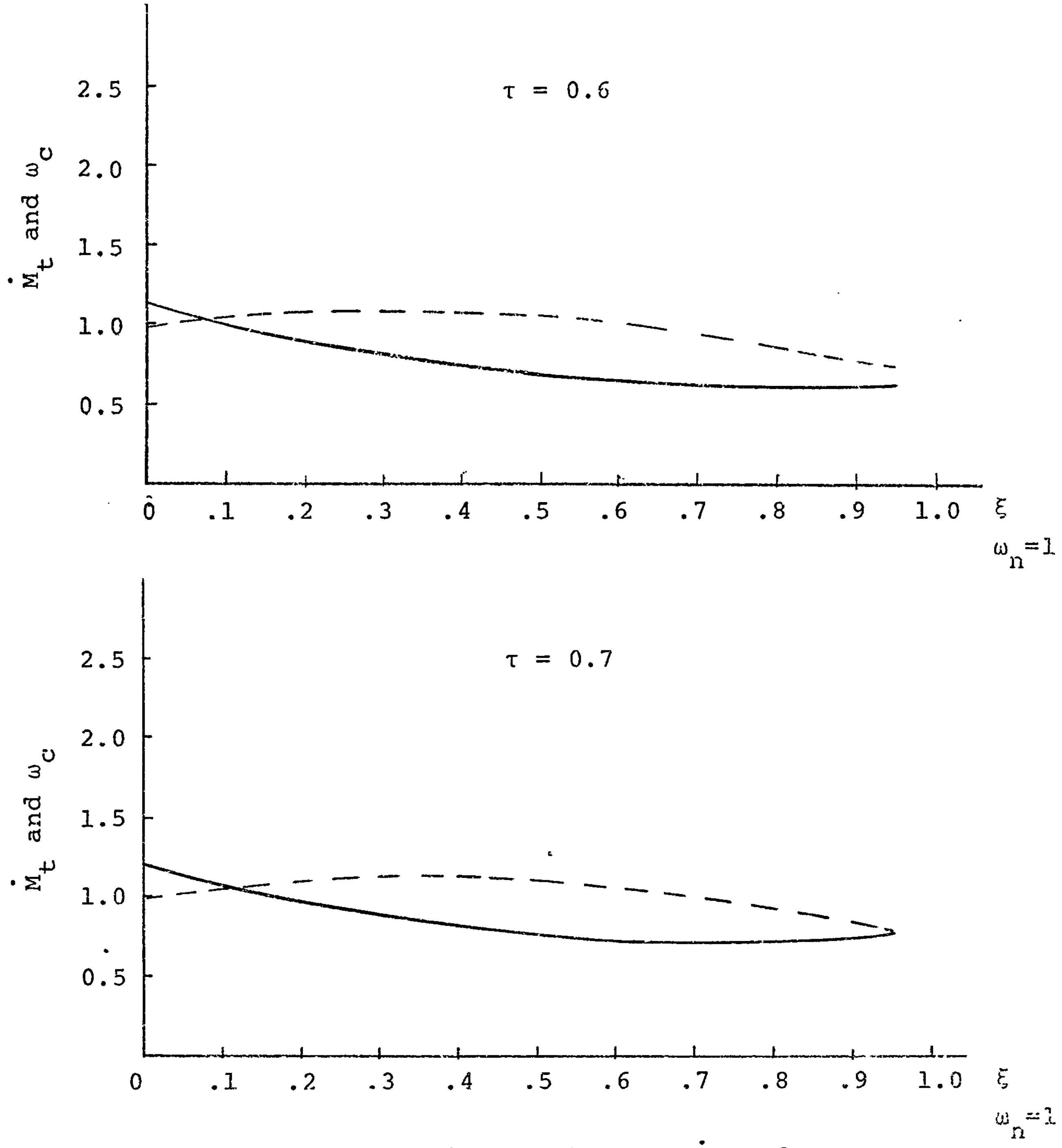


Fig. 2.22 Relationship between $\dot{M}_t$ and ω_c

———— $\dot{M}_t$ — — — — — ω_c.

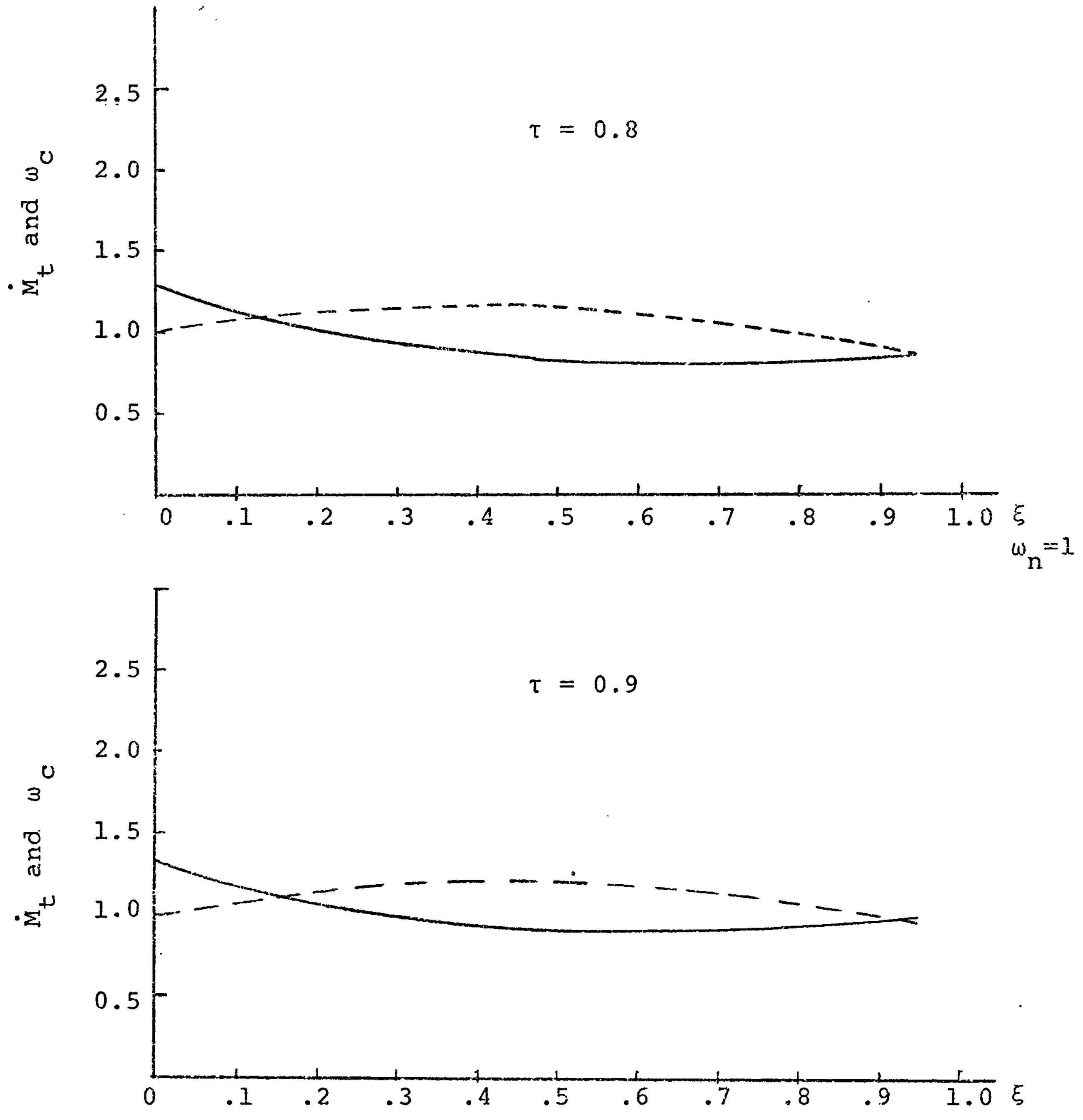


Fig. 2.23 Relationship between $\dot{M}_t$ and ω_c

$\dot{M}_t$ ———— ω_c — — — — — .

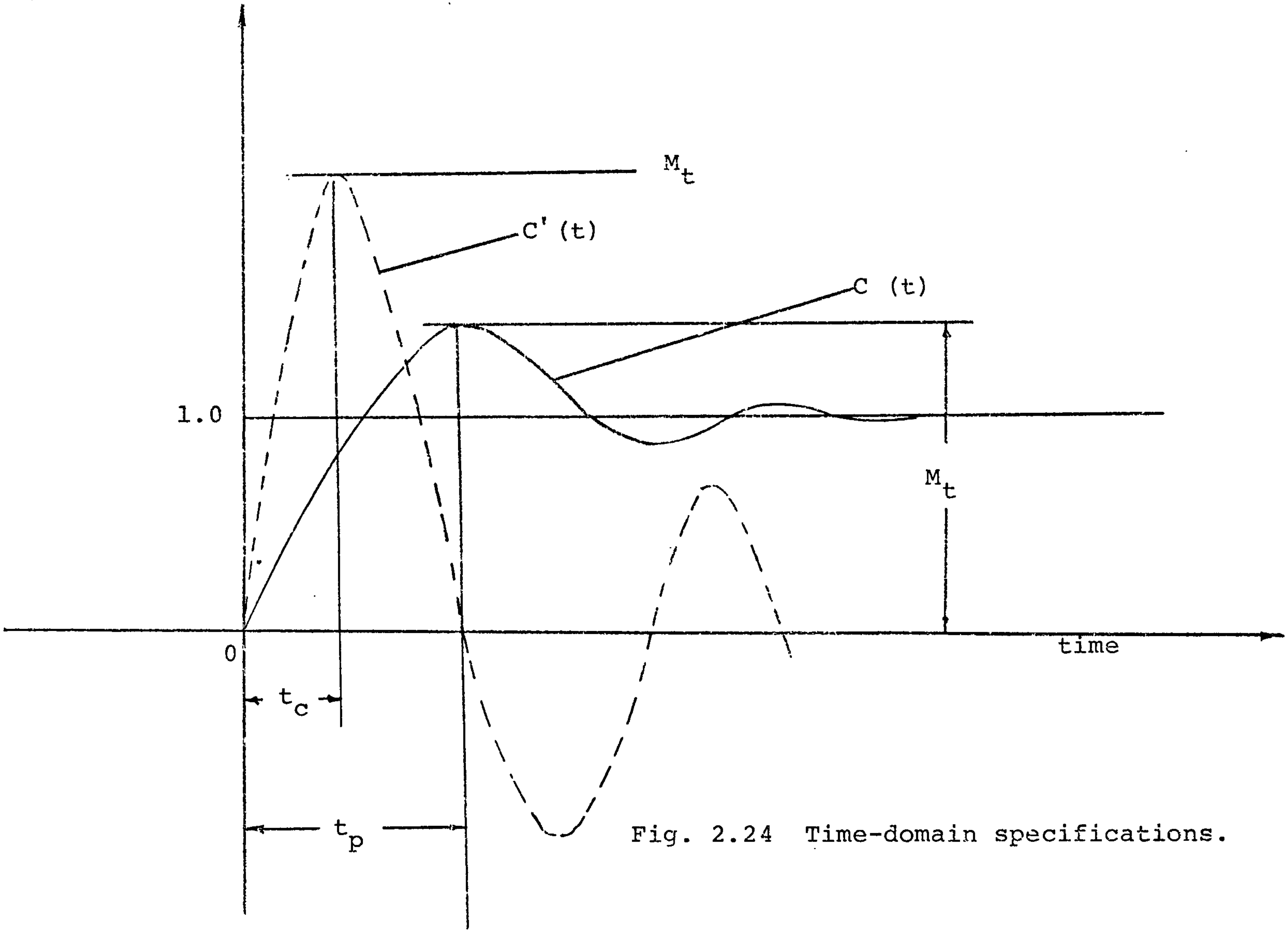


Fig. 2.24 Time-domain specifications.

Based on section 2.3, we knew that

$$\omega_c = \omega_n[\sqrt{(2\xi^2-2\xi\tau)^2 + 1} - (2\xi^2 - 2\xi\tau)]^{1/2} \quad (2.45)$$

Based on section 2.2, we have

$$t_p = \frac{\Pi + \tan^{-1}\frac{\tau\sqrt{1-\xi^2}}{\tau\xi - 1}}{\omega_n\sqrt{1-\xi^2}} \quad (2.46)$$

According to Axelby's rule, the approximate criterion is

$$t_p = \frac{3}{\omega_c} \quad (2.47)$$

In order to inspect this criterion, we plot Eq.(2.45) and Eq. (2.46) in Fig. 2.25, Fig. 2.26, Fig. 2.27, Fig. 2.28, and Fig. 2.29 by using t_p, and $\frac{\omega_c}{3}$ as the vertical coordinate. From these plots we found that the equality in Eq.(2.47) holds approximately in a region of $\xi < 0.7$.

3. The relationships between t_c and $\frac{1}{\omega_c}$.

From section 2.5 we find the time t_c at which the maximum value occurs is

$$t_c = \frac{\tan^{-1}\frac{(1-2\xi\tau)\sqrt{1-\xi^2}}{\xi - 2\tau\xi^2 + \tau}}{\omega_n\sqrt{1-\xi^2}} \quad (2.48)$$

Also, we know the relationship between crossover frequency, ξ, and τ which is written as follows:

$$\omega_c = \omega_n[\sqrt{(2\xi^2-2\xi\tau)^2 + 1} - (2\xi^2 - 2\xi\tau)]^{1/2} \quad (2.49)$$

The approximate criterion of Axelby's rule is

$$t_c = \frac{1}{\omega_c} \quad (2.50)$$

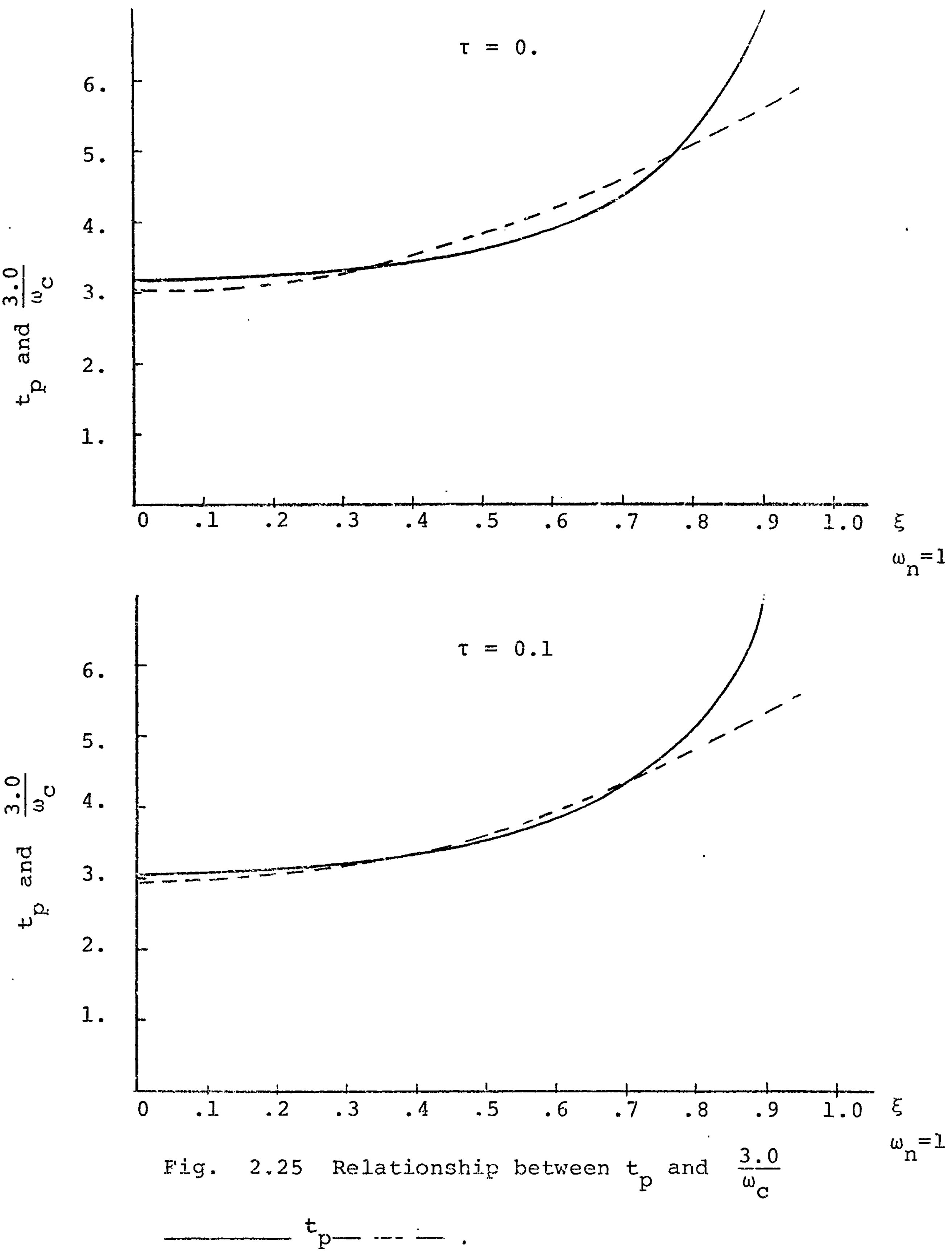


Fig. 2.25 Relationship between t_p and $\frac{3.0}{\omega_c}$

———— t_p — --- — .

