ANALYSIS OF A BOUNDARY VALUE PROBLEM ASSOCIATED WITH LAMINAR FLUID FLOW

A Dissertation

Presented to

the Faculty of the Department of Mathematics

University of Houston

Houston, Texas

In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

---

by

John L. Engvall

August 1972

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ABSTRACT

This dissertation considers the linear operator eigenvalue problem $L(u) + su - a(u) = 0$ subject to the boundary condition $u = c$ for some real valued scalar c. It is shown that under certain conditions the problem is equivalent to solving the equation $L(v) + sv = 0$ subject to the boundary condition $v = 0$. The analysis is applied to a linearized mathematical model for solving velocity profiles for laminar fluid flow in the entrance region of ducts. Existence and uniqueness of a solution are guaranteed for a large class of initial profiles. Numerical results are presented for a rectangular duct with 2 to 1 aspect ratio.

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INTRODUCTION

The following work has been motivated by the study of linear operator, eigenvalue, and boundary value problems. Problems arise in mathematics and engineering concerning solutions of linear operator equations of the type $L(u) + su - a(u) = 0$ where L and "a" are linear operators defined on a space H and s is a field element to be determined. An analytical approach, slightly different from those for solving classical boundary value problems, will be applied to such problems herein. A portion of mathematical literature considers operator equations where the function u is defined on a set A in a space S with topological structure so that an interior A^o and a boundary d can be defined. Classical problems in physics and engineering often require that the operator equation be satisfied on the interior of the set A, and that the solution u satisfy the boundary condition $u = 0$ on the boundary d of a.

In this investigation S is a topological space, A a closed subset of S, d a subset of A, and H a vector space of real valued functions on A including the constant functions C. The operator "a" is restricted such that $a(u)$ is in C for every u in H, and it will be assumed that C is contained in the null space of both operators L and "a". This is a common occurence in linear operator theory; for example, differential operators have this property. The problem considered in this thesis differs from the classical boundary value problem as follows. The solutions u are restricted to those such that the operator equation is satisfied on the entire closed set A, and the classical *boundary* condition is changed to $u = c$, a specified constant on the *set* d.

The results in Chapter 1 show, in this general setting, that if L has a given property, the problem can be reduced to solving the operator-eigenvalue equation $L(v) + sv = 0$, subject to $v = 0$ on d. The special property needed, $L(u) = 0$ at some point on d, is motivated by geometrical consideration and operator properties realized in certain physical problems. An immediate result of this reduction is that for negative definite operators L only positive solutions for s exist. This again is significant for certain applied problems. In this chapter primarily algebraic properties are considered. There is no attempt to prove existence of solutions to either problem. Properties of a topological nature and possible functional relationships of L and "a" are investigated in Chapter 2.

In Chapter 2 and all other chapters the space S is Euclidean 2-space, d is a simple closed curve such that d is the boundary of the bounded set A, and $\int dA = 1$. The space H is required to be a subspace of $L^2(A)$, the square integrable functions on A. In addition to topological restriction of the spaces, the operators L and "a" are related through

$$\int L(u)\,dA = a(u) \tag{1}$$

and L is required to satisfy

$$\int (vL(u) - uL(v))\,dA = \oint_d (v\,du/dn - u\,dv/dn)\,ds , \tag{2}$$

where the line integral is around the simple closed curve and the derivatives are with respect to the outward normal. This is a generalized form of Green's identity. In Chapter 2 another operator equation

$$L(w) - a(w) = 0 \tag{3}$$

is considered where w is subject to $w = 0$ on d. An expression for the inner product, $\int uw\,dA$, is derived for u a solution to

$$L(u) + su - a(u) = 0 \quad \text{on } A \tag{4}$$

subject to $u = c$, constant on d.

In addition to being nicely related to w, the solutions to Equation 4 have easily obtainable pairwise inner products and each is orthogonal to unity. These relationships are shown in Chapter 3 where all results of the previous chapters are required. If u_2 and u_3 are solutions to Equation 4 associated with distinct eigenvalues and having respective boundary values c_2 and c_3, then

$$\int u_2 u_3 dA = c_2 c_3 \tag{5}$$

and

$$\int u_2 dA = 0 \tag{6}$$

In Chapter 4 it is shown that the solutions to

$$L(v) + sv = 0 \tag{7}$$

subject to $v = 0$ on d, always determine a subset of solutions to Equation 4. Also, it is shown that if v is subject to a constant boundary condition, $v = c$ on d, then the two problems are equivalent. The foregoing development is applied to examples where

$$L(u) = \nabla^2(u) \tag{8}$$

and

$$a(u) = \oint du/dn\ ds\ ; \tag{9}$$

that is to say, $\nabla^2(u(x,y))$ is the two-dimenstional Laplacian operator, and a(u) is the line integral over d of the derivative of u with respect to the outward normal to the curve d. Two geometries, a rectangle and a circular sector, are considered. The choice of operators is associated with the classical problem of determining the velocity profile of an incompressible fluid flowing through a duct of constant cross section. The fully developed profile is known for ducts of various shapes, but a closed form solution for the entrance region velocity profile has never been obtained for any geometry. In 1960 Han presented numerical results for a rectangular duct [1]*, and in 1964 Sparrow, Lin and Lundgren presented numerical solutions for rectangular ducts using a technique to reduce the problem to a two-dimensional boundary value problem [2]. More recently numerical and experimental results have been presented by Fleming for the triangular duct [3], and Wiginton and Dalton for the rectangular duct [4].

In Chapter 5 the simplified flow equations are presented, and the analysis of previous chapters is applied to the problem with an arbitrary cross section of the duct. Reduction of the problem to two dimensions requires the assumption that the velocity in the axial, or z, direction can be expressed

$$U = \Sigma c_i u_i \exp(s_i z) \tag{10}$$

where u_i, s_i are solutions to Equation 4 using the operators defined in Equations 8 and 9, and the velocity is subject to $U = 0$ on the boundary. A proof is presented in Chapter 5 showing that for u_i satisfying

*Numbers in brackets refer to references in the bibliography

$u_i = 0$ on d and satisfying Equation 4, the corresponding coefficient c_i in Equation 10 is zero for a large class of initial profiles including the uniform profile. Thus the development in this thesis appears nonapplicable to the problem. However the analysis can be extended to velocity profiles subject to the condition that for fixed z , U is constant and nonzero on the boundary.

In Chapter 6 and all succeeding chapters the region A^o is open, bounded, simply connected and the boundary d must be a simpled closed piecewise smooth curve. Under these assumptions, the existence, uniqueness and continuity of a solution to the fluid flow problem is guaranteed for a class of initial profiles characterized in Chapter 6.

The simplifying process for reducing the problem to two dimensions also involves transforming from the physical axial coordinate to a stretched axial coordinate in a highly nonlinear manner. The inverse transformation is a very time consuming process, but it has been simplified recently [5,6] by both Wiginton and Fleming. A similar simplification is possible for the constant nonzero boundary condition as shown in Chapter 8.

In Chapters 9 and 10 the analysis is applied to the rectangular duct. A solution is presented for the boundary condition

$$U(x,y,0) = 1 \quad \text{on} \quad A \tag{11}$$

and for the boundary condition

$$U(x,y,0) = 1 \quad \text{on} \quad A^{o}$$
$$U(x,y,0) = 0 \quad \text{on} \quad d\ .$$

Numerical results of these applications are presented in Chapter 11.

CHAPTER 1

ALGEBRAIC AND OPERATOR RESTRICTIONS

In this chapter we consider a reduction for the operator equation

$$L(u) + su - a(u) = 0$$

where s is a nonzero real scalar to be determined. The domain for the operators L and a is a real vector space defined as follows. Let A be a closed subset of a topological space S where A^{o} and d are respectively the interior and an arbitrary subset of the set A. Let H be a real vector space of real valued functions on A. Let $L: H \to H$ and $a: H \to H$ be linear operators. We will consider only those spaces H containing the collection C of constant functions from A into R, and those operators L and "a" having the additional properties:

P1. $L(u + c) = L(u)$

P2. $a(u + c) = a(u)$

P3. $a(u)$ belongs to C

for every u in H and every c in C.

THEOREM 1. *The statements* S1 *and* S2 *are equivalent.*

S1. There exists real nonzero s , u in H , x in d , such that $L(u) = 0$ at x and $L(u) + su - a(u) = 0$ and $u = 0$ on d .

S2. There exists real nonzero t, v in H , and x in d , such that $L(v) = 0$ at x and $L(v) + tv = 0$, and $v = 0$ on d , and $a(v) = 0$.

PROOF: Suppose there exists s , x and u satisfying S1. Then

$$L(u(x)) + su(x) - a(u(x)) = -a(u(x)) = 0$$

since $L(u) = 0$ at x and $u = 0$ on all of d . Moreover, $a(u)$ is in C , hence $a(u) = 0$ on A . Thus,

$$L(u) + su = 0$$

and S2 is satisfied for $v = u$ and $t = s$.

Conversely, suppose there exist s, v, and x satisfying S2. Then $s = t$ and $u = v$ satisfy S1. This completes the proof.

At this point we emphasize that the domain of u is a closed subset of S and we require the operator equation to be satisfied on all of this domain quite contrary to the usual approach used for boundary value problems. Also note that $L(u) = 0$ at x in d necessitates $a(u) = 0$.

Another similar result can be stated as a reduction theorem.

THEOREM 2. *The statements* S3 *and* S4 *are equivalent:*

S3. There exist real nonzero s , u in H , x in d , and a nonzero scalar c , such that

$$L(u) = 0 \quad \text{at } x ,$$

$$L(u) + su - a(u) = 0 ,$$

$$u = c \text{ on } d .$$

S4. There exist real nonzero scalars s_2 and c_2 , v in H, y in d , such that

$$L(v) + sv = 0$$

$$L(v) = 0 \text{ at } y ,$$

$$a(v) = c_2 ,$$

$$v = 0 \text{ on } d .$$

PROOF: Suppose there exist s, c, u, and x satisfying the conditions in S3. Then $L(u) = 0$ at x implies that $a(u) = su(x) = sc$. Define v in H by $v = u - a(u)/s$ which is equivalent to $v = u - c$, hence $v = 0$ on d . It follows from property P2 that $a(v) = a(u - c) = a(u)$ so that $a(v)$ is nonzero, and it follows from property P1 that $L(v) = L(u - c) = L(u)$. Substituting these equations and expanding the equation

$$L(u) + su - a(u) = 0$$

we have

$$L(u - a(u)/s) + su - a(u) = 0 ,$$

hence

$$L(u - a(u)/s) + s(u - a(u)/s) = 0 .$$

and the desired equality

$$L(v) + sv = 0$$

has been obtained. Since $L(v) = L(u)$, it follows that $L(v) = 0$ at x in d , hence all of the hypotheses in S4 are satisfied when the elements are constructed as follows:

$$v = u - c$$
$$s_2 = s$$
$$c_2 = sc$$
$$y = x .$$

Conversely, suppose there exist c_2, s_2, v, and y such that the conditions in S4 are satisfied. For any nonzero scalar c , define the real nonzero scalar $b = cs_2/a(v)$, and let u in H be defined by

$$u = b(v + a(v)/s_2) .$$

Then if $s = s_2$,

$$\begin{aligned} L(u) + su - a(u) &= L(bv) + s_2(bv + ba(v)/s_2) - a(bv) \\ &= b(L(v) + s_2 v) + b(a(v) - a(v)) \\ &= 0 ; \end{aligned}$$

shown in the proof is one-to-one and onto so that the solution sets are of the same cardinality. If all the solutions to one problem are known, then all the solutions to the other problems are known.

These results are independent of the choice of the subset d , and the choice of the function space H is arbitrary to within inclusion of the constant functions. Moreover, no inner product, norm, metric, or topological structure has been assumed for H . Some questions might be formulated concerning the equivalence of the statements in Theorems 1 and 2 for different requirements on the continuity of u, L, and "a" if H does possess a topological structure. In application L and "a" may be unbounded linear operators, hence continuity of u on all of A is still insufficient to draw many conclusions about the behavior of $L(u) + su - a(u)$. In particular, u_n converging to u in H , $L(u) + su - a(u) = 0$ is not sufficient to guarantee convergence of $L(u_n) + su_n - a(u_n)$ to zero. The existence of an inner product structure on H involving integration with respect to a measure μ may lead one to solve $L(u) + su - a(u) = 0$ almost everywhere μ .

The properties required for L and "a" rule out invertible operators, but do not force either operator to be continuous.

The problem has been intentionally stated in a fairly flexible setting with the exception of the required properties for the operator "a" . There are problems of the form $L(u) + su = 0$ where certain properties of L make possible closed form solutions. To force $a(u)$ to be constant is extremely restrictive, but $L(u) - a(u)$ may possess few of the desirable characteristics of L , and the analytical and numerical tools for solving Equation 1-2 may fail to provide any information concerning Equation 1-1. If the geometry can be chosen such that $L(u) = 0$ for some x in d , then the problem is reduced to solving Equation 1-2, and then choosing those solutions such that $a(u)$ is zero or nonzero depending on the requirement for u on d . An immediate application of this theory is that for negative definite operators L only positive values of s_2 are possible solutions for Equation 1-2, and the related solution to 1-1 has the property that $s = s_2$ hence only positive values of the eigenvalues are possible.

EXAMPLE 1. Let S be Euclidean 2-space R^2 , and A the rectangle enclosed by the straight lines joining the points (0,0), (r,0), (r,1/r), (0,1/r), and (0,0) respectively, where r is a positive real scalar. Define H as the vector space spanned by the constant function I and the functions f_i where

$$f_i = \sin(p_i(\pi/r)x) \sin(q_i(\pi r)y)$$

where p_i, q_i take on all possible integral values. Find a nonzero scalar s and u in H such that

$$L(u) + su - a(u) = 0$$

subject to $u = 0$ on the boundary d, where

$$L(u) = d^2u/dx^2 + d^2u/dy$$

and

$$a(u) = \oint_d (du/dn)\, ds .$$

That is to say L is the two-dimensional Laplacian operator and "a" is the line integral of the derivative of u with respect to the outward normal to the boundary f.

