Systemic Risk Assessment and Mitigation in Financial Networks Based on Optimization Modeling and Techniques

A Dissertation

Presented to

the Faculty of the Department of Industrial Engineering

University of Houston

In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

in Industrial Engineering

by

Aein Khabazian

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Systemic Risk Assessment and Mitigation in Financial Networks

Based on Optimization Modeling and Techniques

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Dedicated to

My dear parents Ahmad and Fakhri

My devoted brother Amir and my sister Arezoo

with all my love

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Abstract

The financial crisis in 2007-2008 has inspired intensive research on the risk assessment and control in financial networks that consist of nodes representing the financial institutions and the links between them representing the interconnection among the financial institutions. Many models and techniques have been proposed to estimate the risk and identify strategy to mitigate the risk in financial networks. Among others, the clearing agent model introduced by Eisenberg and Noe (2001) and its variants have been widely adopted in risk analysis and control. A key concern in this model is the unavailability of complete information regarding the interbank liabilities and the market shock to which the asset values of the financial institutions subject. Most works in the literature assume that full information is known or use an entropy optimization approach based on the so-called Kullback-Leibler divergence to estimate the liability matrix. It has been observed, however, such an approach has led to a significant underestimation of the risk in the financial system.

In this thesis, we propose to assess the systemic risk, and develop mitigation and control strategies under partial information of the underlying financial network. First, we study the vulnerability of the financial network where the asset vector subjects to market shocks. We develop a new extended sensitivity analysis to characterize the conditions under which a bank is solvent, default or bankrupted, and estimate the probability of insolvency and the probability of bankruptcy under mild conditions on the market shock and the network structure. Particularly, we show that while an increment in the social asset may not able to improve the stability of the financial system, a larger asset inequality in the system will reduce its stability. Moreover, under certain assumption on the market shock and the network structure, we show that the least stable network can be attained at some network with a monopoly node, which also has the highest probability of insolvency. The probability of bankruptcy in the network when all the nodes receive shocks is estimated. We also study the vulnerability of a well-balanced network with a monopoly node and explore the domino effect of bankruptcy in it. Numerical experiments are presented to verify the theoretical conclusions.

In the second part, we study the case where only partial information regarding the liability matrix is revealed and the asset vector is fixed. We first propose two bi-level linear optimization models to identify the least and most stable network structures under which the overall repayment in the system is minimal and maximal, respectively. Then we combine several classical optimization methods with new optimization techniques to develop an integrated approach to identify the least and most stable structure in the network. Numerical experiments illustrate that the contagious risk in the identified least stable network is much more significant than what underestimated in the current literature, and the system with the identified most stable network structure is the most resilient one.

In the third part, we propose a new mitigation strategy based on merging and acquisitions to stabilize a financial system. For this, we first introduce some measurement to estimate the benefits of mergers in the merging process based on the extended Eisenberg-Noe model which takes the leverage ratio requirement and liquidation costs into account. We consider subsidized merging where the social planner provides some bail-outs to cover part of the liabilities of the insolvent bank, and develop a goal programming approach to maximize the total merger gain of the merging banks and minimize the bail-out cost for the social planner. We use major European banks linked to the adverse economic scenario used in 2016 EU-wide stress testing for demonstration. The results show that our subsidized merger policy may significantly reduce the bail-out cost compared to the generic public bail-out. Several issues are of interests for future research which is discussed in the last section.

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Chapter 1

Introduction

1.1 Background and Motivation

A typical financial network comprises multiple financial institutions such as firms, traders, and banks. They interact with each other directly through loan contractual obligations, or interconnect through overlapping portfolios. Each of these institutions has two categories of assets: outside assets and interbank assets. Outside assets are claims on non-financial entities, such as mortgages and commercial loans, and interbank assets are claims on other banks in the network. The bank also has some liabilities including obligations to non-financial entities and obligations to other banks in the network. The difference between the assets and liabilities yields the equity of the bank. The data about the asset and liabilities of each bank can be obtained from the balance sheet which is shown in Figure 1.1.

Balance Sheet

As of August 06,2013

Assets		Liabilities	
Cash on Hand	15,000.00	Account Payable	2,200.00
Account Receivable	6,795.00	Taxes Payable	883.00
Inventory	3,000.00	Current Loans Payable	5000.00
Equipment	1,763.00	Long Terms Loans Payable	1,578.00
Assets	**26,558.00**	Credit Cards Payable	900.00
		Other Liabilities	6,117.00
		Liabilities	**16,678.00**
		Owner's Equity	
		Owner's Capital	4,800.00
		Retained Earnings	5,080.00
		Equity	9,880.00
		Liability and Equity	26,558.00

Figure 1.1: A sample of balance sheet

The balance sheets of the banks are strongly connected to each other. Such tight linkages among financial institutions have various consequences in the global financial network. It speeds up the transaction process and affects the asset prices by acquiring and processing the related information more efficiently. There is also a trade-off between the stabilizing effect of interconnections due to diversification and the amplifying effect under which shocks can spread. In fact for the case that a sufficiently large market shock triggered the system, it may cause some banks to default. If the payment shortfall is large enough, it can cause the related bank to default as well, and so on. In such a case, the connectivity of the financial institutions may help to diversify the risk in the system. On the other hand, it may provide a link along which the failure of one institution can spread throughout the system, creating a cascade of defaults (called domino effect), leading to a catastrophic disaster. This is usually referred as the so-called systemic risk.

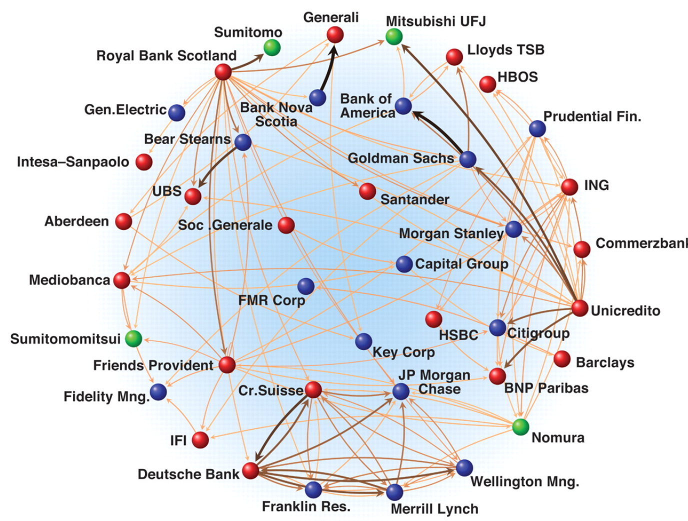


Figure 1.2: A sample of the international financial network, where the nodes represent major financial institutions and the links represent the strongest existing relations among them [66].

The financial crisis in 2007-2008 is piece of evidence of this disaster that has triggered not only the entire U.S. financial industry but also several international financial markets around the world. Motivated by the recent financial crisis, a large literature has been established in the study of systemic risk in financial networks. With the unfolding of the crisis, many concerns were raised that called for the reasons and causes of crisis, lessons that might be drawn from it, as well as policies that should be employed to mitigate and manage the crisis. For example, Flood et al. [38] state that one of the main reasons that the financial crisis happened is the inadequacies of the information infrastructure supporting the US financial system. They identify some of the key reasons for data anarchy in the financial industry, including multiple heterogeneous silos, the data quality gap, lack of standards and the inherent complexity and uncertainty involved. To address this and other shortcomings, the Dodd-Frank Wall Street Reform Act has created an Office of Financial Research (OFR) with a mandate to establish a sound data-management infrastructure for systemic risk monitoring.

Historical accounts of financial crises suggest that fear and greed are the key drivers of these disruptive events. For example, fears of insolvency in the banking industry in August 2007, along with the sudden breakdown of interbank lending and short-term financing, were the initial flash points of the crisis. These fears caused mortgage-related securities such as collateralized debt obligations (CDOs) to lose value and become highly illiquid. The failure of large credit default swap (CDS) counter-parties, the inaccuracy of AAA bond ratings, regulatory lapses and forbearance, political efforts to promote the "homeownership society", and the implicit government guarantees of Fannie Mae and Freddie Mac can also be cited as significant factors in creating the crisis [6].

Counter-party credit risk has proven to be one of the major drivers of the credit crisis. For example, we can recall the credit events occurred in one month of 2008, involving Fannie Mae, Freddie Mac, Lehman Brothers, and Kaupthing. It is also observed by Nathanaël [61] that during the crisis roughly two thirds of the credit risk losses have been due to mark to market of counter-party risk, with only one third due to actual defaults. This shows that counter-party risk has been one of the most important sources of systemic risk, as market participants were highly interconnected through overlapping credit exposures. This led the Basel Committee to revisit the guidelines and moving towards a new set of rules called "Basel III". Such rules require banks to be subject to a capital charge for mark-to-market losses due to changes in the credit spread of a counter-party [19].

The impact of such crisis has implications beyond the United States and extend to several investment banks around the globe (e.g., Australia, China, France, and the UK).

The 2008 financial crisis timeline began in March 2008. It was triggered by the proliferation of mortgage loans, the famous subprime loans, granted to low-income households. As a consequence, major banks found themselves totally or nearly bankrupted. The first major institution to go under was Countrywide Financial Corp., the largest American mortgage lender. The next victim, in March, was the Wall Street investment house Bear Stearns, which had a thick portfolio of mortgage-based securities. Then, Fannie and Freddie suffered the same losses as other mortgage companies. With Bear Stearns disposed of, the markets bid down share prices of Lehman Brothers and Merrill Lynch. Under pressure from the Treasury, Merrill Lynch agreed on September 14 to sell itself to Bank of America for $50 billion, half of its market value within the past year. Lehman Brothers, however, could not find a buyer, and the government refused a Bear Stearns-style subsidy. Lehman declared bankruptcy the day after Merrill's sale. Finally, AIG, an insurance company, was dragged down by its subsidiary.

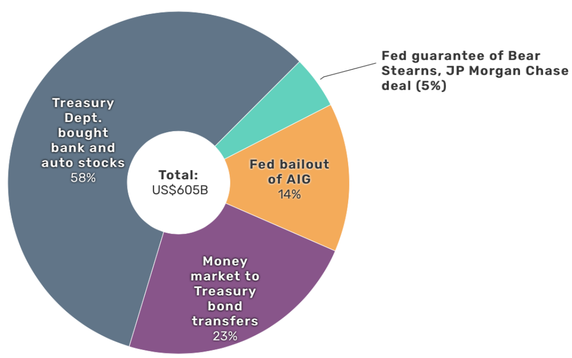


Figure 1.3: A breakdown of how much the 2008 financial crisis initially cost. [55]

As in the U.S., the financial crisis spilled into Europe's overall economy. Asia's major economies were swept up by the financial crisis. Japan's and China's economy also got affected by the financial crisis and by year's end, all of the world's major economies were in recession or struggling to stay out of one [50].

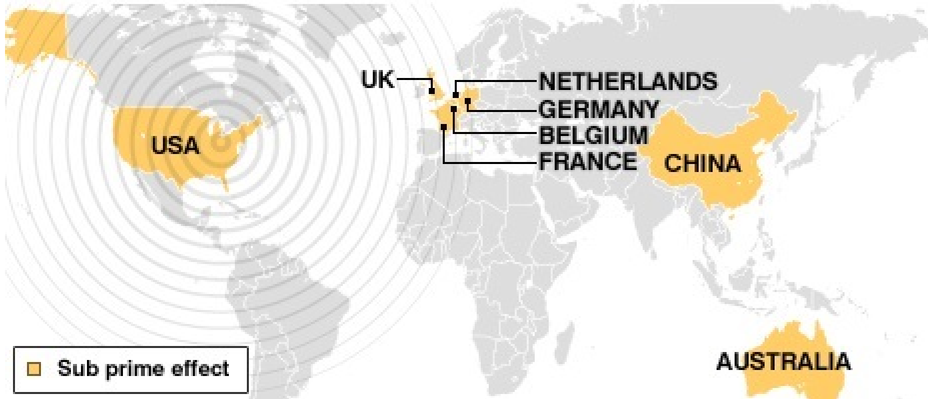


Figure 1.4: Global effect of financial crisis 2007-2008 [12]

That crisis called for a massive government intervention. For example, the U.S. and European banks came together in an attempt to strengthen the money markets by allowing the banks to use funds. The interest rates were reduced in order to assist them in encouraging lending. The UK government provide its own bail-out to eight of the UK's largest banks and building societies. Governments came up with the plan to nationalize banks from Iceland to France since the credit crunch triggered the economies around the world. Central banks in the U.S., Canada and some parts of Europe took the unprecedented step in an effort to ease the crisis. Some of the major institutions failed who were acquired by other institutions, and some of them were subject to government bail-out. These included Lehman Brothers, Merrill Lynch, Fannie Mae, Freddie Mac, Washington Mutual, Wachovia, and AIG (see page 136 in [18]). The shortage of liquidity and the stock market crash contributed to a rapid spread of financial distress from one institution to another in a domino effect.

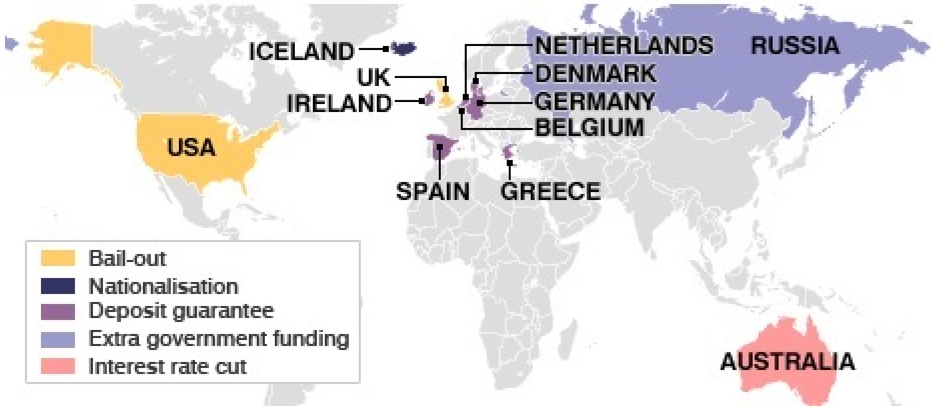


Figure 1.5: Global actions after the financial crisis 2007-2008 [12]

The crisis exposed the fact that very limited information about the interbank liabilities between financial institutions are available to regulators and market participants. It also exposed that there was not enough theoretical understanding about the relation between interconnectedness and stability of the financial system. Moreover, the Lehman bankruptcy consequeces explain the fact that multiple factors cause such contagion occur, e.g., information contagion that spread fears to other money-market funds, a funding run as creditors pulled back lending, and potential fire sales. Network opacity heightened uncertainty in the lead-up to the Lehman bankruptcy. According to the FCIC report, there was no reliable information on who would be owed how much and when payments would have to be made. Such information would be critically important to analyze the possible impact of a Lehman bankruptcy on derivatives counter-parties and the financial markets.

The cost of such financial crises has been immense. It has drawn significant research on the causes of crisis, lessons that might be drawn from it, as well as policies that should be employed to mitigate and manage the crisis. And yet, financial crises have continued to happen, with its new challenges for its management. While it may not be possible to avoid a financial crisis, it will certainly be possible to improve our management of the crisis to minimize its costs and consequences on the financial system.

1.2 Problem Description

In this dissertation, we focus on the vulnerability analysis of the financial network based on a linear optimization model introduced in [30] (E-N model). We remark that several other network contagion models are also developed to study the systemic risk in financial network. For example, the model proposed by Rogers and Veraart [65] is a generalization of the E-N model that takes into account bankruptcy and liquidation costs. Different from Rogers and Veraart [65] , Glasserman and Young [44] define the insolvency cost of a default node in the system via using a function proportional to the amount of default. An alternative to the E-N model is the E-G-J model introduced by Elliot et al. in [31] where the authors consider cross-holdings as direct claims on value of organizations. Default Cascades model (DC) is also proposed by Battiston et al. [10, 9] where they study the propagation of losses from defaulted banks to counter-parties directly in terms of a contagion process on the equity of banks.

In E-N model, a financial system is considered as a network where its nodes represent each financial institution and the links between them represent their interbank liabilities to other nodes in the network. These nominal liabilities denote the promised payments due to other nodes in the system. In the original formulation of the E-N model banks are assumed to have no external liabilities. Bankruptcy costs and liquidation costs are not considered in their model.

Remark that in [30], they proposed various optimization models to measure the systemic risk in a financial network, and these optimization models variate in objective functions but do share the same set of constraints and the same optimal solution. As such, we consider only the linear optimization model in [30] where the objective is to maximize the total payment in the system. Unfortunately, for Eisenberg-Noe's linear optimization model, as pointed out in [35], only limited information on the interbank liabilities such as total liability and total claim of a financial institution is available while the asset of a financial institution is typically subject to market shock. Such uncertainties in this model, poses a tremendous challenge in estimating the market shock impact on the financial network and the real risk threatening the stability of the system.

1.3 Contribution of Dissertation

1.3.1 Issues in the existing analysis

One challenge is that the asset values of the financial institutions fluctuate constantly due to some market shock triggering the system. However, the asset vector in the E-N model is static. Based on this, we cannot estimate the impact of market shock on system's stability and also the resiliency of the financial institutions.

Another challenge is that the complete information about interbank liabilities is usually not exposed and only partial information such as the total liabilities and the total claims of a bank are available. In this regard, most works in the literature first compute the liability matrix by solving some entropy optimization problem based on the KL divergence method, and then analyze the contagious risk based on the estimated liability matrix. As observed in [35], this has led to a significant underestimation of the risk in the financial system.

These challenges have motivated some research questions which this dissertation attempts to answer.

1.3.2 Questions to be addressed

- Can we identify the bankruptcy/default in the network without knowing the full liability matrix? To address this question, we conduct a new sensitivity analysis to characterize the feasibility of Eisenberg-Noe's model. Note that mathematically speaking, the vulnerability of the financial network described in [30] can be identified via analyzing the (in)feasibility of the corresponding constraint set. As such, we consider only Eisenberg-Noe's linear optimization model, and then relax the problem by removing the non-negativity constraints in their original model and explore various properties of the relaxed model. Under the assumption that only a single bank is subject to a market shock, we give a precise estimate on the amount of shock under which a bank will be bankrupted, solvent and default.

- What is the worst-case scenario from the market fluctuation? Does an increase in the asset inequality have a negative effect on the stability of the network? To answer these questions, we consider the generic scenario where all the banks are subject to market shocks. We first show that while a larger social asset may not improve the stability of a financial network, a larger asset inequality between a default node and a strictly solvent node in the network will reduce the stability of the network itself.

- What is the worst-case scenario in terms of asset distribution? What is the minimum probability of bankruptcy in the system? In this regards, we study the network with a monopoly node where a monopoly node owns an asset equals the total social asset and dominates the entire network, an extreme scenario of the asset distribution, and show that the least stable network can be attained at some network with a monopoly node. We also estimate the probability of insolvency in the system under certain assumptions on the market shock and network structure, and show that the network with a monopoly node has the highest probability of insolvency and thus is the most vulnerable one. By using duality theory in the linear optimization, we derive lower bounds for the probability of bankruptcy in the network. We also study the contagious effect of bankruptcy under the network with a monopoly node and tridiagonal structure, and we identify that if the network has a tridiagonal structure and a solvent monopoly node, then the bankruptcy of every non-monopoly node will still have a significant domino effect.

- What's the worst-case structure of the financial network? Does the network structure affect the cascading failures? To answer these questions, we explore the uncertainty in liability matrix with fixed asset vector to identify the worst-case network structure in which the overall payment is minimal. For this, we introduce a bi-level (worst-case) linear optimization model (WCLO) to account for the uncertainties in the liability matrix. Then we propose two update schemes to update the liability matrix and the payment vector alternatively to reduce the overall payment in the system depending on the number of default nodes in the system. We also introduce a new scheme to further reduce the overall payment in the system based on the linear approximation and line search techniques. By combining these three different updating schemes, we develop an integrated algorithm for the proposed WCLO model, characterize the obtained solution and explore the network structure in it.

- What's the best-case structure of the financial network? How can we stabilize the network without changing the total liabilities and total claims obtained from balance sheet data? For this, we propose a bi-level optimization model to identify the most stable network structure under which the overall payment in the system is maximal and design an integrated approach for it. Note that such an algorithm is developed under the assumption that the total liability and total claims remain invariant. This provides means for us to rearrange the liabilities to improve the resiliency of the system without changing the total liabilities and total claims of any bank.

  We then compare the contagious risk under the least stable network, the most stable network and the one based on the KL-divergence. Numerical experiment shows that the propagation of risk under the least stable network is much more significant than the other two networks. We also compare the systemic loss for three different networks under random shocks. Our numerical experiment shows that the most stable network has the minimal systemic loss, while the least stable network has the maximal loss.

