FIXED POINT THEOREMS IN ANALYSIS

A Thesis

Presented to

the Faculty of the Department of Mathematics

University of Houston

In Partial Fulfillment

of the Requirements for the Degree

Master of Science

by

Betty Jane Barr

January 1969

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ABSTRACT

A survey of fixed point theorems in analysis is given from their initiation to the present. The many and varied operators and spaces which occur are classified into a logical arrangement. An analysis is made of certain classical as well as very recent general theorems, and a generalization of one of the modern theorems is proved. The application of fixed point theorems to the proof of existence and uniqueness of solutions to differential and integral equations is illustrated by several examples. Many conjectures and suggestions for further research are interspersed with the ordered arrangement of the fixed point theorems in analysis which appear in the literature.

TABLE OF CONTENTS

LIST OF FIGURES

FIGURE PAGE

CHAPTER I

INTRODUCTION

An element x is said to be a fixed point of a function f from a set A into itself provided that $f(x) = x$. Fixed points are of interest both to the topologist and to the analyst. The topologist is usually concerned with the topological properties of spaces on which continuous mappings have fixed points, and in particular with the structure of the fixed point set. The analyst, on the other hand, seeks to establish the existence of fixed points for various types of functions and families of functions. He tries to find a constructive method for obtaining or approximating a fixed point in order to apply his results to differential and integral equations. The analyst rarely works in a space which is more general than a topological vector space or a metric space. The topologist, however, may be concerned with a general topological space.

The study of fixed points has a long and full history. An excellent survey of fixed points from the topological point of view is given by Van Der Walt [34].* Cronin [10] presents an introduction to fixed points and topological

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*Throughout this paper a bracketed number refers to the corresponding reference in the bibliography.

degree. One purpose of this thesis is to provide a review from the viewpoint of the analyst.

In 1912 Brouwer proved his classical theorem on the existence of a fixed point for a continuous map of the closed unit ball in Euclidean n-space into itself. In 1922 Banach formulated his contraction principle, and in 1927 Schauder proved the existence of a fixed point for a continuous map from a convex set C into a compact subset of C in a Banach space. This was generalized by Tychonoff in 1935 to locally convex topological vector spaces. Kakutani, in 1938, established a fixed point theorem for groups of equicontinuous linear mappings on compact, convex subsets of locally convex topological vector spaces.

Since these first fundamental theorems, an enormous number of fixed point theorems have been proved by introducing variations in the types of spaces and types of operators considered. Ideally, these theorems would range from the very specific to the most general in some simple order, allowing classification as one or two very general theorems and their numerous corollaries. Unfortunately, such a simple ordering is not yet possible.

The current research on fixed points, being most prolific, has resulted in a great many types of spaces and an even greater variety in conditions on operators without any easily discernable relationship linking them all together.

Nevertheless, some idea of the general state of affairs is essential to further progress, in particular in the direction of finally obtaining the ideal--the most general fixed point theorem. In addition, such a knowledge would be most beneficial to those who seek to apply fixed point theorems in the area of differential and integral equations.

Therefore, this investigation attempts to present a view of the current situation in the study of fixed points in analysis. Chapter II contains the definitions of the types of spaces, operators, and sets encountered in the research on fixed points, in some cases attempting to alleviate ambiguity by coining a new term for different concepts which appear in the literature with the same name. Also in Chapter II is found a statement of the various relations known to exist between the spaces and operators, as well as the facts assumed without proof which are used in the succeeding chapters. Proofs of the classical Brouwer, Banach-Cacciopoli, Schauder-Tychonoff, and Kakutani theorems are presented in Chapter III. Chapter IV contains what appear to be the most general of the modern theorems, those of Petryshyn and Guedes de Figueiredo. Chapter V is devoted to examples of the various applications of fixed point theorems. Suggestions for further research, as well as a classification and statement of the fixed point theorems found in the literature are presented in Chapter VI.

CHAPTER II

BACKGROUND

This chapter provides the necessary background for the succeeding chapters. It includes definitions of the terms encountered in the literature on fixed points and a statement of the lesser known facts which are assumed without proof in the remainder of the investigation. In addition, the final section of the chapter gives a number of the relations known to exist between the spaces and operators. Some of these relations are illustrated, for ease of reference, in Figures 1 and 2 at the end of the chapter.

DEFINITIONS

I. Spaces

2.1 A pair (V,U) is said to be a <u>topological space</u> provided that V is a nonempty set and U is a collection of subsets of V satisfying:

1. V and $\emptyset$ belong to U.
2. The intersection of any finite number of members of U is a member of U.
3. The union of any collection of members of U is a member of U.

2.2 A topological space (V,U) is said to be a Hausdorff space provided that for any two distinct points x,y in V there exist sets U_x, U_y in U such that x is in U_x, y is in U_y, and $U_x \cap U_y = \emptyset$.

2.3 A triple $(V,+,\cdot)$ is said to be a vector space (linear space) over the field F provided that V is a set, $+$ is a binary operation on V, and $\cdot$ is a function from $F \times V$ into V satisfying the following:

1. $(V,+)$ is an abelian group.
2. $a(bx) = (ab)x$ for all a,b in F and x in V.
3. $1x = x$ for all x in V.
4. $(a+b)x = ax + bx$ for all a,b in F and x in V.
5. $a(x+y) = ax + ay$ for all a in F and x,y in V.

Note that in the remainder of this investigation, as in the literature on fixed points, F is assumed to be either the real or the complex numbers.

2.4 A topological space X on which a structure of vector space over F is defined is a topological vector space (linear topological space) provided that:

1. X is a Hausdorff space.
2. The map $(x,y) \to x + y$ from $X \times X$ into X is continuous.
3. The map $(a,x) \to ax$ from $F \times X$ into X is continuous.

2.5 A metric space (X,d) is a pair where X is a nonempty set and d is a nonnegative real-valued function on

$X \times X$ which satisfies:

1. $d(x,y) = 0$ if and only if $x = y$.
2. $d(x,y) = d(y,x)$.
3. $d(x,y) \leq d(x,z) + d(z,y)$.

2.6 A metric space X in which every fundamental sequence converges to a point in X is said to be complete.

2.7 A vector space X is a normed space if there exists a real number $||x||$ associated with each x in X which satisfies:

1. $||x|| > 0$ if $x \neq 0$.
2. $||ax|| = |a|\ ||x||$.
3. $||x + y|| \leq ||x|| + ||y||$.

2.8 A normed linear space which is complete is a Banach space.

2.9 An infinite dimensional Banach space X is said to be a PB space if it has the property that there exist a sequence $\{X_n\}$ of finite dimensional subspaces of X and a sequence $\{P_n\}$ of projections such that $\overline{\cup X_n} = X$, and for all n $P_nX = X_n$, $X_{n+1} \supsetneqq X_n$, and for some $K > 0$, $||P_n|| \leq K$. Note that in the remainder of this investigation, in the context of a PB space, $\{X_n\}$ and $\{P_n\}$ will denote the subspaces and projections of the definition.

2.10 A Banach space X is said to be a GB space if there exist a family of finite dimensional subspaces $\{F_\alpha\}$

and a family of projections $\{P_\alpha\}$ such that $P_\alpha X = F_\alpha$, $||P_\alpha|| = 1$, given any two subspaces there is a third which contains both, and the union of the $\{F_\alpha\}$ is dense in X. Note that in the remainder of this investigation, in the context of a GB space, $\{F_\alpha\}$ and $\{P_\alpha\}$ will denote the subspaces and projections of the definition.

2.11 A vector space X is an <u>inner product space</u> provided that there is a function defined on $X \times X$ whose range is contained in the field of scalars and which satisfies the following:

1. $(ax,y) = a(x,y)$.
2. $(x+y,z) = (x,z) + (y,z)$.
3. $(x,y) = \overline{(y,x)}$.
4. $(x,x) > 0$ if $x \neq 0$.

2.12 A <u>Hilbert space</u> is a complete inner product space.

2.13 A space which contains a countable, dense set is said to be <u>separable</u>.

2.14 A <u>locally convex space</u> is a topological vector space which has a basis of convex sets.

2.15 A metric space X is said to be <u>ε-chainable</u> if for every a,b in X there exists a finite set of points $\{a = x_0, x_1, \ldots, x_m = b\}$ such that $d(x_{i-1}, x_i) < \varepsilon$ for $i = 1, \ldots, m$.

2.16 A normed linear space is <u>strictly convex</u> if it has the property that if $||x + y|| = ||x||+||y||$ and $y \neq 0$ then there is a number t such that $x = ty$.

2.17 If V is a vector space over F, then the conjugate (dual) space of V is the vector space V^* whose members are the continuous linear functionals defined on V with range contained in F. If V is a normed space, then V^* is a Banach space under the norm $||T|| = \sup\{||Tx||:||x|| \leq 1\}$.

2.18 Let X be a Banach space and X^* and X^{**} its first and second conjugate spaces. If x_0 is in X then F, defined by $F(f) = f(x_0)$, is a continuous linear functional defined on X^*. X is said to be reflexive if every element of X^{**} is of this form.

2.19 A normed linear space is uniformly convex if for any $k > 0$ there exists $h > 0$ such that $||x - y|| < k$ if $||x|| < 1 + h$, $||y|| < 1 + h$ and $||x + y|| > 2$.

2.20 A locally convex topological vector space is said to satisfy condition E if the closed convex hull of any compact set is compact.

2.21 The X^* topology of a locally convex topological vector space X is the topology obtained by taking as a basis all sets of the form

$N(p,A,K) = \{q \text{ in } X: |f(p) - f(q)| < K \text{ for } f \text{ in } A\}$

where A is a finite subset of X^* and $K > 0$.

2.22 $C^n[a,b]$ is the complete metric space consisting of the set of all n-times differentiable real-valued functions on $[a,b]$ with $d(f,g) = \sup\{|f(x) - g(x)| : x \in [a,b]\}$. C[a,b] is the complete metric space of continuous real-valued functions on $[a,b]$ under the above metric.

2.23 $L_2[0,1]$ is the collection of all square integrable functions on $[0,1]$.

2.24 l_2 is the Hilbert space whose elements are sequences of real numbers, $x = (x_1,\ldots,x_n,\ldots)$, which satisfy the condition $\sum_n |x_n|^2 < \infty$, and where $(x,y) = \sum_n x_n y_n$.

2.25 Euclidean n-space (E^n) is the normed linear space of n-tuples of real numbers over the reals with norm $||(a_1,\ldots,a_n)|| = (\sum_1^n |a_i|^2)^{\frac{1}{2}}$.

2.26 Unitary n-space is the normed linear space of n-tuples of complex numbers over the complex numbers with the norm $||(a_1,\ldots,a_n)|| = (\sum_1^n |a_i|^2)^{\frac{1}{2}}$.

II. Operators

2.27 A map T from a Banach space X into itself is said to be accretive if for all u,v in X, w in J(u-v), $w(Tu-Tv) \geq 0$, where for each x in X, J(x) is the convex subset of X^* given by
$J(x) = \{w \text{ in } X^* : w(x) = ||x||^2,\ ||w|| = ||x||\}$.

2.28 Let T be a bounded linear operator mapping a Banach space X into itself. Then T is said to be <u>asymptotically convergent</u> if $\{T^k x\}$ converges for each x in X. The map T is said to be <u>asymptotically regular</u> if for each x in X, $\{T^{n+1}x - T^n x\} \to 0$. The map T is <u>weakly asymptotically regular</u> if the above convergence is weak.

2.29 Let X be a normed linear space. An operator T is said to be <u>bounded</u> if there exists a scalar M such that $||Tx|| \leq M||x||$ for all x in X. T is <u>locally bounded</u> if $\{Tx_n\}$ is bounded whenever $\{x_n\}$ is fundamental.

2.30 Let X be a normed linear space, T a mapping from $D(T) \subseteq X$ into X. T is said to be <u>closed</u> if $\{x_n\} \subseteq D(T)$, $\{x_n\} \to x$, $\{Tx_n\} \to y$, then $Tx = y$. T is <u>demiclosed</u> if $\{x_n\} \subseteq D(T)$, $\{x_n\} \to x$, $\{Tx_n\} \overset{w}{\to} y$, then $Tx = y$. T is <u>strongly closed</u> if $\{x_n\} \subseteq D(T)$, $\{x_n\} \overset{w}{\to} x$, and $\{Tx_n\} \to y$, then $Tx = y$.

2.31 An operator T mapping a metric space X into itself is said to be <u>compact</u> if it maps every bounded set onto a set with a compact closure.

2.32 An operator T mapping a Hilbert space X into itself is said to be <u>demicompact</u> if it has the property that whenever $\{x_n\}$ is a bounded sequence and $\{Tx_n - x_n\}$ is strongly convergent, then there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which is strongly convergent.

2.33 An operator A in a PB space X is said to be <u>P compact</u> if P_nA is continuous in X_n for all large n and if for any $p > 0$ and any bounded sequence $\{x_n\}$ with x_n in X_n the sequence $\{P_nAx_n - px_n\}$ is strongly convergent then there exists a strongly convergent subsequence $\{x_{n_i}\}$ and x in X such that $\{x_{n_i}\} \to x$ and $\{P_{n_i}Ax_{n_i}\} \to Ax$.

2.34 An operator A in a PB space is said to be <u>quasi-compact</u> if A satisfies the following:

1. A is bounded.
2. $\{x_n\} \to x$ implies that $\{P_mAx_n\} \to P_mAx$ for $m = 1,2,\ldots.$
3. If for some $h > 0$ the sequence $\{Ax_n + hx_n\}$ where $\{x_n\}$ is bounded is strongly convergent, then there exists a strongly convergent subsequence $\{x_{n_i}\}$.
4. If for some $h > 0$ the sequence $\{P_nAx_n + hx_n\}$ where $\{x_n\}$ is bounded is strongly convergent with x_n in X_n then there exists a strongly convergent subsequence $\{x_{n_i}\}$.

2.35 In a topological space X a function is said to be <u>continuous</u> if the inverse image of open sets is open.

2.36 Let X be a normed linear space, T an operator in X. T is <u>completely continuous</u> if it is both continuous and compact. T is <u>demicontinuous</u> if $\{x_n\} \to x$ implies that $\{Tx_n\} \overset{W}{\to} Tx$. T is <u>weakly continuous</u> if $\{x_n\} \overset{W}{\to} x$ implies that $\{Tx_n\} \overset{W}{\to} Tx$. T is <u>strongly continuous</u> if $\{x_n\} \overset{W}{\to} x$ implies that $\{Tx_n\} \to Tx$. T is <u>finitely continuous</u> if it is demicontinuous on finite dimensional subspaces of X. T is <u>hemicontinuous</u> if it is demicontinuous on line segments in X.

2.37 Let X be a metric space, T an operator on X. T is a <u>strict contraction with constant k</u> (class P_0) if $0 < k < 1$ and $d(Tx,Ty) \leq kd(x,y)$ for all x,y in its domain. T is <u>strictly nonexpansive</u> if $d(Tx,Ty) < d(x,y)$ for all x,y in its domain. T is <u>nonexpansive</u> (class P_1) if $d(Tx,Ty) \leq d(x,y)$ for all x,y in its domain. T is <u>I locally contractive</u> if for all x in X there exists $k > 0$ and a, $0 \leq a < 1$, which may depend on x, such that p,q in $N_k(x)$ implies that $d(Tp,Tq) \leq ad(p,q)$. T is <u>uniformly (k,a) locally contractive</u> if it is locally contractive and both k and a do not depend on x. T is <u>k-contractive</u> if there exists a $k > 0$ such that $0 < d(p,q) < k$ implies $d(Tp,Tq) < d(p,q)$ and T satisfies condition 3.

T is <u>II locally contractive</u> if it is continuous and there exists a real-valued function ϕ defined on the nonnegative reals which is upper semicontinuous and satisfies $\phi(0) = 0$, $\phi(r) < r$ for $r > 0$, such that there exists a positive integer $n(x)$ where x is in X such that $d(T^{n(x)}p, T^{n(x)}q) \leq \phi(d(p,q))$ for all p,q in $\{T^k x\}$. T is <u>locally iteratively contractive</u> if it is continuous and there exists a real-valued function ϕ defined on the nonnegative reals which is upper semicontinuous and satisfies $\phi(0) = 0$, $\phi(s) \leq \phi(t)$ whenever $s \leq t$, and $\sum_{j=0}^{\infty} \phi^j t < \infty$ for all $t > 0$, such that there exists a positive integer $n(x)$ where x is in X such that $d(T^{n(x)}p, T^{n(x)}q) \leq \phi(d(p,q))$ for all p,q in $\{T^k x\}$.

2.38 Let X be a normed linear space, T an operator on X. T is <u>strictly pseudocontractive with constant k</u> (class P_2) if $k < 1$ such that $||Tx - Ty||^2 \leq ||x - y||^2 + k||(I-T)x - (I-T)y||^2$ for all x,y in X. T is <u>I pseudocontractive</u> (class P_3) if $||Tx - Ty||^2 \leq ||x - y||^2 + ||(I-T)x - (I-T)y||^2$ for all x,y in X. T is <u>II pseudocontractive</u> if for all x,y in X and all $r > 0$, $||x - y|| \leq ||(1+r)(x-y) - r(Tx-Ty)||$.

