FINITE PROPAGATION SPEEDS IN A THEORY OF LINEAR ISOTROPIC HEAT CONDUCTION

A Thesis

Presented to

the Faculty of the Department of Mathematics

University of Houston

In Partial Fulfillment

of the Requirements for the Degree

Master of Science

by

John M. Finn

August 1971

ACKNOWLEDGMENTS

For many stimulating conversations and incisive criticism during the course of this study, I am indebted to my advisor, Dr. Lewis Wheeler. I also wish to acknowledge support of a different sort, under an NDEA Title IV Fellowship and NSF Grant GK-4893.

FINITE PROPAGATION SPEEDS IN A THEORY OF LINEAR ISOTROPIC HEAT CONDUCTION

An Abstract of a Thesis Presented to
the Faculty of the Department of Mathematics
University of Houston

In Partial Fulfillment
of the Requirements for the Degree
Master of Science

---

by
John M. Finn
August 1971

ABSTRACT

Recently a new theory of heat conduction has appeared in the literature. The *raison d'être* of this theory is that in the classical theory heat propagates in a body with infinite speed. The present paper deals with the linearized form of the theory, which gives rise to an integro-partial differential equation.

Two problems for this equation, called history-value problems, are posed. It is shown that, under certain conditions, solutions to these history-value problems on a bounded region of space are unique. Next, it is shown that if the data of the problem have bounded support, then for any time the solution has bounded support. This proves the hypothesis of finite wave speeds. This result is then used to prove that solutions to the history-value problems on an unbounded region of space are unique.

TABLE OF CONTENTS

Chapter		Page
I.	INTRODUCTION	1
II.	NOTATION	4
III.	COMPARISON WITH THE CLASSICAL THEORY	5
IV.	HISTORY VALUE PROBLEMS. UNIQUENESS FOR BOUNDED REGIONS	12
V.	FINITE PROPAGATION SPEEDS. UNIQUENESS FOR UNBOUNDED REGIONS	28
REFERENCES	. . . .	44

I. INTRODUCTION

The classical theory of heat conduction, as well as other theories dealing with diffusive phenomena, leads in the case of a homogeneous isotropic medium to the parabolic partial differential equation

$$\dot{\theta} = \alpha\Delta\theta. \tag{1.1}$$ [1]

It is a familiar fact that this equation, as is characteristic of those of the parabolic type, has solutions whose physical interpretation implies an infinite propagation speed in the sense that a disturbance in any part of the body will be accompanied by an instantaneous change in the temperature throughout the body. This physically untenable attribute of solutions of (1.1) has stimulated recent researches [1,2] aimed at a theory of heat conduction that gives rise to finite wave speeds.

The theory presented by Gurtin and Pipkin in [1] pertains to materials with memory and as a consequence they propose to supplant (1.1) with an integro-differential equation. By means of linearizations based upon the assumption that $|\theta-\theta_0|$ and $|\nabla\theta|$ are small (θ_0 is a constant

---

[1] Here θ stands for the temperature, α the thermal diffusivity, and Δ the Laplace operator, while the superposed dot refers to partial time differentiation.

temperature), they arrive at

$$c\ddot{\theta}(\underset{\sim}{x},t) + \beta(0)\dot{\theta}(\underset{\sim}{x},t) + \int_0^\infty \beta'(s)\dot{\theta}(\underset{\sim}{x},t-s)ds$$
$$= a(0)\Delta\theta(\underset{\sim}{x},t) + \int_0^\infty a'(s)\Delta\theta(\underset{\sim}{x},t-s)ds + \dot{r}(\underset{\sim}{x},t) \tag{1.2}$$

for the temperature in a homogenous isotropic material. In (1.2), $c > 0$ denotes the heat capacity, β and a the respective energy and heat flux relaxation functions, and r the body heating.

The authors of [1] give a twofold motivation for their claim to have found a mathematical model that predicts finite propagation speeds for thermal disturbances. First, they point out that, in the case of one dimensional temperature fields where β and r are identically zero, (1.2) reduces to the equation governing the longitudinal motion of a viscoelastic bar with mass density c and stress relaxation function a. It is well known that this equation has associated with it the finite propagation speed $[a(0)/c]^{\frac{1}{2}}$. Second, they arrive at finite speeds for the propagation of singular surfaces,[2] through an analysis that obviates the linearization used to obtain (1.2).

The objective of this thesis is to establish by means different from those employed in [1] that the linear theory has the finite propagation speed property. Two

---

[2]This analysis has been extended by Chen [3], who studied the amplitude of temperature rate waves in the one-dimensional case.

approaches are utilized. The first, which is primarily for motivation, is to compare solutions of (1.1) and (1.2) which are of the form of spherical waves with harmonic time dependence. This treatment, however, does not furnish a rigorous confirmation of the finiteness of propagation speeds with sufficient generality to suit our purposes. There is little doubt, however, that more general results can be obtained, notwithstanding some hypotheses of unjustifiable complexity and restrictiveness, by the use of Fourier integrals. The second approach is to adapt the method of energy integrals[3] to (1.2). This technique is used to prove that certain types of problems suggested for (1.2) are well posed from the point of view of uniqueness. These problems, because of the form of (1.2) are boundary-history-value problems, or more simply history-value problems, in contrast with the usual boundary-initial-value problems posed for (1.1). The method is then used to establish the finiteness of the wave speed and delivers an upper bound for it.

---

[3] See, for example, Courant & Hilbert [4, p. 642].

II. NOTATION

We designate by E_n the euclidean space of ordered n-tuples of real numbers. The open spherical ball of radius r in E_n about $\underset{\sim}{x}$ will be denoted by $B_r(\underset{\sim}{x})$, and its bounding surface by $S_r(\underset{\sim}{x})$. Hence

$$
\begin{aligned}
B_r(\underset{\sim}{x}) &= \{\underset{\sim}{y}\varepsilon E_n: \ |\underset{\sim}{x}-\underset{\sim}{y}| < r\}, \\
S_r(\underset{\sim}{x}) &= \{\underset{\sim}{y}\varepsilon E_n: \ |\underset{\sim}{x}-\underset{\sim}{y}| = r\}.
\end{aligned}
\tag{2.1}
$$

An open connected set in E_3 will be called a region. If a region R has the property that for any bounded set $S \subseteq R$, there exists a bounded set R^*, with $S \subseteq R^* \subseteq R$, and the boundary of R^* consists of a finite number of "closed regular surfaces" (in the sense of Kellogg [5, p. 112]), then R will be called a regular region. For a set $S \subseteq E_n$, its interior, boundary, and closure will be written in the usual manner, namely $\overset{\circ}{S}$, ∂S, and $\overline{S}$ respectively.

Finally, we use the conventional notation to designate the degree of smoothness of a function. Thus we write $\phi\varepsilon C^m(S)$ if ϕ is defined and m-times continuously differentiable on $S \subseteq E_n$. If ϕ is continuous on S we write $\phi\varepsilon C(S)$ instead of $\phi\varepsilon C^\circ(S)$. For the restriction of a function ϕ defined on a set S to a subset T of S we write $\phi|T$.

III. COMPARISON WITH THE CLASSICAL THEORY

The purpose of this section is to compare the classical theory of heat conduction with the linear theory presented by Gurtin and Pipkin in [1]. In particular it will be shown that:

a) In either theory if the temperature is time independent, then it is a harmonic function in its spacial coordinates, i.e., if

$$\dot{\theta} = 0, \text{ then } \Delta\theta = 0.$$

b) For a certain family of solutions (with harmonic time dependence) there is an upper bound for the propagation speeds of temperature functions that satisfy (1.2), whereas there is no such bound for the propagation speeds of functions satisfying (1.1).

The field equations of the classical heat conduction theory are:

$$\begin{aligned} \dot{e}(\underset{\sim}{x},t) &= -\nabla\cdot\underset{\sim}{q}(\underset{\sim}{x},t) + r(\underset{\sim}{x},t), \\ \underset{\sim}{q}(\underset{\sim}{x},t) &= -\kappa_0\nabla\theta(\underset{\sim}{x},t), \qquad (3.1) \\ e(\underset{\sim}{x},t) &= b + c\theta(\underset{\sim}{x},t), \end{aligned}$$

where e is the internal energy, $\underset{\sim}{q}$ the heat flux vector, r the body heating, $\kappa_0 > 0$ the thermal conductivity, $c > 0$ the heat capacity, and where b is a constant pertaining to the zero point of the temperature scale.

The field equations established by Gurtin and Pipkin are

$$\begin{aligned} \dot{e}(\underset{\sim}{x},t) &= -\nabla\cdot\underset{\sim}{q}(\underset{\sim}{x},t) + r(\underset{\sim}{x},t), \\ \underset{\sim}{q}(\underset{\sim}{x},t) &= -\int_0^\infty a(s)\nabla\theta(\underset{\sim}{x},t-s)\,ds, \\ e(\underset{\sim}{x},t) &= b + c\theta + \int_0^\infty \beta(s)\theta(\underset{\sim}{x},t-s)\,ds. \end{aligned} \tag{3.2}$$

The relaxation functions β and a were mentioned in the introduction. As in [1], we assume that $\beta(s)$, $a(s)\to 0$ as $s\to\infty$.

By taking the divergence in $(3.1)_2$ and the time derivative in $(3.1)_3$ and combining the three equations (3.1), assuming that $r = 0$, one obtains (1.1), where $\alpha = \kappa_0/c$. The analogous operations on (3.2) give

$$c\dot{\theta}(\underset{\sim}{x},t) + \int_0^\infty \beta(s)\dot{\theta}(\underset{\sim}{x},t-s)\,ds = \int_0^\infty a(s)\Delta\theta(\underset{\sim}{x},t-s)\,ds + r(\underset{\sim}{x},t). \tag{3.3}$$

For convenience, in the remainder of this section we set r equal to zero. This amounts to the assumption that all the heat flowing between the body and the external world occur by conduction through its surface.[1] It is clear that both (1.1) and (3.3) reduce to

---

[1]This rules out phenomena such as heating by electromagnetic radiation (microwave cooking).

$$\Delta\theta = 0$$

if $\dot{\theta} = 0$ and the equilibrium conductivity

$$\kappa = \int_0^\infty a(s)\, ds \tag{3.4}$$

is not zero. Thus the two theories reduce to identical forms in the steady state case.

It is worth mentioning that we cannot make this conclusion from (1.2). Under the hypothesis that $\dot{\theta} = 0$ (1.2) becomes

$$a(0)\Delta\theta + \Delta\theta\int_0^\infty a'(s)ds = 0.$$

But since $a(\infty) = 0$, this becomes merely an identity. This is due to the fact that (1.2) is found by time differentiation in (3.3). This can be seen by putting (3.3) into the form

$$c\dot{\theta}(\underset{\sim}{x},t) + \int_{-\infty}^{t} \beta(t-s)\dot{\theta}(\underset{\sim}{x},s)ds = \int_{-\infty}^{t} a(t-s)\Delta\theta(\underset{\sim}{x},s)ds + r(\underset{\sim}{x},t). \tag{3.5}$$

Differentiation in this equation and another change of variable give (1.2). With this in mind it is easy to anticipate that we could not prove that $\dot{\theta} = 0$ implies $\Delta\theta = 0$. In fact by taking the time derivative in (1.1), we obtain

$$\ddot{\theta} = \alpha\Delta\dot{\theta},$$

from which no such conclusion can be reached.