The operators L and "a" satisfy the conditions necessary for Theorem 1 and $L(u) = 0$ for any of the corner points of the rectangle. Hence the problem is reduced to solving

$$L(u) + su = 0$$

subject to

$$u = 0 \text{ on } d,$$

and

$$a(u) = 0 .$$

A countably infinite set of solutions can be seen upon inspection to the functions f_i with either or both p_i, q_i even since $a(F_i)$ is zero and nonzero when either p_i or q_i is odd. The eigenvalues are given by

$$s_i = (p_i\pi/r)^2 + (q_i\pi r)^2 .$$

EXAMPLE 2. Solve the same problem as in Example 1 except where u is subject to being a nonzero constant value on the boundary d .

In this event the solutions v_i to Equation 1-2 are the F_i where either or both p_i, q_i are odd. The eigenvalues are functionally the same form. If a particular nonzero value c is required for u on the boundary d , then define

$$b_i = s_i c/a(v_i)$$

and it follows from Theorem 2 that the solutions to the problem are

$$u_i = b_i(v_i + a(v_i)/s_i) .$$

Examples 1 and 2 will be further expanded and analyzed in Chapter 8.

EXAMPLE 3. Let S be as in Example 1 and let A be the circular sector defined in polar coordinates by $\rho \in [0,r]$, $\theta \in [0,\theta_o]$ where r is a positive real number and the angle θ_o is between zero and $\pi/2$ radians. Let H be the vector space spanned by the constant function 1 and the functions

$$F_i = \sin(p_i\pi r\rho)\ \sin(q_i\pi\theta)$$

where p_i, q_i take on all possible positive integral combinations. Find a nonzero scalar s and a function u in H such that Equation 1-1 is satisfied when L is the two-dimensional Laplacian operator given in polar coordinates by

$$L(u) = \frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial u}{\partial \theta} \qquad \text{1-3}$$

and a(u) is given by

$$a(u) = \oint \frac{\partial u}{\partial n}\, ds$$

where du/dn is the derivative of u with respect to the outward normal.

Since L(u) and a(u) are not defined at the origin, define both to be zero at this point.

The operators L and a satisfy the conditions necessary for Theorem 1, and $L(u) = 0$ at either of the points $(r,0)$ or (r,θ_o), thus the problem is reduced to solving Equation 1-2 subject to $u = 0$ on d and $a(u) = 0$.

The problem is solvable by separation of variable. Suppose

$$u = G(\theta)X(\rho)$$

then substitution of u into Equation 1-2 yields

$$\chi''G + \frac{1}{\rho}\chi'G + \frac{1}{\rho^2}\chi G'' + \alpha^2\chi G = 0 \,. \qquad 1\text{-}4$$

or

$$(\rho^2\chi'' + \rho\chi' + \rho^2\alpha^2\chi)/\chi = -G''/G \,. \qquad 1\text{-}5$$

In this event both sides of Equation 1-5 must be constant, say k^2,

$$G'' = -k^2G$$

and

$$\rho^2\chi'' + \rho\chi' + (\rho^2\alpha^2-k^2)\chi = 0 \qquad 1\text{-}6$$

subject to

$$G(0) = G(0_o) = 0$$

and

$$\chi(0) = \chi(r) = 0 \,.$$

Equation 1-6 is known as Bessel's equation of order k with parameter α [17]. Solutions to this equation are known as Bessel functions of the first kind of order k denoted $J_k(\alpha\rho)$.

A countably infinite set of solutions to Equation 1-4 is given by $P_{mk} = \sin(m\pi\theta_o\theta)\, J_k(\alpha\rho)$ where m and k take on all positive integral values.

The previous examples are useful in solving a particular fluid flow problem by a method introduced by Sparrow [1]. In deriving this method Equation 1-1 must be solved where L and "a" are given as in Example 1. Only positive eigenvalues are meaningful for the particular application, hence the form

$$L(u) + s^2 u - a(u) = 0$$

is used. Miklin [2] has shown that the two-dimensional Laplacian is a negative definite operator on the space $M = \{u:A \quad R:u$ is continuously twice differentiable with respect to each variable and $u = 0$ on $d\}$.
In this event only positive solutions exist as is shown by

THEOREM 3. *Suppose* L *and* "a" *satisfy properties* P1 *through* P3 *and if* $L(u) = 0$ *at* x *for some* x *in* d, *and* L *is negative definite on some subspace* M *in a Hilbert space* H *such that* M *contains the constant functions,* $f(x,y) = c$. *Then* $L(u) + su - a(u) = 0$, u *in* M, s *nonzero,* $u = c$ *constant on* d, u *not identically zero on* A *implies that* s *is greater than zero.*

PROOF: The solution u is constant on d hence via Theorem 1 or 2 the problem is reduced to solving

$$L(v) + sv = 0$$

subject to $v = 0$ on d.
Moreover u being nonzero on A implies v is not identically zero on A (i.e., v is a constant translate of

CHAPTER 2

ORTHOGONALITY OF U AND W IN L_2

Most of the preceding results depend only on algebraic properties. It is of interest to consider a topological structure for H , a restriction of d , and topological properties of the operators L and "a" in place of the property $L(u(x_o,y_o)) = 0$. The results can be combined with those of the preceding section, but the proofs are independent.

Let A be a closed subset of Euclidean 2-space R^2 bounded by a simple closed curve d , and let H be a collection of square integrable functions defined on A . Let d be further restricted so that the area of A , $\int dA$, is unity.[1] The following equations in P4 and P5 are a generalization of Green's identity. These properties will be assumed in all succeeding chapters unless specified differently. Let $L: H \to H$ and $a: H \to H$ be linear operators such that for every u and v in H the following properties are satisfied:

P4. $\int L(u)\,dA = a(u)$ and

P5. $\int (vL(u) - uL(v))\,dA = \oint (v\partial u/\partial n - u\partial v/\partial n)\,ds$.

---

[1] This can be omitted if $a(u) = 0$.

Note if L is invariant under constant translates then L of a constant is zero hence for $v = 1$

$\int L(u) = \oint_d \partial u/\partial n ds$ implies $a(u) = \oint_d \partial u/\partial n ds)$

Equation 1-2 has been analyzed extensively for the case when L is the Laplacian operator. Under certain conditions to be discussed in Chapter 6, the collection of solutions to Equation 1-2 is countable, and a large class of functions can be expressed as a series

$$f = \Sigma b_i v_i \text{ .}$$

The development in this thesis will characterize a class of functions expressable as a series

$$g = \Sigma c_i u_i$$

where the u_i are the corresponding solutions to Equation 1-1. An application to a fluid flow problem is analyzed where the fully developed profile W is a solution to the equation

$$L(w) - a(w) = 0 \text{ .}$$

The pairwise inner products of the functions u_i and the inner products of the u_i with W will be useful in determining velocity profiles in the entrance region of ducts.

Returning to the more general operator problem we derive expressions for these inner products.

THEOREM 4. *If* s *is a nonzero scalar and* u *and* w *are elements of* H *such that*

(a) $L(u) + su - a(u) = 0$, and

(b) $L(w) - a(w) = 0$ where $u = c$ on d and $w = 0$ on d

then

$$\int wu\,dA = (a(u)/s)\int w\,dA + (c/s)\oint_d \partial w/\partial n\,ds$$

PROOF: First note

$$\begin{aligned} s\int u\,dA &= \int a(u)\,dA - \int L(u)\,dA \\ &= a(u)\ dA - \int L(u)\,dA \\ &= a(u) - \int L(u)\,dA \\ &= 0 \end{aligned}$$

hence

$$\int u\,dA = 0 .$$

Multiplying Equation (a) by w and equation (b) by u , subtracting, and integrating over A we have

$$\int (wL(u) - uL(w)\,dA + \int swu\,dA + \int (ua(w) - wa(u)\,dA = 0$$

hence

$$-s\int wu\,dA = \oint_d (w\partial u/\partial n - u\partial w/\partial n)\,ds + a(w)\int u\,dA - a(u)\int w\,dA .$$

But $w = 0$ and $u = c$ on d implies

$$\oint_d (w\partial u/\partial n - u\partial w/\partial n(ds = -c\oint_d \partial w/\partial n ds$$

and

$$\int udA = 0$$

hence

$$s\int wudA = a(u)\int wdA + c\oint_d \partial w/\partial n ds$$

and

$$\int wudA = 1/sa(u)wdA + c/sa(w) .$$

This completes the proof.

COROLLARY 1. *If* s *is a nonzero scalar, and* u *and* w *are elements of* H *such that* $L(u) + su = 0$, $L(w) - a(w) = 0$, $a(u) = 0$, $u = w = 0$ *on* d *then* $\int wudA = 0$ *(the property* $\int dA = 1$ *can be omitted.).*

PROOF: First note that even if $\int dA \neq 1$ we have

$$s\int udA = -\int L(u)dA = a(u) = 0 .$$

This is the only part of the proof of Theorem 4 where $\int dA = 1$ is needed, and by Theorem 4

$$s\int wudA = a(u)\int wdA = 0 .$$

COROLLARY 2. *If* $L(u) = 0$ *for some* x *in* d *and* $L(u) + su - a(u) = 0$, $u = c$ *on* d *, for* c *a non-zero scalar then* $\int wudA = c[a(w)/s + \int wdA)]$.

PROOF: If $L(u(x_o,y_o)) = 0$ then $c = u = a(u)/s$ on d implies $a(u) = cs$ hence, via Theorem 4,

$$\int wu\,dA = sc/s\int w\,dA + (c/s)a(w)$$

$$\int wu\,dA = c\int w\,dA + (c/s)a(w) \ .$$

These results apply to the previous examples with

$$L = \nabla^2 , \quad a = \oint_d \partial/\partial n\,ds ,$$

and $\quad w = w_f$, the solution to

$$L(w) - a(w) = 0 ,$$

subject to

$$w = 0 \text{ on } d .$$

Once the desired boundary condition is defined, Corollary 2 can be used to compute $\int wu\,dA$. For example, if u is a solution to Example 1, then $a(u) = 0$, hence $\int wu\,dA = 0$.

THEOREM 5. *If* s_1 *and* s_2 *are distinct nonzero scalars and* u_1 *and* u_2 *are elements of* H *such that* (u_1,s_1) *and* (u_2,s_2) *satisfy* $L(u) + su - a(u) = 0$ *subject to* $u = 0$ *on* d *then* $<1,u_i> = 0$ *and* $<u_1,u_2> = 0$.

PROOF:

$$L(u_1) + s_1u_1 - a(u_1) = 0 \qquad \text{2-1}$$

$$L(u_2) + s_2u_2 - a(u_2) = 0 \qquad \text{2-2}$$

hence

$$\int u_2 L(u_1) - u_1 L(u_2) dA + (s_1 - s_2) \int u_1 u_2 dA$$
$$= \int (a(u_1) u_2 - a(u_2) u_1) dA \qquad 2\text{-}3$$

Note

$$\int u_i dA = (1/s_i)(\int a(u_i) dA - \int L(u_i) dA)$$
$$= (1/s_i)(a(u_i) \int dA - a(u_i))$$

but $\int dA = 1$

hence

$$\int u_i dA = 0 \ . \qquad 2\text{-}4$$

Substituting Equation 2-4 into 2-3

$$\int u_1 u_2 dA = (1/(s_1 - s_2)) \int (u_2 L(u_1) - u_1 L(u_2)) dA$$
$$= (1/(s_1 - s_2)) \oint_d (u_2 \partial u_1/\partial n - u_1 \partial y_2/\partial n) ds$$

but

$$u_1 = u_2 = 0 \quad \text{on} \quad d$$

hence $\int u_1 u_2 dA = 0$.

A generalization of this theorem follows in Chapter 3; however, this form will be useful later.

CHAPTER 3

L_2 INNER PRODUCTS OF THE u_i WHEN $L(u) = 0$ ON d

The operator and algebraic properties of Chapter 1 can be combined with the topological and geometric properties of Chapter 2 to obtain stronger results relating to the pairwise L_2 inner products of the functions u_i. Henceforth it will be assumed that A is a closed set in the plane R^2 bounded by the simple closed curve d, H is a vector space of real valued square integrable functions over A including the set of constant functions C, and unless otherwise specified the area of A is assumed to be unity. In all succeeding chapters we assume that L and "a" are linear operators as previously defined and that unless specified otherwise properties P1 through P5 are satisfied. From this point forward $< \ >$ will denote the L_2 inner product unless another inner product is specified.

THEOREM 6. *If* $L(u) = 0$ *for some* x *in* d, *and if* (u_1, s_1) *and* (u_2, s_2) *are solutions to Equation 1-1 where* $u_1 = c_1$, $u_2 = c_2$ *on* d *and if the* s_i *are distinct nonzero scalars then* $\langle u_1, 1\rangle = 0$ *and* $\langle u_1, u_2\rangle = -c_1c_2$.

PROOF: Substituting u_i into Equation 1-1, integrating over A, and using property P4 we see that

$$a(u_i)(1 - \int dA) = s_i \int u_i dA ,$$

but $\int dA = 1$, s_i is nonzero hence $\int u_i dA = <1,u> = 0$.