2.39 An operator T from a Banach space X into its dual X^* is said to be demi-invertible if T^{-1} exists and is a demicontinuous map from X^* into X.

2.40 Let X be a topological vector space, C a compact convex subset of X, T map C into X. T is inner if $T(C) \subseteq C$. T is inward if for all x in C, $T(x)$ is in inw (x). T is outward if for all x in C, $T(x)$ is in ouw (x). T is weakly inward if for all x in C $T(x)$ is in weak inw (x). T is weakly outward if for all x in C $T(x)$ is in weak ouw (x). If X is also strictly convex and normed, T is nowhere normal outward if $T(x)$ belongs to the normal outward set of x for no x in C.

2.41 A map T from a bounded, closed convex subset C of a GB space X into X is said to be a G operator if:

1. $P_\alpha T$: $C \cap F_\alpha \to F_\alpha$ is continuous for all α.
2. The solvability of $P_\alpha Tx = x$ in F_α for all but a finite number of α implies the solvability of $Tx = x$ in X.

2.42 An operator T mapping a Banach space X into its dual X^* is said to satisfy the k-condition if T is demi-invertible and there exists a constant $k > 0$ such that for all x in X $|(Tx-T(0))(x)| \geq k||x||^2$.

2.43 An operator T mapping a metric space X into itself is said to satisfy <u>condition 3</u> if there exists an x in X such that $\{T^n(x)\}$ has a convergent subsequence.

2.44 Let X be a Banach space, T a mapping of X into X^*. T is said to be <u>monotone</u> if for all x,y in X, $\mathrm{Re}(Tx-Ty)(x-y) \geq 0$. T is <u>strongly monotone</u> if for all x,y in X and some $a > 0$ $\mathrm{Re}(Tx-Ty)(x-y) \geq a\,||x - y||^2$. T is <u>semi-monotone</u> if it is obtained from a map f $f\colon\ X \times X \to X^*$, that is $T(x) = f(x,x)$, such that f is monotone in the first variable and strongly continuous in the second.

2.45 Let X be a Hilbert space, T a mapping of X into itself. T is <u>monotone increasing on rays</u> if $\mathrm{Re}(T(sx),x)$ is a monotone increasing function of the real variable s for all x and all sufficiently large s. T belongs to <u>class M</u> if it is finitely continuous and for all x,y in $X(Tx-Ty,\ x-y) \geq 0$. T belongs to <u>class M_0</u> (<u>strongly monotone</u>) if for all x,y in X and some $a > 0$ $\mathrm{Re}(Tx-Ty,x-y) \geq a||x - y||^2$. T belongs to <u>class M_1</u> if there exists a continuous, strictly increasing function c(r) on the nonnegative reals with $c(0) = 0$ such that $(Tx-Ty,x-y) \geq c(||x-y||)$ for all x,y in X. T is in class M_2 if there exists a constant a, $0 < a < 1$ such that $(Tx-Ty,x-y) \geq a||Tx - Ty||^2$ for all x,y in X. T is in class M_3 (monotone) if $Re(Tx-Ty,x-y) \geq 0$ for all x,y in X.

2.46 In a Banach space X with duality map J a nonlinear map T is said to be J monotone if $(J(x-y))(Ax-Ay) \geq 0$ for all x,y in X.

2.47 A map T in a normed linear space X is said to be a reasonable wanderer if starting at x_0 in X,

$$\sum_{n=0}^{\infty}||T^{n+1}x_0 - T^n x_0||^2 < \infty.$$

2.48 In a vector space a map T is said to lie on Ray(U) if there exists $t > 0$ such that $T = I + t(U-I)$.

2.49 In a metric space X a map T is said to be Lipschitz (belong to class Lip) with constant L if there exists $L > 0$ such that $d(Tx,Ty) \leq Ld(x,y)$ for all x,y in X.

2.50 Let C be a closed, convex subset of a Hilbert space X. Then for each x in X the map $R_C x$ is defined as the closest point to x in C.

2.51 Let $\mu(r)$ be a nondecreasing continuous real-valued function defined for $0 \leq r < \infty$ such that $\mu(0) = 0$ and $\mu(r) \to \infty$ as $r \to \infty$. Then μ is a gauge function.

2.52 Let X be a Banach space. A duality map in X with gauge function μ is a map J from X to the power set of X^* such that $J(0) = \{0\}$ and for $x \neq 0$,

$Jx = \{y' \text{ in } X^* : y'(x) = ||x||\,||y'||, ||y'|| = \mu(||x||)\}$.

2.53 Let 0 be an interior point of a convex subset C of a topological vector space X. For each x in X let $A(x) = \{a : a > 0, x \in aC\}$. Define the functional q on X by $q(x) = \inf A(x)$. q is the Minkowski functional on C.

2.54 In a metric space X the Picard iterates of an operator T are given by $x_1 = Tx_0$, $x_2 = Tx_1 = T^2x_0, \ldots, x_n = Tx_{n-1} = T^nx_0, \ldots$, where x_0 is an arbitrary point in X.

III. Sets

2.55 A set B in a topological vector space X is bounded if given any neighborhood V of the origin there exists a positive real number k such that $aB \subseteq V$ provided $|a| \leq k$.

2.56 A set C is compact if every open cover has a finite subcover.

2.57 A set C in a vector space X is convex if for any x,y in C and $a,b \geq 0$ such that $a + b = 1$, ax + by is in C.

2.58 Let X be a normed linear space, S a subset of X. We denote the diameter of S by $\delta(S) = \sup \{||x - y||: x,y \text{ in } S\}$. A point x in S is a diametral point of S provided that $\sup \{||x - y||: y \text{ in } S\} = \delta(S)$.

2.59 Let M be any set in the metric space X and let $\varepsilon > 0$. The set A in X is said to be an ε net with respect to M if for each x in M there exists an a in A such that $d(a,x) < \varepsilon$.

2.60 A family G of functions on a topological vector space X is equicontinuous on a subset C of X if for every neighborhood V of the origin in X there exists a neighborhood U of the origin such that if k_1, k_2 are in C with $k_1 - k_2$ in U then $T(k_1) - T(k_2)$ is in V for each T in G.

2.61 Let T be a map of the metric space X, $T(Y) \subseteq Y \subseteq X$. Then y in Y is said to belong to the T closure of Y, $y \in Y^T$, if there exists an x in Y and a sequence of positive integers $\{n_i\}$ $(n_1<n_2<\ldots)$ such that $\{T^{n_i}x\} \to y$. The set X^T is called the T closure.

2.62 Let X be a topological vector space, C a compact convex subset of X, x in C. Then inw (x) = $\{z : z = (1-a)x + ay \text{ for } y \text{ in } C \text{ and } a \geq 0\}$.

<u>ouw (x)</u> = {z : z = (1-a)x + ay for y in C and $a \leq 0$}.

<u>weakly inw (x)</u> = $\overline{\text{inw }(x)}$.

<u>weakly ouw (x)</u> = $\overline{\text{ouw }(x)}$.

If X is also strictly convex and normed, then the <u>normal outward set</u> of x is the set of all points y distinct from x such that

$$||y - x|| = \inf_{z \in C} ||y - z||.$$

2.63 Let X be a vector space, T an operator in X. Then for x in X the <u>linear variety spanned by</u> $\{T^n x\}$ is given by

$$\underline{L(x)} = \{y : y = \sum_1^m a_i T^i x, \sum_1^m a_i = 1, m = 1,2,\ldots\}.$$

2.64 Let X be a normed linear space, A,B subsets of X with B bounded. Define

$r_x(B) = \sup \{||x - y|| : y \text{ is in } B\}$.

$r(B,A) = \inf \{r_x(B) : x \text{ is in } A\}$.

$C(B,A) = \{x \text{ in } A : r_x(B) = r(B,A)\}$.

$r(B) = \inf \{r_x(B) : x \text{ is in } B\}$.

$B_C = \{x \text{ in } B : r_x(B) = r(B)\}$.

Then a convex set K in X is said to have a <u>normal structure</u> if for each bounded convex subset L of K which contains more than one point, there exists at least one point in L which is not a diametral point. A bounded closed convex subset K of X is said to have complete normal structure (C.N.S.) if every closed convex subset W of K which contains more than one point satisfies:

(*) For every decreasing net $\{W_\alpha : \alpha \varepsilon A\}$ of subsets of W which have the property that $r(W_\alpha, W) = r(W,W)$ for all α, it is the case that the closure of $\bigcup_\alpha C(W_\alpha, W)$ is a nonempty proper subset of W. If condition (*) in the above is replaced by a similar condition where only countable nets, that is sequences, are considered, K is said to have countable normal structure.

2.65 A subset M of a metric space X is totally bounded if X contains a finite ε-net with respect to M for each $\varepsilon > 0$.

2.66 An n-simplex is a set which consists of $n + 1$ linearly independent points $p_0, p_1, \ldots, p_n$ of a Euclidean space of dimension greater than n together with all points of the type $x = \sum_{i=0}^{n} a_i p_i$ where $a_i \geq 0$ for each i and $\sum_{i=0}^{n} a_i = 1$.

2.67 The Hilbert cube is the subset of l_2 consisting of all sequences $[\{x_n\}]$ such that $|x_n| \leq (1/n)$ for each n.

2.68 Let X be a topological vector space, S a subset of X. Then $C(S)$ is the set of all bounded and continuous scalar valued functions defined on S.

2.69 Let X be a metric space. The ball $B_r(a)$ is the set $B_r(a) = \{x \text{ in } X : d(x,a) \le r\}$. The sphere $S_r(a)$ is the set $S_r(a) = \{x \text{ in } X : d(x,a) = r\}$. The neighborhood $N_r(a)$ is the set $N_r(a) = \{x \text{ in } X : d(x,a) < r\}$. If $a = 0$ it is customary to write $B_r(0) = B_r$, and so forth.

2.70 A set A is said to have the fixed point property for a specified class of mappings provided that every map of A into itself which belongs to the class has a fixed point in A.

FACTS

2.1 If f is a function of the two real variables x and y and its two first partial derivatives f_x and f_y exist in a region R and the mixed partial f_{xy} exists in R and is continuous at the point (x_0,y_0) of R, then the mixed partial derivative f_{yx} exists at (x_0,y_0) and is equal to f_{xy} at that point [28].

2.2 Let $X(t) = (X_1,X_2,\ldots,X_n)$ be an $n \times n$ matrix with columns $X_1,\ldots,X_n$ whose entries are scalar valued differentiable functions of the variable t. Then

$$\frac{d}{dt}\det X(t) = \det(\frac{d}{dt}X_1, X_2, \ldots, X_n)$$
$$+ \det(X_1, \frac{d}{dt}X_2, \ldots, X_n)$$
$$+ \ldots + \det(X_1, \ldots, \frac{d}{dt}X_n) \quad [19].$$

2.3 Unitary n-space is isometric to Euclidean 2n space [13].

2.4 (Weierstrass Approximation Theorem) If f is continuous on a closed and bounded set I and if $\varepsilon > 0$, then there exists a polynomial P(x) such that for all x in I, $|f(x) - P(x)| < \varepsilon$ [28].

2.5 A closed and bounded subset of a finite dimensional normed linear space is compact [26].

2.6 If $\{S_n(x)\}$ is a sequence of functions defined for x in a deleted neighborhood J of x = c and if

1. $S_n = \lim_{x \to c} S_n(x)$ exists and is finite for each n.
2. $f(x) = \lim_{n \to \infty} S_n(x)$ exists and is finite for each x in J.
3. The convergence in 2 is uniform.

Then:

4. $\lim_{n \to \infty} S_n$ exists and is finite.
5. $\lim_{x \to c} f(x)$ exists and is finite.

6. The limits in 4 and 5 are equal [28].

2.7 (Cauchy-Schwartz inequality) In an inner-product space $|(f,g)| \leq ||f||\,||g||$ with equality if and only if f and g are linearly dependent [28].

2.8 If m functions of n variables where $m \leq n$ are functionally dependent in a region R then every m^{th} order Jacobian of the m functions with respect to m of the variables vanishes identically in R [28].

2.9 If $f(x,y)$ and $f_x(x,y)$ exist and are continuous on $a \leq x \leq b$, $c \leq y \leq d$, then the function

$F(x) = \int_c^d f(x,y)dy$ is differentiable for

$a \leq x \leq b$ and $F_x(x,y) = \int_c^d f_x(x,y)dy$ [28].

2.10 (Zorn's Lemma) A partially ordered system has a maximal element if every totally ordered subset has an upper bound [13].

2.11 (Tychonoff) A Cartesian product of compact spaces is compact in its product topology [13].

2.12 Let X,Y be topological spaces, $F: X \to Y$. Then f is continuous if and only if $A \subseteq X$ implies that $f(\overline{A}) \subseteq \overline{f(A)}$ [13].

2.13 A compact subset of a topological vector space is bounded [13].

2.14 Let V be a convex set containing zero as an interior point in a topological vector space X and let $q = q_V$ be the Minkowski functional on V. Then

1. $q(x) \geq 0$.
2. $q(ax) = aq(x)$ for $a \geq 0$.
3. The set of interior points of V is characterized by the condition $q(x) < 1$ and the set of boundary points by the condition $q(x) = 1$ [13].

2.15 In a topological vector space X, the intersection of convex sets is convex, if K_1, K_2 are convex and T is a linear map of X, then aK_1, $K_1 + K_2$, and TK_1 are all convex [13].

2.16 The Hilbert cube is compact [13].

2.17 If C is a compact subset of a metric space X and x is in X then there exists a point c in C such that $||x - c|| = \inf_{y \in C} ||y - x||$ [33].

2.18 Let S be a compact subset of a topological vector space X and let K be a bounded set in $C(S)$. If K is compact, then for every $\varepsilon > 0$ there exists a neighborhood U of the origin in X such that $|f(t) - f(s)| < \varepsilon$ for all f in K and all s,t in S such that $t - s$ is in U [13].

2.19 If p and q are distinct points of a locally convex topological vector space X then there exists a continuous linear functional f defined on X such that $f(p) \neq f(q)$ [13].

RELATIONS

I. Spaces

2.1 A normed vector space is a locally convex topological vector space [12].

2.2 A finite dimensional normed linear space is reflexive [13].

2.3 A Hilbert space is reflexive [13].

2.4 A finite dimensional Banach space is strictly convex if and only if it is uniformly convex, but an infinite dimensional space can be strictly convex without being uniformly convex [23].

2.5 Hilbert space is uniformly convex [23].

2.6 Any uniformly convex Banach space is reflexive [23].

2.7 The following are GB spaces:

1. Hilbert spaces.
2. Banach spaces with monotone Schauder bases.
3. C[0,1] [21].

2.8 A complete locally convex topological vector space satisfies condition E [18].

2.9 Every bounded decreasing net of nonempty closed convex subsets of X has a nonempty intersection is a necessary and sufficient condition that a Banach space be reflexive [25].

2.10 In compact metric spaces condition 3 is always satisfied [15].

2.11 Separable GB spaces are PB spaces [21].

2.12 The normed linear space X is reflexive if and only if the unit ball is weakly compact [33].

II. Operators

2.13 A linear operator in a normed space is bounded if and only if it is continuous [26].

2.14 The class of P compact operators with $p < 0$ contains, among others, the following when defined in a PB space X:

1. Closed, precompact.
2. Completely continuous and strongly continuous.
3. Quasicompact.
4. Continuous, demicontinuous, and weakly continuous monotone increasing operators in a Hilbert space [29].

2.15 The class of P compact operators with $p > 0$ includes the following in a PB space:

1. Closed, precompact.
2. A if $-A$ is quasicompact.
3. Continuous, demicontinuous, and weakly continuous monotone decreasing operators in a Hilbert space [29].

2.16 The following are true in a Hilbert space X:

1. U is in P_3 if and only if I-U is in M_3.
2. U is in P_2 if and only if I-U is in M_2.
3. U is in P_0 (P_1) implies that I-U is in $M_2 \subsetneq M_3$.
4. U is in P_3 implies that Ray(U) $\subsetneq P_3$; U is in P_2 implies that Ray(U) $\subsetneq P_2$ [9].

2.17 U is strictly pseudocontractive if and only if there is a W in Ray(U) such that W is nonexpansive [9].

2.18 Let X be a GB space. Then the following are G operators:

1. X separable, every completely continuous operator mapping a closed bounded convex set into X.
2. X separable, every strongly continuous operator mapping a closed bounded convex set into X.
3. X separable, P compact operators.
4. X separable and reflexive, every weakly

continuous operator mapping a closed bounded convex set into X.

5. X reflexive, X^* strictly convex, and the duality map J both continuous and weakly continuous, every operator of the form I-A where A is J monotone and demicontinuous, and every nonexpansive operator [21].