We may note at this point that (1.1) is, in a sense, a limiting case of (3.3) (with $r = 0$). If we put $\beta(s) = 0$ and $a_\mu(s) = \kappa\mu e^{-\mu s}$ for $s \geq 0$, then by a familiar theorem about Laplace Transforms,[2]

$$\begin{aligned}
&\lim_{\mu\to\infty} \int_0^\infty a_\mu(s)\Delta\theta(\underset{\sim}{x},t-s)ds \\
&\quad = \lim_{\mu\to\infty} \left\{ \kappa\mu \mathcal{L}\,[p_t\theta](\mu) \right\} \qquad (3.6) \\
&\quad = \kappa(p_t\theta)(\underset{\sim}{x},0) = \kappa\Delta\theta(\underset{\sim}{x},t),
\end{aligned}$$

where p_t is the operator defined by

$$(p_t\theta)(\underset{\sim}{x},s) = \Delta\theta(\underset{\sim}{x},t-s).$$

In order to compare propagation characteristics of solutions of (1.1) and (3.3) in a context where time dependence is present, we seek solutions to these equations of the form

$$\theta = u(r,\omega)\cos\,(\omega t-kr), \qquad (3.7)$$

where $r = |\underset{\sim}{x}|$. For (1.1) we find that (3.7) is a solution if

$$u(r,\omega) = Ar^{-1}\exp(-kr) \qquad (3.8)$$

and

$$k = (\omega/2\alpha)^{\frac{1}{2}}. \qquad (3.9)$$

For (3.3) we find that (3.7) is a solution if

---

[2] See [6, p. 350].

$$u(r,\omega) = Ar^{-1}\exp(-k'r), \tag{3.10}$$

$$k = \mathrm{Im}\left\{\frac{i\omega[c + B(\omega)]}{A(\omega)}\right\}^{\frac{1}{2}},$$

and (3.11)

$$k' = \mathrm{Re}\left\{\frac{i\omega[c + B(\omega)]}{A(\omega)}\right\}^{\frac{1}{2}}.$$

The functions A and B are defined by

$$A(\omega) = \int_0^\infty a(s)e^{-i\omega s}ds,$$

$$B(\omega) = \int_0^\infty \beta(s)e^{-i\omega s}ds. \tag{3.12}$$

Note that A is the Fourier Transform of the function

$$\hat{a}(s) = a(s) \qquad s \geq 0$$
$$= 0 \qquad s < 0$$

and that B is the Fourier Transform of the function

$$\hat{\beta}(s) = \beta(s) \qquad s \geq 0$$
$$= 0 \qquad s < 0.$$

The <u>phase velocity</u> associated with a solution of the form (3.7) is defined by

$$V(\omega) = \omega/k. \tag{3.13}$$

So the phase velocity for the solution (3.8)-(3.9) of (1.1) is

$$V_1(\omega) = (2\alpha\omega)^{\frac{1}{2}}, \tag{3.14}$$

whereas the phase velocity corresponding to the solution (3.10)-(3.11) of (3.3) is

$$V_2(\omega) = \frac{1}{\operatorname{Im}\left\{\frac{c + B(\omega)}{-i\omega A(\omega)}\right\}^{\frac{1}{2}}}. \tag{3.15}$$

We immediately note that as $\omega\to\infty$, $V_1(\omega)\to\infty$, indicating that there is no bound for the speed of propagation of solutions of (1.1). Taking the limit of $V_2(\omega)$ as $\omega\to\infty$,

$$V_2(\infty) = \lim_{\omega\to\infty} V_2(\omega) = \frac{1}{\operatorname{Im}\left\{\frac{\lim\limits_{\omega\to\infty}[c + B(\omega)]}{\lim\limits_{\omega\to\infty}[-i\omega A(\omega)]}\right\}^{\frac{1}{2}}}. \tag{3.16}$$

But $\lim\limits_{\omega\to\infty} B(\omega) = 0$ by the Riemann-Lebesque Theorem.[3] Also

$$-i\omega A(\omega) = -i\omega F[\hat{a}] = -a(0)-F[\hat{a}']. \tag{3.17}$$

Using the Riemann-Lebesque Theorem again, we find

$$\lim_{\omega\to\infty} [-i\omega A(\omega)] = -a(0) \tag{3.18}$$

and thus

$$V_2(\infty) = \frac{1}{\operatorname{Im}\{-c/a(0)\}^{\frac{1}{2}}} = \left[\frac{a(0)}{c}\right]^{\frac{1}{2}} \tag{3.19}$$

---

[3] See [7, p. 11]. We assume, for arguments' sake, that a and β are absolutely integrable.

We conclude that for solutions of (3.3) (or of (1.2)) of the form (3.7), the associated phase velocities are bounded, although not necessarily by $[a(0)/c]^{\frac{1}{2}}$.

It is interesting to note that if we let $\beta(s) = 0$ for all $s \geq 0$, (1.2) reduces to

$$c\ddot{\theta}(\underset{\sim}{x},t) = a(0)\Delta\theta(\underset{\sim}{x},t) + \int_0^\infty a'(s)\Delta\theta(\underset{\sim}{x},t-s)ds \quad (3.20)$$

which is the equation governing the one-dimensional longitudinal motion of a viscoelastic bar. As mentioned in [1], the speed of propagation of solutions of this equation is $[a(0)/c]^{\frac{1}{2}}$, the value found in (3.19).

Moreover, a much wider class of solutions to (3.3) can be formed by taking an integral superposition of solutions of the form (3.7), leading to Fourier integrals.

IV. HISTORY-VALUE PROBLEMS

UNIQUENESS FOR BOUNDED REGIONS

In this section we give sufficient conditions for uniqueness of solutions of "history-value problems" for (3.3). The method of proof is based on an identity similar to the "energy identity" for the scalar wave equation.[1] A similar technique of proof has been used by Volterra [8] to prove uniqueness of solutions to an integro differential equation of a different type.

At this stage let us state precisely what we mean by a solution of the first history-value problem.

DEFINITION 4.1. A function θ: $\overline{R}\times(-\infty,\infty) \longrightarrow (-\infty,\infty)$ is a solution of the first history-value problem for the region R corresponding to history ψ: $\overline{R}\times(-\infty,0] \longrightarrow (-\infty,\infty)$, boundary value f: $B_1\times[0,\infty) \longrightarrow (-\infty,\infty)$, normal derivative g: $B_2\times[0,\infty) \longrightarrow (-\infty,\infty)$, where B_1 and B_2 are complementary subsets of ∂R, heat capacity c, relaxation functions a,β: $[0,\infty) \longrightarrow (-\infty,\infty)$ and body heating function r: $R\times[0,\infty) \longrightarrow (-\infty,\infty)$ if

a) $\theta\varepsilon C^2(R\times(-\infty,\infty))\cap C^1(\overline{R}\times(-\infty,\infty))$,

b) $a\varepsilon C^2[0,\infty)$, $\beta\varepsilon C^1[0,\infty)$, $0 < \kappa<\infty$, where $\kappa = \int_0^\infty a(s)ds$,

---

[1] [4, p. 440-445].

and $a(s), \beta(s) \longrightarrow 0$ as $s \longrightarrow \infty$,

c) θ satisfies (3.3) on $R \times (0,\infty)$, and

d) $\theta = \psi$ on $\bar{R} \times (-\infty, 0]$,

$\theta = f$ on $B_1 \times [0,\infty)$,

$$\frac{\partial \theta}{\partial n} = \nabla\theta \cdot \underset{\sim}{n} = g \quad \text{on } B_2{}^* \times [0,\infty).$$

Here $B_2{}^*$ is the set of all points $\underset{\sim}{x} \varepsilon B_2$ at which a normal $\underset{\sim}{n}(\underset{\sim}{x})$ can be defined.

To be brief we will say that θ is a solution to the first history-value problem on R corresponding to ψ, f, g, c, a, β, and r.

We are now in a position to prove

LEMMA 4.1. Let R be a bounded regular region, and assume that θ is a solution to the first history-value problem on R corresponding to $\psi \equiv 0$, f, g, c, a, β, and r. Then

$$\frac{c}{2}\int_R \dot{\theta}^2(\underset{\sim}{x},t)\,d\underset{\sim}{x} + \beta(0)\int_0^t\int_R \dot{\theta}^2(\underset{\sim}{x},\sigma)\,d\underset{\sim}{x}\,d\sigma$$

$$+ \int_0^t\int_0^\sigma\int_R \beta'(\sigma-s)\,\dot{\theta}(\underset{\sim}{x},\sigma)\,\dot{\theta}(\underset{\sim}{x},s)\,d\underset{\sim}{x}\,ds\,d\sigma + \frac{a(0)}{2}\int_R |\nabla\theta(\underset{\sim}{x},t)|^2\,d\underset{\sim}{x}$$

$$+ \gamma(0)\int_0^t\int_R |\nabla\theta(\underset{\sim}{x},\sigma)|^2\,d\underset{\sim}{x}\,d\sigma$$

$$+ \int_0^t\int_0^\sigma\int_R \gamma'(\sigma-s)\,\nabla\theta(\underset{\sim}{x},\sigma)\cdot\nabla\theta(\underset{\sim}{x},s)\,d\underset{\sim}{x}\,ds\,d\sigma$$

$$+ \int_0^t\int_R a'(t-s)\,\nabla\theta(\underset{\sim}{x},t)\cdot\nabla\theta(\underset{\sim}{x},s)\,d\underset{\sim}{x}\,ds$$

$$= a(0)\int_0^t\int_{\partial R} \dot{f}(\underset{\sim}{x},\sigma)g(\underset{\sim}{x},\sigma)d\underset{\sim}{x}d\sigma$$

$$+ \int_0^t\int_0^\sigma\int_{\partial R} a'(\sigma-s)\dot{f}(\underset{\sim}{x},\sigma)g(\underset{\sim}{x},s)d\underset{\sim}{x}dsd\sigma$$

$$+ \int_0^t\int_R\dot{\theta}(\underset{\sim}{x},\sigma)\dot{r}(\underset{\sim}{x},\sigma)d\underset{\sim}{x}d\sigma \qquad \text{for all } t \geq 0, \qquad (4.1)$$

where

$$\gamma = -a'. \qquad (4.2)$$

Proof: By hypothesis, for $(x,\sigma) \in R\times[0,\infty)$

$$c\dot{\theta}(\underset{\sim}{x},\sigma) + \int_0^\sigma\beta(s)\dot{\theta}(\underset{\sim}{x},\sigma-s)ds = \int_0^\sigma a(s)\Delta\theta(\underset{\sim}{x},\sigma-s)ds + r(\underset{\sim}{x},\sigma)$$

or

$$c\dot{\theta}(\underset{\sim}{x},\sigma) + \int_0^\sigma\beta(\sigma-s)\dot{\theta}(\underset{\sim}{x},s)ds = \int_0^\sigma a(\sigma-s)\Delta\theta(\underset{\sim}{x},s)ds + r(\underset{\sim}{x},\sigma). \qquad (4.3)$$