To determine pairwise inner products Theorems 1 and 2 can be used to represent

$$u_2 = b_2(v_2 + a(v_2)/s_2) \qquad 3\text{-}1$$

$$u_1 = b_1(v_1 + a(v_1)/s_1) \qquad 3\text{-}2$$

where v_1, v_2 satisfy Equation 1-2 subject to

$$v_i = 0 \text{ on } d ,$$

and

$$c_i = b_i a(v_i)/s_i , \quad i = 1, 2 .$$

Properties P1 through P3 imply

$$L(u_i) = b_i L(v_i) \qquad 3\text{-}3$$

and

$$a(u_i) = b_i a(v_i) \qquad 3\text{-}4$$

The hypotheses state that

$$L(u_1) + s_1 u_1 - a(u_1) = 0 \qquad 3\text{-}5$$

and

$$L(u_2) + s_2 u_2 - a(u_2) = 0 \qquad 3\text{-}6$$

Multiplying Equation 3-5 by u_2 , Equation 3-6 by u_1 , subtracting and integrating over A yields

$$\int(u_1 L(u_2) - u_2 L(u_1))dA + (s_2 - s_1)\int u_1 u_2 dA$$
$$+ \int(a(u_1)u_2 - a(u_2)u_1)dA = 0$$

but $\int u_i dA = 0$, $a(u_i)$ is constant, hence

$$(s_1 - s_2)\int u_1 u_2 dA = \int (u_1 L(u_2) - u_2 L(u_1))dA \qquad 3\text{-}7$$

Substituting Equations 3-1 through 3-4 into the righthand side of 3-7 we have

$$\begin{aligned}(s_1 - s_2)\int u_1 u_2 dA = \int\{&b_1(v_1 + a(v_1)/s_1)b_2 L(v_2) \\ &- b_2(v_2 + a(v_2)/s_2)b_1 L(v_1)\}dA\end{aligned} \qquad 3\text{-}8$$

Hence

$$\begin{aligned}(s_1 - s_2)\int u_1 u_2 dA = \; &b_1 b_2 \int (v_1 L(v_2) - v_2 L(v_1))dA \\ &- b_1 b_2 \int\{a(v_2/s_2)L(v_1) - a(v_1/s_1)L(v_2)\}dA\end{aligned}$$

Invoking Property P5 one obtains

$$\begin{aligned}(s_1 - s_2)\int u_1 u_2 dA = \; &b_1 b_2 \oint_d (v_1 \partial v_2/\partial n - v_2 \partial v_1/\partial n)ds \\ &- b_1 b_2 (a(v_2/s_2)a(v_1) - a(v_1/s_1)a(v_2)\end{aligned} \qquad 3\text{-}9$$

Both v_1 and v_2 are zero on d hence the term involving the line integral is zero. The known relationship $b_i = -c_i s_i/a(v_i)$ can be used to reduce Equation 3-9 to

$$(s_2 - s_1)\int u_1 u_2 dA = \frac{s_1 s_2 c_1 c_2}{a(v_1)a(v_2)}\left(\frac{s_1 a(v_2)a(v_1) - s_2 a(v_1)a(v_2)}{s_1 s_2}\right) \qquad 3\text{-}10$$

yielding the desired result,

$$\int u_1 u_2 dA = -c_1 c_2$$

CHAPTER 4

PROBLEM REDUCTION WHEN $L(u) \neq 0$ ON d

The requirement that $L(u) = 0$ for some x in d occurs for certain applied problems. The previous theorems will be applied to such a problem. However before pursuing that goal, an analysis similar to that of Chapter 1 will be presented for those problems where $L(u) \neq 0$ on d.

Let L, "a", H, and C be defined as in Chapter 1 and satisfy properties P1 through P3. The object of this chapter is to reduce the problem of solving the operator-eigenvalue equation

$$L(u) + su - a(u) = 0 \qquad 4\text{-}1$$

subject to

$$L(u) \neq 0 \text{ on } d \qquad 4\text{-}2$$

and

$$u = c \text{ (possibly zero) on } d \qquad 4\text{-}3$$

to the problem of solving

$$L(v) + sv = 0 \qquad 4\text{-}4$$

and to determine what additional properties v must satisfy.

THEOREM 7. *The statements* S1 *and* S2 *are equivalent.*

S1. There exists a real nonzero scalar s, and a u in H such that

$$L(u) + su - a(u) = 0$$

$$L(u) \neq 0 \text{ on } d$$

and $u = 0$ on d.

S2. There exist real nonzero scalars s and f and a v in H, such that

$$L(v) + sv = 0$$

$$v = f \text{ on } d$$

and $$sf + a(v) = 0$$

THEOREM 8. *The statements* S3 *and* S4 *are equivalent.*

S3. There exist real nonzero scalars s and c, u in H such that

$$L(u) + su - a(u) = 0,$$

$$L(u) \neq 0 \text{ on } d,$$

and $u = c$ on d.

S4. There exist real nonzero scalars s and f, and v in H such that

$$L(v) + sv = 0,$$

$$v = f \text{ on } d,$$

and $$sf + a(v) \neq 0.$$

The proofs can be carried out together. The eigenvalues are invariant and a constructive proof will relate the functions u and v.

PROOF: Suppose there exists a nonzero scalar s and a function u in H satisfying Equations 4-1 and 4-2. Let

$$v = b(u - a(u)/s) \qquad 4\text{-}5$$

where b is a scalar to be determined. Substituting 4-5 into 4-4

$$L(v) + sv = L(bu - a(u)/s) + s(bu - a(u)/s) .$$

Using property P1

$$L(v) + sv = b(L(u) + su - a(u)) = 0$$

since u satisfies Equation 4-1. Hence s and v satisfy Equation 4-4. If $u = 0$ on d let b be defined such that $v = f$ on d, that is, let

$$-ba(u)/s = f \neq 0$$

then

$$sf + a(v) = s(-b(a(u)/s) + a(v)$$

Applying Properties P2 and P3 to Equation 4-5

$$a(v) = ba(u)$$

hence

$$sf + a(v) = -ba(u) + ba(u) = 0$$

and we have shown that S1 implies S2. If $u = c = 0$ on d, then define b such that

$$b(c - a(u)/s) = f = 0$$

If $L(u) \neq 0$ on d then $sc - a(u) \neq 0$, hence $b \neq 0$. Then

$$\begin{aligned} sd + a(v) &= sbc - ba(u) + ba(u) \\ &= sbc \neq 0 \end{aligned}$$

and we have shown that S3 implies S4.

Now suppose that there exist real nonzero scalars s and f, and a function v in H, such that (s,v) satisfy Equation 4-4 and

$$v = f \text{ on } d .$$

Let

$$u = (v + a(v)/s) ,$$

then

$$\begin{aligned} L(u) + su - a(u) &= L(v) + s(v + a(v)/s) - ba(v) \\ &= L(v) + sv + (a(v) - a(v)) \\ &= 0 ; \end{aligned}$$

moreover,

$$L(u) = L(v) \neq 0 \text{ on } d$$

hence s, u satisfy 4-1 and 4-2. If $f + a(v)/s = 0$ then $u = 0$ on d, hence S2 implies S1. If $f + a(v)/s \neq 0$ then $u = c \neq 0$ on d thus S4 implies S3. This completes the proof of both Theorem 7 and Theorem 8.

These theorems show that the problem $L(u) + su - a(u) = 0$ subject to $L(u) = 0$ on d, u constant on d is equivalent to the problem of solving $L(v) + sv = 0$ subject to v nonzero and constant on d.

The zero or nonzero character of u on d is determined by the value of $sf + a(v)$ rather than $a(v)$ as in Chapter 1.

CHAPTER 5

VELOCITY PROFILES FOR FLUID FLOW

Following the simplifications by Sparrow, we assume that the velocity profile for laminar flow of a fluid flowing in a duct of constant cross section can be expressed as

$$V(x,y,z) \quad = \quad U(x,y,z) + W(x,y) \qquad 5\text{-}1$$

where U and W are determined by the requirements

$$\partial V/\partial z \quad = \quad \nabla^2 V - \oint_d \partial V/\partial n ds \qquad 5\text{-}2$$

subject to

$$V(x,y,0) \quad = \quad V_o \text{ constant on } d\,, \qquad 5\text{-}3$$

$W = 0$ on the boundary of the duct d, and the $\int wdA = 1$. The operator ∇^2 is the two-dimensional Laplacian $\partial^2/\partial x^2 + \partial^2/\partial y^2$ hereafter denoted by L, and $\oint_d \partial v/\partial n$ is the line integral of the derivative of V with respect to the outward normal around the boundary of a cross section for z constant. The operator $\oint_d \partial V/\partial n ds$ will be denoted by a (V). Substituting Equation 5-1 in Equation 5-2 yields

$$(L(U) - \partial U/\partial z - \oint_d \partial U/\partial n ds) + (L(W) - \oint_d \partial W/\partial n ds) \quad = \quad 0 \qquad 5\text{-}4$$

Again following Sparrow's analysis, assume

$$L(W) - a(W) = 0 = L(u) - \partial U/\partial z - a(u) \qquad 5\text{-}5$$

subject to $W = 0$ on d, and assume U can be expressed

$$U = \Sigma c_i u_i(x,y) \exp(-s_i z) \qquad 5\text{-}6$$

for some countable collection (s_i, u_i) where the s_i are real positive scalars and each u_i is constant on the boundary d. The operators ∇^2 and $\oint_d \partial/\partial n ds$ satisfy the properties required for L and "a" respectively in Theorems 1 through 8, and we assume that the coordinate transformation to dimensionless variables forces $\int dA = 1$. This assumption reduces the problem to solving the operator eigenvalue equation

$$L(u_i) + s_i u_i - a(u_i) = 0 \qquad 5\text{-}7$$

subject to u_i constant on d. It has already been shown that the general problem

$$L(u) + su - a(u) = 0 \qquad 5\text{-}8$$

subject to u constant on d implies that s is positive if either $L(u) = 0$ for some x in d or $\langle u,1\rangle = 0$ (See Theorems 3 and 3A.), hence the requirement that the $s_i > 0$ needs no intuitive justification in this event. It is not apparent what the cardinality of the solution set to Equation 5-8 might be if, indeed, any solutions exist. For the present purpose we will assume that countable solution set (s_i,u_i) exists such that U can be expressed as in Equation 5-6. The results of Theorem 6 indicate

$$\langle u_i,1\rangle = 0$$

and

$$\langle u_i,u_j\rangle = t_i t_j$$

where $u_i = t_i$ on d. Theorems 1 and 2 indicate that the problem can be reduced to solving the problem

$$L(v) + sv = 0$$

subject to $v = 0$ on d. It is not apparent how to choose the boundary values t_i for the corresponding functions u_i. Before developing the properties and relationships of the various coefficients or considering specific geometries, it will be shown that the choice of the values t_i presents no problem.

THEOREM 9. *If*

$$U = \Sigma c_i u_i \exp(-s_i z)$$

where (s_i,u_i) *are solutions to*

$$L(u_i) + s_i u_i - a(u_i) = 0$$

satisfying

$$L(u_i) = 0 \text{ *for some* } x \text{ *in* } d \text{ *and*}$$

$$u_i = t_i \text{ *on* } d\ ,$$

$$u_i = b_i(v_i + a(v_i)/s_i)$$

and if v_i *are the corresponding solutions to*

$$L(v_i) + s_i v_i = 0$$

subject to $v_i = 0$ *on* d , *then for any solution set*

$$u_i^* = b_i^*(v_i + a(v_i)/s_i)$$

there exists a set of scalars c_i^* *such that*

$$U = \Sigma c_i^* u_i^* \exp(-s_i z)$$

provided that the b_i^* *have the same zero-nonzero character as the* b_i .

PROOF: If for any integer k , $b_k = 0$, then the term $c_k u_k \exp(-s_k z)$ is zero, hence we will define all such coefficients $c_k^* = 0$. Similarly if $c_k = 0$ define $c_k^* = 0$. Otherwise define

$$c_k^* = c_k b_k / b_k^*$$

As noted in Theorems 1 and 2 the u_i are reducible to the form

$$u_i = b_i(v_i + a(v_i)/s_i)$$

and the individual terms are equivalent since $c_i b_i = c_i^* b_i^*$ hence

$$U = \Sigma c_i u_i \exp(-s_i z) = \Sigma c_i^* u_i^* \exp(-s_i z) .$$

This completes the proof.

For notational convenience one can assume all of the b_i to be unity or for computational simplicity one can assume values of the b_i such that the L^2 norm of each u_i is unity. Unless otherwise specified we will assume the former and in any event t_i will denote the value of u_i on the boundary.

The functions u_i can be divided into Category I where $t_i = 0$ and Category II where $t_i \neq 0$. The following theorem asserts that a function u_i from Category I, need not be considered in the series representation of U given by Equation 5-6 if $\int V_o u_i = 0$.

Substituting Equations 5-3 and 5-6 into Equation 5-1 and evaluating at $z = 0$ one obtains

$$V_o = \Sigma c_i u_i + W \tag{5-9}$$

Multiplying Equation 5-9 by u_j and integrating over the cross section yields

$$\int V_o u_j dA = \sum_{i \neq j} -c_i t_i t_j + \int u_j W dA + c_j \int u_j^2 dA \tag{5-10}$$

where Theorem 4 can be used to evaluate the last term and $\int u_i u_j$ has obtained by means of Theorem 6. This expression will be useful in defining the coefficients c_i and in other relationships.

THEOREM 10. *If* u_i *satisfies Equation 5-7,* $L(u_i) = 0$ *for some* x *in* d *, and* $u_i = 0$ *on* d *, then for any initial velocity profile* V_o *such that*

$$\int V_o u_i dA = 0$$

the coefficient c_i *in Equation 5-6 is zero, provided*

$$\int \Sigma c_j u_j dA = \Sigma \int c_j u_j dA .$$

PROOF: If s_j and u_j satisfy Equation 5-7 and u_j belongs to Category I then application of Corollary 1 to Theorem 4 and Theorem 6 yields respectively

$$\int W u_j dA = 0$$

and

$$\int u_i u_j dA = 0$$

for every $i \neq j$. By hypothesis

$$\int V_o u_j = 0$$

hence Equation 5-10 is reduced to

$$0 = c_j \int u_j^2 dA$$

where $u_j \neq 0$, implying $c_j = 0$, which is the desired result.

COROLLARY. *If* u_i *is of Category I and satisfies Equation 5-7,* $L(u_i) = 0$ *for some* x *in* d *,* $V_o = 1$ *, and* $\int \Sigma c_j u_j dA = \Sigma \int c_j u_j dA$ *then* $c_i = 0$.

In this chapter we have assumed the existence of a countable set $\{v_i, s_i, c_i\}$ such that

$$V = \Sigma c_i u_i \exp(-s_i z) + W ,$$

and

$$\int \Sigma c_i v_i = \Sigma c_i \int v_i dA .$$

We will also assume that W and the v_i are known functions. Conditions sufficient to guarantee existence and uniqueness are presented in Chapter 6 and Chapter 7. Orthogonality of the functions v_i is also proven therein.

