A MATHEMATICAL ANALYSIS TO DETERMINE COMPONENT REQUIREMENTS AND PROBABLE ERROR OF AN ANALOG COMPUTER FOR SEISMIC RECORD EVALUATION

An Abstract of a Thesis

Presented to

the Faculty of the Department of Physics

University of Houston

In Partial Fulfillment

of the Requirements for the Degree

Master of Science

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by

Robert William Brennen

April 1955

ABSTRACT

The basic design of an analog computograph has been worked out to help evaluate reflection seismograms, by simultaneously calculating and plotting points on a seismic profile map, these points corresponding to the reflections detected on the seismic record. The computer is designed for the more common assumptions made in computing that subsurface wave velocities may be approximated by a linear increase of velocity with depth. This thesis has a description of the planned computer following a short section on seismograph theory.

In order to make possible an analysis to find component tolerances, it was necessary to establish a relationship between the equation variables and the machine variables. The maximum possible error in output due to an error in any one of the components was then calculated.

Equations have been derived in this thesis for quickly calculating the minimum requirements of a computer as described herein, and from these, an analysis has been made of the computer described. In event the general equations thus derived do not apply, a special analysis was made for the problem at hand. The computer was analyzed for (1) minimum dead zone of the position servos, (2) linearity and resolution of potentiometers, (3) conformity to function of function generators, (4) minimum

load resistance of potentiometers, (5) phase shifts, (6) changes of gain of amplifiers, and (7) temperature effects in resistances.

From the equations developed in the thesis, component requirements and the probable error of a plotted point were calculated for some given conditions. The final section includes some recommendations for improvements in the computer design. It is estimated that with following these recommendations, and using commercially available components, a probable error of as little as 0.02% of the maximum may be possible.

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CHAPTER I

THE PROBLEM AND DEFINITIONS OF TERMS USED

For many years, one of the drawbacks to accurate interpretation of reflection seismograms as used in geophysical work has been the large amount of time consumed by the tedious calculations of "point-plotting" methods. Charts[1], slide rules[2,3], and mechanical devices[4,5] have been developed to expedite calculations, but there is still need for a more rapid means of obtaining a profile map from the seismic record. In order to fill this need, an analog computograph has been designed to simultaneously calculate and plot points on a seismic profile map.

I THE PROBLEM

Statement of the Problem. It was the purpose of this study to determine the effect of errors in the various computer components upon the error of the solution of the equation involved, and the probable error of the computer as a whole, such that component tolerances in respect to linearity, temperature variations, etc., may be determined.

Importance of the Study. Before a computer of this type is constructed, specific knowledge should be obtained as to its feasibility, necessary component tolerances, and expected accuracy of the result. This study was made to help supply information from which component tolerances, expected accuracy, and feasibility, insofar as availability of components is concerned, may be determined.

II Definition of Terms

Reflection Seismic Prospecting. In the reflection seismic method of geophysical prospecting, energy is artificially introduced into the earth by controlled explosions. These initiate elastic waves, comparable to sound waves, the velocity of which is governed by the elastic constants of the earth materials. When such waves traveling downward into the earth strike interfaces at which these constants change, because of a change of material, the waves are at least partially reflected back toward the surface. The depth of a reflecting interface is determined from the time required for the reflected wave to make the trip down and back to detecting instruments near the source.

Shot. The term "shot" refers to the explosion used to create elastic waves in the ground.

Shot-point. The "shot-point" is the location of the explosion.

Detector. The detecting instrument, or "detector", designates the unit placed on or in the surface of the ground to detect the ground motion where it is located. The unit is also referred to as "receiver", "geophone", "seismometer", "geotector", and more colloquially as "jug".

Seismogram. The photographic paper record is called the "seismogram".

Travel Time. The time required for the wave to make the trip down to the reflecting surface, and back up to a particular detector is here called the "travel time". It is measured as a distance on a seismogram.

Point-plotting. "Point-plotting" is the method of determining a seismic profile by calculating and plotting individual depth points for the travel time of each reflection detected on the seismogram.

Analog Computer. An analog computer is a computer which deals with continuous physical variables; one physical system is used as a model for another system, more difficult to construct or measure, that obeys equations of the same form.[6,7] In these devices, physical quantities, such as voltages and shaft rotations, are made to obey mathematical relations comparable to those of the original problem. The physical quantities, or machine variables, which may be conveniently varied and measured in the laboratory, must then behave in a manner analogous to that of the original variables. The magnitudes and the scales of the analogous variables can be reduced or increased in order to facilitate their measurement.

Machine Variables. A computer cannot establish mathematical relations between the original equation variables. The machine variables are physical quantities, simulating the equation variables. The machine variable may be the fraction of the reference voltage of the computer, or it may be the fraction of the maximum rotation of a potentiometer. The relations between the problem variables and the machine variables are of the form:

$$X = a_x x,\ T = a_t t,\ \text{etc.},$$

where the capital letters are the machine variables, the lower case letters are the problem variables, and the a's are constant scale factors. The use of lower case letters shall be used exclusively for equation variables, except in the case of angles, where the angle in the machine coincides with the angle in the equation. The scale factor associated with a variable q is chosen by the rule:

$$a_q = \frac{1}{\text{maximum expected value of } |q|}.$$

III Organization of Remainder of the Thesis

After a brief review of seismograph theory and a statement of the depth equation used, in chapter two, chapter three describes the operation of the computer, and includes a listing of the limits of the variables, and certain relationships between computer components. A study of the effect of change of the equation variables and the machine variables is then made in chapter four. Chapter five is a study of the sources of error in the computer. These are studied, where possible, from a general point of view, and the results are then tabulated for the individual components. Chapter six makes a study of the probable error of the computer as a whole. In Chapter seven, the accuracy requirements of the computer are discussed, and component tolerances are given for a set of specified conditions. The final chapter summarizes the results and includes some recommendations for improvements.

CHAPTER II

REFLECTION SEISMOGRAPH THEORY

A brief review of seismograph theory as utilized in the computer will be presented here. For more complete information, the reader is referred to any of a number of texts on geophysical prospecting.[8,9,10]

The Method. When the ground is set in motion by energy from the explosion, the receivers produce electrical impulses which are all recorded on a single record. The record may be on photographic paper, or more recently, on magnetic tape. At the present, however, a photographic record, or seismogram, is normally produced for final analysis and interpretation.

Interpretation. The seismograph record may be reduced by various means to give a picture of the underlying structure. One of the more accurate, but also time consuming, methods is that of point-plotting, or calculating and plotting points for each travel time.

In calculating these points, a linear increase of velocity with depth is often assumed. If such an assumption may be made, the velocity, v, at any depth, z, may be written

$$v = f(z) = v_o + kz$$

where v_o = initial velocity

$k = dv/dz$, a constant.

For such a condition, it may be shown that the wave front at any particular time, t, is a circle with parameters[11]

$y = (v_o/k)(\cosh kt - 1)$

= the vertical depth below the shot-point of the center of the circle

$r = (v_o/k) \sinh kt$

= the radius of the circle.

For the case of a wave reflected from an underlying strata parallel to the surface, one half the travel time may be set into the above equations to find the parameters y and r at the instant the detected portion of the wave front was reflected. Hereafter, the symbols y and r will represent the parameters at the point of reflection. The depth of the reflecting layer is then

$$z = y + (r^2 - x^2)^{1/2}$$

$$= y + r \cos A$$

$$= h(\cosh kt - 1 + \sinh kt \cos A)$$

where x = 1/2 the distance between shot-point and detector

$A = \arcsin x/r$

= angle between r and vertical

$h = v_o/k$

t = 1/2 the travel time from shot-point to detector.

Time Correction. The travel time, or half travel time as used in the equation, must be corrected for differences in surface elevation and for the time of travel in the weathered layer.[12] This, in effect, reduces the shot-point and detectors to the same datum plane. Let

t_r = 1/2 the travel time as measured on the seismograph record

t_w = time correction for elevation and weathering. This may be positive or negative.

The corrected half travel time is then

$$t = t_r + t_w$$

Shot, Detector Layout. In most field procedures the detectors are laid out with equal spacing in a straight line along the earth's surface. The position of the shot may be along the same line, or it may be offset to one side. The shot-point, or its perpendicular projection on the line of detectors, may be beyond one end of the detector spread, or located anywhere within the spread.

If the detectors are numbered consecutively from one end of the spread (nearest the shot-point), and the detectors have equal spacings, 2s, between detectors, then the distance from the n^{th} detector to the number one detector is $2(n - 1)s$. Letting

$2u$ = the in-line (parallel) distance from the shot-point to the number one detector,

$2w$ = the off-line (perpendicular) distance from the shot-point to the line of detectors,

the half distance from the shot-point to the n^{th} detector may be solved from the right triangle with legs of length, w, and $[u + (n - 1)s]$.

For a horizontal reflecting surface, the reflection point as detected by the n^{th} receiver will be on the vertical line intersecting the midpoint of a straight line between the shot-point and the receiver. If a horizontal plane rectangular coordinate system is established with one axis lying on the line of detectors, and the other axis passing through the shot-point, then the coordinates of the projected reflection points as detected by the n^{th} receiver are w, and $[u + (n - 1)s]$, the same as the legs of the triangle to be solved for x. This indicates that the reflection points on a single bed as detected by the various detectors lie on a straight line and are at a distance $[u + (n - 1)s]$ from the projection of the shot-point on that line.

Summary of Variables. A restatement of the depth equation with a list and an explanation of the variables is here made for reference purposes.

z = depth of reflecting layer

$= y + r \cos A$

$= h(\cosh kt - 1 + \sinh kt \cos A)$

$= h\{(\cosh kt - 1) + [\sinh^2 kt - (x/h)^2]^{1/2}\}$

y = depth of center of wave front

$= h(\cosh kt - 1)$

r = radius of wave front

$= h \sinh kt$

t = 1/2 travel time corrected to same datum plane as shot-point

t_r = 1/2 travel time measured on seismogram

t_w = time correction for elevation and weathering

k = rate of increase of velocity with depth

v = velocity of the wave at any point

v_o = original velocity of the wave

$h = v_o/k$

x = 1/2 the distance from shot-point to receiver

A = angle between vertical and radius, r

$= \arcsin x/r$

u = 1/2 the in-line distance from shot-point to number one detector

w = 1/2 the off-line distance from the shot-point to number one detector

s = 1/2 the distance between consecutive detectors

n = the number of any detector as numbered from the end detector nearest the shot-point.

Another variable that will be made use of later is

u' = the in-line distance from the shot-point to the n^{th} detector

$= u + (n - 1)s.$

CHAPTER III

THE COMPUTER

The computer here described was devised to solve for and plot points on a profile map, these points being computed as a function of the travel time of the wave reflections. The computer is based upon the assumption that the sub-surface wave velocity conditions may be approximated by a linear increase of velocity with depth.

After a seismologist has analyzed the seismogram and indicated the reflection travel times to be computed, the record is rolled onto a drum similarly to the way in which a sheet of paper is rolled onto a typewriter. The zero time line on the record is set to coincide with the zero time line on the drum, and the record is then rolled in and stopped with the indicator over each travel time to be computed. A selector switch is set for the number of the trace to be computed, and when the plotting head comes to rest, a button is pressed to make a mark on the plotting paper. A gear box with a changeable ratio is used so as to accommodate seismograms with different time scales.

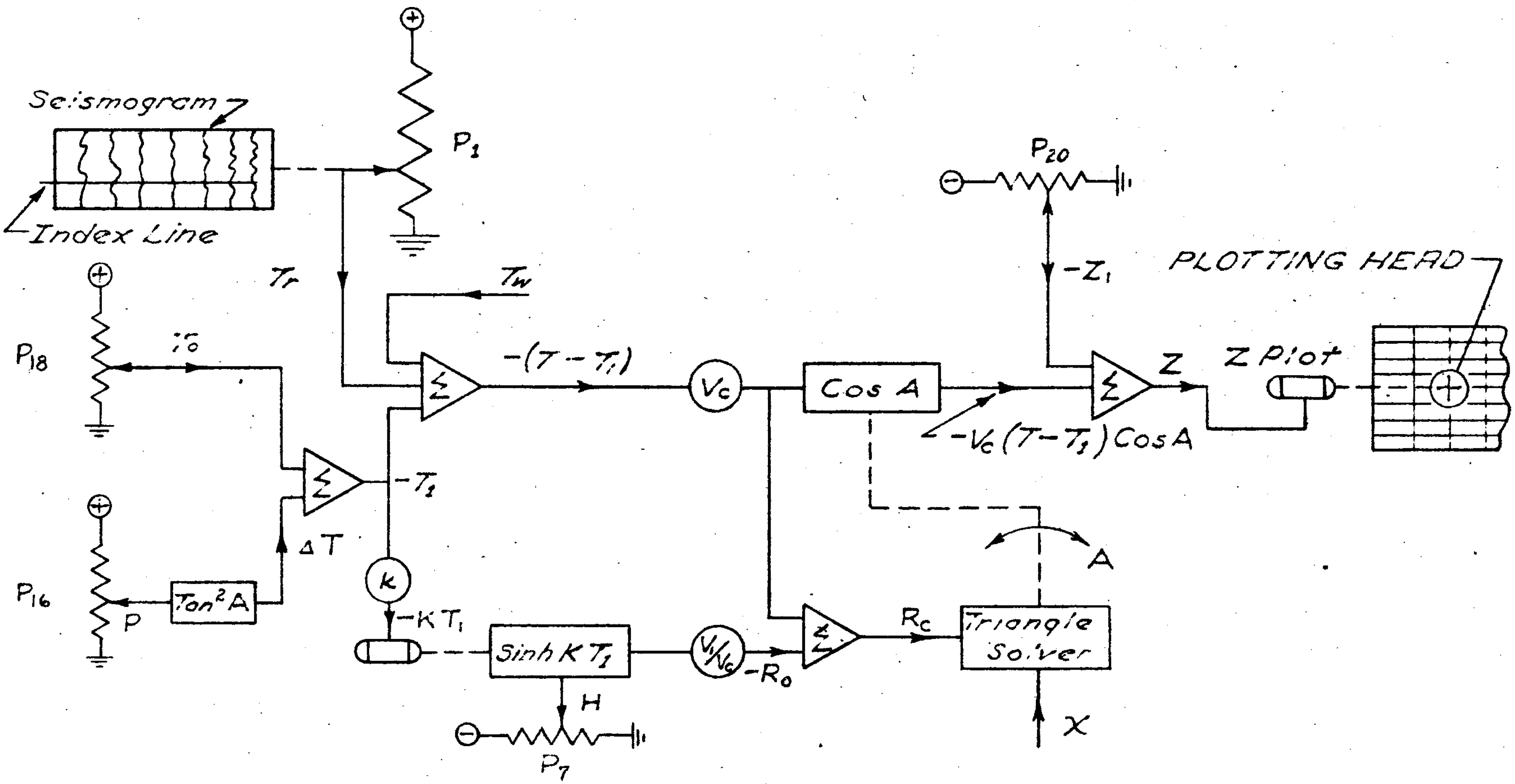
Mathematical Operations. In general, the operation of the computer is this: as a preliminary adjustment, the computer is adjusted for the parameters t_w, k, h, u, w, s, and n. The recorded time variable, t_r, is then entered as an input, and corrected for elevation and weathering conditions. The corrected time is multiplied by k, and the product, kt, is utilized as an input to function generators, generating the functions (cosh kt - 1) and sinh kt. The functions are multiplied by h. The function r = h sinh kt, and x are introduced into a triangle solver to produce the output $(r^2 - x^2)^{1/2} = r \cos A$. The two functions $y = h(\cosh kt - 1)$ and $r \cos A = h \sinh kt \cos A$ are then added to produce the desired depth, $z = y + r \cos A$, which appears on the output as a position of a plotting head, or a point on a profile map.