Note that by hypothesis and (4.3) r is differentiable. By differentiation in (4.3) one obtains

$$c\ddot{\theta}(\underset{\sim}{x},\sigma) + \beta(0)\dot{\theta}(\underset{\sim}{x},\sigma) + \int_0^\sigma\beta'(\sigma-s)\dot{\theta}(\underset{\sim}{x},s)ds$$
$$= a(0)\Delta\theta(\underset{\sim}{x},\sigma) + \int_0^\sigma a'(\sigma-s)\Delta\theta(\underset{\sim}{x},s)ds + \dot{r}(\underset{\sim}{x},\sigma). \qquad (4.4)$$

This is (1.2), for a function θ which is zero on $R\times(-\infty,0]$. Now multiply both sides of (4.4) by $\dot{\theta}(\underset{\sim}{x},\sigma)$ and use the identities

$$\dot{\theta}(\underset{\sim}{x},\sigma)\ddot{\theta}(\underset{\sim}{x},\sigma) = \frac{1}{2}\frac{\partial}{\partial\sigma}[\dot{\theta}^2(\underset{\sim}{x},\sigma)]$$

and

$$\dot{\theta}(\underset{\sim}{x},\sigma)\Delta\theta(\underset{\sim}{x},s) = \nabla\cdot[\dot{\theta}(\underset{\sim}{x},\sigma)\nabla\theta(\underset{\sim}{x},s)] - \nabla\dot{\theta}(\underset{\sim}{x},\sigma)\cdot\nabla\theta(\underset{\sim}{x},s)$$

to find that

$$\frac{c}{2}\frac{\partial}{\partial\sigma}[\dot{\theta}^2(\underset{\sim}{x},\sigma)] + \beta(0)\dot{\theta}^2(\underset{\sim}{x},\sigma) + \int_0^\sigma \beta'(\sigma-s)\dot{\theta}(\underset{\sim}{x},\sigma)\dot{\theta}(\underset{\sim}{x},s)ds$$

$$= a(0)\nabla\cdot[\dot{\theta}(\underset{\sim}{x},\sigma)\nabla\theta(\underset{\sim}{x},\sigma)] - a(0)\nabla\dot{\theta}(\underset{\sim}{x},\sigma)\cdot\nabla\theta(\underset{\sim}{x},\sigma)$$

$$+ \int_0^\sigma a'(\sigma-s)\nabla\cdot[\dot{\theta}(\underset{\sim}{x},\sigma)\nabla\theta(\underset{\sim}{x},s)]ds$$

$$- \int_0^\sigma a'(\sigma-s)\nabla\dot{\theta}(\underset{\sim}{x},\sigma)\cdot\nabla\theta(\underset{\sim}{x},s)ds$$

$$+ \dot{\theta}(\underset{\sim}{x},\sigma)\ \dot{r}(\underset{\sim}{x},\sigma) \quad \text{for all } (\underset{\sim}{x},\sigma)\varepsilon R\times[0,\infty). \tag{4.5}$$

By the smoothness hypotheses,

$$\nabla\dot{\theta}(\underset{\sim}{x},\sigma)\cdot\nabla\theta(\underset{\sim}{x},\sigma) = \frac{1}{2}\frac{\partial}{\partial\sigma}|\nabla\theta(\underset{\sim}{x},\sigma)|^2$$

and

$$\frac{\partial}{\partial\sigma}\int_0^\sigma a'(\sigma-s)\nabla\theta(\underset{\sim}{x},\sigma)\cdot\nabla\theta(\underset{\sim}{x},s)ds = a'(0)|\nabla\theta(\underset{\sim}{x},\sigma)|^2$$

$$+ \int_0^\sigma a''\ (\sigma,s)\nabla\theta(\underset{\sim}{x},\sigma)\cdot\nabla\theta(\underset{\sim}{x},s)ds$$

$$+ \int_0^\sigma a'(\sigma-s)\nabla\dot{\theta}(\underset{\sim}{x},\sigma)\cdot\nabla\theta(\underset{\sim}{x},s)ds.$$

Accordingly, (4.5) becomes

$$\frac{c}{2}\frac{\partial}{\partial\sigma}[\dot{\theta}^2(\underset{\sim}{x},\sigma)] + \beta(0)\dot{\theta}^2(\underset{\sim}{x},\sigma) + \int_0^\sigma \beta'(\sigma-s)\dot{\theta}(\underset{\sim}{x},\sigma)\dot{\theta}(\underset{\sim}{x},s)ds$$

$$= a(0)\nabla\cdot[\dot{\theta}(\underset{\sim}{x},\sigma)\nabla\theta(\underset{\sim}{x},\sigma)] + \int_0^\sigma a'(\sigma-s)\nabla\cdot[\dot{\theta}(\underset{\sim}{x},\sigma)\nabla\theta(\underset{\sim}{x},s)]ds$$

$$- \frac{a(0)}{2}\frac{\partial}{\partial\sigma}|\nabla\theta(\underset{\sim}{x},\sigma)|^2 + a'(0)|\nabla\theta(\underset{\sim}{x},\sigma)|^2$$

$$+ \int_0^\sigma a''(\sigma-s)\nabla\theta(\underset{\sim}{x},\sigma)\cdot\nabla\theta(\underset{\sim}{x},s)ds$$

$$- \frac{\partial}{\partial\sigma}\int_0^\sigma a'(\sigma-s)\nabla\theta(\underset{\sim}{x},\sigma)\cdot\nabla\theta(\underset{\sim}{x},s)ds$$

$$+ \dot{\theta}(\underset{\sim}{x},\sigma)\dot{r}(\underset{\sim}{x},\sigma). \tag{4.6}$$

Let $t \geq 0$ and integrate both sides of (4.6) over the set $R\times[0,t]$. By hypothesis it can be seen that $\theta(\underset{\sim}{x},0) = \dot{\theta}(\underset{\sim}{x},0) = 0$ on R. Thus

$$\frac{c}{2}\int_R \dot{\theta}^2(\underset{\sim}{x},t)d\underset{\sim}{x} + \beta(0)\int_0^t\int_R \dot{\theta}^2(\underset{\sim}{x},\sigma)d\underset{\sim}{x}d\sigma$$

$$+ \int_0^t\int_0^\sigma\int_R \beta'(\sigma-s)\dot{\theta}(\underset{\sim}{x},\sigma)\dot{\theta}(\underset{\sim}{x},s)d\underset{\sim}{x}dsd\sigma$$

$$= -\frac{a(0)}{2}\int_R |\nabla\theta(\underset{\sim}{x},t)|^2 d\underset{\sim}{x} - \gamma(0)\int_0^t\int_R |\nabla\theta(\underset{\sim}{x},\sigma)|^2 d\underset{\sim}{x}d\sigma$$

$$- \int_0^t\int_0^\sigma\int_R \gamma'(\sigma-s)\nabla\theta(\underset{\sim}{x},\sigma)\cdot\nabla\theta(\underset{\sim}{x},s)d\underset{\sim}{x}dsd\sigma$$

$$- \int_0^t\int_R a'(t-s)\nabla\theta(\underset{\sim}{x},t)\cdot\nabla\theta(\underset{\sim}{x},s)d\underset{\sim}{x}ds$$

$$+ a(0)\int_0^t\int_R \nabla\cdot[\dot{\theta}(\underset{\sim}{x},\sigma)\nabla\theta(\underset{\sim}{x},\sigma)]d\underset{\sim}{x}d\sigma$$

$$+ \int_0^t\int_0^\sigma a'(\sigma-s)\int_R \nabla\cdot[\dot{\theta}(\underset{\sim}{x},\sigma)\nabla\theta(\underset{\sim}{x},s)]d\underset{\sim}{x}dsd\sigma$$

$$+ \int_0^t\int_R \dot{\theta}(\underset{\sim}{x},\sigma)\dot{r}(\underset{\sim}{x},\sigma)d\underset{\sim}{x}d\sigma,$$

where the substitution $\gamma = -a'$ has been made.

Now using the Divergence Theorem[2], we arrive at (4.1).

With the help of this lemma we are able to prove uniqueness of solutions to the first problem for a bounded region. But first we make the following

DEFINITION 4.2. If θ is a solution to the first history-value problem on a region R corresponding to ψ, f, g, c, a, β, and r, where $\psi \equiv 0$, f, g, and r are zero on $\bar{R} \times [0,T]$ for some $T \geq 0$, then we say that θ is a solution to the first history-value problem on R with null data up to time T, corresponding to c, a, and β. If θ is a solution with null data up to time T for every $T \geq 0$, we say that θ is a solution to the first history-value problem on R with null data, corresponding to c, a, and β.

THEOREM 4.1. Let R be a bounded regular region, and suppose that θ is a solution to the first history-value problem on R with null data up to time $T > 0$, corresponding to c, a,

---

[2]See [5, p. 113].

and β. Suppose further that c, $a(0)$, $\beta(0)$, and $\gamma(0) = -a'(0)$ are positive. Then

$$\theta = 0 \qquad \text{on } \bar{R}\times[0,T].$$

<u>Proof</u>: By Lemma 4.1 and by hypothesis

$$\frac{c}{2} \int_R \dot{\theta}^2(\underset{\sim}{x},t)d\underset{\sim}{x} + \beta(0) \int_0^t \int_R \dot{\theta}^2(\underset{\sim}{x},\sigma)d\underset{\sim}{x}d\sigma$$

$$+ \int_0^t \int_0^\sigma \int_R \beta'(\sigma-s)\dot{\theta}(\underset{\sim}{x},\sigma)\dot{\theta}(\underset{\sim}{x},s)d\underset{\sim}{x}dsd\sigma$$

$$+ \frac{a(0)}{2} \int_R |\nabla\theta(\underset{\sim}{x},t)|^2 d\underset{\sim}{x} + \gamma(0) \int_0^t \int_R |\nabla\theta(\underset{\sim}{x},\sigma)|^2 d\underset{\sim}{x}d\sigma$$

$$+ \int_0^t \int_0^\sigma \int_R \gamma'(\sigma-s)\nabla\theta(\underset{\sim}{x},\sigma)\cdot\nabla\theta(\underset{\sim}{x},s)d\underset{\sim}{x}dsd\sigma$$

$$+ \int_0^t \int_R a'(t-s)\nabla\theta(\underset{\sim}{x},t)\cdot\nabla\theta(\underset{\sim}{x},s)d\underset{\sim}{x}ds = 0 \tag{4.8}$$

for all $t\varepsilon[0,T]$. Define four functions on $[0,T]$;

$$\begin{aligned} f_1(t) &= \int_R \dot{\theta}(x,t)d\underset{\sim}{x}, \\ f_2(t) &= \int_0^t f_1(s)ds, \\ f_3(t) &= \int_R |\nabla\theta(\underset{\sim}{x},t)|^2 d\underset{\sim}{x} \\ f_4(t) &= \int_0^t f_3(s)ds. \end{aligned} \tag{4.9}$$

Note that each of these functions is nonnegative.