The coefficients c_i can be expressed in terms of V_o, W, and the set $\{s_i, v_i\}$. Let t denote the value of V_o on d. Then evaluating Equation 5-9 at a boundary point yields

$$t = \Sigma c_i t_i .$$

The v_i are orthogonal hence the relation

$$V - t - W = \Sigma c_i v_i$$

implies that the coefficients can be expressed

$$c = \langle V_o - t - W, v_j \rangle / \langle v_j, v_j \rangle \qquad \text{5-11}$$

The coefficients c_j can also be expressed as a function of the u_j. Making use of Theorems 4, 5, and 6, Equation 5-10 can be rewritten

$$<v_o,u_j> - t_j(\int WdA + a(w)/s_j) =$$

$$(-1)\left(\sum_{\substack{i\neq j\\ i=1}}^{\infty} c_i t_i t_j\right) + c_j<u_j,u_j> \qquad 5\text{-}12$$

If u_j is of Category I then $t_j = 0$ hence

$$c_j = <v_o,u_j>/<u_j,u_j> \qquad 5\text{-}13$$

If u_j is of Category II then $t_j \neq 0$. Multiplying Equation 5-12 by $1/t_j$ and expanding $<u_j,u_j>$, and substituting $\int WdA = 1$ we have

$$<v_o,u_j>/t_j - (1 + a(W)/s_j) = (1/t\ )(-1)\sum_{i\neq j} c_i t_i t_j$$

$$+ \left(\frac{c_j}{t_j}\right)\left[\int v_j^2 dA + 2\,\frac{a(v_j)}{s_j}\int v_j dA + \left(\frac{a(v_j)}{s_j}\right)^2\right]$$

But $t_j = a(v_j)/s_j$ hence

$$<v_o,u_j>/t_j - (1 + a(W)/s_j) =$$

$$-\sum_{i=1}^{\infty} c_i t_i + (c_j/t_j)\left[\int v_j^2 dA + 2t_j\int v_j dA + 2\left(\frac{a(v_j)}{s_j}\right)^2\right] \qquad 5\text{-}14$$

But

$$L(v_j) + s_j v_j = 0$$

implies

$$\int v_j dA = -\int L(v_j)/s_j dA = -a(v_j)/s_j = -t_j \qquad 5\text{-}1$$

and

$$\Sigma c_i t_i \;=\; t$$

so that Equation 5-14 simplifies to

$$c_j \;=\; \{<v_o,u_j> - t_j(1 + a(W)/s_j) \;-\; tt_j\}/\{\int v_j^2 dA\}$$

when u_j is of Category II.

Computationally the problem is reduced to calculating

$$<1,W>,\; a(W),\; <1,v_i>, \text{ and } <v_i,v_i>$$

for each value of i . For a particular geometry the problem can be simplified as shown in Chapter 8.

It is perhaps appropriate to summarize the steps necessary to take and the questions to be answered in actual practice. First, if u_i is to satisfy

$$L(u) + su - a(u) \;=\; 0$$

on all of A , is there any validity for assuming $L(u) = 0$ for some x in d and if not, is there reason for $<u_i,1> = 0$. If so then the problem is to solve for v_i such that

$$L(v_i) + s_i v_i \;=\; 0 \;.$$

If this equation has a nontrivial solution then one must consider the cardinality of the solution set and the computation of $a(v_i)$ so that the set can be categorized and the u_i computed. In particular, does the set u_i contain sufficient solutions such that

$$V_o = \Sigma c_i u_i + W$$

and finally is the solution unique? That is if r_i, f_i, g_i satisfy the same properties as s_i, u_i, v_i respectively and

$$V_o = \Sigma b_i f_i + W$$

then is

$$\Sigma b_i f_i \exp(-r_i z) = \Sigma c_i u_i \exp(-s_i z) \quad ?$$

For convergence in $L^2(A)$ the answer is yes, and for pointwise convergence the answer is a qualified yes. If V_o is sufficiently well behaved so that all series are uniformly convergent (or at least integration distributes over the sums) then the series are equivalent in the case where no multiple eigenvalues occur and the eigenvalues are in algebraic ascending order. If multiple eigenvalues occur then parentheses must be inserted to include those terms of like eigenvalues. A proof of this will be presented, but first the existence of a solution will be considered in terms of the properties of V_o .

CHAPTER 6

THE EXISTENCE AND CONTINUITY OF SOLUTIONS

Solutions to Equation 5-4 are now the principle concern of this analysis. All analysis to this point, with the exception of examples, presents results for an arbitrary geometry. The examples cited, the steady state solution W for the geometries cited, and the nature of the exponential decay functions in Equation 5-6 all indicate that classical results from the analysis of Fourier series should be considered in conjunction with the expected properties of V_o. Continuity of V_o on the interior of A is a natural requirement for laminar fluid flow. If V satisfies Equation 5-2 at $z = 0$ then $\nabla^2 V_o$ exists which implies two or more derivatives of V_o exist with respect to each of the cross sectional coordinates. Equation 5-2 also requires that a one-sided derivative exists at each point of the boundary. For the geometry of a rectangular duct, solutions for the classical uniform initial profile with bound boundary values of zero and unity will be derived. Although uniform initial profiles present some ideal abstract properties, a much more general type of profile V_o will be considered.

The first properties considered are:

H0. V_o is constant on d .

H1. V_o is continuous and bounded on the interior of A .

H3. V_o is integrable over a and $\int V_o dA = 1$, such that for any sequence of points x_i and y_i converging to a boundary point of A , then $V_o(x_i, y_i)$ converges to c . We denote V^* the function $V^* = V_o$ in the interior of A and $V^* = c$ on the boundary.

H4. $\partial V^*/\partial n$ exists on the boundary and $\oint_d \partial V^*/\partial n ds$ exists.

H5. V^* is of bounded variation over $\bar{A}$.

H6. V_o is continuously twice differentiable with respect to x and with respect to y on the interior of A . Moreover, $\frac{\partial^2 V}{\partial x \partial y}$ exists and is continuous on A^o .

The function $V^* - c$ has the properties H1 through H6 and $V^* - c = 0$ on the boundary d . We denote $B = V^* - c - W$, and $B_o = V_o - c^* - W$ where c^* is the value of V_o on d . The functions B_o and B belong to the separable Hilbert space of square integrable functions that vanish on d if W the solution to $L(W) = a(W)$, satisfies these properties.

The governing equation for W is similar to Poison's equation, $L(u) = g(x,y)$, as we will see. The existence of a solution is a classical result in mathematical physics requiring minimal restrictions of the geometry. In particular, if D^{o} is a bounded, simply connected, open subset of the plane such that the boundary d of D^{o} is a simple closed piecewise smooth curve, then proofs of the following results can be found in Courant and Hilbert [12]. Denote the closure of D^{o} by D and the two-dimensional Laplacian by L .

C1. For any real valued function $g(x,y)$ defined on D , such that g with its first derivatives are continuous, then there exists a real valued function u defined on D which is continuous, has first and second continuous derivatives, and satisfies the equations

$$L(u) = g(x,y) \quad \text{on} \quad D^{o}$$
$$u = 0 \quad \text{on} \quad d .$$

C2. The set of solutions s_i and v_i to the equation

$$L(v) + sv = 0$$

subject to $v = 0$ on d is countably infinite, multiplicities of the s_i are finite, the s_i are unbounded, and the v_i can be assumed orthonormal. (The fact that they are all positive has already been shown. The eigenfunctions are orthogonal to within multiple eigenvalues hence can be assumed an orthonormal sequence.)

C3. Any function w satisfying $w = 0$ on d, and having continuous first and second derivatives may be expanded in terms of the eigenfunctions v_i in an absolutely and uniformly convergent series

$$w = \Sigma d_i v_i .$$

Suppose $g = 1$ on D. Then g satisfies all hypotheses, hence there exists a solution u such that u has continuous first and second derivatives and

$$L(u) = 1$$
$$u = 0 \text{ on } d .$$

By Green's Theorem

$$\int L(u)\,dA = \oint_d \partial u/\partial n ds$$

If the area of D is unity then

$$\int L(u)\,dA = \int 1 dA = 1$$

hence $W = u/\int u dA$ is the required solution. Result C3 implies the existence of coefficients d_i such that

$$W = \Sigma d_i v_i$$

and

$$\int W dA = 1 = \Sigma d_i \int v_i dA .$$

The series for W is uniformly convergent and W has continuous first and second derivatives.

If V_o is discontinuous on d then the functions B and B_o differ everywhere by $c - c^*$ except on the boundary. If series solutions exist for Equation 5-6, then for positive z the two solutions may differ everywhere.

It should be pointed out, that there are some results obtainable for positive definite operators in general that apply to this problem. Suppose that M is a dense linear manifold in a Hilbert space H, and L is a negative definite self-adjoint operator from M into H. Define H_L to be the domain of L with the scalar product

$$[u,v] = \langle -Lu,v \rangle$$

where $\langle \ \rangle$ is the inner product in H. Miklin [6] proves the following:

R1. H_L is an inner product space and

$$||u||_H \leq 1/b \; ||u||_L$$

where b is the scalar associated with the negative definite operator L.

R2. H_L can be completed by elements from H. That is the completion $\overline{H}_L$ of H_L is a subspace of H.

R3. M is dense in H_L.

R4. If bounded sets in H_L are compact in H, then

(a) the operator $-L$ has a countable set of eigenvalues (i.e., the set s such that $L(v) + sv = 0$ is countable).

(b) The only condensation point of the set of eigenvalues is infinity.

(c) The set of eigenvectors of $(-L)$ is complete in H_L as well as in H.

R5. The two-dimensional Laplacian operator is negative definite self adjoint on the linear manifold of all real valued functions from R^2 into R, such that f is zero on the boundary and outside a boundary region and continuously twice differentiable on the interior of the region.

If the two-dimensional Laplacian satisfies the hypothesis of result R4, then for any function B in the space H, there exist a solution

$$B = \Sigma c_i v_i$$

where the convergence of the series is in H and in H_L. In particular when B is in M, then the solution exists.

The solution may not converge pointwise but it does converge in $L^2(A)$. Moreover, in a Hilbert space H, if a set of vectors b_i is convergent to a vector b, then for any vector c in H it follows that

$$<\Sigma b_i, c> = \Sigma <b_i, c> ,$$

and the summations are independent of the ordering of the sum. (See Wilansky [7]).

Miklin [10] gives additional results concerning generalized operator equations of this type and Sobolev [11], proves several useful results applicable to this problem, including the fact that for very general geometry and continuous V_o the problem is soluable. Henceforth, the operator L will denote the Laplacian operator, and the geometry of the boundary will be as set forth in defining the domain D.

Define the equation

$$L(u) + su = 0 \qquad 6\text{-}1$$

and the domain of L to be restricted to those real valued functions D^o such that $L(u)$ is defined and $u = 0$ on d.

The following results are proven in [11].

T1. The solutions s_i to Equation 6-1 are countably infinite.

T2. For any square integrable function f defined on D then there exist c_i such that

$$f = \Sigma c_i v_i$$

convergent in $L^2(D)$. (Note f need not be zero on d .)

T4. The functions are complete in the space of continuous real valued functions on D . That is, given f continuous, there exist a real sequence c_i such that

$$f = \Sigma c_i v_i$$

where convergence is in $L^2(D)$.

If $f = 0$ on d , and if f has second order derivatives continuous everywhere including d , then the series converges uniformly. Under these conditions the derived series converges in $L^2(D)$. That is to say, if

$$f_n = \Sigma c_i v_i$$

then the sequence

$$\int \left(\frac{\partial(f - f_n)}{\partial x}\right)^2 + \left(\frac{\partial(f - f_n)}{\partial y}\right)^2 dA$$

tends to be zero.

These results make it possible to impose some very weak conditions on V_o and still guarantee existence of solutions to Equation 5-4. The bounded simply connected set A with boundary a piecewise smooth simple closed curve d , satisfies the hypothesis of the domain D , so that for any initial profile V_o and steady state solution W such that $V_o - W$ is square integrable then there exist real c_i such that

$$V_o - W = \Sigma c_i v_i$$

where convergence is in $L^2(A)$, and the c_i are given by

$$c = \langle V_o - W, v_i\rangle / \langle v_i, v_i\rangle \quad .$$

If conditions H2 and H0 hold then V^* can be defined and

$$B = V^* - c - W \qquad 6\text{-}2$$

and

$$B_o = V_o - c^* - W \qquad 6\text{-}3$$

are both square integrable if V_o is, hence can be expressed as a series in v_i . A discontinuous profile is certainly legitimate if the normal derivatives are defined in the limit along the normal to the boundary at each point of the boundary. Derivation of the flow equations and nondimensionalizing requires that

$$\int W dA = \int V_o dA = 1 \ . \qquad 6\text{-}4$$

Since V_o and V^* differ only on d

$$\int V_o dA = \int V^* dA = 1 \qquad 6\text{-}5$$

denote

$$B_o = \Sigma c_i^* v_i \qquad 6\text{-}6$$

$$B = \Sigma c_i v_i \qquad 6\text{-}7$$

Substituting Equation 6-6 into 6-3 and Equation 6-7 into 6-2 yields respectively,

$$\int B_o = \int(\Sigma c_i^* v_i)\,dA = -c \qquad 6\text{-}8$$

$$\int B = \int(\Sigma c_i v_i)\,dA = -c^* \qquad 6\text{-}9$$

If H6 holds, then the series in Equation 6-7 is uniformly convergent, thus B is continuous in the product topology and

$$\Sigma c_i \int v_i\,dA = -c\ . \qquad 6\text{-}10$$

If V_o is discontinuous on d, then the series in Equation 6-6 is not uniformly convergent. At this point the difference between a continuous and discontinuous initial profile becomes evident. With no further hypotheses the difference will be that the solution to Equation 5-4 for the continuous profile V_o yields a uniformly convergent solution and for the discontinuous profile yields a solution convergent in $L^2(A)$ hence possibly pointwise convergent nowhere. In order to obtain the series for Equation 5-4 for the discontinuous profile an equation analogous to 6-10 must be obtained. By hypothesis V_o is integrable and square integrable and W is continuously twice differentiable on A, hence shares these properties. The function B_o is absolutely integrable and by the Schwartz inequality

$$\left|\int f\,dA\right|^2 \le \int f^2\,dA\ . \qquad 6\text{-}11$$

If $f = |B_o|$ then

$$\int |B_o| dA \leq \int B_o^2 dA . \tag{6-12}$$

Similarly, if $f_n = \sum_{i=1}^{n} c_i v_i$, then

$$\int |B_o - f_n| dA \leq \int (B_o - f_n)^2 dA = p_n \tag{6-13}$$

The p_n converge to zero, hence

$$\int B_o dA = \int \sum_{i=1}^{\infty} c_i^* v_i dA = \sum_{i=1}^{\infty} c_i^* (\int dA) \tag{6-14}$$

Substitution of Equation 6-14 into 6-8 yields

$$\sum_{i=1}^{\infty} c_i^* \int v_i dA = -c^* \tag{6-15}$$

Recall that

$$\int v_i dA = -a(v_i)/s_i \tag{6-16}$$

Substituting Equation 6-16 into 6-9 and 6-15 yields respectively

$$\Sigma c_i a(v_i)/s_i = c \tag{6-17}$$

and

$$\Sigma c_i^* a(v_i) s_i = c^* . \tag{6-18}$$

Substitution of Equation 6-6, 6-7, 6-17, 6-18 into Equations 6-1 and 6-2 yields

$$\Sigma c_i (v_i + a(v_i)/s_i) = V^* - W \tag{6-19}$$

$$\Sigma c_i^* (v_i + a(v_i)/s_i) = V_o - W \ . \tag{6-20}$$

where the convergence is uniform in 6-17 and in $L^2(A)$ in Equation 6-19. The equations 6-17 and 6-18 are of the desired form to satisfy Equation 5-6. The principle question remaining is: Does the series in Equation 6-17 converge pointwise? Kolmogorov and Fomin [12] show that there exists a subsequence of the functions

$$f_n = \sum_{i=1}^{n} c_i^* v_i$$

that converges to B_o almost everywhere. There exist then a sequence of integers i_k such that

$$\lim_{k} \sum_{j=1}^{i_k} c_j^* v_j = B_o$$

almost everywhere.

