IDENTIFICATION OF SYSTEMS BY QUASILINEARIZATION

A Thesis

Presented to

The Faculty of the College of Engineering

The University of Houston

In Partial Fulfillment

of the Requirements for the Degree

Master of Science in Mechanical Engineering

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by

George Adam Zupp, Jr.

January 1968

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ABSTRACT

The theoretical development and application of a quasilinearization technique to problems of system identification is presented. A one-degree-of-freedom system is used for the examples for the identification problems. Several numerical experiments are presented relating the effects of noisy input data to the accuracy obtained in computing unknown parameters of the governing differential equations. Also presented is the application of quasilinearization to the identification problems of automobile coasting dynamics. Conclusions from these experiments are included.

PREFACE

In this thesis, equations are numbered by chapters, with the chapters and equation numbers separated by a period and enclosed by parentheses. Numbers enclosed in parentheses but not separated by a period refer to references. When differential equations appear, differentiation with respect to time is indicated by the dot notation. The Greek letter Σ is used to denote summation.

I wish to express my appreciation to the following for their help in the preparation of this thesis:

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CHAPTER I

INTRODUCTION

In many instances, measured observations of various physical phenomena can be considered as solutions to the differential equation(s) which describe the phenomena. By knowing the initial conditions and the coefficients for the appropriate equation(s), it is possible to gain insight into engineering problem areas. The process of determining these coefficients and initial conditions is called "System Identification." This thesis presents a development and an application of some quasilinearization techniques for solving problems of system identification.

The origin of quasilinearization can be found in the theory of dynamic programming (1). The theory of quasilinearization gives a systematic iterative approach to solving nonlinear or linear differential equations subject to initial and multi-point boundary-value conditions and unknown parameters. The development of quasilinearization involves the solution of a set of linear differential equations with varying coefficients in which the solution converges under appropriate conditions to the solution of the nonlinear equations. Since these equations are linear, and the principle of superposition applies, the boundary conditions at each iteration will be satisfied.

The application of quasilinearization techniques is illustrated by an engineering analogy of a one-degree-of-freedom system. The examples presented are linear and nonlinear oscillators and the dynamic phenomenon of a coasting automobile.

The data given for the linear and nonlinear oscillators illustrate the relation between the accuracy of the specified boundary conditions

and the resulting predicted accuracy of the initial conditions and coefficients of the pertinent differential equation(s). The automobile problem illustrates how the aerodynamic drag coefficient and the rolling friction coefficient can be predicted with data which is generally the most economical to obtain.

CHAPTER II

SYSTEM IDENTIFICATION

System identification is the process of fitting the governing differential equation(s) to a given set of boundary conditions. This process involves the determination of the coefficients and initial conditions of the differential equation(s) that satisfy the given set of boundary conditions. The coefficients and initial conditions of the differential equation(s) are referred to as the unknown parameters of the mathematical model.

To illustrate the concept of system identification, consider the simple example where the governing differential equations are

$$\begin{aligned} \frac{dy}{dt} &= ay \\ \frac{da}{dt} &= 0 \end{aligned} \tag{2.1}$$

where "a" is an unknown parameter.

The solution of equation (2.1) can be determined by straightforward integration. If the range of interest is from 0 to t then it can be written as

$$y = y_o \, e^{at} \tag{2.2}$$

where the term y_o is the initial condition. If the desired solution must satisfy certain specified boundary conditions, say

$$y = y_1 \quad @ \quad t = t_1$$

$$y = y_2 \quad @ \quad t = t_2$$

then it is found that

$$a = \frac{1}{t_1 - t_2} \ln \frac{y_1}{y_2} \tag{2.3}$$

$$y_o = y_2 \bar{e} \left(\frac{t_2}{t_1 - t_2} \right) \ln \frac{y_1}{y_2} \tag{2.4}$$

Thus system identification for a system that can be described by a differential equation as simple as (2.1) is straightforward. It should be noted that even through equation (2.1) is apparently linear, the identification problem is nonlinear.

Consider a more complex differential equation such as

$$\frac{d^2y}{dt^2} + \lambda^2 y = 0 \tag{2.5}$$

Again the apparent solution is straightforward when

$$y = A \sin \lambda t + B \cos \lambda t \tag{2.6}$$

where A and B are arbitrary parameters. If at $t = 0$, $y = y_o$, and $\dot{y} = \dot{y}_o$, then

$$A = \dot{y}_o / \lambda$$

and

$$B = y_o$$

Equation (2.6) can now be written as

$$y = y_o \cos \lambda t + \dot{y}_o / \lambda \sin \lambda t \tag{2.7}$$

In order to determine y_o, $\dot{y}_o$, and λ, three independent boundary conditions for y(t) are necessary, say

$$y = y_1 \quad @ \quad t = t_1$$

$$y = y_2 \quad @ \quad t = t_2$$

$$y = y_3 \quad @ \quad t = t_3$$

Writing equation (2.7) at the data points t_1, t_2, and t_3 gives

$$\begin{aligned} y_1 &= y_o \cos \lambda t_1 + \dot{y}_o/\lambda \sin \lambda t_1 \\ y_2 &= y_o \cos \lambda t_2 + \dot{y}_o/\lambda \sin \lambda t_2 \\ y_3 &= y_o \cos \lambda t_3 + \dot{y}_o/\lambda \sin \lambda t_3 \end{aligned} \tag{2.8}$$

which are independent if the spacing of the data points is properly chosen.

Because of the presence of the sine and cosine functions, the set of equations (2.8) is nonlinear. In order to determine y_o, $\dot{y}_o$, and λ, an iterative technique for solving nonlinear equations is required. One such technique is the Newton-Raphson method. This method is discussed in the next chapter.

The governing differential equations will always be written as a set of first order equations and unknown constant parameters are incorporated by adding a null equation.