Description. With the mathematical operations to be performed having been related, a more detailed description of the method of performing these operations may now be given. One should strive to keep in mind the general relationship between the equation variables and the machine variables, and that there is a proportionality constant between the two. For example, a rotation proportional to 2t is also proportional to t, so that though the input to a potentiometer is a rotation proportional to 2t, the output machine variable may be labeled as T, analogous to t; the proportionality constant is twice what it would be for 2t.

In this computer, referring to Fig. 1, the input is a shaft rotation proportional to the recorded time, $2t_r$, but as a machine variable is T_r. This rotation is transferred to the shaft of a potentiometer, P1, with an output voltage T_r, where T_r is equal to a fraction of the reference voltage. A weathering correction voltage, T_w, from potentiometer P8 and switch S1a, is added to T_r by a summing amplifier. (It should be noted here, that the operational amplifiers used for summing change the sign of the voltages.) The output voltage, T, proportional to the corrected time, t, serves as a source voltage for another potentiometer, the fractional rotation of which is analogous to the equation variable k, thus multiplying voltage T by a factor K. The resulting voltage is fed into a servo which then produces a rotation proportional to kt. This shaft rotation adjusts the position of the wipers of two function potentiometers, the resistance of which increases as $[\cosh a(kt)_{max} - 1]$ and $\sinh a(kt)_{max}$, respectively, where a is the fractional rotation of the potentiometer.

Potentiometer P7 is adjusted to multiply the output functions by a factor H, proportional to h, with the resulting output voltages from the function potentiometers being Y and R, proportional to y and r.

The radius vector, r, must be resolved into the vertical component, r cos A. This may be done by feeding



NOTE:

$t = t_r + t_w$

$t_1 = t_o + \Delta t$

$\Delta t = p \tan^2 A$

$A = \sin^{-1} x/r_c$

$r_c = v_c(t - t_1) + \frac{v_1}{v_c}\, h \sinh k t_1$

Fig. 3

METHOD OF SOLVING FOR $z = z_1 + v_c(t - t_1)\cos A$

voltages R and X (the source of X is discussed later) into a triangle solver to obtain the output $R \cos A = (R^2 - X^2)^{1/2}$. One method of solution is by the use of a variable transformer, known as an A. C. resolver, and a servo unit to position the resolver rotor (see Fig. 2). The resolver has two sets of windings, one set free to rotate inside the other. If an alternating voltage, E, is applied to the primary, then voltages appearing across the two secondary windings will be proportional to E sin A and E cos A, where A is the angle between the primary and secondary windings.

Voltage R is fed into resolver RS1, and the secondary voltage R sin A is compared with the X voltage. The difference, or error signal, is amplified, and then drives a motor until a balance is reached, hence positioning the resolver rotor for the correct angle, $A = \arcsin x/r$. The voltage from the other secondary winding of RS1 is then proportional to r cos A, and is added to the voltage from the other function potentiometer to obtain a voltage $-Z = -(Y + R \cos A.)$ This summed voltage drives a position servo which in turn positions the plotting head at a distance z on the plotting scale being used.

The voltage representing the x distance is obtained through operation on three voltages. Switch S1b taps off a voltage proportional to $(n - 1)s$, where n is the number

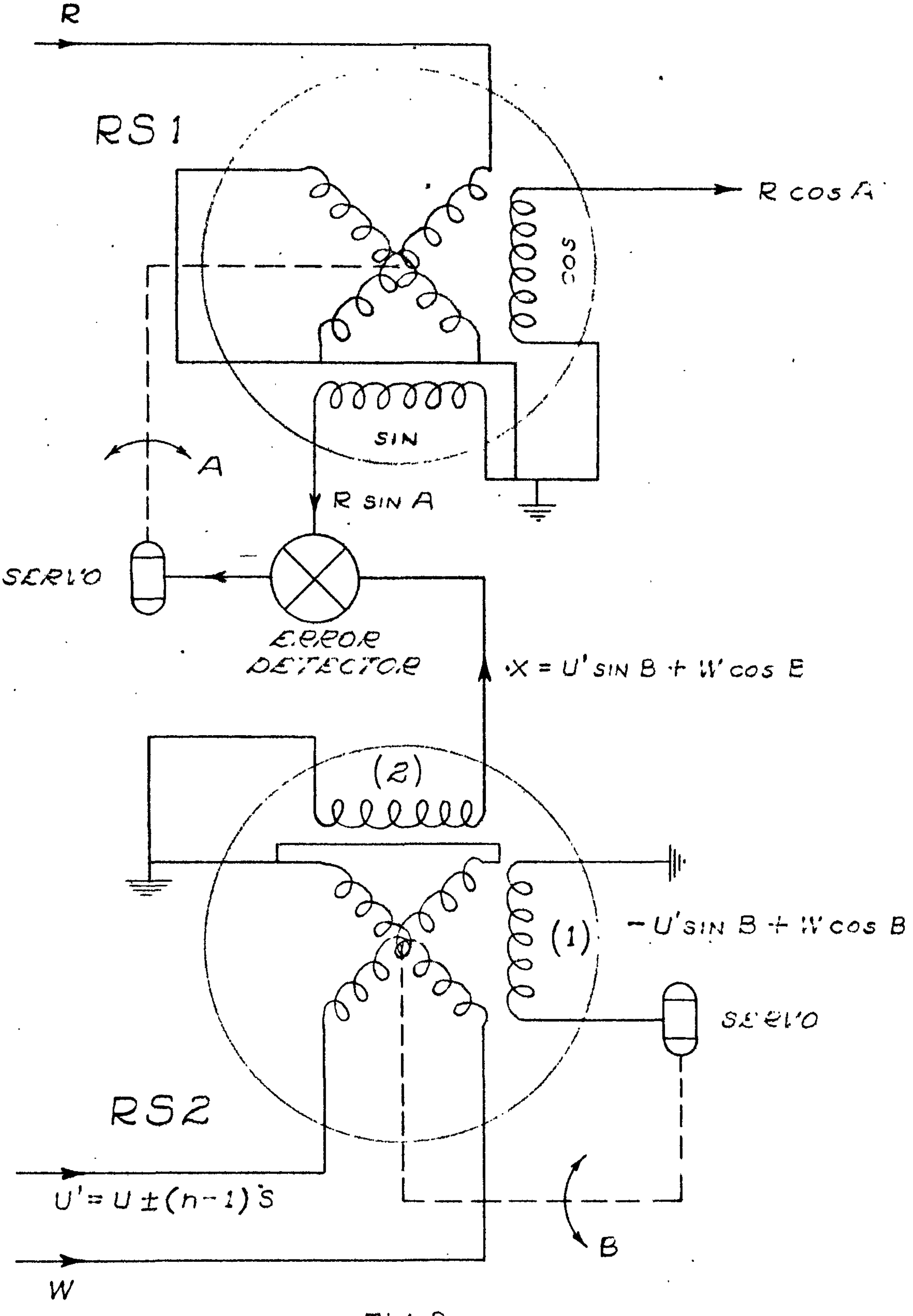


Fig. 2

METHOD OF TRIANGLE SOLUTION

of the trace being computed, and s is one half the distance between geophones. Potentiometer P9 is used to adjust the machine variable, S, for the correct value. Potentiometer P12 taps off a voltage proportional to u, one half the in-line distance from the shot-point to the number one detector; this voltage is added to the voltage from switch S1b to give a voltage $[U + (n - 1)S]$, proportional to the total in-line distance from the shot-point to the receiver. A voltage proportional to w, the off-line distance of the shot-point to the detectors, is obtained from potentiometer P10.

The quantities W and $[U + (n - 1)S]$ represent the legs of a right triangle, from which the hypotenuse, X, is desired. Letting $U' = [U + (n - 1)S]$, if U' and W are in-phase voltages connected to the primary coils of a resolver, then the outputs on the secondary coils are:[13]

(1) $-U' \sin B + W \cos B$

(2) $U' \cos B + W \sin B$.

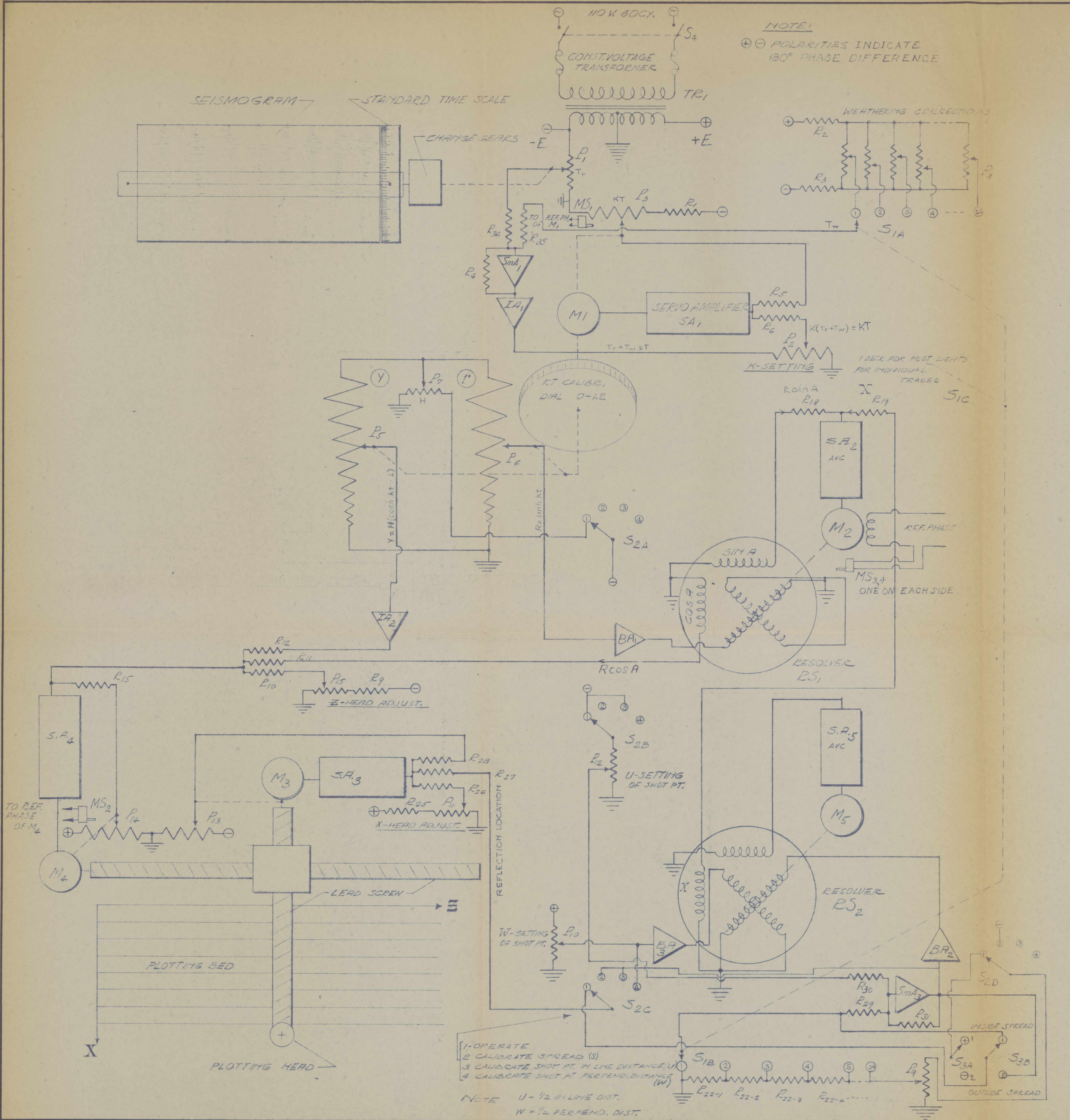
The voltage on coil (1) is used as the error signal for the servo amplifier, so that $-U' \sin B + W \cos B = 0$, or $\tan B = W/U'$. The output on the other coil is then $U' \cos B + W \sin B = X$, the desired quantity.

For plotting the horizontal distances (x plot) the reflection points must be located with reference to the shot-point, or in case of off-line shooting, with reference to the perpendicular projection of the shot-point on the line of detectors. If the shot-point projection lies outside the receiver layout (beyond one of the two end detectors), the shot-point projection is located at the edge of the paper, and the reflection points are then positioned progressively across the paper. The servo driving the plotting head is positioned by using the voltage [U + (n - 1)S] as an input. If the projection of the shot-point is inside the receiver layout (between the two end detectors), the number one reflection is located near the edge of the paper, and the shot-point is then located at the proper distance toward the center. In this case, the voltage (n - 1)S, from switch S1b, is used for positioning the plotting head to locate the reflection points. In order to locate the position of the shot-point projection, the calibrating switch (S2 in Fig. 3) is turned to the "CALIBRATE SHOT-POINT IN-LINE DISTANCE" position; the U voltage then positions the servo, and the location of the plotting head may be observed. After the position of the shot-point is established, the switch is returned to the "OPERATE" position, and the plotted points will assume the proper location with respect to this projection.