Halmos [14] states the following results:
If A is a measureable set of finite measure, and if f_n is a sequence of finite valued measureable functions which converge almost everywhere to a finite valued measureable function f, then for every ε greater than zero there exist a measureable set F such that the measure of F is less than ε and the sequence f_n converges uniformly to f on $A - F$.

This type of convergence will be referred to as almost uniform convergence. The subsequence f_{i_k} above converges almost uniformly to B_o .

For certain spaces the convergence of the series in Equation 6-6 can be expressed without considering a subsequence of the f_n . Define the function I_o to be 1 on A^o and zero on d .

H7. The function I_o can be expanded in a series

$$I_o = \Sigma b_i v_i$$

uniformly convergent on any compact subset of A^o the interior of A , and absolutely convergent to I_o on A .

If the hypothesis H7 holds, then

$$B_o = B + (c - c^*) I_o$$

hence can be written as the sum of two series uniformly convergent on compact subsets of A^o say

$$B_o = \Sigma (c_i + b_i) v_i$$

and

$$\int B_o = \int \Sigma (c_i + b_i) v_i = \Sigma \int (c_i + b_i) v_i = -c^*$$

In either case, a solution to Equation 5-4 is possible, convergence in $L^2(A)$ is guaranteed, almost uniform convergence of a subsequence is guaranteed, and for the case when H7 holds the series is uniformly convergent on compact subsets of A^o .

For the special case of a rectangular duct I_o can be expressed by $I_o = f(x)g(y)$ where f and g are series uniformly convergent to 1 on any compact subset of the open interval hence H7 holds.

Several results can be obtained concerning the continuity of the solution. Consider first the following generalized form of

Abel's Convergence Test: Let S and T be point sets, let u_i, v_i be real valued functions such that

$$v_i: S \to R$$

$$h_i: T \to R$$

and

$$\sum_{i=1}^{\infty} v_i(x)$$

converges uniformly for x in S and for fixed z in T the $h_i(z)$ are monotone, and $h_i(t)$ are uniformly bounded for all positive integers and all $t \in T$. Then $\Sigma v_i h_i$ converges uniformly. (See Carslaw [15].)

The set S could be composed of n-dimensional vectors, and the set could be composed of one point. For example, if S is a subset of R^2 and Σv_i is convergent at a point (x_o, y_o) then $\Sigma v_i h_i$ is uniformly convergent on T. Similarly, if for fixed x_o, Σv_i is uniformly convergent for every (x_o, y) in S, then $\Sigma v_i h_i$ is uniformly convergent for (x_o,y) in S and t in T. Finally, if Σv_i is uniformly convergent for all (x,y) in S, then $\Sigma v_i h_i$ is uniformly convergent on $S \times T$.

Define $h_i = \exp(-s_i z)$, and the set

$$T = [0,z_o] .$$

For fixed x_o and y_o then $\Sigma v_i h_i$ is uniformly convergent for x in T, hence continuous in z. If X is any subset such that $\Sigma c_i v_i$ converges uniformly on X, then the series converges uniformly on $X \times T$. If X is compact, then $X \times T$ is compact, hence the limit function is continuous on $X \times T$ provided the v_i are all continuous.

CHAPTER 7

UNIQUENESS OF SOLUTIONS

Suppose a solution U exists such that

$$V = \Sigma c_i u_i \exp(-s_i z) + W \qquad 7\text{-}1$$

where

$$U = \Sigma c_i u_i \exp(-s_i z)$$

and the u_i satisfy

$$L(u_i) + s_i u_i - a(u_i) = 0 \qquad 7\text{-}2$$

and v_i are the corresponding solutions to

$$L(v_i) + s_i v_i = 0 . \qquad 7\text{-}3$$

Let us first insist that the eigenvalues are in ascending algebraic order since infinite reordering of a series can change the sum. That is, we assume $j > i$ implies $s_j \geq s_i$. Suppose that another solution U^* exists such that

$$V^* = U^* + W , \qquad 7\text{-}4$$

and

$$V^*(x,y,0) = V(x,y,0) \qquad 7\text{-}5$$

where

$$U^* = \Sigma c_i^* u_i^* \exp(-s_i z) \tag{7-6}$$

and v_i^* are the corresponding solutions to

$$L(v_i^*) + s_i^* v_i^* = 0 \tag{7-7}$$

Assume that the s_i^* are also in algebraic ascending order. We claim that $v = v^*$. First consider the case where none of the eigenvalues s_i are multiple eigenvalues. Note that

$$V^*(x,y,0) = V(x,y,0) \tag{7-8}$$

implies

$$\Sigma c_i u_i(x,y,0) = \Sigma c_i^* u_i^*(x,y,0) \tag{7-9}$$

and

$$\Sigma c_i v_i(x,y,0) = \Sigma c_i^* v_i^*(x,y,0) \tag{7-10}$$

since for any boundary point x_o, y_o

$$V^*(x_0,y_0,0) = V(x_0,y_0,0) \tag{7-11}$$

whence

$$\Sigma c_i \frac{a(v_i)}{s_i} = \Sigma c_i^* \frac{a(v_i)}{s_i^*} . \tag{7-12}$$

But the v_i form an orthogonal sequence of functions since

$$L(v_i) + s_i v_i = 0$$

$$L(v_j) + s_j v_j = 0$$

implies

$$(s_i - s_j)v_iv_j = v_iL(v_j) - v_jL(v_i)$$

hence

$$(s_i - s_j)\int v_iv_j dA = \int(v_iL(v_j) - v_jL(v_i))dA$$

which by property P3 yields

$$(s_i - s_j)\int v_iv_j dA = \oint(v_i\partial v_j/\partial n)ds$$

and $v_i = v_j = 0$ on the boundary d . Finally we have

$$(s_i - s_j)\int v_iv_j dA = 0$$

and the eigenvalues are single valued, hence $s_i \neq s_j$ implies

$$\int v_iv_j dA = 0 . \qquad 7\text{-}13$$

If $s_i = s_i^*$ for every i then

$$\int v_iv_j^* = 0 \qquad 7\text{-}14$$

via the same proof. Let us further assume that the coefficients c_i and c_i^* are all nonzero.

If $v_i = bv_i^*$, then

$$<v_i,v_i> = b^2<v_i^*,v_i^*>, \qquad 7\text{-}15$$

from Equation 10

$$c_i<v_i,v_i> = c_i^*<v_i^*,v_i^*> \qquad 7\text{-}16$$

hence

$$c_i = bc_i^* \frac{<v_i^*, v_i^*>}{v_i, v_i} \qquad 7\text{-}17$$

thus

$$c_i v_i = \frac{bc_i^* <v_i^*, v_i^*> bv_i^*}{<v_i, v_i>} \qquad 7\text{-}18$$

Substituting Equation 15 into Equation 18

$$c_i v_i = \frac{b^2 c_i^* <v_i^*, v_i^*> v_i^*}{b^2 <v_i^*, v_i^*>} \qquad 7\text{-}19$$

$$cv_i = c_i^* v_i^* \ .$$

Thus when $s_i = s_i^*$ for every value of i we have eigenvalence of each individual term of the series. Now suppose for some j , $s_j \neq s_i$ for every i . The multiplying Equation 10 by v_j and integrating over A , we have

$$c_j <v_j, v_j> = 0$$

which implies $c_j = 0$ contrary to the hypothesis that all c_i are nonzero. This completes the proof for the case where all eigenvalues are single valued.

Now suppose that one or more of the s_i are finitely multiple eigenvalues. In this event we will denote the eigenfunctions by ${}_n v_i$ where n varies from 1 to the number of linearly independent eigenfunctions associated with s_i . Without loss of generality we can assume that the ${}_n v_i$ are mutually orthogonal. If for some j , $s_j \neq s_i^*$ for each value of i , then $c_i = 0$ contrary to our hypothesis. Henc $s_i = s_i^*$ for every i . Hence, for all values of i , all multiplicities will be of the same corresponding order. If s_i is a single valued eigenvalue then $c_i v_i = c_i^* v_i^*$ as before. If s_i is a multiple eigenvalue, denote the series

$$U = \sum_i \left((\exp -s_i z) \left(\sum_{j=1}^{q_i} {}_j c_i \; {}_j v_i \right) \right) , \qquad 7\text{-}20$$

where q_i is the order of the eigenvalue s_i and ${}_j v_i$, $j = 1$, q_i are the corresponding eigenvectors. We know that $s_i = s_i^*$ for each i so that $q_i = q_i^*$. It will be shown that for each value of i ,

$$\sum_j {}_j c_i \; {}_j v_i = \sum_j {}_j c_i^* \; {}_j v_i^* \qquad 7\text{-}21$$

We have already shown that the choice of the norm of the vectors v_i does not affect the value of $c_i v_i$ when all eigenvalues are single valued. The only property used in that proof is that v_i is orthogonal to all other v_j hence the vectors ${}_j v_i$ have the same property. That is $\langle {}_j v_i , {}_k v_l \rangle$ is nonzero if and only if $k = j$ and $i = l$. Hence we can assume that the ${}_j v_i$ and the ${}_j v_i^*$ are all of norm 1. Moreover, the vector subspace spanned by the eigenvectors associated with a given eigenvalue is unique and independent of the way in which the basis eigenfunctions are chosen, hence for a fixed k , there exist coefficients ${}_i b_j$ such that

$$ {}_iv_k = \sum_{j=1}^{l} {}_ib_j \; {}_jv_k^* \qquad 7\text{-}22 $$

and we have the following relations: for $m \neq i$

$$ \langle {}_iv_k, {}_mv_k \rangle = 0 = \sum_{j=1}^{q_j} {}_ib_j \; {}_mb_j \qquad 7\text{-}23 $$

$$ \langle {}_iv_k, {}_iv_k \rangle = 1 = \sum_{j=1}^{q_i} {}_ib_j^2 \qquad 7\text{-}24 $$

The scalars ${}_ib_j$ form a square matrix B where i,j denote respectively the row and column. The rows of the matrix B are pairwise orthogonal vectors in l^2 norm, and each has norm one, hence

$$ (B^T)^{-1} = (B^T)^T = B $$

hence

$$ B^{-1} = ((B^T)^{-1})^{-1} = B^T $$

implies that the columns of B are pairwise orthogonal and have norm one. Thus

$$ \sum_{i=1}^{q_j} {}_ib_j^2 = 1 \qquad 7\text{-}25 $$

and when $j \neq m$

$$\sum_{i=1}^{q_i} {}_ib_j \; {}_ib_m = 0 \qquad 7\text{-}26$$

When $z = 0$ we get an equation similar to Equation 10 by substituting $z = 0$ into Equation 20 and we have

$$\sum_m \sum_{j=1}^{q_m} {}_jc_m \; {}_jv_m = \sum_m \sum_{j=1}^{q_m} {}_jc_m^* \; {}_jv_m^* \qquad 7\text{-}27$$

Multiplying Equation 21 by ${}_iv_k$ and integrating, yields

$$ {}_ic_k = \sum_{j=1}^{q_k} {}_jc_k^* \langle {}_jv_k^* \; {}_iv_k \rangle \qquad 7\text{-}28$$

Substituting Equation 22 into 28, yields

$$ {}_ic_k = \sum_{j=1}^{q} {}_jc_k^* \; {}_ib_j \qquad 7\text{-}29$$

where $q = q_k$.

Hence

$$\sum_i^q {}_ic_k \; {}_iv_k = \sum_i \left(\sum_{j=1}^{q} {}_jc_k^* \; {}_ib_j \right) {}_iv_k \qquad 7\text{-}30$$

Substituting Equation 22 into Equation 28

$$\sum_{i=1}^{q} {}_i c_k \; {}_i v_k = \sum_{i=1}^{q} \left(\sum_{j=1}^{q} {}_j c_k^* \; {}_i b_j \right) \left(\sum_{p=1}^{q} {}_i b_p \; {}_p v_k \right) \tag{7-31}$$

$$\sum_{i=1}^{q} {}_i c_k \; {}_i v_k = \sum_{p=1}^{q} {}_p v_k \; {}_p c_k^* \left(\sum_{i=1}^{q} {}_i b_p^2 \right) + \sum_{j \neq p} {}_j c_k \left(\sum_{i=1}^{q} {}_i b_j \; {}_i b_p \right) \tag{7-32}$$

Substituting Equation 25 and 26 into Equation 32 we have

$$\sum_i {}_i c_k \; {}_i v_k = \sum_i {}_i c_k^* \; {}_i v_k^*$$

and the proof is complete.