Thus equation (2.5) will be written as

$$\frac{dy}{dt} = z$$

$$\frac{dz}{dt} = - \xi y$$

$$\frac{d\xi}{dt} = 0.$$

where $\xi = \lambda^2$

From the discussion of equations (2.1) and (2.5), it is obvious that the problem of system identification can be difficult. A discussion of quasilinearization theory and application to the identification of systems is presented in the following chapters.

CHAPTER III

QUASILINEARIZATION

Newton-Raphson Method

The Newton-Raphson algorithm is an iterative procedure for solving nonlinear simultaneous algebraic equations (2). The theoretical development of the Newton-Raphson method is based on a Taylor's series expansion of several variables.

Consider the vector $\vec{R}(\vec{x})$ which represents a set of residue equations of the form

$$\left.\begin{aligned} R_1(\vec{x}) &= R_1(x_1, x_2, \ldots x_m) = 0 \\ R_2(\vec{x}) &= R_2(x_1, x_2, \ldots x_m) = 0 \\ &\vdots \\ R_m(\vec{x}) &= R_m(x_1, x_2, \ldots x_m) = 0 \end{aligned}\right\} \qquad (3.1)$$

Expanding $\vec{R}(\vec{x})$ in a Taylor's series about an approximate solution $\vec{x}_n$ gives

$$\vec{R}(\vec{x})_{n+1} = \vec{R}(\vec{x})_n + \left.\frac{\partial \vec{R}(\vec{x})}{\partial \vec{x}}\right|_n (\vec{x}_{n+1} - \vec{x}_n) + \frac{1}{2!}\left.\frac{\partial^2 \vec{R}(\vec{x})}{\partial \vec{x}^2}\right|_n (\vec{x}_{n_{f1}} - \vec{x}_n)^2 \ldots \qquad (3.2)$$

where n denotes the iteration.

If $(\vec{x}_{n+1} - \vec{x}_n)$ is sufficiently small such that the higher order terms of equation (3.2) can be neglected, then equation (3.2) reduces to

$$\vec{R}(\vec{x})_{n+1} = \vec{R}(\vec{x})_n + \left.\frac{\partial \vec{R}(\vec{x})}{\partial \vec{x}}\right|_n (\vec{x}_{n+1} - \vec{x}_n) \qquad (3.3)$$

In the final iteration the residue vector $\vec{R}(\vec{x})$ must go to zero for the vector $(\vec{x})$ to be a solution, therefore equation (3.3) reduces to

$$\vec{R}\,(\vec{x})_n + \frac{\partial \vec{R}(\vec{x})}{\partial \vec{x}}_n \left(\vec{X}_{n+1} - \vec{X}_n \right) = 0 \qquad (3.4)$$

or rewriting in matrix form gives

$$\begin{bmatrix} R_1(\vec{x}) \\ R_2(\vec{x}) \\ \vdots \\ R_m(\vec{x}) \end{bmatrix} + \begin{bmatrix} \frac{\partial R_1(\vec{x})}{\partial x_1} & \frac{\partial R_1(\vec{x})}{\partial x_2} & \cdots & \frac{\partial R_1(\vec{x})}{\partial x_m} \\ \frac{\partial R_2(\vec{x})}{\partial x_1} & \frac{\partial R_2(\vec{x})}{\partial x_2} & \cdots & \frac{\partial R_2(\vec{x})}{\partial x_m} \\ \vdots & \vdots & & \vdots \\ \frac{\partial R_m(\vec{x})}{\partial x_1} & \frac{\partial R_m(\vec{x})}{\partial x_2} & \cdots & \frac{\partial R_m(\vec{x})}{\partial x_m} \end{bmatrix} \begin{bmatrix} \vec{X}_{1_{n+1}} - \vec{X}_{1_n} \\ \vec{X}_{2_{n+1}} - \vec{X}_{2_n} \\ \vdots \\ \vec{X}_{m_{n+1}} - \vec{X}_{m_n} \end{bmatrix} = 0 \qquad (3.5)$$

Denoting the Jacobian matrix by J,

$$\begin{bmatrix} \frac{\partial R_1(\vec{x})}{\partial x_1} & \frac{\partial R_1(\vec{x})}{\partial x_2} & \cdots & \frac{\partial R_1(\vec{x})}{\partial x_m} \\ \frac{\partial R_2(\vec{x})}{\partial x_1} & \frac{\partial R_2(\vec{x})}{\partial x_2} & \cdots & \frac{\partial R_2(\vec{x})}{\partial x_m} \\ \vdots & \vdots & & \vdots \\ \frac{\partial R_m(\vec{x})}{\partial x_1} & \frac{\partial R_m(\vec{x})}{\partial x_2} & \cdots & \frac{\partial R_m(\vec{x})}{\partial x_m} \end{bmatrix} = J_n \qquad (3.6)$$

$$\vec{x} = \vec{x}_n$$

the equation (3.5) can be written in vector form as

$$\vec{X}_{n+1} = \vec{X}_n - J_n^{-1}\, \vec{R}(\vec{x})_n \qquad (3.7)$$

Equation (3.7) is the recurrence relation for the Newton-Raphson algorithm. This algorithm converges rapidly when the initial guess vector $(\vec{x})$ is in the convergence space. Problems do arise when there are irregularities in the individual equations. The Newton-Raphson method possesses the properties of quadratic convergence and monotonicity for many problems.