Let

$$d = \min\{c/2,\ \beta(0),\ a(0)/2,\ \gamma(0)\} \tag{4.10}$$

and

$$\phi(t) = d \sum_{i=1}^{4} f_i(t). \tag{4.11}$$

By hypothesis, d is positive and therefore ϕ is non-negative on [0,T].

By our continuity hypotheses there exists a positive number K such that

$$|a'(t)|,\ |\beta'(t)|,\ |\gamma'(t)| \leq K \quad \text{on } [0,T].$$

From (4.8) one can conclude that

$$\begin{aligned}\phi(t) \leq K \int_0^t \int_0^\sigma \int_R |\dot{\theta}(\underset{\sim}{x},\sigma)\dot{\theta}(\underset{\sim}{x},s)|d\underset{\sim}{x}dsd\sigma \\ + K \int_0^t \int_0^\sigma \int_R |\nabla\theta(\underset{\sim}{x},\sigma)\cdot\nabla\theta(\underset{\sim}{x},s)|d\underset{\sim}{x}dsd\sigma \\ + K \int_0^t \int_R |\nabla\theta(\underset{\sim}{x},t)\cdot\nabla\theta(\underset{\sim}{x},s)|d\underset{\sim}{x}ds.\end{aligned} \tag{4.12}$$

Since each f_i is continuous on [0, T], there exists a positive number α_0 such that for each i=1,...,4

$$f_i(t) \leq \alpha_0 \quad \text{for all } t\varepsilon[0,T]. \tag{4.13}$$

Define the sequence $\{\alpha_n\}_{n=0}^{\infty}$ by

α_0 : from (4.13)

$$\alpha_{n+1} = \frac{2K}{d}\left[\frac{2T}{(n+2)^2} + \frac{1}{n+2}\right]\alpha_n, \quad n=0,1,2,... \tag{4.14}$$

Note that

$$\lim_{n\to\infty}\left|\frac{\alpha_{n+1}\ t^{n+1}}{\alpha_n\ t^n}\right| = \lim_{n\to\infty}\ \frac{2K}{d}\left[\frac{2T}{(n+2)^2} + \frac{1}{n+2}\right]t = 0.$$

Therefore, by the Ratio Test, $\sum_{n=0}^{\infty} \alpha_n t^n$ converges for all $t\varepsilon[0,T]$, which implies that

$$\lim_{n\to\infty} \alpha_n t^n = 0 \qquad \text{for all } t\varepsilon[0,T].$$

In order to show that each f_i is zero on $[0,T]$ it is sufficient to show that for each $i=1,...,4$ and $n=0,1,2,...$

$$f_i(t) \le \alpha_n t^n \qquad \text{on } [0,T] \tag{4.15}$$

This assertion can be proved by induction. The first step, for $n = 0$, is already shown in (4.13). The inductive hypothesis is that (4.15) holds for $n = j$.

By the Schwarz inequality

$$\int_R |\dot\theta(\underset{\sim}{x},\sigma)\dot\theta(\underset{\sim}{x},s)|d\underset{\sim}{x} \le \left[\int_R \dot\theta^2(\underset{\sim}{x},\sigma)d\underset{\sim}{x}\right]^{\frac{1}{2}}\left[\int_R \dot\theta^2(\underset{\sim}{x},s)d\underset{\sim}{x}\right]^{\frac{1}{2}}$$

$$= [f_1(\sigma)f_1(s)]^{\frac{1}{2}},$$

so that the inductive hypothesis implies

$$\int_0^t\int_0^\sigma\int_R |\dot\theta(\underset{\sim}{x},\sigma)\dot\theta(\underset{\sim}{x},s)|d\underset{\sim}{x}dsd\sigma \le \alpha_j\int_0^t\int_0^\sigma \sigma^{j/2}s^{j/2}dsd\sigma$$

$$= \frac{2\alpha_j t^{j+2}}{(j+2)^2} \le \frac{2\alpha_j T}{(j+2)^2}\, t^{j+1}. \tag{4.16}$$

Similarly

$$\int_0^t\int_0^\sigma\int_R |\nabla\theta(\underset{\sim}{x},\sigma)\cdot\nabla\theta(\underset{\sim}{x},s)|d\underset{\sim}{x}dsd\sigma \le \frac{2\alpha_j T}{(j+2)^2}\, t^{j+1}. \tag{4.17}$$

Also, since

$$\int_R [s^{j/2}|\nabla\theta(\underset{\sim}{x},t)| - t^{j/2}|\nabla\theta(\underset{\sim}{x},s)|]^2 d\underset{\sim}{x} \ge 0$$

one finds that

$$\int_R |\nabla\theta(\underset{\sim}{x},t)\cdot\nabla\theta(\underset{\sim}{x},s)|d\underset{\sim}{x} \le \int_R |\nabla\theta(\underset{\sim}{x},t)||\nabla\theta(\underset{\sim}{x},s)|d\underset{\sim}{x}$$

$$\le \frac{1}{2}\int_R (\frac{s}{t})^{j/2}|\nabla\theta(\underset{\sim}{x},t)|^2 d\underset{\sim}{x} + \frac{1}{2}\int_R (\frac{t}{s})^{j/2}|\nabla\theta(\underset{\sim}{x},s)|^2 d\underset{\sim}{x}$$

$$= \frac{1}{2}(\frac{s}{t})^{j/2} f_3(t) + \frac{1}{2}(\frac{t}{s})^{j/2} f_3(s). \tag{4.18}$$

Again by the inductive hypothesis

$$\int_R |\nabla\theta(\underset{\sim}{x},t)\cdot\nabla\theta(\underset{\sim}{x},s)|\,d\underset{\sim}{x} \le \alpha_j t^{j/2} s^{j/2}.$$

and hence

$$\int_0^t \int_R |\nabla\theta(\underset{\sim}{x},t)\cdot\nabla\theta(\underset{\sim}{x},s)|\,d\underset{\sim}{x}\,ds \le \alpha_j \int_0^t t^{j/2} s^{j/2}\,ds$$

$$= \frac{2\alpha_j}{j+2}\, t^{j+1}. \tag{4.19}$$

By (4.12), (4.16), (4.17), and (4.19)

$$\phi(t) \le \frac{4KT\alpha_j}{(j+2)^2} t^{j+1} + \frac{2K\alpha_j}{j+2}\, t^{j+1}. \tag{4.20}$$

It follows that for $i=1,...,4$

$$f_i(t) \le \frac{2K}{d}\left[\frac{2T}{(j+2)^2} + \frac{1}{j+2}\right]\alpha_j t^{j+1} = \alpha_{j+1} t^{j+1}. \tag{4.21}$$

Therefore, $f_1 = f_2 = 0$ on $[0,T]$, which by (4.9) and the smoothness hypotheses implies that

$$\dot\theta = 0 \qquad \text{and} \qquad \nabla\theta = \underset{\sim}{0}$$

on $\bar{R}\times[0,T]$. Therefore θ is a constant on $\bar{R}\times[0,T]$, and since $\theta(\cdot,0) = 0$ on $\bar{R}$, one concludes that

$$\theta = 0 \qquad \text{on} \qquad \bar{R}\times[0,T],$$

the desired result.

COROLLARY 1. Let R be an unbounded regular region, and suppose that θ is a solution to the first history-value problem on R with null data, corresponding to c, a, and β. Suppose further that c, a(0), $\beta(0)$, and $\gamma(0)$ are positive. Then

$$\theta = 0 \qquad \text{on} \qquad \bar{R}\times[0,\infty).$$

Proof: See Definition 4.2.

COROLLARY 2. Let R be a bounded regular region and suppose that θ_1, θ_2 are solutions of the first history-value problem on R, corresponding to ψ, f_1, g_1, c, a, β, r_1, and ψ, f_2, g_2, c, a, β, r_2 respectively. If c, a(0), $\beta(0)$, and $\gamma(0)$ are positive and if $f_1 = f_2$, $g_1 = g_2$, and $r_1 = r_2$ on $[0,T]$, then $\theta_1 = \theta_2$ on $\bar{R}\times[0,T]$. If, under the same hypotheses $f_1 = f_2$, $g_1 = g_2$, and $r_1 = r_2$ on $[0,\infty)$, then $\theta_1 = \theta_2$ on $\bar{R}\times[0,\infty)$.

Proof: Let $\theta = \theta_1 - \theta_2$. By hypothesis and the linearity of (3.3), θ is a solution to the first history-value problem on R with null data on $[0,T]$ (or on $[0,\infty)$), corresponding to c, a, and β. Using Theorem 4.1 or Corollary 1, the proof is immediate.

Physically it would perhaps be more meaningful to pose history-value problems for (3.3) giving the normal component of the heat flux vector, instead of the normal derivative of θ (these differ only by the multiplicative constant $-\kappa_0$ in the classical theory), on B_2. By (3.2) these are related by

$$q_n(\underset{\sim}{x},t) = \underset{\sim}{q}(\underset{\sim}{x},t)\cdot\underset{\sim}{n}(\underset{\sim}{x}) = -\int_0^\infty a(s)\frac{\partial\theta}{\partial n}(\underset{\sim}{x},t-s)\,ds. \qquad (4.22)$$

DEFINITION 4.3. A function θ: $\bar{R}\times(-\infty,\infty)\to(-\infty,\infty)$ is a solution of the second history-value problem for the region R corresponding to history ψ: $\bar{R}\times(-\infty,0]\to(-\infty,\infty)$, boundary value f: $B_1\times[0,\infty)\to(-\infty,\infty)$, normal component of heat flux h: $B_2\times[0,\infty)\to(-\infty,\infty)$, where B_1 and B_2 are complementary parts of ∂R, heat capacity c, relaxation functions a,β: $[0,\infty)\to(-\infty,\infty)$, and body heating function r: $\bar{R}\times[0,\infty)\to(-\infty,\infty)$ if

(a) $\theta \in C^2(R\times(-\infty,\infty)) \cap C^1(\bar{R}\times(-\infty,\infty))$,

(b) $a \in C^2[0,\infty)$, $\beta \in C^1[0,\infty)$, $0 < \kappa < \infty$, and $a(s)$, $\beta(s)\to 0$ as $s \to \infty$,

(c) θ satisfies (3.3) on $R\times(0,\infty)$, and

(d) $\theta = \psi$ on $\bar{R}\times(-\infty,0]$,

$\theta = f$ on $B_1\times[0,\infty)$,

$q_n = h$ on $B_2{}^*\times[0,\infty)$.

Here q_n is given by (4.22) and $B_2{}^*$ is the set of points $\underset{\sim}{x} \in B_2$ at which a normal $\underset{\sim}{n}(\underset{\sim}{x})$ can be defined.

To be brief we will say that θ is a solution to the second history-value problem on R corresponding to ψ, f, h, c, a, β, and r. If ψ, f, h, and r are identically zero on [0,T], we will say that θ is a solution to the second history-value problem on R with null data up to T, corresponding to c, a, and β.