<u>General</u>. The full diagram of the computer is shown in Fig. 3. Potentiometers P11 and P15 apply small correcting voltages to position the plotting head over the

Fig. 3

DIAGRAM OF ANALOG COMPUTOGRAPH
FOR SEISMIC RECORD EVALUATION



PART	DESCRIPTION	No. REQ.	MATERIAL	NOTES
AMPLIFIERS - SERVO				
SA1		1		
SA2	WITH AVC	1		
SA3		1		
SA4		1		
SA5	WITH AVC	1		
AMPLIFIERS - SUMMING				
SmA1		1		
SmA3		1		
AMPLIFIERS - ISOLATION				
IA1		1		
IA2		1		
AMPLIFIERS - BOOSTER				
BA1		1		
BA2		1		
BA3		1		
SERVO MOTORS DIEHL FPE 25-11				
M1-M5		1		
POTENTIOMETERS				
P1		1		
P2		1		
P3		1		
P4		1		
P5	FUNCTIONAL 50K; ±0.1% DEVIAT.	1	cosh x-1	
P6	FUNCTIONAL 50K; ±0.1% DEVIATION	1	sinh x	
P7		1		
P8		1		
P9		1		
P10		1		
P11		1		
P12		1		
P13		1		
P14		1		
P15		1		
MICROSWITCHES				
MS1		1		
MS2		1		
MS3		1		
MS4				
TR1	TRANSFORMER 110V 60CY PRI.	1	SEC. CONSTANT VOLTAGE	
SWITCHES				
S1	24 POSIT. - 3 CIRCUITS - SHORTING - INSTR. TYPE (3 DECKS)	1		
S2	4 POSIT. - 4 CIRCUITS	1		
S3	DPDT TOGGLE	1		
S4	DPST	1		
RS1	RESOLVER DIEHL FJE 43-11	1		
RS2	RESOLVER DIEHL FJE 43-11	1		
RESISTORS				
R1		1		
R2		1		
R3		1		
R4		1		
R5		1		
R6		1		
R9		1		
R10		1		
R11		1		
R12		1		
R15		1		
R18		1		
R19		1		
R22		23		
R25		1		
R26		1		
R27		1		
R28		1		
R29		1		
R30		1		
R31		1		

SCHEMATIC DIAGRAM

DESIGN: RB; GM; LJ
DRAWN gl
CHECKED
DATE 9 JUNE 1954

Fig 3
ANALOG COMPUTOGRAPH
FOR SEISMIC RECORD EVALUATION
X-133

REVISION

SCALE:

RUSKA INSTRUMENT CORPORATION
HOUSTON, TEXAS

SHEET

correct x or z reference line on the plotting paper. These potentiometers should normally be adjusted above ground to allow some leeway in adjustment, and to prevent potentiometers P13 and P14 from driving against the mechanical stops.

On servo units in which voltages of a wrong phase may be applied, such that a balance cannot be reached, limiting switches are used to prevent the motors from running away, or damaging the potentiometer stops. These switches are arranged to open the reference phases of the two phase motors when the potentiometer rotors move too far. Servos 3 and 5 are arranged to prevent improper phasing, therefore have no limit switches.

A constant voltage transformer is used as a voltage source. Since the end of all voltages is to drive a servo unit, and since all voltages stem from the same supply, the computer is not critical to small voltage fluctuations. The constant voltage transformer is used to prevent changes of sensitivity of the servo units that take place with large voltage variations.

The lead screws shown on the diagram indicate schematically a means of positioning the plotting head, but are not necessarily the means used.

Method of Setting Machine Variables. The time variables, t_r and t_w, are the only ones that are set into the machine through accurately calibrated dials. All other controls may be approximately calibrated to facilitate setting, but the accurate settings of parameters should be done by utilizing the plotting head movement as a check. The values of the machine variables H and K may be adjusted by cross checking between the output at two different depths as previously calculated. Thus, in effect, it becomes a process of empirically solving for two unknowns from two equations. The values of u, w, and s may be calibrated by switching S2 to the proper position, and observing the motion of the plotting head in the x direction.

Limits of Variables. In order that certain resistance ratios, and amplifier gains may be determined, and that errors in the computer may be calculated, it is necessary that limits be set on the equation variables. These limits have been set at approximately the maximum values reached in practice. The set values may be found in Table I.

Relations of Components. In order to fulfill the requirements set forth by the limits on the variables, certain resistance ratios, and the gain of isolation and booster amplifiers must be established. These values, within practical limits, may be a matter of choice and expediency, for there is no unique set of values which

TABLE I

LIMITS OF EQUATION VARIABLES

Equation Variable	Minimum	Maximum	Units
z	0	20,000	ft
t	0	2.4	sec
t_r	0	2.4	sec
t_w	-0.1	0.1	sec
k	0.5	3.0	1/sec
kt	0	1.2	-
v_o	5500	9500	ft/sec
v	5500	20,000	ft/sec
h	3000	18,000	ft
x	0	3000	ft
u	0	1500	ft
w	0	1500	ft
s	0	150	ft
n	0	24	-
y	0	7.1×10^3	ft
r	0	14.7×10^3	ft
A	0	30	degrees

will fulfill the requirements. The relations chosen are given below.

Comparative Resistance Values:

P8 = 2R2 = 2R3

R4 = R35 = R36

R1 = 9P3

R5 = 2R6

R11 = R12 = 0.74 R15 = 0.74 R10

R18 = 0.2 R19

R26 = R27 = R28

R29 = R31 = 0.5 R30

R9 = 9 P15

R25 = 9 P11

Gain of Amplifiers. The gain of all isolation and booster amplifiers is 1.0, except for BA3 and IA2, which have gains of 0.5 and 0.54 respectively.

Miscellaneous. Trimmer potentiometers should be provided on the grounded end of R35 and R19 in order to allow an initial adjustment of the correct relationships between T_r and T_w, and between X and R cos A. Also, the gain of IA2 should have a screwdriver adjustment to obtain the proper summing of Y and R cos A. Elsewhere, small errors in original adjustments will be compensated in calibration.

The functional relationship on P5 and P6 is effective over 300° of rotation; P3 is directly ganged to P5 and P6, so that but 5/6 of the 360 degree potentiometer is used.

CHAPTER IV

EFFECT OF ERRORS IN THE VARIABLES

The effect of errors in the equation variables and machine variables is studied in the following section. For each of the variables, an equation is derived for the error in output due to an error in the variable. From this, and the limits that have been set for the equation variables, the values of the equation variables have been determined for the maximum error in output for an error in the given variable. The values of the maximum partial derivatives with respect to the equation variables and the machine variables, and values of the partials for a special problem are listed in Table II at the end of the chapter.

Output Error--General. Let the equation to be solved be expressed as

$$z = f(q_1, q_2, q_3, \ldots)$$

If there is an error dq_1 in each of the variables, the error in z is approximately

$$dz = \frac{\partial z}{\partial q_1} dq_1 + \frac{\partial z}{\partial q_2} dq_2 + \ldots$$

In order to find the error in the machine, one must find the partial derivative of each of the equation variables, and the effect of the machine components upon the variables.

Effect of Machine Components on the Variables. The machine variables are related to the equation variables by the relation

$$Q_i = \frac{1}{(q_i)_{max}} q_i$$

so that $$dq_i = (q_i)_{max} \, dQ_i$$

and $$dz = (q_1)_{max} \frac{\partial z}{\partial q_1} dQ_1 + (q_2)_{max} \frac{\partial z}{\partial q_2} dQ_2 + \dots$$

The output error due to any one machine variable is

$$dz = (q_i)_{max} \frac{\partial z}{\partial q_i} dQ_i$$

$$= \frac{\partial z}{\partial Q_i} dQ_i$$

Note that $$\frac{\partial z}{\partial Q_i} = (q_i)_{max} \frac{\partial z}{\partial q_i}.$$

This will be used in the next section.

In this computer, if the machine variable Q is considered as the fraction of the maximum output of a component, then $Q_{max} = 1$. Therefore, dQ is the fractional error of the component. Since components are normally classified according to the maximum percent error, then it remains to find the partial derivatives and their maximum values, in order to find the maximum error due to any one component.

THE PARTIAL DERIVATIVES

Equation Variable t. Writing the depth equation in the form

$$z = h\left\{(\cosh kt - 1) + \left[\sinh^2 kt - \left(\frac{x}{h}\right)^2\right]^{1/2}\right\}$$

and taking the partial with respect to t,

$$\frac{\partial z}{\partial t} = kh\left\{\sinh kt + \frac{\sinh kt \cosh kt}{\left[\sinh^2 kt - \left(\frac{x}{h}\right)^2\right]^{1/2}}\right\}$$

Since $$kh = v_0$$

and $$\left[\sinh^2 kt - \left(\frac{x}{h}\right)^2\right]^{1/2} = \sinh kt \cos A$$

then $$\frac{\partial z}{\partial t} = v_0\left(\sinh kt + \frac{\cosh kt}{\cos A}\right)$$

Referring to the maximum values on the limits, in Table I

$$\frac{\partial z}{\partial T} = 2.4\, v_0\left(\sinh kt + \frac{\cosh kt}{\cos A}\right)$$

$$\frac{\partial z}{\partial T_r} = 2.4\, v_0\left(\sinh kt + \frac{\cosh kt}{\cos A}\right)$$

$$\frac{\partial z}{\partial T_w} = 0.1\, v_0\left(\sinh kt + \frac{\cosh kt}{\cos A}\right)$$

To set up conditions for a maximum value of this partial, it is necessary that

$v_0 \rightarrow$ max. = 9500 ft/sec

A $\rightarrow$ max. = 30°

x $\rightarrow$ max. = 3000 ft

$\therefore$ r = x/sin A = 6000 ft

In order to let the term (cosh kt/cos A) have greater effect, let kt become a maximum for the given conditions, then h will become a corresponding minimum, since h $\propto$ 1/k.

Conditions must also be such that

$$v \doteq 20{,}000 \text{ ft/sec}$$

or, for $$v_0 = 9500 \text{ ft/sec}$$

then $$\frac{dv}{dz} z = kz = v - v_0 = 10{,}500 \text{ ft/sec}$$

so that $$k \doteq \frac{10{,}500}{z} \frac{1}{\text{sec}}$$

Also, $$k = \frac{v_0}{h} = \frac{9500}{h}$$

so that $$z \doteq \frac{10{,}500}{9500} h = 1.1 h$$

But $$z = h f(kt)$$

where $$f(kt) = \cosh kt - 1 + \sinh kt \cos A$$

Letting $$f(kt) = 1.1$$

from a table of hyperbolic functions it may be determined that

$$\sinh kt = 0.87$$

$$\cosh kt = 1.33$$

and $$kt = 0.79$$

Then $$h = \frac{r}{\sinh kt} = 6.9 \times 10^3 \text{ ft}$$

$$z = 7.6 \times 10^3 \text{ ft}$$

$$k = \frac{v_0}{h} = 1.4 \frac{1}{\text{sec}}$$

$$t = 0.56 \text{ sec}$$

thus showing that all parameters are within tolerance.

The maximum value of the partial of z with respect to the equation variable is

$$\left(\frac{\partial z}{\partial t}\right)_{max} = 2.3 \times 10^4 \text{ ft/sec}$$

<u>Equation</u> <u>Variable</u> <u>k</u>. In a similar manner to that used for t, it may be shown that

$$\frac{\partial z}{\partial k} = ht\left(\sinh kt + \frac{\cosh kt}{\cos A}\right)$$

and
$$\frac{\partial z}{\partial K} = 3ht\left(\sinh kt + \frac{\cosh kt}{\cos A}\right)$$

To obtain a maximum of this partial,

$h \longrightarrow$ max. = 18,000 ft

$x \longrightarrow$ max. = 3,000 ft

$t \longrightarrow$ max. allowable with other specifications

$\therefore$ $k \longrightarrow$ min. = 0.5 sec^{-1}

$z \longrightarrow$ max. = 20,000 ft

Now
$$f(kt) = \frac{z}{h} = 1.11$$

For the general case of

$$f(kt) = f = (\cosh kt - 1) + \left[\sinh^2 kt - \left(\frac{x}{h}\right)^2\right]^{1/2}$$

$$(\cosh kt - 1 - f)^2 = \sinh^2 kt - \left(\frac{x}{h}\right)^2$$

and since
$$\cosh^2 kt - \sinh^2 kt = 1$$

solving for cosh kt gives

$$\cosh kt = \frac{(f+1)^2 + 1 + (x/h)^2}{2(f+1)}$$

For the specific case of f(kt) = 1.11, and x/h = 1/6,

$$\cosh kt = 1.300$$

and from a table of hyperbolic functions, kt = 0.756, and sinh kt = 0.831. Now

$$t = \frac{(kt)}{k} = 1.51 \text{ sec}$$

$$A = \sin^{-1}\left(\frac{x}{h \sinh kt}\right) = 11.6^{\circ}$$

$$v_0 = kh = 9000 \text{ ft/sec}$$

$$v = v_0 + kz = 19{,}000 \text{ ft/sec}$$

thus completing the list.