- How can we stabilize a financial system? Does the merging strategy reduce the bail-out cost comparing to the generic public bail-out? To address these questions, we propose a new mitigation strategy based on merging and acquisitions to stabilize a financial system. First we extend the E-N clearing model for financial networks by taking into account bankruptcy and liquidation costs, and leverage ratio requirement. Second we propose a new way to estimate the merging gain of a merging pair in the merging process. Under the assumption that all the banks after merging become solvent, we give an explicit formulae to compute the merging gain for all the merging pairs in the network. Third, we introduce a goal programming approach to maximize the total merging gain and minimize the bail-out cost based on the estimated merging gains. We also explore the relationship between the optimal solutions of the two optimization models for the maximal merging gain and the minimal bail-out cost, respectively, and show that under certain conditions, the optimal solution of these two models can be obtained by solving only a single optimization model. Forth, we introduce a new integer linear optimization model (ILP) to manage the trade-off between the merging gain and the bail-out cost, and develop an effective Lagrangian search method for it.

1.4 Outcome

Journal Publication

- Aein Khabazian and Jiming Peng, "Vulnerability Analysis of the Financial Network", *Management Science*, 1–20, 2018.
- John Birge, Aein Khabazian, and Jiming Peng "Optimization Modeling and Techniques for Systemic Risk Assessment and Control in Financial Networks", *Recent Advances in Optimization and Modeling of Contemporary Problems*, 64-84, 2018.
- Aein Khabazian and Jiming Peng "Stabilizing Financial Networks via Merging", **Submitted** to *IMA Journal of Management Mathematics*, 2018.
- Aein Khabazian and Jiming Peng "A Bi-level Linear Optimization Model for Assessing Systemic Risk under Uncertain Liabilities", **Submitted** to *SIAM Journal on Financial Mathematics (SIFIN)*, 2019.

Conference Presentation

- Aein Khabazian, Jiming Peng "Assessing Systemic Risk in Financial Network", INFORMS Annual Conference, Philadelphia, Pennsylvania, November 2015.
- Aein Khabazian, Jiming Peng "Vulnerability Analysis of the Financial Network", INFORMS Annual Conference, Nashville, Tenseness, November 2016.

- Jiming Peng, Aein Khabazian "Assessing Systemic Risk under Uncertain Liabilities", INFORMS Annual Conference, Nashville, Tenseness, November 2016.

- Aein Khabazian, Jiming Peng "Assessment and Control of Systemic Risk Under Uncertain Liabilities", INFORMS Annual Conference, Houston, Texas, October 2017.

- Aein Khabazian, Jiming Peng "Stabilizing Financial Networks via Merging", INFORMS Annual Conference, Phoenix, Arizona, November 2018.

1.5 Outline of Dissertation

This dissertation is organized into six chapters as follows. In Chapter 2, we will review some of the existing literatures on systemic risk assessment and network resilience in financial network. In Chapter 3, we study the vulnerability of the financial system via the sensitivity analysis and characterize the worst-case scenario in terms of the asset distribution. In Chapter 4, we explore the uncertainties in the liability matrix and develop an algorithm to identify the worst-case and best-case structure. It is also shown that under the identified least stable network the contagious effect of failures are significant. In Chapter 5, we propose a new mitigation policy based on merging to stabilize a financial system. The results show that our subsidized merger policy may significantly reduce the bail-out cost compared to the generic public bail-out. In Chapter 6, we discussed some future research directions.

Chapter 2

Literature review

As pointed out in the introduction a large literature has been established in the study of systemic risk. Most existing works in the literature fit into the one of the following three streams:

2.1 The Assessment of Systemic Risk

Works in the risk assessment can be mainly categorized into two groups. The first group focuses on the development of various risk measurements and models. For example, in their seminal paper, Eisenberg and Noe [30] introduce the clearing payment system framework considering bankruptcy law to assess the systemic risk in inter-banking networks. They describe the contagious effect of failure where the payment shortfall originating at a single institution can transmit to other financial institutions. They also study the existence and uniqueness of the clearing payment vector and propose an algorithm to compute the clearing payment vector by tracking the sequence of default. Rogers and Veraart [65] extend the Eisenberg and Noe's model by taking into account bankruptcy and liquidation costs. Different from Rogers and Veraart [65], Glasserman and Young [44] define the insolvency cost of a default node in the system via a function proportional to the amount of default. The existence and uniqueness of the clearing payment vector is also studied in [44]. Elliot et al. [31] also develop a new model by considering cross-holdings as direct claims on the value of organizations. Several different clearing algorithms have been studied by Barucca et al. [7], Gai and Kapadia [42], and Rogers and Veraart [65]. For more details in this direction, we refer to the monograph [49] and references therein.

The clearing agent model, of course, is not the only systemic risk measure. Huang, Zhou and Zhu [48] use ex ante measures of default probabilities of individual banks and forecasted asset return correlations to estimate expected credit losses above a given share of the financial sector's total liabilities. Jonghe [29] presents estimates of tail betas for European financial firms as their systemic risk measure. Adrian and Brunnermeier [3] propose a new reduced-form measure of systemic risk, ΔCoVaR, that captures the (cross-sectional) tail-dependency between the whole financial system and a particular institution. Acharya et al. [2] focus on high-frequency marginal expected shortfall as a systemic risk measure. They develop a way to measure each bank's contribution to systemic risk.

Works in the second group focuses on empirically measuring risk based on market data. For example, Furfine [40] measures the contagious effect of one or small number of insolvent institutions through the network by using federal funds exposure data. Sheldon and Maurer [67] and Upper and Worms [70] used information from the banking systems in Switzerland and Germany in their analysis. Sheldon and Maurer [67] and Cocco et al. [26] further observed that the liability matrix from the banking system is usually sparse, which is very different from the complete liability matrix estimated based on KL divergence. Empirical studies by Mistrulli et al. [59] and Degryse and Nguyen [28] also observed such a difference between the estimated liability matrix and the real interbank structure.

2.2 The Stability and Resilience of Financial Networks

Works in the financial network resiliency and stability concentrate on exploring the impact of market shocks and network structure on the stability of a financial system. For example, Acemoglu et al. [1] , Chen et al. [22], Cont et al. [27], Elsinger et al. [34, 33, 32, 35] Glasserman and Young [44] and Liu and Staum [57] study the contagions in a financial system under different settings.

In a series of papers, Elsinger et al. [34, 33, 32, 35] study the financial stability of a banking system considering the cascading impact of failures over the entire network. In their model, the joint impact of two major sources of risk, the correlated exposure and domino effect, is considered. Specifically, Elsinger et al. [34, 33] estimate the systemic risk in the financial network in the UK and Austria based on data from banks in these two countries. As pointed out in [35], the data from banks usually reveal only partial information regarding the interbank liabilities, while the assets are subject to market fluctuation. To account for the uncertainties in the assets, Elsinger et al. [34, 33] use stochastic optimization and scenario generation to estimate the worst-case scenario for the underlying linear optimization model. In this regard, they suggest to compute the liability matrix by solving some entropy optimization problem based on the so-called Kullback-Leibler (KL) divergence. Liu and Staum [57] apply the standard sensitivity analysis in Eisenberg-Noe's LP model to estimate the impact of the market shock to a single financial institution.

Some other scholars have argued that the presence of inequality played an important role in the crisis. For example, Treeck [69] discusses that an increase in income inequality was a main cause for the rapid growth of the US non-prime mortgage market and the global balance of payment imbalances contributing to the Great Recession.

Several researchers study the impact of interbank liability structure on risk exposure. In their pioneering work, Allen and Gale [5] first show that there exists a relation between the specific pattern of interbank lending and the extent of contagion in financial system. Gai and Kapadia [42] discuss contagious effect of failure in a random network and analyze the knock-on effects of distress. They also observe that the impact of shock depends on the network connectivity and the location of a node that is triggered by that shock. Battiston et al. [8, 9] study how the credit risk diversification and network density affect systemic risk. Acemoglu et al. [1] study the way that the network structure and the magnitude of negative shock to a single financial institution in the network can affect the stability of financial networks. Elliot et al. [31] also discuss the trade-off between diversification and integration of the network and its impact of the stability of the financial system. Glasserman and Young [44] study the contagion effects via network spillovers under assumptions on the shock distribution and show that pure contagion effects are usually low for realistic interbanking networks. Chen et al. [24] explore the optimality conditions in Eisenberg-Noe's model to design a partition algorithm that can separate the default institutions and the solvent ones in the network. They also estimate the impact of both market and liquidity channel in risk transmission using sensitivity analysis.

2.3 Policies and Strategies to Mitigate the Risk

This stream of related studies is exploring the impact of mitigation policies. For example, Pokuta et al. [64] extend the Eisenberg-Noe's model as a network flow problem and study the effect of the bail-out strategy on the financial network. Capponi et al. [21] study the impact of liability concentration on the loss profile of the system using the concept of majorization of the liability matrix. They show that the size of risk exposure to individual counter-parties in financial networks depends on the state of the system (i.e., balancing or unbalancing) and the liability concentration. In [20], they develop a multi-period clearing framework and investigate the impact of regulatory and preventive policies in order to either limit the loss exposure towards financial institutions or increase the resilience of system against financial failures. Bernard et al. [14] propose three intervention policies that a social planner may use to stabilize the system, including bail-out, bail-in, and subsidized bail-in. Kallio and Khabazian [51] propose a different strategy to mitigate the risk in the network by formulating several coalitions of banks such that all the banks in the same coalition collaborate with each other and pay the same fraction of their liabilities. Their approach can serve as a decision support mechanism for such negotiations. Another set of studies captures the effect of capital and liquidity requirements in order to mitigate systemic risk (e.g., [25, 47, 41]).

Chapter 3

Vulnerability Analysis of the Financial Network

3.1 Introduction

A typical financial network comprises multiple financial institutions interacting with each other through borrowing and lending or interconnecting indirectly through the market by holding similar shares or portfolios. In this network, a financial institution that cannot make none of its required payments goes bankrupt, an institution that can payback partial of its liabilities default and when it can fulfill its liabilities is solvent. The presence of tight linkages in these financial institutions has various consequences in the global financial market. On the one hand, it influences asset prices by acquiring and processing the related information more efficiently, and as a result, large numbers of transactions can be proceeded smoothly without any interruptions and the trading performance is improved. On the other hand, whenever some institution bankrupts in the system, it may lead to a catastrophic disaster by spreading this failure quickly over the entire system. This is usually referred as the so-called systemic risk. As evidence we recall the 2008 financial crisis in U.S. that have triggered not only the entire U.S. financial industries but also several international financial markets around the world [60]. Another example is the European sovereignty debt crisis that causes the European financial business to face serious loss of confidence [56].

The catastrophic consequences from these widespread phenomena have prompted extensive study on the sources, effects and results of the crises, and developing tools to mitigate and manage the systemic risk to increase the resilience of financial networks to encounter economic crisis. To this end, a growing number of literatures has been devoted to assessing the systemic risk and studying the contagion effect in a financial network. In a seminal paper, Eisenberg and Noe [30] introduce the clearing payment system framework considering bankruptcy law to assess the systemic risk in inter-banking networks. They show that the failure of a single institution can transmit to other financial institutions, leading to a contagion risk. Existence and uniqueness of the clearing payment vector are studied in the paper. Moreover, they also propose an algorithm to compute the clearing payment vector by tracking the sequence of default.

In a series of papers, Elsinger et al. [34, 33, 32, 35] study the financial stability of a banking system considering the cascading impact of failures over the entire network. In their model, the joint impact of two major sources of risk, the correlated exposure and domino effects, is considered. Specifically, Elsinger et al. [34, 33] estimate the systemic risk in the financial network in UK and Austrian based on data from banks in these two countries. As pointed out in [35], the data from banks usually reveals only partial information regarding the interbank liabilities, while the assets are subject to market fluctuation. To account for the uncertainties in the assets, Elsinger et al. [34, 33] use stochastic optimization and scenario generation to estimate the worst-case scenario for the underlying linear optimization model. They also suggest to compute the liability matrix by solving some entropy optimization problem based on the so-called Kullback-Leibler divergence.

There exist several works focusing on the impact of interbank liability structure on the risk exposure. In their pioneering work, Allen and Gale [5] first establish the connection between the specific pattern of interbank lending and the extend of contagion in a financial system. Gai and Kapadia [42] discuss how contagion spreads in a random network, and analyze the knock-on effects of distress. Battiston et al. [8, 9] study the effect of credit risk diversification and network density on systemic risk. Liu and Staum [57] apply the standard sensitivity analysis in Eisenberg-Noe's LP model to estimate the impact of the market shock to a single financial institution. Acemoglu et al. [1] study the stability of financial networks depending on the network structure and magnitude of negative shock to a single financial institution in the network. Glasserman and Young [44] study the contagion effects via network spillovers under assumptions on the shock distribution and show that pure contagion effects are usually low for realistic interbanking networks. Very recently, Chen et al. [24] explore the optimality conditions in Eisenberg-Noe's model to design a partition algorithm that can separate the default institutions and the solvent ones in the network. They also use sensitivity analysis to estimate the impact of both market and liquidity channel in risk transmission.

Several studies have explored the impact of mitigation policies. For example, Pokuta et al. [64] extend the Eisenberg-Noe's model as a network flow problem, and study the influence of the bail-out strategy on the financial network. Capponi and Chen [20] and Capponi et al. [21] investigate the impact of regulatory and preventive policies in order to either limit the loss exposure towards financial institutions or increase the resilience of system against financial failures. Treeck [69] discuss the relationship between income inequalities and financial crisis based on historical data.

Different from the above-mentioned works, in this paper we focus on the vulnerability analysis of the financial network based on a linear optimization (LO) model introduced in [30]. It is worthwhile mentioning that in [30], Eisenberg and Noe proposed various optimization models to measure the systemic risk in a financial network, and these optimization models variate in objective functions but do share the same set of constraints and the same optimal solution. As such, in this work we consider only the LO model in [30]. Note that one key issue in the vulnerability analysis of a financial network is to identify conditions under which a bank in the network will default or be bankrupted and estimate the contagious risk caused by the default or bankruptcy of that bank. A rich literature has been established on contagious risk analysis in financial networks. For more details, we refer to the survey paper [35], and the references therein. Unfortunately, as pointed out in [35], most existing works have underestimated the contagious risk in the financial system. In [35], the authors further speculate that the incomplete information on the financial network may be one major reason for the underestimation of the risk. However, we noticed that another possible reason for the underestimation of the risk in a financial system is the restrictive small shock assumption widely used in the existing literature. To see this, let us take a closer look at the reference [57] where the authors estimate the contagious risk under two assumptions: One assumption is that the complete information of a financial network is available and another assumption is that the market shock will not change the set of default banks and the set of solvent banks. As shown in [57], under these two assumptions, the contagious risk in the network can be estimated via the solution to the dual problem of the LP problem in [30]. Note that the assumption that the sets of default banks and solvent banks remain invariant implicitly implies that the market shock is insignificant or small, which is very different from what happened during the financial crisis in 2007-2008 when the large market shock had led to the bankruptcy of financial institution such as Lehman Brothers. Clearly, the small shock assumption cannot be used in the analysis of bankruptcy in a financial network. A challenge here is how to assess the systemic risk of a financial network when only limited and incomplete information regarding the financial network is available and the market shock is significant.

One main motivation of this work is to address the above challenge via developing a new theoretical framework to analyze the vulnerability of a financial network under mild assumptions on market shocks. To start, we mention that based on the LO model in [30], the bankruptcy in a financial network corresponds to the infeasibility of the model itself. For a given linear optimization problem in which all the data are available, we can refer to the well-known Farkas lemma to detect its feasibility [71]. Unfortunately, for the LO model in [30], as observed in [35], only limited information on the interbank liabilities such as the total liability and total claim of a financial institution is available while the asset of a financial institution is typically subject to market shock. In other words, both the data matrix and right hand side of the constraints in the underlying linear optimization model are uncertain. The presence of uncertainty in Eisenberg-Noe's model poses a tremendous challenge in estimating the market shock impact on the financial network and detecting the model's infeasibility.

Our first contribution in this work is to conduct a new extended sensitivity analysis to characterize the (in)feasibility of the LO model in [30]. To achieve such a goal, we propose to relax the LO model in [30] by removing the non-negativity constraints in it. Then, we explore various properties of the relaxed model. Particularly, we derive a simple characterization for the feasibility of the relaxed model in terms of the summation of all the assets in the network (the total asset). Under the assumption that only a single shock is received by some bank in the network, we give a precise estimate on the amount of shock under which the receiving bank will be bankrupted, default or solvent. We also assess the impact of the single shock to all the non-receiving nodes in the system.

For the more generic scenario where all the banks are subject to market shocks, we first show that while a larger total asset may not improve the stability of a financial network, a larger asset inequality between a default node and a strictly solvent node in the network will reduce the stability of the network itself. To the best of our knowledge, our result is the first quantitative analysis showing that the asset inequality has a negative effect on the stability of the network. Then we study the network with a monopoly node where the monopoly owns an asset equals the total asset and dominates the entire network, representing an extreme scenario of the asset distribution. We show that the least stable asset distribution can be attained at some network with a monopoly node. We also estimate the probability of insolvency in the system under certain assumptions on the market shock and network structure, and show that the network with a monopoly node has the highest probability of insolvency and thus is the most vulnerable one. By using duality theory in linear optimization and stochastic optimization, we derive lower bounds for the probability of bankruptcy in the network. Particularly, we show that if the monopoly node in a financial network is liability free, then the probability of bankruptcy in the system is larger than 50%.

We also estimate the impact of bankruptcy in a financial network with a monopoly node. We first show that the bankruptcy of the monopoly node in a network will cause all other nodes in the network to be bankrupted, a catastrophical disaster to the entire system due to the domino effect of bankruptcy. In other words, the monopoly node is too big to fail (TBTF). We further explore the domino effect of bankruptcy in a financial network under a tridiagonal structure. We show that even when the monopoly node in such a financial network is solvent, the bankruptcy of every non-monopoly node in the network will cause other nodes following it to bankrupt consecutively.

The paper is organized as follows. In section 3.2, we first relax Eisenberg-Noe's model and explore various properties of the relaxation model. Then, we give a simple characterization on the (in)feasibility of the relaxation model. In section 3.3, we consider a scenario of the network where only a single node receives the shock. We first characterize different conditions under which the receiving node is solvent, default or bankrupted. We also develop a new algorithm to estimate the indirect impact of a single shock to the non-receiving nodes in the system. In section 3.4, we consider a scenario where all the nodes are subject to market shocks. We first study the impact of asset inequality on the stability of the system. Then, we estimate the probability that some bank in the network will be insolvent or be bankrupted under assumptions on the market shocks and network structure. In Section 3.5, we assess the impact of bankruptcy in a financial network with a monopoly node. Finally we conclude the paper in Section 3.6 by discussing some future research directions.

3.2 A Relaxation Model and Its Properties

As pointed out in the introduction, though many optimization models were proposed to measure the systemic risk of a financial network in [30]. To study the vulnerability of the network, it suffices to investigate the linear optimization model. Consider a financial network consisting of n banks denoted by n nodes where each bank borrows from one another and thus it owes liabilities to others. A clearing agent is in charge of the process of settling the liabilities among these nodes. The value of one node's payment to settle its obligations depends on the payment of other nodes to this node. Let $L \in \Re^{n\times n}$ be the interbank liability matrix where l_{ij} is the liabilities of node i toward node j. Since each nominal claim is nonnegative and no node has a nominal claim against itself, therefore we have $l_{ij} \geq 0$ and $l_{ii} = 0$, $\forall i, j = 1, ..., n$. Let α be the exogenous operating cash flow that consists of external investment plus the liquid and illiquid assets. The total liabilities of node i is equal to $\bar{p}_i = \sum_{j=1}^{n} l_{ij}$. In Eisenberg-Noe's model, the interbank payment made by node i to node j is $x_i l_{ij}$ which can be obtained by solving the following linear optimization problem:

$$\begin{aligned} \max \quad & \bar{p}^T x \\ s.t. \quad & (\overline{P} - L^T)x \leq \alpha, \\ & 0 \leq x \leq e, \end{aligned} \tag{3.2.1}$$

where $\overline{P}$ =diag($\bar{p}$) and e is the all-ones vector. Note that the LP model has been used in [24, 57, 64] with different notations and objective coefficients. In this model, for any strictly positive coefficients in the objective function, the optimal solutions are the same (see section 3.2 in [30]). Therefore, we choose a particular one as (3.2.1) which is more convenient to analyze.

The Eisenberg-Noe's model has attracted the attention from various researchers in recent years and many results have reported in the literature. As pointed out in the introduction, many existing results focus on how the contagion risk spreads over the network when some bank in the network defaults, or the influence of the market shock to the network when the shock happens to a single bank. For more details, we refer to recent works [1, 21, 24, 27, 34, 33, 35, 44, 57] and the references therein.

In this paper, we try to estimate the impact of the market shock when all the banks are subject to market shocks from a certain distribution. We focus primarily on identifying conditions under which some banks in the network will be bankrupted or default. For such a purpose, we first suggest to remove the non-negativity constraint in (3.2.1), resulting in the following relaxation.

$$\begin{aligned} \max_{x} \quad & \bar{p}^T x \\ s.t. \quad & (\overline{P} - L^T)x \leq \alpha; \\ & x \leq e. \end{aligned} \tag{3.2.2}$$

We note that at the optimal solution of (3.2.2) for each node i we have three conditions, i.e., $x_i^* \leq 0$, $x_i^* \in (0,1)$, and $x_i^* = 1$, which are associated with the status of the bank i depending on whether it is bankrupted, default, or solvent. In this model, when we have $x_i^* \leq 0$, it means that node i cannot repay any of its liabilties and the its payements flows backward.