CHAPTER 8

COMPUTING $\varepsilon(z)$

In order to relate V to real 3 space (still non-dimensional) one must transform from the stretched axial coordinate z to the axial coordinate z^*. The non-stretched coordinate is given by

$$z^*(z) = \int_o^z \varepsilon(z)\,dz \qquad 8\text{-}1$$

where

$$\varepsilon(z) = \frac{\partial/\partial z \int (V^2 - V^3/2)\,dA}{-\int V \partial v/\partial z\,dA} \qquad 8\text{-}2$$

The function V is given by

$$V(x,y,z) = U(x,y,z) + W(x,y) \qquad 8\text{-}3$$

where the following relations hold:

$$L(U) = \partial U/\partial z + a(U)\ , \qquad 8\text{-}4$$

$$L(W) = a(W)\ , \qquad 8\text{-}5$$

$$\oint_d W dV/dn ds = 0 \qquad 8\text{-}6$$

since $W = 0$ on d. Moreover,

$$\int U dA = 0 \qquad 8\text{-}7$$

since

$$\int V dA = \int U dA + \int V dA$$

and

$$1 = \int W dA = \int V dA \qquad 8\text{-}8$$

Notice that

$$\partial V/\partial z = \partial U/\partial z$$

since W is not a function of z. We will let $U(a,b,z)$ denote a boundary value of U. That is (a,b) is in d, and denote the denominator of Equation 8-2 by

$$D = -\int V \partial U/\partial z dA \qquad 8\text{-}9$$

Suppose U and W can be represented by series of orthogonal functions g_i as follows:

$$U = \Sigma c_i g_i \exp(-s_i z_i)$$

and

$$W = \Sigma b_i g_i \ .$$

In this event the last term of Equation 8-9 can be simplified to the form

$$\int V \partial U/\partial z dA = \Sigma a_i ||g_i||^2 s_i \exp(-s_i z)\{b_i/c_i + \exp(-s_i z)\} \qquad 8\text{-}10$$

Wiginton and Wendt present the above simplifications [5] where the assumption is made that $U = 0$ on the boundary d simplifying Equation 8-9 to the expression in Equation 8-10.

In the case where V is a nonzero constant on d, the computations are similar but the expressions for U and W will involve different functions. Suppose that

$$W = \Sigma d_i v_i \qquad 8\text{-}11$$

and

$$U = \Sigma c_i u_i \exp(-s_i z) \qquad 8\text{-}12$$

where

$$u_i = v_i + a(v_i)/s_i \qquad 8\text{-}13$$

and the v_i satisfy

$$L(v_i) + s_i v_i = 0 \text{ ,}$$

$$v_i = 0 \text{ on } d \text{ .}$$

From Equation 8-13, one can compute

$$\langle u_i, u_i \rangle = \int v_i^2 dA + 2(a(v_i)/s_i)\int v_i dA + (a(v_i)/s_i)^2 \qquad 8\text{-}14A$$

and Theorem 6 gives

$$\langle u_1, u_2 \rangle = -a(v_1)a(v_2)/s_1 s_2) \qquad 8\text{-}14B$$

when $s_1 \neq s_2$. An application of Theorem 4 yields

$$\int W u_i dA = (a(v_i)/s_i)(1 + a(W)/s_i) \qquad 8\text{-}15$$

(note $\int UdA = 0$ and $\int WdA = 1$.)

Differentiating 8-11,

$$\partial U/\partial z = \Sigma -s_i c_i u_i \exp(-s_i z) \qquad 8\text{-}16$$

hence

$$\int W \partial U/\partial z dA = \Sigma(-s_i c_i \exp(-s_i z) \int W u_i dA \ . \qquad 8\text{-}17$$

Substituting 8-15 into 8-17

$$\int W \partial U/\partial z dA = \Sigma -s_i c_i a(v_i)/s_i \exp(-s_i z)$$
$$+ a(W) \Sigma -c_i a(v_i)/s_i \exp(-s_i z) \qquad 8\text{-}18$$

However

$$\Sigma -s_i c_i a(v_i)/s_i \exp(-s_i z) = \partial U(a,b,z)/\partial z \qquad 8\text{-}19$$

and

$$U(a,b,z) = \Sigma -c_i a(u_i)/s_i \exp(-s_i z) \qquad 8\text{-}20$$

Substituting Equations 8-19 and 8-20 into 8-18

$$\int W \partial U/\partial z dA = \partial U(a,b,z)/\partial z - U(a,b,z) a(W) \qquad 8\text{-}21$$

Similarly one can derive the expression

$$\int U \partial U/\partial z dA = \sum_i \sum_{j \neq 1} c_i c_j (-s_j) \exp(-s_i z) \exp(-s_j z) \langle u_i, u_j \rangle$$
$$+ \sum_k c_k^2 (-s_k)(\exp(-s_k z))^2 \langle u_k, u_k \rangle \qquad 8\text{-}22$$

Substituting Equations 8-14A and 8-14B and 5-15 into 8-22

$$\int U \frac{\partial U}{\partial z} dA = \sum_i \sum_{j \neq 1} -c_i c_j (-s_j) \exp(-s_i z) \exp(-s_j z) a(v_i) a(v_j)/s_j s_j)$$
$$+ \sum_k c_k^2 (-s_k)(\exp(-s_k z))^2 \{\langle v_k, v_k \rangle - (a(v_k)/s_k)^2$$

$$8\text{-}23$$

The last term can be incorporated into the double series so that the double sum is over all indices. The double series can be written as the product of two series as follows:

$$\int U\frac{\partial U}{\partial z}dA = -\sum_i c_i (a(v_i)/s_i)\ \exp(-s_i z)\Sigma(-1)c_i a(v_i)\ \exp(-s_i z)$$
$$+ \sum_k c_k^2(-s_k)(\exp(-s_k z))^2\{\langle v_k, v_k\rangle\} \qquad 8\text{-}24$$

Substituting Equations 8-19 and 8-20 into 8-24 yields

$$\int U\frac{\partial U}{\partial z}dA = -U(a,b,z)\partial U(a,b,z)/\partial z$$
$$+ \sum_k c_k^2(-s_k)(\exp(-s_k z))^2\{\langle v_k, v_k\rangle\} \qquad 8\text{-}25$$

Let us denote

$$T(z) = \sum_k c_k^2(-s_k)(\exp(-s_k z))^2\{\langle v_k, v_k\rangle\} \qquad 8\text{-}26$$

Equation 8-9 can be simplified by substituting Equations 8-25, 8-26, and 8-21 to obtain

$$D = -\{\partial U(a,b,z)/\partial z - U(a,b,z)a(W)$$
$$U(a,b,z)\partial U(a,b,z)/\partial z + T(z)\} \qquad 8\text{-}27$$

The value for a(W) must be computed only once. For each z the denominator can be obtained by computing only three quantities as opposed to an integration over an area. Equation 8-9 can be rewritten

$$\varepsilon(z) = \frac{\frac{3}{2}\int V^2 \frac{\partial V}{\partial z}\, dA}{D} - 2 .$$

where D is a function of z given by Equation 8-27.

In order to compare with other numerical results the quantity $z^*/(D_h R_e)$ is used as the independent variable in the axial direction for all graphs. For an aspect ratio of r^2 we have

$$z^*/(D_h R_e) = \left(\frac{r^2}{4} + .5 + \frac{1}{4r^2}\right) \int_0^z \varepsilon(t)\, dt$$

CHAPTER 9

FLOW IN A RECTANGULAR DUCT WITH UNIFORM V_o

The purpose of this chapter is to present a detailed application of the theory presented in Chapters 5 and 6. The particular geometry chosen is that of a rectangle as described in Example 2 of Chapter 1. A countable set of solutions v_i to the equation

$$L(v) + sv = 0 \qquad 9\text{-}1$$

subject to $v = 0$ on d are given in this example and the corresponding solutions u_i satisfying

$$L(u) + su - a(u) = 0 \qquad 9\text{-}2$$

subject to u constant on d are given by

$$u_i = v_i + a(v_i)/s_i \quad . \qquad 9\text{-}3$$

It was pointed out in Chapter 5 that the computation of the coefficients c_i in Equation 5-6 can be simplified for particular applications. This is generally true if the solution W to Equation 5-5 can be represented as a series

$$W = \sum_i d_i v_i \quad . \qquad 9\text{-}4$$

For if this is the case then

$$V_o = V(x,y,0) = \Sigma c_i(v_i + a(v_i)/s_i) + \Sigma d_i v_i \ . \qquad 9\text{-}5$$

On the boundary $v_i = 0$ hence

$$V_o(a,b,0) = \Sigma c_i a(v_i)/s_i \ . \qquad 9\text{-}6$$

If $V_o(x,y,0)$ is a constant (not $\int V_o dA = 1$ and $\int dA = 1$ forces $V_o = 1$ in this case.) then

$$0 = \Sigma(c_i + d_i)v_i \qquad 9\text{-}7$$

The v_i are orthogonal, hence

$$-c_i = d_i \qquad 9\text{-}8$$

For nonconstant initial profiles we still have

$$V_o(x,y,0) - V_o(a,b,0) = \sum_i (c_i + d_i)v_i \qquad 9\text{-}9$$

where the v_i are orthogonal hence

$$c_j + d_j = \langle V_o(x,y,0) - V(a,b,0), v_j\rangle / \langle v_j, v_j\rangle \ . \qquad 9\text{-}10$$

The only required computations are

$$\langle V_o(x,y,0), v_j\rangle$$

$$\langle 1, v_j\rangle$$

and $$\langle v_j, v_j\rangle$$

for each j . This is a significant computational savings over the quantities required for Equation 5-11.

The nature of the solution for the rectangular duct is such that a double indexing system is most convenient. The solutions to Equation 9-1 as given in Example 2 are

$$v_{mn} = \sin(\pi mx/R)\ \sin(n\pi Ry) \ . \qquad 9\text{-}11$$

where m and n are odd integers. The solution to

$$L(W) - a(W) = 0 \qquad 9\text{-}12$$

subject to

$$W = 0 \text{ on } d \qquad 9\text{-}13$$

and

$$\int W dA = 1 \qquad 9\text{-}14$$

is

$$W = \Sigma d_{mn} v_{mn}$$

where the sum is over all odd m and n. All summations over double indices will be over the odd values only. The d_{mn} are given by

$$d_{mn} = (\pi^2 T_{Rmn})/4S_R \qquad 9\text{-}15$$

where

$$T_{Rmn} = [mnm^2/R^2 + n^2R^2]^{-1} \qquad 9\text{-}16$$

and

$$S_R = \Sigma T_{Rmn}/(mn) \ . \qquad 9\text{-}17$$

The corresponding eigenvalues for Equation 9-1 are given by

$$s_{mn} = (m\pi/R)^2 + (n\pi R)^2 \text{ .} \qquad 9\text{-}18$$

The values for a(v), derived in Appendix 1, are given by

$$a(v_{mn}) = -4(nR^2/m + m/(nR^2)) \text{ .} \qquad 9\text{-}19$$

For the initial profile

$$V_o = 1 \qquad 9\text{-}20$$

the solution is given by

$$V = \Sigma c_{mn} u_{mn} \exp(-s_{mn} z) + W \qquad 9\text{-}21$$

where the c_{mn} are defined by Equation 9-8

$$c_{mn} = -d_{mn} \qquad 9\text{-}22$$

and

$$u_{mn} = v_{mn} + a(v_{mn})/s_{mn} \text{ .} \qquad 9\text{-}23$$

Theoretically the solution is very simple, but computationally the solution can be very time consuming. Recall that the final solution in terms of the coordinate z^* involves the numerical integration of $V^2 \partial V/\partial z$ over the cross section of the duct. This requires the computation of V over a grid of points in the x-y plane; and the series solution is "slowly" convergent especially near $z = 0$. For example, several hundred terms are required before the series agrees with V_o to six significant figures. It is advisable to simplify the solution to require a minimal number of computations. The expression for u_{mn} can be simplified by substituting Equations 9-18 and 9-19 into 9-23 to obtain

$$u_{mn} = v_{mn} + \frac{(-4)\dfrac{n^2R^4 + m^2}{mnR^2}}{\pi^2\dfrac{(m^2 + n^2R^4)}{R^2}}$$

Hence

$$u_{mn} = v_{mn} - 4/(\pi^2 mn) \ . \qquad 9\text{-}24$$

It is economical to group

$$A = \Sigma - d_{mn}v_{mn} \exp(-s_{mn}z) \qquad 9\text{-}25$$

and

$$B = \Sigma(-4d_{mn}/(\pi^2 mn)) \exp(-s_{mn}z) \qquad 9\text{-}26$$

together so that

$$U(x,y,z) = A(x,y,z) + B(z) \qquad 9\text{-}27$$

where B must be computed only once for each z . The function W is computed once for the chosen x-y grid and the function A must be computed at each grid point in the x-y plane for a discrete set of values of z , so that $\int \varepsilon(z)dz$ can be computed.

The necessary equations for computing $\varepsilon(z)$ were presented in Chapter 6, but the term involving T(z) given by Equation 8-26 and the following individual paramters should be established:

$$\langle v_{mn}, v_{mn} \rangle = 1/4 \qquad 9\text{-}28$$

for all m, n; $a(v_{mn})$ can be given by Equation 9-19 or alternately

$$a(v_{mn}) = -4s_{mn}/(mn\pi^2) , \qquad 9\text{-}29$$

and

$$\int v_{mn} dA = 4/(\pi^2 mn) . \qquad 9\text{-}30$$

Substitution of Equation 9-28 into 8-26 yields

$$T(z) = \Sigma d_{mn}^2 (-s_{mn})(\exp(-s_{mn} z))^2 \qquad 9\text{-}31$$

All other terms can be computed directly.

It has been pointed out that the velocity profiles for laminar flow in a rectangular duct have been measured experimentally and approximated numerically using a uniform value, $V_o = 1$, for the initial profile. Previous numerical studies require approximation of the eigenvalues for Equation 9-2 and numerical approximation of the coefficients c_i in Equation 9-5 in addition to the numerical approximation to z^*, whereas this approach requires only the approximation to z^*.