The first two terms on the right-hand side of (4.1) can be put into the form

$$\int_0^t \int_{\partial R} \dot{\theta}(\underset{\sim}{x},\sigma)\ [a(0)\frac{\partial\theta}{\partial n}(\underset{\sim}{x},\sigma) + \int_0^\sigma a'(\sigma-s)\frac{\partial\theta}{\partial n}(\underset{\sim}{x},s)ds]d\underset{\sim}{x}d\sigma. \tag{4.23}$$

If, as in the case of Theorem 4.1, we assume that

$\theta(\underset{\sim}{x},t) = 0$ for all $t \leq 0$, then (4.22), after a substitution, yields

$$q_n(\underset{\sim}{x},t) = -\int_0^t a(t-s)\frac{\partial\theta}{\partial n}(\underset{\sim}{x},s)ds. \tag{4.24}$$

From (4.24) we find by differentiation that

$$\dot{q}_n(\underset{\sim}{x},t) = -a(0)\frac{\partial\theta}{\partial n}(\underset{\sim}{x},t) - \int_0^t a'(t-s)\frac{\partial\theta}{\partial n}(\underset{\sim}{x},s)ds. \tag{4.25}$$

Therefore we recognize the expression in (4.23) to be

$$-\int_0^t \int_{\partial R} \dot{\theta}(\underset{\sim}{x},\sigma)\dot{q}_n(\underset{\sim}{x},\sigma)d\underset{\sim}{x}d\sigma. \tag{4.26}$$

Using this fact, a uniqueness theorem for the second history-value problem, on a bounded regular region, can be proved.

THEOREM 4.2. Let R be a bounded regular region and suppose that θ_1, θ_2 are solutions to the second history-value problem on R corresponding to ψ, f_1, h_1, c, a, β, r_1 and ψ, f_2, h_2, c, a, β, r_2 respectively. If c, a(0), $\beta(0)$, and $\gamma(0)$ are positive and $f_1 = f_2$, $h_1 = h_2$, and $\gamma_1 = \gamma_2$ on $[0,T]$ (on $[0,\infty)$), then $\theta_1 = \theta_2$ on $[0,T]$ (on $[0,\infty)$).

Proof: Let $\theta = \theta_1 - \theta_2$. Then θ is a solution to the second history-value problem with null data. By replacing the first two terms on the right-hand side of (4.1) with (4.26), we see that (4.8) holds. As in Theorem 4.1, then, $\theta = 0$.

Remark: Theorem 4.2 could also have been proved by noting that (4.24), the hypothesis that a is not identically zero ($\kappa > 0$), and Titchmarsh's Theorem on convolutions,[3] imply that if

$$q_n = 0 \qquad \text{on} \qquad B_2 \times [0,\infty),$$

then

$$\frac{\partial \theta}{\partial n} = 0 \qquad \text{on} \qquad B_2 \times [0,\infty).$$

[3] See [9, p. 22].

That is, a solution to the second history-value problem with null data is a solution to the first history-value problem with null data.

V. FINITE PROPAGATION SPEEDS
UNIQUENESS FOR UNBOUNDED REGIONS

The method used here to prove the finite propagation speed hypothesis depends upon a lemma which is a generalization of Lemma 4.1. This lemma has a counterpart in a result given by Zaremba [10], and discussed by Fritz John in [11].

LEMMA 5.1. Let R be a regular region in E_3 (not necessarily bounded) and assume that θ is a solution to the first history-value problem on R corresponding to ψ, f, g, c, a, β, and r. Also suppose that $\tau \varepsilon C^1(\overline{R})$ is a given function such that the set

$$P(\tau) = \{\underset{\sim}{x} \varepsilon R \,|\, \tau(\underset{\sim}{x}) > 0\} \tag{5.1}$$

is bounded, and that $\psi|(P(\tau)\times(-\infty,0]) = 0$. Then

$$\frac{c}{2}\int_R \dot{\theta}^2(\underset{\sim}{x},\tau(\underset{\sim}{x}))d\underset{\sim}{x} + \beta(0)\int_R\int_0^{\tau(\underset{\sim}{x})}\dot{\theta}^2(\underset{\sim}{x},\sigma)d\sigma d\underset{\sim}{x}$$

$$+ \int_R\int_0^{\tau(\underset{\sim}{x})}\int_0^{\sigma}\beta'(\sigma-s)\dot{\theta}(\underset{\sim}{x},\sigma)\dot{\theta}(\underset{\sim}{x},s)dsd\sigma d\underset{\sim}{x}$$

$$+ a(0)\int_R\dot{\theta}(\underset{\sim}{x},\tau(\underset{\sim}{x}))\nabla\theta(\underset{\sim}{x},\tau(\underset{\sim}{x}))\cdot\nabla\tau(\underset{\sim}{x})d\underset{\sim}{x}$$

$$+\int_R \int_0^{\tau(\underset{\sim}{x})} a'(\tau(\underset{\sim}{x})-s)\dot{\theta}(\underset{\sim}{x},\tau(\underset{\sim}{x}))\nabla\theta(\underset{\sim}{x},s)\cdot\nabla\tau(\underset{\sim}{x})\,ds\,d\underset{\sim}{x}$$

$$+\frac{a(0)}{2}\int_R |\nabla\theta(\underset{\sim}{x},\tau(\underset{\sim}{x}))|^2 d\underset{\sim}{x} + \gamma(0)\int_R \int_0^{\tau(\underset{\sim}{x})} |\nabla\theta(\underset{\sim}{x},\sigma)|^2 d\sigma\, d\underset{\sim}{x}$$

$$+\int_R \int_0^{\tau(\underset{\sim}{x})} \int_0^{\sigma} \gamma'(\sigma-s)\nabla\theta(\underset{\sim}{x},\sigma)\cdot\nabla\theta(\underset{\sim}{x},s)\,ds\,d\sigma\,d\underset{\sim}{x}$$

$$+\int_R \int_0^{\tau(\underset{\sim}{x})} a'(\tau(x)-s)\nabla\theta(\underset{\sim}{x},\tau(\underset{\sim}{x}))\cdot\nabla\theta(\underset{\sim}{x},s)\,ds\,d\underset{\sim}{x}$$

$$= a(0)\int_{\partial R} \int_0^{\tau(\underset{\sim}{x})} \dot{f}(\underset{\sim}{x},\sigma)g(\underset{\sim}{x},\sigma)\,d\sigma\,d\underset{\sim}{x}$$

$$+\int_{\partial R} \int_0^{\tau(\underset{\sim}{x})} \int_0^{\sigma} a'(\sigma-s)\dot{f}(\underset{\sim}{x},\sigma)g(\underset{\sim}{x},s)\,ds\,d\sigma\,d\underset{\sim}{x}$$

$$+\int_R \int_0^{\tau(\underset{\sim}{x})} \dot{\theta}(\underset{\sim}{x},\sigma)\dot{r}(\underset{\sim}{x},\sigma)\,d\sigma\,d\underset{\sim}{x} \tag{5.2}$$

where again

$$\gamma = -a'. \tag{5.3}$$

<u>Remark</u>: Lemma 4.1 is a special case of this result, when R is bounded and $\tau(\underset{\sim}{x})$ has the constant value T.

<u>Proof</u>: As in the proof of Lemma 4.1, for $\underset{\sim}{x} \varepsilon P(\tau)$

$$\frac{c}{2}\frac{\partial}{\partial\sigma}[\dot{\theta}^2(\underset{\sim}{x},\sigma)] + \beta(0)\dot{\theta}^2(\underset{\sim}{x},\sigma) + \int_0^{\sigma}\beta'(\sigma-s)\dot{\theta}(\underset{\sim}{x},\sigma)\dot{\theta}(\underset{\sim}{x},s)\,ds$$

$$= a(0)\nabla\cdot[\dot{\theta}(\underset{\sim}{x},\sigma)\nabla\theta(\underset{\sim}{x},\sigma)] + \int_0^{\sigma} a'(\sigma-s)\nabla\cdot[\dot{\theta}(\underset{\sim}{x},\sigma)\nabla\theta(\underset{\sim}{x},s)]\,ds$$

$$- \frac{a(0)}{2} \frac{\partial}{\partial\sigma} |\nabla\theta(\underset{\sim}{x},\sigma)|^2 - \gamma(0)|\nabla\theta(\underset{\sim}{x},\sigma)|^2$$

$$- \int_0^\sigma \gamma'(\sigma-s)\nabla\theta(\underset{\sim}{x},\sigma)\cdot\nabla\theta(\underset{\sim}{x},s)ds$$

$$- \frac{\partial}{\partial\sigma}\int_0^\sigma a'(\sigma-s)\nabla\theta(\underset{\sim}{x},\sigma)\cdot\nabla\theta(\underset{\sim}{x},s)ds + \dot{\theta}(\underset{\sim}{x},\sigma)\dot{r}(\underset{\sim}{x},\sigma). \quad (4.6)$$

Now integrate with respect to σ from 0 to $\tau(\underset{\sim}{x})$. The smoothness hypotheses imply that

$$\int_0^{\tau(\underset{\sim}{x})} \nabla\cdot[\dot{\theta}(\underset{\sim}{x},\sigma)\nabla\theta(\underset{\sim}{x},\sigma)]d\sigma = \nabla\cdot\int_0^{\tau(\underset{\sim}{x})} \dot{\theta}(\underset{\sim}{x},\sigma)\nabla\theta(\underset{\sim}{x},\sigma)d\sigma$$

$$- \dot{\theta}(\underset{\sim}{x},\tau(\underset{\sim}{x}))\nabla\theta(\underset{\sim}{x},\tau(\underset{\sim}{x}))\cdot\nabla\tau(\underset{\sim}{x})$$

and that

$$\int_0^{\tau(\underset{\sim}{x})}\int_0^\sigma a'(\sigma-s)\nabla\cdot[\dot{\theta}(\underset{\sim}{x},\sigma)\nabla\theta(\underset{\sim}{x},s)]dsd\sigma$$

$$= \int_0^{\tau(\underset{\sim}{x})} \nabla\cdot\int_0^\sigma \dot{\theta}(\underset{\sim}{x},\sigma)\nabla\theta(\underset{\sim}{x},s)dsd\sigma$$

$$= \nabla\cdot\int_0^{\tau(\underset{\sim}{x})}\int_0^\sigma a'(\sigma-s)\dot{\theta}(\underset{\sim}{x},\sigma)\nabla\theta(\underset{\sim}{x},s)dsd\sigma$$

$$- \int_0^{\tau(\underset{\sim}{x})} a'(\tau(\underset{\sim}{x})-s)\dot{\theta}(\underset{\sim}{x},\tau(\underset{\sim}{x}))\nabla\theta(\underset{\sim}{x},s)\cdot\nabla\tau(\underset{\sim}{x})ds$$

and that $\theta(\underset{\sim}{x},0) = \dot{\theta}(\underset{\sim}{x},0) = 0$ for all $\underset{\sim}{x}\varepsilon P(\tau)$.