2.19 In a reflexive Banach space strong continuity implies complete continuity [21].

2.20 If X is a convex, complete metric space then every map f of X into itself which is (k,a) uniformly locally contractive is also a strict contraction with the same constant a [14].

2.21 If U is a nonexpansive map of a Banach space X and T = I - U, then T is an accretive map in X [7].

2.22 Let X be a Hilbert space, U an operator in X, T = I - U. U is II pseudocontractive if and only if T is accretive [7].

2.23 A closed linear map whose domain is a complete metric space and whose range is a subset of a complete metric space is continuous [33].

2.24 If T is a continuous linear operator with closed domain then T is closed [33].

2.25 A compact linear operator is continuous and thus completely continuous [33].

III. Sets

2.26 A subset of a reflexive space is weakly compact if and only if it is bounded [13].

2.27 A necessary and sufficient condition that a subset of a complete metric space be compact is that it be closed and totally bounded [26].

2.28 A bounded, closed convex subset of a uniformly convex Banach space has complete normal structure [2].

2.29 A compact convex subset of a Banach space has complete normal structure [2].

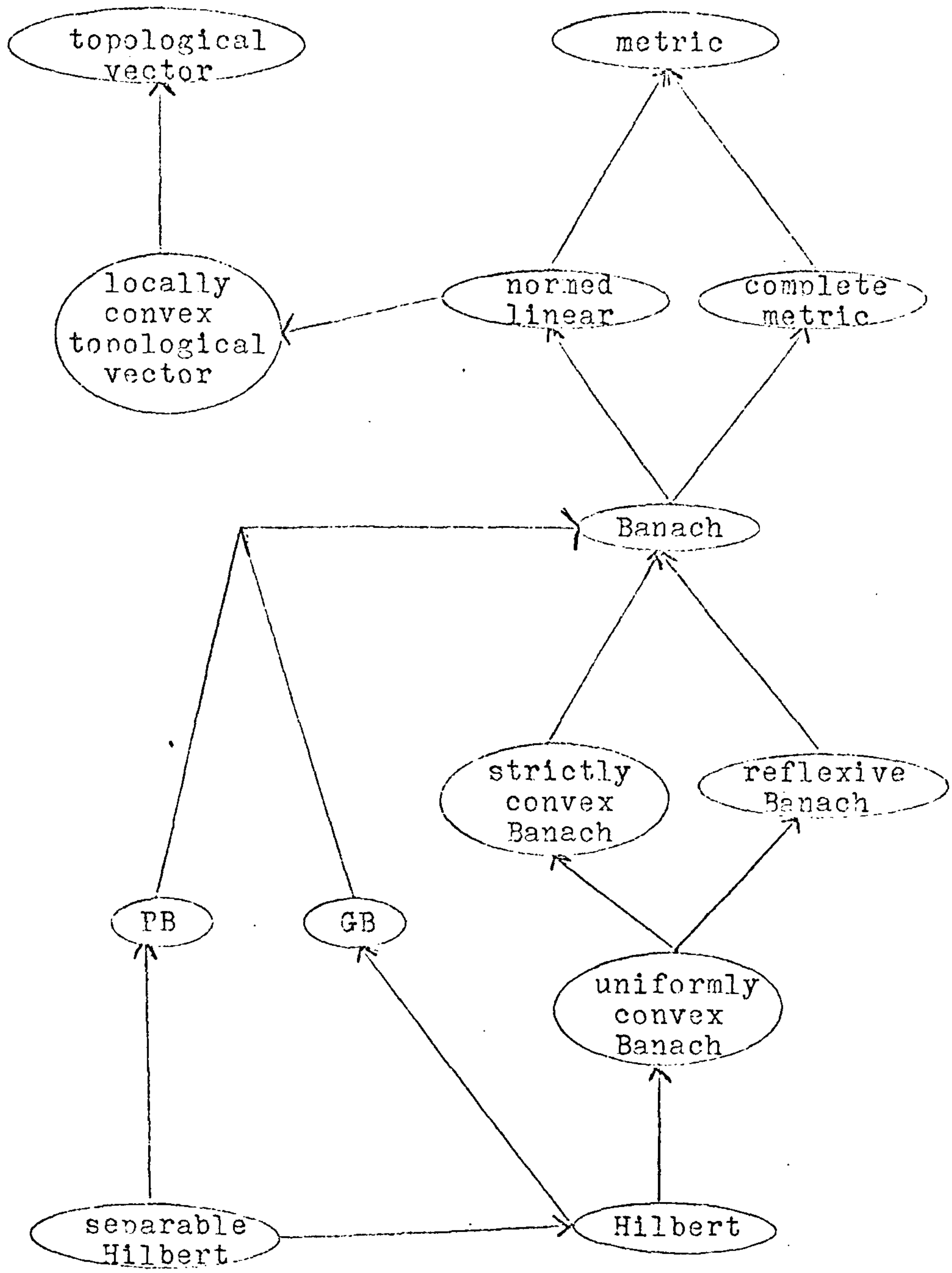

FIGURE 1

SPACES

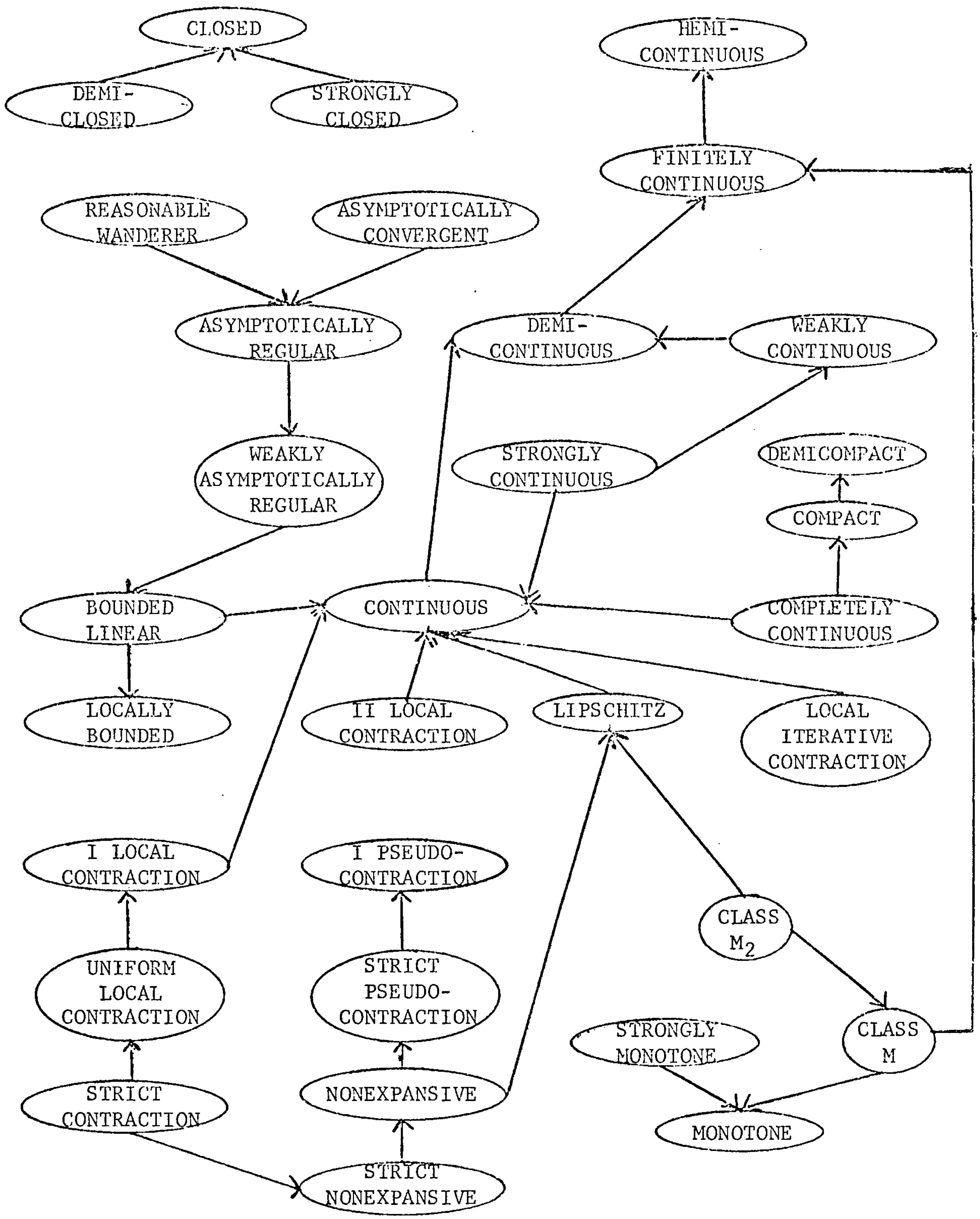

FIGURE 2

OPERATORS

CHAPTER III

CLASSICAL THEOREMS

It is the purpose of this chapter to give complete proofs for several well known and very fundamental theorems. These theorems initiated the study of fixed points, and practically all succeeding research has been and continues to be directed toward their generalization or modification.

THE BROUWER THEOREM

The classical Brouwer theorem remains of fundamental importance in fixed point theory. Even the most modern theorems, with the notable exception of those dealing with contractions, ultimately rely on Brouwer's result. It is usually stated in one of two equivalent forms:

1. A continuous map of the closed unit ball in E^n into itself has a fixed point.
2. A continuous map of an n-simplex in E^n into itself has a fixed point.

Proofs of the theorem range from the purely topological using algebraic topology and the concept of degree of a function as in Dugundji [12], to varying mixtures of topology and analysis, using results from combinatorial topology as in Kantorovitch [24] and Graves [20], to the purely analytical, using theorems of differential and integral equations as in Dunford and Schwartz [13]. Since this thesis is concerned with the viewpoint of analysis, this latter approach is used below.

Lemma 3.1. Let f be an infinitely differentiable function of the $n + 1$ variables $x_0, \ldots, x_n$ with values in E^n. Let D_i denote the determinant of the $n \times n$ matrix

$$M = (f_{x_0}, \ldots, f_{x_{i-1}}, f_{x_{i+1}}, \ldots, f_{x_n})$$

whose columns are the n partial derivatives $f_{x_0}, \ldots, f_{x_{i-1}}, f_{x_{i+1}}, \ldots, f_{x_n}$.

Then

$$\sum_{i=0}^{n} (-1)^i \frac{\partial}{\partial x_i} D_i = 0$$

Proof: For every pair i,j of unequal integers between 0 and n, let C_{ij} denote the determinant of the matrix whose first column is $f_{x_i x_j}$ and whose remaining columns are $f_{x_0}, \ldots, f_{x_n}$ arranged in order of increasing indices and where f_{x_i} and f_{x_j} are omitted. Since $f_{x_i x_j} = f_{x_j x_i}$, $C_{ij} = C_{ji}$. Furthermore, using fact 2.2 of Chapter II and the rules for interchanging columns in determinants, we have

$$\frac{\partial}{\partial x_i} D_i = \det(f_{x_0 x_i}, f_{x_1}, \ldots, f_{x_{i-1}}, f_{x_{i+1}}, \ldots, f_{x_n})$$

$$+ \det(f_{x_0}, f_{x_1 x_i}, \ldots, f_{x_{i-1}}, f_{x_{i+1}}, \ldots, f_{x_n}) + \ldots$$

$$+ \det(f_{x_0}, f_{x_1}, \ldots, f_{x_{i-1}}, f_{x_{i+1}}, \ldots, f_{x_n x_i})$$

$$= \sum_{j<i} (-1)^j C_{ij} + \sum_{j>i} (-1)^{j-1} C_{ij}.$$

Hence,

$$(-1)^i \frac{\partial}{\partial x_i} D_i = \sum_{j=0}^{n} (-1)^{i+j} C_{ij} \sigma(i,j)$$

where

$$\sigma(i,j) = \begin{cases} 1 \text{ if } j < i \\ 0 \text{ if } j = i \\ -1 \text{ if } j > i. \end{cases}$$

Thus,

$$\sum_{i=0}^{n} (-1)^i \frac{\partial}{\partial x_i} D_i = \sum_{i,j=0}^{n} (-1)^{i+j} C_{ij} \sigma(i,j).$$

However, by interchanging the summation indices,

$$\sum_{i=0}^{n} (-1)^i \frac{\partial}{\partial x_i} D_i = \sum_{j=0}^{n} (-1)^j \frac{\partial}{\partial x_j} D_j$$

$$= \sum_{i,j=0}^{n} (-1)^{i+j} C_{ji} \sigma(j,i).$$

Thus

$$\sum_{i,j=0}^{n} (-1)^{i+j} C_{ij}\sigma(i,j) = \sum_{i,j=0}^{n} (-1)^{j+i} C_{ji}\sigma(j,i).$$

But $C_{ij} = C_{ji}$ and $\sigma(i,j) = -\sigma(j,i)$. Hence

$$\sum_{i,j=0}^{n} (-1)^{i+j} C_{ij}\sigma(i,j) = (-1) \sum_{i,j=0}^{n} (-1)^{i+j} C_{ij}\sigma(i,j).$$

Therefore,

$$\sum_{i=0}^{n} (-1)^{i} \frac{\partial}{\partial x_i} D_i = \sum_{i,j=0}^{n} (-1)^{i+j} C_{ij}\sigma(i,j) = 0,$$

which completes the proof.

Theorem 3.2 (Brouwer): If T is a continuous mapping of the closed unit ball B of E^n into itself, then there is a point y in B such that $T(y) = y$.

Proof: The case of complex scalars is a consequence of the case of real scalars. This follows from the fact that unitary n-space is isometric to Euclidean 2n-space, and the unit balls in these spaces correspond in a natural way.

Further, it is sufficient to consider the infinitely differentiable case. The Weierstrass Approximation Theorem (Fact 2.4) for continuous functions of n variables implies that the continuous map T of B into itself is the uniform limit of a sequence $\{T_k\}$ of infinitely differentiable mappings of B into itself. Suppose that the theorem has been proved for infinitely differentiable maps. Then for each integer k there is a point y_k in B such that $T_k(y_k) = y_k$. Since B is a closed and bounded subset of E^n it is compact, and thus some subsequence $\{y_{k_i}\}$ converges to a point y in B. Since $\{T_{k_i}(x)\}$ converges to Tx uniformly on B,

$$T(y) = \lim_{i\to\infty} T_{k_i}(y) = \lim_{i\to\infty} (\lim_{i\to\infty} T_{k_i}(y_{k_i})) = \lim_{i\to\infty} \lim_{i\to\infty} y_{k_i}$$

$$= y.$$

Thus we suppose that T is an infinitely differentiable map of B into itself and, by way of contradiction, that $T(x) \neq x$ for all x in B. Let $a = a(x)$ be the larger root of the quadratic equation $|x + a(x-T(X))|^2 = 1$. Then

$$1 = (x+a(x-T(x)),\ x+a(x-T(x)))$$

$$= |x|^2 + 2a(x,\ x-T(x)) + a^2|x - T(x)|^2.$$

We show that such an a does exist for x in B. By the quadratic formula

$$|x - T(x)|^2 a = (x,\ T(x) - x)$$

$$+ \{(x,\ x-t(x))^2 + (1-|x|^2)|x - T(x)|^2\}^{\frac{1}{2}}. \qquad (1)$$

Since $|x - T(x)| \neq 0$ for x in B, the discriminant $(x,\ x-T(x))^2 + (1-|x|^2)|x - T(x)|^2$ is positive when $|x| \neq 1$. If $|x| = 1$ then if $(x,\ x-T(x)) = 0$ we would have

$$|(x,T(x))| = |(x,x)| = |x|^2 = 1.$$

However $1 = |(x,T(x))| \leq |x||T(x)| \leq 1 \cdot 1 = 1$. Thus we must have $|(x,T(x))| = |x||T(x)|$ and hence $T(x)$ and x are linearly dependent, that is $T(x) = kx$ for some scalar k. Then $1 = |(x,T(x))| = |(x,kx)| = |k||x|^2 = |k|$, and hence $T(x) = x$, contrary to our assumption. Thus $(x,x-T(x)) \neq 0$ for $|x| = 1$. Therefore the discriminant is always positive in B, and a does exist.

Furthermore, since the square root function is an infinitely differentiable function of t for positive t, and since $|x - T(x)| \neq 0$ for x in B, it follows from (1) that $a(x)$ is an infinitely differentiable function of x in B. Moreover, from (1) we have $a(x) = 0$ for $|x| = 1$.