To illustrate the application of the Newton-Raphson algorithm, consider the nonlinear algebraic equation (2.8) of chapter II.

$$R_1(\vec{x}) = y_o \cos \lambda t_1 + \dot{y}_{o/\lambda} \sin \lambda t_1 - y_1 = 0$$
$$R_2(\vec{x}) = y_o \cos \lambda t_2 + \dot{y}_{o/\lambda} \sin \lambda t_2 - y_2 = 0$$
$$R_3(\vec{x}) = y_o \cos \lambda t_3 + \dot{y}_{o/\lambda} \sin \lambda t_3 - y_3 = 0$$

The Jacobian matrix is

$$\begin{bmatrix} [\cos \lambda t_1] & \left[-\left(y_o t_1 + \frac{y_o}{2}\right) \sin \lambda t_1 + \frac{\dot{y}_o t_1}{\lambda} \cos \lambda t_1\right] & \left[\frac{1}{\lambda} \sin \lambda t_1\right] \\ [\cos \lambda t_2] & \left[-\left(y_o t_2 + \frac{y_o}{2}\right) \sin \lambda t_2 + \frac{\dot{y}_o t_2}{\lambda} \cos \lambda t_2\right] & \left[\frac{1}{\lambda} \sin \lambda t_2\right] \\ [\cos \lambda t_3] & \left[-\left(y_o t_3 + \frac{y_o}{2}\right) \sin \lambda t_3 + \frac{\dot{y}_o t_3}{\lambda} \cos \lambda t_3\right] & \left[\frac{1}{\lambda} \sin \lambda t_3\right] \end{bmatrix} = J$$

Using the recurrence equation (3.7), $(\vec{x})$ and $\vec{R}(\vec{x})$ are defined by

$$\vec{X} = \begin{bmatrix} y_o \\ \dot{y}_o \\ \xi \end{bmatrix} \quad ; \quad \vec{R}(\vec{x}) = \begin{bmatrix} R_1(\vec{x}) \\ R_2(\vec{x}) \\ R_3(\vec{x}) \end{bmatrix}$$

Two initial guess vectors for $(\vec{x})$ are selected to show the rate of convergence. Presented in table (3.1) are values of y_o, $\dot{y}_o$, and ξ for each iteration of the initial estimate vectors of (0, 1, 1) and (.5, .5, .5). The boundary conditions to be satisfied are

$$y_1 = 1.00 \text{ at } t_1 = .0$$
$$y_2 = 1.36 \text{ at } t_2 = .5$$
$$y_3 = 1.38 \text{ at } t_3 = 1.0$$

TABLE 3.1.- NEWTON-RAPHSON ITERATIVE SOLUTION TO EQUATION (2.8)

State Variable	Iteration					
	1st	2nd	3rd	4th	5th	6th
y_o	.0000	1.0000				
$\dot{y}_o$	1.0000	1.0000				
ξ	1.0000	1.0000				
y_o	.5000	.98409	.741428	1.001728	.9999	1.0000
$\dot{y}_o$	.5000	1.00000	1.00000	1.00000	1.00000	1.0000
ξ	.5000	1.00000	.92949	1.00861	1.00000	1.0000

Newton-Raphson-Kantorovich Method

The Newton-Raphson-Kantorovich algorithm is an extension of the Newton-Raphson technique to "function space" (3). The concept of "function space" as explained by Lanczos (4) involves the replacement of a continuous function by a vector such that the vector describes the continuous function by a set of discrete points.

The development of the Newton-Raphson-Kantorovich algorithm starts with the consideration of the first-order, nonlinear vector differential equation

$$\dot{\vec{Z}} = \vec{f}\,(\vec{Z},t) \tag{3.8}$$

where (Z) is composed of M dependent variables, and t is the independent time variable. Expanding equation (3.8) in function space gives

$$\dot{\vec{Z}}_{n+1} = \vec{f}\,(\vec{Z},t)_n + \left.\frac{\partial \vec{f}(\vec{Z},t)}{\partial \vec{Z}}\right|_n (\vec{Z}_{n+1} - \vec{Z}_n) + \ldots \tag{3.9}$$

Truncating the higher order terms and rewriting gives

$$\dot{\vec{Z}}_{n+1} = \vec{f}\,(\vec{Z},t) + \left.\frac{\partial \vec{f}(\vec{Z},t)}{\partial \vec{Z}}\right|_n \vec{Z}_{n+1} - \left.\frac{\partial f(\vec{Z},t)}{\partial \vec{Z}}\right|_n \vec{Z}_n \tag{3.10}$$

Letting

$$A_n = \left.\frac{\partial \vec{f}(\vec{Z},t)}{\partial \vec{Z}}\right|_n \quad , \quad B_n = \vec{f}(\vec{Z},t)_n - \left.\frac{\partial \vec{f}(\vec{Z},t)}{\partial \vec{Z}}\right|_n Z_n$$

equation (3.10) can be written in the form

$$\dot{\vec{Z}}_{n+1} = A_n \vec{Z}_{n+1} + B_n \tag{3.11}$$

Note that the vectors $(\dot{\vec{Z}})$ and $(\vec{Z})$ are linear with respect to the $(n+1^{st})$ solution. Also the A_n and B_n terms in equation (3.11) are known functions

that are calculated from the known nth solution. Functions A_n and B_n reflect, in the (n + 1st) solution, the nonlinearity of the original differential equation. Using the quasilinear equation (3.11) successive approximation can be made to the nonlinear solutions until the desired accuracy is achieved.

Equation (3.11) is linear with varying coefficients and is easily solved through superposition. The particular solution is a solution of equation (3.11) with appropriate initial conditions. The vector $\vec{Z}^{(o)}$ will denote this particular solution. The homogeneous solutions are governed by

$$\dot{\vec{Z}}^{(i)}_{n+1} = A_n \vec{Z}^{(i)}_{n+1} \quad 1 \leq i \leq M \tag{3.12}$$

These solutions are generated with linearly independent initial condition vectors $Z^{(i)}_{n+1}$ (t = o). The total solution of equation (3.11) is

$$\vec{Z}_{n+1} = \vec{Z}^{(o)}_{n+1} + \sum_{i=1}^{M} \alpha_i \vec{Z}^{(i)}_{n+1} \tag{3.13}$$

where the α_i are taken to insure satisfaction of the boundary conditions.