Our next result establishes the equivalence between two problems (3.2.1) and (3.2.2) under the assumption that problem (3.2.1) is feasible.

Proposition 3.2.1. Let $x^{(1)}$ be the optimal solution of problem (3.2.1) and $x^{(2)}$ be the optimal solution of the problem (3.2.2). Then we have

$$x^{(2)} = x^{(1)}.$$

Proof. The proof of the proposition is a minor modification of the proof of Theorem 2.1 in [64]. For self-completeness, we give the detailed proof here. Note that to prove the above proposition, it suffices to show that $x^{(2)} \geq 0$. Let X^1, X^2 be the feasible set to problems (3.2.1) and (3.2.2), respectively. Clearly both X^1 and X^2 are bounded and convex. Moreover, we have $X^1 \subset X^2$. Since problem (3.2.1) is feasible and thus, X^1 is nonempty, it follows that X^2 is also nonempty. For the bounded nonempty set X^2, let $\bar{x}$ be the vector whose element $\bar{x}_i, i = 1, \ldots, n$ is defined by

$$\bar{x}_i := \sup_{x \in X^2} x_i. \tag{3.2.3}$$

It follows that for every $i = 1, \ldots, n$, we have

$$\bar{x}_i := \sup_{x \in X^2} x_i \geq \sup_{x \in X^1} x_i \geq 0,$$

where the inequality follows from the fact $X^1 \subset X^2$.

We next show that $\bar{x}$ is the unique optimal solution of problem (3.2.2). For every

$i \in \{i = 1, 2, ..., n\}$, there is $x^i \in X^2$ such that $x^i_i = \bar{x}_i$, thus

$$\bar{p}_i \bar{x}_i - \sum_j l_{ji} x^i_j = \bar{p}_i x^i_i - \sum_j l_{ji} x^i_j$$

holds. Because $l_{ji} \geq 0$, we have

$$\bar{p}_i \bar{x}_i - \sum_j l_{ji} \bar{x}_j \leq \bar{p}_i \bar{x}_i - \sum_j l_{ji} x^i_j,$$

and we can conclude by the fact $x^i \in X^2$ that

$$\bar{p}_i \bar{x}_i - \sum_j l_{ji} \bar{x}_j \leq \alpha_i$$

i.e., $\bar{x}$ is a feasible solution to problem (3.2.2). Clearly, $\bar{x}$ maximizes $\sum_i \bar{p}_i x_i$ and therefore, $\bar{x}$ is the unique optimal solution to (3.2.2). This completes the proof of the proposition.

As discussed in Eisenberg and Noe's paper, the optimal solution of problem (3.2.1) is a clearing vector for the financial system that satisfies the so-called limited liability and absolute priority. By following a similar process as in [30], we can obtain the following result.

Corollary 3.2.1. At the optimal solution of problem (3.2.2), we have either

$$[(\overline{P} - L^T)x^*]_i = \alpha_i \text{ or } x^*_i = 1, \ \forall i = 1, \ldots, n.$$

From Propositions 3.2.1 and Corollary 3.2.1 we can see that there is no differences regarding the properties of the optimal solutions to problems (3.2.1) and (3.2.2), and the optimal solution of problem (3.2.2) can also be used as a clearing vector for the financial system. However, as we shall see in our later analysis, a key difference between problems (3.2.1) and (3.2.2) lies in the fact that a simple characterization of the (in)feasibility of problem (3.2.2) can be derived, which further facilitates the feasibility analysis of model (3.2.1). Before stating our main result in this section, we need the following definition

Definition 3.2.1. Two banks i and j in the financial network are said to be connected if there exists a path $i = i_0 \to i_1 \to \cdots \to i_{K-1} \to i_K = j$ such that

$$l_{i_{k-1}i_k} > 0, \quad \forall k = 1, \ldots, K.$$

A financial network is said to be fully connected if every pair of banks in the network is connected.

In the paper, we make the following assumption:

Assumption 3.2.1. The financial network is fully connected.

We remark that Assumption 3.2.1 is rather mild because if the financial network is not fully connected, then we can divide it into two independent subnetworks such that there exists no connections between these two subnetworks. Correspondingly, we can solve two smaller linear optimization problems to obtain the clearing vectors for these two independent systems.

We next present a technical result that will be used in the analysis later on.

Lemma 3.2.1. Suppose that the financial network is fully connected. Let λ^* be a solution of the following system of linear inequalities

$$(\overline{P} - L)\lambda \leq 0, \ \lambda \geq 0. \tag{3.2.4}$$

Then, it must hold $\lambda^* = ce$ for some $c \geq 0$.

Proof. Without loss of generality, we assume $\lambda \neq 0$. Let $\lambda_{i*} = \max_{i=1,\ldots,n} \lambda_i$, and define the index set $I_{i*} = \{j \neq i* : l_{i*j} > 0\}$. From the definition of $\bar{p}_{i*}$, we have $\bar{p}_{i*} - \sum_{j \in I_{i*}} l_{i*j} = 0$. It follows from (3.2.4) that

$$0 \geq \bar{p}_{i*}\lambda_{i*} - \sum_{j \in I_{i*}} l_{i*j}\lambda_j \geq \lambda_{i*}(\bar{p}_{i*} - \sum_{j \in I_{i*}} l_{i*j}) = 0. \tag{3.2.5}$$

From (3.2.4) and (3.2.5) we can conclude

$$\lambda_j = \lambda_{i*} = \max_{i=1,\ldots,n} \lambda_i, \quad \forall j \in I_{i*}.$$

Similarly, for every $i \in I_{i*}$ and the index set $I_i = \{j \neq i : l_{ij} > 0\}$, we have

$$\lambda_j = \lambda_{i*}, \quad \forall j \in I_i. \tag{3.2.6}$$

Now let us choose arbitrary any index $j*$. Since the network is fully connected, there exists a path $i* = i_1 \to i_2 \cdots \to i_{K-1} \to i_k = j*$. By following a similar vein as in the proof of (3.2.6), we can conclude $\lambda_{j*} = \lambda_{i*}$. This completes the proof of the lemma.

Now we are ready to state the main result in this section.

Theorem 3.2.1. Problem (3.2.2) is infeasible if and only if

$$\sum_i \alpha_i < 0. \tag{3.2.7}$$

Proof. First, we note that the dual problem of (3.2.2) reads as

$$\begin{aligned} \min_{\lambda} \quad & \alpha^T \lambda + e^T \bar{p} - e^T (\overline{P} - L)\lambda \qquad (3.2.8) \\ s.t. \quad & (\overline{P} - L)\lambda \leq \bar{p}; \\ & \lambda \geq 0. \end{aligned}$$

Using the Farkas Lemma, we see that problem (3.2.2) is infeasible if and only if the following problem has a solution.

$$\begin{aligned} & \alpha^T \lambda - e^T (\overline{P} - L)\lambda < 0, and \qquad (3.2.9) \\ & (\overline{P} - L)\lambda \leq 0; \qquad (3.2.10) \\ & \lambda \geq 0. \end{aligned}$$

It remains to show that (3.2.9) has a nontrivial solution if and only if (3.2.7) holds.

If $e^T\alpha < 0$, from the definition of liability matrix L we have $(\overline{P} - L)e = 0$. This implies that for any $c > 0$, ce is a feasible solution of system (3.2.9), and therefore, problem (3.2.2) is infeasible.

On the other hand, suppose that system (3.2.9) has a nontrivial feasible solution. From Lemma 3.2.1 we can conclude that $\lambda = ce$ for some $c \geq 0$. It follows from (3.2.9) that $c\alpha^T e < 0$, and thus it must hold $\alpha^T e < 0$. This completes the proof of the theorem.

Theorem 3.2.1 indicates that we can use the total asset as an indicator for the infeasibility of the relaxed problem (3.2.2) and such an indicator is independent of the liability matrix. We next explore the financial meaning of Theorem 3.2.1. For this, let us consider a scenario where the clearing agent has a superpower to redistribute the assets of all the banks in the system. Under the assumption that the total asset is fixed, the agent would like to use its power to maximize its revenue, leading to the following two-stage linear optimization problem

$$\begin{aligned} \max_{\alpha} \ \max_{x} \quad & \bar{p}^T x && (3.2.11) \\ s.t. \quad & (\bar{P} - L^T)x \leq \alpha; \\ & x \leq e; \\ & \sum_i \alpha_i = \bar{\alpha}. \end{aligned}$$

We have

Proposition 3.2.2. If the total asset is non-negative ($\bar{\alpha} \geq 0$), then the optimal solution to problem (3.2.11) can be attained at some α^* such that $x_i^*(\alpha^*) = 1, \ \forall i = 1, \ldots, n$.

Proof. We first consider the special case when all the nodes in the system are well-balanced. In this case, we have $(\bar{P} - L^T)e = 0$. This implies that for every asset vector α^* satisfying $\alpha_i^* \geq 0, \ \forall i = 1, \ldots, n$, we have $x^*(\alpha^*) = e$ and thus, all the nodes in the system are solvent.

Next we consider the generic case $\sum_i \bar{p}_i = \sum_i (L^T e)_i$. In this case, one can show that if we choose the asset vector α^* by $\alpha_i^* = \bar{\alpha}/n + \bar{p}_i - (L^T e)_i, \ \forall i = 1, \dots, n$, then we still have $x^*(\alpha^*) = e$. This shows that all the nodes in the system are solvent.

From Theorem 3.2.1 and Proposition 3.2.2 we can conclude that the value of the total asset $\bar{\alpha}$ indicates whether we can redistribute the assets to make all the nodes in the network solvent or there exists no ways to make all the nodes solvent. Since problem (3.2.2) is a relaxation of problem (3.2.1), the condition $\bar{\alpha} < 0$ can also be viewed as a sufficient condition for the infeasibility of problem (3.2.1). It is interesting to note that Theorem 3.2.1 holds true for the generic class of financial networks that are fully connected, which shows that the vulnerability of the financial network may not depend on the liability matrix L.

Our next theorem explore various properties at the optimal solution of problem (3.2.2).

Theorem 3.2.2. Suppose that the total asset of the financial network is nonnegative ($\bar{\alpha} = \alpha^T e \geq 0$). Let x^* be the optimal solution of problem (3.2.2). Then the following conclusions hold.

(i) There exists at least one index i such that $x_i^* = 1$;

(ii) For every $i = 1, \dots, n$, if $\alpha_i < \bar{p}_i - (L^T e)_i$, then $x_i^* < 1$;

(iii) For every $i = 1, \dots, n$, if $x_i^* = 1$, then $\alpha_i \geq \bar{p}_i - (L^T e)_i$;

Proof. We start with the proof of Conclusion (i). Let us first consider the case when $\bar{\alpha} > 0$. Suppose to the contrary that Conclusion (i) does not hold, i.e., at the optimal solution x^* of problem (3.2.2), we have

$$x_i^* < 1, \quad \forall i = 1, \dots, n.$$

From Proposition 3.2.1, the following condition holds

$$(\overline{P} - L^T)x^* = \alpha.$$

Because $e^T(\overline{P} - L^T) = 0$, it follows that $e^T\alpha = \bar{\alpha} = 0$, which contradicts to the assumption $\bar{\alpha} > 0$. To prove the conclusion when $\bar{\alpha} = 0$, let us define a new vector α^ϵ where

$$\alpha_1^\epsilon = \alpha_1 + \epsilon, \quad \alpha_i^\epsilon = \alpha_i, \forall i = 2, \ldots, n.$$

It is easy to see that

$$e^T\alpha^\epsilon = \bar{\alpha} + \epsilon > 0, \quad \forall\ \epsilon > 0.$$

Now let us consider a variant of problem (3.2.2) where α is replaced by α^ϵ and let us denote its optimal solution by $x^*(\epsilon)$. Since $e^T\alpha^\epsilon > 0$, there exists some index i such that $x_i^*(\epsilon) = 1$. Now let us choose a sequence of $\epsilon^k \to 0$. By restricting us to a subsequence if necessary, we can see that there exists an index i such that $x_i^*(\epsilon^k) = 1$. Recall that problem (3.2.2) is a linear program, by using the continuity of the solution sets for linear programs [58], we have

$$x_i^* = x_i^*(0) = \lim_{k\to\infty} x_i^*(\epsilon^k) = 1.$$

This completes the proof of conclusion (i).

To prove the second conclusion, we note that at x^*, the following inequality

$$\bar{p}_i x_i^* - \sum_{j\neq i} l_{ji} x_j^* \leq \alpha_i, \quad \forall\ i = 1, \ldots, n,$$

holds. Therefore, for every $i = 1, \ldots, n$, we have

$$\bar{p}_i x_i^* \leq \alpha_i + \sum_{j\neq i} l_{ji} x_j^* \leq \alpha_i + \sum_{j\neq i} l_{ji} = \alpha_i + (L^T e)_i < \bar{p}_i, \tag{3.2.12}$$

where the second inequality follows from the constraint $x^* \leq e$, and the last inequality from the assumption $\alpha_i < \bar{p}_i - (L^T e)_i$. From (3.2.12) we immediately obtain

$$x_i^* < 1.$$

The third conclusion follows directly from Conclusion (ii). This completes the proof of the theorem.

Conclusion (i) in Theorem 2.2 shows that when the total asset is non-negative, at least one bank remains solvent in the system. Based on the second conclusion in Theorem 2.2, we can estimate the upper bound for the asset value of bank i under which bank i will be insolvent in the system. We note this upper bound can be obtained from the current market information.

We remark that though the results of Theorem 3.2.2 are established for problem (3.2.2), from Proposition 3.2.1 one can easily see that these results also hold true for problem (3.2.1) when it is feasible. We note that similar results like Conclusions (ii) and (iii) in Theorem 3.2.2 have been obtained in [24] where the authors used them to design an effective algorithm, and estimate the impact of a single shock to the system based on the classical sensitivity analysis in linear optimization and probability theory. In this paper, we shall use these results to estimate the impact of market shocks on the financial network.

We next provide a numerical example to verify the conclusions in the theorem. For convenience, we introduce the following definition.

Definition 3.2.2. A financial network is said to be well-balanced, if for each node i, its total liability equals its total claim, i.e.,

$$\bar{p}_i = (L^T e)_i, \quad \forall i = 1, \ldots, n.$$

Example 3.2.1. We consider a complete financial network with four banks in which the total asset equals zero ($\bar{\alpha} = e^T \alpha = 0$). The liability matrix is extracted from the liability matrix (see Table 6 in [24]) by considering the first four banks in the network. The asset vector and the optimal solution are also listed below.

In the above example, we first consider an asset vector α^1 satisfying $\bar{\alpha}^1 = 0$. All the nodes in the example are well-balanced, i.e., $\bar{p}_i - (L^T e)_i = 0$ for $i = 1, \ldots, 4$. One can see that only nodes 2 and 4 can fulfill their liabilities while nodes 1 and 3 default. This is also consistent with the conclusions in Theorem 3.2.2. Then, we reduce the value of α_1 slightly

Table 3.1: An Example for the Relaxed Model of Financial Network.

Liability Matrix							
Node	1	2	3	4	$\bar{p}$	α^1	α^2
1	0	4857	9971	11306	26134	-6700	-9400
2	4857	0	10625	12047	27529	1790	32790
3	9971	10625	0	24734	45330	740	-8100
4	11306	12047	24734	0	48087	4170	12358.2
Claims	26134	27529	45330	48087	147080	0	27648.2

Optimal Solution				
x_1^*	x_2^*	x_3^*	x_4^*	
0.7269	1	0.9562	1	α^1
0.5329	1	0.7186	1	$\alpha_{1-}^2 = -9400$
-1.2097	1	-0.1232	0.1597	$\alpha_{1+}^2 = -37048.2$

such that $\alpha_1^1 = -7600.1$ and thus $\bar{\alpha} < 0$. In this case, problem (3.2.2) becomes infeasible. One can show that by changing the coefficient matrix L, problem (3.2.2) is still infeasible, and we cannot find any feasible solution as long as the total asset is below zero ($e^T\alpha^1 < 0$).

We also consider problem (3.2.2) with another asset vector α^2 such that $\bar{\alpha}^2 > 0$. By reducing the value of α_{1-}^2 to $\alpha_{1+}^2 = -37048.2$, we obtain a new total asset value 0, and thus, problem (3.2.2) remains feasible. We observe that at the optimal solutions, we have $x_1^2 < 0$ and $x_3^2 < 0$, which implies that problem (3.2.1) is infeasible. By checking the optimal solutions again, we can also find that as a consequence of the change in α_1^2, the solvent node (4) has changed to a default one ($x_4^*(\alpha_{1-}^2) = 1$ and $x_4^*(\alpha_{1+}^2) = 0.15 < 1$). This demonstrates that the sets of default nodes and solvent nodes may change whenever the asset vector changes.

3.3 The Vulnerability of A Financial Network under A Single Shock

In this section, we consider a scenario where only a single node receives the market shock and study the impact of the shock on the whole system. The section consists of two subsections. In the first subsection, we estimate the direct impact of the shock on the receiving node, and in the second subsection we study the indirect impact of the shock on other nodes in the system.

3.3.1 The impact of a single shock on the receiving node

In this subsection, we consider a scenario where only a single node receives the market shock and present a deterministic characterization on conditions under which the receiving node (i) will be solvent, default or bankrupted. For such a purpose, we consider the following modified variant of problem (3.2.2)

$$\begin{aligned} \max \quad & \sum_{i=1}^{n} \bar{p}_i x_i \\ s.t. \quad & (\bar{P} - L^T)x \leq \alpha + s; \\ & x \leq e, \end{aligned} \tag{3.3.1}$$

where s has only a single nonzero element $s_i \neq 0$ at some index i. We can interpret s as the influence of the market shock on bank i. Our purpose is to characterize the behavior of the optimal solution $x^*(s)$ of problem (3.3.1) in terms of s. Particularly, we are mainly interested in conditions under which the i-th element $x_i^*(s)$ at the optimal solution will satisfy one of the following conditions:

$$x_i^*(s) \leq 0, \tag{3.3.2}$$

$$x_i^*(s) \in (0, 1), and \tag{3.3.3}$$

$$x_i^*(s) = 1. \tag{3.3.4}$$

Note that the above conditions are associated with the status of the bank i depending on whether it is bankrupted, default, or solvent. We also call node i insolvent when $x_i^*(s) < 1$. Next, we first present a technical result.

Lemma 3.3.1. Suppose that $x^*(s)$ be the optimal solution of problem (3.3.1) where s has only a single nonzero element $s_i \neq 0$ for some i. Then the following conclusions hold.

(i) For every index $j \in \{1, \ldots, n\}$, $x_j^*(s)$ is nondecreasing in terms of s_i;

(ii) For index i, $x_i^*(s)$ is locally strictly increasing in terms of s_i if $x_i^*(s) < 1$;

Proof. We start with the proof of Conclusion (i). Let $x^*(s^1)$ and $x^*(s^2)$ denote the optimal solutions of problem (3.3.1) corresponding to s^1 and s^2 respectively, where $s_i^1 < s_i^2$. We want to show that $x_j^*(s^1) \leq x_j^*(s^2)$ for every index $j \in \{1, \ldots, n\}$. Assume to the contrary that there is an index k such that $x_k^*(s^1) > x_k^*(s^2)$. Since we assume that $s^1 < s^2$, it follows that

$$X_s^1 \subseteq X_s^2, \tag{3.3.5}$$

where X_s^1 and X_s^2 are the feasible sets corresponding to s^1 and s^2 respectively. According to the proof of Proposition 3.2.1, for every index i, we have

$$x_i^*(s) = \sup_{x \in X_s} x_i. \tag{3.3.6}$$

where X_s is the feasible set of problem (3.3.1). From (3.3.5) and (3.3.6) we can conclude that $x_k^*(s^1) \leq x_k^*(s^2)$ which contradicts the assumption and finishes the proof of the first conclusion.

To prove the second conclusion it suffices to show that for $s_i^1 < s_i^2$, $x_i^*(s^1) \neq x_i^*(s^2)$. Suppose to the contrary that $x_i^*(s^1) = x_i^*(s^2)$. Now let us define the index sets

$$\mathcal{I}_1 = \{i : \bar{p}_i x_i^* - \sum_{j \neq i} l_{ji} x_j^* = (\alpha + s)_i\}, \quad \mathcal{I}_2 = \{i : x_i^* = 1\}.$$

From Proposition 3.2.1, we can conclude that x^* is the unique solution of the following linear equation system

$$\begin{aligned} \bar{p}_i x_i^* - \sum_{j \neq i} l_{ji} x_j^* &= (\alpha + s)_i, \quad \forall i \in \mathcal{I}_1, \ \textit{and} \qquad (3.3.7) \\ x_i^* &= 1, \qquad \forall i \in \mathcal{I}_2. \qquad (3.3.8) \end{aligned}$$

Let us denote the coefficient matrix in the above system by A. Clearly A is a so-called M-matrix. On the other hand, since $\mathcal{I}_2$ is nonempty by Theorem 3.2.2, under the assumption that the financial network is fully connected, one can show that A is nonsingular. Since we assume that $x_i^*(s) < 1$, we have

$$x^*(s_i) = A^{-1}\alpha + s_i A_{:i}^{-1},$$

where $A_{:i}^{-1}$ is the i-th column of A^{-1}. Note that the inverse matrix of an M-matrix is nonnegative (see [13]). Using the fact that A_{ii} is the only positive element in the i-th row of A and A^{-1} is nonnegative, we can conclude $A_{ii}^{-1} > 0$. This implies that for $s_i^1 < s_i^2$, $x_i^*(s^1) < x_i^*(s^2)$ which contradicts to the assumption. This completes the proof of the lemma.