Now for each real number t define $f(t,x) = x + ta(x)(x-T(x))$. Then f is an infinitely differentiable function of the $n + 1$ variables $t, x_1, \ldots, x_n$ with values in E^n. Since $a(x) = 0$ for $|x| = 1$, we have $f_t(t,x) = a(x)(x-T(x)) = 0$ for $|x| = 1$. Also $f(0,x) = x$, and from the definition of a, $|f(1,x)| = |x + a(x)(x-T(x))| = 1$ for all x in B. Denote the determinant of the matrix $M(t,x)$ whose columns are the vectors $f_{x_1}(t,x), \ldots, f_{x_n}(t,x)$ by $D_0(t,x)$ and consider

$$I(t) = \int_B D_0(t,x)\, dx_1 \ldots dx_n.$$

Now $D_0(0,x) = 1$ since $M(0,x)$ is the identity. Hence

$$I(0) = \int_B dx_1 \ldots dx_n \neq 0.$$

Since $f(1,x)$ satisfies the nontrivial functional dependence $|f(1,x)| = 1$, it follows from fact 2.8 that the Jacobian determinant $D_0(1,x)$ is identically zero, hence $I(1) = 0$. Thus, if we can show that $I'(t) = 0$, we will have the contradiction that $I(t)$ is constant in B. Now

$$I'(t) = \int_B \frac{d}{dt} D_0(t,x)\, dx_1 \ldots dx_n.$$

From the lemma,

$$\sum_{i=0}^{n} (-1)^i \frac{\partial}{\partial x_i} D_i = 0, \text{ or}$$

$$\frac{\partial}{\partial x_0} D_0 = \sum_{i=1}^{n} (-1)^{i-1} \frac{\partial}{\partial x_i} D_i.$$

Hence, letting $x_0 = t$, we get

$$\frac{\partial}{\partial t} D_0(x,t) = \sum_{i=1}^{n} (-1)^{i-1} \frac{\partial}{\partial x_i} D_i(x,t),$$

where $D_i(x,t)$ is the determinant of the matrix whose columns are the vectors

$f_t(t,x)$, $f_{x_1}(t,x), \ldots, f_{x_{i-1}}, f_{x_{i+1}}, \ldots, f_{x_n}(t,x)$.

Thus $I'(t)$ is a sum of integrals of the form

$$\pm \int_B \frac{\partial}{\partial x_i} D_i(x,t) dx_1 \ldots dx_n. \qquad (2)$$

We will now express the integrand in terms of $n - 1$ coordinates, omitting the i^{th}, and then perform the integration on this coordinate. Let B_i denote the unit ball in the space of $n - 1$ variables $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$. Let x_i^+ denote the positive square root $\{1 - (x_1^2 + \ldots + x_{i-1}^2 + x_{i+1}^2 + \ldots + x_n^2)\}^{\frac{1}{2}}$ and x_i^- denote the corresponding negative square root. Let p_i^+ denote the point whose j^{th} coordinate is x_j if $i \neq j$ and is x_i^+ if $i = j$. Let p_i^- denote the point whose j^{th} coordinate is x_j if $i \neq j$, and is x_j^- if $i = j$. Then, carrying out the integration on x_i, (2) reduces to

$$\pm \int_{B_i} D_i(t,p_i^+)\, dx_1 \ldots dx_{i-1} dx_{i+1} \ldots dx_n$$

$$\pm \int_{B_i} D_i(t,p_i^-)\, dx_1 \ldots dx_{i-1} dx_{i+1} \ldots dx_n.$$

However $|p_i^+| = |p_i^-| = 1$, and since $f_t(t,x) = 0$ for $|x| = 1$, it follows from the definition of D_i that these integrals are zero. Hence $I'(t) = 0$ and we have the desired contradiction.

Thus T does have a fixed point in B, and the proof of the Brouwer theorem is complete.

THE BANACH-CACCIOPOLI THEOREM

Probably the most fruitful of the early theorems from the standpoint of applications is the Banach-Cacciopoli contraction principle. The basic idea of the proof, that of taking Picard iterates, has been widely used in the more modern theorems. These have sought to relax the conditions imposed on the operator while retaining the practical usefulness of the older theorem by giving a constructive method of approximating the fixed point.

Theorem 3.3 (Banach-Cacciopoli): Let (R,d) be a complete metric space, C a closed subset of R, A a mapping of C into itself for which there exists a k, $0 \leq k < 1$, such that $d(Ax,Ay) \leq kd(x,y)$ for any two points x,y in C. Then A has a unique fixed point in C which may be obtained as the limit of the sequence of Picard iterates [26].

Proof: Let x_0 be an arbitrary point in C. Set $x_1 = Ax_0$, $x_2 = Ax_1 = A^2x_0$, and so in general let $x_n = Ax_{n-1} = A^nx_0$. We shall show that this sequence of Picard iterates is fundamental.

First, note that $d(A^nx_0, A^mx_0) \leq k^n d(x_0, x_{m-n})$ for any m,n with $n \leq m$. This follows by induction on n. If $n = 0$, $d(x_0, A^mx_0) = d(x_0, x_m) \leq d(x_0, x_m)$.

Suppose that for $n = j$ and for any m with $j \leq m$ we have

$$d(A^jx_0, A^mx_0) \leq k^j d(x_0, x_{m-j}).$$

Now consider $n = j + 1$. If $j + 1 \leq m$ then $j \leq m - 1$ and hence $d(A^{j+1}x_0, A^mx_0) \leq kd(A^jx_0, A^{m-1}x_0) \leq k(k^j d(x_0, x_{m-1-j}))$

$$= k^{j+1} d(x_0, x_{m-(j+1)}),$$ completing the

the induction. Hence

$$d(x_n,x_m) = d(A^n x_0, A^m x_0) \le k^n d(x_0, x_{m-n})$$

$$\le k^n\{d(x_0,x_1) + d(x_1,x_2) + \ldots + d(x_{m-n-1}, x_{m-n})\}$$

$$\le k^n d(x_0,x_1)\left(1 + k + k^2 + \ldots + k^{m-n-1}\right)$$

$$\le \frac{k^n d(x_0,x_1)}{(1-k)}.$$

Since $k < 1$ this quantity becomes arbitrarily small for sufficiently large n. Thus the sequence is fundamental, and since R is complete, $\lim_{n\to\infty} x_n = x$ exists. Moreover, since A maps C into itself, $\{x_n\} \subseteq C$. Since C is closed, x is in C. By virtue of the continuity of A,

$$Ax = A(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} Ax_n = \lim_{n\to\infty} x_{n+1} = x.$$

Thus the existence of a fixed point and the convergence of the Picard iterates to it are established.

To see that the point is unique, suppose that $Ax = x$ and $Ay = y$. Then $d(x,y) = d(Ax,Ay) \le kd(x,y)$ where $k < 1$. Thus we must have $d(x,y) = 0$, and therefore $x = y$.

THE SCHAUDER-TYCHONOFF THEOREM

The initial work on generalizing the Brouwer Theorem was done in the direction of weakening the conditions on the space under consideration. In 1922 Birkhoff and Kellogg [3] extended Brouwer's result to continuous self mappings of compact convex subsets of certain function spaces such as $L_2[0,1]$ and $C^n[0,1]$. Schauder [34] obtained the following results in 1930:

1. A compact, convex subset of a Banach space has the fixed point property for continuous mappings.
2. A convex, weakly compact subset of a separable Banach space has the fixed point property for weakly continuous mappings.

At present, the most general theorem for continuous mappings, from the point of view of vector spaces, is that proved by Tychonoff in 1935 for compact, convex subsets of locally convex topological vector spaces. Its heavy reliance on the ideas used by Schauder in his proofs cause it to be frequently referred to as the Schauder-Tychonoff theorem. The proof given below is based on that found in [13].

Lemma 3.4: The Hilbert cube has the fixed point property for continuous mappings.

Proof: Let T be a continuous map from the Hilbert cube C into itself, and let P_n: $C \to C$ be the map given by $P_n(x_1, \ldots, x_n, x_{n+1}, \ldots) = (x_1, \ldots, x_n, 0, 0, \ldots)$. The set $C_n = P_n(C)$ is clearly homeomorphic to the closed unit ball in E^n. Since P_n and T are both continuous, P_nT: $C_n \to C_n$ is continuous, and thus by the Brouwer Theorem has a fixed point y_n in $C_n \subseteq C$. Thus

$$|y_n - T(y_n)| = |P_nT(y_n) - T(y_n)| \leq \sqrt{\sum_{i=n+1}^{\infty} 1/i^2}.$$

Now since C is compact, $\{y_n\}$ has a convergent subsequence, say $\{y_{n_i}\} \to y$ in C. Let $\varepsilon > 0$. Then there exists an integer N such that $\sum_{i=n+1}^{\infty} 1/i^2 < \varepsilon^2/9$ for $n \geq N$. Thus

$$|y_n - T(y_n)| < (\varepsilon^2/9)^{\frac{1}{2}} = \varepsilon/3.$$

Also, since $\{y_{n_i}\} \to y$ and T is continuous, $\{Ty_{n_i}\} \to Ty$, so there exists an integer N_1 such that $|Ty_{n_i} - Ty| < \varepsilon/3$ for $n_i \geq N_1$. Likewise, there exists an integer N_2 such that for $n_i \geq N_2$, $|y_{n_i} - y| < \varepsilon/3$. Let N_3 be the maximum of N, N_1, N_2. Then since

$$\sum_{i=N_3+2}^{\infty} 1/i^2 \leq \sum_{i=N+1}^{\infty} 1/i^2 < \varepsilon^2/9, \text{ we have for } n_i = N_3+1,$$

$$||T(y) - y|| \leq ||T(y) - T(y_{n_i})|| + ||T(y_{n_i}) - y_{n_i}||$$

$$+ ||y_{n_i} - y|| < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.$$

However, ε was arbitrary, and thus we must have $T(y) = y$.

Therefore, T has a fixed point in C.

Lemma 3.5: Any closed convex subset K of the Hilbert cube C has the fixed point property for continuous mappings.

Proof: Since C is compact and K is closed, K is compact. Thus by fact 2.17 to each point p in C there is a point N(p) in K such that $||N(p) - p|| = d = \inf_{k \in K} ||k - p||$.

To see that this point is unique, observe first that if p is in K, then $0 = ||p - p|| < ||p - k||$ for all k in $K/\{p\}$. Thus we may suppose that p is not in K, and suppose that k_1, k_2 are in K such that $||k_1 - p|| = ||k_2 - p|| = d$. Since K is convex, $\frac{1}{2}(k_1+k_2)$ is in K. Thus $||p - \frac{1}{2}(k_1+k_2)|| \le \frac{1}{2}||p - k_1|| + \frac{1}{2}||p - k_2|| = d$, while on the other hand $||p - \frac{1}{2}(k_1+k_2)|| \ge d$. Consequently, $||p - \frac{1}{2}(k_1+k_2)|| = d$. Therefore, $||p - \frac{1}{2}(k_1+k_2)|| = \frac{1}{2}||p - k_1|| + \frac{1}{2}||p - k_2||$. But since a Hilbert space is strictly convex, and since $p - k_1 \neq 0$, $p - k_2 = t(p-k_1)$ for some $t \ge 0$. Thus $d = ||p - k_2|| = t||p - k_1|| = td$. Hence $t = 1$, and we have $k_1 = k_2$. Thus the function N(p) from C to K is well-defined.

We now show that N is continuous. Suppose that there exists a point p at which N is not continuous. Then we can construct a sequence $\{p_n\}$ which converges to p and such that no subsequence of $\{N(pn)\}$ converges to N(p). However, since K is compact, $\{N(p_n)\}$ has a convergent subsequence $\{N(p_{n_i})\} \to q$ in K and $q \neq N(p)$. However,

$$||p_{n_i} - N(p_{n_i})|| \le ||p_{n_i} - N(p)|| \le ||p_{n_i} - p|| + ||p - N(p)||.$$

Hence,

$$\begin{aligned} ||p - q|| &= ||p - p_{n_i} + p_{n_i} - N(p_{n_i}) + N(p_{n_i}) - q|| \\ &\le ||p - p_{n_i}|| + ||p_{n_i} - N(p_{n_i})|| + ||N(p_{n_i}) - q|| \\ &\le 2||p - p_{n_i}|| + ||p - N(p)|| + ||N(p_{n_i}) - q||. \end{aligned}$$

However, since $\{p_{n_i}\} \subseteq \{p_n\} \to p$, $||p - p_{n_i}||$ can be made arbitrarily small, and since $\{N(p_{n_i})\} \to q$, $||N(p_{n_i}) - q||$ can be made arbitrarily small. Thus $||p - q|| \leq ||p - N(p)||$ and we must have $q = N(p)$ since q is in K and N(p) is the unique nearest point of K to p. We have a contradiction, and therefore N is continuous.

Now if $T: K \to K$ is continuous, then $TN: C \to K$ is continuous, and thus by lemma 3.4 TN has a fixed point in C. Suppose that $p^* = TN(p^*)$. Then since $TN(p^*)$ is in K, p^* is in K and thus $N(p^*) = p^*$. Therefore $T(p^*) = p^*$.

Lemma 3.6: Let K be a compact convex subset of a locally convex topological vector space X. Let $T:K \to K$ be continuous. If K contains at least two points, then there exists a proper closed convex subset K_1 of K such that $T(K_1) \subseteq K_1$.

Proof: Without loss of generality, we let K have the X* topology since the identity map from X with the original topology to X with the X* topology is continuous, and thus, since K is compact, a homeomorphism of K. Therefore, changing to the X* topology does not affect the hypothesis of the lemma.

We will say that a set of continuous linear functionals F is determined by another set G if for each f in F and $\varepsilon > 0$, there exists a neighborhood

$N(0,\gamma,\delta) = \{x: \ |g(x)| < \delta, \ g \text{ in } \gamma\}$ where γ is a finite subset of G, with the property that if p,q are in K and $p - q$ is in $N(0,\gamma,\delta)$ then $|f(Tp) - f(tq)| < \varepsilon$. Clearly, if F is determined by G, then $g(p) = g(q)$ for all g in G implies that $f(p) = f(q)$ for all f in F.

We begin the proof by showing that each continuous linear functional f is determined by some denumerable set of functionals $G = \{g_m\}$. Since X is a topological group, K a compact subset of X, and $\{fT\}$ a bounded, conditionally compact subset of C(K), it follows from fact 2.18 that for each integer n there exists a neighborhood $N(0,\gamma_n,\delta_n)$ where γ_n is a finite set of continuous linear functionals and $\delta_n > 0$ such that if p,q are in K and $p - q$ in $N(0,\gamma_n,\delta_n)$, then $|f(Tp) - f(Tq)| < 1/n$. Let $G = \bigcup_{n=1}^{\infty} \gamma_n$. Then f is determined by G.

Thus if F is a denumerable subset of X*, and if for each f in F, G_f is the denumerable set which determines f, then the set $G = \bigcup_{f\varepsilon F} G_f$ is a denumerable set which determines F. Moreover, each continuous linear functional f can be included in a denumerable self-determined set G of continuous functionals, since if f is determined by the denumerable set G_1, G_1 by the denumerable set G_2, G_2 by G_3, and so forth, then $G = \{f\} \cup \bigcup_{i=1}^{\infty} G_i$ is self-determined and denumerable.

We will show that if $G = \{g_i\}$ is a denumerable, self-determined set such that for some g_j in G, p,q in K, $g_j(p) \neq g_j(q)$, then $G' = \{k_i g_i\}$, where $\{k_i\}$ is a set of positive scalars, has the same properties as G. Clearly, G' is denumerable and $k_j g_j(p) \neq k_j g_j(q)$. To see that it is self-determined, let $k_i g_i$ be in G' and $\varepsilon > 0$. Then since $\varepsilon/k_i > 0$, g_i in G and G is self-determined, there exists a neighborhood $N(0,\gamma,\delta)$ where $\gamma = \{g_{m_h}\}$ is a finite subset of G and if s,t are in K with s - t in $N(0,\gamma,\delta)$ then $|g_i(Ts) - g_i(Tt)| < \varepsilon/k_i$. Let $\gamma' = \{k_{m_h} g_{m_h}\}$ and $\delta' = \min \{k_{m_h}\}\delta$, and consider the neighborhood $N(0,\gamma',\delta')$. If s,t are in K and s - t is in $N(0,\gamma',\delta')$, then $|k_{m_h} g_{m_h}(s-t)| < \delta'$ for all $k_{m_h} g_{m_h}$ in γ'. Thus, for all g_{m_h} in γ $|g_{m_h}(s-t)| < (\delta'/k_{m_h}) = (\min \{k_{m_h}\}/k_{m_h})\delta \leq \delta$. Hence s - t is in $N(0,\gamma,\delta)$ and thus $|g_i(Ts) - g_i(Tt)| < (\varepsilon/k_i)$. Therefore $|k_i g_i(Ts) - k_i g_i(Tt)| < \varepsilon$. Thus G' is self-determined.

Now suppose that K contains two distinct points p and q. Then there exists an f in X* such that $f(p) \neq f(q)$. Let $G = \{g_i\}$ be a denumerable self-determined set of continuous linear functionals containing f. Since K is compact, $g_i(K)$ is a compact set of scalars for each i and hence closed and bounded. Since, by the above, we can multiply each g_i by an appropriate constant without changing the properties of G, we may assume that $|g_i(K)| \leq (1/i)$.