Therefore,

$$\left(\frac{\partial z}{\partial k}\right)_{max} = 5.9 \times 10^{4} \text{ ft-sec}$$

Equation Variable kt. In a similar manner as used previously,

$$\frac{\partial z}{\partial (kt)} = h\left(\sinh kt + \frac{\cosh kt}{\cos A}\right)$$

and

$$\frac{\partial z}{\partial (KT)} = 1.2\, h\left(\sinh kt + \frac{\cosh kt}{\cos A}\right)$$

For a maximum, the same conditions apply as for k, so that

$$\left(\frac{\partial z}{\partial (kt)}\right)_{max} = 3.9 \times 10^{4} \text{ ft}$$

Equation Variable h. It may also be shown that

$$\frac{\partial z}{\partial h} = \cosh kt - 1 + \frac{\sinh kt}{\cos A}$$

and

$$\frac{\partial z}{\partial H} = 18{,}000\left(\cosh kt - 1 + \frac{\sinh kt}{\cos A}\right)$$

To obtain a maximum, then

$$kt \longrightarrow \text{max.} = 1.2$$

$$x \longrightarrow \text{max.} = 3000 \text{ ft}$$

$$A \longrightarrow \text{max.} = 30^\circ$$

Checking other parameters gives

$$r = \frac{x}{\sin A} = 6000 \text{ ft}$$

$$h = \frac{r}{\sinh kt} = 3.97 \times 10^3 \text{ ft}$$

$$z = 8.42 \times 10^3 \text{ ft}$$

Also $$V = V_o + kz \doteq 20,000 \text{ ft/sec}$$

$$\frac{V_o}{k} + z = h + z \doteq \frac{20,000}{k} \text{ ft/sec}$$

so that $$k \doteq 1.62$$

thus satisfying all requirements for the variables.

Therefore $$\left(\frac{\partial z}{\partial h}\right)_{max} = 2.6$$

<u>Equation</u> <u>Variable</u> <u>x</u>. Taking the partial of z with respect to x gives

$$\frac{\partial z}{\partial x} = \frac{-x}{(h^2 \sinh^2 kt - x^2)^{1/2}}$$

$$= \frac{-x}{r \cos A}$$

$$= \frac{-r \sin A}{r \cos A}$$

$$= -\tan A$$

The partial with respect to X is

$$\frac{\partial z}{\partial X} = -3000 \tan A$$

For the limits specified, the partial has a maximum at $A = 30°$,

$$\left(\frac{\partial z}{\partial x}\right)_{max} = 0.58$$

Equation Variable (n - 1)s. Considering (n - 1)s as a single variable,

$$\frac{\partial z}{\partial[(n-1)s]} = \frac{\partial z}{\partial x}\,\frac{\partial x}{\partial[(n-1)s]}(n-1)ds$$

Now
$$x = \{w^2 + [u + (n-1)s]^2\}^{1/2}$$

$$\frac{\partial x}{\partial[(n-1)s]} = \frac{u + (n-1)s}{x}$$

$\therefore$
$$\frac{\partial z}{\partial[(n-1)s]} = -\frac{u + (n-1)s}{x}\tan A$$

and
$$\frac{\partial z}{\partial[(n-1)S]} = -3000\,\frac{u + (n-1)s}{x}\tan A$$

For $w = 0$,
$$\frac{\partial z}{\partial[(n-1)s]} = -\tan A$$

with a maximum in the computer at $A = 30°$, the same as for x.

Equation Variable u. Similarly,

$$\frac{\partial z}{\partial u} = -\frac{u + (n-1)s}{x}\tan A$$

and
$$\frac{\partial z}{\partial U} = -1500\,\frac{u + (n-1)s}{x}\tan A$$

with a maximum at $w = 0$, and $A = 30°$ of

$$\left(\frac{\partial z}{\partial u}\right)_{max} = -\tan A_{max} = -0.58$$

Equation Variable w. Likewise,

$$\frac{\partial x}{\partial w} = \frac{w}{x}$$

so that
$$\frac{\partial z}{\partial w} = -\frac{w}{x}\tan A$$

$$\frac{\partial z}{\partial W} = -1500\frac{w}{x}\tan A$$

This has a maximum at (n - 1)s = u = 0, and A = 30°,
$$\left(\frac{\partial z}{\partial w}\right)_{max} = -\tan A = -0.58$$

<u>Equation</u> <u>Variable</u> <u>z</u>. Clearly,
$$\frac{\partial z}{\partial z} = 1$$

then
$$\frac{\partial z}{\partial Z} = 2.0 \times 10^4$$

for all values of z.

<u>Equation</u> <u>Function</u> <u>cos</u> <u>A</u>. Considering (cos A) as a variable, and (Cos A) as the machine variable,
$$\frac{\partial z}{\partial(\cos A)} = h \sinh ht$$
$$= r$$
$$\frac{\partial z}{\partial(\mathrm{Cos}\, A)} = r$$

The maximum value of r has been listed as 1.5×10^4 ft.

<u>Equation</u> <u>Function</u> <u>sin</u> <u>A</u>. It has been shown previously, that
$$x = r \sin A$$

and
$$\frac{\partial z}{\partial x} = -\tan A$$

Now
$$\frac{\partial z}{\partial(\sin A)} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial(\sin A)}$$
$$= -r\tan A$$

and
$$\frac{\partial z}{\partial(\sin A)} = -0.5\, r \tan A$$

To obtain a maximum, let A = 30°, with a corresponding maximum of r = x/sin A = 6000 ft. Then

$$\left[\frac{\partial z}{\partial(\sin A)}\right]_{max} = 3.5 \times 10^3$$

<u>Equation Function (cosh kt - 1)</u>. Considering (cosh kt - 1) as a single variable,

$$\frac{\partial z}{\partial(\cosh kt - 1)} = h$$

with a maximum at h = 18,000 ft. Designating the machine variable as M(cosh kt - 1), then

$$\frac{\partial z}{\partial M(\cosh kt - 1)} = 0.81\, h$$

<u>Equation Function sinh kt</u>. Also,

$$\frac{\partial z}{\partial(\sinh kt)} = h \cos A$$

The partial with respect to the machine variable is

$$\frac{\partial z}{\partial M(\sinh kt)} = 1.5\, h \cos A$$

This has a maximum value at h = 18,000 ft, and A = 0.

TABLE II

THE PARTIAL DERIVATIVES WITH RESPECT TO THE VARIABLES

Equation Variable, q	$(\partial z/\partial q)_2^{\div}$	$(\partial z/\partial Q)_2^{\div}$ (ft)	$(\partial z/\partial q)_{max}$	$(\partial z/\partial Q)_{max}$ (ft)
t, t_r	2.0×10^4	4.9×10^4	2.3×10^4	5.5×10^4
t_w	2.0×10^4	2.0×10^3	2.3×10^4	2.3×10^3
k	5.1×10^4	1.5×10^5	5.9×10^4	1.8×10^5
kt	2.9×10^4	3.5×10^4	3.9×10^4	4.7×10^4
h	2.3	4.4×10^4	2.6	4.6×10^4
x	-0.15	-4.6×10^2	-0.58	-1.7×10^3
(n - 1)s	-0.15	-4.6×10^2	-0.58	-1.7×10^3
u	-0.15	-2.3×10^2	-0.58	-8.7×10^2
w	-	-	-0.58	-8.7×10^2
cos A	1.3×10^4	1.3×10^4	1.5×10^4	1.5×10^4
sin A	2.0×10^3	1.0×10^3	3.5×10^3	1.7×10^3
(cosh kt - 1)	8.8×10^3	7.1×10^3	1.8×10^4	1.5×10^4
sinh kt	8.7×10^3	1.3×10^4	1.8×10^4	2.7×10^4
r	0.99	1.3×10^4	1.0	1.5×10^4
y	1.0	7.1×10^3	1.0	7.1×10^3
z	1.0	2.0×10^4	1.0	2.0×10^4

÷ These values are computed for variables specified in Appendix 1.

CHAPTER V

ERRORS IN THE COMPUTER, AND COMPONENT TOLERANCES

With a knowledge of the operation of the computer, and a knowledge of the effects of errors in the machine variables, the effects of various sources of error may be calculated.

In general, the possible errors in the designed computer may be listed as:

(1) Errors in the position servos, including gearing;
(2) Errors due to non-linearity and resolution of linear elements;
(3) Errors due to lack of conformance to function of non-linear elements;
(4) Errors due to loading of electrical components;
(5) Errors due to phase shifts;
(6) Errors due to changes of gain of amplifiers;
(7) Errors due to temperature effects;
(8) Errors due to calibration manually.

This paper will be concerned chiefly with the first seven of the above listed sources of error, for except for the resolution of adjusting potentiometers, it is assumed that the error due to manual calibration is limited only by the reading accuracy of the output.

I Servo Errors

In as much as the inherent error, or dead zone, of the servo units is dependent on the design, it is here assumed that the obtainable accuracy of position of a servo is limited only by the resolution of the voltage actuating element which it controls.

The servo units will be considered from the standpoint of finding the minimum dead zone in terms of angular rotation or in terms of the fraction of the total rotation of the driven element.

It has been pointed out that the error in z caused by a fractional error, dQ, in the voltage from the controlled component is

$$dz = q_{max} \frac{\partial z}{\partial q} dQ$$

$$= \frac{\partial z}{\partial Q} dQ$$

Therefore, in order to limit the output error to a small value dz_0, the servo must position the driven component such that the ratio of the error in the output voltage from the component to its maximum value must be

$$dQ \leq \frac{dz_0}{\left(\frac{\partial z}{\partial Q}\right)_{max}}$$

The above equation is sufficient for servos driving linear elements. However, on servos driving the resolvers, the value of dQ is a non-linear function of the angle; these will be considered separately.

For the linear elements, letting

$\theta =$ the angle of rotation,

the allowable error in angle is

$$d\theta \leq \frac{\theta_{max}\, dz_o}{\left(\frac{\partial z}{\partial Q}\right)_{max}}$$

or, if the component affects the output in the x direction, to limit the error to a value dx_o for any one component,

$$d\theta \leq \frac{\theta_{max}\, dx_o}{\left(\frac{\partial x}{\partial Q}\right)_{max}}$$

For example, servo number 3 must be considered from two standpoints: first, it determines the precision of plotting in the x direction, and second, it helps to determine the accuracy of calibration of the x sources.

<u>Servo</u> <u>Number</u> <u>2</u>. The first of the servos to be considered driving a non-linear element is servo number 2, driving resolver RS1 with an output machine variable

$$Q = R\cos A = \frac{1}{r_{max}}\, r\cos A$$

For an error in angle A,

$$dQ = -\frac{r\sin A}{r_{max}}\, dA$$

$$= -\frac{x}{r_{max}}\, dA$$

so that for $$|dQ| \leq \frac{dz_o}{r_{max}\left(\frac{\partial z}{\partial r}\right)_{max}}$$

then $$dA \leq \frac{dz_o}{x_{max}\left(\frac{\partial z}{\partial r}\right)_{max}} \text{ radians}$$

or $$dA \leq \frac{180}{\pi}\,\frac{dz_o}{x_{max}\left(\frac{\partial z}{\partial r}\right)_{max}} \text{ degrees}$$

Servo Number 5. The output of resolver RS2, driven by servo number 5, is

$$X = U' \cos B + W \sin B$$

If an error dB appears in angle B, the error is

$$dX = (U' \sin B + W \cos B)\, dB$$

The term in parentheses is equal to the voltage on the other coil of the resolver and is set equal to zero, so that the error becomes small.

Turning to another method of approach, and letting

$$dX = X(B + dB) - X(B)$$

then $$dX = U' \cos (B + dB) + W \sin (B + dB)$$
$$-(U' \cos B + W \sin B)$$

$$dX = U'[\cos (B + dB) - \cos B] + W[\sin (B + dB) - \sin B]$$

But $$\cos (B + dB) = \cos B \cos dB - \sin B \sin dB$$
$$\approx \cos B \cos dB$$

and $$\sin (B + dB) = \sin B \cos dB + \cos B \sin dB$$
$$\approx \sin B \cos dB$$

$$\therefore \quad dX = U' \cos B (\cos dB - 1) + W \sin B (\cos dB - 1)$$

$$dX = (\cos dB - 1)(U' \cos B + W \sin B)$$
$$= (\cos dB - 1) X$$

Since $$X_{max} = 1$$

$$dX_{max} = (\cos dB - 1)\frac{(\cos dB + 1)}{(\cos dB + 1)}$$
$$= \frac{\cos^2 dB - 1}{\cos dB + 1}$$
$$= \frac{\sin^2 dB}{\cos dB + 1}$$
$$\approx \frac{\sin^2 dB}{2}$$

for dB small, so that very nearly,

$$dB = \sin^{-1} (2\ dX_{max})^{1/2}$$

In order to limit the output error to a value dz_o, then

$$dX_{max} \leq \frac{dz_o}{\left(\frac{\partial z}{\partial X}\right)_{max}}$$

$$\therefore \qquad dB \leq \frac{180}{\pi} \left[\frac{2\ dz_o}{\left(\frac{\partial z}{\partial X}\right)_{max}} \right]^{1/2} \text{ degrees}$$

<u>Results</u>. Calculations have been made for the results regarding the angular error for the servos, and have been collected into Table III. This table shows the maximum angular error, or dead zone, per unit error in z. The allowable dead zone of the servo, for an output error dz_o, is then, in terms of the output angle,

$$d\theta \leq dz_o \text{ x tolerance per foot error in z.}$$

TABLE III

REQUIREMENTS FOR ANGLE OF DEAD ZONE OF SERVO DRIVEN COMPONENTS

Servo	Equation Variable	Tolerance per Foot Error in x	Tolerance per Foot error in z
1	kt	-	$2.1 \times 10^{-5}\ \theta_{max}$
2	r cos A	-	1.9×10^{-2} degrees
3	x	$3.3 \times 10^{-4}\ \theta_{max}$	$5.8 \times 10^{-4}\ \theta_{max}$
4	z	-	$5.0 \times 10^{-5}\ \theta_{max}$
5	x	-	$2.1/(dz_o)^{1/2}$ degrees

II & III Linearity, Resolution, and Conformity to Function

The general requirements for linearity, resolution, and conformity to function are now discussed. Following a description of a general method of finding the requirements of components, and special discussions where necessary, the output error due to errors in the components, and the maximum allowable errors in the components per unit error in output, z, are shown in Tables IV, V and VI.

General Requirements. The requirements of a component in the computer depend upon the analogous variable which it generates, and the manner in which it is used. Except for functional components, potentiometers driven by servo units and potentiometers requiring an accurately calibrated scale must, of necessity, be linear. On the other hand, potentiometers which are adjusted by hand by observation of the output, need not have any high degree of linearity, but the resolution† must be small enough to allow accurate adjustment of the machine variables.