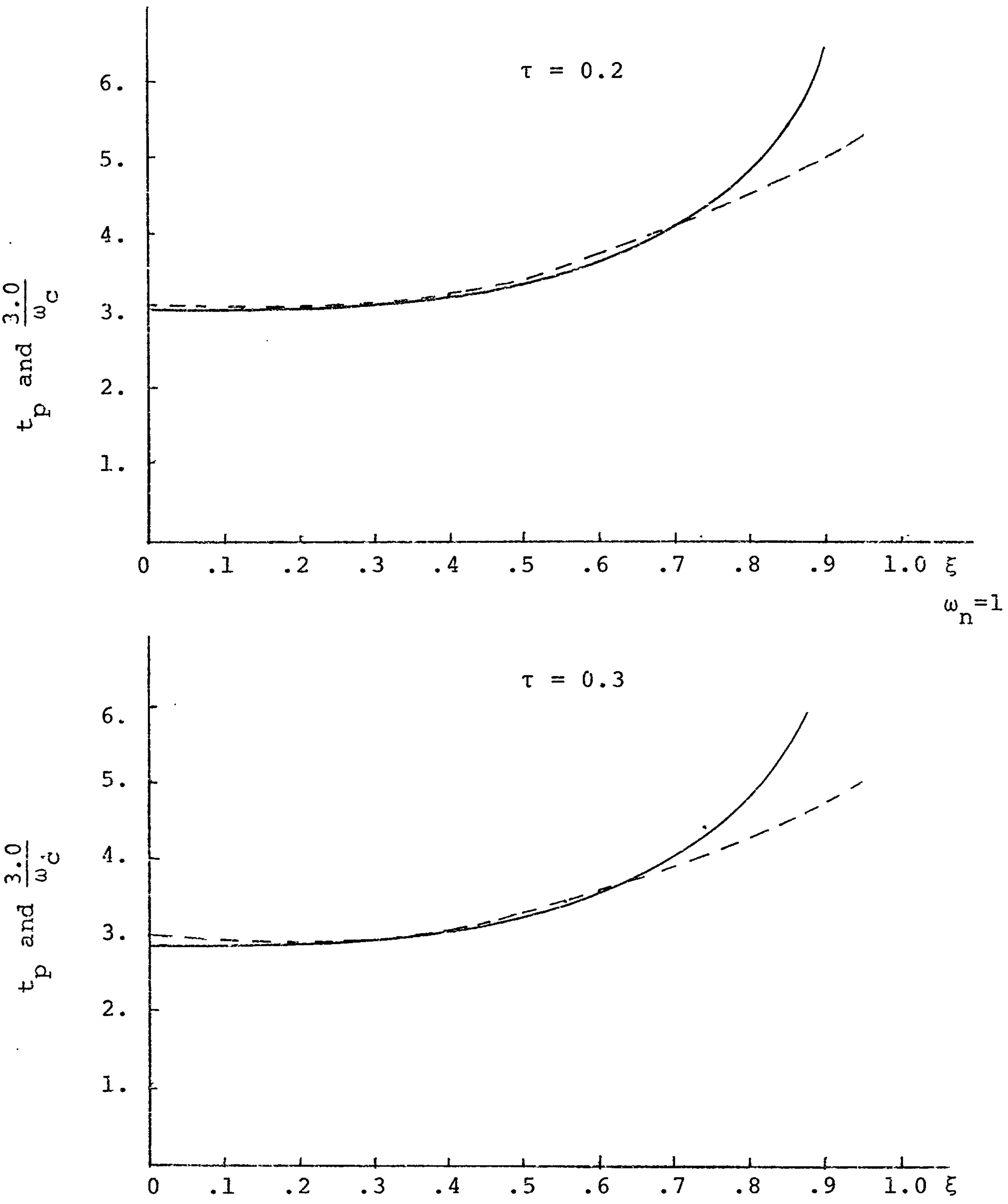


Fig. 2.26 Relationship between t_p and $\frac{3.0}{\omega_c}$

——————— t_p — — — — — — $\frac{3.0}{\omega_c}$

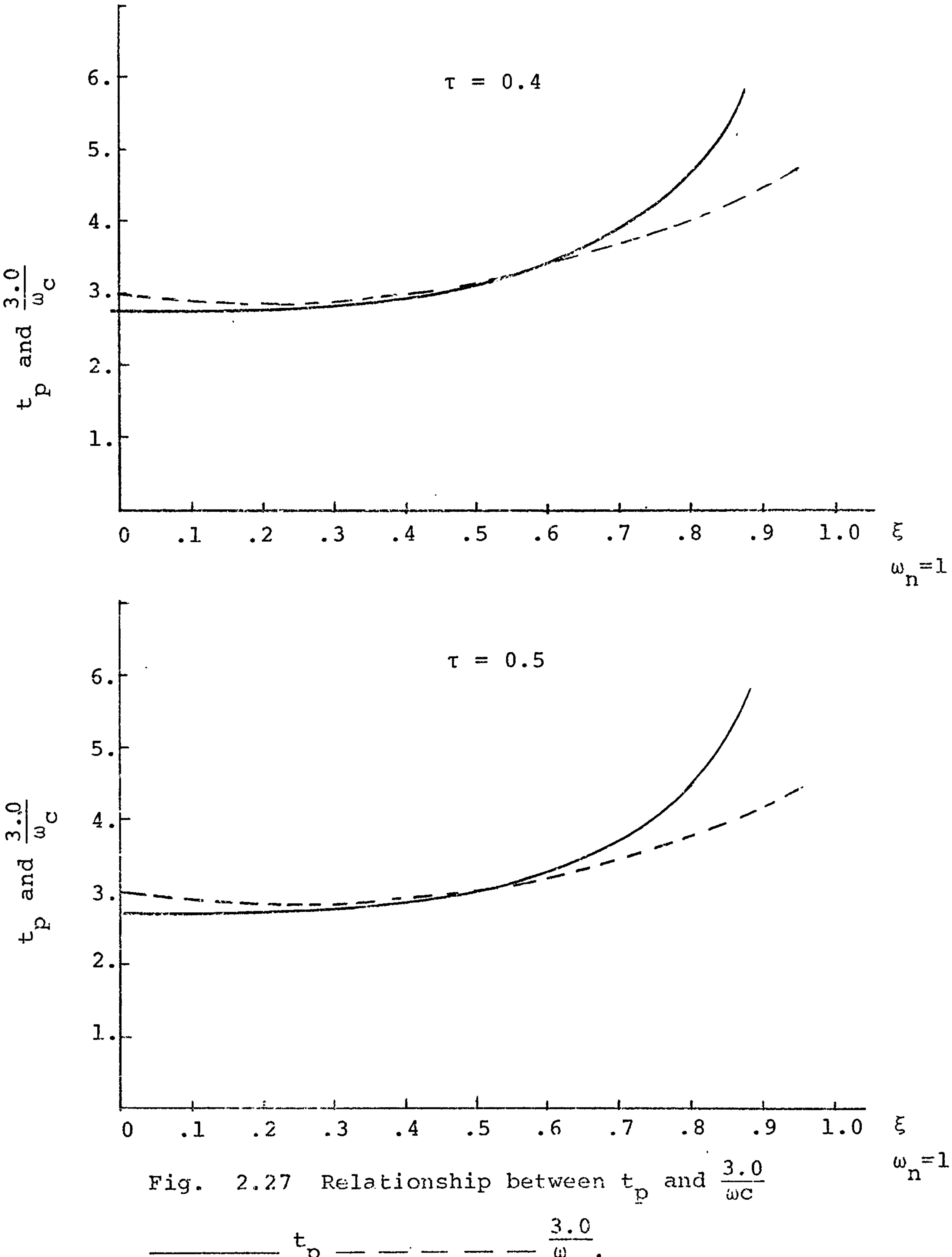


Fig. 2.27 Relationship between t_p and $\frac{3.0}{\omega c}$

——— t_p — — — — — $\frac{3.0}{\omega_c}$.

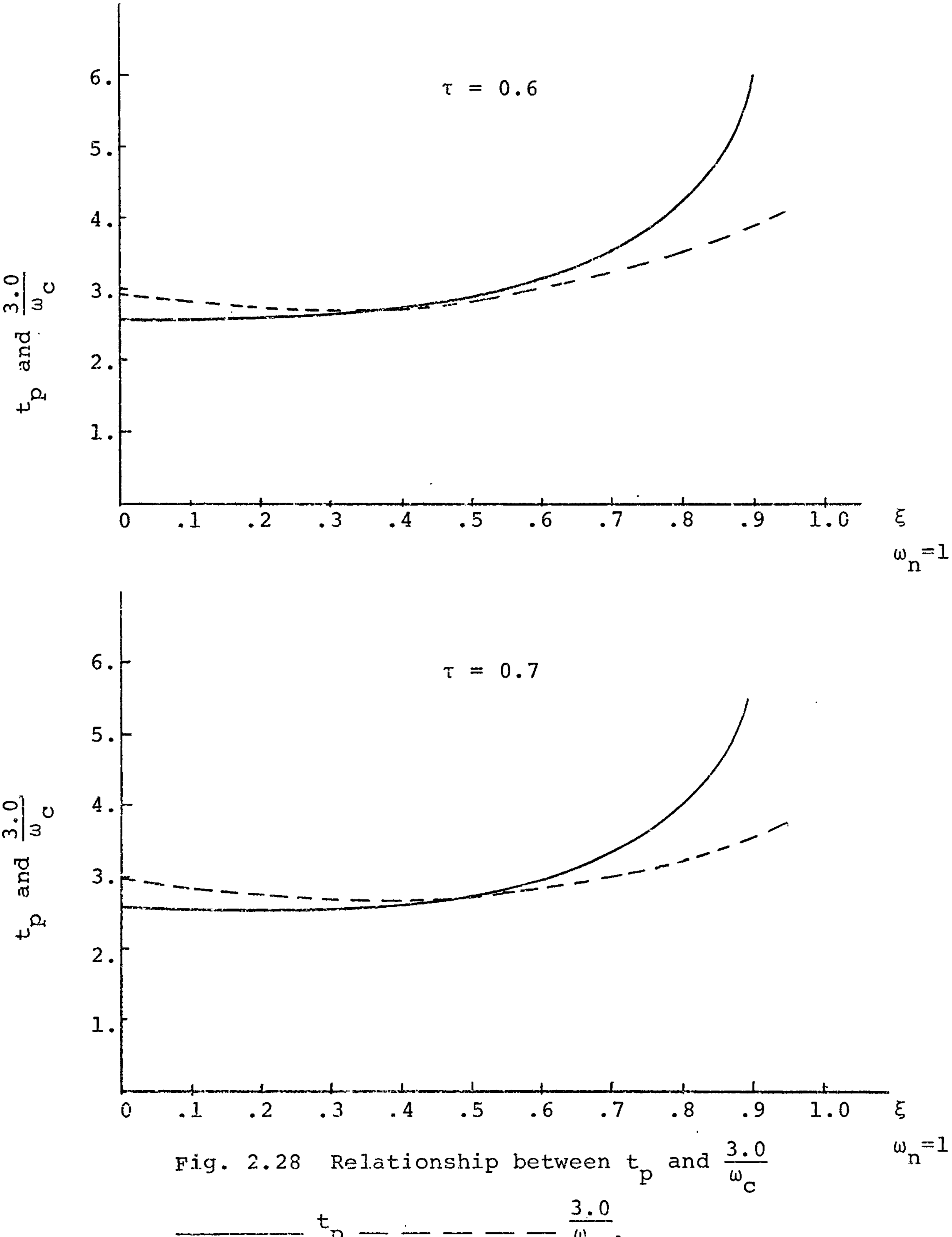


Fig. 2.28 Relationship between t_p and $\frac{3.0}{\omega_c}$

———— t_p — — — — — $\frac{3.0}{\omega_c}$.

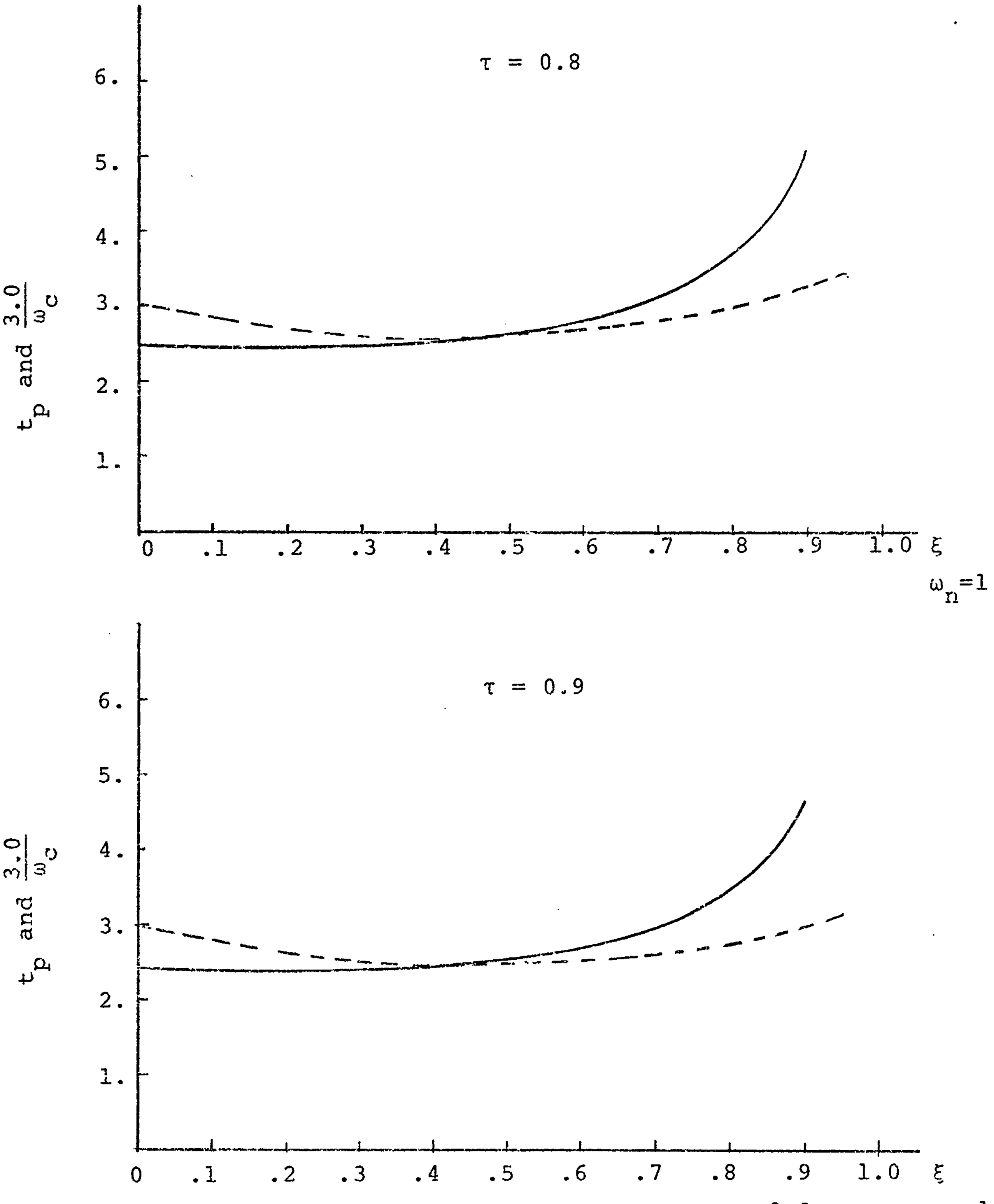


Fig. 2.29 Relationship between t_p and $\frac{3.0}{\omega_c}$

———— t_p — — — — — — $\frac{3.0}{\omega_c}$.

If we use Eq.(2.49) and Eq.(2.50) to plot the curves shown in Fig. 2.30, Fig. 2.31, Fig. 2.32, Fig. 2.33, and Fig. 2.34, it is observed that this approximate formula suggested by Axelby is not highly accurate.