CHAPTER 10

SOLUTION WITH ANOTHER INITIAL PROFILE

The solution in the previous chapter becomes fully developed at a larger axial distance than that predicted by experimental data and by other numerical approximations [3, 4] using a boundary condition of $V_o = 0$. One can theoretically assume an initial profile of unity on the interior of the duct and zero on the boundary hence a discontinuous function of x and y. In practice, the solution is computed by a truncated series, hence a continuous initial condition and in fact, all solutions for $z > 0$ are usually plotted continuously to zero without regard to the possible discontinuity of the infinite series for these values. Computation of $\varepsilon(z)$ can be inaccurate if numerical integration is used to compute $\int_A V^2 \partial V/\partial z dA$ and continuity of the solution is assumed.

The mathematical technique developed in this paper is not applicable for zero boundary conditions as indicated by previous proofs. This is not necessarily a defect since experimental data do not indicate zero boundary velocities, and in fact, for small aspect ratio a constant nonzero boundary velocity for fixed z is indicated by experiment [8]. Indeed, the problem of defining the boundary velocity of a fluid in a solid body has been considered by many men from the various sciences in the last 200 years. Reference [9] includes a short historical note naming many of these men, including Navier who proposed that the velocity v on the boundary is proportional to the derivative of v with respect to the outward normal. By assuming a constant nonzero boundary velocity, we have obtained a closed form solution to within one numerical approximation for $\varepsilon(z)$, the boundary velocity is strongly associated with $\oint \partial V/\partial n ds$, and for small aspect ratios one might expect the solution to be similar to one satisfying Navier's hypothesis.

If the coefficients for W are again denoted by d_i then

$$W = \Sigma d_i v_i .$$

Define

$$c_i = -d_i(1 + s_i/a(W))$$

Then on the boundary at $z = 0$

$$V = \Sigma - d_i(1 + s_i/a(W))a(v_i)/s_i$$

Let t denote the value of V on the boundary at $z = 0$. Then

$$t = \sum - \frac{d_i a(v_i)}{s_i} - \frac{d_i a(v_i)}{a(W)}$$

Let $V^* = U^* + W$ denote the solution in the previous chapter using a uniform initial profile. Then on the boundary at $z = 0$

$$V^* = \Sigma - d_i a(v_i)/s_i = 1 ,$$

but

$$a(W) = \Sigma d_i a(v_i) .$$

Hence

$$t = 1 + \frac{\Sigma - d_i a(v_i)}{\Sigma d_i a(v_i)}$$

implies

$$t = 0 .$$

It should be remarked that numerical investigation of this solution indicates that $V = 0$ on the boundary at $z = 0$, but very rapidly jumps to positive values on the boundary for positive values of z . The profiles agree well with experimental data and differ very little from a continuous solution obtained independently by approximating a smooth curve through the experimental data given at the smallest value of z .

CHAPTER 11

NUMERICAL RESULTS

The theory developed in this thesis has been applied to the problem of laminar flow in the entrance region of a rectangular duct with aspect ratio $r^2 = 2$ as described in Example 2 of Chapter 1. Both Wiginton and Fleming have independently obtained essentially the same method for simplifying the numerical solution [3,4,5] using Sparrow's mathematical model [2], and both have obtained numerical results for rectangular ducts with aspect ratios of 2 and 5. These methods require numerical approximations for the eigenvalues and the coefficients in the series expansion for the velocity profile. The results compare quite favorably with experimental data [8]. Fleming's results were chosen for comparison because of the accurate graphs presented in his thesis containing both his results and experimental data. Comparisons of results for an aspect ratio of $r^2 = 2$ are shown in Figures 1 through 4. The dotted lines represent Fleming's results and the circled points are experimental data. The solid lines represent the results from two independent applications of this analysis to be explained later.

The uniform profile considered in Chapter 9 was first used to obtain numerical results. The results did not compare favorably with experimental data. Values for the centerline velocity were too small and values for the boundary velocity were too large.

The uniform profile with a value of zero on the boundary considered in Chapter 10 is one of the initial profiles that yields the results shown by the solid line in Figures 1 through 4. In order to obtain the transformation from stretched to nonstretched coordinates one must compute the integral of $V^2\partial V/\partial z$ over the area of the duct. A 36 point grid over one-fourth of the duct was used to compute this integral. Profiles were computed for 150 values of t with a step size of $\Delta z = .0005$. Extrapolation of $\varepsilon(z)$ was required from $z = .002$ to $z = 0$ because of the inadequacy of the 36 point grid to yield an accurate value for the numerical integration. Denote

$$f(x,y,z) = V^2\partial V/\partial z .$$

The principle cause of the inaccuracy is that for any small value z_o of z , and for fixed x_o , $f(x_o,y,z_o)$ is positive and near constant over most of the interior and near zero for $y = 0$ but takes on very large negative values over some small interval (y_1,y_2) near the boundary. As pointed out by other authors, the error produced by the extrapolation is very likely in the fourth or fifth decimal place hence insignificant over most of the range of z^* . It is quite possible that the difficulty in computing $\varepsilon(z)$ stems from the discontinuous initial profile. For example, if there exists a sequence of points (x_i, y_i, z_i) where the z_i converge to zero and $\partial V(x_i, y_i, z_i)/\partial z$ is unbounded, then any numerical integration may prove inaccurate. Although the uniform initial profile with a zero boundary value yields excellent results, to within the problems of computing $\varepsilon(z)$, it is quite possible that the actual entering profile is a continuous function of x and y having large derivatives with respect to x and y near the boundary. It is of interest to consider the sensitivity of the numerical solution with respect to changes in the initial conditions.

Experimental velocity profiles along the y and x axes are given in Figures 1 and 3 respectively for an axial distance of about .0006. These curves appear to intersect the boundary at a value of about .2. The curves were translated by -.2 and parameters s and t determined such that $h(y) = 1 - \exp(-sy)$ and $g(x) = 1 - \exp(-tx)$ were smooth curves that approximated the profiles. An initial profile

$$V_o = hg + .2$$

was assumed and series coefficients c_i for Equation 5-6 were determined by using a least squares approximation to the function $V_o - .2 = hg$.

The numerical solution using these coefficients is essentially the same as the solution using the uniform initial profile with $V_o = 0$ on the boundary. The two solutions plotted together appear as one curve.

The first curve in Figure 1 and in Figure 3 is plotted for that profile that best agrees with the experimental data. The number in parenthesis denotes in each case the value of the axial coordinate where this occurs. Although the difference is in the fifth decimal place it represents an error of about 10 percent. The error is probably due to the fact that this is in the range of extrapolation for $\varepsilon(z)$ and some error could be caused by experimental error in measurements for this small a value. Profiles for smaller values of z^* agreed very well with those of Fleming's results for this first profile and it is obvious his results would better agree with the experimental data had he chosen a larger value of z^*. There is no claim that this analysis is more accurate for extremely small values of z^*. This does show, however, the profiles generated in the stretched coordinate system and the profiles measured experimentally are in very good agreement.

For each velocity profile the solid and dotted lines intersect at least once. This behavior is expected since both solutions are constrained by $\int VdA = 1$ for every value of z. Hence if f and g are two solutions satisfying this constraint, then $\int (f - g)dA = 0$. Any solution of the form

$$V = \sum_{i=1}^{n} c_i u_i + W$$

has the property

$$\int V dA = \int W dA = 1$$

since

$$\int u_i dA = 0$$

for each i .

The two curves differ near the boundary, with the solid curve comparing more favorably with the experimental data in each case. For velocities along the x axis the solid curves appear to be in agreement with the experimental data to within the accuracy obtainable from graphs. The only exception to this is near the center of the duct when $z^*/(D_h R_e) = .0098$ where the solid curve is slightly beneath the experimental data points.

The centerline velocity predicted by this analysis seems to lag slightly behind that predicted by experiment. In fact, for any profiles that differ, a profile that agrees very closely to the experimental data is predicted for some larger value of $z^*/(D_h R_e)$ with one exception. This occurs when $z^*/(D_h R_e) = .0036$. No profile is computed that matches the experimental data near $yr = .2$. Also note that the solid curve attains values less than or equal to that predicted by experiment for the profile along the x axis and the y axis. Either the experimental data is in slight error for one of the profiles or, due to the constraints $\int v dA = 1$, the profile predicted by this analysis must be in error of similar magnitude and opposite sign for other portions of the duct.

Both curves compare favorably on the interior of the duct for profiles along the y axis, the accuracy appearing about the same except when $z^*/(D_h R_e) = .0095$ where Fleming's results are more accurate. However, the slip flow model appears to provide better accuracy near the boundary in all cases, and equivalent accuracy on the interior of the duct for all but one profile.

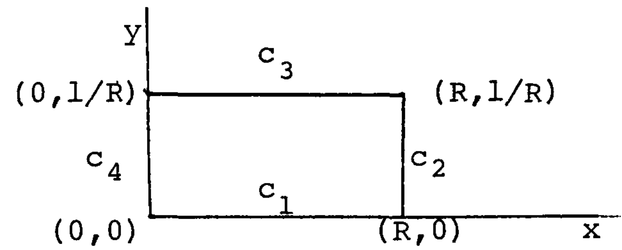
APPENDIX 1

$\oint \partial u/\partial n ds$ FOR RECTANGLE

Let $u = \sin(p\pi/Rx)\sin(q\pi Ry)$.

Compute $\oint_c \partial u/\partial \vec{n} ds$

over the rectangle



Note for any line integral $\int_c f(x,y)ds$ where c is defined by $y = g(x)$, we can rewrite

$$\int_c f(x,y)\,ds = \int_{x_o}^{x_1} f(x,g(x)) \frac{ds}{dx}\,dx$$

For example,

$$\int_{c_3} f(x,y)\,ds = \int_1^o f(x,c)\frac{ds}{dx}\,dx = \int_1^o f(x,c)(-1)\,dx$$

$$= \int_o^1 f(x,c)\,dx$$

Similarly

$$\int_{c_4} f(x,y) = \int_o^1 f(0,y)\,dy$$

Also note on c_3,

$$\partial f/\partial\vec{n} = \partial f/\partial(-y) = \partial f/\partial y$$

and on c_4,

$$\partial f/\partial\vec{n} = \partial f/\partial(-x) = -\partial f/\partial x$$

hence

$$\int_c \partial u/\partial n ds = \int_o^R (-1) q\pi R \sin\left(\frac{p\pi x}{R}\right) dx + \int^{1/R} \frac{p\pi}{R} \cos(p\pi) \sin(q\pi y) dy$$

$$+ \int_o^R q\pi R \cos(q\pi) \sin(p\pi q) dx + \int_o^{1/R} (-1)\left(\frac{p\pi}{R}\right) \sin(q\pi y) dy$$

$$= q\pi R(\cos(q\pi)-1) \int_o^R \sin\left(\frac{p\pi}{R}\right) x ds$$

$$+ p\pi R(\cos(p\pi)-1) \int^{1/R} \sin q\pi R y dy$$

$$= q\pi R \cos(q\pi - 1) \frac{(1-\cos p\pi)}{p\pi/R}$$

$$+ p \frac{\pi}{R} (\cos p\pi - 1) \left(\frac{1 - \cos q\pi}{q\pi R}\right)$$

$$= \pi R^2/p(\cos(q\pi)-1)(1 - \cos(p\pi))$$

$$+ p/qR^2(1 - \cos(p\pi))(\cos(q\pi)-1)$$

$$= (\cos(q\pi)-1)(1 - \cos(p\pi))(qR^2/p + p/qR^2)$$

$$= \oint_c \partial u/\partial\vec{n} ds$$

Checking this result by Green's integral formula

$$\oint \partial u/\partial\vec{n} ds = \int \nabla^2 u dA$$

In this case

$$\nabla^2 u = (-1)((p\pi/R)^2 + (q\pi R)^2) u$$

hence

$$\oint \nabla^2 u dA \quad = \quad (-1)(p\pi/R)^2 + (q\pi R)^2 \int_o^K \sin(\tfrac{p\pi}{R}x)dx \int_o^{1/R} \sin(q\pi R)y)dy$$

$$= \quad ((p\pi/R)^2 + (q\pi R)^2)(-1)\left(\frac{1 - \cos p\pi}{p\pi/R}\right)\left(\frac{1 - \cos q\pi}{q\pi R}\right)$$

$$= \quad (p/qR^2 + qR^2/p)(1 - \cos p\pi)(\cos(q\pi)-1)$$

This completes the proof.

Summary for $\oint_c \frac{du}{dn} ds$:

If

$$u \quad = \quad \sin \frac{p\pi x}{R} \sin q\pi Ry$$

then

$$\oint \partial u/\partial n ds \quad = \quad 0$$

if either or both p and q are even

$$\oint \partial u/\partial n ds \quad = \quad (p/qR^2 + qR^2/p)(1 - \cos p\pi)(\cos q\pi - 1)$$

$$= \quad -4(p/qR^2 + qR^2/p)$$

when p and q are odd.