Function generators must accurately duplicate the analogous variables in order to limit errors on the output depth scale. Since these units are normally quoted

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† Resolution of a potentiometer is defined as the smallest fractional change of resistance as the contact moves along the resistance coil. It is equal to 1/total no. turns of wire.

as to the percent deviation of the function from the maximum value, the requirements may then be computed on the same basis as for linear potentiometers.

Linearity. If a component is generating a machine variable Q, representing equation variable q, and a nonlinearity dQ occurs, then the error in output depth, measured in units of depth, is

$$dz = \frac{\partial z}{\partial Q}\, dQ$$

$$= q_{max} \frac{\partial z}{\partial q}\, dQ .$$

Values of $\partial z/\partial Q$ and $(\partial z/\partial Q)_{max}$ are given in chapter IV. Limiting the error due to each component to a value dz_o, then

$$dQ \leq \frac{dz_o}{\left(\frac{\partial z}{\partial Q}\right)_{max}}$$

The value of the tolerance on a component is

$$\text{percent tolerance per foot error in } z = \frac{100}{\left(\frac{\partial z}{\partial Q}\right)_{max}} \%.$$

Linear and functional component tolerances are listed in Tables IV and V.

Resolution. On adjusting potentiometers dependent only on resolution, if the smallest resistance step corresponds to a change of output 2 dz_o, then it is possible to set the potentiometer to a value which will produce an output within a value dz_o of its desired value.

The resolution must then be

$$dQ \leq \frac{2\,dz_0}{\left(\frac{\partial z}{\partial Q}\right)_{max}}$$

and percent tolerance per foot error in $z = \dfrac{(2)(100)\%}{\left(\frac{\partial z}{\partial Q}\right)_{max}}$

Resolution tolerances are listed in Table VI.

Resistors R22. Resistors R22, which in a sense form a step function generator, and are listed with the function generators in Table V, are here considered in order to determine the necessary precision of the resistors.

Clearly, for an absolute limit of the error to some value, then each resistor must have the same precision as required for the function generator as a whole. However, the resistances of low value will tend to compensate for those of high value so that the probable error is decreased considerably. Consider the equivalent series network in Fig. 4, where

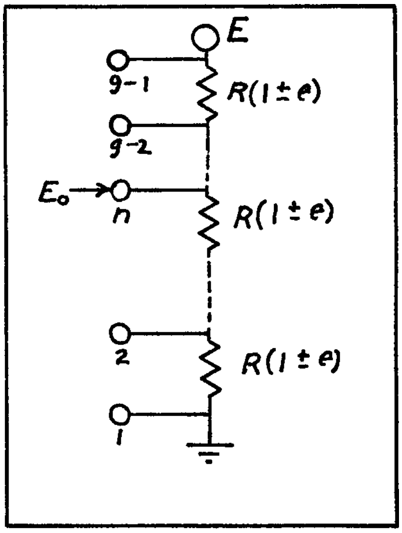

Fig. 4
EQUIVALENT OF
RESISTORS R22

g = the total number of series resistors

$m = n - 1$ = the number of resistors across which the output is measured.

e = the probable fractional error of the resistors

E = the source voltage

E_o = output voltage across m resistors

i = current through the resistors.

The ratio of the output voltage to the source voltage is

$$\frac{E_o}{E} = \frac{iR\,(m \pm \sqrt{m}\,e)}{E}$$

The current is

$$i = \frac{E}{R(g \pm \sqrt{g}\,e)}$$

so that

$$\frac{E_o}{E} = \frac{ER\,(m \pm \sqrt{m}\,e)}{ER\,(g \pm \sqrt{g}\,e)}$$

$$= \frac{m \pm \sqrt{m}\,e}{g \pm \sqrt{g}\,e}$$

$$= \frac{m\left(1 \pm \frac{e}{\sqrt{m}}\right)}{g\left(1 \pm \frac{e}{\sqrt{g}}\right)}$$

Now, if $e \ll 1$, then to a close approximation,

$$\frac{E_o}{E} = \frac{m}{g}\left(1 \pm \frac{e}{\sqrt{m}}\right)\left(1 \mp \frac{e}{\sqrt{g}}\right)$$

Ignoring errors of the second order,

$$\frac{E_o}{E} \approx \frac{m}{g}\left(1 \pm \frac{e}{\sqrt{m}} \mp \frac{e}{\sqrt{g}}\right)$$

$$= \frac{m}{g}\left[1 \pm e\left(\frac{1}{\sqrt{m}} - \frac{1}{\sqrt{g}}\right)\right]$$

$$= \frac{m}{g}\left(1 \pm e\sqrt{\frac{g-m}{gm}}\right)$$

$$= \frac{m}{g} \pm \frac{m}{g}\,e\sqrt{\frac{g-m}{gm}}$$

$$\frac{E_o}{E} = \frac{m}{g} \pm e\sqrt{\frac{m(g-m)}{g^3}}$$

The probable error of E_o with respect to E is then

$$p.e. = \pm \frac{e}{g^{3/2}}\sqrt{m(g-m)}$$

Taking the derivative of this and setting it equal to zero gives

$$\frac{d(p.e.)}{dm} = \pm \frac{e}{2g^{3/2}}\left(\frac{g-2m}{\sqrt{m(g-m)}}\right)$$

$$\therefore \qquad m = \frac{g}{2}$$

This value of m corresponds to a maximum value of the probable error; setting in this value gives

$$(p.e.)_{max} = \pm \frac{e}{2\sqrt{g}}$$

For the case in question, g = 23, so that

$$(p.e.)_{max} = \pm 0.1e$$

This error in terms of the machine variables is

$$p.e. = d[(n-1)S]$$

The maximum probable error in depth then becomes

$$dz = \frac{\partial z}{\partial[(n-1)S]}\, d[(n-1)S]$$

$$= \pm 1.7 \times 10^2\, e$$

The required probable error of the resistors to limit the probable error in depth to a value dz_o is

$$e \leq 5.9 \times 10^{-3}\, dz_o$$

Following the practice of considering the probable error of a component as 1/3 the limiting error,[14] the precision tolerance is 3e per foot error in z, or

$$\text{Tolerance per Foot Error in } z = 1.8\%$$

TABLE IV

LINEARITY REQUIREMENTS FOR POTENTIOMETERS

Component	Equation Variable	Per Cent Tolerance per Foot Error in x	Per Cent Tolerance per Foot Error in z
P1	t_r	-	0.0018
P3 +	kt	-	0.0021
P8	t_w	-	0.043
P13	x	0.033	0.058
P14	z	-	0.050

\+ Note: The tolerance for P3 must be multiplied by 5/6 for linearity computations, since but 300° of 360° rotation is used.

TABLE V

REQUIREMENTS FOR FUNCTIONAL COMPONENTS

Component	Equation Variable	Per Cent Tolerance per Foot Error in x	Per Cent Tolerance per Foot Error in z
P5	(cosh kt - 1)	-	0.0068
P6	sinh kt	-	0.0037
RS1	cos A	-	0.0067
RS2	x	-	0.058
R22 +	(n - 1)s	0.59	1.8

\+ These figures are based on probable error of output from the resistors.

TABLE VI

RESOLUTION REQUIREMENTS FOR POTENTIOMETERS

Component	Equation Variable	Per Cent Tolerance per Foot Error in x	Per Cent Tolerance per Foot Error in z
P2	k	-	0.0011
P7	h	-	0.0043
P9	(n - 1)s	0.067	0.12
P10	w	-	0.23
P11	x/10	0.67	-
P12	u	0.13	0.23
P15	z/10	-	0.10

IV Loading of Components

The loading error of the linear potentiometers has been approached from the standpoint of finding a minimum value for the load resistor.-|- The potentiometers that must be considered are those dependent on linearity, plus the function generators not isolated by amplifiers. Resistors R22 may be considered as a potentiometer of resistance 23 R22.

Simple Potentiometer. The deviation, D, due to loading of a potentiometer here refers to the fraction DE of the total voltage impressed across the potentiometer by which the output voltage is in error due to loading effects, or

$$D = a - \frac{E_o}{E}$$

This is given by Korn and Korn as [15]

$$D = -a^2(1-a)\frac{P}{R_L}$$

for $P/R_L \ll 1$

where E_o = output voltage at the wiper

E = voltage impressed across the potentiometer

a = fractional resistance of the potentiometer across which E_o is measured.

-|- The load resistor is the resistor paralleling the portion of the potentiometer between the wiper and ground; it has the effect of reducing the voltage at the wiper, by reducing the effective resistance to ground.

P = potentiometer resistance

R_L = load resistance.

Taking the partial with respect to a, and setting this equal to zero,

$$\frac{\partial D}{\partial a} = -a(3a-2)\frac{P}{R_L} = 0$$

or

$$a = \frac{2}{3}$$

for a maximum deviation.

Therefore

$$D_{max} = -\frac{4}{27}\frac{P}{R_L}$$

But $D = dQ$, the error in the machine variable. Therefore the maximum error in output due to loading of a potentiometer generating the machine variable Q is

$$dz = \left(\frac{\partial z}{\partial Q}\right)_{max} D_{max}$$

$$= \left(\frac{\partial z}{\partial Q}\right)_{max}\left(\frac{-4}{27}\right)\frac{P}{R_L}$$

so that, if the output error is to be limited to a value dz_o, then

$$R_L \geqq \frac{4}{27}\left(\frac{\partial z}{\partial Q}\right)_{max}\frac{P}{dz_o}$$

<u>Potentiometer with Resistor in Series</u>. The loading effects on potentiometer P3 may be found by study of the fractional part of the loading curve for a potentiometer of resistance(P3 + R1). Since P3 uses but 5/6 of the full rotation, and the resistance of P3 is 1/10 of the total resistance, the maximum value of a is then 1/12. Let

D_m = deviation of pot (P3 + R1) at the maximum of a = 1/12

D = deviation of pot (P3 + R1) at a.

The deviation from linearity of the 300° of P3 is approximately twelve times the difference between D and $12aD_m$; this may be seen more readily by referring to Fig. 5. The multiplier of twelve is necessary since D is the fraction of the total.

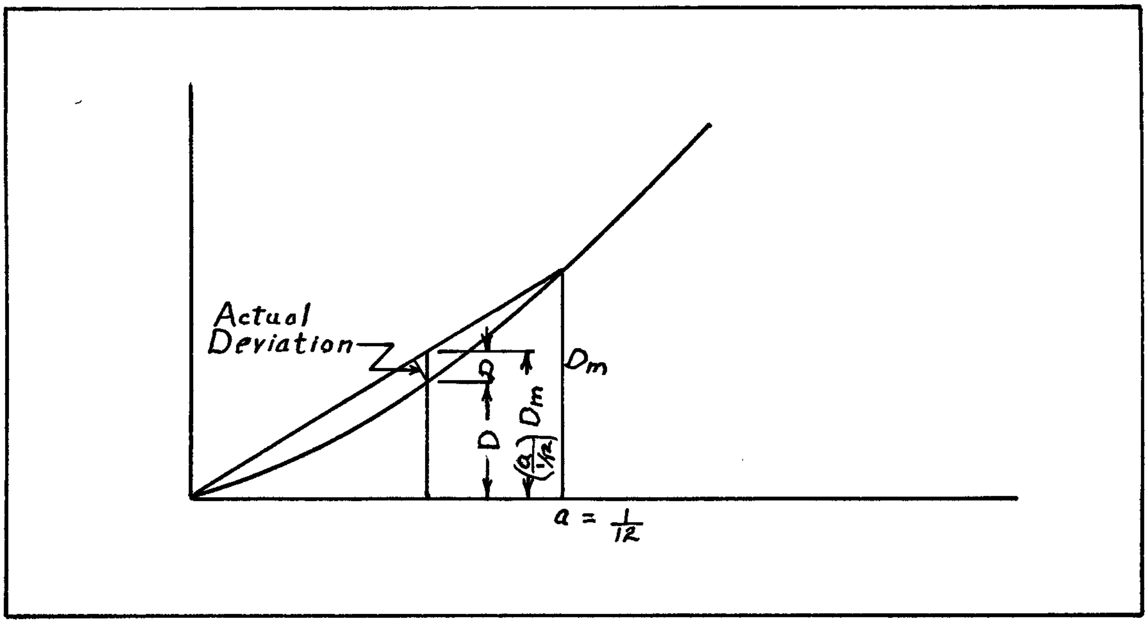

Fig. 5
LOADING OF POTENTIOMETER P3

The deviation of P3 is then

$$\mathcal{D} = 12(12\,a\,D_m - D)$$

Letting

$$p = \frac{P_3 + R_1}{R_L}$$

the deviation at $a_{max} = 1/12$ is

$$D_m = a^2(1-a)p = \frac{11}{(12)^3} p$$

so that
$$\mathcal{D} = 12\, p\, a \left[\frac{11}{(12)^2} - a(1-a)\right]$$

To find the position of the maximum, set the partial equal to zero.

$$\frac{\partial \mathcal{D}}{\partial a} = p\left(36a^2 - 24a + \frac{11}{12}\right) = 0$$

so that

$$a = \frac{24 \pm (576 - 132)^{1/2}}{72}$$

Since $a \leqq 1/12$, inspection indicates that

$$a = \frac{24 - (444)^{1/2}}{72}$$

$$= .040$$

$$\therefore \qquad \mathcal{D}_{max} = 0.018\, p = 0.018 \frac{(P_3 + R_1)}{R_L}$$

And since $R1 = 9\ P3$, and $R_L = R5 + R6 = 1.5\ R5$ (assuming no grid resistor other than the summing resistors),

then $\mathcal{D}_{max} = 0.12\ P3/R5$.

If $\mathcal{D}_{max}$ is held to a value such that the resulting output error is dz_o, then

$$\mathcal{D}_{max} = d(KT) \leqq \frac{dz_o}{\left(\frac{dz}{\partial(KT)}\right)_{max}}$$

and
$$R_5 \geq 0.12 \frac{P_3}{dz_o} \left(\frac{dz}{\partial(KT)}\right)_{max}$$

<u>Potentiometers P8</u>. An equation to find the maximum deviation of potentiometers P8 becomes extremely involved. However, since the load resistor is governed by the requirements of P1, and the portion of the voltage tapped from P8 is small, then by making this branch of the same order of resistance as P1 (e.g., R2 + R3 = P1), an ample safety factor may be obtained. An additional help is that the partial of z with respect to the machine variable is small in comparison to that of P1.