Now we are ready to state the main result in this section.

Theorem 3.3.1. Let $x^*(s)$ be the optimal solution of problem (3.3.1). Then, $x_i^*(s) < 0$ if and only if

$$s_i < \max(-\bar{\alpha}, \Delta_i - \alpha_i), \tag{3.3.9}$$

where

$$\begin{aligned} \Delta_i = -\max \quad & \sum_{j \neq i} l_{ji} x_j \\ s.t. \quad & \bar{p}_j x_j - \sum_{k \neq j, k \neq i} l_{kj} x_k \leq \alpha_j, \ \forall j \neq i; \\ & x_j \leq 1, \ \forall j \neq i. \end{aligned} \tag{3.3.10}$$

Proof. For simplicity, we consider only the special case $s_1 \neq 0$. To prove the sufficiency of the theorem, we consider the following two cases: Case (i): $s_1 < -\bar{\alpha}$. Case (ii): $-\bar{\alpha} \leq s_1 <$

$\Delta_1 - \alpha_1$. In the first case, we have

$$e^T \alpha + s_1 < 0.$$

It follows from Theorem 3.2.1 that problem (3.3.1) is infeasible which implies that problem (3.2.1) is also infeasible.

Now we consider case (ii) where $-\bar{\alpha} \leq s_1 < \Delta_1 - \alpha_1$. In such a case, we have $\bar{\alpha} + s_1 \geq 0$ and thus the relaxed problem (3.3.1) is feasible. It remains to show that $x_1^* < 0$ at the optimal solution x^* of problem (3.3.1). Let us consider the following specific variant of problem (3.3.1)

$$\begin{aligned} \max \quad & \sum_{i=1}^{n} \bar{p}_i x_i \qquad (3.3.11) \\ s.t. \quad & \bar{p}_1 x_1 - \sum_{j \neq 1} l_{j1} x_j \leq \Delta_1; \\ & \bar{p}_i x_i - \sum_{j \neq i} l_{ji} x_j \leq \alpha_i, \ \forall i = 2, \ldots, n; \\ & x \leq e. \end{aligned}$$

One can verify that the above problem and problem (3.3.10) have the same optimal solution, which further implies $x_1^*(s_1) = 0$ when $s_1 = \Delta_1 - \alpha_1$. According to Conclusion (ii) in Lemma 3.3.1 we can conclude that $x_1^*(s_1)$ is locally strictly increasing in terms of s_1. Therefore, we have $x_1^*(s_1) < 0$ whenever $s_1 < \Delta_1 - \alpha_1$. This proves the sufficiency of the theorem.

Now we consider the necessity of the theorem. It suffices to consider the case where problem (3.3.1) is feasible and $x_1^*(s_1) < 0$. Suppose to the contrary that the relation (3.3.9) does not hold, i.e.,

$$s_1 \geq \max(-\bar{\alpha}, \Delta_1 - \alpha_1). \qquad (3.3.12)$$

The above relation indicates

$$s_1 + \bar{\alpha} \geq 0, \quad s_1 + \alpha_1 \geq \Delta_1. \tag{3.3.13}$$

Recall $x_1^*(\Delta_1 - \alpha_1) = 0$, from Lemma 3.3.1 we immediately obtain

$$x_1^*(s_1) \geq 0,$$

which contradicts to the assumption. This completes the proof of the theorem.

We remark that $|\Delta_i|$ represents the maximum amount of repayments that bank i received from other banks in the system under the condition that bank i does not make any payment to other banks in the system, i.e., $x_i^* = 0$. Theorem 3.3.1 provides an estimate on the amount of negative shock that a financial institution can survive bankruptcy under the condition that only the corresponding institution is affected by the shock. We also mention that the estimated amount of negative shock in Theorem 3.3.1 depends only on the current market information, not on the future market fluctuation. Such a result allows the financial institution to estimate the worst-case scenario it can survive under the current market conditions on the assets and interbank liabilities, which may help the financial institution in its decision making to hedge future risk.

We next present a numerical example to verify the theoretical conclusions of Theorem 3.3.1.

Example 3.3.1. Consider a network of four financial institutions where the liability matrix is the same as in Example 3.2.1. The asset vector $\alpha = (-9400, 32790, -8100, 12358.2)^T$ with $\bar{\alpha} = e^T\alpha = 27648.2 > 0$.

Table 3.2: Bankruptcy Example in Financial Network

Optimal Solution				
x_1^*	x_2^*	x_3^*	x_4^*	
0.0001	1	0.46	0.75	$\alpha_1 = -17893$
0	1	0.46	0.75	$\alpha_1 = \Delta_1 = -17894.736$
-0.0001	1	0.46	0.75	$\alpha_1 = -17896$

For the above example, we estimate the maximum amount of negative shock (s) via (3.3.9), where Δ is also obtained by solving problem (3.3.10). The results are shown in the following table.

Table 3.3: Δ and s estimated for Example 3.3.1

Node	Δ	s
1	-17894.736	-8494.736
2	5141.80	-27648.20
3	-23981.652	-15881.652
4	-10635.42	-22993.62

As one can see in the above table, $\Delta_1 = -17894.736$. Based on Theorem 3.3.1, we can conclude that if the negative amount of shock is more than 8494.736 ($s_1 < -8494.736$), then at the optimal solution we have $x_1^* < 0$, i.e., bank 1 will be bankrupted. To verify that, we consider three different scenarios where $s_1 = -8493 > -8494.736$, $s_1 = -8494.736$, and $s_1 = -8496 < -8494.736$ respectively. Then, we change the values of α_1 accordingly as listed in Table 3.2. As one can see from Table 3.2, in all these cases, we have $\bar{\alpha} > 0$ and thus, problem (3.2.2) is feasible. However, we note that when $\alpha_1 = -17896$ (or $s_1 < -8494.736$), it holds $x_1^* = -0.0001 < 0$, which indicates problem (3.2.1) is infeasible.

We also estimate the amount of negative shock that a solvent node in the system can survive. For the solvent node (4) in the above example, from Table 3.3 we see that $s_4 = -22993.62$. This implies that if the negative shock (to node 4) is more than 22993.62 (i.e., $s_4 < -22993.62$), then it will be bankrupted.

We next compare Theorem 3.3.1 with the results in [57] where they used standard sensitivity analysis to estimate the impact on the payments with respect to some changes in the asset vector (called the partial derivative of the repayments with respect to the assets). As shown in [30], problem (3.2.1) can be solved via solving n decomposed problems where the objective is to maximize the payment for every node i subject to the same constraint set as in the original problem (3.2.1) [1]. Therefore, the partial derivatives of the repayments with respect to the assets are precisely the shadow prices for the decomposed problems.

[1] In the proof of Proposition 3.2.1, we constructed a similar decomposition for problem (3.2.2) in the form of (3.2.3).

The matrix of shadow prices computed in [57] is given as

$$\frac{\partial p^*}{\partial \alpha} \approx \begin{bmatrix} 1.09 & 0 & 0.41 & 0 \\ 0 & 0 & 0 & 0 \\ 0.24 & 0 & 1.09 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where $p_i^* = x_i^* * \bar{p}_i$. For example, based on the above estimate, we have that $\frac{\partial p_1^*}{\partial \alpha_1} \approx 1.09$ indicating that a decrease of \$1 in asset value of the first node cause the payment made by that node to drop approximately \$1.09. Unfortunately, such a result remains valid only for small shocks such that when both the set of default nodes and the set of solvent nodes remain invariant, and the result will not hold any more for reasonably large shocks where one solvent node becomes a default one. Take for example, for node 4 we have $\frac{\partial p_4^*}{\partial \alpha_4} \approx 0$ when the node remains solvent. This illustrates that the results in [57] cannot be used to predict when a node will be bankrupted as a negative shock to the asset vector may change the sets of default and solvent nodes. In contrast, Theorem 3.3.1 shows that if $\alpha_4 < -10635.42$, then node (4) will be bankrupted.

In what follows we study the solvency of a financial institution in the system, i.e., to characterize when $x_i^*(s) = 1$ for a given index i.

Theorem 3.3.2. Suppose that $x^*(s)$ be the optimal solution of problem (3.3.1). Then we have $x_i^*(s) = 1$ if and only if

$$s_i \geq \bar{p}_i + \Gamma_i - \alpha_i, \tag{3.3.14}$$

where

$$\begin{aligned} \Gamma_i = -\max \quad & \sum_{j \neq i} l_{ji} x_j \\ s.t. \quad & \bar{p}_j x_j - \sum_{k \neq j, k \neq i} l_{kj} x_k \leq \alpha_j + l_{ij}, \ \forall j \neq i; \\ & x_j \leq 1, \ \forall j \neq i. \end{aligned} \tag{3.3.15}$$

Proof. Let us consider only the special case $s_1 \neq 0$. To prove the sufficiency of the theorem, we assume that condition (3.3.14) holds. Since $s_1 = \bar{p}_1 + \Gamma_1 - \alpha_1$, we can rewrite problem (3.3.1) as follows

$$\begin{aligned} \max \quad & \sum_{i=1}^{n} \bar{p}_i x_i \\ s.t. \quad & \bar{p}_1 x_1 - \sum_{j \neq 1} l_{j1} x_j \leq \bar{p}_1 + \Gamma_1; \\ & \bar{p}_i x_i - \sum_{j \neq i} l_{ji} x_j \leq \alpha_i, \ \forall i = 2, \ldots, n; \\ & x \leq e. \end{aligned} \tag{3.3.16}$$

One can easily verify that problem (3.3.16) and (3.3.15) have the same optimal solution, which implies that we have $x_1^*(s_1) = 1$ if $s_1 = \bar{p}_1 + \Gamma_1 - \alpha_1$. From Conclusion (i) of Lemma 3.3.1 we know that $x^*(s_1)$ is nondecreasing in terms of s_1. Therefore, we have $x_1^*(s_1) = 1$ for $s_1 \geq \bar{p}_1 + \Gamma_1 - \alpha_1$. This proves the sufficiency of theorem.

Now we consider the necessity of the theorem. Suppose that $x_1^*(s_1) = 1$. Suppose to the contrary that inequality (3.3.14) does not hold, i.e.,

$$s_1 < \bar{p}_1 + \Gamma_1 - \alpha_1.$$

From the constraints of problem (3.3.1) we obtain

$$\bar{p}_i - \sum_{j \neq i} l_{ji} x_j^* = \bar{p}_i x_i^* - \sum_{j \neq i} l_{ji} x_j^* \leq \alpha_i + s_i, \tag{3.3.17}$$

where the equality follows from the assumption that $x_i^*(s_i) = 1$. Therefore, we have

$$s_i \geq \bar{p}_i + \Gamma_i - \alpha_i,$$

which contradicts to the assumption. This finishes the proof of the theorem.

We remark that $|\Gamma_i|$ denotes the maximum amount of repayments that solvent bank i

received from other banks in the system. Theorem 3.3.2 provides an estimate on the minimum amount of positive shock or market gain for a financial institution to become solvent. We mention that the estimated positive shock in Theorem 3.3.2 depends on the current asset value and interbank liabilities. This allows the financial institutions to estimate the asset increment (such as the exogenous investment or profit gain) they need in order to become solvent. Theorem 3.3.2 also provides an estimate on the maximum amount of negative shock a solvent node can sustain to remain solvent. The results in Theorem 3.3.2 may help financial institutions in their decision making regarding its investment in the market. We also point out that the results in Theorem 3.3.2 are very different from the results in [57] based on standard sensitivity analysis in linear optimization. Different from our work, [20] estimate the liquidity assistance loan via the proposed multi-period clearing payment system. They also study the impact of two mitigation strategies on the vulnerability of the system in terms of number of default nodes.

We next use the same financial network as in Example 3.3.1 to verify the conclusions in Theorem 3.3.2.

Example 3.3.2. L and α are the same as in Example 3.3.1.

Table 3.4: Solvency Example in Financial Network

Optimal Solution				
x_1^*	x_2^*	x_3^*	x_4^*	
0.99	1	0.82	1	$\alpha_1 = 1780$
1	1	0.82	1	$\alpha_1 = \Gamma_1 + \bar{p}_1 = 1781.714$
1	1	0.82	1	$\alpha_1 = 1782$

Table 3.5: Γ and s estimated for Example 3.3.2

Node	Γ	s
1	-24352.286	11181.714
2	-22270.348	-27531.348
3	-41743.584	11686.416
4	-35845.652	-116.852

In the above example, $\Gamma_1 + \bar{p}_1 = 1781.714$ was obtained by solving problem (3.3.15). For this example, if the first node can manage to increase its asset (by attracting exogenous investment or investing smartly), then its repayment ability will be improved as well. In the case that the increment of asset value reaches $s_1 = 11181.714$, then the first node will become solvent. To verify that, we consider three different scenarios where $s_1 = 11180 < 11181.714$, $s_1 = 11181.714$, and $s_1 = 11182 > 11181.714$ receptively. Then, we change value of α_1 accordingly as listed in Table 3.4. As you can see when $\alpha_1 \geq 1781.714$, it holds $x_1^* = 1$ at the optimal solution.

We also estimate the amount of shock for solvent nodes 2 (or node 4) in the system. As one can see in Table 3.5, $\Gamma_2 + \bar{p}_2 = 5258.652$ (or $\Gamma_4 + \bar{p}_4 = 12241.348$). Based on Theorem 3.3.2, one can show that if the amount of negative shock is less than 27531.348 ($s_2 \geq -27531.348$) (or 116.852 ($s_4 \geq -116.852$)), we have $x_2^* = 1$ (or $x_4^* = 1$) which shows that node 2 (or node 4) remains solvent in the system.

It is also interesting to note that standard sensitivity analysis for linear optimization problem can also be used in some special case in Theorem 3.3.2. Take for example, using the standard sensitivity analysis as in [57], we have $\frac{\partial p_1^*}{\partial \alpha_1} \approx 1.09$ (see discussion in Example 3.3.1). This shows that when α_1 is increased by 11182.7158, the first node can repay it liabilities in full, i.e. $x_1^* = 1$. In such a case, we have

$$[(\overline{P} - L^T)x^*]_1 = \alpha_1, \qquad x_1^* = 1.$$

The above example represents a borderline case where node 1 changes from a default node to a solvent one. In other words, the sets of the default and solvent nodes did not change before α_1 reaches the value 11182.7158. Therefore, the estimate based on the standard analysis is similar to what we obtained from Theorem 3.3.2. For solvent nodes 2 and 4, from the standard sensitivity analysis we have $\frac{\partial p_2^*}{\partial \alpha_2} \approx 0$ and $\frac{\partial p_4^*}{\partial \alpha_4} \approx 0$, which implies that the asset value of those nodes does not have any impact on their repayment ability. In contrast, Theorem 3.3.2 shows that if $\alpha_2 < 5258.651$ (or $\alpha_4 < 12241.348$), then node 2 (or node 4) will default.

Theorems 3.3.1 and 3.3.2 provide an upper bound and a lower bound of the magnitude of shock s_i such that bank (i) is bankrupted or solvent. Combining the results in these two theorems, we immediately obtain the following result.

Corollary 3.3.1. If

$$s_i \in (\max(-\bar{\alpha}, \Delta_i - \alpha_i), \bar{p}_i + \Gamma_i - \alpha_i), \tag{3.3.18}$$

then the relation $0 < x_i^*(s_i) < 1$ holds at the optimal solution $x^*(s_i)$ of problem (3.3.1).

We call interval (3.3.18) as the default window. The default window can be used as an indicator for the resistance of a default bank i to market shock. The larger the window is, more resistant to market shock the bank is. It worths estimate the length of the default window. Since Δ_i and Γ_i are obtained by solving problems (3.3.10) and (3.3.15) respectively, we have

$$\Gamma_i \leq \Delta_i.$$

It follows that

$$\bar{p}_i + \Gamma_i - \alpha_i - \max(-\bar{\alpha}, \Delta_i - \alpha_i) \leq \bar{p}_i + \Gamma_i - \Delta_i \leq \bar{p}_i.$$

In other words, the maximal magnitude of shock that a default bank i can resist is bounded above by its total liability.

3.3.2 The indirect impact of the shock on other nodes in the system

In this subsection, we study the impact of a single shock to other nodes in the system. To start, we point out that [24] and [44] study the contagion impact of a single shock on other non-receiving nodes in the system. Different from the results in these two papers, we will estimate the magnitude of a single shock under which some non-receiving node will be bankrupted. For simplicity of discussion, throughout this subsection we assume that the first node is the receiving node with shock s_1. In such a case, we can consider the decomposed problem as follows

$$\max \quad x_j \tag{3.3.19}$$

$$
\begin{aligned}
s.t. \qquad & \bar{p}_1 x_1 - \sum_{j \neq 1} l_{j1} x_j \leq \alpha_1 + s_1; \\
& \bar{p}_i x_i - \sum_{j \neq i} l_{ji} x_j \leq \alpha_i, \ \forall i = 2, \ldots, n; \\
& x \leq e.
\end{aligned}
$$

Particularly, we are mainly interested in estimating the minimum amount (denoted by $\bar{s}_{1j}$) of s_1 such that bank j become bankrupted, i.e.,

$$
\bar{s}_{1j} := \arg\max x_j^{(j)}(s_1) = 0, \tag{3.3.20}
$$

where $x_j^{(j)}(s_1)$ denotes the objective function value at the optimal solution of problem (3.3.19). One can easily see that $\bar{s}_{11} = \max\{-\bar{\alpha}, \Delta_1 - \alpha_1\}$. Let us define

$$
\bar{s}_1^{max} = \max_{j=1,\ldots,n.} \bar{s}_{1j}.
$$

The following result follows directly from the above definition.

Proposition 3.3.1. Suppose the system is triggered by a single shock s_1. If $s_1 < \bar{s}_1^{max}$, then some node in the system will be bankrupted.

In what follows we consider the issue of how to estimate $\bar{s}_1^{max}$. Let us start by considering the issue of which node is more sustainable under the single shock s_1. For this, we introduce the following definition.

Definition 3.3.1. Let $\bar{s}_{1j}$ be defined by (3.3.20). We say a node i is more sustainable than another node j under the single shock s_1 if $\bar{s}_{1i} \leq \bar{s}_{1j}$.

We next present a result on how to determine whether one node in the system is more sustainable than the receiving node itself . We have

Theorem 3.3.3. Let $x^*(s_1)$ be the optimal solution of problem (3.3.10) with $i = 1$. For every $j > 1$, we have

(i) If $x_j^*(s_1) > 0$, then node j is more sustainable than the receiving node 1;

(ii) If $x_j^*(s_1) \leq 0$, then node j is less sustainable than the receiving node;

(iii) If $x_j^*(s_1) > 0, \forall j > 1$, then $\bar{s}_1^{max} = \bar{s}_{11}$.

Proof. Based on Conclusion (i) in Lemma 3.3.1, we can claim that $x_j^*(s_1)$ is nondecreasing in terms of s_1. By following a similar vein as in the proof of Proposition 3.2.1, we can show that

$$x_j^*(s_1) := \sup_{x(s_1) \in X(s_1)} x_j(s_1), \quad \forall j = 2, \ldots, n,$$

where $X(s_1)$ is the feasible set of problem (3.3.10). Since problem (3.3.10) and (3.3.19) have the same feasible set, $x^*(s_1)$ is a feasible solution of (3.3.19). Clearly, $x^*(s_1)$ maximizes x_j and therefore, $x^*(s_1)$ is the unique optimal solution to (3.3.19). This implies that the optimal solution of problem (3.3.10) can be obtained by solving $n-1$ decomposed problem in the form of problem (3.3.19). Now, because

$$x_j^{(j)}(\bar{s}_{1j}) > 0, \quad \forall j > 1,$$

from (3.3.20) it follows

$$\bar{s}_{1j} \leq \bar{s}_{11}.$$

This proves the first conclusion in the theorem. The second conclusion follows similarly. We consider that $x_j^*(s_1) \leq 0$ which implies that

$$x_j^{(j)}(\bar{s}_{1j}) \leq 0, \quad \forall j > 1.$$

Based on (3.3.20) we have

$$\bar{s}_{1j} \geq \bar{s}_{11}.$$

This proves the second conclusion in the theorem. The last conclusion follows directly from the definition of $\bar{s}_1^{max}$. This completes the proof of the theorem.

Theorem 3.3.3 provides a simple way to determine whether a non-receiving node in the system is more sustainable than the receiving node itself by solving problem (3.3.10). Note that when $\alpha > 0$, one can show that at the optimal solution $x^*(s_1)$ of problem (3.3.10), we have $x_j^* > 0$ for all $j > 1$. It follows from Theorem 3.3.3 that

Corollary 3.3.2. Suppose a system is triggered by a single shock. If all the nodes in the system have positive assets, then every non-receiving node in the system is more sustainable than the receiving node itself.