The initial conditions for the particular and homogeneous solution should show as much information as is known about the desired solution. Thus, the initial condition for the particular solution is taken to be

$$\vec{Z}^{(o)}_{n+1}(o) = \vec{Z}^{(o)}_{n}(o) + \sum_{i=1}^{M} \alpha_i \vec{Z}^{(i)}_{n}(o) \tag{3.14}$$

where the right hand side of equation (3.14) is known from the previous iteration. The initial conditions for the homogeneous solutions are taken to be approximately the same except arbitrary perturbations of elements of these vectors are made to insure

$$\det\left(\vec{Z}^{(i)}_{n+1}(o) \;\middle|\; \ldots \;\middle|\; \vec{Z}^{(m)}_{n+1}(o)\right) \neq 0 \tag{3.15}$$

which is a general Wronskian type statement. These strategies insure $\alpha_k \to o$ as convergence is approached and therefore gives a straightforward indication of convergence.

The number of boundary conditions is denoted by L. The individual boundary values are denoted by S_k, $i \leq k \leq L$. These boundary conditions are on elements of $\vec{Z}_{n+1}$. Introducing an operator Q_k, the boundary conditions are satisfied by

$$S_k = Q_k\left(\vec{Z}^{(o)}_{n+1}(t_k)\right) + \sum_{i=1}^{M} \alpha_i Q_k\left(\vec{Z}^{(i)}_{n+1}(t_k)\right) \qquad L \leq k \leq L \tag{3.16}$$

where t_k is the time at which S_k is measured.

If L = M, the following matrix equation yields the α_i's upon inversion of the coefficient matrix.

$$C\,\vec{\alpha} = \vec{d} \tag{3.17}$$

where

$$C_{ij} = Q_i\left(\vec{Z}^{j}_{n+1}(t_i)\right)$$

defines the elements of the coefficient matrix. The elements of the right hand side vector are defined by the following equation

$$d_i = S_i - Q_i \vec{Z}^{L}_{n+1}(t_i) \tag{3.18}$$

If L > M and all boundary conditions are to be satisfied in a least squares sense, the following matrix equation is used

$$E \vec{\alpha} = \vec{e} \tag{3.19}$$

where

$$E = C^T C \tag{3.20}$$

$$\vec{e} = C^T \vec{d} \tag{3.21}$$

In many disciplines there exist problems where some boundary conditions are to be met exactly and others in some "best fit" sense simultaneously. For these cases the following procedure has been used successfully in some numerical experiments.

The governing matrix equation is taken to be equation (3.19) except the exact equations similar to lines from equation (3.17) are substituted for arbitrary lines of equation (3.19). This lacks in rigor but each least squares equation does involve all "best fit" data. All rigorous procedures to obtain better fits will involve iterate techniques.

Once the vector $(\vec{\alpha})$ is determined, the solution and initial conditions for the nonlinear differential equations can be updated for the next iteration. The updated solution vector can be calculated by equation (3.22). The updated initial condition can be calculated by

$$\vec{Z}(o)\Big|_{n+1} = \vec{Z}^{(o)}(o) + \sum_{i=1}^{M} \alpha_i \vec{Z}^{(i)}(o) \tag{3.22}$$

With the updated solution and initial conditions, the process is repeated until the desired accuracy is achieved.

To illustrate the Newton-Raphson-Kantorovich algorithm, consider the second-order differential equation

$$\ddot{X} + \xi X = 0 \tag{3.23}$$

Where the observed responses or boundary conditions are

$$X = S_1 \quad @ \quad T = T_1$$

$$X = S_2 \quad @ \quad T = T_2$$

$$X = S_3 \quad @ \quad T = T_3$$

The problem now is to determine the initial conditions X_o, $\dot{X}_o$, and ξ such that the specified boundary conditions are satisfied. First reduce equation (3.23) to two first-order differential equations

$$\dot{X} = u$$

$$\dot{u} = - \xi X$$

and, since ξ is a constant,

$$\dot{\xi} = 0$$

such that three first-order differential equations are formed.

$$\dot{\vec{Z}} = \begin{bmatrix} \dot{X} \\ \dot{u} \\ \dot{\xi} \end{bmatrix} \quad ; \quad \vec{Z} = \begin{bmatrix} X \\ u \\ \xi \end{bmatrix} \quad ; \quad \begin{aligned} \dot{Z}_1 &= Z_2 \\ \dot{Z}_2 &= -Z_1 Z_3 \\ \dot{Z}_3 &= 0 \end{aligned}$$

Expanding equation (3.23) according to equation (3.12)

$$\dot{u}_{n+1} = - \xi_n X_n - \xi_n (X_{n+1} - X_n) - X_n (\xi_{n+1} - \xi_n)$$

$$\dot{X}_{n+1} = u_{n+1}$$

$$\dot{u}_{n+1} = - \xi_n X_{n+1} - \xi_{n+1} X_n + X_n \xi_n$$

$$\dot{\xi}_{n+1} = 0$$

Before particular and homogeneous solutions can be determined, an initial guess has to be made for the initial condition vector ($\vec{Z}$). In this case

$$\vec{Z}_{initial} = \begin{bmatrix} X \\ u \\ \xi \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Also, an initial guess for the solution of the governing differential equation (3.23) has to be made. A first approximation can be linear

interpolation between the specified boundary conditions. These assumed solutions and initial conditions are needed to calculate the N subscripted terms in equation (3.11).

Using the initial contitions that result in linearly independent solutions, the homogeneous and nonhomogeneous equations are numerically integrated over the time interval of interest, say from t_i to t_f. Once the homogeneous and nonhomogeneous solutions have been determined, equation (3.13) can be used with standard matrix inversion techniques to determine the unknown α's. After the α's have been determined, the solution to the governing differential equation can be updated by using equation (3.14), also the initial conditions and coefficients can be updated by using equation (3.14). The process is repeated, using the updated solution and initial conditions, until convergence has been achieved.

Presented in table (3.2) are values x_o, and $\dot{x}_o$, and ξ for each iteration for the boundary conditions.