4. The relationship between t_v and ω_c.

From section 2.3 we knew that the time t_v at which the maximum value of error signal occurs is:

$$t_v = \frac{\tan^{-1} \dfrac{\xi B - C\sqrt{1-\xi^2}}{-B\sqrt{1-\xi^2} - \xi C}}{\omega_n \sqrt{1-\xi^2}} \tag{2.51}$$

where $B = (2\xi - \tau)(1 - \xi^2)$

$$C = \sqrt{1-\xi^2}\,(1 - 2\xi^2 + \tau\xi)$$

and the crossover frequency ω_c is:

$$\omega_c = \omega_n \left[\sqrt{(2\xi^2 - 2\xi\tau)^2 + 1} - (2\xi^2 - 2\xi\tau)\right]^{1/2} \tag{2.52}$$

To compare Eq.(2.51) and Eq.(2.52) we plot two curves which are shown in Fig. 2.35, Fig. 2.36, Fig. 2.37, Fig. 2.38, and Fig. 2.39. It is noted that the higher the damping ratio the worse the criterion.

2.7 Illustrative Example

Consider a high order system

$$T(s) = \frac{b_1 s^3 + b_2 s^2 + b_3 s + b_4}{a_1 s^7 + a_2 s^6 + a_3 s^5 + a_4 s^4 + a_5 s^3 + a_6 s^2 + a_7 s + a_8}$$

where

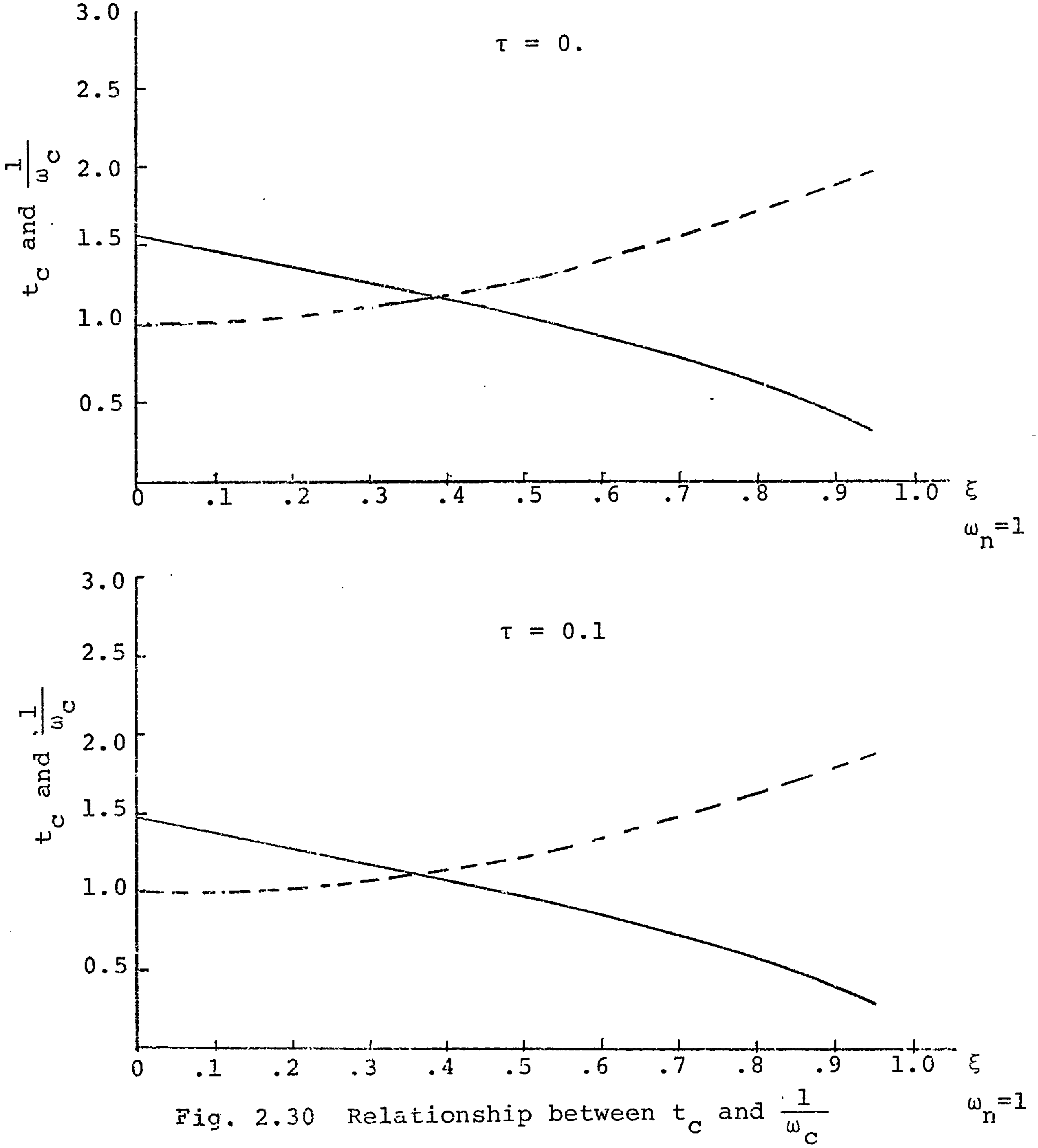
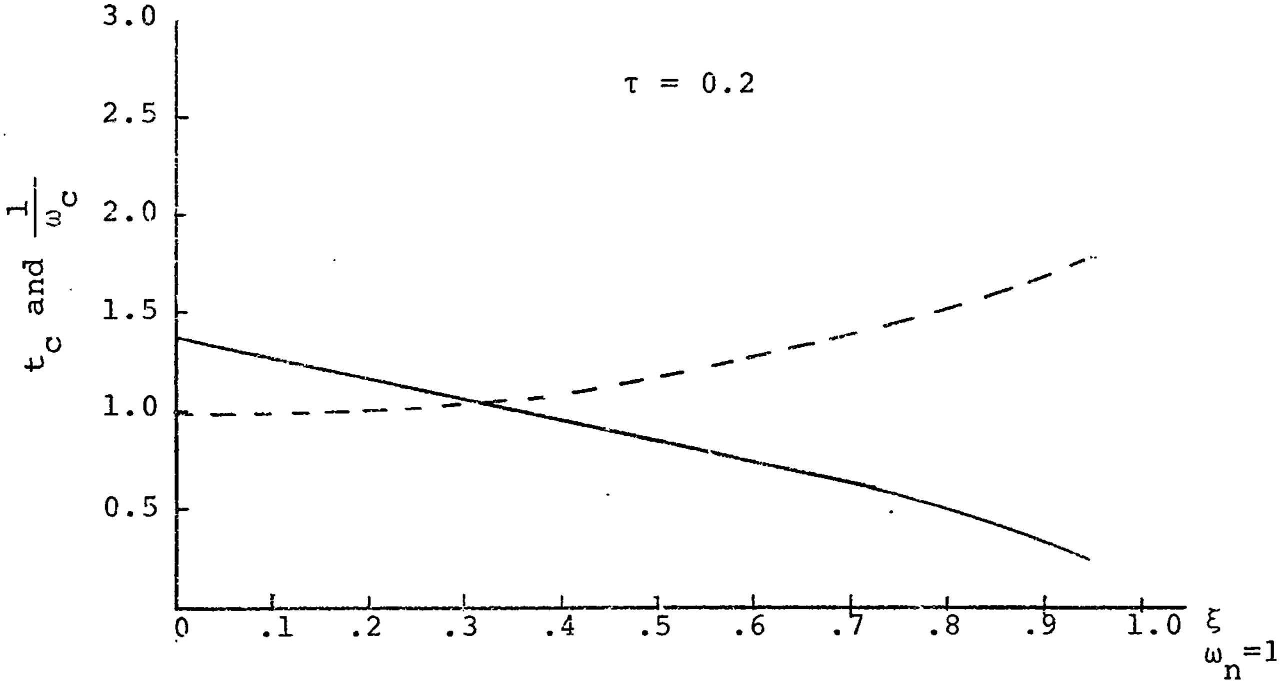


Fig. 2.30 Relationship between t_c and $\frac{1}{\omega_c}$

———— t_c — — — — — $\frac{1}{\omega_c}$.



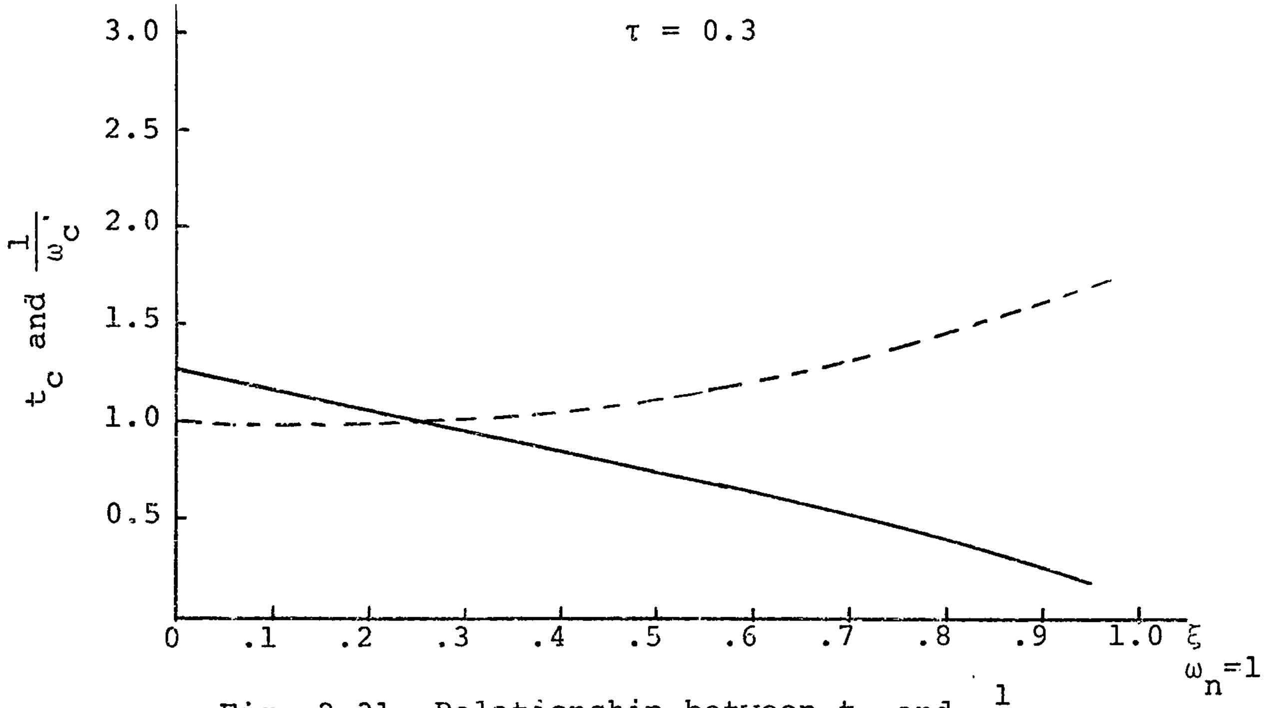


Fig. 2.31 Relationship between t_c and $\frac{1}{\omega_c}$

———— t_c — — — — — $\frac{1}{\omega_c}$.

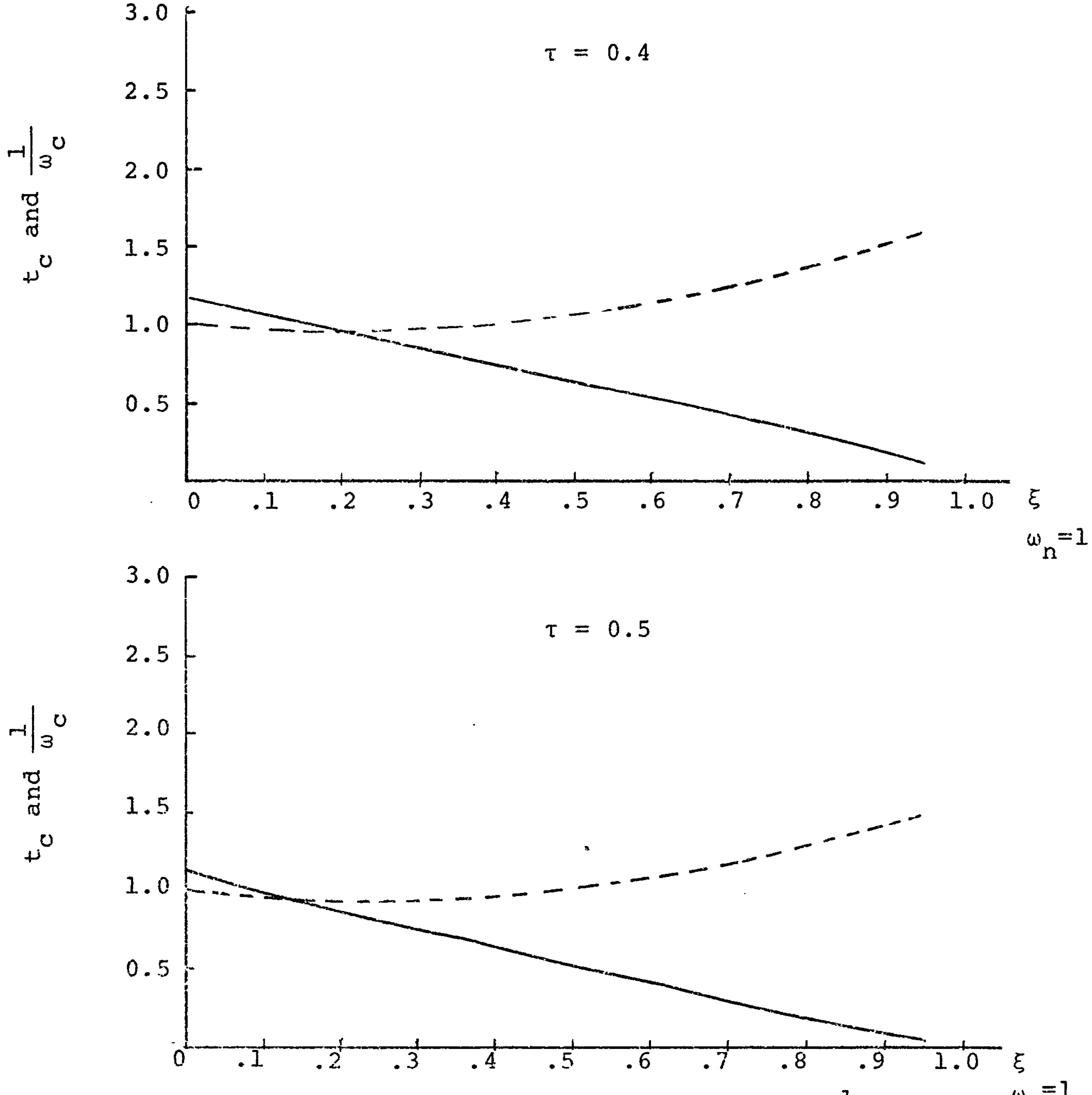


Fig. 2.32 Relationship between t_c and $\frac{1}{\omega_c}$

———— t_c — — — — — $\frac{1}{\omega_c}$

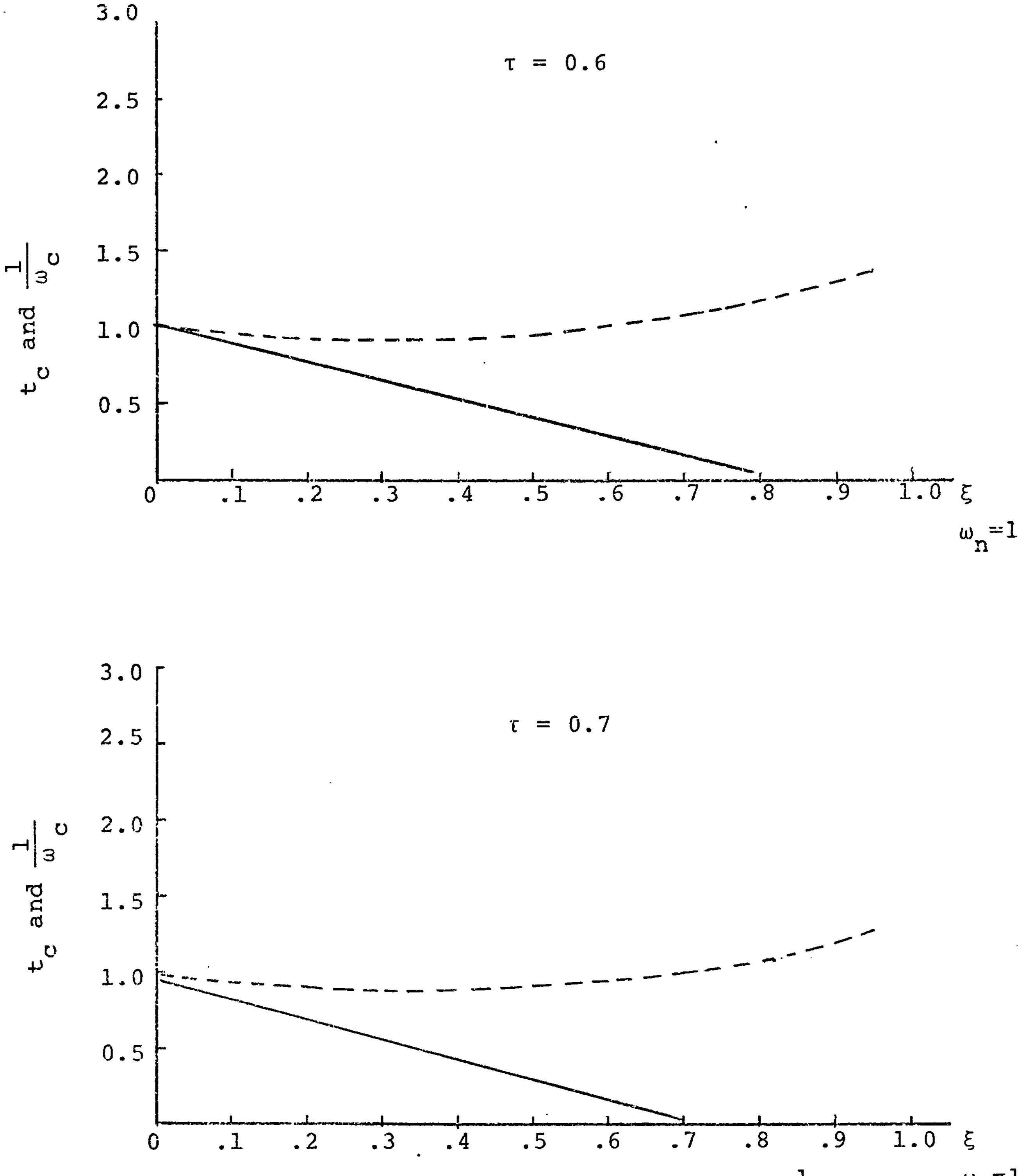


Fig. 2.33 Relationship between t_c and $\frac{1}{\omega_c}$

———— t_c — — — — — $\frac{1}{\omega_c}$.