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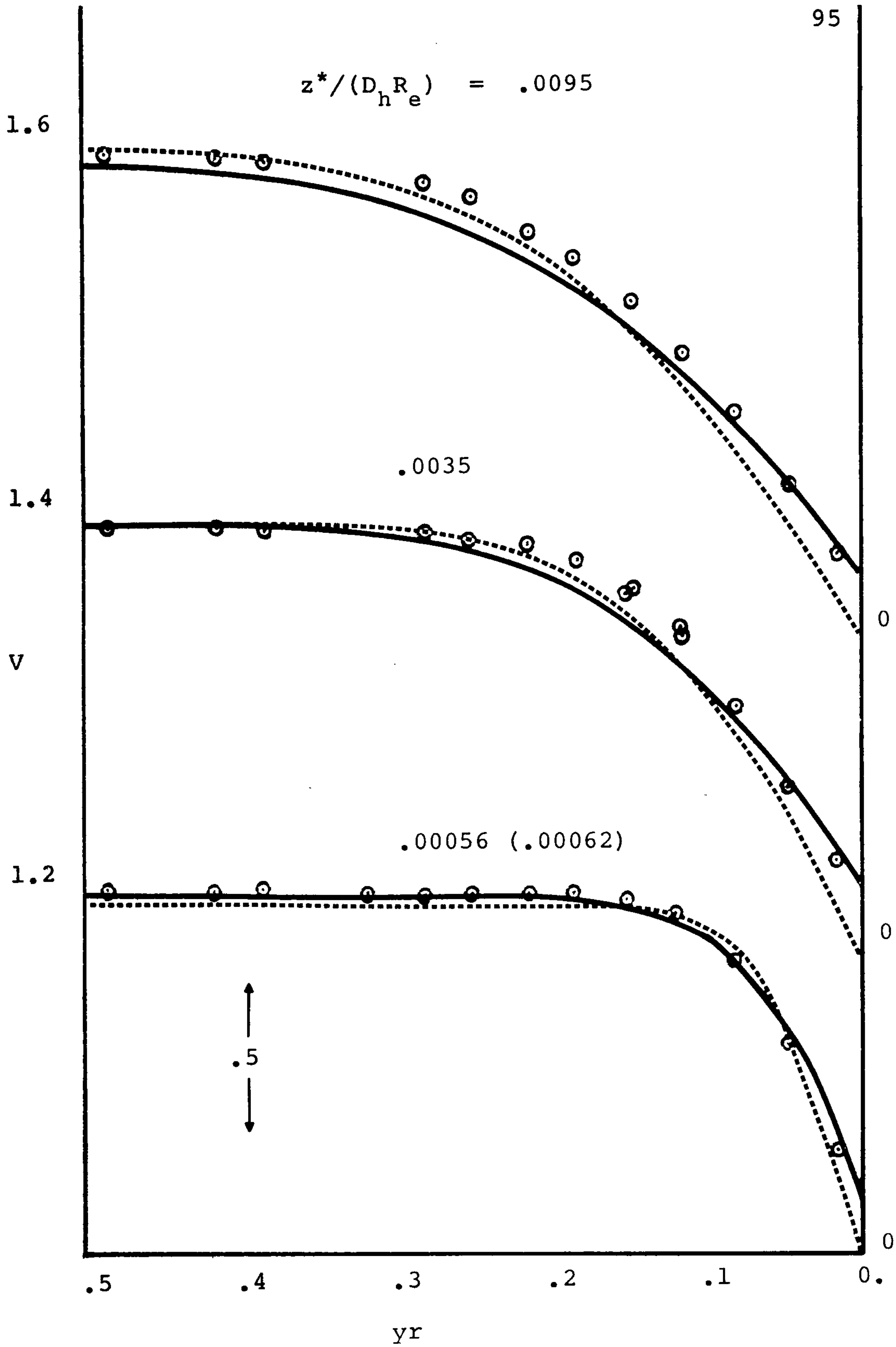

Figure 1. Velocity profiles along the y axis

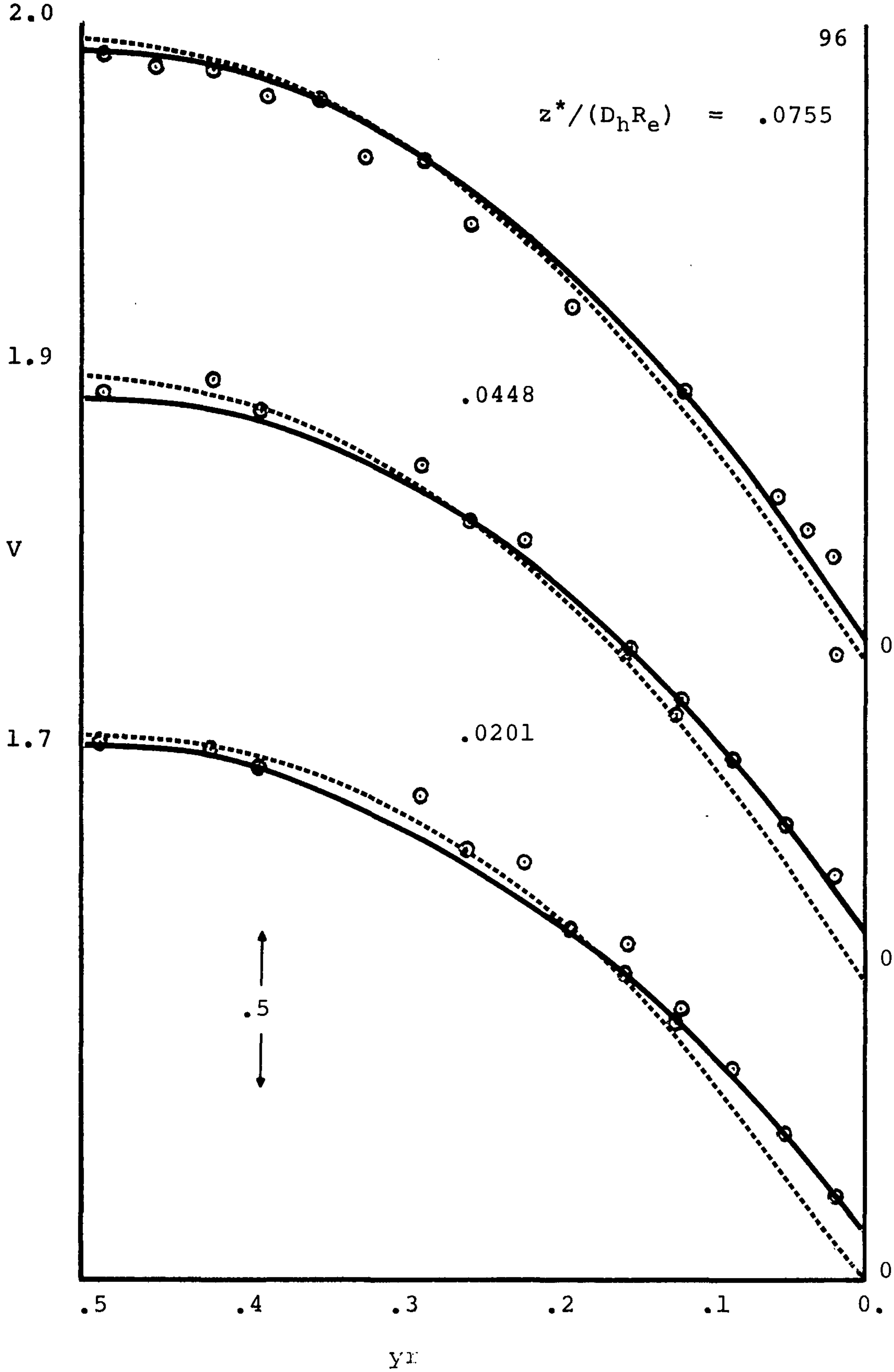

Figure 2. Velocity profiles along the y axis

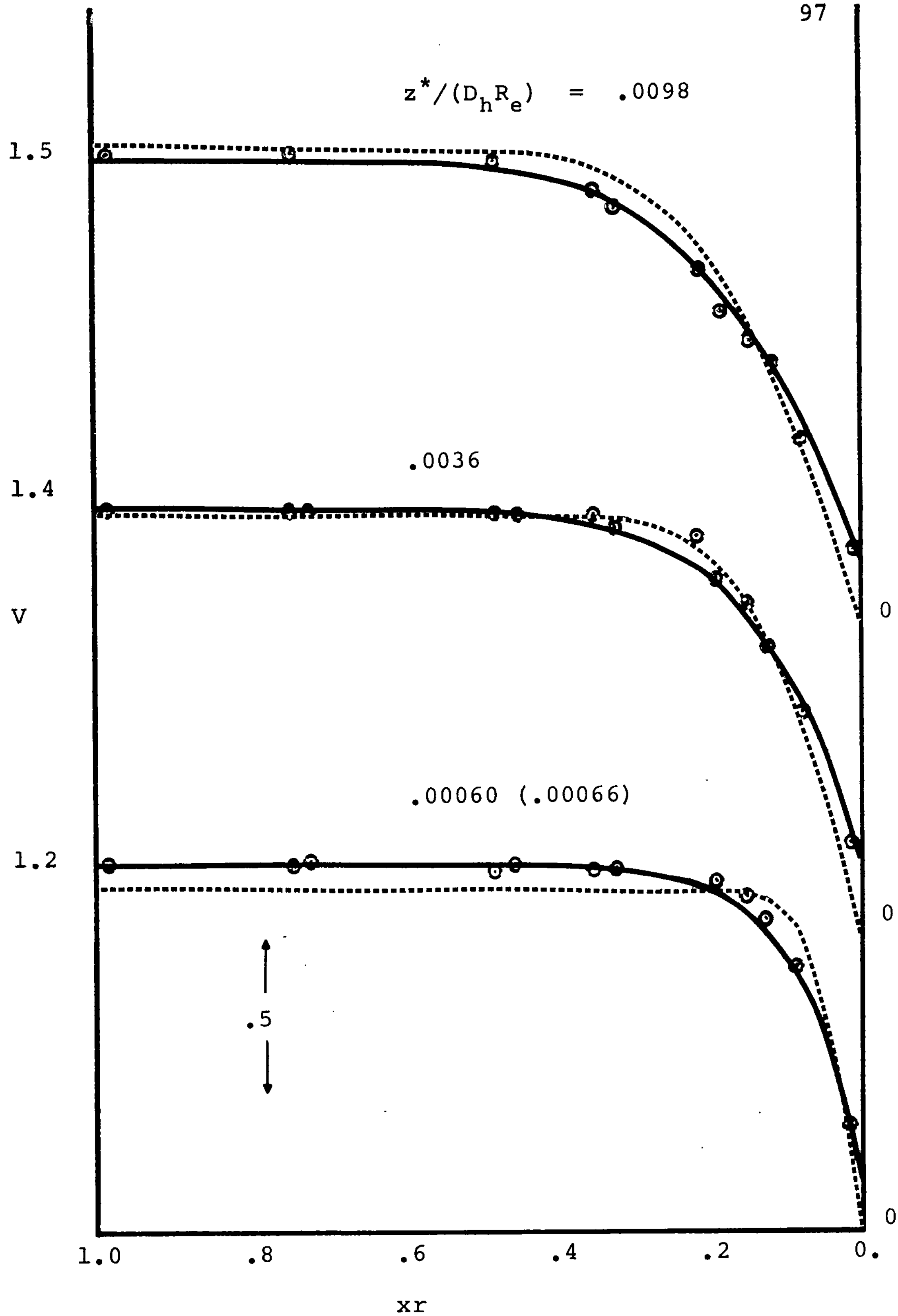

Figure 3. Velocity profiles along the x axis

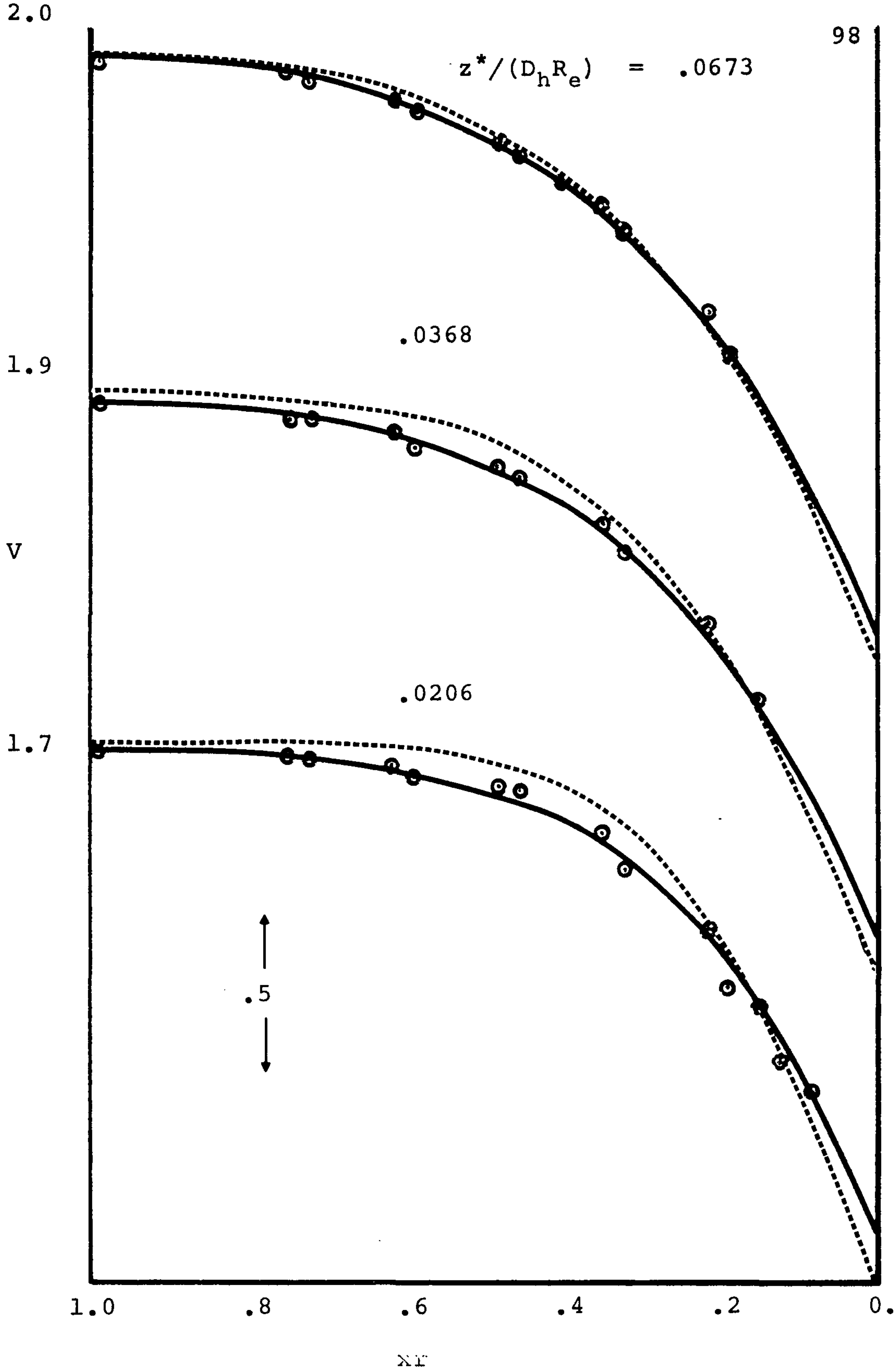

Figure 4. Velocity profiles along the x axis