The above corollary shows that for problem (3.2.1), if the initial asset vector α is positive, and only α_1 is subject to the shock s_1, then node 1 will be bankrupted first. Corollary 3.2 also implies that if the system is triggered by multiple shocks such that only one node receive a negative shock, then the node receiving the negative shock will become bankrupt first.

We next discuss how to estimate $\bar{s}_1^{max}$ when there exists some non-receiving node $j > 1$ that is less sustainable than the receiving node 1. For this, we recall conclusion (i) of Lemma 3.3.1, which shows that $x_j^*(s_1)$ is nondecreasing in terms of s_1. One way to locate s_{1j} is applying a line search procedure based on the monotonicity of $x_j^*(s_1)$. Let us assume that problem (3.2.1) is feasible and thus, we can obtain an upper bound $u_s = 0$ for s_{1j}. From Theorem 3.2.1, we can also obtain a lower bound $l_s = -\bar{\alpha}$ for s_{1j}. We are now ready to describe a bisection search algorithm for locating s_{1j}.

A Bisection Search Algorithm

S.0 **Input:** L, $\bar{p}$, α, $l_s = -\bar{\alpha}$, $u_s = 0$, and a stop criteria ϵ;

S.1 **While** $u - l > \epsilon$ **Do**;

S.1.1 $s_1 := \frac{l_s + u_s}{2}$;

S.1.2 Solve problem (3.3.19);

S.1.3 **If** $x_j^*(s_1) > 0$, **then**

S.1.4 Set $u = s_1$,

S.1.5 **else**

S.1.6 Set $l = s_1$,

S.1.7 **endif**

S.2 **Output:** $s_{1j} = s_1$

For illustration, we use the same financial network as in Example 3.3.1 and adapt the bisection search algorithm to estimate $\bar{s}_{ij}$.

Example 3.3.3. L and α are the same as in example 3.3.1.

Table 3.6: Bankruptcy Example in Financial Network

Optimal Solution				
x_1^*	x_2^*	x_3^*	x_4^*	
0.5329	1	0.7186	1	$\alpha_1 = -9400$
0	1	0.4624	0.7454	$\alpha_{1+} = \Delta_1 = -17894.736$
-0.9551	1	0	0.2830	$\alpha_{1-} = \alpha_1 - \bar{s}_{13} = -33017.514$

For the above example, we first estimate $\Delta_i, \forall i = 1, \ldots, 4$ by solving problem (3.3.10) which is also shown in Table 3.3. We also use the bisection search algorithm to estimate $\bar{s}_{ij}, \forall i, j = 1, \ldots, 4$, as listed in the following table.

Table 3.7: s_{ij} estimated for Example 3.3.3

Node	$\bar{s}_{1j}$	$\bar{s}_{2j}$	$\bar{s}_{3j}$	$\bar{s}_{4j}$
1	-8494.736	-27648.2	-17546.0	-17546.0
2	-27648.2	-27648.2	-27648.2	-27648.2
3	-23617.514	-27648.2	-15881.652	-23617.0
4	-27648.2	-27648.2	-27648.2	-22993.62

From Table 3.6, one can see that under a single shock s_1, the receiving node 1 is the least sustainable node and bankrupts first, while node 2 remains solvent as well as problem (3.2.1) remains feasible. When the amount of the negative shock $|s_1|$ is sufficiently large such that $s_1 \leq \bar{s}_{13}$, then node 3 becomes bankrupted.

From Table 3.7, we can see that if the single shock is received by either node 1 or 3, then the receiving node is the least sustainable node. However, when node 4 is the receiving node, then we have $\bar{s}_4^{max} = \bar{s}_{41}$, which indicates that node 1 is less sustainable than the receiving node 4.

3.4 The Vulnerability of A Financial Network under Multiple Shocks

In this section, we estimate the vulnerability of a financial network in a generic scenario where all the nodes are subject to market shocks. The section consists of three subsections. In the first subsection, we estimate the impact of asset inequality on the stability of the network, and show that, the least stable financial network can be attained at some network with a monopoly node. Then, we characterize conditions for bankruptcy in a financial network with a monopoly node. In the second and third subsections, we estimate the probabilities of insolvency and bankruptcy in the network respectively.

3.4.1 Asset inequality and stability of the financial system

In this subsection we estimate the vulnerability in a financial system when all the nodes are exposed to market shocks. First we point out that, as proved in Lemma 3.3.1, the solution $x^*(s)$ of problem (3.3.1) is component-wise monotone with respect to the market shock when only a single bank receives the shock. In such a case, we can conclude that $x^*(s)$ is also monotone in terms of total asset ($\bar{\alpha}$), i.e., the financial network will be more stable as the total asset increases. Such a result has also been used in [44] to estimate the probability of insolvency in the network caused by the shock to a single node (i). It is of interests to see whether the monotone relationship between the repayments and the shocks still holds when all the nodes receive shocks. For this, we introduce the following two definitions.

Definition 3.4.1. A node (i) in the financial system is said to be strictly solvent node if

$$x_i^* = 1, \quad [(\overline{P} - L^T)x^*]_i < \alpha_i,$$

where, x_i^* is the optimal solution of (3.2.1) or (3.2.2). The asset inequality is defined as a gap between the asset of one strictly solvent node and some default or bankrupted nodes in the system.

Definition 3.4.2. Consider two financial systems with the same network structure L with asset vectors α^1 and α^2 satisfying $e^T\alpha^1 = e^T\alpha^2 = \bar{\alpha}$. The first system is said to be less stable than the second one if

$$\bar{p}^T x^*(\alpha^1) < \bar{p}^T x^*(\alpha^2),$$

where $x^*(\alpha^1)$ and $x^*(\alpha^2)$ denote the optimal solution of (3.2.1) with $\alpha = \alpha^1$ and $\alpha = \alpha^2$, respectively.

We remind the readers that based on Definition 3.4.2, the stability of a financial system is measured in terms of the optimal objective value of problem (3.2.1). This is different from what's in the reference [1] where the authors suggested to use the number of nodes affected by the shock to measure the stability of the system.

We now consider a financial network with total asset $\bar{\alpha}^1$ where problem (3.2.1) is feasible and there exists some default bank, and strictly solvent node. Without loss of generality, let us further assume that bank 1 defaults, i.e., at the optimal solution x^* of problem (3.2.2), we have $0 < x_1^* < 1$, and bank n is strictly solvent, i.e., $x_n^* = 1$, $[(\overline{P} - L^T)x^*]_n < \alpha_n$. Now let us consider a new financial system with total asset $\bar{\alpha}^2 = \bar{\alpha}^1$ such that $\alpha_1^2 = \alpha_1^1 - \epsilon$ and $\alpha_n^2 = \alpha_n^1 + \epsilon$. Since $x_n^*(\alpha^1) = 1$, it holds $x_n^*(\alpha^2) = 1$. It follows from Lemma 3.3.1 that

$$x_1^*(\alpha^2) < x_1^*(\alpha^1), \quad x_i^*(\alpha^2) \le x_i^*(\alpha^1), \forall i = 2, \ldots, n.$$

From the above discussion we immediately obtain the following result.

Proposition 3.4.1. Suppose that the summation of the assets of one default node and another strictly solvent node remain invariant. A larger asset inequality between these two nodes will decrease the stability of the financial network.

We remark that since increasing the asset value α_n will not help to improve the stability of the underlying network, this implies that an increase in the total asset may not improve the stability of the network. The following example demonstrates such an phenomenon.

Example 3.4.1. We consider the same data matrix as in Example 3.2.1 with three different vectors of asset

$$\alpha^1 = (-400, 3790, 100, 1358.2)^T, \alpha^2 = (-500, 4000, 100, 1358.2)^T,$$

$$\alpha^3 = (-1000, 4000, 100, 3358.2)^T$$

such that at least one of the nodes in the system defaults.

Table 3.8: Stability of a Financial System VS Asset Inequality.

Optimal Solution				
x_1^*	x_2^*	x_3^*	x_4^*	
0.9842	1	0.9987	1	α^1
0.980	1	0.9978	1	α^2
0.959	1	0.9932	1	α^3

In the above example, we change the value of α^1 such that the asset value of the first node is decreased while the asset values of all other nodes are increased and thus, the total asset is increased as well. As you can see from Table 3.8, we have

$$x_1^*(\alpha^1) > x_2^*(\alpha^2) > x_3^*(\alpha^3), \quad \bar{\alpha}^1 < \bar{\alpha}^2 < \bar{\alpha}^3.$$

The example shows clearly that an increase in the total asset may not improve the stability of the financial network. With a close look at the example, we find that the asset inequality has also increased as the total asset grows. This illustrates that the asset inequality in the financial network has a negative effect on the stability of the network. Next, we study the stability of a network with a dominant node, which represents an extreme distribution of the assets defined as follows.

Definition 3.4.3. A node i in a financial network is said to be a monopoly node if

$$\alpha_i = \bar{\alpha}, \quad \alpha_j = 0, \quad \forall j = 1, \ldots, i-1, i+1, \ldots, n. \tag{3.4.1}$$

We next show that under the assumption that total asset is fixed, the network is the least stable one when it has a monopoly node. To start, let us consider the following problem

$$\begin{aligned} \min_{\alpha} \max_{x} \quad & \bar{p}^T x \\ s.t. \quad & (\bar{P} - L^T)x \leq \alpha; \\ & x \leq e; \\ & \sum_i \alpha_i = \bar{\alpha}; \\ & \alpha \geq 0. \end{aligned} \tag{3.4.2}$$

For a given α, let $f(\alpha)$ denote the objective function value of problem (3.2.2). Then we can rewrite the above problem as

$$\begin{aligned} \min_{\alpha} \quad & f(\alpha) \\ s.t. \quad & \sum_i \alpha_i = \bar{\alpha}; \\ & \alpha \geq 0. \end{aligned} \tag{3.4.3}$$

Note that by using the duality theorem for linear optimization, we have

$$\begin{aligned} f(\alpha) := \min_{\lambda} \quad & \alpha^T \lambda + e^T \bar{p} - e^T (\overline{P} - L)\lambda \\ s.t. \quad & (\overline{P} - L)\lambda \leq \bar{p}; \\ & \lambda \geq 0. \end{aligned}$$

From the above definition, one can see that $f(\alpha)$ is concave with respect to α. Therefore, the optimal solution of (3.4.3) can be obtained in one of the extreme points of the constrained set, which is precisely a network with a monopoly node. From this, we immediately have the following result.

Proposition 3.4.2. Suppose that the total asset is fixed. The least stable financial network can be attained at some network with a monopoly node. Moreover, if the monopoly node in a network is strictly solvent and there exists some default node in the system, then the stability of network can be improved by redistributing the assets in the network.

Proof. We need only to prove the second conclusion of the proposition. Let assume that the first node is the monopoly node that is strictly solvent. Thus, we have

$$x_1^{(1)} = 1, \quad [(\overline{P} - L^T)x^{(1)}]_1 < \alpha_1.$$

In this case, there exist at least one default node $i > 1$ in the system such that

$$x_i^{(1)} < 1.$$

Now if we slightly increase α_i^1 to $\alpha_i^2 = \alpha_i^1 + \epsilon$ and decrease α_1^1 to $\alpha_1^2 = \alpha_1^1 - \epsilon$ such that $[(\overline{P} - L^T)x^{(1)}]_1 \leq \alpha_1^1 - \epsilon$, we can conclude that

$$x^{(2)} > x^{(1)},$$

where $x^{(1)}$ and $x^{(2)}$ are the optimal solutions of (3.2.1) when $\alpha = \alpha^1$ and $\alpha = \alpha^2$ respectively. Note that the above inequality follows from conclusion (ii) in Lemma 3.3.1 which states that $x_i^{(1)}(s_i)$ is strictly increasing in terms of s_i. This finishes the proof of the proposition.

Proposition 3.4.2 shows that the worst-case scenario for problem (3.2.1) w.r.t. the asset distribution is the network with a monopoly node. We remark that there are several empirical studies showing that the presence of inequality played an important role in the crisis. For example, [68] studies the two-way relation between income inequality and economic fluctuation and their impact on the creation of crisis. [69] also discusses the impact of income inequality on the Great Recession. However, our result is the first one using rigorous analysis to study the impact of asset inequality on the stability of a financial network.

3.4.2 Probability analysis on the vulnerability of a financial network

In what follows we explore the vulnerability of a network with a monopoly node when all the nodes receive shocks under certain assumption of the shock distribution. First, we recall that, as illustrated by Example 3.4.1, the monotonicity between the repayments ($x^*(s)$) and the shocks (s) does not hold in general if all the nodes receive shocks. Fortunately, as shown in Conclusion (ii) of Theorem 3.2.2, a node i is insolvent if $\alpha_i \leq \bar{p}_i - (L^T e)_i$. Since the total liability of a financial institution $\bar{p}_i$ and its total claims $(L^T e)_i$ are usually known in advance, this allows us to estimate the probability of insolvency in the financial network under the following assumption.

Assumption 3.4.1. (i) The financial network is well-balanced and the assets are non-negative;

(ii) All the shocks follow the same independent normal distribution with a zero mean and variance σ^2, i.e., $s_i \sim \mathcal{N}(0, \sigma^2)$.

We have

Theorem 3.4.1. Suppose that Assumption 3.4.1 holds. For a fixed total asset ($\bar{\alpha}$), the following conclusions hold.

(i) The network with a monopoly node has the highest probability of insolvency and is the most vulnerable one. Moreover, it holds

$$P(\exists i : x_i^* < 1) \geq 1 - (0.5)^{n-1}. \tag{3.4.4}$$

(ii) The system is most stable when the assets are evenly distributed, i.e., $\alpha_i = \frac{\bar{\alpha}}{n}, \forall i = 1, \ldots, n$.

Proof. We start with the first conclusion. Under Assumption 3.4.1, from Conclusion (ii) of

Theorem 3.2.2, we obtain that $x_i^* < 1$, if $\alpha_i < \bar{p}_i - (L^T e)_i = 0$. Thus, we have

$$P(\exists i : x_i^* < 1) = P(\exists i : \alpha_i + s_i < 0).$$

Therefore, the most vulnerable network can be identified via solving the following optimization problem

$$\begin{aligned} \min_{\alpha} \quad & \prod_i (1 - P(s_i < -\alpha_i)) = \prod_i F(\alpha_i) \\ s.t. \quad & \sum_i \alpha_i = \bar{\alpha}; \\ & \alpha_i \geq 0, \end{aligned} \tag{3.4.5}$$

where $F(\cdot)$ is the cumulative distribution function of normal distribution with a density function $f(\cdot)$. Note that the equality in the objective function follows from the symmetry of the shock distribution. Since $F(\cdot)$ is strictly monotone, we can rewrite the problem as

$$\begin{aligned} \min_{\alpha} \quad & \sum_i \ln F(\alpha_i) \\ s.t. \quad & \sum_i \alpha_i = \bar{\alpha}; \\ & \alpha_i \geq 0. \end{aligned} \tag{3.4.6}$$

Because the objective and constraint functions in problem (3.4.6) are differentiable, the optimal solution to the above problem must satisfy the Karush-Kuhn-Tucker (KKT) conditions [17], i.e.,

$$\frac{\partial \ln F(\alpha_i)}{\partial \alpha_i} + \lambda - \nu_i = 0, \quad \forall i = 1, \ldots, n; \tag{3.4.7}$$

$$\sum_i \alpha_i = \bar{\alpha}, and \tag{3.4.8}$$

$$\nu_i \alpha_i = 0, \quad \forall i = 1, \ldots, n; \tag{3.4.9}$$

$$\alpha_i \geq 0, \quad \forall i = 1, \ldots, n;$$

$$\nu_i \geq 0, \quad \forall i = 1, \ldots, n,$$

where λ and ν are the Lagrange multipliers. From (3.4.9), it is easy to see that for every index i, either α_i or ν_i must be zero. Let us define the index sets based on the element values of ν by

$$I_0 = \{i : \nu_i = 0\}, \quad I_1 = \{i : \nu_i \neq 0\}. \tag{3.4.10}$$

It follows immediately that

$$\alpha_i = 0, \quad \forall i \in I_1.$$

Using the above relation, we can rewrite the KKT conditions as

$$\begin{aligned} \frac{f(\alpha_i)}{F(\alpha_i)} + \lambda = 0, &\quad \forall i \in I_0; \\ \alpha_i = 0, &\quad \forall i \in I_1; \\ \sum_{i \in I_0} \alpha_i = \bar{\alpha}. & \end{aligned}$$

Since the function $f(\cdot)/F(\cdot)$ is a bijective map, from the above relations we can claim that

$$\alpha_i = \frac{\bar{\alpha}}{k}, \quad \forall i \in I_0,$$

where k is the number of nodes in I_0. Based on this, the objective function of (3.4.5) can be written as a function of k, i.e.,

$$g(k) = F(\bar{\alpha}/k)^k F(0)^{(n-k)} = (2F(\bar{\alpha}/k))^k (1/2)^n.$$

We next show that $g(k)$ is an increasing function with respect to k. For this, we take the first derivative of $\ln g(k)$ with respect to k as

$$\partial \ln g(k)/\partial k = \ln 2F(\bar{\alpha}/k) - (\bar{\alpha}/k) f(\bar{\alpha}/k)/F(\bar{\alpha}/k). \tag{3.4.11}$$

We also have that

$$\lim_{k\to\infty} \ln 2F(\bar{\alpha}/k) - (\bar{\alpha}/k)f(\bar{\alpha}/k)/F(\bar{\alpha}/k) = 0. \tag{3.4.12}$$

Now, if we take the second derivative of $\ln g(k)$ with respect to k, we have

$$\partial^2 \ln g(k)/\partial k^2 = -\bar{\alpha}^3 f(\bar{\alpha}/k)/\sigma^2 F(\bar{\alpha}/k) - \bar{\alpha}^2 f(\bar{\alpha}/k)^2/k^3 F(\bar{\alpha}/k)^2 < 0. \tag{3.4.13}$$

From (3.4.12) and (3.4.13), we can conclude that for small value of k, $\frac{\partial \ln g(k)}{\partial k} > 0$, which implies that $g(k)$ is increasing in k. Based on this, one can conclude that the objective value of (3.4.5) attains its minimum when $k = 1$, i.e., there exists an index i such that

$$\alpha_j^* = \bar{\alpha}, \quad \alpha_i^* = 0, \quad \forall i \neq j.$$

From the above relation and the symmetry of the shock distribution we obtain

$$\begin{aligned} P(\exists i : x_i^* < 1) \qquad &= 1 - \prod_i (1 - P(s_i < -\alpha_i)); \\ &\geq 1 - \prod_{i \neq j} (1 - P(s_i < -\alpha_i)); \\ &= 1 - (0.5)^{n-1}, \end{aligned}$$

where the inequlity follows from the fact that $(1 - P(s_j < -\alpha_j)) \leq 1$.

Similarly, since $k \leq n$, we can conclude that the maximum value of $g(k)$ can be obtained when the assets are evenly distributed as $\alpha_i^* = \frac{\bar{\alpha}}{n}$, $\forall i = 1, \ldots, n$. This finishes the proof of the theorem.

Theorem 3.4.1 shows that for problem (3.2.1), the monopoly case is the most vulnerable in terms of probability of insolvency. We note that Glasserman and Young [44] estimated the probability of a subset of nodes to default caused by the shock received by a single node not in that subset, and obtained a similar result as Conclusion (ii) of Theorem 3.4.1 (see Proposition 1 and Corollary 1 in [44]). Chen et al. [24] also extend the results in

[44] by considering the impact of the liquidity concentration on the bank. It is worthwhile mentioning that the proof of the theorem depends only on the two properties that the function $(f(\cdot)/F(\cdot))$ is bijective, and the function $g(k)$ is decreasing. This implies that the results in Theorem 3.4.1 can be extended to any shocks that follows some distribution with a density function and an accumulative function such that these two properties are satisfied.

3.4.3 Probability of bankruptcy in the system

In this subsection, we estimate the probability of bankruptcy in the system. Throughout this subsection, we make the following assumption.

Assumption 3.4.2. All the shocks follow some independent normal distributions with a zero mean and variance σ_i^2, i.e., $s_i \sim \mathcal{N}(0, \sigma_i^2)$.