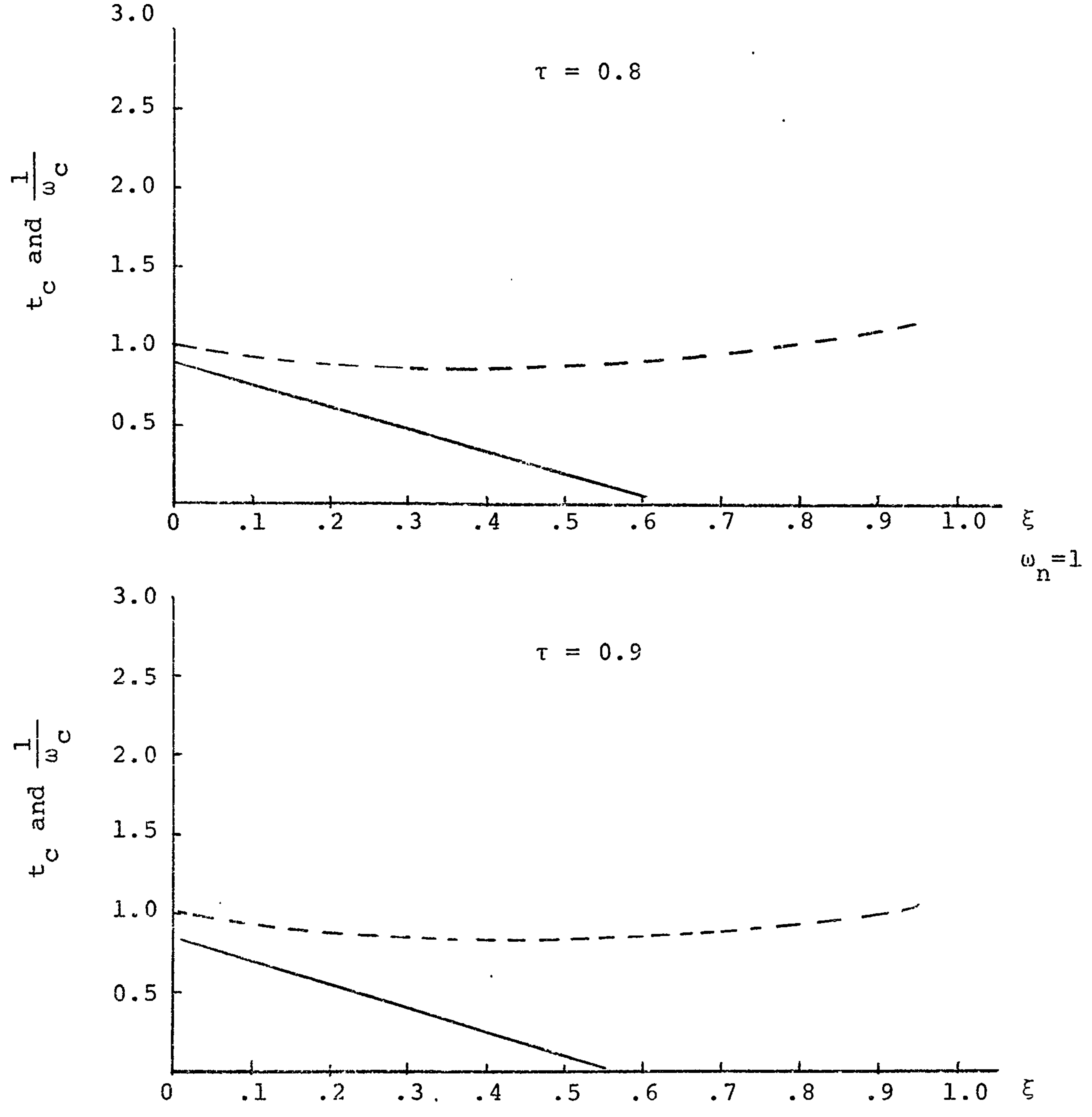


Fig. 2.34 Relationship between t_c and $\frac{1}{\omega_c}$

———— t_c — — — — — $\frac{1}{\omega_c}$.

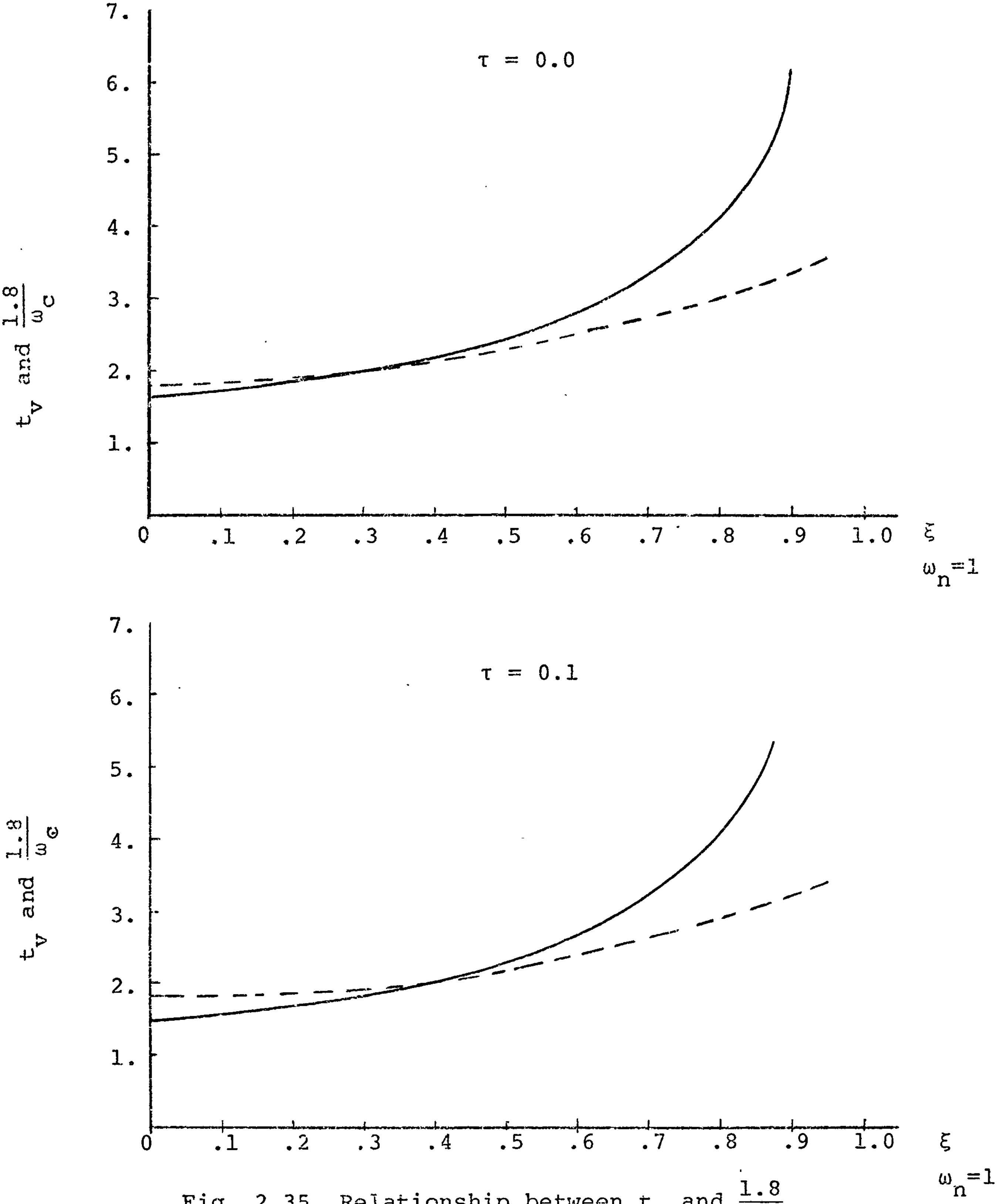


Fig. 2.35 Relationship between t_v and $\frac{1.8}{\omega_c}$

———— t_v — — — — — $\frac{1.8}{\omega_c}$.

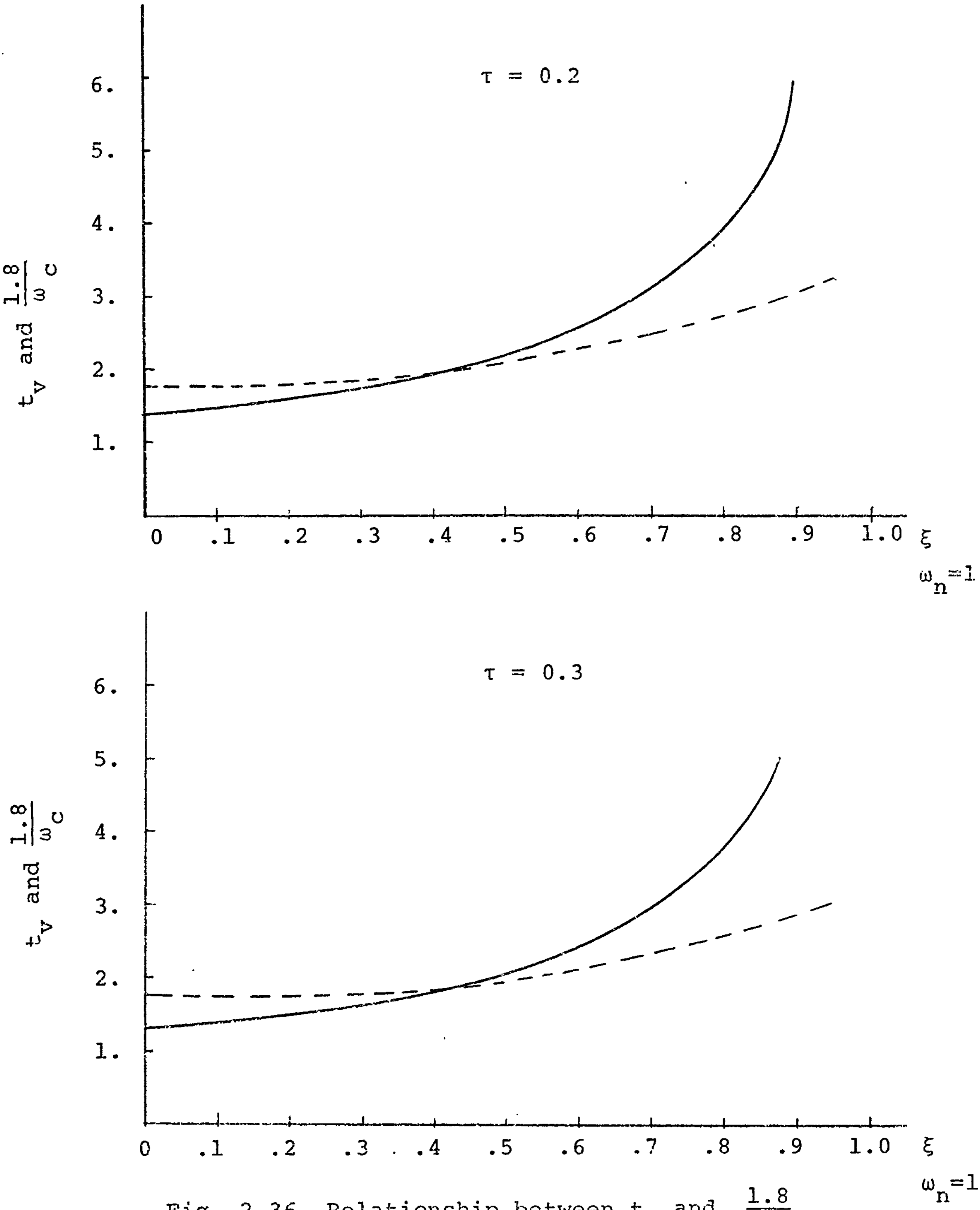
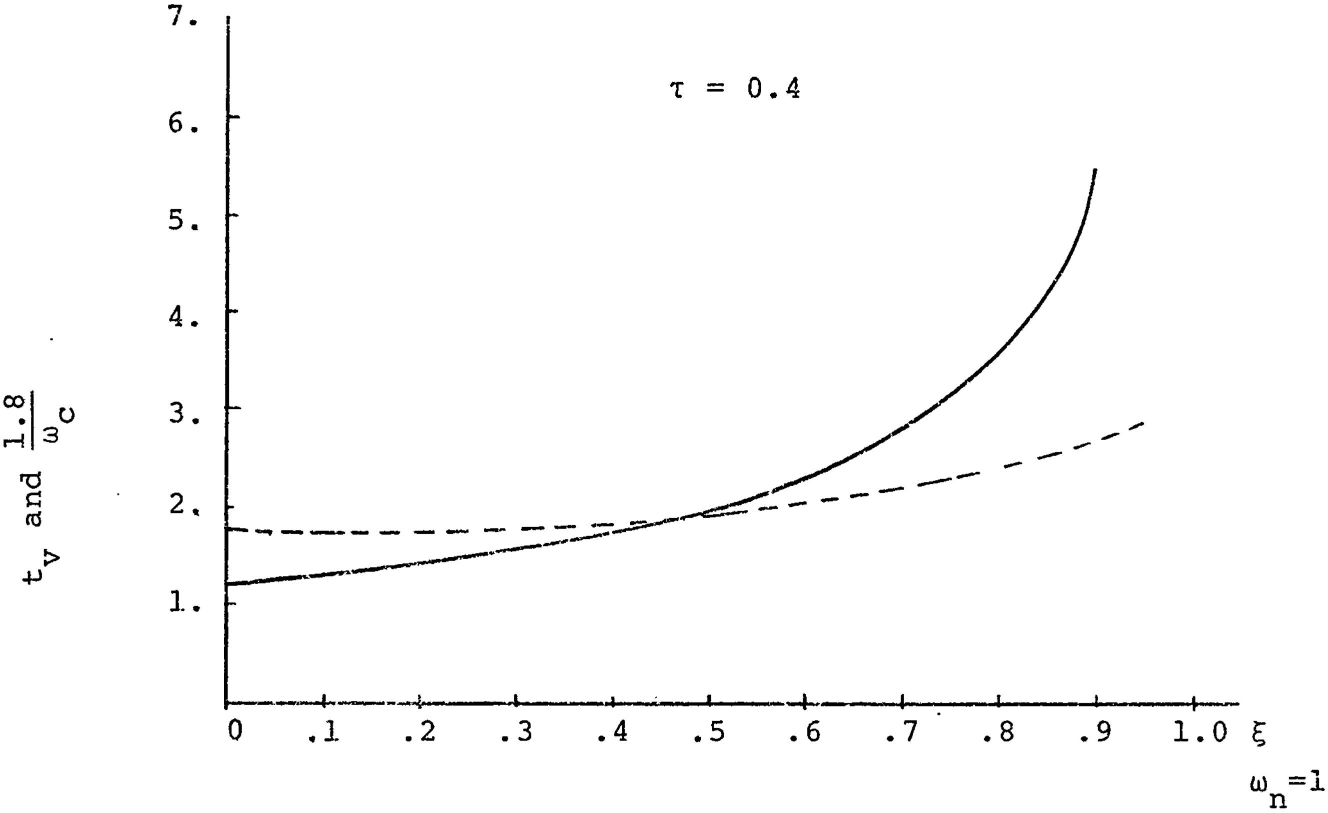


Fig. 2.36 Relationship between t_v and $\frac{1.8}{\omega_c}$

———— t_v — — — — — $\frac{1.8}{\omega_c}$.