Accordingly,

$$\frac{c}{2}\dot{\theta}^2(\underset{\sim}{x},\tau(\underset{\sim}{x})) + \beta(0)\int_0^{\tau(\underset{\sim}{x})}\dot{\theta}^2(\underset{\sim}{x},\sigma)d\sigma$$

$$+ \int_0^{\tau(\underset{\sim}{x})}\int_0^{\sigma}\beta'(\sigma-s)\dot{\theta}(\underset{\sim}{x},\sigma)\dot{\theta}(\underset{\sim}{x},s)dsd\sigma$$

$$= a(0)\nabla\cdot\int_0^{\tau(\underset{\sim}{x})}\dot{\theta}(\underset{\sim}{x},\sigma)\nabla\theta(\underset{\sim}{x},\sigma)d\sigma$$

$$- a(0)\dot{\theta}(\underset{\sim}{x},\tau(\underset{\sim}{x}))\nabla\theta(\underset{\sim}{x},\tau(\underset{\sim}{x}))\cdot\nabla\tau(\underset{\sim}{x})$$

$$+ \nabla\cdot\int_0^{\tau(\underset{\sim}{x})}\int_0^{\sigma}a'(\sigma-s)\dot{\theta}(\underset{\sim}{x},\sigma)\nabla\theta(\underset{\sim}{x},s)dsd\sigma$$

$$- \int_0^{\tau(\underset{\sim}{x})}a'(\tau(\underset{\sim}{x})-s)\dot{\theta}(\underset{\sim}{x},\tau(\underset{\sim}{x}))\nabla\theta(\underset{\sim}{x},s)\cdot\nabla\tau(\underset{\sim}{x})ds$$

$$- \frac{a(0)}{2}|\nabla\theta(\underset{\sim}{x},\tau(\underset{\sim}{x}))|^2 - \gamma(0)\int_0^{\tau(\underset{\sim}{x})}|\nabla\theta(\underset{\sim}{x},\sigma)|^2d\sigma$$

$$- \int_0^{\tau(\underset{\sim}{x})}\int_0^{\sigma}\gamma'(\sigma-s)\nabla\theta(\underset{\sim}{x},\sigma)\cdot\nabla\theta(\underset{\sim}{x},s)dsd\sigma$$

$$- \int_0^{\tau(\underset{\sim}{x})}a'(\tau(\underset{\sim}{x})-s)\nabla\theta(\underset{\sim}{x},\tau(\underset{\sim}{x}))\cdot\nabla\theta(\underset{\sim}{x},s)ds$$

$$+ \int_0^{\tau(\underset{\sim}{x})}\dot{\theta}(\underset{\sim}{x},\sigma)\dot{r}(\underset{\sim}{x},\sigma)d\sigma \qquad (5.4)$$

Now since each of the terms in (5.4) has bounded support, integration over R can be performed. Using the divergence theorem the result follows at once.

By Lemma 5.1 we can show that a perturbation in the data that occurs on a bounded part of the region propagates at a finite speed. More precisely, we can prove the following:

THEOREM 5.1. Suppose that R is an unbounded regular region in E_3, and that θ is a solution to the first history-value problem on R, corresponding to ψ, f, g, c, a, β, and r, with c, a(0), $\beta(0)$, and $\gamma(0) = -a'(0)$ positive. Suppose further that for any $t > 0$ there exists a bounded set $\Lambda_t \subset \bar{R}$ such that

$$\psi = 0 \qquad \text{on} \qquad (\bar{R}-\Lambda_t)\times(-\infty,0],$$

$$f = 0 \qquad \text{on} \qquad (B-\Lambda_t)\times[0,t],$$

$$g = 0 \qquad \text{on} \qquad (B_2^*-\Lambda_t)\times[0,t],$$

and

$$r = 0 \qquad \text{on} \qquad (\bar{R}-\Lambda_t)\times[0,t].$$

Then there exists a bounded set $\Omega_t \subset \bar{R}$ such that

$$\theta = 0 \qquad \text{on} \qquad (\bar{R}-\Omega_t)\times[0,t]. \tag{5.5}$$

Proof: Let $t > 0$. By assumption a', β', and γ' (γ defined in (5.2)) are continuous on $[0,t]$. Hence there exists $K > 0$ such that

$$|a'(s)|, \ |\beta'(s)|, \ |\gamma'(s)| \le K \qquad \text{on } [0,t] \tag{5.6}$$

Let

$$u_0 > \max\{1, \frac{K + a(0)}{c}\}. \tag{5.7}$$

Now choose $\delta > 0$ such that

$$\begin{aligned} &\partial R \cup \Lambda_t \subseteq \overline{B_\delta(\underset{\sim}{0})} \text{ if } \partial R \text{ is bounded} \\ &\Lambda_t \subseteq \overline{B_\delta(\underset{\sim}{0})} \text{ if } \partial R \text{ is unbounded.} \end{aligned} \tag{5.8}$$

Consider the set

$$\Omega_t = \overline{R} \cap \overline{B_{\delta+u_0 t}(\underset{\sim}{0})}. \tag{5.9}$$

Since Ω_t is closed, $\overline{R}-\Omega_t$ is contained in the closure of $R-\Omega_t$. In order to show that (5.5) holds, we let $\underset{\sim}{x}\varepsilon\overline{R}-\Omega_t$. For each $\lambda\varepsilon[0,t]$ define

$$\tau_\lambda(\underset{\sim}{y}) = \lambda-u_0^{-1}|\underset{\sim}{y}-\underset{\sim}{x}|. \tag{5.10}$$

Clearly $\tau_\lambda \varepsilon\ C^1(\overline{R}-\{\underset{\sim}{x}\}) \cap C(\overline{R})$,

$$\tau_\lambda(\underset{\sim}{y}) \leq \lambda \leq t \quad \text{for all } \underset{\sim}{y}\varepsilon\overline{R}, \tag{5.11}$$

and $\quad |\nabla\tau_\lambda(\underset{\sim}{y})| = u_0^{-1} \quad$ for all $\underset{\sim}{y}\varepsilon\overline{R}-\{\underset{\sim}{x}\}$.

Also, recalling (5.1),

$$P(\tau_\lambda) = \overline{R} \cap B_{u_0\lambda}(\underset{\sim}{x}) \text{ for all } \lambda\varepsilon[0,t).$$

Since $\overline{B_\delta(\underset{\sim}{0})} \cap B_{u_0\lambda}(\underset{\sim}{x}) = \emptyset$ for all $\lambda\varepsilon[0,t]$,

it follows that

$$\begin{aligned} P(\tau_\lambda) &\subseteq \overline{R}-(\Lambda_t \cup \partial R) && \text{if } \partial R \text{ is bounded} \\ P(\tau_\lambda) &\subseteq \overline{R}-\Lambda_t && \text{if } \partial R \text{ is unbounded.} \end{aligned} \tag{5.12}$$

By hypothesis, (5.12), and $(5.11)_2$ it is clear that

$$\int_{\partial R} \int_0^{\tau_\lambda(\underset{\sim}{y})} \dot{f}(\underset{\sim}{y},\sigma)g(\underset{\sim}{y},\sigma)d\sigma d\underset{\sim}{y} = 0,$$

$$\int_{\partial R} \int_0^{\tau_\lambda(\underset{\sim}{y})} \int_0^{\sigma} a'(\sigma-s)\dot{f}(\underset{\sim}{y},\sigma)g(\underset{\sim}{y},s)dsd\sigma d\underset{\sim}{y} = 0, \tag{5.13}$$

and

$$\int_R \int_0^{\tau_\lambda(\underset{\sim}{y})} \dot{\theta}(\underset{\sim}{y},\sigma)\dot{r}(\underset{\sim}{y},\sigma)\,d\sigma d\underset{\sim}{y} = 0 .$$

At this point, let $\rho_0 > 0$ be small enough so that $B_{\rho_0}(\underset{\sim}{x}) \subset R$. Then consider the one parameter family of regular regions

$$R_\rho = R - B_\rho(\underset{\sim}{x}) \qquad 0 < \rho < \rho_0 \tag{5.14}$$

The hypotheses of Lemma 5.1 are satisfied on each $R\rho$ with the functions $\theta|(R_\rho \times (-\infty,\infty))$ and $\tau_\lambda|R_\rho$. Using Lemma 5.1 on R_ρ and passing to the limit as $\rho \to 0$, using (5.13), one obtains

$$\begin{aligned}
&\frac{c}{2}\int_R \dot{\theta}^2(\underset{\sim}{y},\tau_\lambda(\underset{\sim}{y}))\,d\underset{\sim}{y} + \beta(0)\int_R \int_0^{\tau_\lambda(\underset{\sim}{y})} \dot{\theta}^2(\underset{\sim}{y},\sigma)\,d\sigma d\underset{\sim}{y} \\
&\quad + \int_R \int_0^{\tau_\lambda(\underset{\sim}{y})} \int_0^\sigma \beta'(\sigma-s)\dot{\theta}(\underset{\sim}{y},\sigma)\dot{\theta}(\underset{\sim}{y},s)\,ds d\sigma d\underset{\sim}{y} \\
&\quad + a(0)\int_R \dot{\theta}(\underset{\sim}{y},\tau_\lambda(\underset{\sim}{y}))\nabla\theta(\underset{\sim}{y},\tau_\lambda(\underset{\sim}{y}))\cdot\nabla\tau_\lambda(\underset{\sim}{y})\,d\underset{\sim}{y} \\
&\quad + \int_R \int_0^{\tau_\lambda(\underset{\sim}{y})} a'(\tau_\lambda(\underset{\sim}{y})-s)\dot{\theta}(\underset{\sim}{y},\tau_\lambda(\underset{\sim}{y}))\nabla\theta(\underset{\sim}{y},s)\cdot\nabla\tau_\lambda(\underset{\sim}{y})\,ds d\underset{\sim}{y} \\
&\quad + \frac{a(0)}{2}\int_R |\nabla\theta(\underset{\sim}{y},\tau_\lambda(\underset{\sim}{y}))|^2 d\underset{\sim}{y} + \gamma(0)\int_R \int_0^{\tau_\lambda(\underset{\sim}{y})} |\nabla\theta(\underset{\sim}{y},\sigma)|^2 d\sigma d\underset{\sim}{y} \\
&\quad + \int_R \int_0^{\tau_\lambda(\underset{\sim}{y})} \int_0^\sigma \gamma'(\sigma-s)\nabla\theta(\underset{\sim}{y},\sigma)\cdot\nabla\theta(\underset{\sim}{y},s)\,ds d\sigma d\underset{\sim}{y} \qquad (5.15) \\
&\quad + \int_R \int_0^{\tau_\lambda(\underset{\sim}{y})} a'(\tau_\lambda(\underset{\sim}{y})-s)\nabla\theta(\underset{\sim}{y},\tau_\lambda(\underset{\sim}{y}))\cdot\nabla\theta(\underset{\sim}{y},s)\,ds d\underset{\sim}{y} = 0.
\end{aligned}$$

Now let

$$\alpha_1(\lambda) = \int_R \dot{\theta}^2(\underset{\sim}{y},\tau_\lambda(\underset{\sim}{y})d\underset{\sim}{y},$$

$$\alpha_2(\lambda) = \int_R \int_0^{\tau_\lambda(\underset{\sim}{y})} \dot{\theta}^2(\underset{\sim}{y},\sigma)d\sigma d\underset{\sim}{y},$$

$$\alpha_3(\lambda) = \int_R |\nabla\theta(\underset{\sim}{y},\tau_\lambda(\underset{\sim}{y}))|^2 d\underset{\sim}{y},$$

$$\alpha_4(\lambda) = \int_R \int_0^{\tau_\lambda(\underset{\sim}{y})} |\nabla\theta(\underset{\sim}{y},\sigma)|^2 d\sigma d\underset{\sim}{y}.$$