<u>Load Resistor Requirements</u>. Using the above equations, Table VII has been prepared showing the required value of the load resistor for each potentiometer as a function of the minimum error to be produced by loading of the potentiometer. The summing resistors for the potentiometer output have also been included, assuming that the amplifier input grids have no connection to ground other than the summing resistors.

TABLE VII

REQUIREMENTS FOR LOAD RESISTORS OF LINEAR POTENTIOMETERS

Potentiometer	Minimum Load Resistor	Summing Resistor
P1	$(8.2 \times 10^3/dz_o)$P1	R36 $\geq (5.5 \times 10^3/dz_o)$P1
P3	$(8.4 \times 10^3/dz_o)$P3	R5 $\geq (5.6 \times 10^3/dz_o)$P3
P8+	$(8 \times 10^3/dz_o)$P8	R35 $\geq (5 \times 10^3/dz_o)$P8
P14	$(3.0 \times 10^3/dz_o)$P14	R15 $\geq (2.3 \times 10^3/dz_o)$P14
P13	$(2.6 \times 10^2/dz_o)$P13	R28 $\geq (1.7 \times 10^2/dz_o)$P13
	$(4.4 \times 10^2/dx_o)$P13	R28 $\geq (3.0 \times 10^2/dx_o)$P13
23 R22	$(5.9 \times 10^3/dz_o)$R22	R29 $\geq (3.5 \times 10^3/dz_o)$R22
	$(1.0 \times 10^4/dz_o)$R22	R29 $\geq (6.1 \times 10^3/dx_o)$R22

+ See discussion above.

V Phase Shifts

Phase shifts will be considered from a general point of view as to the effects in the computer. The places in which trouble may occur may be listed as:

(1) Phase shifts between inputs of two-phase motors

(2) Phase shift of input to servo amplifier

(3) Phase shift of voltages into summing networks

(4) Phase difference between inputs into resolver RS2.

Motor. If there is a phase shift of the control phase of the motor with respect to the reference phase, the result will be a reduction of torque, since only the component 90° from the reference phase is effective in producing torque on the rotor. Thus, a small phase shift at the motor can do no more than effect a slight change of gain, which would not be detected because the servo feedback reduces dependency on gain.

Servo Amplifier. Consider two voltages, E and E_2, into a servo amplifier, where E is the input voltage, and E_2 is the controlled voltage. Let E be of the correct driving phase of the two phase motor, and E_2 be at an angle of $(\pi - C)$ with respect to E. For the motor to drive E_2 until a balance is reached, then the component of E_2 180° out of phase with E must equal -E, or

$$E = -E_2 \cos C$$

The fractional error in the driven component is

$$\frac{dE}{E} = \frac{E - E_2}{E} = \frac{1}{\cos C} - 1$$

$$= \frac{1 - \cos C}{\cos C}$$

$$= \left(\frac{1 - \cos C}{\cos C}\right)\left(\frac{1 + \cos C}{1 + \cos C}\right)$$

$$= \frac{1 - \cos^2 C}{\cos C(1 + \cos C)}$$

$$= \frac{\sin^2 C}{\cos C(1 + \cos C)}$$

$$\approx \frac{1}{2} \sin^2 C$$

for C small.

If the voltages into the servo amplifier have a constant small angle C, then the resulting error may be calibrated out in the original calibration of the computer. However, if one of the input voltages varies in phase, for instance due to resolver phase shifts, then a nonlinearity results in the servo controlled output. The effect of an input of varying phase is discussed in the following section.

Summing Networks. In general, harmful phase shifts into the summing networks are unlikely in this computer because of the extremely small inductive reactances in the resistors at 60 c.p.s., and because the shift would be reasonably constant. However, voltages out of the resolvers present more difficulty, because of inherent phase shifts in resolvers.

In this computer the voltages may be considered from the standpoint of driving components for the servo motors, since all voltages eventually drive a servo. If any one of the source voltages for a summing network should be out of phase, then effectively the voltage is multiplied by the cosine of the phase angle. Thus, a fixed phase angle between inputs to the summing network may be calibrated out. However, a fixed phase difference should not be allowed to become too great, for it has the effect of producing noise in the servo amplifier.

If the phase angle changes by an angle C from the desired phase, then the error is the same as was indicated in the servo amplifier,

$$\frac{dE}{E} = \frac{1}{2} \sin^2 C$$

If a summing network has an output corresponding to an equation variable q, the fractional change of q is

$$\frac{dq}{q_{max}} = \frac{dE}{E_{max}}$$

$$= \frac{1}{2} \sin^2 C$$

so that

$$dq = \frac{1}{2} q_{max} \sin^2 C$$

The error in the output of the computer is then

$$dz = \frac{\partial z}{\partial q} dq$$

$$dz = \frac{1}{2} q_{max} \frac{\partial z}{\partial q} \sin^2 C$$

$$= \frac{1}{2} \frac{\partial z}{\partial Q} \sin^2 C$$

If it is desired that the output error be limited to a value dz_o, then, of necessity,

$$C \leq \sin^{-1}\left[\frac{2\, dz_o}{\left(\frac{\partial z}{\partial Q}\right)_{max}}\right]^{1/2}$$

Since C is small, then the sine is very nearly equal to the angle, so that

$$C \leq \left[\frac{2\, dz_o}{\left(\frac{\partial z}{\partial Q}\right)_{max}}\right]^{1/2} \text{ radians}$$

$$\leq \frac{180}{\pi}\left[\frac{2\, dz_o}{\left(\frac{\partial z}{\partial Q}\right)_{max}}\right]^{1/2} \text{ degrees}$$

Resolver RS2. If the two input voltages to a resolver differ in phase, then effectively, one of the input magnitudes is multiplied by the cosine of the phase difference.[16] Since the inputs to RS2 are calibrated separately before entering the resolver, an error can result from a phase difference in the inputs. The output is proportional to

$$x = u' \cos B + w \sin B$$

Now is W is at an angle C with respect to U', then

$$x = u' \cos B + w \sin B \cos C$$

$$\Delta x = w \sin B \, (1 - \cos C)$$

$$\approx \frac{1}{2} w \sin B \sin^2 C$$

$$\Delta x \approx \frac{1}{2} w \sin B \sin^2 C$$

Since
$$dx \leq \frac{dz_o}{\left(\frac{dz}{dx}\right)_{max}}$$

letting B = 90° and solving for C gives

$$C \leq \sin^{-1}\left[\frac{2\,dz_o}{\left(\frac{\partial z}{\partial x}\right)_{max} w_{max}}\right]^{1/2}$$

$$\leq \sin^{-1}\left[4.8 \times 10^{-2} (dz_o)^{1/2}\right]$$

Thus an error from a phase difference between the two inputs would be very unlikely, since an error of 2° would give an error of less that 1 ft. in z.

Phase Shift---Results. Assuming that there can be no harmful phase shifts except in the resolvers, the resolvers are the only components for which calculations have been made. Table VIII shows the maximum allowable variation of phase angle of output for the resolvers.

TABLE VIII

LIMITS ON CHANGE OF PHASE OF RESOLVERS

Resolver	Equation Variable	Maximum change of phase, degrees
RS1	r cos A	$0.21(dz_o)^{1/2}$
RS2	x	$1.9(dz_o)^{1/2}$

VI Changes of Gain of Amplifiers

It is a well known fact that electronic amplifiers are subject to changes of amplification which may be caused by any number of things, such as variations of electrode voltages, temperature, aging of tubes, etc. These changes of gain can affect the precision of a computer so that the required accuracy must be considered in the design or in the purchase of such units.

Servo Amplifiers. Reduction of gain of the servo amplifiers will result in decreased sensitivity of response, and an increased dead zone of the servo-controlled component. With good amplifier design, good voltage regulation, quality tubes, and a suitable warm-up period before operation is begun, little trouble should be experienced with change of servo amplifier gain.

Isolation and Booster Amplifiers. The isolation and booster amplifiers are feed-back amplifiers with a gain of approximately 1.0 or less, to prevent excessive current loading of the potentiometers or other voltage sources. They should be of a design such that the gain is constant to within the same accuracy that is required of the source potentiometers as given in Tables IV and V. Many different designs are in use, so methods of stabilization will not be considered.

Summing Amplifiers. Chiefly, it is not necessary that the summing amplifiers have an extremely accurate multiplying factor of any particular value for the added voltages, but it is important that this factor should remain constant within limits.

Amplifier SmA3 differs a little in this respect, for in operating with the shot-point inside the spread, the voltage (n - 1)S is calibrated beyond the amplifier, but the plotting voltage is taken prior to entering the amplifier. Thus, to prevent error in the x plot for this particular case, then the gain of the amplifier must be equal to 1.0. This may be accomplished by a trim-pot, or rheostat, on the ground side of R29.

As shown in Fig. 6, a summing amplifier consists of an input summing network;✝ a high gain amplifier, and a feedback resistor providing an error signal at the input grid of the amplifier. The output is[17]

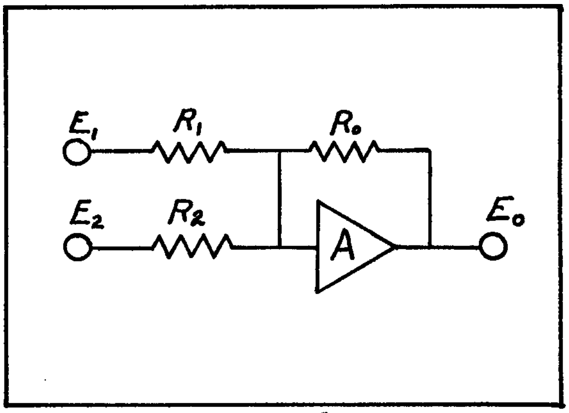

Fig. 6
SUMMING AMPLIFIER

$$E_o = \left(E_1 \frac{R_o}{R_1} + E_2 \frac{R_o}{R_2}\right) \frac{A}{(1 - A) + R_o\left(\frac{1}{R_1} + \frac{1}{R_2}\right)}$$

✝ Appendix 2

where E_1 and E_2 are the input voltages to be summed, R_1 and R_2 are their respective summing resistors, and A is the amplifier gain. Let

$$f(A) = \frac{\quad}{(1-A) + R_o\left(\frac{1}{R_1} + \frac{1}{R_2}\right)}$$

then
$$E_o = -\left(E_1 \frac{R_o}{R_1} + E_2 \frac{R_o}{R_2}\right) f(A)$$

Letting
$$\rho = 1 + R_o\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

so that
$$f(A) = \frac{A}{A-\rho}$$

then very nearly, for $A \gg \rho$

$$f(A) = 1 + \frac{\rho}{A}$$

Also, by the same token, if A is multiplied by some factor c, then
$$f(cA) = 1 + \frac{\rho}{cA}$$

It follows that

$$f(A) - 1 = \frac{\rho}{A}$$

$$f(cA) - 1 = \frac{\rho}{cA}$$

$$f(cA) - 1 = \frac{1}{c}\left[f(A) - 1\right]$$

Now [f(A) - 1] is the difference of the output from what it would be with f(A) = 1, the ideal value. In practice it is not necessary that f(A) = 1, but it must be that changes of amplification, A, such as may be expected in the amplifier with change of tubes and change of transconductance, will not affect the value of f(A) appreciably.

The above expressions may be used to find the minimum design gain of an amplifier. If the gain should decrease by some fraction b, the amplification A becomes (1 - b)A, the value of c becomes

$$c = 1 - b$$

and the fractional error due to the change is

$$\begin{aligned}\frac{dE}{E} &= [f(cA) - 1] - [f(A) - 1] \\ &= \left(\frac{1}{1-b}\right)[f(A) - 1] - [f(A) - 1]\end{aligned}$$

which may be reduced to

$$\frac{dE}{E} = \frac{-\rho b}{A(1-b)}$$

so that if the fractional error associated with the decrease of A is to be limited to a value e, then it must be that

$$\begin{aligned}A &\geq \frac{\rho b}{e(1-b)} \\ &\geq \left[\frac{b}{e(1-b)}\right]\left(1 + \frac{R_o}{R_1} + \frac{R_o}{R_2}\right)\end{aligned}$$

If the output of the amplifier is a machine variable Q, then

$$e = dQ = \frac{dz}{\frac{\partial z}{\partial Q}}$$

and limiting the change of output depth to dz_o, for the change of gain,

$$A \geq \frac{b}{1-b}\left(1 + \frac{R_0}{R_1} + \frac{R_0}{R_2}\right)\left(\frac{\partial z}{\partial Q}\right)_{max}\left(\frac{1}{dz_0}\right)$$

The required gains of the summing amplifiers are shown in Table IX, including a column allowing for a 25% reduction of gain.

TABLE IX

MINIMUM GAIN OF SUMMING AMPLIFIERS

Amplifier	Equation Variable	Minimum A for Reduction bA	Minimum A for 25% Reduction
SmA1	t	$1.7 \times 10^5 \dfrac{b}{(1 - b)dz_o}$	$5.5 \times 10^4/dz_o$
SmA3	x	$4.2 \times 10^3 \dfrac{b}{(1 - b)dz_o}$	$1.4 \times 10^3/dz_o$

VII Temperature Effects in Resistances

The effects of temperature changes in the resistance networks will now be studied. The possible sources of temperature errors are in the summing networks, in potentiometers set in series with other resistances, and the series resistance network of resistors R22. It is here assumed that the current in all resistors is small with respect to the rating, so that the temperature of any resistance is uniform, and no "hot spots" develop. The effects of temperature will be approached from the standpoint of simply determining what the limits must be on temperature change and temperature coefficients. Results are shown in Tables X and XI.