We will show that the map $H:K \to l_2$ defined by

$H(k) = (g_1(k), g_2(k), \ldots)$ is a continuous map of K onto a compact, convex subset K_0 of the Hilbert cube C which contains at least two points. Since $|g_i(K)| \leq 1/i$, $K_0 \subseteq C$. Since f is in G, p and q in K, and $f(p) \neq f(q)$, $H(p) = \{g_i(p)\} \neq \{g_i(q)\} = H(q)$, so K_0 has at least two points. Now let U be a neighborhood of $(\{g_i(k)\})$ in K_0, that is

$$U = \{(\{g_i(h)\}) : (\sum_{i=1}^{\infty} |g_i(k) - g_i(h)|^2)^{\frac{1}{2}} < \varepsilon\}.$$

Now since $|g_i(K)| \leq 1/i$,

$$|g_i(k) - g_i(h)| \leq |g_i(k)| + |g_i(h)| \leq (1/i) + (1/i) = (2/i).$$

Pick N such that $\sum_{i=N+1}^{\infty} (4/i^2) < (\varepsilon^2/2)$. Consider the neighborhood of k given by

$$V = \{h : |g_i(k) - g_i(h)| < (\varepsilon/\sqrt{2N}) \text{ for } i = 1,2,\ldots,N\}.$$

If h is in V, $H(h) = \{g_i(h)\}$ has the property that

$$(\sum_{i=1}^{\infty} |g_i(k) - g_i(h)|^2)^{\frac{1}{2}} = (\sum_{i=1}^{N} |g_i(k) - g_i(h)|^2$$

$$+ \sum_{i=N+1}^{\infty} |g_i(k) - g_i(h)|^2)^{\frac{1}{2}}$$

$$< ((N\varepsilon^2/2N) + (\varepsilon^2/2))^{\frac{1}{2}} = (\varepsilon^2)^{\frac{1}{2}} = \varepsilon.$$

Thus $H(h)$ is in U. Hence H is continuous, and since K is compact, $H(K) = K_0$ is compact. Finally, observe that since each g_i is linear, H is linear so the convexity of K implies the convexity of K_0.

Now let $T_0 = HTH^{-1} : K_0 \to K_0$. We will show that T_0 is well-defined and continuous. Let $\{g_i(k)\}$ be in K_0, and suppose that k_1, k_2 are in $H^{-1}\{g_i(k)\}$, that is $g_i(k_1) = g_i(k_2)$ for all i. Then since G is self-determined, by our earlier remark we have that $g_i(Tk_1) = g_i(Tk_2)$ for all i. Thus $HTk_1 = HTk_2$, that is $HTH^{-1}\{g_i(k)\} = HTH^{-1}\{g_i(k)\}$. Hence T_0 is well defined. Now let b_0 be in K_0 and $\varepsilon > 0$. Choose N such that $\sum_{i=N+1}^{\infty} (1/i^2) < \varepsilon$. Then since G is self-determined, there exists $\delta > 0$ and m such that if

$|g_j(p) - g_j(q)| < \delta$ for $j = 1,\ldots,m$ then

$|g_i(Tp) - g_i(Tq)| < (\varepsilon/N)^{1/2}$ for $i = 1,\ldots,N$. Thus if

$|b - b_0| < \delta$ and if p and q are any points in K with

$b = \{g_i(p)\}$ and $b_0 = \{g_i(q)\}$, then since

$|g_j(p) - g_j(q)| \leq (\sum_{i=1}^{\infty} |g_j(p) - g_j(q)|^2)^{1/2} < \delta$, we have for

$i = 1,\ldots N$,

$|g_i(Tp) - g_i(Tq)| < (\varepsilon/N)^{1/2}$. Thus

$$\begin{aligned} ||T_0(b) - T_0(b_0)||^2 &= ||HTH^{-1}(b) - HTH^{-1}(b_0)||^2 \\ &\leq \sum_{i=1}^{N} |g_i(Tp) - g_i(Tq)|^2 + 2 \sum_{i=N+1}^{\infty} (1/i^2) \\ &< 3\varepsilon. \end{aligned}$$

Thus T_0 is continuous.

Therefore, by lemma 3.5, T_0 has a fixed point k_0 in K_0. Hence, $TH^{-1}(k_0) \subseteq H^{-1}T_0(k_0) = H^{-1}(k_0)$. Setting $K_1 = H^{-1}(k_0)$ we note that K_1 is a proper subset of K since $H(p) \neq H(q)$ implies that either p or q does not belong to K_1. Also, as the inverse image of the closed set $\{k_0\}$, K_1 is closed. $T(K_1) = TH^{-1}(k_0) \subseteq H^{-1}(k_0) = K_1$. Finally, if k_1, k_2 are in K_1 and $a,b \geq 0$ such that $a + b = 1$, then we have

$$\begin{aligned} H(ak_1+bk_2) &= \{g_i(ak_1+bk_2)\} = \{ag_i(k_1)\} + \{bg_i(k_2)\} \\ &= a\{g_i(k_1)\} + b\{g_i(k_2)\} = aH(k_1) + bH(k_2) \\ &= (a+b)k_0 = k_0. \end{aligned}$$

Thus $ak_1 + bk_2$ is in K_1, and hence K_1 is convex.

Therefore, K_1 has the desired properties, and the proof of the lemma is complete.

Theorem 3.7 (Schauder-Tychonoff): A continuous map $T:K \to K$ of a compact, convex subset K of a locally convex topological vector space has a fixed point in K.

Proof: By Zorn's Lemma there exists a minimal convex, closed subset K_1 of K with the property that $TK_1 \subseteq K$. By lemma 3.6 this minimal subset contains only one point. Thus this point is a fixed point of T.

THE KAKUTANI THEOREM

Another older theorem which is of some importance in the development of fixed point theory was proved by Kakutani in 1938. This is one of the earliest of the small but significant number of theorems which are concerned with families of mappings. These establish the existence of a fixed point, not simply for one function, but for a collection of functions which satisfy certain properties. Kakutani's result is particularly interesting because, unlike the other theorems on families, there is no requirement that the functions commute. The reliance of his proof on Zorn's Lemma is typical of the theorems on families.

Theorem 3.8 (Kakutani): Let K be a compact, convex subset of a locally convex topological vector space X and let G be a group of linear mappings which is equicontinuous on K and such that $G(K) \subset K$. Then there exists a p in K such that $G(P) = p$[13].

Proof: By Zorn's Lemma K contains a minimal nonvoid compact convex subset K_1 such that $G(K_1) \subseteq K_1$. If K_1 has just one point we are through. If not, consider $K_1 - K_1 = \{k_1 - k_2 : k_1, k_2 \text{ are in } K_1\}$. Since K_1 is compact, $K_1 \times K_1$ is compact, and since subtraction is continuous $K_1 - K_1$ is compact. Since K_1 has more than one point, $K_1 - K_1$ contains some point other than the origin. Now X is Hausdorff and thus there exists a neighborhood V_1 of the origin such that $K_1 - K_1 \not\subseteq V_1$. By the equicontinuity of G on K_1 and the local convexity of X, there exists a convex neighborhood U_1 of the origin such that k_1, k_2 in K_1 and $k_1 - k_2$ in U_1 imply $G(k_1 - k_2) \subseteq V_1$. Let $U_2 = g(U_1)$. Observe that U_2 has the following properties:

1. U_2 is convex since U_1 is convex and the members of G are linear.
2. U_2 contains the origin since U_1 contains the origin and the members of G are linear.
3. U_2 is open since U_1 is open and G is a group, all of whose members are continuous.
4. $G(U_2) = U_2$ since G is a group.
5. $G(\overline{U_2}) \subseteq \overline{G(U_2)} = \overline{U_2} \subseteq G(\overline{U_2})$, and thus $\overline{U_2} = G(\overline{U_2})$, since each function in G is continuous and the identity map is in G.

Now let $d = \inf\{a : a > 0, K_1 - K_1 \subseteq aU_2\}$. We will show that d exists and is positive. By 2 and 3 above, U_2 is a neighborhood of zero. Moreover, since $K_1 - K_1$ is compact it is bounded. Thus there exists a $b > 0$ such that $K_1 - K_1 \subseteq b'U_2$ for all $|b'| \leq b$. Hence the set is nonempty and therefore d exists. We will show that d is strictly positive. Let $a > 0$ be such that $K_1 - K_1 \subseteq aU_2 = aG(U_1)$. Let $k_1 - k_2$ be in $K_1 - K_1$. Then $k_1 - k_2 = ag(u)$ for some g in G and u in U_1. Since G is a group, g^{-1} is in G and is linear. Thus

$g^{-1}(k_1-k_2) = g^{-1}(ag(u)) = ag^{-1}(g(u)) = au$ which is in aU_1. However, if $a \leq 1$, then since U_1 is a convex neighborhood of zero, au would belong to U_1. Thus $g^{-1}(k_1-k_2)$ is in U_1. Hence, by the linearity of g^{-1}, $g^{-1}(k_1) - g^{-1}(k_2)$ is in U_1. However, since $G(K_1) \subseteq K_1$, $g^{-1}(k_1)$ and $g^{-1}(k_2)$ are in K_1. Thus, by the construction of U_1, $G(g^{-1}(k_1) - g^{-1}(k_2)) \subseteq V_1$. Therefore, $gg^{-1}(k_1) - gg^{-1}(k_2) = k_1 - k_2$ is in V_1. But $k_1 - k_2$ was arbitrary in $K_1 - K_1$. Thus we have $K_1 - K_1 \subseteq V_1$, a contradiction. Therefore we must have $a > 1$ for all a in our set, and thus $d \geq 1 > 0$.

Now let $U = dU_2$. Note that since d is positive, the following properties of U follow from the corresponding properties of U_2:

1. U is open.
2. U is convex.
3. 0 is in U.
4. $G(\overline{U}) = G(\overline{dU_2}) = dG(\overline{U_2}) = d\overline{U_2} = \overline{U}$.

We show next that $K_1 - K_1 \nsubseteq (1-c)\overline{U}$ for $0 < c < 1$. Let x be in $(1-c)\overline{U}$, that is $x = (1-c)y$ where y is in $\overline{U}$. Let qu be the Minkowski functional on U. Then $qu(x) = qu(1-c)y = (1-c)qu(y)$. However, y is in $\overline{U}$ so $qu(y) \leq 1$. Thus $0 < qu(x) = (1-c)qu(y) \leq (1-c) < 1 - \frac{1}{2}c$. Therefore there exists $p < 1 - (c/2)$ such that x is in pU. However, $0 < p < 1 - (c/2) < 1$ implies that $p/(1 - \frac{1}{2}c) < 1$. Thus, since U is convex containing $0, (p/(1-\frac{1}{2}c))U \subsetneq U$. Hence

$pU \subseteq (1-\frac{1}{2}c)U$. Therefore x is in $(1-\frac{1}{2}c)U$. Thus we have $(1-c)\bar{U} \subseteq (1-\frac{1}{2}c)U$. Then $K_1 - K_1 \subseteq (1-c)\bar{U} \subseteq (1-\frac{1}{2}c)U$ $= (1-\frac{1}{2}c)dU_2$ implies that $(1-\frac{1}{2}c)d \geq d$, a contradiction. Thus $K_1 - K_1 \not\subseteq (1-c)\bar{U}$.

Also note that $K_1 - K_1 \subseteq (1+c)U$ for $0 < c < 1$, since $(1+c)d > d$ gives, by definition, $K_1 - K_1 \subseteq (1+c)dU_2 = (1+c)U$.

Now, since 0 is in U, the family of open sets $\{-\frac{1}{2}U + k : k \varepsilon K_1\}$ is an open cover of the compact set K_1. Let $\{-\frac{1}{2}U + k_1, -\frac{1}{2}U + k_2, \ldots, -\frac{1}{2}U + k_n\}$ be a finite subcover. Let $p = (1/n)(k_1 + \ldots + k_n)$. Then since K_1 is convex, p is in K_1. Now if k is in K_1, then $k_i - k$ is in $\frac{1}{2}U$ for some $1 \leq i \leq n$. Since $k_i - k$ is in $(1+c)U$ for $1 \leq i \leq n$ and all $0 < c < 1$, we have

$$p = (1/n)(k_1 + \ldots + k_n - nk + nk) = (1/n)(k_i - k + \sum_{j \neq i} k_j - k) + k.$$

Thus p is in $(1/n)(\frac{1}{2}U + (n-1)(1+c)U) + k$. Let $c = (1/4(n-1))$. Then p is in

$$\begin{aligned}(1/n)(\tfrac{1}{2}U+(n-1)(1+(1/4(n-1)))U) + k &= (1/n)(\tfrac{1}{2}U+nU-(3/4)U) + k \\ &= (1/n)(nU-\tfrac{1}{4}U) + k \\ &= (1-(1/4n))U + k\end{aligned}$$

for each k in K_1.

Thus $p - k$ is in $(1-(1/4n))U$ for each k in K_1. Hence $p - k$ is in $(1-(1/4n)\bar{U}$, that is, p is in $(1-(1/4n))\bar{U} + k$ for each k in K_1. Let $K_2 = K_1 \cap \bigcap_{k \varepsilon K_1} ((1-(1/4n))\bar{U}+k)$. Then p is in K_2 so K_2 is nonvoid. However, if $K_2 = K_1$ then

$K_1 \subseteq \bigcap_{k \in K_1} (1-1/4n)\overline{U} + k$. Hence for any k_1, k_2 in K_1 we have k_1 is in $(1-1/4n)\overline{U} + k_2$, or $k_1 - k_2$ is in $(1-1/4n)\overline{U}$. Thus $K_1 - K_1 \subseteq (1-1/4n)\overline{U}$, where $0 < (1/4n) < 1$, a contradiction. Thus K_2 is a proper subset of K_1. However, K_2 is the intersection of closed sets and hence is closed. Also, since K_1 is convex and U is convex, K_2 is convex. Finally,

$$G(K_2) \subseteq G(K_1) \cap \bigcap_{k \in K_1} G((1-1/4n)\overline{U}+k) \subseteq K_1 \cap \bigcap_{k \in K_1} (1-1/4n)G\overline{U} + Gk$$

$$\subseteq K_1 \cap \bigcap_{k \in K_1} (1-1/4n)\overline{U} + k = K_2.$$

Thus K_2 is a proper, nonempty, closed, convex subset of K_1 with the property that $G(K_2) \subseteq K_2$ which is a contradiction to the minimality of K_1.

Therefore we must have that K_1 consists of a single point, which is thus a common fixed point of G.

CHAPTER IV

MODERN THEOREMS

Current research in the area of fixed point theorems has exploded in so many directions that selecting the most general of the modern theorems is virtually impossible. Nevertheless, two very recently published results are most interesting, both with regard to their generality and to their relationship with each other. Petryshyn's P compact operator [29, 30] and Guedes de Figueiredo's G operator [21] are both attempts to find the most general existence theorem for fixed points by introducing new spaces and operators. An examination of the relations listed in Chapter II reveals that they both have partially accomplished their goal.

Nevertheless, much remains to be done. Necessary and sufficient conditions for a space to be PB or GB should be established. Guedes de Figueiredo mentions that all the Banach spaces which he has investigated were, in fact, GB spaces. He asks for a proof, or a counter-example, of the proposition that all Banach spaces are GB spaces. If they were, the power of his results would indeed be formidable.

However, it is not even clear that every PB space is a GB space because of the requirement that the projections have a norm of unity. Likewise, because of Petryshyn's requirement on countability, it is doubtful that every GB space is a PB space. The similarities in the two definitions seem to indicate that further investigation in this direction would be fruitful. In particular, an interesting conjecture is that a new space could be defined which includes both the PB and the GB spaces and in which, by suitable modifications in the proofs, both Petryshyn's and Guedes de Figueiredo's results would remain valid.

A NEW RESULT

One step toward generalizing their results is taken in Theorem 4.2 which extends Petryshyn's Theorem 1 [29] from a ball to a bounded closed convex set, and which includes Guedes de Figueiredo's Theorem 2 [21] as a corollary.

Lemma 4.1: Let X be a finite dimensional Banach space, let C be a bounded closed convex subset of X which contains the origin as an interior point. Then a function R can be defined on X which has the following properties:

1. R maps X onto C.
2. $R(x) = x$ if x is in C and $R(x) = tx$ for some t, $0 < t < 1$ if x is not in C.
3. $R(x)$ is on the boundary of C if x is not in C.
4. R is continuous.

<u>Proof</u>: For x in X consider the set $S_X = \{tx : t \geq 0\}$. Then S_X is closed and convex, $S_X \cap C$ is nonempty, closed, bounded, and convex. Since X is finite dimensional, $S_X \cap C$ is compact.