Note that each α_i is nonnegative. In order to show that each $\alpha_i(\lambda)$ is zero for all $\lambda\varepsilon[0,t]$, let

$$t^* = \sup\{s\varepsilon[0,t]: \sum_{i=1}^{4} \alpha_i = 0 \text{ on } [0,s]\}, \tag{5.17}$$ [1]

and assume for contradiction that

$$0 \leq t^* < t. \tag{5.18}$$

By (5.15), (5.16), and (5.6)

$$\frac{1}{2} c\, \alpha_1(\lambda) + \beta(0)\alpha_2(\lambda) + \frac{1}{2} a(0)\alpha_3(\lambda) + \gamma(0)\alpha_4(\lambda)$$
$$\leq K \int_R \int_0^{\tau_\lambda(\underset{\sim}{y})} \int_0^{\sigma} |\dot{\theta}(\underset{\sim}{y},\sigma)|\cdot|\dot{\theta}(\underset{\sim}{y},s)|ds d\sigma d\underset{\sim}{y}$$

$$+ a(0) \int_R |\dot{\theta}(\underset{\sim}{y},\tau_\lambda(\underset{\sim}{y}))|\cdot|\nabla\theta(\underset{\sim}{y},\tau_\lambda(\underset{\sim}{y}))|\cdot|\nabla\tau_\lambda(\underset{\sim}{y})|d\underset{\sim}{y}$$

---

[1] Note that t^* exists, i.e. the set in (5.17) is nonempty, since $\alpha_i(0) = 0$, $i=1,\ldots,4$.

$$+ K \int_R \int_0^{\tau_\lambda(\underset{\sim}{y})} |\dot{\theta}(\underset{\sim}{y},\tau_\lambda(\underset{\sim}{y}))| \cdot |\nabla\theta(\underset{\sim}{y},s)| \cdot |\nabla\tau_\lambda(\underset{\sim}{y})| ds d\underset{\sim}{y}$$

$$+ K \int_R \int_0^{\tau_\lambda(\underset{\sim}{y})} \int_0^{\sigma} |\nabla\theta(\underset{\sim}{y},\sigma)| \cdot |\nabla\theta(\underset{\sim}{y},s)| ds d\sigma d\underset{\sim}{y}$$

$$+ K \int_R \int_0^{\tau_\lambda(\underset{\sim}{y})} |\nabla\theta(\underset{\sim}{y},\tau_\lambda(\underset{\sim}{y}))| \cdot |\nabla\theta(\underset{\sim}{y},s)| ds d\underset{\sim}{y} \tag{5.19}$$

For the moment, concentrate on the second and third terms on the right-hand side of (5.19). Since

$$\int_R [|\dot{\theta}(\underset{\sim}{y},\tau_\lambda(\underset{\sim}{y}))| - |\nabla\theta(\underset{\sim}{y},\tau_\lambda(\underset{\sim}{y}))|]^2 d\underset{\sim}{y} \geq 0,$$

it follows from (5.16) that

$$\int_R |\dot{\theta}(\underset{\sim}{y},\tau_\lambda(\underset{\sim}{y}))| \cdot |\nabla\theta(\underset{\sim}{y},\tau_\lambda(\underset{\sim}{y}))| d\underset{\sim}{y} \leq \frac{1}{2}\alpha_1(\lambda) + \frac{1}{2}\alpha_3(\lambda) \tag{5.20}$$

for all $\lambda\varepsilon[0,t]$. Also, as mentioned in (5.11), $|\nabla\tau_\lambda(\underset{\sim}{y})| = u_0^{-1}$. Consequently

$$a(0)\int_R |\dot{\theta}(\underset{\sim}{y},\tau_\lambda(\underset{\sim}{y}))| \cdot |\nabla\theta(\underset{\sim}{y},\tau_\lambda(\underset{\sim}{y}))| \cdot |\nabla\tau_\lambda(\underset{\sim}{y})| d\underset{\sim}{y}$$
$$\leq \frac{a(0)}{2u_0}[\alpha_1(\lambda) + \alpha_3(\lambda)]. \tag{5.21}$$

Similarly,

$$\int_R [|\dot{\theta}(\underset{\sim}{y},\tau_\lambda(\underset{\sim}{y}))| - \int_0^{\tau_\lambda(\underset{\sim}{y})} |\nabla\theta(\underset{\sim}{y},s)| ds]^2 d\underset{\sim}{y} \geq 0$$

implies that

$$\int_R |\dot{\theta}(y,\tau_\lambda(\underset{\sim}{y}))| \int_0^{\tau_\lambda(\underset{\sim}{y})} |\nabla\theta(\underset{\sim}{y},s)| ds d\underset{\sim}{y} \leq \frac{\alpha_1(\lambda)}{2}$$
$$+ \int_R [\int_0^{\tau_\lambda(\underset{\sim}{y})} |\nabla\theta(\underset{\sim}{y},s)| ds]^2 d\underset{\sim}{y} \tag{5.22}$$

But, for $X_1 \le X_2$ the Schwarz inequality implies that

$$[\int_{X_1}^{X_2} |f|ds]^2 \le (X_2 - X_1)\int_{X_1}^{X_2} |f|^2 ds, \tag{5.23}$$

so that if $\lambda\varepsilon[t^*,t]$

$$[\int_0^{\tau_\lambda(\underset{\sim}{y})} |\nabla\theta(\underset{\sim}{y},s)|ds]^2 = [\int_{t^*}^{\tau_\lambda(\underset{\sim}{y})} |\nabla\theta(\underset{\sim}{y},s)|ds]^2$$

$$\le (\tau_\lambda(\underset{\sim}{y}) - t^*) \int_{t^*}^{\tau_\lambda(\underset{\sim}{y})} |\nabla\theta(\underset{\sim}{y},s)|^2 ds$$

$$\le (\lambda - t^*)\int_0^{\tau_\lambda(\underset{\sim}{y})} |\nabla\theta(\underset{\sim}{y},s)|^2 ds. \tag{5.24}$$

Accordingly, by (5.22)

$$\int_R \int_0^{\tau_\lambda(\underset{\sim}{y})} |\dot{\theta}(\underset{\sim}{y},\tau_\lambda(\underset{\sim}{y}))| \cdot |\nabla\theta(\underset{\sim}{y},s)| ds d\underset{\sim}{y}$$

$$\le \frac{1}{2}\alpha_1(\lambda) + \frac{1}{2}(\lambda - t^*)\alpha_4(\lambda)$$

$$\le \frac{1}{2}\alpha_1(\lambda) + \frac{1}{2} t^{\frac{1}{2}}\alpha_4(\lambda)(\lambda - t^*)^{\frac{1}{2}}. \tag{5.25}$$

Again since $|\nabla\tau_\lambda(\underset{\sim}{y})| = u_0^{-1}$,

$$K\int_R \int_0^{\tau_\lambda(\underset{\sim}{y})} |\dot{\theta}(\underset{\sim}{y})\tau_\lambda(\underset{\sim}{y}))| \cdot |\nabla\theta(\underset{\sim}{y},s)| \cdot |\nabla\tau_\lambda(\underset{\sim}{y})| ds d\underset{\sim}{y}$$

$$\le \frac{K\alpha_1(\lambda)}{2u_0} + \frac{Kt^{\frac{1}{2}}\alpha_4(\lambda)}{2u_0}(\lambda - t^*)^{\frac{1}{2}}. \tag{5.26}$$

Therefore, by (5.19), (5.21), and (5.26)

$$\left[\frac{c}{2} - \frac{a(0)}{2u_0} - \frac{K}{2u_0}\right] \alpha_1(\lambda) + \beta(0)\alpha_2(\lambda) + \left[\frac{a(0)}{2} - \frac{a(0)}{2u_0}\right] \alpha_3(\lambda)$$

$$+ \gamma(0)\alpha_4(\lambda) \le K\int_R \int_0^{\tau_\lambda(\underset{\sim}{y})} \int_0^{\sigma} |\dot{\theta}(\underset{\sim}{y},\sigma)| \cdot |\dot{\theta}(\underset{\sim}{y},s)| ds d\sigma d\underset{\sim}{y}$$

$$+ K \int_R \int_0^{\tau_\lambda(\underset{\sim}{y})} \int_0^{\sigma} |\nabla\theta(\underset{\sim}{y},\sigma)| \cdot |\nabla\theta(\underset{\sim}{y},s)| ds d\sigma d\underset{\sim}{y}$$

$$+ K \int_R \int_0^{\tau_\lambda(\underset{\sim}{y})} |\nabla\theta(\underset{\sim}{y},\tau_\lambda(\underset{\sim}{y}))| \cdot |\nabla\theta(\underset{\sim}{y},s)| ds d\underset{\sim}{y}$$

$$+ \frac{Kt^{\frac{1}{2}}\alpha_4(\lambda)}{2u_0} (\lambda - t^*) \qquad (5.27)$$

for all $[t^*,t]$. By (5.7) the coefficients of the α_i in (5.27) are positive. If d is the minimum of these coefficients, the left-hand side of (5.27) can be replaced by $d\sum_{i=1}^{4}\alpha_i$.

Consider how the first term on the right-hand side of (5.27). Define the function W by

$$W(\underset{\sim}{y},\sigma) = \int_0^{\sigma} |\dot{\theta}(\underset{\sim}{y},s)| ds, \qquad (5.28)$$

with which is possible to conclude that

$$\int_0^{\tau_\lambda(\underset{\sim}{y})} \int_0^{\sigma} |\dot{\theta}(\underset{\sim}{y},\sigma)| \cdot |\dot{\theta}(\underset{\sim}{y},s)| ds d\sigma = \int_0^{\tau_\lambda(\underset{\sim}{y})} W(\underset{\sim}{y},\sigma) \frac{\partial}{\partial\sigma} W(\underset{\sim}{y},\sigma) d\sigma$$

$$= W^2(\underset{\sim}{y},\sigma)\Big|_0^{\tau_\lambda(\underset{\sim}{y})} - \int_0^{\tau_\lambda(\underset{\sim}{y})} \frac{\partial}{\partial\sigma} [W(\underset{\sim}{y},\sigma)] W(\underset{\sim}{y},\sigma) d\sigma$$

$$= [\int_0^{\tau_\lambda(\underset{\sim}{y})} |\dot{\theta}(\underset{\sim}{y},s)|ds]^2 - \int_0^{\tau_\lambda(\underset{\sim}{y})} \int_0^{\sigma} |\dot{\theta}(\underset{\sim}{y},\sigma)| \cdot |\dot{\theta}(\underset{\sim}{y},s)| ds d\sigma. \tag{5.29}$$