TABLE 3.2 NEWTON-RAPHSON-KANTOROVICH ITERATIVE SOLUTION TO EQUATION (3.23)

		Iteration				
State Vector	Exact Solution	Initial guess	First	Second	Third	Fourth
X_o	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$\dot{X}_o$	.0000	1.0000	-.00273	-.00273	-.00766	-.00766
ξ	π^2	1.0000	11.300	9.8520	9.8712	9.8712

Also presented on figure (3.1) is the initial guess solution and the predicted solution for each iteration. It is noted that the accuracy of the solution doubles with each successive iteration (quadratic convergence).

Discussed in the following chapter are specific applications of the quasilinearization methods to problems of system identification.

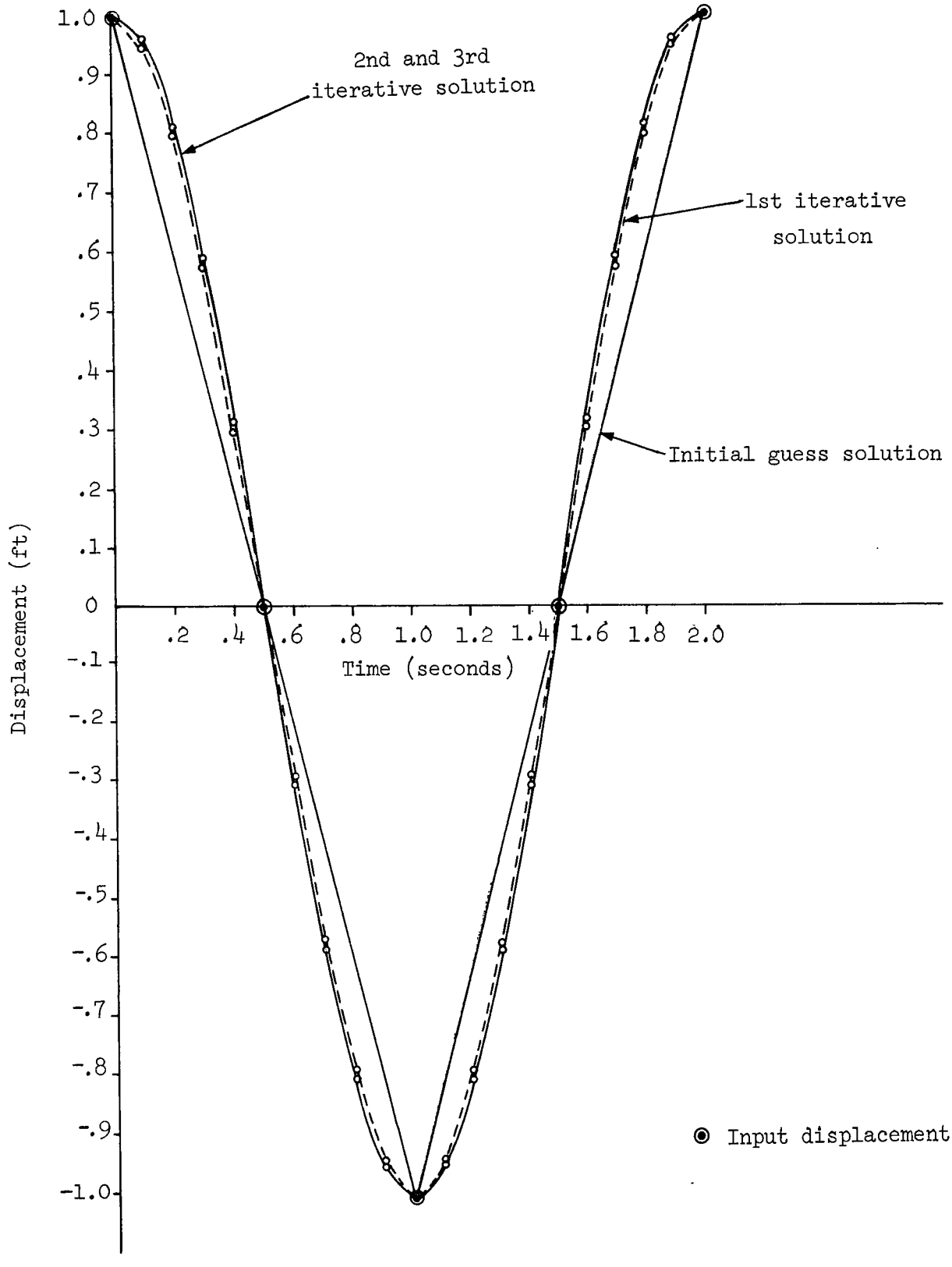


Figure 3.1- Newton-Raphson-Kantorovich iterative solution to equation (3.23)

CHAPTER IV

APPLICATION OF QUASILINEARIZATION

The application of quasilinearization techniques to problems of system identification is discussed in this chapter. The specific examples used are variations of a single-degree-of-freedom system.

The first set of examples will deal with the linear and nonlinear oscillator problem. These examples are designed to show the influence of noisy input boundary conditions on the identification of the unknown parameters of the differential equation.