We begin by showing that if x is in X then there exists a unique point $R(x)$ in $S_X \cap C$ such that $||x - R(x)|| = \inf_{y \in S_X \cap C} ||y - x|| = d$. By the compactness of $S_X \cap C$ we know that such a point does exist. To see the uniqueness, observe that if x is in C then x is in $S_X \cap C$ and $||x - x|| = 0 < ||x - y||$ for all y in $S_X \cap C$ and $y \neq x$. If x is not in C, suppose that $x_1 = tx$ and $x_2 = sx$ are in $S_X \cap C$ and $||x - x_1|| = ||x - x_2|| = d$. Now since x is not in C and C is convex containing zero, we must have $t < 1$ and $s < 1$. Thus $1 - t > 0$ and $1 - s > 0$. Moreover,

$$\begin{aligned} |1 - t|\,||x|| &= ||x - tx|| = ||x - x_1|| = ||x - x_2|| \\ &= ||x - sx|| = |1 - s|\,||x||. \end{aligned}$$

But $||x|| \neq 0$ since x is not in C. Thus we must have $1 - t = |1 - t| = |1 - s| = 1 - s$, and therefore $s = t$. Hence $x_1 = x_2$, and the function $R(x)$ so defined is single valued and maps X onto C.

Observe that if x is in C, $R(x) = x$ and if x is not in C, $R(x) = tx$ where $0 < t < 1$.

To see that if x is not in C then $R(x)$ is in the boundary of C, suppose, by way of contradiction, that $R(x)$ is in the interior of C. Then there exists an $a > 0$ such that $N_a(R(x)) \subseteq \text{Int } C$. Let $R(x) = tx$ and consider the point $y = (t+(\frac{1}{2}a/||x||))x$. Then

$$||Rx - y|| = ||(t+(\tfrac{1}{2}a/||x||))x - tx|| = ||(\tfrac{1}{2}a/||x||)x||$$
$$= \tfrac{1}{2}a < a.$$

Thus y is in $N_a(Rx) \subseteq \text{Int } C$, and since C is convex containing 0 and x is not in C, $(t+\frac{1}{2}a/||x||) < 1$. However,

$$||x - y|| = ||x - (t+(\tfrac{1}{2}a/||x||))x|| = |1 - (t+(\tfrac{1}{2}a/||x||))|\,||x||$$

$$= (1 -(t+(\tfrac{1}{2}a/||x||)))||x|| < (1-t)||x||$$

$= ||x - Rx||$, a contradiction to the definition of $R(x)$. Thus if x is not in C, $R(x)$ is in the boundary of C.

We show that R is continuous in X. Suppose, by way of contradiction, that it is not continuous at a point p in X. Then there exists a sequence $\{p_n\}$ converging to p such that no subsequence of $\{R(p_n)\}$ converges to $R(p)$. If any subsequence $\{p_{n_i}\}$ were contained in C, then since C is closed and $\{p_{n_i}\} \to p$, p would belong to C. However, in such a case $\{R(p_{n_i})\} = \{p_{n_i}\} \to p = R(p)$ which gives a contradiction. Thus all but a finite number of $\{p_n\}$ do not belong to C, so that p is not in the interior of C. However, $\{R(p_n)\} \subseteq C$ which is compact, and therefore a subsequence, which without loss of generality we call $\{R(p_n)\}$, converges to a point q in C and $q \neq R(p)$. Since at most a finite number of the $\{p_n\}$ are in C, a subsequence which we again call $\{p_n\}$ can be chosen so that $\{p_n\} \subseteq X/C$ and $\{R(p_n)\} \to q$. Since $\{R(p_n)\}$ is contained in the boundary of C, q is in the boundary of C.

However, $R(p_n) = t_n p_n$ for some t_n in $[0,1]$. Thus the sequence $\{t_n\} \in [0,1]$ has a convergent subsequence $\{t_{n_i}\} \to t$. Now $\{p_{n_i}\} \to p$ since $\{p_n\} \to p$, and thus $\{t_{n_i} p_{n_i}\} \to tp$. However $\{t_{n_i} p_{n_i}\} = \{R(p_{n_i})\} \to q$ since $\{R(p_n)\} \to q$. Thus we must have $q = tp$. In addition, $R(p) = kp$ for some $0 < k < 1$.

If $t < k$, we show that q is in the interior of C, a contradiction to our observation that q was on the boundary. Since p is not in the interior of C, $R(p) \neq 0$, and thus $k > 0$. Since 0 is in the interior of C, there exists an $a > 0$ such that $N_a(0) \subseteq C$. Observe that

$q = tp = (t/k)kp = (t/k)R(p) = sR(p)$ and $s < 1$ since $t < k$. Then $(1-s)a > 0$. We will consider $N_{(1-s)a}(q)$. Let y be in $N_{(1-s)a}(q)$. Then $||y - q|| = ||y - sR(p)|| < (1-s)a$.

Let $w = (1/1-s)y - (s/1-s)R(p)$. Then

$$||w|| = ||(1/1-s)y - (s/1-s)R(p)|| = |(1/1-s)|\,||y - sR(p)||$$
$$< |(1/1-s)|\,(1-s)a = a.$$

Thus w is in $N_a(0) \subseteq C$. Therefore since $R(p)$ is in C, $y = (1-s)\,w + sR(p)$ where $0 < s < 1$, and C is convex, y is in C. Hence $N_{(1-s)a}(q) \subseteq C$. Thus q is in the interior of C, and we have the desired contradiction. We conclude that $t \nless k$.

Therefore, since $q \neq R(p)$, we must have $t > k$. But then

$$||p - q|| = ||p - tp|| = (1-t)||p|| < (1-k)||p||$$
$$= ||p - kp|| = ||p - R(p)||,$$

a contradiction to the definition of $R(p)$.

Therefore no such p exists and R is continuous in X.

This completes the proof of the lemma.

Theorem 4.2: Let X be a finite dimensional Banach space, let C be a bounded, closed, convex subset of X which has zero as an interior point. Let A be a continuous mapping of C into X, and let k be any constant. Then there exists at least one element u in C such that

$$Au - ku = 0 \tag{1}$$

provided that the mapping A satisfies either the condition
$(\pi,k,\leq)$: If for some x on the boundary of C the equation $Ax = bx$ holds, then $b \leq k$.
or the condition
$(\pi,k,\geq)$: If for some x on the boundary of C the equation $Ax = bx$ holds, then $b \geq k$.

Proof: Suppose that A satisfies $(\pi,k,\leq)$ and let $Tx = Ax - kx + x$. Define $R(x)$ on X as in lemma 4.1. Since both T and R are continuous, it follows that $Sx = RT(x)$ is a continuous mapping of C into itself. Thus by the Schauder-Tychonoff Theorem, S has a fixed point u in C.

We will show that u is also a fixed point of T. Suppose that u is in the interior of C, $u = RT(u) = Su$. By the lemma, if x is not in C, then $R(x)$ is in the boundary of C. Thus we must have $T(u)$ in C, and hence $u = RT(u) = T(u)$. If u is in the boundary of C and u is not a fixed point of T, then $T(u)$ does not belong to C. Thus $u = R(T(u)) = tT(u)$

where $0 < t < 1$. Hence $T(u) = (1/t)u$ where $1 < (1/t)$. By definition of T, $(1/t)u = Tu = Au - ku + u$, so $Au = ((1/t)-1+k)u$ for u in the boundary of C and $a = ((1/t)-1+k) > k$, a contradiction to the condition $(\pi,k,\leq)$. Thus u must be a fixed point of T, and therefore $u = Tu = Au - ku + u$. Hence $Au = ku$ and (1) is satisfied.

Now suppose that A satisfies condition $(\pi,k,\geq)$. Define the operator B by $Bx = 2kx - Ax$. Then B is a continuous map of C into X. Moreover, if for some x in the boundary of C the equation $Bx = ax$ holds, then $ax = 2kx - Ax$, and thus $Ax = (2k-a)x$. Hence by condition $(\pi,k,\geq)$, $2k - a \geq k$, and so $k \geq a$. Therefore B satisfies condition $(\pi,k,\leq)$ so that we can apply the above result to B and conclude that there exists u in C such that $ku = Bu = 2ku - Au$. Thus $Au = ku$, and equation (1) is satisfied, completing the proof of the theorem.

Remark: If in condition $(\pi,k,\leq)$(or in condition $(\pi,k,\geq)$), we require that $a < k$ (or that $a > k$), then an element u which satisfies equation (1) must lie in the interior of C.

PETRYSHYN'S RESULT

Lemma 4.3: Suppose that A is a P compact operator mapping the PB space X into itself. Suppose further that for given $r > 0$ and $k > 0$ the operator A satisfies both of the following conditions:

(h) There exists a number $c(r) > 0$ such that if, for any n, $P_nAx = hx$ holds for x in S_r with $h > 0$, then $h \leq c(r)$.

(π,k) If for some x in S_r the equation $Ax = ax$ holds then $a < k$.

Then there exists an integer $n_0 > 0$ such that if $n \geq n_0$ and $P_nAx = bx$ for some x in $S_r \cap X_n$, then $b < k$.

Proof: Suppose, by way of contradiction, that no such n_0 exists. Then we could find a sequence $\{x_n\}$ with x_n in $X_{m_n} \cap S_r$ and a sequence of numbers $\{b_n\}$ such that

$$P_{m_n}Ax_n = b_nx_n \qquad \text{and} \qquad b_n \geq k. \tag{1}$$

Hence condition (h) implies that $||P_{m_n}Ax_n|| = b_nr \leq c(r)r$. Thus $b_n \varepsilon [k, c(r)]$ for each n. But this is a closed and bounded subset of the real line and hence compact. Thus $\{b_n\}$ has a convergent subsequence, say $\{b_{n_i}\} \rightarrow b \varepsilon [k,c(r)]$. Combining this result with (1) we get

$$||P_{m_{n_i}}Ax_{n_i} - bx_{n_i}|| = ||(b_{n_i}-b)x_{n_i}|| = |(b_{n_i}-b)r| \rightarrow 0. \tag{2}$$

Since A is P compact, (2) implies the existence of a strongly convergent subsequence, which we denote by $\{x_n\}$, and an element x in $S_r \cap X$ such that $\{x_n\} \rightarrow x$ and $\{P_{m_n}Ax_n\} \rightarrow Ax$. This fact and (2) imply that

$$||Ax - bx|| \leq ||Ax - P_{m_n}Ax_n|| + ||P_{m_n}Ax_n - bx_n|| + ||bx_n - bx|| \rightarrow 0.$$

Thus $Ax - bx = 0$ for x in S_r and $b \geq k$, contrary to condition (π,k), which finishes the proof of the lemma.

Theorem 4.4: Suppose that A satisfies the hypothesis of lemma 4.3. Then there exists at least one element u in B_r/S_r such that $Au - ku = 0$.

Proof: By the definition of P compact, there exists N_1 such that $n \geq N_1$ implies that P_nA is continuous in X_n. By lemma 4.3 there exists N_0 such that $n \geq N_0$ and $P_nAx = bx$ for x in $S_r \cap X_n$ implies that $b < k$. Let N be the maximum of N_1 and N_0. Now $B_r \cap X_n$ is the ball of radius r centered at the origin in the finite dimensional space X_n, and for all $n \geq N$. P_nA is continuous in X_n and satisfies condition $(\pi,k,<)$ of theorem 4.2. Thus by that theorem and the remark following it, there exists an element u_n in $B_r \cap X_n$ such that $P_nAu_n - ku_n = 0$, $k > 0$, $||u_n|| < r$. Therefore, again by the P compactness of A, there exists a subsequence $\{u_{n_i}\}$ and an element u in B_r such that $\{u_{n_i}\} \to u$ and $\{P_{n_i}Au_{n_i}\} \to Au$. Therefore

$$||Au - ku|| \leq ||Au - P_{n_i}Au_{n_i}|| + ||P_{n_i}Au_{n_i} - ku_{n_i}|| + ||ku_{n_i} - ku|| \to 0.$$

Thus $Au = ku$. Finally, u is in B_r/S_r since the assumption that u is in S_r would contradict condition (π,k).

GUEDES DE FIGUEIREDO'S RESULT

Lemma 4.5: Let E be a GB space, J a duality map in E with gauge function μ. Then for any x in F_α the following inclusion holds: $P'_\alpha(Jx) \subseteq Jx$, where P'_α is the adjoint of P_α in E*.

Proof: Let y' be in Jx. Then we have
$||x||\,||P'_\alpha y'|| \geq P'_\alpha y'(x) = y'(P_\alpha(x)) = y'(x) = ||x||\mu(||x||)$.
Thus $||P'_\alpha y'|| \geq \mu(||x||)$. On the other hand,
$||P'_\alpha y'|| \leq ||y'|| = \mu(||x||)$. Thus $||P'_\alpha y'|| = \mu(||x||)$ and
$P'_\alpha y'(x) = ||x||\mu(||x||)$. Hence $P'_\alpha y'$ is in Jx, and we have the desired result.

Theorem 4.6: Let E be a finite dimensional Banach space. Let μ be a gauge function and let T: C $\rightarrow$ E be a continuous mapping from a bounded closed convex subset C of E into E, where zero is in the interior of C. Suppose that for every x in the boundary of C there exists v' in E* such that $v'(x) = ||x||\mu(||x||)$ and $v'(Tx) \leq ||x||\mu(||x||)$. Then T has a fixed point in C.

Proof: Let x be in the boundary of C and suppose that Tx = ax. Suppose that a > 1. Then
$0 = v'(ax - Tx) = v'(ax) - v'(Tx) = av'(x) - v'Tx$
$\geq a||x||\mu(||x||) - ||x||\mu(||x||) > 0$, which is a contradiation. Thus $a \leq 1$, and T satisfies condition $(\pi,1,\leq)$ of

theorem 4.2. Therefore T has a fixed point in C.

Theorem 4.7: Let E be a GB space. Let $T:C \to E$ be a G operator where C is a bounded closed convex subset of E, and the origin belongs to the interior of $C \cap F_\alpha$ for all but a finite number of α. Let J be a duality mapping in E with gauge function μ. If $Jx(Tx) \leq ||x||\mu(||x||)$ for every x in the boundary of C, then T has a fixed point in C.

Proof: Consider for each α the mapping $T_\alpha = P_\alpha T$. Such a map is continuous because T is a G operator.

Let x be in the boundary of $C \cap F_\alpha$. We establish the existence of a v' in E^* which satisfies the condition of theorem 4.6. Let y' be any element of $Jx \subseteq E^*$. Set $v' = y'|F_\alpha$. Then v' is in F_α^* and we have
$v'(x) = ||x||\mu(||x||)$. Now
$v'(T_\alpha x) = y'(T_\alpha x) = y'(P_\alpha Tx) = P_\alpha' y' Tx$ where P_α' is the adjoint of P. Since $P_\alpha' y'$ is in Jx by lemma 4.5, by hypothesis we have $v'(T_\alpha x) = P_\alpha' y'(Tx) \leq ||x||\mu(||x||)$. Thus v' satisfies the condition of theorem 4.6.

Hence, by that theorem, for all α the equation $T_\alpha x = x$ is solvable in F_α. Since T is a G operator it follows that T has a fixed point in C, and this completes the proof of the theorem.

CHAPTER V

APPLICATIONS

Fixed point theorems have many applications in the area of functional analysis and applied mathematics. In particular, they have proved to be a useful technique for establishing the existence, and in some cases the uniqueness, of solutions to differential and integral equations. When the proof of a fixed point theorem is constructive, it provides a method for obtaining an approximate solution to an operator equation.

Given a particular problem, the general procedure for proving the existence of a solution is to replace the original equation with an equivalent equation which defines an operator mapping a function space into itself. In order to prove the existence of a solution to the original problem, it is then necessary to find an invariant function, that is a fixed point, of the operator. Thus if the operator and the function space can be shown to satisfy the hypothesis of one of the known fixed point theorems, the existence is proved.

From this standpoint, the fixed point theorem is even more valuable if its proof provides a technique for constructing the fixed point. In this regard, the theorems on contraction mappings are particularly important. While the existence of a fixed point may follow from a more general theorem, the properties of contraction type mappings seem to lend themselves most readily to constructive proofs. The Banach-Cacciopoli Theorem whose proof was given in Chapter III was the earliest of the contraction theorems. Much current research, in particular by Browder and Edelstein, has been devoted to generalizing the idea of a strict contraction while retaining enough control over the operator to be able to construct a fixed point. An indication of the modern interest in contraction theorems may be found by examining the list in Chapter VI.

The procedure used in applying fixed point theorems is illustrated in the proofs of the theorems of Picard, Cauchy, and Poincaré, together with applications to Fredholm and Volterra integral equations.

Theorem 5.1: (Picard) Consider the differential equation $\frac{dy}{dx} = f(x,y)$, where $f(x,y)$ satisfies the Lipschitz condition $|f(x,y_1) - f(x,y_2)| \leq M|y_1 - y_2|$. Then, on the interval $|x - x_0| \leq d$, there exists a unique solution $y(x)$ of the equation which satisfies the condition $y(x_0) = y_0$ [27].

Proof: We replace the differential equation by the integral equation $y = y_0 + \int_{x_0}^{x} f(t,y)dt$.