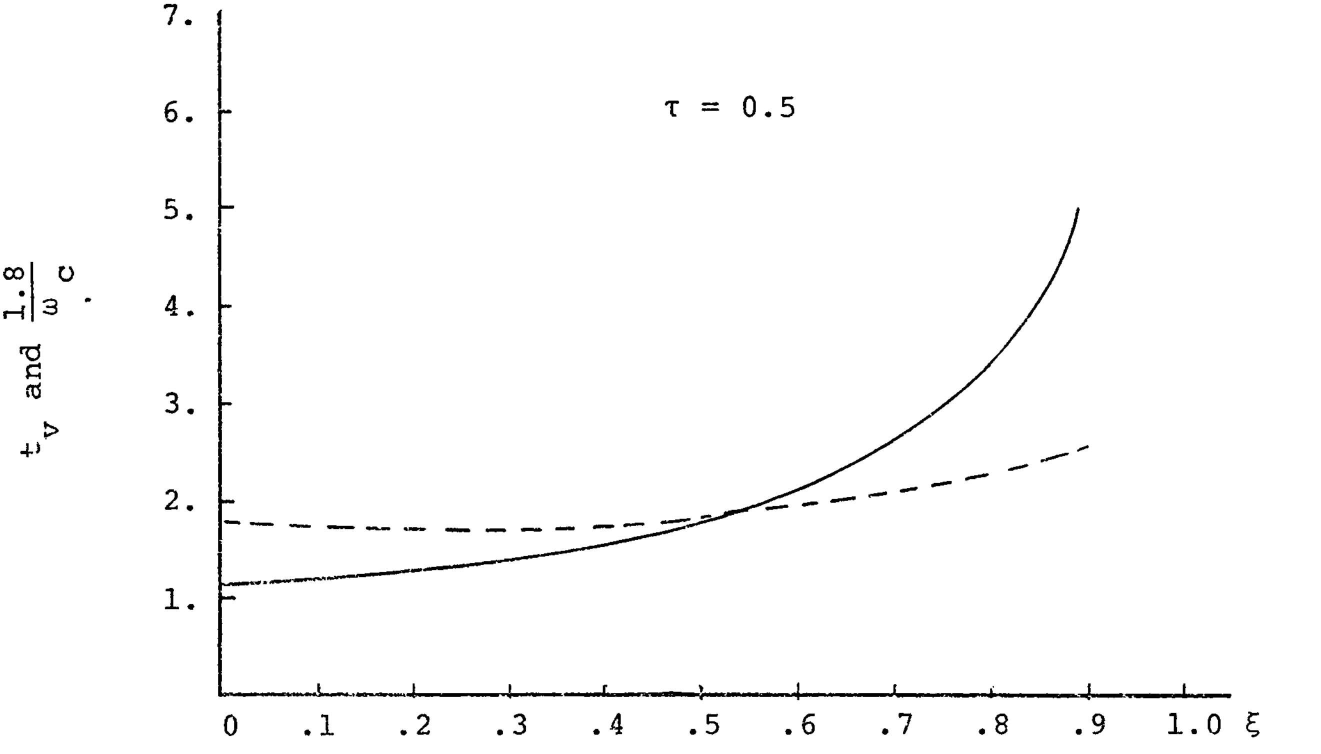
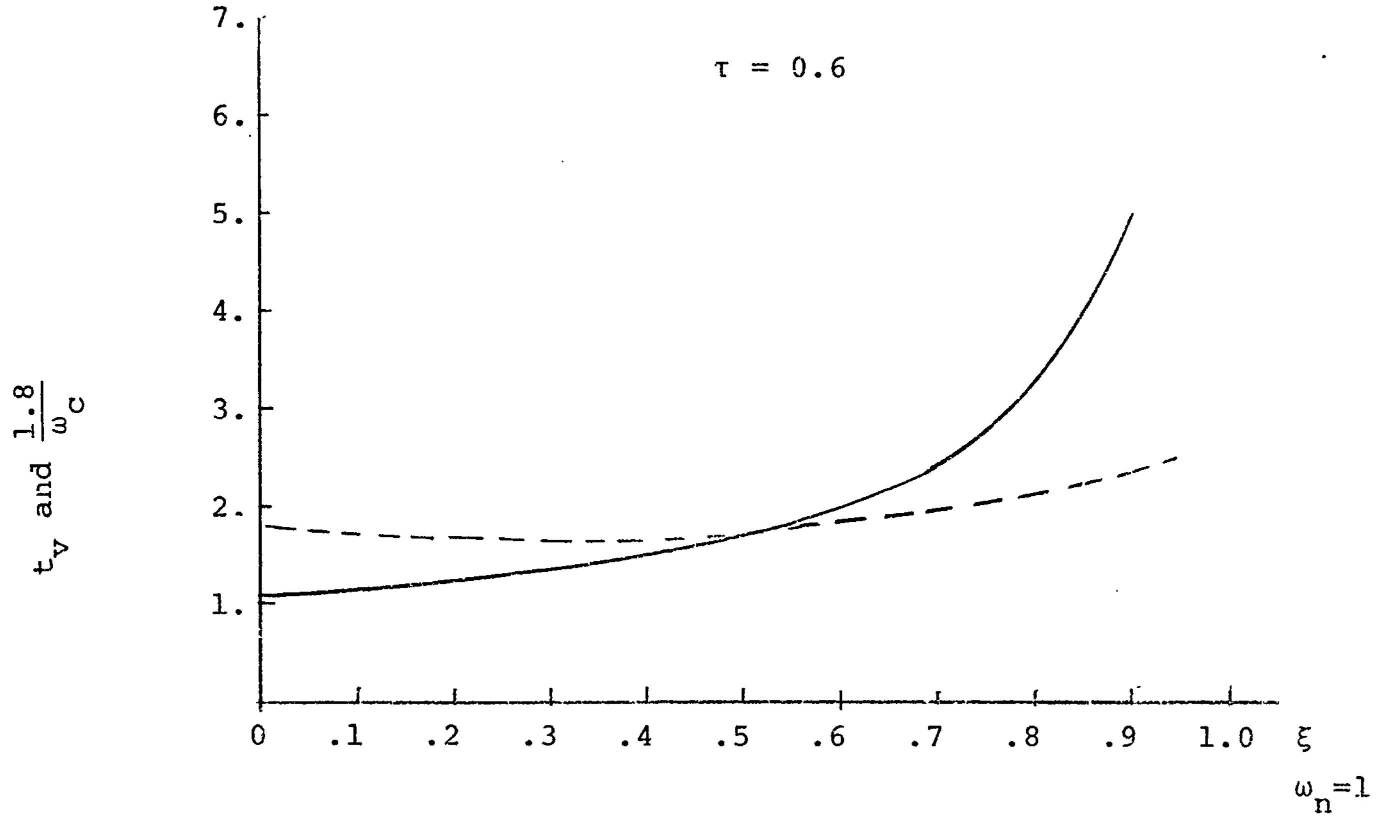


Fig. 2.37 Relationship between t_v and $\frac{1.8}{\omega_c}$

———— t_v — — — — — — $\frac{1.8}{\omega_c}$.



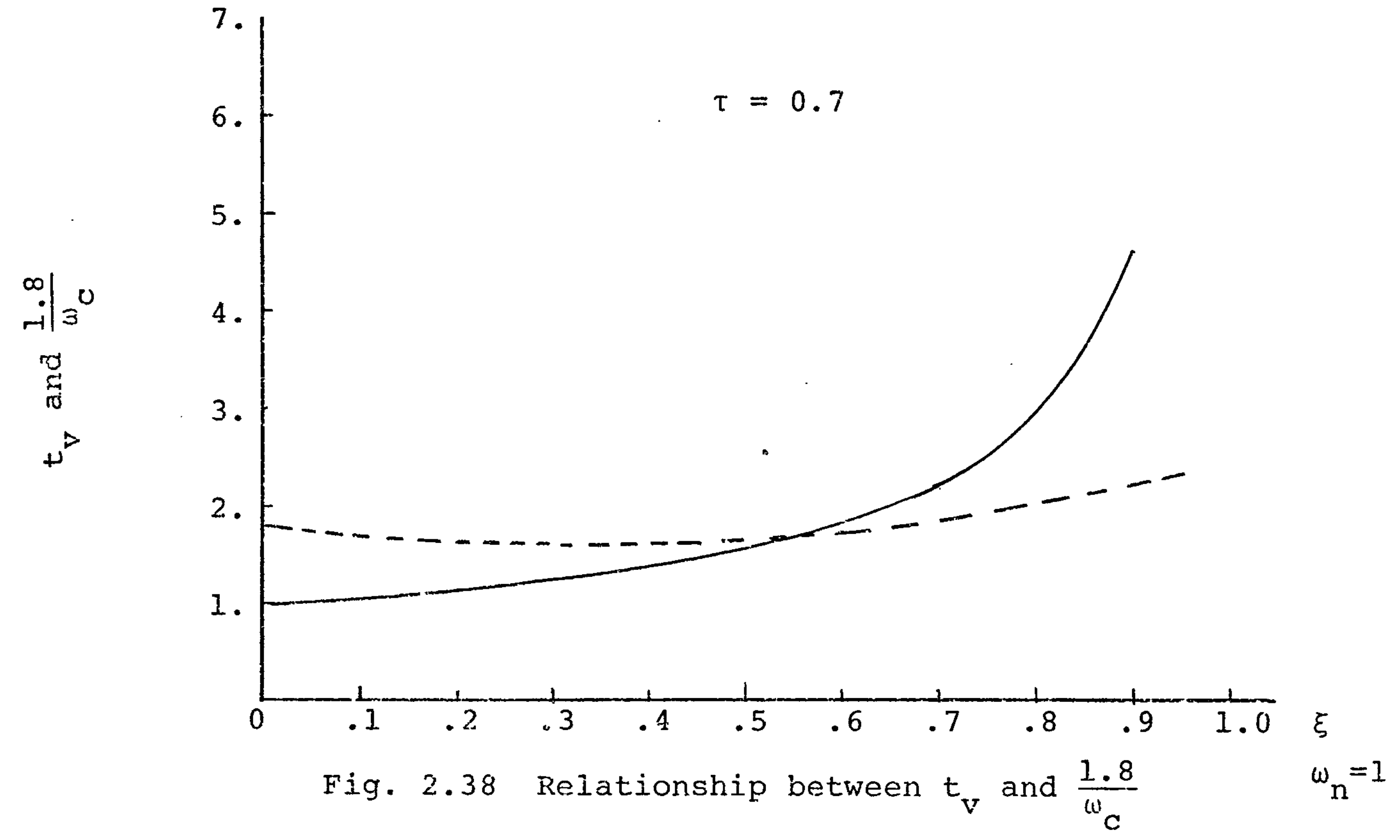
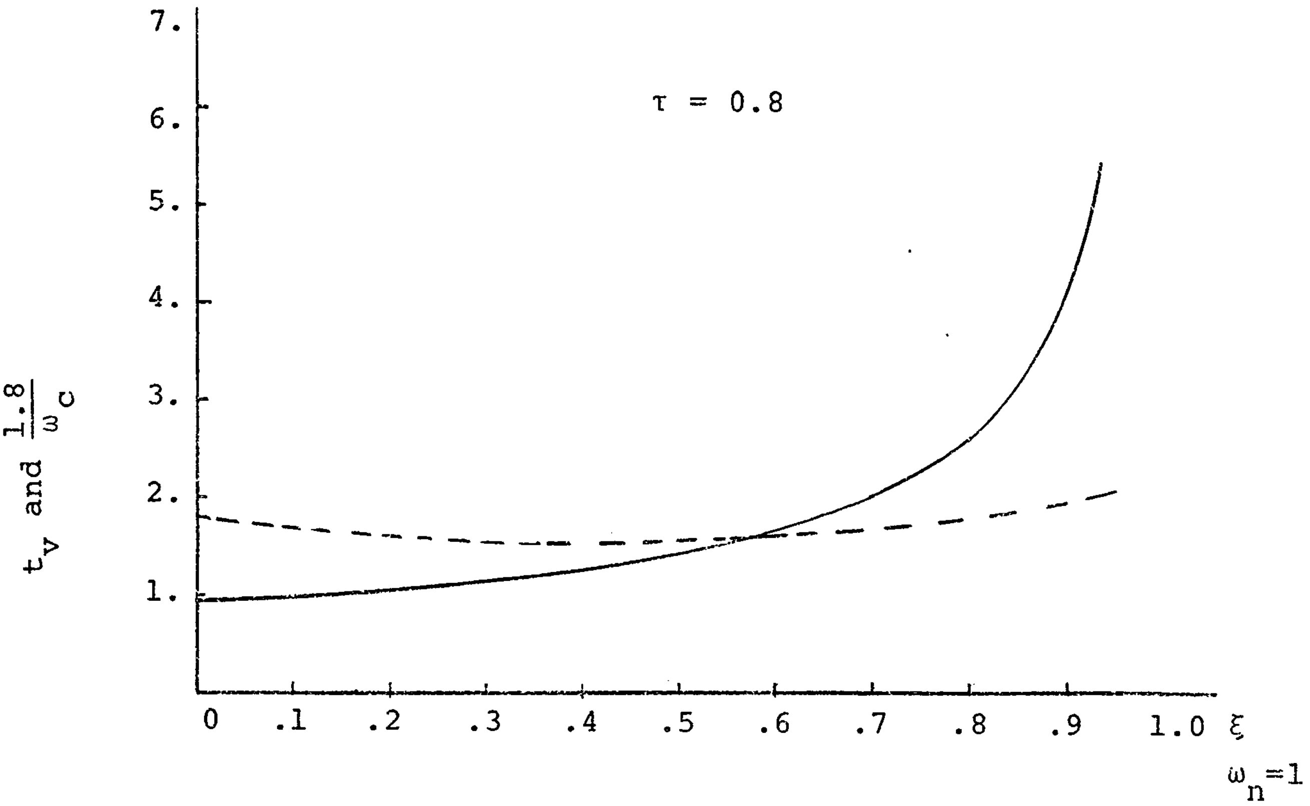


Fig. 2.38 Relationship between t_v and $\frac{1.8}{\omega_c}$

———— t_v — — — — — — $\frac{1.8}{\omega_c}$.



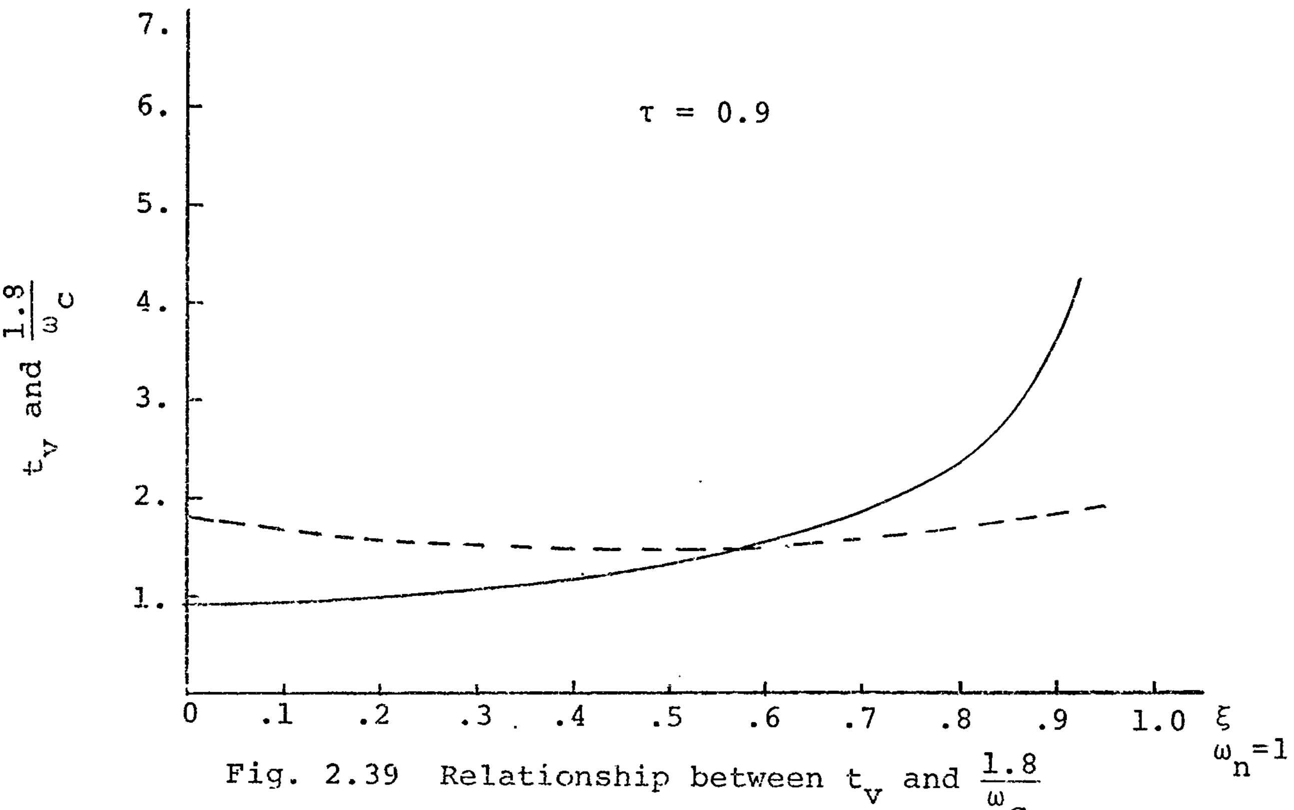


Fig. 2.39 Relationship between t_v and $\frac{1.8}{\omega_c}$

———— t_v — — — — — $\frac{1.8}{\omega_c}$.

$$a_1 = 1 \qquad b_1 = 1464.786701$$
$$a_2 = 112.04 \qquad b_2 = 7958$$
$$a_3 = 3755.92 \qquad b_3 = 533760.7473$$
$$a_4 = 39736.62 \qquad b_4 = 617497.375$$
$$a_5 = 363650.56 \qquad (2.53)$$
$$a_6 = 759894.19$$
$$a_7 = 683656.25$$
$$a_8 = 617497.375$$

It is required to compare the specifications obtained from the original system and the reduced model.