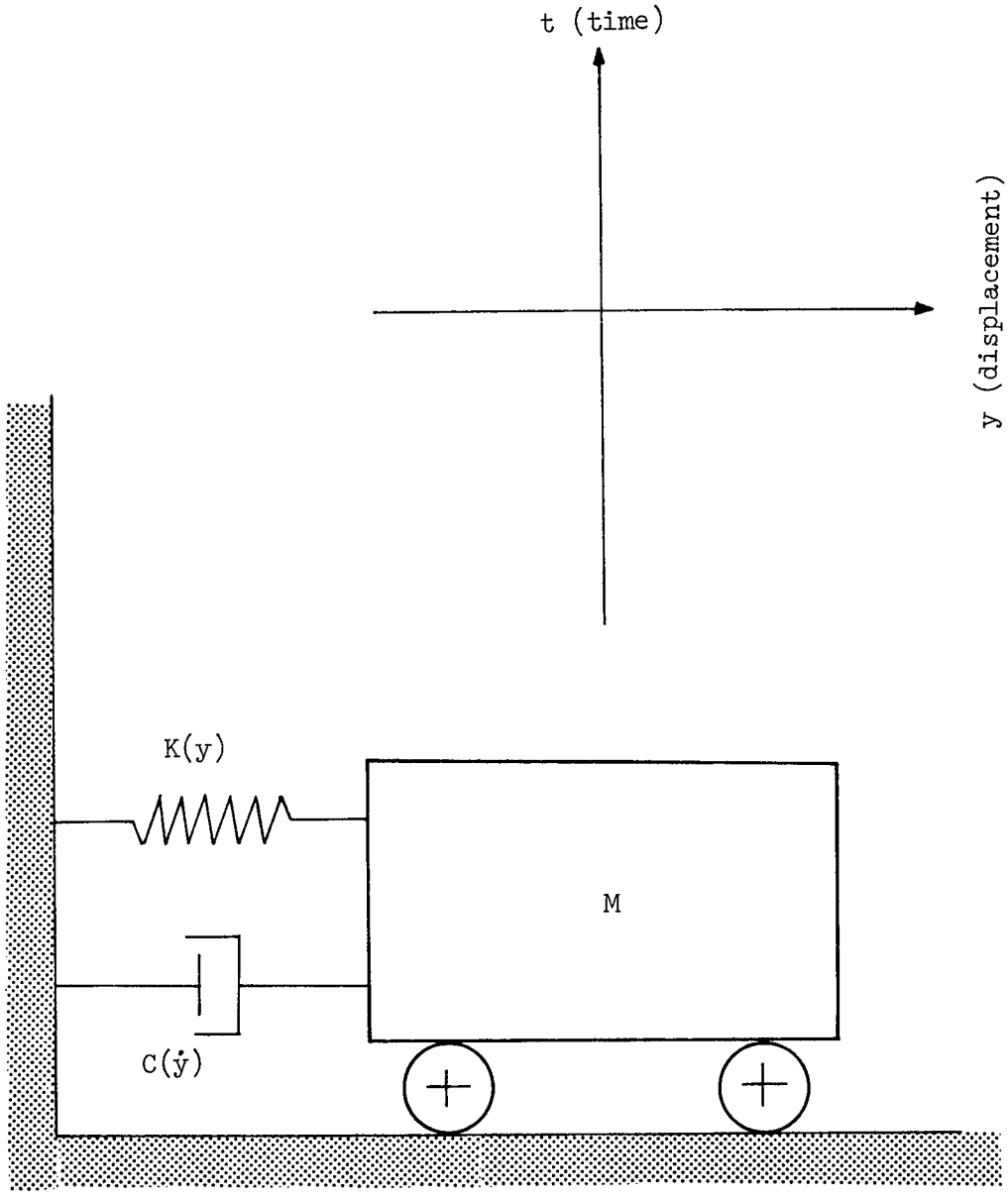
Consider the system shown on figure (4.1) as having a linear and nonlinear spring and a damper characteristic such that the governing differential equation for the nonlinear system is

$$\frac{d^2y}{dt^2} + A\frac{dy}{dt} + B\,y + C\left(\frac{dy}{dt}\right)^3 + D\,y^3 = 0 \qquad (4.1)$$

By letting the parameters C and D equal zero, the nonlinear differential equation (4.1) reduces to the linear equation

$$\frac{dy^2}{dt^2} + A\frac{dy}{dt} + B\,y = 0 \qquad (4.2)$$

To develop precision data for the input boundary conditions, values for the unknown parameters were assigned and equations (4.1) and (4.2) were numerically integrated (5). It is noted that values of velocity and displacement have six-digit accuracy.



M	Mass
K(y)	Spring characteristic
C(ẏ)	Damper characteristic

Figure 4.1- One degree-of-freedom oscillator

Several numerical experiments were conducted to determine the influence of noise and density of the specified boundary conditions on the predicted accuracy of the initial conditions and the parameters A, B, C, and D.

The noise on the specified boundary conditions was generated by rounding off the six-digit accuracy data to four-and-two-digit accuracy. The density of the input boundary conditions was taken from 0.0 to 6.5 seconds in increments of 0.1 seconds (high density) and from 0.0 to 6.5 seconds in increments of 0.5 seconds (low density).

Presented in table (4.1) are comparisons of initial conditions and the parameters A and B for six-, four-, and two-digit accuracy for the specified boundary conditions of the linear differential equation (4.2). There were 14 specified boundary conditions over a range of 0.0 to 6.5 seconds in increments of 0.5 seconds. In table (4.2) is a similar comparison for 66 specified boundary conditions of two-digit accuracy from 0.0 to 6.5 seconds in 0.1 second increments.

Presented in tables (4.3) and (4.4) are similar comparisons for six-, four-, and two-digit accuracy on the specified boundary conditions for the nonlinear differential equation (4.1).

It is noted for these examples that the predicted values for the unknown parameters were better for the linear differential equation with the same accuracy and density of the specified boundary conditions. There was a slight change in the predicted accuracy of the unknown parameters for the linear differential equation when the density was increased from 14 to 66 specified boundary conditions of two-digit accuracy. On the other hand, the increase from 14 to 66 boundary conditions did improve the accuracy of prediction of the unknown parameters of the nonlinear differential equation.