Thus

$$\int_0^{\tau_\lambda(\underset{\sim}{y})} \int_0^{\sigma} |\dot{\theta}(\underset{\sim}{y},\sigma)| \cdot |\dot{\theta}(\underset{\sim}{y},s)| ds d\sigma = \frac{1}{2} [\int_0^{\tau_\lambda(\underset{\sim}{y})} |\dot{\theta}(\underset{\sim}{y},s)|ds]^2$$

$$\leq (\lambda - t^*) \int_0^{\tau_\lambda(\underset{\sim}{y})} |\dot{\theta}(\underset{\sim}{y},s)|^2 ds, \tag{5.30}$$

as in (5.24). Therefore, as in (5.25),

$$\int_R \int_0^{\tau_\lambda(\underset{\sim}{y})} \int_0^{\sigma} |\dot{\theta}(\underset{\sim}{y},\sigma)| \cdot |\dot{\theta}(\underset{\sim}{y},s)| ds d\sigma d\underset{\sim}{y}$$

$$\leq \frac{1}{2} t^{\frac{1}{2}} \alpha_2(\lambda)(\lambda - t^*)^{\frac{1}{2}}. \tag{5.31}$$

Similarly

$$\int_R \int_0^{\tau_\lambda(\underset{\sim}{y})} \int_0^{\sigma} |\nabla\theta(\underset{\sim}{y},\sigma)| \cdot |\nabla\theta(\underset{\sim}{y},s)| ds d\sigma d\underset{\sim}{y}$$

$$\leq \frac{1}{2} t^{\frac{1}{2}} \alpha_4(\lambda)(\lambda - t^*)^{\frac{1}{2}}. \tag{5.32}$$

By the Schwarz inequality, (5.24), and (5.16)

$$\int_R |\nabla\theta(\underset{\sim}{y},\tau_\lambda(\underset{\sim}{y}))| \, |\int_0^{\tau_\lambda(\underset{\sim}{y})} |\nabla\theta(\underset{\sim}{y},s)| ds d\underset{\sim}{y}$$

$$\leq \{\int_R |\nabla\theta(\underset{\sim}{y},\tau_\lambda(\underset{\sim}{y}))|^2 d\underset{\sim}{y}\}^{\frac{1}{2}} \{\int_R [\int_0^{\tau_\lambda(\underset{\sim}{y})} |\nabla\theta(\underset{\sim}{y},s)| ds]^2 d\underset{\sim}{y}\}^{\frac{1}{2}}$$

$$\leq \alpha_3(\lambda)^{\frac{1}{2}} \alpha_4(\lambda)^{\frac{1}{2}} (\lambda - t^*)^{\frac{1}{2}} \tag{5.33}$$

Therefore, by (5.27), (5.31), (5.32), and (5.33),

$$d\sum_{i=1}^{4}\alpha_i(\lambda) \le K(\lambda-t^*)^{\frac{1}{2}}\left\{\frac{1}{2}t^{\frac{1}{2}}\alpha_2(\lambda)+\frac{1}{2}t^{\frac{1}{2}}\alpha_4(\lambda)\right.$$

$$\left.+[\alpha_3(\lambda)\alpha_4(\lambda)]^{\frac{1}{2}}+\frac{1}{2u_0}t^{\frac{1}{2}}\alpha_4(\lambda)\right\}. \qquad (5.34)$$

Each α_i is continuous on $[0,t]$, so there exists a positive number L_0 such that

$$\alpha_i \le L_0 \qquad \text{on } [0,t],\ i=1,\dots,4 \qquad (5.35)$$

Now define the sequence $\{L_n\}_{n=0}^{\infty}$ by

$$L_0 : \text{ from } (5.35)$$

$$L_{n+1} = \frac{K}{d}\left[1+t^{\frac{1}{2}}\left(\frac{2u_0+1}{2u_0}\right)\right]L_n \qquad n=0,1,2,\dots \qquad (5.36)$$

or

$$L_n = \eta^{n+1}L_0 \qquad n=0,1,2,\dots \qquad (5.37)$$

where

$$\eta = \frac{K}{d}\left[1+t^{\frac{1}{2}}\left(\frac{2u_0+1}{2u_0}\right)\right] > 0$$

It will be shown by induction that $\lambda\varepsilon[t^*,t]$ implies

$$\alpha_i(\lambda) \le L_n(\lambda-t^*)^{n/2} \qquad n=0,1,2,\dots \qquad (5.38)$$

for each $i=1,\dots,4$. For $n=0$, the assertation is clear. For $n=j$, (5.34) gives

$$d\sum_{i=1}^{4}\alpha_i(\lambda) \le KL_j(\lambda-t^*)^{\frac{j+1}{2}}[1+t^{\frac{1}{2}}\cdot(1+\frac{1}{2u_0})] \qquad (5.39)$$

Consequently, for each $i=1,\ldots,4$

$$\alpha_i(\lambda) \le \frac{K}{d}\left[1+t^{\frac{1}{2}}\cdot\left(\frac{2u_0+1}{2u_0}\right)\right] L_j\,(\lambda-t^*)^{\frac{j+1}{2}}, \tag{5.40}$$

or

$$\alpha_i(\lambda) \le L_{j+1}\,(\lambda-t^*)^{\frac{j+1}{2}}. \tag{5.41}$$

By (5.37)

$$L_n S^{n/2} = L_0\,(\eta S^{\frac{1}{2}})^n$$

and hence

$$\lim_{n\to\infty} L_n S^{n/2} = 0 \quad \text{if} \quad S < \eta^{-2}. \tag{5.42}$$

Therefore, by (5.38), $\alpha_i(\lambda) = 0$ if $\lambda-t^* < \eta^{-2}$. Hence

$$\alpha_i(\lambda) = 0 \quad \text{for all } \lambda \text{ such that } t^* < \lambda < t^* + \eta^{-2}, \tag{5.43}$$

which contradicts the definition of t^*, (5.17).

Therefore $t^* = t$ and for $i=1,\ldots,4$

$$\alpha_i(\lambda) = 0 \quad \text{for all } \lambda\varepsilon[0,t] \tag{5.44}$$

By (5.16), then,

$$\dot{\theta}(\underset{\sim}{y},\tau_\lambda(\underset{\sim}{y})) = 0,$$

$$\nabla\theta(\underset{\sim}{y},\tau_\lambda(\underset{\sim}{y})) = \underset{\sim}{0} \quad \text{for all } y\varepsilon\overline{R},$$

which proves that $\theta(\underset{\sim}{y},\tau_\lambda(\underset{\sim}{y}))$ is a constant for all $y\varepsilon\overline{R}$. Furthermore, if $\underset{\sim}{y}\varepsilon\overline{R}-P(\tau_\lambda)$, its value is zero. Finally, letting $\underset{\sim}{y} = \underset{\sim}{x}$,

$$\theta(\underset{\sim}{x},\lambda) = 0 \qquad \text{for all } \lambda\varepsilon[0,t].$$

COROLLARY. Suppose that R is an unbounded regular region and that θ_1, θ_2 are solutions to the first history-value problem on R, corresponding to ψ_1, f_1, g_1, c, a, β, r_1 and ψ_2, f_2, g_2, c, a, β, r_2 respectively, with c, a(0), $\beta(0)$, and $\gamma(0)$ positive. If, for any $t > 0$, there exists a bounded set $\Lambda_t \subset \bar{R}$ such that

$$\begin{aligned} \psi_1 &= \psi_2 \quad \text{on } (\bar{R} - \Lambda_t)\times(-\infty,0], \\ f_1 &= f_2 \quad \text{on } (B_1 - \Lambda_t)\times[0,t], \\ g_1 &= g_2 \quad \text{on } (B_2 - \Lambda_t)\times[0,t], \end{aligned} \tag{5.45}$$

and

$$r_1 = r_2 \quad \text{on } (\bar{R} - \Lambda_t)\times[0,t],$$

then there exists a bounded set $\Omega_t \subset \bar{R}$ such that $\theta_1 = \theta_2$ on $(\bar{R} - \Omega_t)\times[0,t]$.

Proof: The function $\theta = \theta_1 - \theta_2$ satisfies the hypotheses of Theorem 5.1. The rest follows at once.

THEOREM 5.2. Let R be a regular region and suppose that θ_1, θ_2 are solutions to the first history-value problem on R, corresponding to ψ, f, g, c, a, β, and r. If c, a(0), $\beta(0)$, and $\gamma(0)$ are positive, then

$$\theta_1 = \theta_2 \qquad \text{on } \bar{R}\times[0,\infty).$$

<u>Proof</u>: The proposition has already been proved in the case that R is bounded (Ch. 4). So assume that R is unbounded. Let $t > 0$. The hypotheses of the Corollary to Theorem 5.1 are satisfied with $\Lambda_t = \emptyset$. Therefore there exists a bounded set $\Omega_t \subset \bar{R}$ such that

$$\theta_1 = \theta_2 \quad \text{on } (\bar{R}-\Omega_t)\times[0,t]. \tag{5.46}$$

Since R is a regular region, there exists a bounded set R^* with $\Omega_t \subseteq R^* \subset R$, and the boundary of R^* consists of a finite number of "closed regular surfaces" (Kellogg). That is, R^* is a bounded regular region. Since θ_1 and θ_2 satisfy the hypotheses of Corollary 2 to Theorem 4.1 on $R^*\times[0,t]$

$$\theta_1 = \theta_2 \quad \text{on} \quad R^*\times[0,t]. \tag{5.47}$$

Since $R-R^* \subseteq R-\Omega_t$, (5.46) and (5.47) imply that

$$\theta_1 = \theta_2 \qquad R\times[0,t].$$

Finally, since t is arbitrary, the conclusion follows.

<u>Remark</u>: Clearly Lemma 5.1, Theorem 5.1, its corollary, and Theorem 5.2 have counterparts for the second history-value problem, the proof being along the lines of Theorem 4.2.

REFERENCES

[1] Morton E. Gurtin & A. C. Pipkin, "A General Theory of Heat Conduction with Finite Wave Speeds," Arch. Rational Mech. Anal. 31, 113-126 (1968).

[2] Josef Meixner, "On the Linear Theory of Heat Conduction," Arch. Rational Mech. Anal. 39, 108-130 (1970).

[3] Peter J. Chen, "On the Growth and Decay of One-Dimensional Temperature Rate Waves," Arch. Rational Mech. Anal. 35, 1-15(1969).

[4] R. Courant and D. Hilbert, Methods of Mathematical Physics, v.2, New York: Interscience 1962.

[5] O. D. Kellogg, Foundations of Potential Theory, New York: Dover 1953.

[6] H. F. Weinberger, A First Course in Partial Differential Equations, Waltham, Mass.: Blaisdell Publishing Co. 1965.

[7] E. C. Titchmarsh, Introduction to the Theory of Fourier Integrals, Second Ed., Oxford at The Clarendon Press 1948.

[8] V. Volterra, "Sulle Equazioni Integro-Differenziali," Rend. Acc. Lincei, Series 5, 18, 167-174(1909).

[9] Jan Mikusiński, Operational Calculus, New York: Pergamon Press 1959.

[10] S. Zaremba, "Sopra un Teorema d'Unicitá Relativo alla Equazione delle Onde Sferiche," Atti della Reale Academia dei Lincei, Series 5, 24, 904(1915).

[11] Bers, L., F. John, & M. Schechter, Partial Differential Equations, New York: Interscience 1964.

H0
510
F515