From the exact response curves in the time domain and the frequency domain, we found a set of specifications for this high order system which are written as follows:

$$M_e = 0.8925$$
$$M_t = 1.497$$
$$M_p = 2.22$$
$$\phi_m = -153^{\circ}$$
$$\dot{M}_t = 0.84899 \qquad (2.54)$$
$$\omega_p = 1$$
$$\omega_c = 1.01$$
$$t_p = 2.8$$
$$t_c = 1.05$$
$$t_v = 1.6$$

The simplified second-order model of this high order system is .

$$T^*(s) = \frac{0.254407s + 1.051966}{s^2 + 0.509768s + 1.051966} \tag{2.55}$$

From this second-order model we know that

$$\begin{aligned} \omega_n &= 1.02565 \\ \tau &= .24804 \\ \xi &= .248508 \end{aligned} \tag{2.56}$$

From Eq.(2.54), we obtain the relationships between time-domain specifications and frequency domain specifications of this high order system which are written as follows.

$$\begin{aligned} &M_t \neq M_p \quad \left|\frac{1}{\sin \phi_m}\right| \neq M_t \\ &\dot{M}_t = 0.84\ \omega_c \\ &M_e = 0.9\ \frac{1}{\omega_c} \\ &\omega_p = 0.99\ \omega_c \\ &t_p = 2.8/\omega_c \\ &t_v = 1.6/\omega_c \\ &t_c = 1/\omega_c \end{aligned} \tag{2.57}$$

By using the equivalent specifications obtained from the reduced model in Eq.(2.56) and the curves shown in Figs. 2.3, 2.10, 2.15, 2.20, 2.26, 2.31, 2.36, we can determine the following equivalent relationships between the time domain and frequency domain specifications of the reduced second order model.

$$
\begin{aligned}
& M_t \neq M_p \quad \left|\frac{1}{\sin \phi_m}\right| \neq M_t \qquad \text{when } \xi < 0.4 \\
& M_t = 0.75\ \omega_c \\
& M_e = 0.88 \frac{1}{\omega_c} \\
& \omega_p = 0.9\ \omega_c \\
& t_p = 3.0/\omega_c \\
& t_v = 1.7/\omega_c \\
& t_c = 1.1/\omega_c
\end{aligned}
\tag{2.58}
$$

Comparing the results obtained in Eq.(2.57) and Eq.(2.58) we observe that the rules, which are used to express the relationships among specifications, hold in both the original system and the reduced system. In other words, we can determine the specifications of a high order system by using the equivalent specifications obtained from the reduced model and the rules developed in this research.

CHAPTER III

MODELLING CONTROL SYSTEMS

3.1 Introduction

Model reference tecnnique is successfully used in many control systems designs, particularly in the field of model reference adaptive control systems. Wilkie and Perkins[11] have proposed a simple method to minimize an algebraic function, thus the minimization of the integral square error can be avoided when a reference model, whose order is equal to that of the system to be designed, can be established. In that paper,[11] they have indirectly exposed the advantages and necessity of a high-order reference model. For example, if a high-order reference model is used, no system simulations are required in the design of a model control system. While in state-space control system design, the need of a high-order standard model is most urgent, particularly on the pole-zero assignment problems. In order to construct a high-order standard model, Shieh and Huang[5] developed another technique to construct a low-order system based on the limited number of industrial specifications which are available, then, expand the low-order model into an approximate high-order one. In their technique, the Newton-Raphson multidimensional method (see Appendix) is applied to solve the resulting in non-linear simultaneous equations. However, if the order of the non-linear equation is higher than two, the initial guess gives some trouble. Shieh and Huang[5] dealt with this problem by using Ausman's approximation[15] and Bode's asymptotes.[16] In this chapter we first review the single-input-single-output low order system, and develop a new method to estimate the initial guess of the Newton-Raphson multidimensional method, then we concentrate on the multivariable control system model construction, and finally, the corresponding state-space equations which are the minimal realizations of the multivariable standard model are constructed by means of the matrix continued fraction.

3.2 The Second-Order Model with Phase-Advance Factor

Consider the well-known second-order model, the transfer function is

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} \tag{3.1}$$

where $R(s)$ = Laplace transform of the system input signal.

$C(s)$ = Laplace transform of the system output signal.

ξ = the damping ratio in complex domain.

ω_n = the system natural angular frequency in the frequency domain.

ξ and ω_n are the specifications used to specify transient response. Obviously, the steady-state is ignored in the formulation of equation (3.1). In a servomechanism design, or a follow-up control system design, the most desirable property is that the controlled system follow the variation in the reference input rapidly and accurately. Therefore, the velocity error constant K_v, a specification of steady-state response, is an important element to be prescribed. Taking the velocity error constant into consideration, the modified form of Eq. (3.1) can be written as

$$\frac{C(s)}{R(s)} = \frac{bs + \omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} \tag{3.2}$$

where b is an unknown constant to be determined, we simply follow the definition of K_v or

$$K_v = \lim_{s \to 0} sG(s)$$

where G(s) is the open-loop transfer function

$$G(s) = \frac{bs + \omega_n^2}{s(s + 2\xi\omega_n - b)} \tag{3.3}$$

We have

$$b = \omega_n(2\xi - \frac{\omega_n}{K_v}) \tag{3.4}$$

the modified second-order model is

$$\frac{C(s)}{R(s)} = \frac{\omega_n(2\xi - \frac{\omega_n}{K_v})s + \omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} \tag{3.5}$$

Thus the follow-up velocity in the model reference design can be prescribed. The appearance of a zero in Eq. (3.5) is very significant since the transient and steady-state response of a system are greatly affected by the zero.[17] For example, the zero in a transfer function can be used to control overshoot and steady-state velocity error, etc.; on the other hand, the zero of Eq. (3.5) will contribute a leading phase angle in the frequency response analysis. Therefore, the pole assignment is not only the determinative factor to be considered in design control systems. In other words, in ignoring the role of zeros in a linear control system it is impossible to obtain the satisfactory dynamic properties.

The control system designs, often the specifications other than K_v and ξ are specified. The equivalent specification in Eq.(3.5) can be obtained from Chapter II.

3.3 The Third-Order Control System Model

In practical problems quite often more than three specifications are given. To meet the specifications perfectly and precisely, a high-order model is necessary. Chen and Shieh[6] developed an original synthesis technique to fit a second-order transfer function based on three industrial specifications. The Newton-Raphson multidimensional method is applied to solve the resulting non-linear simultaneous equations. However, if the order of the non-linear equation is higher than two, the initial guess is not so easy to decide. Here the author develops a new method to estimate the initial value by means of the continue fraction. We will illustrate this method by using an example.

Suppose that the following specifications are given:

$$
\begin{aligned}
\omega_c &= \text{crossover frequency} &&= 4.7 \\
\phi_m &= \text{phase margin} &&= 45.6 \\
M_p &= \text{peak value} &&= 1.5 \\
\omega_p &= \text{peak value frequency} &&= 3.5 \\
\omega_b &= \text{bandwidth} &&= 6.5
\end{aligned}
\qquad (3.6)
$$

The first two are open-loop specifications, while the last three are closed-loop ones. A third-order model is necessary to represent the specifications.

Let us assume a third-order model

$$\frac{C(s)}{R(s)} = T(s) = \frac{b_1 s^2 + b_2 s + a_3}{s^3 + a_1 s^2 + a_2 s + a_3} \tag{3.7}$$

Following the definitions shown in Eq.(3.6), we construct the non-linear equations:

(i) the definition of ω_c is

$$|G(j\omega_c)| = 1 \tag{3.8}$$

where $G(j\omega_c)$ is the frequency response of the open loop system evaluated at ω_c. we write the non-linear equation as:

$$(a_1-b_1)^2\omega_c^4 + [\omega_c^3-(a_2-b_2)\omega_c]^2 - (a_3-b_1\omega_c^2)^2$$
$$-b_2^2\,\omega_c^2 = 0 \tag{3.9}$$

(ii) The definition of ϕ_m can be written as

$$\phi_m = 180° + \underline{/\ G(j\omega_c)} \tag{3.10}$$

this means:

$$b_2\omega_c^2(a_1-b_1) - (a_3-b_1\omega_c^2)(\omega_c^2-a_2+b_2)$$
$$-\tan\phi_m\,[(a_3-b_1\omega_c^2)(a_1-b_1)\omega_c + b_2\omega_c(\omega_c^2-a_2+b_2)] = 0 \tag{3.11}$$

(iii) The definition of ω_b is known as:

$$\left|\frac{C}{R}(j\omega_b)\right| = \frac{1}{\sqrt{2}} \tag{3.12}$$

The corresponding non-linear equation is:

$$(a_3-b_1\omega_b^2)^2+b_2{}^2\omega_b^2-\frac{1}{2}\,[(a_3-a_1\omega_b^2)+(\omega_b^3-a_2\omega b)^2] = 0 \qquad (3.13)$$

(iv) The definition of M_p is:

$$\left|\frac{C}{R}(j\omega_p)\right| = M_p \qquad (3.14)$$

The corresponding non-linear equation is:

$$(a_3-b_1\omega_p^2)^2+b_2^2\,\omega_p^2-M_p{}^2[(a_3-a_1\omega_p^2)^2+(\omega_p^3-a_3\omega_p)^2] = 0 \qquad (3.15)$$

(v) The definition of ω_p is the frequency at which

$$\frac{d\,M_p}{d\,\omega_p} = 0 \qquad (3.16)$$

Following Higgins and Siegel's[18] complex variable differential technique

$$\begin{aligned}&[2a_1a_3\omega_p-2a_1^2\omega_p^3-(a_3-3\omega_p^2)(-\omega_p^3+a_2\omega_p)]\\&[(a_3-b_1\omega_p^2)^2+(b_2\omega_p)^2]+[-2a_3b_1\omega_p+2b_1^2\omega_p^3+b_2^2\omega_p]\\&[(a_3-a_1\omega_p^2)^2+(-\omega_p^3+a_2\omega_p)^2] = 0\end{aligned} \qquad (3.17)$$

Eqs. (3.9), (3.11), (3.13), (3.15), and (3.17) are a set of non-linear simultaneous equations. After substituting the given values into these equations, we can solve for the unknowns a_i and b_i. In order to have a very good initial value, we have to choose three specifications from the given specifications, and construct a second order model first. In this example, assume we choose:

$M_p = 1.5$

$\omega_p = 3.5$

$\omega_b = 6.5$

Using M_p and ω_p to find the dominate poles of the second-order system model by referring to Fig. 3.1

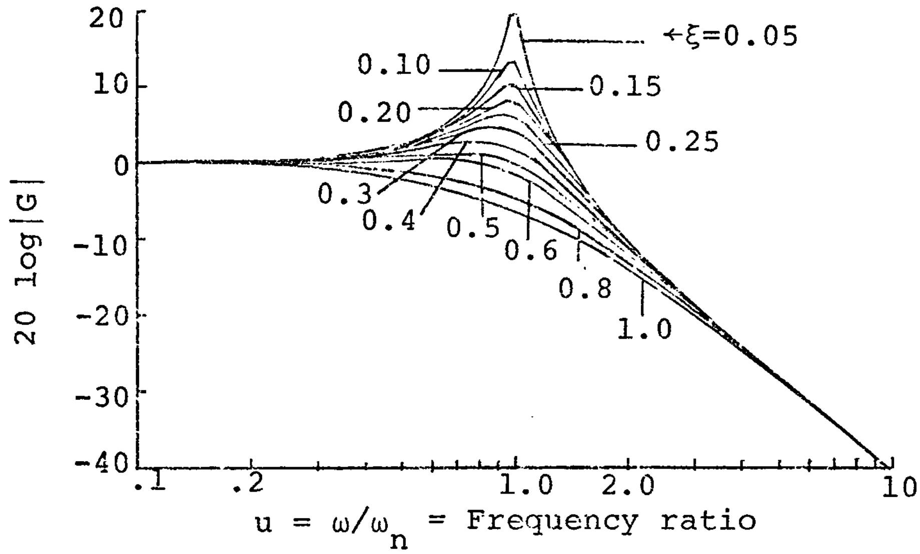


Fig. 3.1 Frequency Response Plots

the dominant poles are

$\lambda_1 = -1.3611+j\,3.6429$

$\lambda_2 = -1.3611-j\,3.6429$

So the second-order model can be constructed as follows:

$$T(s) = \frac{bs + 15.12345679}{s^2 + 2.72222s + 15.12345679}$$

solving the unknown b by using the definition ω_b

$b = 2.645762092$

Since the second-order model is

$$T(s) = \frac{2.645762092S + 15.12345679}{s^2 + 2.72222S + 15.12345679}$$

the continued-fraction expansion is

$$T(s) = \cfrac{1}{h_1 + \cfrac{1}{\frac{h_2}{s} + \cfrac{1}{h_3 + \cfrac{1}{\frac{h_4}{s}}}}} \tag{3.18}$$

where

$h_1 = 1.$ $\qquad h_2 = 197.7953319$

$h_3 = -0.0003918027$ $\qquad h_4 = -195.1495698$

The quotients h_1, h_2, h_3 and h_4 are the significant quotients of this system. We need to insert a group of insignificant quotients into the expanded continued-fraction to obtain a stable approximate high-order model without significantly distorting the characteristic of the original system. Shieh and Huang[5] found the regularity of these insignificant quotients which are as follows:

(1) $h_{2n+j} \cdot h_{2n+j+1} > 0$

(2) $| h_{2n+j} | > | h_{2n+j+1} |$

$j=1,3,5.\ldots,2m-2n-1.$

In practical applications one more rule is suggested to simplify the manipulations. That is

$$\begin{aligned} h_{2n+j} &= h_{2n+j+2} \\ h_{2n+j+1} &= h_{2n+j+3} \end{aligned} \qquad j=1,3,5\ldots$$

In this example, we need to have a third-order model, so h_5 and h_6 should be added to the expanded continued-fraction. Assume

$$h_5 = 100 \quad , \quad h_6 = 0.1$$

Using h_1, h_2, h_3, h_4, h_5, and h_6, we can construct a new third-order model which can be simplified into the original second-order model.

The third-order model is:

$$T(s) = \frac{2.7457621s^2 + 41.573328s + 151.2345679}{s^3 + 12.82218305s^2 + 42.33792928s + 151.2345679}$$

From which the initial guess is easily made:

$$\begin{aligned} a_1 &= 12.82218305 \qquad & b_1 &= 2.7457621 \\ a_2 &= 42.33792928 \qquad & b_2 &= 41.573328 \\ a_3 &= 151.2345679 \end{aligned} \tag{3.19}$$

We use Eq.(3.19) as the first guess, and substituting the data into the multidimensional Newton-Raphson formula. After seven iterations the model is obtained to meet all the specifications. The transfer function is as follows:

$$T(s) = \frac{3.18835518s^2 + 15.56104302s + 29.80626}{s^3 + 4.26715952s^2 + 20.58799529s + 29.80626} \tag{3.20}$$

3.4 High-Order Model Control Systems

In design follow-up control systems or state-space feedback designs more than often a high-order model is necessary. Shieh and Huang[5] developed a method based on the continued-fraction to establish an approximate high-order model from a low-order model which is constructed with the industrial specifications. Sometimes, the dominant poles and the other poles of the system have been assigned, the only thing we have to do is to determine the zeros of the system. Here we present another method by using the Newton-Raphson method.