Theorem 3.4.2. Suppose that Assumption 3.4.2 holds. Then the probability that some bank $(i = 1, \ldots, n)$ will be bankrupted is larger than

$$1 - \prod_i (1 - P(s_i \leq \Delta_i - \alpha_i)). \tag{3.4.14}$$

Proof. Without loss of generality, we assume $x_i^* = 0$. Given any i, we rewrite the reduced problem (3.3.10) w.r.t. the shock s as

$$\begin{aligned} -\Delta_i(s) = \max \quad & \sum_{j \neq i} l_{ji} x_j \\ s.t. \quad & \bar{p}_j x_j - \sum_{k \neq j, k \neq i} l_{kj} x_k \leq \alpha_j + s_j, \ \forall j \neq i; \\ & x_j \leq 1, \ \forall j \neq i. \end{aligned} \tag{3.4.15}$$

The dual problem of (3.4.15) reads as

$$-\Delta_i(s) = \min \quad \sum_{j \neq i} (\alpha_j + s_j - \bar{p}_j + \sum_{k \neq j, k \neq i} l_{jk}) y_j + \sum_{j \neq i} l_{ji} \tag{3.4.16}$$

$$
\begin{aligned}
s.t. \quad & \bar{p}_j y_j - \sum_{k\neq j, k\neq i} l_{jk} y_k \leq l_{ji}, \ \forall j \neq i; \\
& y \geq 0.
\end{aligned}
$$

Let $\mathcal{Y}$ denotes the feasible set of problem (3.4.16). Since all the variables s_j $\forall j = 1, \ldots, n$ are independent random variables $s_j \sim \mathcal{N}(0, \sigma_j^2)$, it follows from Theorem 3.3.1 that

$$
P(x_i^* \leq 0) = P(s_i \leq -\mathbf{E}[\min_{y\in\mathcal{Y}} \sum_{j\neq i} (\alpha_j + s_j - \bar{p}_j + \sum_{k\neq j, k\neq i} l_{jk}) y_j^i + \sum_{j\neq i} l_{ji}] - \alpha_i),
$$

where $\mathbf{E}[\cdot]$ denotes the expected value of $-\Delta_i(s)$. Note that the dual problem (3.4.16) is a linear optimization problem parameterized by the random noise s. From a computational perspective, we can use stochastic programming and scenario generation to estimate the probability $P(x_i^* < 0)$. In what follows we present a simple way to obtain a lower bound on the the probability $P(x_i^* < 0)$. Since the objective function in problem (3.4.16) is linear in terms of s, $-\Delta_i(s)$ is concave with respect to s. It follows immediately

$$
\begin{aligned}
&\mathbf{E}[\min_{y\in\mathcal{Y}} \sum_{j\neq i} (\alpha_j + s_j - \bar{p}_j + \sum_{k\neq j, k\neq i} l_{jk}) y_j + \sum_{j\neq i} l_{ji}] \\
&\quad \geq \min_{y\in\mathcal{Y}} \sum_{j\neq i} \mathbf{E}(\alpha_j + s_j - \bar{p}_j + \sum_{k\neq j, k\neq i} l_{jk}) y_j + \sum_{j\neq i} l_{ji} \\
&\quad = \min_{y\in\mathcal{Y}} \sum_{j\neq i} \mathbf{E}(\alpha_j - \bar{p}_j + \sum_{k\neq j, k\neq i} l_{jk}) y_j + \sum_{j\neq i} l_{ji} \\
&\quad = -\Delta_i,
\end{aligned}
$$

where the first inequality follows from the concavity of $-\Delta_i(s)$ w.r.t. s and the fact that all the parameters in (3.4.16) except s are constant, and the first equality follows from the assumption $\mathbf{E}[s_j] = 0, \ \forall j = 1, \ldots, n$. Thus, we have

$$
P(x_i^* \leq 0) \geq P(s_i \leq \Delta_i - \alpha_i), \tag{3.4.17}
$$

which further implies

$$1 - \prod_i (1 - P(x_i^* \leq 0)) \geq 1 - \prod_i (1 - P(s_i \leq \Delta_i - \alpha_i)).$$

This completes the proof of the theorem.

Theorem 3.4.2 provides an estimation on the lower bound for the probability that some bank will be bankrupted in the system. Such an estimate depends on the current asset vector and interbank liability matrix. Therefore, it may help the clearing agent to identify the future risk of bankruptcy in the system based on the current market information, and implement policies to avoid such a risk in advance.

We next study the vulnerability of a specific network with monopoly node under some assumption about the monopoly node in it. For this, we first introduce the following definition.

Definition 3.4.4. A node i in the financial network is said to be liability-free, if we have $l_{ij} = 0, \ \forall j = 1, \ldots, n, j \neq i$.

We have

Theorem 3.4.3. Given a network with a monopoly node in which the monopoly node is liability-free, and all the other nodes are well-balanced. Suppose that Assumption 3.4.2 hold. Then we have

$$P(\exists j : x_j^* \leq 0) \geq 0.5.$$

Proof. W.l.o.g., we assume that the first node is the monopoly node. In this case, we have $l_{1j} = 0, \forall j = 1, \ldots, n$, and $\alpha_i = 0, \forall i \neq 1$. In such a case, we can rewrite problem (3.2.2) as follows

$$\begin{aligned} \max \quad & \sum_i \bar{p}_i x_i \\ s.t. \quad & \bar{p}_1 x_1 - \sum_{k \geq 2} l_{k1} x_k \leq \alpha_1 \ and \end{aligned} \tag{3.4.18}$$

$$\bar{p}_j x_j - \sum_{k\geq 2} l_{kj} x_k \leq s_j, \ \forall j = 2, \ldots, n; \tag{3.4.19}$$
$$x_j \leq 1, \ \forall j = 1, \ldots, n.$$

By summing up the left-had sides of the constraints (3.4.19), we obtain

$$\sum_{j=2}^{n} \left(\bar{p}_j x_j - \sum_{k\geq 2} l_{kj} x_k \right) = \sum_{k\geq 2} l_{k1} x_k \leq \sum_{j=2}^{n} s_j, \tag{3.4.20}$$

where the first equality follows from the fact that all the non-monopoly nodes are balanced. From (3.4.20), we can conclude that if all the elements of x are positive, then it must hold $\sum_{j=2}^{n} s_j > 0$. In other words, if $\sum_{j=2}^{n} s_j \leq 0$, then x must have some element less than or equal 0. It follows from the symmetry of the shock distribution that

$$P(\exists j : x_j^* \leq 0) \geq P(\sum_{i=1}^{n} s_i \leq 0) = 0.5.$$

This completes the proof of the theorem.

Theorem 3.4.3 indicates that for problem (3.2.1), if the liability matrix L has a row whose elements are zeros and the corresponding node is a monopoly in the system, then even when all the other non-monopoly nodes are well-balanced, under mild assumption on the shock distribution, the probability of bankruptcy in the system is larger than 50%. The theorem also implies that if a monopoly node in the system bankrupts, then there is a high probability of bankruptcy in the reduced system consisting of all the non-monopoly nodes.

We next provide some numerical examples to verify our theoretical results.

Example 3.4.2. We consider the same data matrix as in Example 3.2.1. The vector of asset $\alpha^1 = (7674.5, 7674.5, 7674.5, 7674.5)^T$ is computed based on data from Table 3 in [24] [2]. We also generate a new asset vector $\alpha^2 = (30698, 0, 0, 0)^T$ such that $\bar{\alpha}^1 = \bar{\alpha}^2$.

[2] As reported in a recent FDIC report [36], the total equity (which we call asset in this paper) is about 14.4% of the total liability.

Table 3.9: Bankruptcy in a Financial Network with Random Shock

$0 < x_i^* < 1$	$x_i^* \leq 0$	$\sum_i(\alpha_i + s_i) < 0$	
9493	2	505	α^1
8854	582	564	α^2

Node	α^1		α^2	
	Δ_i	$P(s_i \leq \Delta_i - \alpha_i)$	Δ_i	$P(s_i \leq \Delta_i - \alpha_i)$
1	-23023.5	0.0000013	0.000028	0.0000013
2	-23023.5	0.0000040	-15833.2	0.0107076
3	-23023.5	0.0033758	-21063.5	0.0315362
4	-23023.5	0.0053317	-21873.7	0.0344167

We use MATLAB function NORMRAND to generate random vectors of shock with Normal distribution under the assumption that $s_i \sim \mathcal{N}(0, \sigma_i^2),\ \forall i = 1, \ldots, n$. For each node i, we consider $\sigma_i = 0.25\bar{p}_i$. Then we solve 10000 random instances of problem (3.3.1) by using CVX under MATLAB R2015b. The results are summarized in Table 3.9.

For α^1 where all the nodes have positive assets, as shown in Table 3.9, there are 507 bankruptcy cases observed in our experiments, showing an empirical probability bound $P(x_i^* < 0) = 0.0507$. For each $i = 1, 2, 3, 4$, we also compute the value of Δ_i via solving problem (3.3.10). Based on the values of Δ_is, we have

$$1 - \prod_i (1 - P(s_i \leq \Delta_i - \alpha_i)) = 0.0086 \leq 0.0507.$$

We next consider a monopoly network with the monopoly node 1 corresponding to the asset vector α^2. Note that in the original system, all the banks have nonnegative assets and the network is well-balanced, thus all the banks in the network are solvent. According to Theorem 3.4.1, it has the highest probability of insolvency and is the most vulnerable scenario of a well-balanced network. As shown in Table 3.9, the empirical probability of bankruptcy is 0.1146. Based on the computed values of Δ_is, we derive a lower bound (3.4.14) as

$$1 - \prod_i (1 - P(s_i \leq \Delta_i - \alpha_i)) = 0.0748 \leq 0.1146.$$

One can easily see from Table 3.9 that the values of Δ_is are much larger in a monopoly network.

We also note that based on Table 3.9, for all the generated 10000 scenarios of the monopoly network with α^2, there are insolvent nodes in the system, indicating an empirical probability of insolvency 1. Note that, from Theorem 3.4.1 we have the following theoretical lower bound for the probability of insolvency

$$P(\exists i : x_i^* < 1) \geq 1 - (0.5)^3 = 0.875.$$

Example 3.4.3. We next consider a financial network with a liability-free monopoly node. The liability matrix and the asset vector are given in Table 3.10.

Table 3.10: Bankruptcy in a financial network with liability free monopoly node

Liability Matrix						
Node	1	2	3	4	$\bar{p}$	α
1	0	0	0	0	0	30698
2	4857	0	10625	12047	27529	0
3	9971	10625	0	24734	45330	0
4	11306	12047	24734	0	48087	0
Claims	26134	22672	35359	367810	120946	30698

$0 < x_i^* < 1$	$x_i^* \leq 0$	$\sum_i(\alpha_i + s_i) < 0$
4057	5518	371

We use the same way as in Example 3.4.2 to generate random shocks. Based on empirical results shown in Table 3.10, there are 5889 bankruptcy cases observed, showing an empirical probability of bankruptcy $P(x_i^* < 0) = 0.5889$, which is larger than the theoretical bound provided by Theorem 3.4.3.

3.5 The Domino Effect of Bankruptcy in A Financial Network with A Monopoly Node

In this section, we estimate the domino effect of bankruptcy in a well-balanced network with a monopoly node. This is motivated by several observations. First of all, as shown in the previous section, the network with a monopoly node is the most vulnerable one in terms of asset distribution. Second, massive bankruptcies had been observed during the 2007-2008 financial crisis. Due to the severe consequence of the domino effect of bankruptcy, it is of interests to explore the network structure under which the bankruptcy of some node in the network will cause other nodes to be bankrupted.

The following result regarding the domino effect of the monopoly node's bankruptcy is intuitive. For self-completeness, we also include a proof.

Proposition 3.5.1. Given a fully connected and well-balanced financial network with a monopoly node. If the monopoly node is bankrupted, then all other nodes in the system will be bankrupted as well.

Proof. For simplicity we assume that the first node is the monopoly node. Therefore, we can rewrite problem (3.2.2) as

$$\begin{aligned} \max \quad & \bar{p}^T x \ and & (3.5.1)\\ s.t. \quad & \bar{p}_1 x_1 - \sum_{k>2} l_{k1} x_k \leq \bar{\alpha}, and \\ & \bar{p}_j x_j - \sum_{k>2} l_{kj} x_k - l_{1j} x_1 \leq 0, \ \forall j = 2, \ldots, n-1; & (3.5.2)\\ & x \leq e. \end{aligned}$$

Now let us consider that the monopoly node is bankrupted, i.e., $x_1^* \leq 0$. In this case, we first show that at the optimal solution of (3.5.1) we have

$$x_j^* = c, \quad \forall j \geq 2, \quad c \in \Re. \tag{3.5.3}$$

To show this, suppose to the contrary that (3.5.3) does not hold. Then there exists an index j^* such that $x_{j*}^* = \max_{j=2,\ldots,n} x_j^*$. Based on this, by rewriting feasibility condition (3.5.2) for index j^*, we have

$$\bar{p}_{j^*} x_{j^*}^* - \sum_{k>2} l_{kj^*} x_k^* \leq l_{1j^*} x_1^* \leq 0.$$

Using the connectivity of the underlying network and following a similar vein as in the proof of Lemma 3.2.1, we can conclude that (3.5.3) holds at the optimal solution of problem

(3.5.1). Now it suffices to show that $c \leq 0$. Since the network is fully connected, there exists $m \geq 2$ satisfying $l_{1m} > 0$. Now recall the feasibility condition (3.5.2) for index m, we have

$$l_{1m}x_m^* = (\bar{p}_m - \sum_{k>2} l_{km})x_m^* \leq 0,$$

where the equality follows from the fact that the network is well-balanced. This implies that $x_m^* = c \leq 0$, which completes the proof of the proposition.

Proposition 3.5.1 illustrates that the bankruptcy of the monopoly node will cause all other nodes in the system to be bankrupted, which is consistent with the "too big to fail" theory. Next we study how the bankruptcy of a non-monopoly will affect the system. For this, we consider a well-balanced network with a tridiagonal structure specified below.

Assumption 3.5.1. A financial network is said to have a tridiagonal structure if it satisfies the following relations:

$$l_{ij} > 0, \quad \forall (i,j) \in \{(i,j) : |i-j| \leq 1,\ \forall i,j = 1,\dots,n\}. \tag{3.5.4}$$

We remark that the above structure is identified in recent work by [54]. Next we study the domino effect of bankruptcy in a balanced network with a tridiagonal structure and the first node as a monopoly node.

Proposition 3.5.2. Given a well-balanced network with a tridiagonal structure and a monopoly node. If a non-monopoly node is bankrupted, then all the nodes following it will be bankrupted.

Proof. Let us rewrite problem (3.2.2) as

$$\max \quad \bar{p}^T x \ and \tag{3.5.5}$$

$$\begin{aligned} s.t. \quad & \bar{p}_1 x_1 - l_{21}x_2 \leq \bar{\alpha}; \\ & \bar{p}_j x_j - l_{(j-1)j}x_{j-1} - l_{(j+1)j}x_{j+1} \leq 0,\ \forall j = 2,\dots,n-1; \end{aligned} \tag{3.5.6}$$

$$\bar{p}_n x_n - l_{(n-1)n} x_{n-1} \leq 0;$$
$$x \leq e.$$

Since the network is well-balanced and has a tridiagonal structure, we have $\bar{p}_n = l_{(n-1)n}$ and $\bar{p}_1 = l_{21}$. Based on this, at the optimal solution of (3.5.5), we have

$$x_n^* \leq x_{n-1}^*. \tag{3.5.7}$$

Now, let us consider (3.5.6) for $j = n-1$. In this case, we have,

$$\begin{aligned}
&\bar{p}_{n-1} x_{n-1} - l_{(n-2)(n-1)} x_{n-2} - l_{(n)(n-1)} x_n \\
&= l_{(n-2)(n-1)}(x_{n-1} - x_{n-2}) + l_{(n)(n-1)}(x_{n-1} - x_n) \leq 0.
\end{aligned} \tag{3.5.8}$$

The equality follows from the fact that $\bar{p}_{n-1} = l_{(n-2)(n-1)} + l_{(n)(n-1)}$. From (3.5.7) and (3.5.8), we have

$$x_n^* \leq x_{n-1}^* \leq x_{n-2}^*.$$

Following a similar procedure for $j \leq n-2$, we can obtain

$$x_n^* \leq x_{n-1}^* \leq \cdots \leq x_1^*.$$

This implies that when a non-monopoly node i becomes bankrupted, i.e., $x_i^* \leq 0$, we have

$$x_n^* \leq x_{n-1}^* \leq \cdots \leq x_i^* \leq 0.$$

This finishes the proof of this proposition.

Proposition 3.5.2 shows that if the network has a tridiagonal structure and a solvent monopoly node, then the bankruptcy of every non-monopoly node will still have a significant domino effect. In other words, the solvency of big banks cannot avoid massive bankruptcies if the financial network has a tridiagonal structure and is dominated only by a few big banks.

We remark that during the 2007-2008 financial crisis, the federal government bailed out numerous major banks to stabilize the market. Nevertheless, a large number of banks still bankrupted during the crisis period. Proposition 3.5.2 provides an interesting interpretation to such a phenomenon.

Next, we provide a numerical example to verify the conclusions in the proposition.

Example 3.5.1. We consider a tridiagonal financial network with four banks in which the first node is the monopoly node ($\bar{\alpha} = \alpha_1$). The liability matrix is extracted from the liability matrix (see Table 8 in [24]) by considering the first four banks in the network. The asset vector and the optimal solutions are also listed below.

Table 3.11: Domino Effect of Bankruptcy in a Tridiagonal Financial Network with a Monopoly Node.

Liability Matrix						
Node	1	2	3	4	$\bar{p}$	α
1	0	31855	0	0	31855	32698
2	31855	0	18016	0	49871	0
3	0	18016	0	73185	91201	0
4	0	0	73185	0	73185	0
Claims	31855	49871	91201	73185	246112	32698

Optimal Solution				
x_1^*	x_2^*	x_3^*	x_4^*	
1	1	1	1	α
1	0	0	0	$\alpha_{2-} = -31855$

In this example, a shock of magnitude $s_1 < -\bar{\alpha}$ will cause node 1 to be bankrupted ($x_1^* < 0$). Following this, the whole system will be bankrupted. In other words, the bankruptcy of the monopoly node will be propagated to the whole network. This domino phenomenon is consistent with the so-called "too big to fail" theory, which advocates for government's intervenience in a period of financial crisis.

Example 3.5.1 demonstrates that not only the failure of the monopoly node in the network, but also the failure of a non-monopoly node may lead to a catastrophic disaster. As one can see from Table 3.11, under a negative shock of magnitude $s_2 = -31855$ triggering node 2 we have $x_2^* = 0$. Following this, we have $x_3^* = x_4^* = 0$. This shows that the monopoly network with a tridiagonal structure is very fragile, and the bankruptcy of a non-monopoly node in the network may have a domino effect too.

3.6 Conclusions

In this paper, we study the vulnerability of the financial network via analyzing the infeasibility of Eisenberg-Noe?s linear optimization model and its relaxation. We show that as long as the total asset is nonnegative, the relaxation model is feasible. Under the assumption that only a single bank is exposed to market shock, we characterize conditions under which a single bank is solvent, default, or bankrupted.

For the generic scenario where all banks are triggered by market shocks, we show that both the total asset and the asset inequality may affect the stability of financial network. Particularly, we show that while a larger total asset may not improve the stability of the network, a larger asset inequality will reduce the stability of the network. We estimate the probability of insolvency and the probability of bankruptcy under certain assumptions on network structure and shock distribution. Particularly, we carry out a deterministic analysis showing that the least stable network can be attained at some network with a monopoly node, and show that the such a network has the highest probability of insolvency and is the most vulnerable network. We also study the contagious effect of bankruptcy under the network with a monopoly node and tridiagonal structure.

Several issues are of interests for future research. The first issue is how to identify the structure of the liability matrix such that the resulting system is the most or least stable one. Progress in such a topic will provide insights for the clearing agent on which kind of policies should be implemented in advance to prevent a catastrophic disaster. The second issue is, though we have provided a lower bound for the probability of bankruptcy in the network, we do observe in our experiments that there is a large gap between the empirical bound and the theoretical lower bound. Further study is needed to close such a gap. Finally we point out that after the financial crisis in 2007-2008, new regulations have been implemented/enforced for the financial market. It will be interesting to incorporate the new regulations in the assessment of systemic risk.

Chapter 4

A Bi-level Linear Optimization Model for Assessing Systemic Risk under Uncertain Liabilities

4.1 Introduction

A financial network consists of nodes representing financial institutions and links representing the interactions of institutions through borrowing and lending or interconnecting indirectly through the market by holding similar shares or portfolios. The strong interconnections among financial institutions has various consequences in the global financial market. In one hand, it may help to diversify risk exposures for each financial institution. On the other hand, the interconnections may create a channel along which the failure of financial institutions can quickly spill over through the entire system and leading to a catastrophic disaster. This is usually referred as the so-called systemic risk. One such example is the 2008 financial crisis in the United States where the entire U.S. financial industries and several international financial markets were affected by large exposures of banks failure [60]. Another example is the European sovereignty debt crisis that causes the European financial business to face serious loss of confidence [56].

The catastrophic disaster from the financial crisis has inspired intensive study on the sources, effects, and results of the crises, and developing tools to mitigate and manage the systemic risk to increase the resilience of financial networks to encounter economic crisis. In a seminal paper, Eisenberg and Noe [30] introduce a clearing payment system framework to assess the systemic risk in inter-banking networks, where the clearing payment satisfies the bankruptcy law. In their framework, a financial institution is called solvent when it can fulfill its total liabilities, and default when it is able to meet a fraction of its liabilities.

They describe the cascading impact of failure where the payment shortfalls originating at a single institution can transmit to other financial institutions. They also study the existence and uniqueness of the clearing payment vector and present an algorithm to compute it.