We shall consider the right hand side of the integral equation as an operator defined on $C[a,b]$ where $a < x_0 < b$:

$$A(g) = y_0 + \int_{x_0}^{x} f(t,g)dt.$$

Since the operation of integration is a continuous function of the upper limit, the operator transforms points of C into C. Estimating $d(Ag_1,Ag_2)$ we have

$$d(Ag_1,Ag_2) = \max|Ag_1 - Ag_2| = \max\left|\int_{x_0}^{x} f(t,g\ ) - f(t,g_2)dt\right|$$

$$\leq M \max |g_1 - g_2|\ |x - x_0|.$$

If we take $|x - x_0| \leq k/M$ where $k < 1$, then

$$d(Ag_1,Ag_2) \leq kd(g_1,g_2),$$

and hence a unique solution of the equation $A(g) = g$ exists by the Banach-Cacciopoli Theorem. This solves the given differential equation. It follows from the same theorem that the solution can be approximated by iterating the operator A starting with any continuous function.

Theorem 5.2: (Cauchy) Consider the differential equation $\frac{dy}{dx} = f(x,y)$ where $f(x,y)$ is analytic at the point (x_0,y_0). Then there exists a unique solution $y(x)$ which can be expanded in powers of $x - x_0$ in some neighborhood of the point x_0, and which satisfies the condition $y(x_0) = y_0$ [27].

Proof: Let $f(x,y) = \Sigma a_{\alpha\beta}x^{\alpha}y^{\beta}$ in the domain $|x - x_0| < \varepsilon$ $|y - y_0| < \varepsilon$. Let $M = \max \left|\frac{\partial f}{\partial y}\right|$ for $|x - x_0| \leq \varepsilon' < \varepsilon$,

$|y - y_0| \le \varepsilon' < \varepsilon$. Consider the set of analytic functions C which are holomorphic in the circle $(x-x_0)^2 + (y-y_0)^2 = r^2$ of radius $r = \min\{k/M, \varepsilon^2\}$ where k is some fixed number less than one. If we define $d(g_1,g_2) = \max|g_1 - g_2|$ for g_1,g_2 in C, then C is a complete metric space.

We replace the differential equation by the integral equation:

$$y = y_0 + \int_{x_0}^{x} f(t,y)dt.$$

Consider the right side of the equation as an operator A defined on C.

Estimating $d(A(g_1),A(g_2))$ we have

$$d(A(g_1),A(g_2)) = \max \left| \int_{x_0}^{x} (f(t,g_1) - f(t,g_2))dt \right|$$

$$\le \int_{x_0}^{x} \max \left|\frac{\partial f}{\partial g}\right| |g_1 - g_2| dt \le M \max|g_1 - g_2||x - x_0|.$$

Taking $|x - x_0| \le \kappa/M$, we have $d(A(g_1),A(g_2)) \le \kappa d(g_1,g_2)$. Consequently we can apply the Banach-Cacciopoli Theorem to prove A has a fixed point in C and thus has the desired expansion. This completes the proof.

Theorem 5.3: (Poincaré) Suppose that in the equation $\frac{dy}{dx} = f(x,y,\lambda)$, the function $f(x,y,\lambda)$ can be expanded in a power series $\Sigma a_{\alpha\beta\gamma} x^\alpha y^\beta \lambda^\gamma$ in x, y and λ which converges in the region $|x| < \varepsilon$, $|y| < \varepsilon$, $|\lambda| < \varepsilon$. Then there exists a solution of the form

$y = u_1(x) + \lambda u_2(x) + \ldots + \lambda^n u_n(x) + \ldots$ [27].

Proof: Let $M = \max \left|\frac{\partial f}{\partial y}\right|$ and consider the set of functions $C = \{g(x,\lambda) = \Sigma c_{\alpha\beta} x^\alpha \lambda^\beta\}$, which are analytic in the domain $|x| < \min\{k/M, \varepsilon\}$, $|\lambda| < \min\{k/M, \varepsilon\}$ where $k < 1$. This set is a complete metric space if the distance is taken as $\max|g_1(x,\lambda) - g_2(x,\lambda)|$.

Consider the operator $A(g) = \int_0^x \Sigma a_{\alpha\beta\gamma}\, t^\alpha g^\beta \lambda^\gamma dt$. Then $A(g)$ is also a function in the set C. Estimating $d(A(g_1), A(g_2))$ as in Cauchy's Theorem, we obtain $d(A(g_1), A(g_2)) \leq kd(g_1, g_2)$, which, again using the Banach-Cacciopoli Theorem, proves Poincaré's Theorem.

Theorem 5.4: There exists a unique solution for the Fredholm nonhomogeneous linear integral equation of the second kind: $f(x) = \lambda \int_a^b K(x,y) f(y) dy + \phi(x)$

where $K(x,y)$ and $\phi(x)$ are given continuous functions for $a \leq x \leq b$, $a \leq y \leq b$ and $f(x)$ is the function sought, provided that $|\lambda| < 1/M(b-a)$ where $|K(x,y)| \leq M$ [26].

Proof: Consider the mapping $g = Af$, that is $g(x) = \lambda \int_a^b K(x,y) f(y) dy + \phi(x)$, of the complete metric space $C[a,b]$ into itself. We obtain

$d(g_1, g_2) = \max|g_1(x) - g_2(x)| \leq |\lambda| M(b-a) \max|f_1 - f_2|$.

Consequently, the mapping A is a strict contraction for

$|\lambda| < 1/M(b-a)$, and thus by the Banach Theorem, the Fredholm equation has a unique continuous solution. The successive approximations to this solution:

$f_0(x), f_1(x), \ldots, f_n(x), \ldots$ have the form

$f_n(x) = \lambda\int_a^b K(x,y)f_{n-1}(y)dy + \phi(x)$. An arbitrary continuous function may be chosen for $f_0(x)$.

<u>Theorem 5.5</u>: There exists a unique solution to a nonlinear equation of the form $f(x) = \lambda\int_a^b K(x,y,f(y))dy + \phi(x)$ where K and ϕ are continuous, and K satisfies the condition $|K(x,y,z_1) - K(x,y,z_2)| \leq M|z_1 - z_2|$, provided that $|\lambda| < 1/M(b-a)$ [26].

<u>Proof</u>: For $|\lambda| < 1/M(b-a)$, the mapping $g = Af$ of the complete metric space $C[a,b]$ into itself given by the formula $g(x) = \lambda\int_a^b K(x,y,f(y))dy + \phi(x)$ is a strict contraction since we have $\max|g_1(x) - g_2(x)| \leq |\lambda|M(b-a)\max|f_1 - f_2|$.

<u>Theorem 5.6</u>: For the Volterra type integral equation $f(x) = \lambda\int_a^x K(x,y)f(y)dy + \phi(x)$ where K and ϕ are continuous, the existence of a solution is guaranteed regardless of the value of the parameter λ [26].

Proof: Consider the mapping

$$g(x) = \lambda\int_a^x K(x,y)f(y)dy + \phi(x) = Af,$$

defined on the complete metric space $C[a,b]$. If f_1, f_2 are two continuous functions defined on the closed interval $[a,b]$

then $|Af_1 - Af_2| = \lambda\int_a^x K(x,y)(f_1(y) - f_2(y))dy \leq |\lambda|Mm(x-a)$

where $M = \max|K(x,y)|$ and $m = \max|f_1 - f_2|$.

Thus, by substituting Af and integrating, we get $|A^2f_1 - A^2f_2| \leq |\lambda|^2M^2m((x-a)^2/2)$. In general $|A^nf_1 - A^nf_2| \leq |\lambda|^nM^nm((x-a)^n/n!) \leq |\lambda|^nM^nm((b-a)^n/n!)$.

For arbitrary λ, the number n can be chosen so large that $M^n|\lambda|^n((b-a)^n/n!) < 1$. Thus the mapping A^n will be a strict contraction. Consequently, the Volterra integral equation has a solution for arbitrary λ and this solution is unique.

CHAPTER VI

THE PRESENT STATUS AND SUGGESTIONS FOR FURTHER RESEARCH

The purpose of this chapter is to state those fixed point theorems which have appeared in the literature, arranging them to give a clear view of the present state of affairs. This listing demonstrates two main points--there is a tremendous amount of current interest in fixed points with research in a great many directions, and the ideal fixed point theorem which would include most of the other theorems as corollaries has not yet been proved.

The theorems are arranged, first, according to type of operator. Theorems on the same type of operator are then listed according to type of space in order of decreasing generality. It was discovered in this investigation that a number of the theorems which have appeared independently in the literature follow, in fact, as corollaries from others and these are listed as such. Caution should be used, however, in drawing conclusions about this duplication. Certain of the more general theorems use the results of the corollaries in their proof. In addition, some of the corollaries are more useful for applications than the general theorem since they provide a constructive method of approximating the fixed point. This, as was pointed out in the preceding chapter, is particularly true of the contraction theorems. The theorems on families of operators are listed separately.

Incorporated in the presentation of the theorems are appropriate suggestions for further research. Some of the theorems whose hypotheses are related seem to indicate that a new theorem which includes all of them could be proved. An investigation of the "sharpness" of each theorem would be profitable, attempting to weaken each part of the hypothesis or strengthen the conclusion, or to construct an example showing the theorem cannot be improved.

EXISTENCE THEOREMS FOR SINGLE MAPPINGS

I. P Compact Operators

A. Suppose that T is a P compact operator in a PB space X. Suppose further that for given $r > 0$ and $k > 0$ the operator T satisfies both of the following conditions:

(h) There exists a number $c(r) > 0$ such that if, for any n $P_nTx = hx$ holds for x in S_r with $h > 0$, then $h \leq c(r)$.

(πk) If for some x in S_r the equation $Tx = ax$ holds then $a < k$.

Then there exists at least one element u in

B_r/S_r such that $Tu - ku = 0$ [30].

Comment: Using theorem 4.2, it might be possible to generalize this result to a closed, bounded, convex set C which contains the origin as an interior point. An easier characterization of a PB space and a P compact operator would be most useful.

Corollaries:

1. The assertion of the above remains true if condition (h) is replaced by any one of the following stronger conditions whose degree of generality increases in the given order:

   a. T is bounded [29].

   b. For any given $r > 0$ the set $T(S_r)$ is bounded [30].

   c. X is a Hilbert space and for any given $r > 0$ $(Tx,x) \leq c||x||$ for all x in S_r and some $c > 0$ [30].

2. The following theorems, in a PB space X, are corollaries [29]:

   a. (Schauder) If T is a completely continuous mapping of B_r into B_r, then T has a fixed point in B_r.

   b. (Rothe) If T is a completely continuous mapping of B_r into X such that $T(S_r) \subseteq B_r$, then T has a fixed point in B_r.

c. (Altman) If T is a completely continuous mapping of B_r into X such that $||Tx - x||^2 \leq ||Tx||^2 - ||x||^2$ for all x in S_r, then T has a fixed point in B_r.

d. (Krasnoselsky) If X is a Hilbert space, T a completely continuous mapping of B_r into X such that $(Tx,x) \leq ||x||^2$ for all x in S_r, then T has a fixed point in B_r.

e. (Kaniel) If T is a quasicompact mapping of B_r into X such that $Tx + \lambda x \neq 0$ for all x in S_r, and any $\lambda > \mu > 0$, then there exists an element u in B_r such that $Tu + \mu u = 0$.

B. Suppose that T is a P Compact operator in a PB space X. Suppose further that there exists a sequence of spheres $\{S_{r_p}\}$ with $r_p \to \infty$ as $p \to \infty$, and two sequences of positive numbers $c_p = c(r_p)$ and $k_p = k(r_p)$ with $k_p \to \infty$ as $r_p \to \infty$ such that the following conditions hold:

(Λf) Whenever for any given f in B_r and any n the equation $P_n Tx - \lambda x = P_n f$ holds for x in S_{r_p} with $\lambda > 0$, then $\lambda \leq c_p$.

(πp) $||Tx - \eta x|| \geq k_p$ for any $\eta \geq \mu > 0$ and any x in S_{r_p}. Then for every f in X there exists an element u in X such that $Tu - \mu u = f$ [30].

Corollary:

1. The above assertion remains valid if condition (Λf) is replaced by the stronger condition that T is bounded [29].

C. Let X be a PB space, T a bounded P compact operator in X. Then for any f in X there exists a ball B_r and a number $\mu(r) > 0$ such that the equation $Tx - \mu x = f$ has a solution u in B_r for every constant $\mu \geq \mu(r)$ [29].

D. Let T and U be two bounded P compact operators mapping the ball B_r in a PB space X into X. If X is a Hilbert space and if for all x in S_r, $(Tx,x) \leq ||x||^2$ and $||Tx - Ux|| \leq ||x - Tx||$, then U has a fixed point in B_r [29].

II. G Operators

A. Let X be a GB space, C a bounded closed convex subset of X containing zero as an interior point, $T:C \to X$ a G operator. Let J be a duality map in X with gauge μ. If $Jx(Tx) \leq ||x||\mu(||x||)$ for all x on the boundary of C, then T has a fixed point in C [21].

<u>Comment</u>: A simpler characterization of G operators as well as GB spaces is needed, and an investigation of their relation to PB spaces and operators should be made.

Corollaries:

The following theorems, in a GB space X, are corollaries. Let C be a bounded closed convex set in X which has zero as an interior point [21].

1. (Schauder) If X is separable, T a completely continuous operator mapping C into itself, then T has a fixed point in C.
2. (Schauder) Let X be reflexive and separable, T a weakly continuous mapping of C into itself. Then T has a fixed point in C.
3. (Rothe) If X is separable, T a completely continuous operator mapping C into X such that the boundary of C is mapped by T into C, then T has a fixed point in C.
4. (Petryshyn) If X is separable, $T:C \to X$ is P compact, and if J is a duality mapping in X with gauge μ such that $Jx(Tx) \leq ||x||\mu(||x||)$ for every x in the boundary of C, then T has a fixed point in C.
5. If X is separable and reflexive, $T:C \to X$ weakly continuous, and if J is a duality mapping in X with gauge μ such that for every x in the boundary of C $Jx(Tx) \leq ||x||\mu(||x||)$, then T has a fixed point in C.

6. (Browder-Guedes de Figueiredo) Let X be reflexive, X^* strictly convex, and the duality map J both continuous and weakly continuous. If T is demicontinuous mapping C into X such that $T = I - A$ where A is J monotone, and if $Jx(Tx) \leq ||x||\mu(||x||)$ for all x in the boundary of C, then T has a fixed point in C.

7. (Browder) Let X be reflexive, X^* strictly convex and the duality map J both continuous and weakly continuous. If $T:C \to C$ is nonexpansive, then T has a fixed point in C.

Comment: It would be interesting to find practical applications for each of the above theorems.

III. Class M

A. Let B_r be a closed ball in a Hilbert space X, $T = I - f$ where f is in Class M. If T maps S_r into B_r then T has a fixed point in B_r [5].

B. Let C be a bounded closed convex subset with nonempty interior of a Hilbert space X, $T = I - f$ where f is in Class M. Suppose that there exists an $\varepsilon > 0$ such that $||Tu - u|| \geq \varepsilon$ for all u in the boundary of C. Suppose that $T_1 = I - f_1$ where f_1 is in Class M and for u in the boundary of C $||Tu - T_1u|| \leq ||Tu - u||$. Then if T has a fixed point in C, T_1 will have a fixed point in C [5].

C. Let B_r be a closed ball in a Hilbert space X, $T = I - f$ where f is in Class M. Suppose that for each $d > 0$ there exists $\varepsilon(d) > 0$ such that for u in S_r, $d \leq t \leq 1, ||fu - tf(-u)|| \geq \varepsilon(d)$. Then T has a fixed point in B_r [5].

IV. Hemicontinuous

A. Let T be a hemicontinuous map of the ball B_r in a Hilbert space X into X. Suppose that for all u,v in B_r, $\mathrm{Re}(Tu-Tv, u-v) \leq ||u - v||^2$ while for $||u|| = r$, $\mathrm{Re}(Tu,u) \leq ||u||^2$. Then T has a fixed point in B_r [4].

V. Weakly Continuous

A. Let C be a closed convex subset of a Banach space X and $T:C \to C$ a weakly continuous mapping such that T[C] is separable and the weak closure of T[C] is weakly compact. Then T has a fixed point in C [34].

Corollary:

1. A convex, weakly compact subset of a separable Banach space has the fixed point property for weakly continuous maps [34,27].

B. Let T be a weakly continuous operator on the separable Hilbert space X. If there exists a positive constant r such that $\mathrm{Re}(Tx,x) \leq ||x||^2$ for all x in S_r then T has a fixed point in B_r [32].

C. Let T be a weakly continuous operator on a separable Hilbert space which is monotone increasing on rays. Then T has a fixed point [32].

VI. Continuous

A. (Schauder-Tychonoff) Let T be a continuous map of a compact convex subset C of a locally convex topological vector space into itself. Then T has a fixed point in C [34,3].