Suppose that the following specifications are given:

(i) The poles of a closed-loop system (Type 1 system) are assigned at

$$\lambda_1 = -1.36 + j3.64 \qquad \lambda_3 = -5$$
$$\lambda_2 = -1.36 - j3.64 \qquad \lambda_4 = -10 \tag{3.21}$$

The dominant poles are λ_1 and λ_2.

(ii) ω_b = The bandwidth frequency = 6

(iii) ω_c = The crossover frequency = 4

(iv) ϕ_m = The phase margin = 37

A fourth-order transfer function is required to meet those specifications. Assume the transfer function is:

$$T(s) = \frac{1}{\Delta_o(s)} q(s) = \frac{a_{21} + a_{22}s + a_{23}s^2 + a_{24}s^3}{a_{11} + a_{12}s + a_{13}s^2 + a_{14}s^3 + a_{15}s^4} \tag{3.22}$$

Following the basic definitions of the industrial specifications shown in Eq.(3.21) yields a set of non-linear equations as follows:

(i) The pole assignment implies:

$$\Delta_o(s) = a_{11} + a_{12}s + a_{13}s^2 + a_{14}s^3 + a_{15}s^4$$
$$= (s-\lambda_1)(s-\lambda_2)(s-\lambda_3)(s-\lambda_4) \tag{3.22a}$$
$$= 755 + 362.5s + 105.9s^2 + 17.72s^3 + s^4$$

This means

$$a_{11} = 755 \qquad a_{12} = 362.5$$

$$a_{13} = 105.9 \qquad a_{14} = 17.72 \tag{3.22b}$$

$$a_{15} = 1.$$

(ii) The definition of ω_b gives

$$|T(j\omega_b)| = \frac{1}{\sqrt{2}} \tag{3.22c}$$

The non-linear equation is

$$(a_{11}-a_{23}\omega_b^2)^2 + \omega_b^2(a_{22}-a_{24}\omega_b^2)^2 -$$

$$\frac{1}{2}\left[(a_{11}+\omega_b^4-a_{13}\omega_b^2)^2+\omega_b^2(a_{12}-a_{14}\omega_b^2)^2\right] = 0 \tag{3.22d}$$

(iii) The definition of ω_c is:

$$|G(j\omega_c)| = 1 \tag{3.22e}$$

where $G(j\omega_c)$ is the frequency response of the open-loop system evaluated at ω_c. The corresponding non-linear equation is:

$$(a_{11}-a_{23}\omega_c^2)^2 + \omega_c^2(a_{22}-a_{24}\omega_c^2)^2-\omega_c^4[\omega_c^2-(a_{13}-a_{23})]^2$$

$$- \omega_c^2[(a_{12}-a_{22})-(a_{14}-a_{24})\omega_c^2]^2 = 0 \tag{3.22f}$$

(iv) The definition of ϕ_m can be represented as:

$$\phi_m = \Pi + \underline{/G(j\omega_c)} \tag{3.22g}$$

The corresponding equation is

$$\omega_c^2(a_{22}-a_{24}\omega_c^2)(\omega_c^2-a_{13}+a_{23})-(a_{11}-a_{23}\omega_c^2)[(a_{12}-a_{22})$$

$$- (a_{14}-a_{24})\omega_c^2]-\tan\phi_m[\omega_c(a_{11}-a_{23}\omega_c^2)(\omega_c^2-a_{13}+a_{23})$$

$$+\ \omega_c(a_{22}-a_{24}\omega_c^2)(a_{12}-a_{22}-a_{14}\omega_c^2+a_{24}\omega_c^2)] = 0 \qquad (3.22h)$$

Eqs.(3.22d), (3.22f), and (3.22h) are a set of high-order non-linear equations. Using the given values in Eq.(3.22) and Eq.(3.22b) and applying the Newton-Raphson multidimensional method, we can solve a_{2i}. However, the Newton-Raphson multidimensional methodonly converges for a small range of starting values or the initial values. The following new procedure is suggested in this chapter to determine good starting values.

The matrix equation proposed by Chen and Shieh[6] can be applied to determine the starting values and can be written as follows

$$[a_2] = [H][a_1] \qquad (3.23a)$$

where

$$[a_2]^T = [a_{21}, a_{22}, a_{23}, a_{24}, \ldots a_{2p}]$$

$$[a_1]^T = [a_{11}, a_{12}, a_{13}, \ldots, a_{1p}]$$

$$[H] = [H_2]^{-1}[H_1]$$

$$[H_2] = \begin{bmatrix} h_1 & 0 & & 0 \\ 1 & h_2 & 0 & 0 \\ 0 & 1 & h_3 & \cdot \\ 0 & \cdot & 1 & \cdot \\ \cdot & \vdots & \vdots & \cdot \\ 0 & 0 & 1 & h_p \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & h_1 & 0 & \cdots & 0 \\ 0 & 1 & h_2 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & h_{p-1} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & h_1 \end{bmatrix}$$

$$[H_1] = \begin{bmatrix} 1 & 0 & 0 & \cdot & 0 \\ 0 & h_2 & 0 & \cdot & 0 \\ 0 & 1 & h_3 & \cdot & \cdot \\ 0 & 0 & 1 & \cdot & \cdot \\ \cdot & \cdot & 0 & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & 1 \end{bmatrix} \cdots \begin{bmatrix} 1 & 0 & \cdot & 0 & 0 \\ 0 & 1 & \cdot & 0 & 0 \\ 0 & 0 & \cdot & \cdot & 0 \\ 0 & 0 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & 1 & h_{p-1} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & \cdot & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & 0 & h_2 \end{bmatrix}$$

The a_{1i} and a_{2i} are the coefficients of the transfer function $g(s)$ in Eq.(3.22). The h_i in Eq.(3.23a) are obtained by expanding the $g(s)$ into the continued fraction of the second Cauer form shown below:

$$g(s) = \frac{a_{21} + a_{22}s + \ldots + a_{2,n}s^{n-1}}{a_{11} + a_{12}s + a_{13}s^2 + \ldots + a_{1,n+1}s^n}$$

$$= \cfrac{1}{h_1 + \cfrac{1}{\cfrac{h_2}{s} + \cfrac{1}{h_3 + \cfrac{1}{\cfrac{h_4}{s} + \ddots}}}} \qquad (3.23b)$$

or by the process of evaluating starting values for the Newton-Raphson multidimensional method described as follows. First a low-order transfer function $g^*(s)$ is constructed by using some dominant poles and specifications assigned. This $g^*(s)$ can be considered as a reduced model of the expected function $g(s)$. Next, expanding the $g^*(s)$ into the continued-fraction of the second Cauer form yields a set of quotients h_i. Finally, using these h_i's as dominant quotients in Eq.(3.23a) and constructing the vector a_1 with the coefficients a_{1i} in Eq.(3.22b), we can solve the vector a_2 in Eq.(3.23a). The elements in the vector a_2 are the required starting values. In this example:

The $g^*(s)$ is

$$g^*(s) = \frac{b_1 s + b_2}{(s-\lambda_1)(s-\lambda_2)} = \frac{b_1 s + b_2}{s^2 + 2.72s + 15.1} \tag{3.24a}$$

The system is of type 1, therefore $b_2 = 15.1$. By using $\omega_c = 4$ and following the definition of ω_c, we have $b_1 = 1.68157$.

The $g^*(s)$ is

$$g^*(s) = \frac{1.68157s + 15.1}{s^2 + 2.72s + 15.1} \tag{3.24b}$$

Expanding Eq.(3.24b) into the continued-fraction of the second Cauer form yields a set of quotients h_i i.e.,

$h_1 = 1$ $\qquad$ $h_2 = 14.54127$

$h_3 = -0.08075$ $\qquad$ $h_4 = -12.8596$

Substituting these h_i values and the a_{1i} values in Eq.(3.22b) into Eq.(3.23a), we have the required starting values $[a^o_{2i}]$, i.e.,

$a^o_{21} = 755$ $\qquad$ $a^o_{22} = 310.5788$

$a^o_{23} = 40.323$ $\qquad$ $a^o_{24} = 1.68165$

Using these values as starting values for the Newton-Raphson method, we can obtain the required values of a_{2i} at the fourth iteration. The required $g(s)$ is:

$$g(s) = \frac{1.5803859s^3 + 39.03793s^2 + 320.08482s + 755}{s^4 + 17.72s^3 + 105.9s^2 + 362.5s + 755} \tag{3.24c}$$

3.5 Modelling Multivariable Control Systems in the Frequency Domain

Let a multivariable system having m inputs and ℓ outputs be described by the following transfer function matrix:

$$[Y(s)] = [G_o(s)][R(s)]$$

where

$$[G_o(s)] = \frac{1}{\Delta_o(s)} [Q(s)] = \frac{1}{\Delta_o(s)} \{q_{i,j}(s)\}$$

$$\Delta_o(s) = \sum_{i=1}^{n+1} a_i s^{i-1} \ , \ a_1 \neq 0 \quad \text{and} \quad a_{n+1} = 1 \tag{3.25}$$

The dimensions of $[Y(s)]$, $[G(s)]$ and $[R(s)]$ are $\ell \times 1$, $\ell \times m$, and $m \times 1$, respectively. Let us define $q = \min(\ell, m)$ and $r_o = \text{rank}[G_o(s)]$, which can be obtained by applying Gilbert's theorem[12] or by checking the rank of the Hankel matrix[13] obtained from $[G_o(s)]$. Each $q_{i,j}(s)$ and $g_{i,j}(s)$ is the element which is located respectively at the ith row and the jth column in the matrices $[Q(s)]$ and $[G_o(s)]$. The transfer function $g_{i,j}(s)$ of a subsystem which is the transfer function of the ith output to the jth input is:

$$g_{i,j}(s) = \frac{1}{\Delta_o(s)} q_{i,j}(s) = \frac{a_{21} + a_{22}s + a_{23}s^2 + \ldots\ldots + a_{2,n}s^{n-1}}{a_{11} + a_{12}s + \ldots\ldots + a_{1,n+1}s^n} \tag{3.26}$$

Using the basic definition of the given industrial specifications and considering the poles assigned, in general, we can construct n non-linear equations. The Newton-Raphson multidimensional method can be applied to find the solutions $a_{2,j}$ of the non-linear equations. The procedures are just like those for the single-input-single-output system. The method proposed to construct the multivariable system model may be summarized as follows:

Step 1. Determine the degree of the common denominator $\Delta_o(s)$ and the order of the standard multivariable model required. The desired $\Delta_o(s)$ and the characteristic polynomial $\Delta(s)$. (i.e., the least common denominator polynomial of all the minors of $G_o(s)$ are:

$$\Delta_o(s) = (s-\lambda_1)(s-\lambda_2)(s-\lambda_3)\cdots(s-\lambda_n) = \sum_{j=1}^{n+1} a_{1,j}s^{j-1} \qquad (3.27)$$

$$a_{1,n+1} = 1, \quad \text{and} \quad a_{1,1} \neq 0$$

$$\Delta(s) = \{\Delta_o(s)\}^q = (s-\lambda_1)^q(\lambda-\lambda_2)^q \cdots (s-\lambda_n)^q \qquad (3.28)$$

The eigenvalues λ_j of Eq.(3.27) are assigned in the s-plane. The choice of the λ_j is a design freedom and a certain amount of experience is helpful.

Step 2. Determine the coefficients $a_{2,j}$ of each subsystem $g_{i,j}(s)$ by using the basic definitions of the industrial specifications and the $a_{1,j}$ obtained in Eq.(3.27).

Step 3. Combine each subsystem $g_{i,j}(s)$ to form a composite system $\{g_{i,j}(s)\}$ and examine the properties of $\{g_{i,j}(s)\}$. If the characteristic polynomial $\Delta(s)$ of the composite system $\{g_{i,j}(s)\}$ has the form shown in Eq.(3.28) or $\Delta(s) = \{\Delta_o(s)\}^q$, then the standard model $[G_o(s)]$ is obtained. An illustrative example will be shown as follows.

Consider a multivariable system with three inputs and two outputs. It is required to construct a standard model by using a set of industrial specifications and some pole assignments.

The multivariable system in the frequency domain is:

$$[Y(s)] = [G_o(s)][U(s)]$$

where

$$[G_o(s)] = \{g_{i,j}(s)\} = \frac{1}{\Delta_o(s)}\begin{bmatrix} q_{11}(s) & q_{12}(s) & q_{13}(s) \\ q_{21}(s) & q_{22}(s) & q_{23}(s) \end{bmatrix} \quad (3.29)$$

The eigenvalues of the system $[G_o(s)]$ are assigned as follows

$$\begin{aligned} \lambda_1 &= -1 + j \\ \lambda_2 &= -1 - j \end{aligned} \quad (3.30)$$

For this second order system, the pole assignment implies that the damping ratio = $\frac{1}{\sqrt{2}}$ and the natural angular frequency = $\sqrt{2}$.

The least common denominator $\Delta_o(s)$ of the $[G_o(s)]$ is

$$\Delta_o(s) = (s - \lambda_1)(s - \lambda_2) = 2 + 2s + 1s^2$$

$$\text{or} \quad a_{11} = 2 \quad \text{and} \quad a_{12} = 2 \quad (3.31)$$

The transfer function and the specifications for each subsystem $g_{i,j}(s)$ are:

(1) $g_{11} = \frac{1}{\Delta_o(s)}[a_{21} + a_{22}s]$ (3.32a)

The final value of unit step response is unity.

K_v = The velocity error constant = 20 (3.32b)

By using the basic definition of each specification, we have the following equations

$$a_{21} = a_{11} \quad (3.32c)$$

$$(a_{12} - a_{22})\, K_v = a_{11} \quad (3.32d)$$

(2) $g_{12}(s) = 0$ (3.33)

(3) $g_{13}(s) = \frac{1}{\Delta_o(s)}[b_{21} + b_{22}s]$ (3.34a)

The final value of unit step response is zero

ω_b = The bandwidth frequency = 6.5 rad/sec (3.34b)

The equations to be solved are:

$$b_{21} = 0. \quad (3.34c)$$

$$b_{21}^2 + b_{22}^2\omega_b^2 = \frac{1}{2}[(a_{11} - \omega_b^2)^2 + a_{12}^2\omega_b^2] \quad (3.34d)$$

and

(4) $g_{21}(s) = \frac{1}{\Delta_o(s)}[c_{21} + c_{22}s]$ (3.35a)

The final value of unit step response is zero

ϕ_m = The phase margin = 45° (3.35b)

The corresponding equations are:

$$c_{21} = 0 \quad (3.35c)$$

$$(a_{11} - c_{21} - \omega_c^2)^2 + (a_{12} - c_{22})^2\omega_c^2 = (c_{21}^2 + c_{22}^2\omega_c^2) \quad (3.35d)$$

and

$$c_{22}(a_{11} - c_{21} - \omega_c^2)\omega_c - c_{21}(a_{12} - c_{22})\omega_c =$$
$$\tan\phi_m[c_{21}(a_{11} - c_{21} - \omega_c^2) + c_{22}(a_{12} - c_{22})\omega_c^2] \quad (3.35e)$$

where ω_c is a crossover frequency.

(5) $g_{22}(s) = \frac{1}{\Delta_o(s)}[d_{21} + d_{22}s]$ (3.36a)

The final value of unit step response is unity

ω_c = The crossover frequency = 5 rad/sec (3.36b)

The equations to be solved are:

$$d_{21} = a_{11} \quad , \quad (3.36c)$$

$$(a_{11} - d_{21} - \omega_c^2)^2 + (a_{12} - d_{22})^2\omega_c^2$$

$$(d_{21}^2 + d_{22}^2\omega_c^2) \tag{3.36d}$$

and

(6) $g_{23}(s) = 0$ (3.37)

Substituting the coefficients a_{11} and a_{12} assigned and the specifications given into Eq.(3.32) to Eq.(3.37) yields the following coefficients for each numerator polynomial.