In a series of papers, Elsinger et al. [34, 33, 32, 35] investigate the contagious effect of failures on the stability of a banking system considering the joint impact of two major sources of risk, the correlated exposure and domino effects. As pointed out in [35], the detailed information regarding the interbank liabilities is never publicly available. To deal with the incomplete information about interbank liabilities, they suggest to first solve an entropy optimization problem based on the so-called Kullback-Leibler (KL) divergence to compute the liability matrix. Such an approach has been adopted in numerous works [20, 24, 44, 57]. However, as observed by Cocco et al. [26] and Sheldon and Maurer [67], the KL-divergence method always lead to a fully connected network structure which is different from liability matrix in banking systems. For example, [67] use the KL-divergence to impute the missing network data considering the balance sheet information of the banking system in Swiss and German, and observe that the usage of the interbank liabilities estimated based on the KL-divergence in the risk assessment model usually leads to substantial underestimation of the real risk in the system. Similar observation were also made in [70]. Sheldon and Maurer [67] and Cocco et al. [26] further observe that the liability matrix from the banking system is usually sparse, which is very different from the network based on the KL-divergence. Empirical studies by Mistrulli et al. [59] and Degryse and Nguyen [28] also noticed such a difference.

Several researchers study the contagious risk in financial systems corresponding to certain network structure. For example, Allen and Gale [5] first show that the extend of contagion in a financial system is influenced by the topology of its interbank lending. Gai and Kapadia [42] study the cascading impact of failures in a random network and analyze the knock-on effects of distress. Liu and Staum [57] apply the standard sensitivity analysis for linear optimization to assess the contagious risk in a financial network. Battiston et al. [8, 9] study the systemic risk considering the effect of credit risk diversification and network density. Acemoglu et al. [1] study the effect of types of network structure on the stability of a financial system under certain assumptions on the balance sheets and discrete shocks. Chen et al. [24] study the impact of both network and liquidity channel in risk transmission. They also discuss how to estimate the liability matrix using the incomplete information from the EBA data. Glasserman and Young [44] use node-level data (asset size, leverage, and total liabilities) to bound contagion and amplification effect in the financial network regardless of the network structure. Glasserman and Young [44] study the cascades of failure via network spillovers under continuous shocks and show that the contagious risk is usually small. Unfortunately, as pointed out in a recent survey by Elsinger et al. [35], most existing works have substantially underestimated the contagious risk in the financial system. They further speculate that unavailability of the complete information on the financial network may be one major reason for the underestimation of the risk in the current literature.

The main goal of our work is to address the issue of risk underestimation in the literature on risk assessment for a financial network in which only partial information on the interbank liabilities is available. It should be pointed out that two recent papers have studied such an issue. For example, in [21] Capponi et al. compare the stability of two financial systems with the same asset vector but different liability matrices. They show that when one liability matrix is majorized by another one, the systemic loss corresponding to the majorized matrix dominates the loss in the system with another liability matrix. Feinstein et al. [37] perform a sensitivity analysis of the clearing payment vector with respect to the interbank liabilities based on the clearing agent model. They show that perturbations to the interbank liabilities could lead to an underestimation of the risk of contagion.

Different from these two papers, in this work we introduce a bi-level (worst-case) linear optimization model (WCLO) to assess the systemic risk in a financial system where only partial information such as the total liabilities and the total claims of the banks are known. Then we propose various update schemes to update the liability matrix and the payment vector alternatively to reduce the overall payment in the system iteratively. By combining these different updating schemes, we develop an integrated algorithm to identify the least stable network structure and show that the least stable network structure identified in our work has different characteristics from the least stable network structure identified in [1]. Our preliminary experiments demonstrate that the contagious risk in the identified least stable network is much more significant that what has been underestimated in the current literature [44, 57].

We also propose a bi-level optimization model to identify the most stable network structure under which the overall payment in the system is maximal and design an integrated approach for it. We compare the contagious risk under the least stable network, the most stable network and the one based on the KL-divergence. Numerical experiments show that the contagious risk under the least stable network is the most significant and is much more substantial than the other two cases, while the contagious risk in the most stable network is very close to the one based on the Kl-divergence. Since most existing works on systemic risk assessment are based on a liability matrix computed based on the KL-divergence, our results demonstrate clearly that the usage of the liability matrix based on the KL-divergence is one major reason for the underestimation of the contagious risk in the financial network in the current literature.

Our approach is very different from the recent works by Capponi et al. [21] where the authors used the concept of majorization to study the stability of the system in terms of liability concentration. As we shall see later, if one liability matrix is majorized by another one, then the total claims from these two liability matrices may not be the same. In other words, the work [21] compares the loss in two financial systems with different total claims (or equities) for the nodes in the system while we compare the loss in two system under the same total liabilities and claims for all the nodes in the system. In [37], Feinstein et al. use directional derivatives to quantify the sensitivity of the payment vector to estimation errors in the liability matrix. In their framework, they assume that the estimation error is sufficiently small so that the set of default nodes and solvent nodes remain invariant, while we study the generic case where different liability matrices may lead to different sets of default banks and solvent banks in the system.

The paper is organized as follows. In Section 4.2, we present the relaxed Eisenberg and Noe's model and provide a motivational numerical example for this paper. Then, we introduce the WCLO model to identify the least stable network structure. In Section 4.3, we present an integrated algorithm to solve the proposed WCLO model and explore the properties at the obtained solution. In Section 4.4, we introduce another bi-level LO model to identify the most stable network structure and design an integrated approach for it. We also propose a new strategy to mitigate the contagious risk in the system based on the identified most stable network structure. In Section 4.5, we evaluate the contagious effect of the failure through the identified least stable and most stable network structure, and compare with that of the network with a liability matrix estimated based on the KL-divergence. Finally we conclude the paper in Section 4.6 by discussing some future research directions.

4.2 The Clearing Payment Model and Its Worst-case Scenario

We first describe the linear optimization model introduced in [30]. Consider a financial network with n banks (represented by n nodes) interconnected to each other. A clearing agent is in charge of the process of settling the liabilities among these nodes. The ability of one node to settle its obligations depends on the repayment of other nodes to this node and also its own asset. Let $L \in \Re^{n\times n}$ be the interbank liability matrix where l_{ij} is the liability of node i toward node j. Since each nominal claim is nonnegative and no node has a nominal claim against itself, we have $l_{ij} \geq 0$ and $l_{ii} = 0$, $\forall i, j = 1, ..., n$. Let α be the exogenous asset. The total liability of node i is equal to $p_i = \sum_{j=1}^{n} l_{ij}$. The payment made by node i to node j, i.e., $x_i l_{ij}$ is obtained by solving the following problem [30]:

$$\begin{aligned} \max_{x} \quad & p^T x \; and \\ s.t. \quad & (P - L^T)x \leq \alpha; \end{aligned} \tag{4.2.1}$$

$$0 \leq x \leq e.$$

where P =diag(p) and e is the all-ones vector. In [52], we consider the following relaxation of model (4.2.1): In [52], we consider the following relaxation of model (4.2.1):

$$\max_{x} \quad p^T x \; and \tag{4.2.2}$$

$$s.t. \quad (P - L^T)x \leq \alpha; \tag{4.2.3}$$

$$x \leq e.$$

Here, at the optimal solution of (4.2.2) we have three conditions, $x_i^* \leq 0$, $x_i^* \in (0, 1)$, and $x_i^* = 1$, which are associated with the status of the bank i depending on whether it is bankrupted, default, or solvent.

As proved in [52], the relaxed model (4.2.2) enjoys several appealing properties as summarized in the following proposition, which is a combination of Propositions 2.1, Corollary 2.1, and Theorem 2.1 in [52].

Proposition 4.2.1. Let $x^{(1)}$ be the optimal solution of problem (4.2.1) and $x^{(2)}$ be the optimal solution of problem (4.2.2). Then we have:

(i) $x^{(2)} = x^{(1)}$;

(ii) $[(P - L^T)x^{(2)}]_i = \alpha_i \; or \; x_i^{(2)} = 1, \; \forall i = 1, \ldots, n$;

(iii) If the network is fully connected, then problem (4.2.2) is feasible if and only if $\sum_i \alpha_i \geq 0$.

In [52], we analyze the vulnerability of the financial system based on problem (4.2.2) with uncertain asset vector α, and identify the most vulnerable scenario of the system with respect to the asset distribution. Particularly, when only a single node i in the system receives a shock s_i, the available asset would be $\alpha_i + s_i$, and based on this, we have the following result (see Theorems 3.1 and 3.2 in [52]).

Theorem 4.2.1. Let x^* be the optimal solution of problem (4.2.2). Then,

(i) $x_i^* \leq 0$ if and only if $s_i \leq \max(-e^T\alpha, \Delta_i - \alpha_i)$, where

$$\begin{aligned} \Delta_i = -\max \quad & \sum_{j \neq i} l_{ji} x_j \ and \\ s.t. \quad & p_j x_j - \sum_{k \neq j, k \neq i} l_{kj} x_k \leq \alpha_j, \ \forall j \neq i; \\ & x_j \leq 1, \ \forall j \neq i. \end{aligned} \tag{4.2.4}$$

(ii) $x_i^* = 1$ if and only if $s_i \geq p_i + \Gamma_i - \alpha_i$, where

$$\begin{aligned} \Gamma_i = -\max \quad & \sum_{j \neq i} l_{ji} x_j \ and \\ s.t. \quad & p_j x_j - \sum_{k \neq j, k \neq i} l_{kj} x_k \leq \alpha_j + l_{ij}, \ \forall j \neq i; \\ & x_j \leq 1, \ \forall j \neq i. \end{aligned} \tag{4.2.5}$$

In this paper, we focus on the stability of the financial system under uncertainties in liability matrix L. As pointed out in a recent survey [35], the full liability matrix L is usually not exposed and only partial information such as the total liabilities and the total claims of a bank (corresponding to the summations of all the elements in a row and a column of L) is available. To estimate the systemic risk in the system, most works in the literature first compute the liability matrix by solving some entropy optimization problem based on the KL-divergence, and then analyze the contagious risk based on the estimated liability matrix. As observed in [35], this has led to a significant underestimation of the risk in the financial system. We next present one numeric example to illustrate such an issue.

Example 4.2.1. Consider a complete financial network with the first 8 banks extracted from one example in [24] (see Table 6 in [24]). The liability matrix, total liabilities, total claims and asset vector (α) are listed in Table 4.1. For demonstration, we use an asset vector such that only one bank (node 1) has a negative value.

In all our experiment, we use CVX [45] under MATLAB R2015 to solve the linear optimization models. We list the optimal primal and dual solutions of problem (4.2.2) in Table 4.1. From Table 4.1 one can see that the last two nodes are solvent while the first six nodes default and can repay more than 90% of their liabilities. We note that the contagious risk from the default of node 1 has caused banks 2,3,4,5 and 6, to default too. On the other hand, by checking the optimal solution to the dual problem of (4.2.2) (which represents the contagious impact factors in the system according to [57]), we find that there is no significant contagious effect in the system, which indicates the system is rather stable.

Table 4.1: The liability matrix based on KL-divergence

Node	1	2	3	4	5	6	7	8	p	α
1	0	4857	9971	11306	6753	5413	712	2213	41225	-1690
2	4857	0	10625	12047	7196	5768	758	2358	43609	0
3	9971	10625	0	24734	14774	11841	1557	4841	78343	0
4	11306	12047	24734	0	16751	13427	1766	5489	85520	0
5	6753	7196	14774	16751	0	8020	1055	3279	57828	0
6	5413	5768	11841	13427	8020	0	845	2628	47942	10
7	712	758	1557	1766	1055	845	0	349	7042	1000
8	2213	2358	4841	5489	3279	2628	349	0	21157	2500
claim	41225	43609	78343	85520	57828	47942	7042	21157	382666	1820

Optimal Solutions to the Primal and Dual Problems

x_1^*	x_2^*	x_3^*	x_4^*	x_5^*	x_6^*	x_7^*	x_8^*
0.9059	0.9428	0.9428	0.9428	0.9428	0.9430	1.00	1.00
λ_1^*	λ_2^*	λ_3^*	λ_4^*	λ_5^*	λ_6^*	λ_7^*	λ_8^*
12.96	12.95	12.84	12.81	12.91	12.94	0.00	0.00

We next resolve problem (4.2.2) with the worst-case liability matrix identified by the proposed algorithm in this work. Note that for the identified worst-case instance, the total liabilities, the total claims and the asset vector all remain the same as in the first instance, and only the elements of the liability matrix have been changed.

Table 4.2: The identified worst-case liability matrix

Node	1	2	3	4	5	6	7	8
1	0	41225	0	0	0	0	0	0
2	41225	0	2384	0	0	0	0	0
3	0	2384	0	75959	0	0	0	0
4	0	0	61519	0	24001	0	0	0
5	0	0	14440	9561	0	33827	0	0
6	0	0	0	0	33827	0	0	14115
7	0	0	0	0	0	0	0	7042
8	0	0	0	0	0	14115	7042	0

Optimal Solutions to the Primal and Dual Problems

x_1^*	x_2^*	x_3^*	x_4^*	x_5^*	x_6^*	x_7^*	x_8^*
0.0019	0.0428	0.7517	0.7606	0.8310	0.8810	1.00	1.00
λ_1^*	λ_2^*	λ_3^*	λ_4^*	λ_5^*	λ_6^*	λ_7^*	λ_8^*
81.96	80.97	45.39	43.24	34.17	25.11	0.00	0.00

From Table 4.2, one can see that the repayment rates for the first two nodes are close to zero and the rest of the nodes can repay more than 70% of their liabilities in the system. By checking the optimal solution to the dual problem of (4.2.2), we find that there is a very high contagious risk in the system. Further, by using the results in Theorem 4.2.1, one can see that bank 1 will be bankrupted if it receives a very small negative shock $s_1 = -3.13$. This demonstrates the high vulnerability of the system under the identified worst-case scenario. Since a financial crisis usually happens under some extreme circumstances, for the first time, the above example provides an interesting illustration close to what observed during a period of financial crisis.

Motivated by the above example, in this paper we first consider the issue of how to identify the worst-case network structure under which the system is the least stable one. For convenience, we adopt a similar measurement as in [21] and [52] for the stability of a financial system through the optimal objective value of problem (4.2.2).

Definition 4.2.1. Consider two financial systems with the same asset vector (α) with liability matrices L^1 and L^2 satisfying $e^T L^1 = e^T L^2$, and $L^1 e = L^2 e$. The first system is said to be less stable than the second one if $p^T x^*(L^1) < p^T x^*(L^2)$, where $x^*(L^1)$ and $x^*(L^2)$ denote the optimal solution of (4.2.2) when $L = L^1$ and $L = L^2$ receptively.

We remind the readers that the above definition is different from what used in [1] where the number of nodes affected by the shock is used to measure the stability of the system. Definition 4.2.1 implies that the stability of a financial network can be influenced by both the asset vector (α) and liability matrix (L). In [52], we have analyzed the stability of the financial system with respect to uncertainties in the asset vector α. In this work, we assess the stability of the financial system with respect to uncertainties in the liability matrix L under the assumption that the asset vector α is fixed. For this, we introduce the following worst-case linear optimization model (WCLO):

$$\begin{aligned} \min_{\Delta L \in \mathcal{U}_L} \quad \max_{x} \quad & p^T x \\ s.t. \quad & (P - L^T - \Delta L^T)x \le \alpha, \quad x \le e, \end{aligned} \tag{4.2.6}$$

where L is the input liability matrix, and the uncertainty set $\mathcal{U}_L$ is defined as follows:

$$\mathcal{U}_L = \left\{ \Delta L : \Delta L e = \Delta L^T e = 0, \Delta l_{ii} = 0, \forall i = 1, \ldots, n, -l_{ij} \le \Delta l_{ij}, \forall i, j \right\}.$$

We remark that the first constraint in the uncertainty set follows from the assumption that total claims and total liabilities for each node must remain the same. The second one is due to the fact that $l^+_{ii} = l_{ii} = 0$ ($L^+ = L + \Delta L$) and the last constraint follows from the assumption that $l^+_{ij} \ge 0$.

4.3 Identifying the Least Stable Network Structure

In this section, we develop an integrated approach for the WCLO problem (4.2.6). The section consists of four subsections. In the first subsection, we consider the case where there exists only one default node in the system, and discuss how to update the liability matrix to reduce strictly the stability of the underlying network, or verify the optimality of the current solution. In the second subsection, we consider the case where there exist multiple default nodes in the system and develop an algorithm that updates the liability matrix and contagious factors alternatively to reduce the stability of the system. In the third subsection, we present an algorithm based on the linear approximation and line search technique to further reduce the stability of the system. In the fourth subsection, we combine all these three algorithms to develop the integrated approach for problem (4.2.6). We establish the convergence of the integrated approach and explore the network structure at the obtained solution.

At first, we point out that by applying the duality theorem for linear optimization to the sub-problem in (4.2.6), we can rewrite problem (4.2.6) as the following single-level non-convex quadratic optimization problem with non-convex quadratic constraints.

$$\begin{aligned} \min_{\Delta L \in \mathcal{U}_L} \quad \min_{\lambda} \quad & (\alpha^T - e^T(P-L))\lambda + e^T \Delta L \lambda + e^T p \; and \\ s.t. \quad & (P - L - \Delta L)\lambda \leq p, \quad \lambda \geq 0. \end{aligned} \tag{4.3.1}$$

Since $e^T \Delta L = 0$, the above problem reduces to

$$\begin{aligned} \min_{\lambda, \Delta L} \quad & (\alpha^T - e^T(P-L))\lambda + e^T p \; and \\ s.t. \quad & (P - L - \Delta L)\lambda \leq p; \\ & \lambda \geq 0, \quad \Delta L \in \mathcal{U}_L. \end{aligned} \tag{4.3.2}$$

Note that problem (4.3.2) involves some non-convex quadratic (bilinear) constraints and thus finding a global solution to it is very difficult. Several approaches for generic bilinear optimization problems have been proposed in the literature, see [4, 39, 46] and the references therein. One popular approach is to update the different sets of variables alternatively. For example, we can first fix the dual variables λ ($\lambda = \lambda^*$) and update $L^+ = L + \Delta L$ to reduce the objective function value in (4.3.2). Then we fix L and update λ via solving problem (4.3.2). There are numerous issues in such an alternative approach need to be addressed with respect to problem (4.3.2). One is that when we fix the dual variables and update the liability matrix, since the objective function in (4.3.2) is independent of ΔL, it is critical to choose a suitable objective function depending on ΔL that will reduce the objective function value in (4.2.2) after updating the liability matrix. There are also other issues such as how to characterize the relationship between the solutions before and after one update of the liability matrix in the alternative approach and what to do if the alternative approach can not be applied.

For convenience of discussion, let us introduce the following index sets:

$$\mathcal{I} = \{1, \ldots, n\}, \quad \mathcal{I}_1 = \{i : \lambda_i^* > 0\}, \quad \mathcal{I}_2 = \mathcal{I} - \mathcal{I}_1, \tag{4.3.3}$$

where λ^* and μ^* are the optimal solution that can be obtained from the dual problem of (4.2.2):

$$\begin{aligned} \min_{\lambda,\mu} \quad & \alpha^T \lambda + e^T \mu \\ s.t. \quad & (P - L)\lambda + \mu = p, \quad \lambda, \mu \geq 0. \end{aligned} \tag{4.3.4}$$

Note that these two index sets are associated with the status of bank i, i.e., whether node i defaults in the system. We next consider the following assumption:

Assumption 4.3.1. The index set $\mathcal{I}_1$ is precisely the set of default nodes.

Proposition 4.3.1. Suppose that Assumption 4.3.1 holds, then at the optimal solution of (4.3.4) we have:

$$\lambda_i^* \mu_i^* = 0, \quad \forall i \in \mathcal{I}.$$

Proof. From the feasibility condition in (4.3.4) we have that $\lambda_i^* \mu_i^* \geq 0$. Thus, it suffices to show that both μ_i^* and λ_i^* cannot be non-zero at the same time. Suppose to contrary that there exists $i \in \mathcal{I}_1$ such that $\mu_i^* > 0$ and $\lambda_i^* > 0$. In this case, based on the complementary condition, we have:

$$[(P - L^T)x^*]_i = \alpha_i, \quad x_i^* = 1.$$

This contradicts the assumption that the set of default nodes and solvent nodes are well-separated, which finishes the proof of this proposition.

From Proposition 4.3.1, it follows immediately that for every $i \in \mathcal{I}$, if $x_i^* < 1$, then $i \in \mathcal{I}_1$. Similarly one can also see that at the optimal solution of (4.2.2), it holds

$$\lambda_i^* = 0, \quad x_i^* = 1, \quad \forall i \in \mathcal{I}_2.$$

Using these index sets, we can rewrite matrix ΔL as

$$\Delta L = \begin{bmatrix} \Delta L_{\mathcal{I}_{11}} & \Delta L_{\mathcal{I}_{12}} \\ \Delta L_{\mathcal{I}_{21}} & \Delta L_{\mathcal{I}_{22}} \end{bmatrix}, \tag{4.3.5}$$

where $\Delta L_{\mathcal{I}_{11}}, \Delta L_{\mathcal{I}_{12}}, \Delta L_{\mathcal{I}_{21}}$ and $\Delta L_{\mathcal{I}_{22}}$ denote the elements of matrix with indexes $(i, j) \in \mathcal{I}_1 \times \mathcal{I}_1, \mathcal{I}_1 \times \mathcal{I}_2, \mathcal{I}_2 \times \mathcal{I}_1$ and $\mathcal{I}_2 \times \mathcal{I}_2$ respectively.