TABLE 4.1.- TABULATED VALUES FOR THE LINEAR DIFFERENTIAL EQUATION PARAMETERS FOR THE LOW DENSITY INPUT BOUNDARY CONDITIONS

Remarks: From t = 0 to t = 6.5

14 Specified Boundary Conditions

Differential equation:

$$\ddot{y} + A\dot{y} + By = 0$$

	Exact	6 digit Input	4 digit Input	2 digit Input
y_o	0.0000	.000000	.000000	-.001870
$\dot{y}_o$	1.00000	1.00002	1.00001	1.00012
A	.100000	.099999	.100030	.102050
B	3.00000	3.00015	3.00015	3.00104

TABLE 4.2.- TABULATED VALUES FOR THE LINEAR DIFFERENTIAL EQUATION PARAMETERS FOR THE HIGH DENSITY INPUT BOUNDARY CONDITIONS

Remarks: From t =0 to t = 6.5

66 Specified Boundary Condition

Differential equation:

$$\ddot{y} + A\dot{y} + By = 0$$

	Exact	2 digit Input
y_o	0.00000	-.000550
$\dot{y}_o$	1.00000	.996580
A	.10000	.0985520
B	3.00000	3.00177

TABLE 4.3.- TABULATED VALUES FOR THE NONLINEAR DIFFERENTIAL EQUATION PARAMETERS FOR THE LOW DENSITY INPUT BOUNDARY CONDITIONS

Remark: 14 Specified Boundary Conditions from t = 0 to t = 6.5

Differential Equation:

$$\ddot{y} + A\dot{y} + B y + C \dot{y}^3 + D y^3 = 0$$

	Exact	6 digit input	4 digit input	2 digit input
y_o	0.00000	-.000001	.000010	-.001527
$\dot{y}_o$	1.00000	1.00002	.999846	1.02469
A	.1000	.100027	.100171	.059754
B	3.0000	3.00015	2.99986	2.99733
C	.2000	.199912	.199583	.303364
D	4.000	4.00063	4.00348	4.03972

TABLE 4.4.- TABULATED VALUES FOR THE NONLINEAR DIFFERENTIAL EQUATION PARAMETERS FOR THE HIGH DENSITY INPUT BOUNDARY CONDITIONS

Remark: 66 Specified Boundary Conditions from t = 0 to t = 6.5

Differential Equation:

$$\ddot{y} + A\dot{y} + B y + C \dot{y}^3 + D y^3 = 0$$

	Exact	2 digit Input
y_o	0.0000	-.000498
$\dot{y}_o$	1.00000	1.00487
A	.10000	.089051
B	3.0000	3.00770
C	.20000	.224580
D	4.00000	3.94190

In table (4.5) are comparisons of predicted differential equation parameters for six- and two-digit accuracy using a density of eight specified boundary conditions over the range of 0.0 to 6.5 seconds. The reduction in density had little effect on the predicted accuracy of the nonlinear differential equation parameters for the six-digit accuracy boundary conditions. However, there was a significant loss of accuracy in the predicted differential equation parameters for the two-digit accurate boundary conditions.

Another application of quasilinearization techniques is in the area of evaluating aerodynamic and rolling friction coefficients on automobiles. The mathematical model used in this thesis to describe the dynamics of an automobile coasting along a level surface is a one-degree-of-freedom system. The external forces acting on this model are aerodynamic forces (F_A) and rolling friction forces (F_f). The governing differential equation is

$$M \frac{d^2y}{dt^2} + F_A + F_f = 0 \tag{4.3}$$

where M is the mass of the vehicle.

Defining the aerodynamic force term to be

$$F_A = 1/2\ \rho\ A_f\ C_D \left(\frac{dy}{dt}\right)^2 \tag{4.4}$$

where ρ = air density ($slug/ft^3$)

A_f = vehicle frontal area (ft^2)

C_D = dimensionless aerodynamic drag coefficient

TABLE 4.5.- TABULATED VALUES FOR THE NONLINEAR DIFFERENTIAL EQUATION PARAMETERS FOR EIGHT SPECIFIED INPUT BOUNDARY CONDITIONS

Remark: 8 Specified Boundary Conditions from t = 0 to t = 6.5			
Differential Equation: $\ddot{y} + A\dot{y} + B y + C \dot{y}^3 + D y^3 = 0$			
	Exact	6 Digit Input	2 Digit Input
y_o	0.0000	.000000	-.000134
$\dot{y}_o$	1.0000	1.000037	1.09951
A	.1000	.099978	-.070835
B	3.0000	3.00024	.667966
C	.2000	.200023	.667966
D	4.0000	4.00037	3.23572

The rolling friction force term will be constant such that the rolling friction coefficient (μ) is defined as

$$\mu = \frac{F_f}{MG} \tag{4.5}$$

where

G = acceleration of gravity (ft/sec^2)

Equation (4.3) then becomes

$$\frac{d^2y}{dt^2} + \frac{1}{2M} \rho A_f C_D \left(\frac{dy}{dt}\right)^2 + \mu G = 0 \tag{4.6}$$

Using equation (4.6) as the governing differential equation for describing the dynamics of a coasting automobile, the aerodynamic drag and the rolling friction coefficients were determined for the Sunbeam Alpine, Porsche, and XKE Jaguar automobiles. Coasting data (6) for these cars are shown in table (4.6) along with the calculated aerodynamic drag and rolling friction coefficients.

Figure (4.2) shows the predicted velocity history and the governing differential equation for the Sunbeam Alpine. The air density in the calculation of the aerodynamic drag coefficients was assumed to be $0.00238\ slug/ft^3$.

TABLE 4.6.- TABULATED COASTING DATA

	Sunbeam Alpine A_f = 17.3 ft^2 C_D = .51 M = 80.43 slugs μ = .0207		Porsche 911 A_f = 18.4 ft^2 C_D = .27 M = 76.39 slugs μ = .0201		XKE Jaguar A_f = 18.2 C_D = .37 M = 105.59 slugs μ = .0198	
Sec	Exp. Measured Velocity (ft/sec)	Predicted Velocity	Exp. Measured Velocity (ft/sec)	Predicted Velocity	Exp. Measured Velocity ft/sec	Predicted Velocity
0	117.3	116.9	117.3	116.9	117.3	117.0
5	104.9	105.5	108.4	108.7	108.5	108.9
10	95.3	95.6	100.9	101.2	101.2	101.4
15	87.3	86.8	93.9	94.2	94.6	94.6
20	79.2	79.0	88.0	87.7	88.6	88.2
25	71.9	72.0	82.1	81.7	82.1	82.2
30	65.7	65.5	76.3	76.0	76.4	76.6
35	59.8	59.7	70.5	70.7	71.6	71.3
40	54.0	54.2	65.4	65.6	66.0	66.3