$a_{21} = 2.$ $\quad a_{22} = 1.9$ (3.38a)

$b_{21} = 0.$ $\quad b_{22} = 4.60134$ (3.38b)

$c_{21} = 0.$ $\quad c_{22} = 1.17157$ (3.38c)

$d_{21} = 2.$ $\quad d_{22} = 7.21$ (3.38d)

The standard multivariable system in the frequency domain is:

$$[G_o(s)] = \frac{1}{s^2 + 2s + 2}\begin{bmatrix} 1.9s + 2. & 0 & 4.60134s \\ 1.17157s & 7.21s + 2. & 0 \end{bmatrix} \tag{3.39}$$

3.6 Modelling Multivariable Control Systems in the Time Domain

The standard multivariable model $[G_o(s)]$ constructed in section 3.5 can be transformed into state-space equations which are controllable and observable. When the number of the inputs m is equal to the outputs ℓ, ℓ=m, the matrix $[G_o(s)]$ in Eq.(3.25) can be represented by a matrix continued fraction of the second cauer form as follows:

$$[G_o(s)] = [A_{21} + A_{22}s + \cdots + A_{2,n}s^{n-1}][A_{11} + A_{12}s + \cdots + A_{1,n+1}s^n]^{-1}$$

$$=[H_1+[H_2 \frac{1}{s} +[H_3+[H_4 \frac{1}{s} +[\ldots]^{-1}]^{-1}]^{-1}]^{-1}]^{-1} \tag{3.40}$$

where $A_{1,i} = a_i[I]$ and $A_{2,i} = [Q_i]$, $i=1,2,\ldots\ldots\ldots$

The corresponding state equation of the $[G_o(s)]$ in Eq.(3.40) is:

$$\begin{aligned} \dot{X} &= A_1X + B_1U \\ Y &= C_1^T X \end{aligned} \tag{3.41}$$

where

$$A_1 = - \begin{bmatrix} (H_1)H_2 & (H_1)H_4 & (H_1)H_6 & \cdots & (H_1)H_{2k} \\ (H_1)H_2 & (H_1+H_3)H_4 & (H_1+H_3)H_6 & \cdots & (H_1+H_3)H_{2k} \\ \cdot & \cdot & \cdot & & \cdot \\ (H_1)H_2 & (H_1+H_3)H_4 & (H_1+H_3+H_5)H_6 & & (H_1+\ldots+H_{2k-1})H_{2k} \end{bmatrix}$$

$$= - \begin{bmatrix} H_1 & 0 & 0 & \cdots & 0 \\ H_1 & H_3 & 0 & \cdots & 0 \\ H_1 & H_3 & H_5 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ H_1 & H_3 & H_5 & \cdots & H_{2k-1} \end{bmatrix} \begin{bmatrix} H_2 & H_4 & H_6 & \cdots & H_{2k} \\ 0 & H_4 & H_6 & \cdots & H_{2k} \\ 0 & 0 & H_6 & \cdots & H_{2k} \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & H_{2k} \end{bmatrix}$$

$B_1^T = [I, I, \ldots I]$ and

$C_1^T = [H_2, H_4, \ldots, H_{2k}]$

The matrices H_i, $i = 1, 2, \cdots, 2k$ and $k \leq n$ in Eqs.(3.40) and (3.41) can be obtained by either of the matrix Routh algorithms as follows:

(i)

$$H_i = A_{i,1} [A_{i+1,1}]^{-1}, \quad i = 1,2\cdots,2k \quad \text{and} \quad k \leq n$$

$$\text{rank } [A_{i,1}] = q \qquad q = \text{Min}(\ell,m) \tag{3.42a}$$

$$A_{i,j} = A_{i-2,j+1} - H_{i-2}A_{i-1,j+1} \; ; \; \begin{matrix} j = 1,2,\cdots \\ i = 3,4,\cdots \end{matrix}$$

(ii)

$$H_i = [A_{i+1,1}]^{-1} A_{i,1}, \quad i = 1,2\cdots,2k \quad \text{and} \quad k \leq n$$

$$\text{rank } [A_{i,1}] = q \qquad q = \text{Min}(\ell,m)$$

$$A_{i,j} = A_{i-2,j+1} - A_{i-1,j+1} H_{i-2} \; ; \; \begin{array}{l} j = 1,2,\cdots \\ i = 3,4,\cdots \end{array} \qquad (3.42b)$$

It has been shown by using Gilbert's[11] theorems that Eq.(3.41) is a minimal realization of the matrix $[G_o(s)]$, if rank $[H_i] = q$ $i = 1, 2, \cdots, 2k$ and $k \leq n$. This can be further verified by Rosenbrock's approach as follows. The system matrix of Eq.(3.41) is

$$P(s) = \left[\begin{array}{c|c} sI - A_1 & B_1 \\ \hline -C_1^T & 0 \end{array}\right] \qquad (3.43)$$

From Eq.(3.41) we observe that if rank $[H_i] = q$ then there is no pole at the origin or $s \neq 0$, and the dimension of A_1 is $k \times q$. By performing row and column operations, the matrices $[sI - A_1 \,\vdots\, B_1]$ and $[sI - A_1^T \,\vdots\, -C_1]$ in Eq.(4.43) can be transformed to the smith form $[I_{kxq} \,\vdots\, 0_q]$. If $k \times q = r_o = \text{rank } [G_o(s)]$, then Eq.(3.41) is a minimal realization of $[G_o(s)]$ with minimal dimension $k \times q$. This is necessary and sufficient condition for the existence of the matrix continued fraction. The necessary and sufficient condition for the existence of Eqs,(3.40) and (3.41) is that the ratio (k) of the rank (r_o) and the dimension (q) of the matrix $[G_o(s)]$, $k = \frac{r_o}{q}$, is an integer and rank $[A_{i,1}]$ in Eq.(3.42) is q. Eq.(3.43) can be further transformed to a strictly equivalent system as follows:

$$P_1(s) = \left[\begin{array}{c|c} SI - A_2 & B_2 \\ \hline -C_2^T & 0 \end{array}\right] \tag{3.44}$$

where

$$A_2 = -\begin{bmatrix} H_2(H_1) & H_2(H_1) & H_2(H_1) & \cdots H_2(H_1) \\ H_4(H_1) & H_4(H_1+H_3) & H_4(H_1+H_3) & \cdots H_4(H_1+H_3) \\ H_6(H_1) & H_6(H_1+H_3) & H_6(H_1+H_3+H_5) & \cdots H_6(H_1+H_3+H_5) \\ \cdot & \cdot & \cdot & \cdot \\ H_{2k}(H_1) & H_{2k}(H_1+H_3) & H_{2k}(H_1+H_3+H_4) & \cdot\ H_{2k}(H_1+H_3+\cdots+H_{2k-1}) \end{bmatrix}$$

$$B_2 = \begin{bmatrix} H_2 \\ H_4 \\ \cdot \\ \cdot \\ \cdot \\ H_{2k} \end{bmatrix}$$

and

$$C_2^T = [\ I,\ I,\ I,\ \cdots,\ I\ ]$$

The equivalent state equation of Eq.(3.41) is:

$$\begin{aligned} \dot{Z} &= A_2\ Z + B_2 U \\ Y &= C_2^T\ P \end{aligned} \tag{3.45}$$

An equivalent state equation of Eq.(3.41) in the phase-variable coordinates is expressed as follows

$$\dot{P} = A_3 P + B_3 U$$
$$Y = C_3^T P \qquad (3.46)$$

where

$$A_3 = \begin{bmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ . & . & . & & . \\ -R_{11} & -R_{12} & -R_{13} & \cdots & -R_{1,k} \end{bmatrix} B_3 = \begin{bmatrix} 0 \\ 0 \\ . \\ I \end{bmatrix}$$

$$C_3^T = [R_{21}, R_{22}, \cdots, R_{2,k}]$$

The $R_{i,j}$ in Eq.(3.46) can be obtained by the reverse process of the matrix Routh algorithm[9].

$$R_{2k+1,\,1} = [I]$$
$$R_{p,1} = H_p R_{p+1,1} \quad , \quad p = 2_k, 2_{k-1}, \cdots, 2, 1$$
$$R_{i-2,j+1} = R_{i,j} + H_{i-2}R_{i-1,j+1} \qquad (3.47)$$
$$i = 2_{k+1}, 2_k, \cdots, 3 \; ; \; j = 1, 2, \cdots k.$$

From Eq.(3.46) we have the two-factored polynomial matrices for the matrix $G_o(s)$ as follows

$$[G_o(s)] = [R_2(s)][R_1(s)]^{-1}$$
$$= [R_{21}+R_{22}s+\cdots+R_{2,k}s^{k-1}]\,[R_{11}+R_{12}s+\cdots+R_{1,k}s^k]^{-1} \qquad (3.48)$$

The $[R_2(s)]$ and $[R_1(s)]$ in Eq.(3.48) are the relative right prime polynomial matrices.[13] This can be verified by performing row and column operations on Eq.(3.43). The system matrix $[P(s)]$ in Eq.(3.43) is transformed to the following unimodular equivalent matrices.[10]

$$\left[\begin{array}{c|c} SI - A_1 & B_1 \\ \hline -C_1^T & 0 \end{array}\right] \quad \text{and} \quad \begin{bmatrix} I_{r_0} - q & 0 & 0 \\ 0 & R_1(s)q & I_q \\ 0 & -R_2(s)q & 0 \end{bmatrix} \tag{3.49}$$

From Eq.(3.49) we have the factored polynomial matrices shown in Eq.(3.48)

In the same fashion Eq.(3.46) can be expressed by other phase-variable form:

$$\begin{aligned} \dot{F} &= A_4F + B_4U \\ Y &= C_4^TF \end{aligned} \tag{3.50}$$

where

$$A_4 = \begin{bmatrix} 0 & 0 & 0 & \cdots & -T_{11} \\ I & 0 & 0 & \cdots & -T_{12} \\ 0 & I & 0 & \cdots & -T_{13} \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & -T_{1,k} \end{bmatrix}, \quad B_4 = \begin{bmatrix} T_{21} \\ T_{22} \\ T_{23} \\ \cdot \\ T_{2,k} \end{bmatrix}$$

$$C_4^T = [0,\ 0,\ 0,\ \cdots\ I]$$

$T_{i,j}$ in Eq.(3.50) can be obtained by the reverse-matrix Routh algorithm as follows

$$T_{2k+1,1} = [I]$$

$$T_{p,1} = T_{p+1,1}\,H_p, \quad p = 2_k,\ 2_{k-1}, \cdots, 2,\ 1$$

$$T_{i-2,j+1} = T_{i,j} + T_{i-1,j+1} H_{i-2} : \quad i = 2_{k+1}, 2k, \cdots 3 ; \quad j = 1, 2, \cdots, k. \tag{3.51}$$

The factored matrix polynomial of the $[G_o(s)]$ is:

$$[G_o(s)] = [T_{11}+T_{12}s+\cdots+T_{1,k+1}s^k]^{-1} [T_{21}+T_{22}s+\cdots+T_{2,k}s^{k-1}] \tag{3.52}$$

when $\ell > m$, Eqs. (3.41) and (3.46) are the minimal realizations of the $[G_o(s)]$ and Eq.(3.48) is the factored form of the $[G_o(s)]$ because the rank $[G_o(s)] = r_o$ = the dimension of $A_1 = k \times m$ and k is an integer. The H_i can be evaluated from Eq.(3.42a) except that the pseudo-inversion should be applied. In other words, the pseudo inverse of $[A_{i+1,1}] = [A_{i+1,1}]^{-1} = [A^T_{i+1,1}A_{i+1,1}]^{-1}[A^T_{i+1,1}]$ in Eq.(3.42a) is the left inverse of the matrix $[A_{i+1,1}]$. Eq.(3.47) is used to determine the matrix coefficients $R_{i,j}$ in Eq.(3.48). When ℓxm, Eqs.(3.45) and (3.50) are the minimal realizations of the $[G_o(s)]$ and Eq.(3.52) is the factored form of the $[G_o(s)]$. The matrices H_i in Eq.(3.45) and (3.50) and the matrices $T_{i,j}$ in Eq.(3.52) can be obtained from Eqs.(3.42b) and (3.51) except that the pseudo inverse $[A_{i+1,1}] = [A_{i+1,1}]^{-1} = [A^T_{i+1,1}][A_{i+1,1}A^T_{i+1,1}]^{-1}$ in Eq.(3.42b) the right inverse of the matrix $[A_{i+1,1}]$. It is noted that when $\ell \neq m$, the matrices H_i obtained from Eqs.(3.42a) and (3.42b) are not the same matrix quotients H_i in Eq.(3.40) because some H_i in Eq.(3.40) are the right inverse and the others are the left inverse of a non-square matrix $[G_o(s)]$. While all the matrices H_i in Eqs.(3.42a) and (3.42b) are obtained by either straight left inverse of a non-square matrix $[G_o(s)]$ if $\ell > m$ or straight right inverse of a non-square matrix $[G_o(s)]$ if $\ell < m$. The example in section 3.5 will be continued as follows:

Since the standard multivariable system in the frequency domain is:

$$[G_o(s)] = \frac{1}{s^2 + 2s + 2} \begin{bmatrix} 1.9s + 2 & 0 & 4.60134s \\ 1.1715s & 7.21s + 2 & 0 \end{bmatrix} \tag{3.53}$$

This multivariable system can be transformed into state space. The rank $[G_o(s)] = r_0 = 4$ and $q = \min(\ell, m) = 2$. The ratio, $\frac{r_o}{q} = 2 = k$, is an integer; therefore, we have $2k = 4$ matrices H_i as follows:

$$H_1 = \begin{bmatrix} 1. & 0. \\ 0. & 1. \\ 0. & 0. \end{bmatrix} \qquad H_3 = \begin{bmatrix} -0.01178 & -0.03007 \\ -0.14547 & -0.66365 \\ -0.024615 & -0.1183 \end{bmatrix} \tag{3.54}$$

$$H_2 = \begin{bmatrix} 20. & 0. & 0. \\ -4.4974 & -0.38388 & 0. \end{bmatrix} \quad H_4 = \begin{bmatrix} -18.1 & 0. & 4.60134 \\ 5.66897 & 7.59388 & 0. \end{bmatrix}$$

From Eqs. (3.45) and (3.50), we have the corresponding state equations.

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \end{bmatrix} = - \begin{bmatrix} 20. & 0. & 20. & 0. \\ -4.4974 & -0.38388 & -4.4974 & -0.038388 \\ -18.1 & 0. & -18. & 0. \\ 5.66897 & 7.59388 & 4.4974 & 2.38388 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix}$$

$$
+\begin{bmatrix} 20. & 0. & 0. \\ -4.4974 & -0.38388 & 0. \\ -18.1 & 0. & 4.60134 \\ 5.66897 & 7.59388 & 0. \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}
$$

$$
\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} 1. & 0. & 1. & 0. \\ 0. & 1. & 0. & 1. \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \\ Z_4 \end{bmatrix} \tag{3.55a}
$$

and

$$
\begin{bmatrix} \dot{f}_1 \\ \dot{f}_2 \\ \dot{f}_3 \\ \dot{f}_4 \end{bmatrix} = \begin{bmatrix} 0. & 0. & -2. & 0. \\ 0. & 0. & 0. & -2. \\ 1. & 0. & -2. & 0. \\ 0. & 1. & 0. & -2. \end{bmatrix} \begin{matrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{matrix} + \begin{bmatrix} 2. & 0. & 0. \\ 0. & 2. & 0. \\ 1.9 & 0. & 4.60134 \\ 1.17157 & 7.21 & 0. \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}
$$

$$
\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} 0. & 0. & 1. & 0. \\ 0. & 0. & 0. & 1. \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} \tag{3.55b}
$$

CHAPTER IV

CONCLUSIONS

In this thesis, first, the relationships between the time-domain specifications and the frequency-domain specifications were studied. Next, a method was presented to construct a transfer function for a single variable control system in which industrial specifications may be used. Two methods were proposed for estimating the initial values which will cause the Newton-Raphson multidimensional method to converge very rapidly. Finally, a method was presented in which industrial specifications can be used to formulate a standard multivariable system with various numbers of inputs and outputs. Various forms of the state equation which are the minimal realizations of the standard transfer function matrix, are obtained by an algebraic method.

By using these methods proposed in this thesis, standard specifications used in industry can be interpreted into mathematical terms of single and multivariable systems. Thus, allowing more effective engineering design of electrical control systems.

APPENDIX

The Newton-Raphson Multidimensional Method

Non-linear simultaneous equations can be described by the following equations:

$$f_i(x_1, x_2, \cdots, x_n) = 0 \quad , i = 1, 2, \cdots, n \tag{1}$$

In vector notation

$$f(\vec{x}) = 0 \tag{2}$$

If we make an initial guess by letting:

$$\vec{x}\Big|_o = [\vec{x}_{10}] \tag{3}$$

and find that Eq.(2) is satisfied, then Eq.(3) is the solution. If Eq.(3) is not satisfactory, we will have:

$$f_i(x_1, x_2, \cdots, x_n)\big|_o = r_i\big|_o \quad , i = 1, \cdots, n \tag{4}$$

where the vector

$$\vec{r} = [r_1, r_2, \cdots, r_n]^\tau \tag{5}$$

is called the "residue vector". The Newton-Raphson formula is:

$$\vec{x}_1 = \vec{x}_0 - \left(\frac{\partial \vec{r}}{\partial x}\right)_0^{-1} \vec{r}_0 \tag{6}$$

where $\left(\frac{\partial \vec{r}}{\partial x}\right)_0$ is the Jacobian evaluated at t = 0. In general, the Newton-Raphson formula is as follows:

$$\vec{X}_{n+1} = \vec{X}_n - \left(\frac{\partial \vec{r}}{\partial X}\right)_n^{-1} \vec{r}_n \tag{7}$$

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