Summing Networks. To study the temperature effects of a summing amplifier or a servo amplifier having an output voltage[†]

$$-E_0 = E_1 \frac{R_0}{R_1} + E_2 \frac{R_0}{R_2} + \ldots$$

the input voltages may be held constant and the resistances allowed to vary with temperature. The change of output with temperature is

$$-dE_0 = \frac{\partial E_0}{\partial R_0} \frac{dR_0}{dT} dT_0 + \frac{\partial E_0}{\partial R_1} \frac{dR_1}{dT} dT_1 + \ldots$$

and the change of resistance with temperature is

[†] Appendix 2

$$\frac{dR_i}{d\mathcal{T}} = R_i \lambda_i$$

where $\mathcal{T}$ = temperature

λ_i = temperature coefficient of resistance of resistor R_i

$d\mathcal{T}_i$ = change of temperature of resistor R_i

The change of output then becomes

$$-dE_o = E_1 \frac{R_o}{R_1} \lambda_o \, d\mathcal{T}_o + E_2 \frac{R_o}{R_2} \lambda_o \, d\mathcal{T}_o + \ldots - E_1 \frac{R_o}{R_1} \lambda_1 \, d\mathcal{T}_1 - \ldots$$

$$= E_1 \frac{R_o}{R_1} (\lambda_o \, d\mathcal{T}_o - \lambda_1 \, d\mathcal{T}_1) + E_2 \frac{R_o}{R_2} (\lambda_o \, d\mathcal{T}_o - \lambda_2 \, d\mathcal{T}_2) + \ldots$$

Letting $d\lambda_1 = \lambda_o - \lambda_1$; $d\lambda_2 = \lambda_o - \lambda_2$; etc.

and $d\mathcal{T} = d\mathcal{T}_o = d\mathcal{T}_1 = d\mathcal{T}_2$

then
$$-dE_o = E_1 \frac{R_o}{R_1} \, d\lambda_1 \, d\mathcal{T} + E_2 \frac{R_o}{R_2} \, d\lambda_2 \, d\mathcal{T} + \ldots$$

Further letting

Q = output machine variable

Q_i = input machine variable into the i^{th} resistor

then
$$Q = Q_1 + Q_2 + Q_3 + \ldots$$

$$= \frac{E_o}{E_{o\,max}} = \frac{E_1}{E_{o\,max}} \frac{R_o}{R_1} + \frac{E_2}{E_{o\,max}} \frac{R_o}{R_2} + \ldots$$

and
$$Q_i = \frac{E_i}{E_{o\,max}} \frac{R_o}{R_i}$$

From the above, then

$$dQ = \frac{dE_o}{E_{o\,max}} = \frac{E_1}{E_{o\,max}} \frac{R_o}{R_1} \, d\lambda_1 \, d\mathcal{T} + \frac{E_2}{E_{o\,max}} \frac{R_o}{R_2} \, d\lambda_2 \, d\mathcal{T} \ldots$$

$$dQ = Q_1\, d\lambda_1\, d\tau + Q_2\, d\lambda_2\, d\tau$$

and

$$dQ_i = Q_i\, d\lambda_i\, d\tau$$

Also

$$\frac{\partial z}{\partial Q_i} = \frac{\partial z}{\partial Q}$$

so that

$$dQ_i = \frac{dz}{\left(\frac{\partial z}{\partial Q_i}\right)}$$

$$= \frac{dz}{\left(\frac{\partial z}{\partial Q}\right)}$$

Therefore, for a single resistance ratio

$$d\lambda_i\, d\tau = \frac{dz}{Q_i\left(\frac{\partial z}{\partial Q}\right)}$$

To limit the output error to a value dz_o, for a change of the i^{th} resistance ratio, since $(Q_i)_{max} = 1$, then

$$d\lambda_i\, d\tau \leq \frac{dz_o}{\left(\frac{\partial z}{\partial Q}\right)_{max}}$$

Series Resistors and Potentiometers. Potentiometers set in series with another resistance between the supply and ground will now be considered. First, as a special case, potentiometer P3 will be considered.

Potentiometer P3 has been chosen as a 360° pot using but 300° to represent the function kt. For this discussion let

R_a = resistance of 300° used for function

R_b = resistance of remainder of pot.

a = fractional rotation of 300° used for function.

With no loading effects, the output from P3 may be written

$$E_o = \frac{a E R_a}{R_a + R_b + R_1} = \frac{a E R_a}{P_3 + R_1}$$

and $$dE_o = \frac{\partial E_o}{\partial R_a}\frac{dR_a}{dT}\,dT_a + \frac{\partial E_o}{\partial R_b}\frac{dR_a}{dT}\,dT_b + \frac{\partial E_o}{\partial R_1}\frac{dR_1}{dT}\,dT_1$$

$$dE_o = aE\left[\frac{R_a(R_b + R_1)}{(P_3 + R_1)^2}\lambda_a\,dT_a - \frac{R_a R_b}{(P_3 + R_1)^2}\lambda_b\,dT_b - \frac{R_a R_1}{(P_3 + R_1)^2}\lambda_1\,dT_1\right]$$

But $$\lambda_a = \lambda_b = \lambda_3$$

and $$dT_a = dT_b = dT_3$$

so that $$dE_o = \left(\frac{a E R_a}{P_3 + R_1}\right)\left(\frac{R_1}{P_3 + R_1}\right)\left(\lambda_3\,dT_3 - \lambda_1\,dT_1\right)$$

$$= E_o \frac{R_1}{P_3 + R_1}\left(\lambda_3\,dT_3 - \lambda_1\,dT_1\right)$$

or, letting $$d\lambda = \lambda_3 - \lambda_1$$

and $$dT = dT_1 = dT_3$$

then $$\frac{dE_o}{E_o} = \frac{R_1}{P_3 + R_1}\,d\lambda\,dT$$

It can be shown using similar reasoning that for a potentiometer P, in series with resistor R, the change of output with temperature is

$$\frac{dE_o}{E_o} = \frac{R}{P + R}\,d\lambda\,dT$$

where $d\lambda$ = difference between temperature coefficients of potentiometer and resistor. If the potentiometer generates a machine variable Q, then

$$dQ = \frac{dE_o}{E_{o\,max}} = \frac{R}{P + R}\,d\lambda\,dT$$

so that to limit the change of output to dz_o,

$$d\lambda \, d\tau \leq \frac{P+R}{R} \; \frac{dz_o}{\left(\frac{\partial z}{\partial Q}\right)_{max}}$$

Series Resistance Networks. For a network of series resistors such as R22, the maximum possible change with temperature is, using the above reasoning,

where m = number of resistors in series

$d\lambda$ = the maximum variation of temperature coefficient of the resistors.

Therefore, in order to assure an error no greater than dz_o, then the product of the variation of temperature coefficients and change of temperature in resistors R22 should be

$$d\lambda \, d\tau \leq \frac{24}{23} \; \frac{dz_o}{\left\{\frac{\partial z}{\partial[(n-1)S]}\right\}_{max}}$$

Potentiometers P8. Assuming no loading effects, the output at the junction of R2 and P8 may be determined from the equivalent diagram in Fig. 7 as

$$E_o = \frac{(R_3 + P_8/24)\,2E}{R_2 + P_8/24 + R_3} - E$$

$$= \frac{E\,(R_3 + P_8/24 - R_2)}{R_2 + P_8/24 + R_3}$$

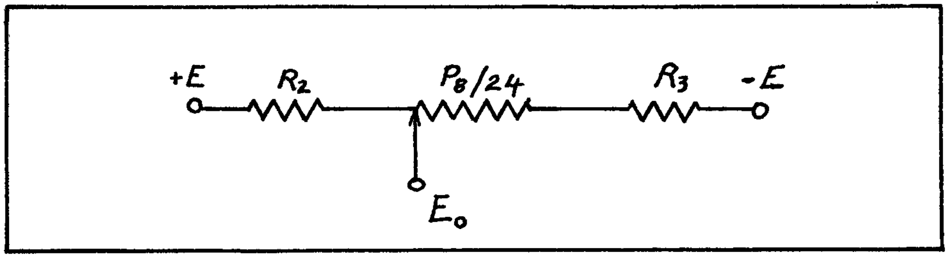

Fig. 7
EQUIVALENT DIAGRAM OF POTENTIOMETERS P8

The change of output with temperature is

$$dE_o = \frac{\partial E_o}{\partial R_2}\frac{dR_2}{dT}dT_2 + \frac{\partial E_o}{\partial R_3}\frac{dR_3}{dT}dT_3 + \frac{\partial E_o}{\partial (P_8/24)}\frac{d(P_8/24)}{dT}dT_p$$

$$= \frac{E\left[-2R_2(R_3 + P_8/24)\lambda_2 dT_2 + 2R_2R_3\lambda_3 dT_3\right]}{(R_2 + P_8/24 + R_3)^2}$$

$$+ \frac{E\left[2R_2(P_8/24)\lambda_p dT_p\right]}{(R_2 + P_8/24 + R_3)^2}$$

where λ_p and dT_p are the temperature coefficient and change of temperature of (P8/24).

Since $$R_2 = R_3 = \frac{P_8}{2}$$

and letting $$dT = dT_2 = dT_3 = dT_p$$

the fractional change of output may be reduced to

$$\frac{dE_o}{E_o} = \frac{(24)(12)}{25}dT\left[(\lambda_3 - \lambda_2) + \frac{1}{12}(\lambda_p - \lambda_2)\right]$$

Further letting $$\lambda_2 \approx \lambda_3 \approx \lambda_p$$

and $$d\lambda = \lambda_3 - \lambda_2$$

then $$\frac{dE_o}{E} \approx 12\, dT d\lambda$$

The change of depth, z, for a change of temperature in this circuit generating machine variable T_w then becomes

$$dz = \frac{\partial z}{\partial T_w}dT_w$$

$$dz = 12 \frac{\partial z}{\partial T_w} d\tau d\lambda$$

Therefore it is necessary that

$$d\tau d\lambda \leq \frac{dz_0}{12\left(\frac{\partial z}{\partial T_w}\right)_{max}}$$

with the previous specification that all of the temperature coefficients of resistance in the circuit are approximately equal, i.e., the resistors are wound with the same material. (Resistors of the same material can have different temperature coefficients because of various factors, as discussed by Blackburn[18].)

Calculations on Temperature Effects. Using the values of $(\partial z/\partial Q)_{max}$ for all the variables fed into the summing networks or generated by the voltage dividers, calculations have been made and Tables X and XI have been prepared showing the maximum value of $d\tau d\lambda/dz$ for each of the resistance pairs.† Since some of the pairs affect the x output, the tables include columns showing also the maximum $d\tau d\lambda/dx$.

If it is desired to limit the output error in z to a value dz_o, for any one of the resistance pairs, then the design should be such that

$$d\tau d\lambda \leq \left(\frac{d\tau d\lambda}{dz}\right)_{max} dz_0$$

† A resistance pair here indicates the feedback resistor and one of the input resistors of a summing network, or the series resistors in the case of a voltage divider.

TABLE X

TEMPERATURE REQUIREMENTS IN SUMMING NETWORKS

Amplifier	Equation Variable	Resistors	$(d\tau d\lambda/dx)_{max}$	$(d\tau d\lambda/dz)_{max}$
SmA1	t_r	R4, R36	-	1.8×10^{-5}
SmA1	t_w	R4, R35	-	4.3×10^{-4}
SA1	kt	R5, R6	-	2.1×10^{-5}
SA4	r	R15, R11	-	6.7×10^{-5}
SA4	y	R15, R12	-	1.4×10^{-4}
SA4	z/10	R15, R10	-	5.0×10^{-4}
SA2	x	R18, R19	-	5.8×10^{-4}
SmA3	u	R31, R30	6.7×10^{-4}	1.2×10^{-3}
SmA3	(n - 1)s	R31, R29	3.3×10^{-4}	5.8×10^{-4}
SA3	x	R28, R27	3.3×10^{-4}	-
SA3	x/10	R28, R26	3.3×10^{-3}	-

TABLE XI

TEMPERATURE REQUIREMENTS OF SERIES RESISTORS

Resistors	Equation Variable	$(d\tau d\lambda/dx)_{max}$	$(d\tau d\lambda/dz)_{max}$
P3, R1	kt	-	2.4×10^{-5}
P8, R2, R3	t_w	-	3.6×10^{-5}
P15, R9	z/10	-	5.6×10^{-3}
P11, R25	x/10	3.7×10^{-3}	-
R22	(n - 1)s	3.4×10^{-4}	6.0×10^{-4}

CHAPTER VI

CALCULATION OF PROBABLE ERROR OF A PLOTTED POINT

In order to further help the engineer, calculations have been made concerning the probable error of a plotted point. It shall here be assumed that amplifiers and temperature variations have been chosen so as to produce no error in output. It will further be assumed that all other components have been selected to give the same maximum error dz_o, with the exception of resistors R22; these have been chosen with a maximum probable error of $(1/3)dz_o$. The probable error due to a component shall here be assumed to be 1/3 of its limiting error, which is specified by the manufacturer. The same assumption will here be made concerning the error due to loading of a potentiometer, servo dead zone, resolver phase shifts and resolution of pots not dependent on linearity. The maximum probable error due to each error source is then $\pm(1/3)dz_o$. The equation variables and the corresponding error sources are listed in Table XII.