$$\ddot{y} + 1.303 \times 10^{-4}(\dot{y})^2 + .6673 = 0$$

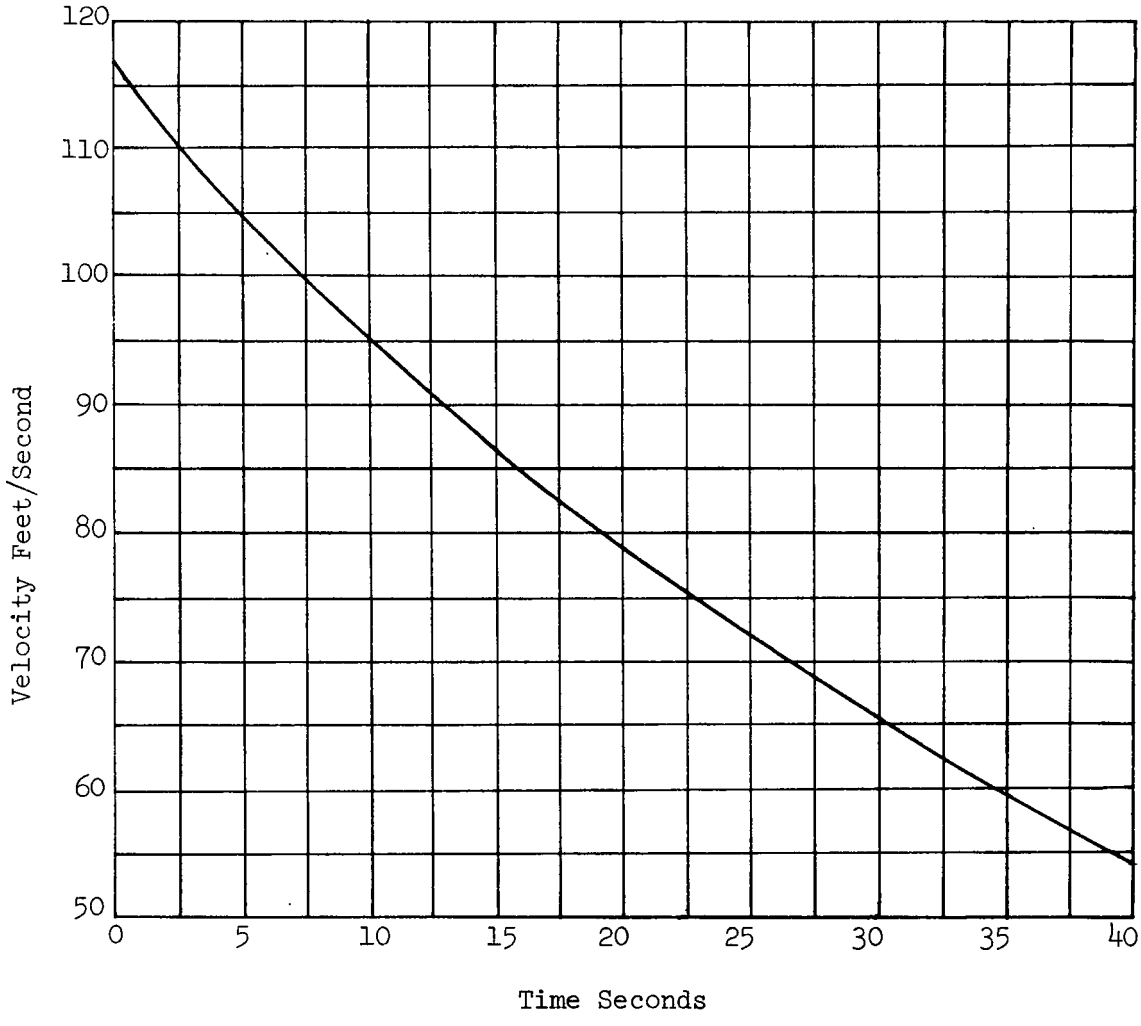


Figure 4.2.- Velocity Decay History of a 1966 Sunbeam Alpine

CHAPTER V

CONCLUSIONS

Quasilinearization theory gives a systematic iterative approach to fitting a governing differential equation(s) to a set of measured observations. The advantage of this technique is that it allows the selection of the state variable that describes the measured observation. These measurements can be made from the standpoint of economy, accuracy, or convenience.

Several numerical experiments were conducted to relate the accuracy of measured observations with the predicted accuracy of the unknown parameters of the governing differential equation. The results presented in chapter IV for the models studied indicate that the accuracy of the predicted parameters of the governing differential equation(s) can be less than the accuracy of the specified boundary conditions. However, the accuracy of the predicted parameters of the governing differential equations can be better than the accuracy of observed responses due to the averaging effect of the least squares procedure. These results indicate that the predicted accuracy of the unknown parameters of the governing differential equation(s) can be improved by increasing the density of the specified boundary conditions as shown by the nonlinear model. The experiments with the linear model indicate that there exists some "optimum" density for a given differential equation. This "optimum" is the point where any increases in density of the specified boundary conditions of the same accuracy results in no increase in the accuracy of the predicted parameters of the differential equation.

The experiments conducted to determine the aerodynamic drag and rolling friction coefficients of a coasting automobile illustrate the application of quasilinearization techniques to solving engineering problems of one-degree-freedom systems. With available data developed outside the laboratory, the coasting dynamics of an automobile was determined. The aerodynamic drag coefficient would be extremely difficult and expensive to experimentally determine in the laboratory. The same can be said for the rolling friction coefficient and again the laboratory experiments could only approximate the field conditions.

The ability of the quasilinearization techniques to fit the governing differential equation(s) to the measured response gives an economical method of determining coefficients and initial conditions.

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