Corollaries:

1. A continuous map T of a convex compact set C in a Banach space into itself has a fixed point in C [24,34,31,27].
2. A continuous self-map of the Hilbert cube has a fixed point [13].
3. Any convex, closed subset of the Hilbert cube has the fixed point property for continuous maps [13].
4. If T is a continuous map of a k-dimensional simplex into itself, then T has a fixed point [20].
5. (Brouwer) A continuous operator T mapping the closed unit ball in E^n into itself has a fixed point [24,13,31,27]
6. Any closed, bounded convex subset of E^n has

the fixed point property for continuous maps [31].

B. Let X be a locally convex topological vector space satisfying condition E. Suppose C is a nonvoid closed convex subset of X and T is a continuous map of C into itself such that T[C] is compact. Then T has at least one fixed point [18].

Corollary:

1. A continuous operator T mapping a closed convex set C in a Banach space into a compact set $A \subseteq C$ has a fixed point [24,27].

C. Let T be a continuous self map of the Banach space X which satisfies $||T^k x - T^k y|| \leq ||T^k|| \, ||x - y||$ for $k = 1,2,\ldots.$ Suppose that $\sum_{i=1}^{\infty} ||T^k|| < \infty$. Then T has a unique fixed point x_0 in X and the sequence of Picard iterates converges to x_0 [35].

Comment: Picard iterates are used so frequently in fixed point theorems it would be interesting to take a slightly different point of view: Find necessary and sufficient conditions on the space and operator for the Picard iterates to converge. This could be accomplished by fixing the space and varying the operator then fixing the operator and varying the space.

D. Let T be a continuous map in a separable real

Hilbert space X such that for some $c_0 > 0$ $((I-T)x-(I-T)y, x-y) \geq c_0||x - y||^2$ for all x,y in X. Then T has a fixed point in X [9].

E. Let X be a finite dimensional Banach space, T a continuous map of B_r into X and μ any constant. Then there exists u in B_r such that $Tu - \mu u = 0$ provided that T satisfies either

($\pi\mu\leq$) If for some x in S_r the equation $Tx = \alpha x$ holds then $\alpha \leq \mu$.

or

($\pi\mu\geq$) If for some x in S_r the equation $Tx = \alpha x$ holds then $\alpha \geq \mu$ [29].

Corollary:

1. Let T be a continuous operator on the finite dimensional Hilbert space X. If there exists a positive constant r such that $Re(tx,x) \leq ||x||^2$ for all x in S_r then T has a fixed point in B_r [32].

F. Let X be a finite dimensional Banach space. Let μ be a gauge function and let $T:C \to X$ be a continuous mapping of the bounded closed convex set C in X into X. (0 is in the interior of C.) Suppose that for all x in the boundary of C there exists v' in X* such that $v'(x) = ||x||\mu(||x||)$ and $v'(Tx) \leq ||x||\mu(||x||)$. Then T has a fixed point in C [21].

Comment: As noted in Chapter IV, both E and F are corollaries to the result proved in that chapter. It would be an interesting study to attempt to relate all the esoteric conditions to one another and perhaps synthesize them.

VII. Contraction Operators

A. ε-Contractions

1. Let X be a metric space, T an ε-contractive self map of X such that there exists an x_0 in X for which the sequence $\{T^n(x_0)\}$ has a subsequence converging to a point x^* in X. Then:

   a. x^* is a periodic point of T, that is there exists $k > 0$ such that $T^k(x^*) = x^*$.

   b. If X is ε-chainable and x^* has a compact neighborhood $N_p(x^*)$ of radius $p \geq \varepsilon$, then x^* is the unique fixed point of T [15].

B. Local Iterative Contractions

1. Let T be a local iterative contraction on a complete metric space X. Then T has a unique fixed point in X. For arbitrary x_0 in X, the Picard iterates of x_0 converge in the metric to the fixed point of T [35].

C. II Local Contractions

1. Let T be a II local contraction on the complete metric space X and ϕ satisfy the additional condition: $\liminf_{t\to\infty} (t-\phi(t)) = a > 0$.

   Then T has a unique fixed point in X and the Picard iterates converge in metric to the fixed point [35].

D. (ε,λ) Uniform Local Contractions

1. Let X be a complete metric ε-chainable space, T a map of X into itself which is (ε,λ) uniformly locally contractive. Then there exists a unique fixed point of T in X [14].

2. If T is a 1-1(ε,λ) uniformly locally expansive map of a metric space Y onto an ε-chainable complete metric space X contained in Y, then T has a unique fixed point in X [14].

E. II Pseudocontractions

1. Let X be a uniformly convex Banach space, B a closed ball in X, G an open set containing B. Let T be a II pseudocontraction mapping G into X such that T maps the boundary of B into B. Suppose also that T is demicontinuous and that either

   a. T is uniformly continuous in the strong topology on bounded subsets of X.

or

b. X^* is uniformly convex.

Then T has a fixed point in B [7].

F. I Pseudocontractions

1. Let T be I pseudocontractive and Lipschitz mapping a Hilbert space X into X such that for some $r > 0$ $(Tx,x) \leq ||x||^2$ for all x in S_r. Then T has a fixed point in B_r [9].

2. Let T be I pseudocontractive and Lipschitz mapping a Hilbert space X into itself such that for some $r > 0$, $Tu - \lambda u \neq 0$ for all u in S_r and $\lambda > 1$. Then T has a fixed point in B_r [9].

Corollary:

a. Let B_r be a closed ball centered at the origin of a Hilbert space X, T a nonexpansive map of B_r into X such that for all u in S_r, $\lambda > 1$, $Tu - \lambda u \neq 0$. Then T has a fixed point in B_r [5].

<ins>Comment</ins>: It should be noted that Browder uses the word "pseudocontractive" for both I pseudocontractive and II pseudocontractive. While all examples considered appear to be both or neither, it would be interesting to have a proof of their exact relationship.

G. Strict Pseudocontractions

1. Let T be a strict pseudocontraction with constant k mapping a ball B_r about the origin in a Hilbert space X into X such that for all u in S_r and any $\lambda > 1$, $Tu - \lambda u \neq 0$. Let R be the retract of X onto B_r. Then for any x_0 in B_r and any γ such that $0 < 1 - k < \gamma < 1$, the sequence $\{x_n\} = \{(RT)^n x_0\}$ given by $x_n = \gamma RTx_{n-1} + (1-\gamma)x_{n-1}$ converges weakly to a fixed point of T in B_r. If T is demicompact then the convergence is strong [9].

2. Let C be a bounded closed convex subset of the Hilbert space X and let T be a map of C into C such that T is a strict pseudocontraction with constant k. Then for any x_0 in C and any fixed γ such that $1 - k < \gamma < 1$, the sequence $\{x_n\} = \{T^n x_0\}$ determined by $x_n = \gamma Tx_{n-1} + (1-\gamma)x_{n-1}$ converges weakly to a fixed point of T in C. If T is demicompact then the convergence is strong [9].

H. Nonexpansive Operators

1. Let X be a Banach space, T a nonexpansive self map of X. For given f in X let $T_f(u) = T(u) + f$ and suppose T_f is weakly asymptotically regular. Let $\{x_n\} = \{T_f^n x_0\}$ be the sequence of Picard

iterates starting at x_0 and suppose that there exists an infinite subsequence $\{x_{n_i}\} \to y$ in X. Then y is a solution of $u - Tu = f$ and the whole sequence converges strongly to y [8].

2. Let X be a strictly convex Banach space, T a nonexpansive self map of X with nonempty T-closure.

   a. If there exists an x in T-closure such that $[\{T^n(x)\}]$ is finite dimensional, then there exists x^* in the closed convex hull of $\{T^n(x)\}$ such that $T(x^*) = x^*$.

   b. If there exists an x in the T-closure such that $\{1/n \sum_1^n T^n(x)\}$ contains a subsequence which converges weakly to some x^* in X, then x^* is a fixed point of T.

   c. If X is also reflexive, and if there exists an x in T-closure such that $\{T^n(x)\}$ is bounded then T has a fixed point in X [16].

Comment: The definition of weakly asymptotically regular is sufficiently similar to the condition on 2.b above that a relationship appears to exist between the two. It would be interesting to discover it.

3. Let C be a nonempty closed convex subset of a reflexive Banach space with normal structure

and suppose that T is a nonexpansive self map of C. If there exists p in C such that $\{T^n(p)\}$ is bounded, then T has a fixed point in C [25].

Corollaries:

a. If the condition that $\{T^n(p)\}$ be bounded is replaced by the stronger condition that C is bounded, then the result remains true [25].

b. Every nonexpansive self map of a nonempty bounded closed convex subset C of a uniformly convex Banach space has a fixed point [8,6].

c. Let X be a Hilbert space, C a bounded closed convex subset of X, T a nonexpansive self map of C.

   (1) T has a fixed point x^* in C and if x^* is a unique fixed point then $T^n x_0 \overset{w}{\to} x^*$ for any x_0 in C [9,5].

   (2) If $T_\lambda = \lambda I + (1-\lambda)T$ for $0 < \lambda < 1$, then for any x_0 in C, $T_\lambda^n x_0 \overset{w}{\to} y$ where y is a fixed point of T in C. If, in addition, T is demicompact, then the convergence is strong [9].

4. Let C be a closed convex subset of a strictly convex Banach space, T a nonexpansive self map of C, and suppose that $T(0) \subseteq C_1 \subseteq C$ where C_1

is compact. Then the sequence $\{F^n(x)\}$ where $F:C \to C$ is defined by $F(x) = \frac{1}{2}(T(x) + x)$ converges to a fixed point of T [17].

5. Let X be a uniformly convex Banach space, B a closed ball in X, G an open subset of X containing B. Suppose T is a nonexpansive map of G into X which maps the boundary of B into B. Then T has a fixed point in B [7].
6. Let X be a uniformly convex Banach space, C a closed bounded convex subset of X, G an open subset of X which contains C and such that C has positive distance from X/G. Suppose T is a nonexpansive map of G into X which maps the boundary of C into C. Then T has a fixed point in C [7].
7. Suppose T is a nonexpansive map of a bounded closed convex subset C of a Hilbert space X into X. Suppose further that if u lies on the boundary of C and if $u = R_C(Tu)$, then u is a fixed point of T. Then T has a fixed point in C and for any λ, $0 < \lambda < 1$ and any x_0 in C, the sequence $\{x_n = \lambda R_C T x_{n-1} + (1-\lambda)x_{n-1}\}$ converges weakly to a fixed point of T in C. If in addition T is demicompact, the convergence is strong [9].

8. Let B be a closed ball centered at the origin of a Hilbert space X, T a nonexpansive map of B into X such that for u in the boundary of B, $Tu = -T(-u)$. Then T has a fixed point in B [9].

9. Let C be a bounded closed convex subset of a real Hilbert space and let T mapping C into itself be representable as $T = S + U$ where S satisfies $||Sx - Sy|| \leq q||x - y||$ for all x,y in C. Then T has at least one fixed point if either of the following conditions is satisfied:

   a. If $q < 1$ then U is completely continuous.

   b. If $q = 1$ then U is strongly continuous [36].

I. Strict Nonexpansive Operators

1. Let X be a metric space, T a strictly nonexpansive self map of X such that there exists an x in X for which $\{T^n(x)\}$ has a subsequence converging to a point x^* in X. Then x^* is a fixed point of T [15].

Corollary:

a. If an operator T in a complete metric space X maps a closed set A onto a compact set $B \subseteq C$ and T is strictly nonexpansive on C, then T has a unique fixed point in C [24,27].

J. Strict Contractions

1. Let T be a continuous self map of the complete metric space X. Suppose that $d(Tx,Ty) \le \lambda(d(x,y))d(x,y)$ where $\lambda(p)$ is a nonincreasing function in p and satisfies $0 \le \lambda(p) < 1$ for $p > 0$. Then T has a unique fixed point in X [35].

Comment: The requirement that T be continuous is superfluous since the condition $\lambda(p) < 1$ gives $d(Tx,Ty) \le d(x,y)$.

2. T is a continuous map of a closed subset of a complete metric space X into itself such that T^n is a strict contraction for some integer $n > 0$, then the sequence of Picard iterates converges to a unique fixed point of T in X [26,18].

Corollary:

a. If T is a strict contraction with constant k mapping a closed subset C of a complete metric space X into itself, then T has a unique fixed point in C. Moreover, we can obtain the fixed point x^* as the limit of a sequence $\{x_n\}$ where $x_{n+1} = T(x_n)$ and x_0 is any element in C. The ratio of convergence is given by $d(x_n,x^*) \le (k^n/1-k)d(x,x_0)$ [24,26,27].

<u>Comment</u>: The existence of a fixed point in this case also follows as a corollary to 1.

3. Let U be an open subset of the Banach space X, T a strict contraction with constant k mapping U into itself. Suppose that there exists a ball $B_r(x_0)$ contained in U such that $||Tx_0 - x_0|| \leq (1-k)r$. Then T has a unique fixed point in B_r [31].

Corollary:

a. In R^n let T be a strict contraction with constant k defined on a closed neighborhood $N_r(y_0)$ and suppose that $||T(y_0) - y_0|| \leq (1-k)r$. Then T has a unique fixed point in $N_r(y_0)$ [20].

<u>Comment</u>: A close examination of the relations between all the types of contractions in order to find one kind of contraction which includes all, or most, of the others needs to be made. It would perhaps be fruitful to construct examples of situations in which a contraction does not have a fixed point. An investigation of the use of Picard iterates and alternative schemes for approximating the fixed point of a contraction type operator could be made, with special attention given to algorithms for speeding convergence.

VIII. Nowhere Normal Outward Maps

A. Let X be a strictly convex normed linear space, K a compact convex subset of X and T a nowhere normal outward map from K into X. Then T has a fixed point [22].

IX. Weakly Outward and Inward Maps

A. Let X be a topological vector space such that continuous linear functionals distinguish points. Let K be a compact convex subset of X.

1. If $T:K \to X$ is weakly inward then T has a fixed point in K [22].

2. If $T:K \to X$ is weakly outward then T has a fixed point and $K \subseteq T(K)$ [22].

Comment: In all of the preceding theorems a technique which makes the results easier to apply in a practical situation is needed. In particular, an easy method of determining whether the complicated hypotheses are fulfilled would be extremely useful.

FIXED POINTS OF FAMILIES

I. Continuous

A. Let X be a topological vector space, C a nonvoid compact convex subset of X. Suppose G is a set of continuous maps of C into itself such that:

1. If g is in G, x,y in C and a,b $\geq$ 0 such that

$a + b = 1$ then $g(ax+by) = ag(x) + bg(y)$.

2. There exists a natural number n and subsets G_i $(0 \leq i \leq n-1)$ of G such that $\{1\} = G_n \subsetneq G_{n-1} \subseteq \ldots \subseteq G_0 = G$ where 1 is the identity map of C. To each pair g', g'' in G_{i-1} there exists an f in G_i such that $g'g'' = g''g'f$.

Then there exists x_0 in C such that $g(x_0) = x_0$ for all g in G [18].

Comment: Can any other characterization of the set G be given? How does G relate to the set of all continuous functions? Is G a solvable group? What are the implications if G is abelian or consists of linear functions, or both?

B. Let C be a compact convex subset of a topological vector space X. Let F be a commuting family of continuous linear maps which map C into itself. Then F has a common fixed point in C [13].

C. Let C be a compact convex subset of a locally convex topological vector space X and let G be a group of linear maps which is equicontinuous on C and such that $g(C) \subseteq C$. Then G has a common fixed point in C [13].

Comment: It should be determined if a weaker condition such as locally compact and bounded could replace the requirement of compactness.

II. Contraction

A. Let X be a Banach space and let C be a nonempty compact convex subset of X. If F is a nonempty commutative family of nonexpansive maps of C into itself then the family has a common fixed point in C [11].

B. Suppose C is a weakly compact, convex subset of a Banach space X and suppose C has complete normal structure. Let F be a commutative family of nonexpansive maps of C into itself. Then the family has a common fixed point in C [2].

Corollaries:

1. If F is countable, or if C is separable, then complete normal structure in the above may be replaced by countable normal structure [2,1].
2. Let X be a uniformly convex Banach space, F a commuting family of nonexpansive maps of a given bounded closed convex subset C of X into C. Then the family has a common fixed point in C [6].

C. Let C be a nonempty closed bounded convex subset of a Banach space X, M a compact subset of C. Let F be a nonempty commutative family of nonexpansive mappings of C into itself with the property that for some f_1 in F and for all x in C the closure of the set $\{f_1^n(x)\}$ contains a point of M. Then the family has a common fixed point in M [1].

D. Suppose C is a nonempty weakly compact convex subset of a strictly convex Banach space X. Suppose F is a nonempty commutative family of nonexpansive mappings of C into itself such that for each f in F the f closure is nonempty. Then the family has a common fixed point in X [1].

Comment: A close examination of the structure of each of the above families is needed. Examples of each type should be constructed, both to exhibit the theorem and to show that it can or cannot be improved as well as to make it easier to answer certain conjectures which arise.

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