Twenty eight individual error sources are listed in Table XII, each having a maximum probable error of $\pm(1/3)dz_o$. The probable error of output is then, in general, since the maximum values of the partial derivatives do not occur with any one set of values on the

TABLE XII

SOURCES OF COMPUTER ERROR

Equation Variable	Sources of Error
t_r	P1, P1-load
t_w	P8, P8-load
k	P2
kt	P3, P3-load, Servo #1-position
h	P7
sinh kt	P6
(cosh kt - 1)	P5
cos A	RS1, RS1-phase, Servo #2-position
x	RS2, RS2-phase, P13, P13-load, Servo #3-position, Servo #5-position
u	P12
(n - 1)s	P9, R22, R22-load
w	P10
z	P14, P14-load, Servo #4-position

variables,

$$p.e. < \frac{1}{3} \sqrt{28\, dz_0^2}$$

$$< 1.8\, dz_0$$

Probable Error of a Hypothetical Point. The probable error of a hypothetical point at the maximum depth and maximum value of kt, as set forth in Appendix 1 will now be computed. The probable error due to a component of equation variable q, is $(1/3)dz_0$ multiplied by the ratio of $(\partial z/\partial q)/(\partial z/\partial q)_{max}$, at the value of q in question. The values of the partial derivatives are listed in Table II. The probable error due to each machine variable and the corresponding number of error sources are listed in Table XIII. Using the values here given, the probable error of the result for this particular point is then

$$p.e. = \frac{dz_0}{3} \left[7(.87^2 + (.86)^2 + 3(.74)^2 + (.88)^2 + (.48)^2 + (.47)^2 + 10(.26)^2 + 3 \right]^{1/2}$$

From these calculations, it may be said that for the design assumptions that the maximum allowed output error for each error source is dz_0, then the probable error in plotting of a point is of the order of dz_0, the value being dependent on the values of the input variables,

TABLE XIII

PROBABLE ERRORS DUE TO COMPONENTS FOR HYPOTHETICAL POINT

Equation Variable	Number of Error Sources	Probable Error due to Machine Variable
t_r	2	0.87 dz_o
t_w	2	0.87 dz_o
k	1	0.86 dz_o
kt	3	0.74 dz_o
h	1	0.88 dz_o
sinh kt	1	0.48 dz_o
(cosh kt - 1)	1	0.47 dz_o
cos A	3	0.87 dz_o
x	6	0.26 dz_o
u	1	0.26 dz_o
(n - 1)s	3	0.26 dz_o
w	1	0
z	3	1.0 dz_o

CHAPTER VII

COMPUTER REQUIREMENTS

The accuracy desired in a computer of this type may differ through the industry according to personal or company preferences. First of all, one may point out that, in making velocity assumptions, the data fed into the machine may well be off by 5%, and then argue that, since this is true, there is no need for any great accuracy in the computer. He may contend that the main requirement is not a high degree of absolute accuracy but that the computer repeat precisely the results for any given set of values, and that it be free from random errors. Certainly, it would be undesirable and impractical for the machine to plot points for a reflecting layer out of the proper relationship with another layer.

Only partially opposed to the above idea, others may argue that in spite of the fact that the original data may be in error, the computer should accurately reproduce the results obtained by calculations made by hand, and precisely repeat the results for any given set of values.

In agreement with the latter point of view calculations have been made on component tolerances to give a total probable error of the same order as the error that may be expected from errors in reading the time on the seismogram. However, more attention has been given to factors influencing repeatability and random errors, in an attempt to reduce these.

On the standard seismogram, the two way travel time may be estimated to one milli-second, a variation of $dt = 5 \times 10^{-4}$ sec. on the one way travel time. It was shown in Table II that the maximum rate of change of depth, z, with respect to time, t, is 2.3×10^{4} ft/sec. The maximum error in depth for a one milli-second error in the two way travel time is then

$$\left(\frac{\partial z}{\partial t}\right)_{max} dt = 12 \text{ ft}$$

If each component is chosen to give a maximum error of 25 ft. (a probable error of 8 ft.) in the depth, then from the calculation in Chapter VI, the probable error of the computer will have a maximum of the same order as the maximum error due to one milli-second error in the recorded time. On components not affecting repeatability, except for summing resistors, calculations have been made on a basis of a 25 ft. maximum error due to each component. Since little extra expense would be involved, summing resistors have been calculated for a 5 ft. maximum error. On items that can affect the repeatability (i.e., servo dead zone, isolation and booster amplifiers, summing amplifiers, and temperature effects) calculations have been made on the basis of a 5 ft. maximum error due to each component. On components that do not affect z, but do affect x, calculations have been made for a maximum of 5 ft. error in x. The precision of resistors R22 was computed for a probable error of 8 ft. In regard to random errors, the sources of such errors are the same as for sources affecting repeatability with the possible addition of the resolvers. The calculations were made from the information given in Tables III through XI. The results are shown in Tables XIV, XV, and XVI.

Because of the difficulties that may be encountered by use of 300° rotation in potentiometer P3, as planned, calculations have also been made for a potentiometer using a full 10 turns.

TABLE XIV

TOLERANCE OF COMPONENTS NOT AFFECTING REPEATABILITY

Component	Characteristic	Tolerance
Potentiometers		
P1	Linearity	0.045%
P3 (10 turns)	Linearity	0.052%
P3 (300° function)	Linearity	0.045%
P8	Linearity	1.1 %
P13	Linearity	1.5 %
P14	Linearity	1.2 %
P5	Function	0.17 %
P6	Function	0.09 %
P2	Resolution	0.028%
P7	Resolution	0.11 %
P9	Resolution	3.0 %
P10	Resolution	5.8 %
P11	Resolution	3.4 %
P12	Resolution	5.8 %
P15	Resolution	2.5 %
Resistors		
R22	Precision	14 %
Resolvers		
RS1	Accuracy	0.17 %
RS1	Variation of Phase	1.0°
RS2	Accuracy	1.5 %

TABLE XIV (continued)

TOLERANCE OF COMPONENTS NOT AFFECTING REPEATABILITY

Component	Characteristic	Tolerance
Resolvers		
RS2	Variation of Phase	9.5°
Summing Resistors		
R5	Minimum Value	2.2×10^2 P3
R15	Minimum Value	92 P14
R28	Minimum Value	6.8 P13
R29	Minimum Value	1.4×10^2 R22
R35	Minimum Value	2.0×10^2 P8
R36	Minimum Value	2.2×10^2 P1

TABLE XV

TOLERANCE OF COMPONENTS AFFECTING REPEATABILITY

Component	Characteristic	Tolerance
Servos		
#1 (10 turns)	Dead Zone	0.37°
#1 (300° function)	Dead Zone	0.032°
#2	Dead Zone	0.095°
#3 (1 turn)	Dead Zone	1.0°
#4 (10 turns)	Dead Zone	0.90°
#5	Dead Zone	4.7°
Summing Amplifiers		
SmA1	Minimum Gain	1.1×10^4
SmA3	Minimum Gain	2.8×10^2
Isolation Amplifiers		
IA1	Maximum Change of Gain	0.009%
IA2	Maximum Change of Gain	0.034%
Booster Amplifiers		
BA1	Maximum Change of Gain	0.018%
BA2	Maximum Change of Gain	0.29 %
BA3	Maximum Change of Gain	0.58 %

TABLE XVI

MAXIMUM CHANGE OF TEMPERATURE OF RESISTORS

Resistors	$(d\tau/d\lambda)_{max}$	$d\tau_{max}$ for $d\lambda = \pm 20 \times 10^{-6}$
R4, R36	9.0×10^{-5}	4.5 °C
R4, R35	2.2×10^{-3}	1.1×10^{2}
R5, R6	1.0×10^{-4}	5.0
R15, R11	3.4×10^{-4}	17
R15, R12	7.0×10^{-4}	35
R15, R10	2.5×10^{-3}	1.2×10^{2}
R18, R19	2.9×10^{-3}	1.4×10^{2}
R31, R30 ÷	3.4×10^{-3}	1.7×10^{2}
R31, R29 ÷	1.6×10^{-3}	80
R28, R27 ÷	1.6×10^{-3}	80
R28, R26 ÷	1.6×10^{-2}	8.0×10^{2}
P3, R1	1.2×10^{-4}	6.0
P8, R2, R3	1.8×10^{-4}	9.0
P15, R9	2.8×10^{-2}	1.4×10^{3}
P11, R25 ÷	1.8×10^{-2}	9.0×10^{2}
R22 ÷	1.7×10^{-3}	85

÷ These values are computed for a 5 foot error in x.

CHAPTER VIII

SUMMARY AND DISCUSSION OF RESULTS

In this paper, a combination seismograph computer and plotting device has been described with the theory on which it is based. Calculations are included to find the errors in the position of the plotting head due to errors in the machine variables, and the maximum of such errors. These calculations were then used to furnish information to help in the selection of computer components, and to obtain some idea of the probable error of output of the computer.

Component Tolerances. In order to help in the selection of components, calculations were made concerning requirements in the angle of dead zone of servo-controlled components, linearity, loading, resolution, functional conformity, and temperature requirements of potentiometers and resolvers, and requirements regarding change of gain of amplifiers. Many of these calculations have been made in such a manner as to give the maximum proportionality between an error in the component and the resulting error in the computer output; thus, to limit the output error due to a component to a value dz_o, then the maximum allowable error in the component is the product of the proportionality constant and dz_o. The results of these calculations may be found in Tables III through XI.

Probable Error. The calculations made concerning the expected probable error of the computer were made at the maximum depth, assuming that each component was selected for a single value of limiting error for that component. Indications are that the probable error of the computer will be of an order a little greater than the maximum probable error of output for each component.

Recommendations. From consideration of the results of these calculations, some recommendations can be made, in the event any great accuracy is desired, to obtain the requirements set forth:

(1) Replace the 300° rotation function potentiometers with multiple turn function potentiometers. The requirement on the smallness of the angle of dead zone may otherwise make the requirements on gearing impossible.

(2) Replace the series arrangement of P3 and R1 with a single potentiometer of the same number of turns as the function potentiometer. This may also necessitate a change of the summing resistor.

(3) Make P1, P2, P7, P9, P10, P12, and P15 of several turns to facilitate setting.

(4) Design summing amplifier SmA1 to directly feed

potentiometer P2, thus eliminating isolation amplifier IA1.

(5) Make P5 and R12 of values such that isolation amplifier IA2 may be eliminated. This must also take into consideration potentiometer P7, since P5 and P6 load this potentiometer; this loading effect, however, is not critical.

(6) Include phase adjustments in all booster amplifiers in order to properly adjust the output phase of the resolvers.

(7) Make all summing network resistors and series resistances of low temperature coefficient types, or relatively closely matched coefficients for each network.

(8) Enclose summing network resistors R4, R36, R35, R5 and R6 in a temperature controlled compartment.

(9) Enclose P3 and R1 (if this arrangement is used) in a temperature controlled compartment. Potentiometers P8 and resistors R2 and R3 should also be enclosed.

Shielding. Another important possible source of error that has not been considered is that of reactive pickup of voltages. The amount of shielding necessary is dependent on design and arrangement of components, and can be found only through considerable experimentation.

<u>Possible Accuracy</u>. A comparison of available components with the calculations made indicates that considerable accuracy may be reached with a computer of this type. The chief limitations are in the function potentiometers, the resolvers, the isolation and booster amplifiers, and possibly the servo systems. It would appear, however, that a probable error of as little as 0.2% would be possible. The cost of such a unit would be high, so that the real limiting factor would be the amount of money available.

In conclusion, the designed computer gives promise of fast, and accurate plotting of seismic profile maps, for the assumption that the wave velocity increases linearly with depth.

APPENDIX 1

DETERMINATION OF VARIABLES FOR HYPOTHETICAL POINT

In the set-up of a hypothetical problem of a point to be plotted, a freedom of choice exists only so far in the selection of variables. The others may then be computed to conform to those chosen.

For this problem let

z = 20,000 ft

v = 20,000 ft/sec

kt = 1.20

x = 2000 ft

u = 1500 ft

w = 0

$(n - 1)s$ = 1500 ft

The equation for z may be rewritten

$$h^2\left[\sinh^2 kt - (\cosh kt - 1)^2\right] + 2hz(\cosh kt - 1) - (z^2 + x^2) = 0$$

Setting in the values just given, and solving for h gives a single positive root of

$$h = 8.70 \times 10^3 \text{ ft}$$

Also, since $\qquad k = \dfrac{v_0}{h}$

and $\qquad v = v_0 + kz = v_0\left(1 + \dfrac{z}{h}\right)$ ft/sec

then $\qquad v_0 = \dfrac{hv}{h + z} = 6.06 \times 10^3$ ft/sec

$$k = \frac{v_0}{h} = 0.697 \ \text{sec}^{-1}$$

$$t = \frac{kt}{k} = 1.72 \ \text{sec}$$

$$r = h \sinh kt = 13.1 \times 10^3 \ \text{ft}$$

$$y = h(\cosh kt - 1) = 7.06 \times 10^3 \ \text{ft}$$

$$A = \sin^{-1}\left(\frac{x}{r}\right) = 8.7^\circ$$

thus completing the list.

APPENDIX 2

SUMMING NETWORKS

The output of a summing network may be derived as follows: Consider a network as shown in Fig. 8 having input voltages E_o, E_1, E_2, etc., summed through resistors R_o, R_1, R_2, R_3, etc., with a load resistor R_L, and a voltage E_g at the junction. The sum of all currents flowing to the common terminal must be zero, or

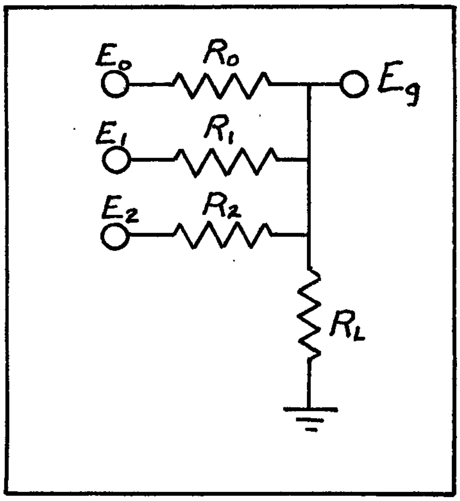

Fig. 8
SUMMING NETWORK

$$\frac{E_o - E_g}{R_o} + \frac{E_1 - E_g}{R_1} \; . \; . \; . - \frac{E_g}{R_L} = 0$$

$$\frac{E_o}{R_o} + \frac{E_1}{R_1} + \frac{E_2}{R_2} + \; . \; . \; . - E_g\left(\frac{1}{R_o} + \frac{1}{R_1} + \; . \; . \; . \frac{1}{R_L}\right) = 0$$

so that

$$E_g = \left(\frac{E_o}{R_o} + \frac{E_1}{R_1} + \frac{E_2}{R_2} + \; . \; . \; . \;\right)\left(\frac{1}{\frac{1}{R_o} + \frac{1}{R_1} + \; . \; . \; . \frac{1}{R_L}}\right)$$

<u>Summing Servo Amplifier</u>. If the common terminal is connected to the grid of a servo amplifier, and E_o is the feedback voltage from the servo, then $E_g = 0$. The output voltage becomes

$$-E_o = E_1 \frac{R_o}{R_1} + E_2 \frac{R_o}{R_2} + E_3 \frac{R_o}{R_3} + \ . \ . \ .$$

It may be noted that this equation is identical with that of a summing amplifier.[19]

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