Next we consider a special case where all the banks in the system are solvent. In this case, one can easily see that for every $\Delta L \in \mathcal{U}_L$, we have $f(L + \Delta L) = f(L)$, where $f(L)$ and $f(L + \Delta L)$ denote the optimal value of (4.2.2) with liability matrix L and $L + \Delta L$ respectively. From this, we immediately obtain the following result.

Proposition 4.3.2. The structure of the liability matrix does not have any impact on the stability of the financial network when all the nodes in the system are solvent.

4.3.1 Dealing with the case that $|\mathcal{I}_1| = 1$

In this subsection, we study the case that only one node defaults in the system, i.e., $|\mathcal{I}_1| = 1$. For simplicity, we assume that $|\mathcal{I}_1| = \{1\}$. In order to find an update scheme $L^+ = L + \Delta L$ to reduce the stability of the system, we solve a series of optimization models:

$$\begin{aligned} \Delta L^j = \arg\max \quad & \delta_j \; and && (4.3.6) \\ s.t. \quad & [(P - L - \Delta L)^T x^*(L)]_j = \alpha_j + \delta_j; && (4.3.7) \\ & \Delta L \in \mathcal{U}_L, \end{aligned}$$

where $x^*(L)$ denotes the optimal solution of (4.2.2) when the liability matrix is L. We note that for every fixed j, problem (4.3.6) might have multiple optimal solutions, however, its optimal value δ_j is unique. Now we are ready to state the following result.

Theorem 4.3.1. Suppose that $|\mathcal{I}_1| = \{1\}$. Let $f(L)$ and $f(L^+)$ denote the optimal value of (4.2.2) with liability matrix L and L^+, respectively. Then the following conclusions hold:

(i) Suppose that there exists $j \in \mathcal{I}_2$ such that the objective function at the optimal solution of problem (4.3.6) has a positive value and let $L^+ = L + \Delta L^j$. Then it holds $f(L^+) < f(L)$;

(ii) If the objective function at the optimal solution of problem (4.3.6) has a non-positive value for every $j \in \mathcal{I}_2$, then for every $\Delta L \in \mathcal{U}_L$, it holds $f(L^+) \geq f(L)$.

Proof. Clearly, we have $x_1^*(L) < 1$ and $x_i^*(L) = 1, \forall i \neq 1$. Since $\Delta L^T e = 0$, we obtain

$$p_1 x_1^*(L) - \sum_{i \neq 1} l_{i1}^+ = [(P - L)^T x^*(L)]_1 = \alpha_1. \tag{4.3.8}$$

Now suppose that for some $j \in \mathcal{I}_2$, the objective function value at the optimal solution of problem (4.3.6) is positive. Let $x^*(L^+)$ be the optimal solution of (4.2.2) with liability matrix L^+. We first show that:

$$x_1^*(L^+) \leq x_1^*(L). \tag{4.3.9}$$

Suppose to the contrary that $x_1^*(L^+) > x_1^*(L)$. It follows from (4.3.7) that

$$p_1 x_1^*(L^+) - \sum_{i \neq 1} l_{i1}^+ x_i^*(L^+) > p_1 x_1^*(L) - \sum_{i \neq 1} l_{i1}^+ = \alpha_1,$$

which contradicts to the fact that $x^*(L^+)$ is a feasible solution of (4.2.2) with liability matrix L^+. Therefore, the relation (4.3.9) holds. We next show that there exists some index $j \in \mathcal{I}_2$

satisfying $x_j^*(L^+) < x_j^*(L)$. Suppose to the contrary that $x_j^*(L^+) = x_j^*(L) = 1, \forall j \in \mathcal{I}_2$. It follows

$$p_j - l_{j1}^+ x_1^*(L^+) - \sum_{i \neq 1,j} l_{ji}^+ \leq \alpha_j < p_j - l_{j1}^+ x_1^*(L) - \sum_{i \neq 1,j} l_{ji}^+.$$

The above relation holds true only when $x_1^*(L^+) > x_1^*(L)$, which contradicts to (4.3.9). From the above discussion we can conclude

$$x_j^*(L^+) \leq x_j^*(L), \quad \forall j = 1, \ldots, n,$$

with at least one inequality holds strictly. This proves the first conclusion of the theorem.

To prove the second conclusion, we observe that if for every $j \in \mathcal{I}_2$, the objective function at the optimal solution of problem (4.3.6) has a non-positive value, then we have

$$[(P - L^+)^T x^*(L)]_j \leq \alpha_j, \quad \forall j \in \mathcal{I}_2, \forall \Delta L \in \mathcal{U}_L.$$

Combining the above relation with (4.3.8) we can conclude that $x^*(L)$ is a feasible solution of (4.2.2) with liability matrix L^+. It follows from (4.2.2) that $f(L^+) \geq f(L)$. This completes the proof of the theorem.

Theorem 4.3.1 shows that if only one node defaults in the system, by solving at most $n-1$ optimization problems in the form of (4.3.6), we can either find an update scheme $L^+ = L + \Delta L$ to strictly reduce the stability of the system, or provide a certificate that the current system is the least stable one. We next describe update scheme I that combines the producer discussed in this subsection.

Algorithm 1 Update Scheme I

Inputs: L, α
Output: L^+

```
for j ∈ I_2 do
    Solve problem (4.3.6) to find ΔL;
    if the optimal objective value of (4.3.6) is positive then
        Update the liability matrix via L+ = L + ΔL;
    end if
end for
```

4.3.2 An alternative update scheme

In this subsection, we study the financial system with multiple default nodes and introduce a scheme for problem (4.3.2) that updates the liability matrix and the dual variables λ alternatively. A key component in such an alternative update scheme is how to update L for a temporarily fixed λ^*. Note that for fixed $\lambda = \lambda^*$, the feasible set of problem (4.3.2) reduces to a polyhedron as

$$(P - L - \Delta L)\lambda^* \leq p \text{ } and \tag{4.3.10}$$

$$\Delta L \in \mathcal{U}_L. \tag{4.3.11}$$

For convenience, we denote such a set by $\mathcal{U}_L^*$. Since $\lambda = \lambda^*$ is the optimal solution to the dual problem of (4.2.2) defined below:

$$\min_{\lambda} \quad (\alpha^T - e^T(P - L))\lambda + e^T p \text{ } and \tag{4.3.12}$$

$$s.t. \quad (P - L)\lambda \leq p, \quad \lambda \geq 0. \tag{4.3.13}$$

we have

Theorem 4.3.2. Suppose that λ^* be the optimal solution of (4.3.12). Let $\Delta L \in \mathcal{U}_L^*$ and $L^+ = L + \Delta L$, then we have:

$$f(L^+) \leq f(L),$$

where $f(L)$ and $f(L^+)$ denote the objective value of (4.2.2) with liability matrix L and L^+, respectively.

Proof. Because $\Delta L \in \mathcal{U}_L^*$, λ^* is a feasible solution of (4.3.12) with liability matrix L^+. It follows immediately $f(L^+) \leq f(L)$, which completes the proof of this theorem.

Theorem 4.3.2 shows that for any $\Delta L \in \mathcal{U}_L^*$, the stability of the system decreases under the updated liability matrix L^+. We next develop the following optimization model to identify the updated liability matrix L^+ under which the stability of the system strictly decreases.

$$\begin{aligned} \max_{\Delta L} \quad & e_{\mathcal{I}_1}^T \Delta L \lambda^* \\ s.t. \quad & \Delta L \in \mathcal{U}_L^*; \end{aligned} \tag{4.3.14}$$

where $e_{\mathcal{I}_1}$ is a vector such that its i-th element equal 1 for all $i \in \mathcal{I}_1$, and the rest of its elements equal 0. We have:

Theorem 4.3.3. Let ΔL^* denote the optimal solution of problem (4.3.14) and $L^+ = L + \Delta L^*$. Let $f(L)$ and $f(L^+)$ denote the objective value of (4.2.2) with liability matrix L and L^+, respectively. If Assumption 4.3.1 holds and $[\Delta L^* \lambda^*(L)]_i > 0, \forall i \in \mathcal{I}_1$, then we have:

$$f(L^+) < f(L).$$

Proof. Suppose to the contrary that $f(L^+) = f(L)$. Since $\lambda^*(L)$ is the feasible solution of (4.3.12) with liability matrix L^+, we can conclude that $\lambda^*(L) = \lambda^*(L^+)$. We have that $[\Delta L^* \lambda^*(L)]_i > 0, \forall i \in \mathcal{I}_1$, which implies that

$$(P - L^+)\lambda^*(L) < p.$$

From (4.3.12) we can see that for arbitrary ϵ it holds

$$\epsilon(\alpha^T - e^T(P - L))\lambda^*(L) < 0, \quad \epsilon(P - L)\lambda^*(L) < \epsilon p.$$

By choosing a sufficiently small ϵ satisfying

$$\epsilon(P - L)\lambda^*(L) \leq \epsilon p \leq (1 + \epsilon)\Delta L^* \lambda^*(L),$$

we have that

$$(P - L^+)(1+\epsilon)\lambda^*(L) \le p.$$

Therefore, $(1+\epsilon)\lambda^*(L)$ is a feasible solution for problem (4.3.12) with liability matrix L^+. One can see the fact $(1+\epsilon)(\alpha^T - e^T(P-L))\lambda^*(L) < (\alpha^T - e^T(P-L))\lambda^*(L)$ contradicts to the assumption that $\lambda^*(L)$ is the optimal solution of (4.3.12). Therefore, we have

$$f(L^+) < f(L).$$

This finishes the proof of the theorem.

In Theorem 4.3.3 we show that if the objective function value of (4.3.14) is positive, by updating the liability matrix via solving problem (4.3.14) the stability of the system strictly decreases. It can also be shown that under such update scheme the set of default nodes with liability matrix L is the subset of the set of default nodes with liability matrix L^+, i.e., $\mathcal{I}_1(L) \subseteq \mathcal{I}_1(L^+)$.

Next we present a new update scheme such that the contagious impact factors (denoted by $\lambda^*(L^+)$) after the update dominates the contagious factors (denoted by $\lambda^*(L)$) before the update, and the repayment ratios after the update (denoted by $x^*(L^+)$) is dominated by the repayment ratios (denoted by $x^*(L)$) before the update. For this, we introduce the following definition:

Definition 4.3.1. Given two vectors $\lambda, \lambda^+ \in \mathbb{R}^n$. λ^+ is said to dominate λ (or λ is dominated by λ^+) if the following relations hold:

$$\lambda_i^+ \ge \lambda_i, \quad \forall i = 1, \ldots, n.$$

Let us first introduce several technical results that will be used in our later analysis.

As shown in [52] (See Proposition 2.1 and its proof in [52]), the optimal solution to problem (4.2.2) can be obtained via solving the following decomposed problems:

$$\max \quad p_i x_i \ and \tag{4.3.15}$$
$$(P - L^T)x \leq \alpha;$$
$$x \leq e, \tag{4.3.16}$$

all $i \in \mathcal{I}$. The dual of the above problem can be written as

$$\begin{aligned} \min_{\lambda} \quad & (\alpha^T - e^T(P - L))\lambda + p_i \ and \\ s.t. \quad & (P - L)\lambda \leq p_i e_i, \quad \lambda \geq 0, \end{aligned} \tag{4.3.17}$$

where e_i is the unit vector which has a unique positive element at its i-th element equal 1. For every $i \in \mathcal{I}$, let $\hat{\lambda}^i$ denote the optimal solution to problem (4.3.17) and let $\hat{\lambda} = \sum_{i \in \mathcal{I}} \hat{\lambda}^i$. Since $x_i^* = 1$ for every $i \in \mathcal{I}_2$, it follows directly

$$\hat{\lambda}^i = 0, \quad \forall i \in \mathcal{I}_2.$$

Because $\hat{\lambda}$ is a feasible solution of problem (4.3.2), we thus have $\hat{\lambda} = \lambda^*$. Recall definition (4.3.3), we have:

$$[(P - L)\lambda^*]_i = p_i, \quad \forall i \in \mathcal{I}_1.$$

Combining the above two relations, we can claim that for every $i \in \mathcal{I}_1$, it must hold:

$$[(P - L)\hat{\lambda}^i]_i = p_i, \quad [(P - L)\hat{\lambda}^i]_j = 0, \quad \forall j \neq i \in \mathcal{I}_1. \tag{4.3.18}$$

We next propose to solve the following optimization model in order to find an update scheme for ΔL to reduce the stability of the system and keep the dominance relations.

$$\begin{aligned}
\max_{\Delta L} \quad & e_{\mathcal{I}_1}^T \Delta L \lambda^* \ and && (4.3.19)\\
s.t. \quad & \Delta L \in \mathcal{U}_L^*; \\
& \Delta L \hat{\lambda}^i \geq 0, \quad \forall i \in \mathcal{I}_1, && (4.3.20)
\end{aligned}$$

where $e_{\mathcal{I}_1}$ is a vector such that its i-th element equal 1 for all $i \in \mathcal{I}_1$, and the rest of its elements equal 0. We have:

Theorem 4.3.4. Let ΔL^* denote the optimal solution of problem (4.3.19) and $L^+ = L + \Delta L^*$. Let $x^*(L), \lambda^*(L)$ and $x^*(L^+), \lambda^*(L^+)$ denote the optimal solution and the optimal dual solution of (4.2.2) with liability matrix L and L^+, respectively. If $e_{\mathcal{I}_1}^T \Delta L^* \lambda^* > 0$, then the following conclusions hold:

(i) For all $i \in \mathcal{I}$ we have $\lambda_i^*(L^+) \geq \lambda_i^*(L)$, and there exists $i \in \mathcal{I}_1$ such that $\lambda_i^*(L^+) > \lambda_i^*(L)$;

(ii) For all $i \in \mathcal{I}$ we have $x_i^*(L^+) \leq x_i^*(L)$, and there exists $i \in \mathcal{I}_1$ such that $x_i^*(L^+) < x_i^*(L)$.

A detailed proof of the above theorem is provided in Appendix A.1.

Theorem 4.3.4 shows that if we update the liability matrix via solving problem (4.3.19), then there exists some dominance relation between the payment ratios and the contagious risk factors before and after the update, and there exists at least one default node whose repayment ratio strictly decreases while its contagious impact factor strictly increases. It should be pointed out that in our update scheme, we keep the total liabilities and claims for all the nodes in the system invariant. We note that [21] study the dominance relation among the clearing payment ratios and the loss in two financial systems based on liability concentration. In their study, they assume that the liability matrices in two financial systems satisfy the following relation $L^1 = L^2 M$, where M is a doubly stochastic matrix. One can easily verify that for a generic doubly stochastic matrix M, $(L^1)^T e \neq (L^2)^T e$ if all the nodes in the second systems have different total claims, i.e., $(L^2)^T e \neq te$ for some scalar t.

We also mention that a simple sufficient condition to ensure constraint (4.3.20) is $\Delta l_{ij} \geq 0, \forall i, j \in \mathcal{I}_1$. Finally, though we introduce two schemes for updating the liability matrix via solving two optimization models (4.3.14) and (4.3.19), respectively, we do observe in our experiments that model (4.3.14) works better than (4.3.19). Based on this, we present the following algorithm which combines several procedures described in this subsection.

Algorithm 2 Update Scheme II

Inputs: L, α
Output: L^+
Step 1. Solve problem (4.3.4);
Step 2. Solve problem (4.3.14) to find ΔL;
Step 3. Update the liability matrix via $L^+ = L + \Delta L$.

It is easy to see that update scheme II works well when the objective function at the optimal solution of (4.3.14) is positive. In the next subsection, we discuss how to further improve the optimal objective value of (4.2.2) when the alternative update scheme does not work.

4.3.3 An update scheme based on linear approximation and line search

In this subsection, we consider the financial network where we could not find any meaningful solution ΔL satisfying condition (4.3.10). To deal with such a scenario, we develop a new updating scheme based on linear approximation to update the liability matrix.

To start, we mention that for all non-solvent nodes, the following equation system:

$$p_j \lambda_j - \sum_{k \in \mathcal{I}_1, k \neq j} l_{jk} \lambda_k = p_j, \quad \forall j \in \mathcal{I}_1, \tag{4.3.21}$$

holds. The following assumption will be used throughout this section.

Assumption 4.3.2. Submatrix $(P - L)_{\mathcal{I}_{11}}$ is diagonally row dominant and with at least one row that is strictly dominant.

We remark that both Assumptions 4.3.1 and 4.3.2 are rather mild. To see this, we recall that matrix $P - L$ itself is diagonally dominant. Thus, if $L_{\mathcal{I}_{12}} \neq 0$, then Assumption 4.3.2 always hold. Under Assumption 4.3.2, one can show that submatrix $(P - L)_{\mathcal{I}_{11}}$ is nonsingular. Therefore, we have

$$\lambda_{\mathcal{I}_1} = [(P - L)^{-1}_{\mathcal{I}_{11}} p_{\mathcal{I}_1}].$$

Assumption 4.3.1 excludes the boundary case where a solvent node belongs to the index set $\mathcal{I}_1$. Now let consider the following update of the liability matrix:

$$L^+ = L + \beta \Delta L,$$

where β is the step size ($\beta \in (0, 1]$) and ΔL is in the form specified by (4.3.5). Under Assumption 4.3.1, one can show that for sufficiently step size β, the set of default nodes and solvent nodes will not change. Let $f(L + \beta\Delta L)$ denote the objective function value of sub-problem (4.3.1). We thus have

$$f(L + \beta\Delta L) = (\alpha^T - e^T(P - L - \beta\Delta L)_{\mathcal{I}_1})(P - L - \beta\Delta L)^{-1}_{\mathcal{I}_{11}} p_{\mathcal{I}_1}.$$

Consequently, we can rewrite problem (4.3.1) as the following:

$$\begin{aligned} \min_{\Delta L} \quad & (\alpha^T - e^T(P - L - \beta\Delta L)_{\mathcal{I}_1})(P - L - \beta\Delta L)^{-1}_{\mathcal{I}_{11}} p_{\mathcal{I}_1}, \ \textit{and} \\ s.t. \quad & \Delta L \in \mathcal{U}_L. \end{aligned} \tag{4.3.22}$$

Note that the objective function in (4.3.22) is highly nonlinear, and it is hard to solve. Next, we use a local linear approximation to $f(L + \beta\Delta L)$ based on directional derivative of f in terms of matrix ΔL (denoted by $\mathcal{D}_{\Delta L} f(L)$) as follows (see Definition 2.5 in [37]):[1]

$$\mathcal{D}_{\Delta L} f(L) = (\alpha^T - e^T(P - L))_{\mathcal{I}_1}(P - L)^{-1}_{\mathcal{I}_{11}} \Delta L_{\mathcal{I}_{11}} (P - L)^{-1}_{\mathcal{I}_{11}} p_{\mathcal{I}_1}.$$

[1]For self-completeness, we also give a simple way to estimate the directional derivative ($\mathcal{D}_{\Delta L} f(L)$) in Appendix A.2.

Based on this, we propose to solve the following linear approximation model:

$$\min_{\Delta L} \quad (\alpha^T - e^T(P-L))_{\mathcal{I}_1}(P-L)^{-1}_{\mathcal{I}_{11}}\Delta L_{\mathcal{I}_{11}}(P-L)^{-1}_{\mathcal{I}_{11}}p_{\mathcal{I}_1} \; and \tag{4.3.23}$$

$$s.t. \quad \Delta L \in \mathcal{U}_L;$$

$$||\Delta L||_1 \leq 2\min\{l_{ij} > 0, \forall i, j \in \mathcal{I}\}. \tag{4.3.24}$$

It is easy to see that if the objective function at the optimal solution (ΔL^*) of (4.3.23) has a negative value, then we can apply a line search procedure to find a suitable step size β to update the liability matrix such that the objective function value in (4.3.22) is reduced. Note that constraint (4.3.24) is included to ensure the corresponding direction is the steepest descent direction within a certain neighborhood of the current iterate.

We also point out that in [37], Feinstein et al. develop a similar optimization model to find the worst-case perturbation of the liability matrix such that the payment rate is minimal. In their model, they also impose the constraints that the total liabilities and total claims of each bank remain unchanged by the perturbation. They further assume that the sets of solvent and default nodes also remain unchanged under the perturbation. Under these assumptions, they develop high-order schemes to approximate the nonlinear objective function in (4.3.22). However, in our approach, we assume that the sets of solvent and default nodes may change after one update of the liability matrix. Also, we solve only the simple linear approximation model (4.3.23). As we will see in our later analysis, such a simple approach can help to explore the structure of the identified least stable network.

We next use the following line search algorithm to find the best value for step size β such that the objective function value in (4.3.22) decreases.

Algorithm 3 A Line Search Procedure

Inputs: $\tau = \frac{1}{2}$, L, ΔL
Output: L^+
set $\beta^{(0)} = 1$, $k = 0$;
while $f(L + \beta^{(k)}\Delta L) \geq f(L)$ **do**
    set $\beta^{(k+1)} = \tau\beta^{(k)}$;
    set $k = k + 1$;
end while
set $\beta = \beta^{(k)}$;
set $L^+ = L + \beta\Delta L$;

Combining the linear approximation model (4.3.23) and the line search procedure, we obtain the following update scheme:

Algorithm 4 Update Scheme III

Inputs: L, α
Output: L^+
Step 1. If the optimal value of (4.3.23) is negative go to Step 2; otherwise, stop;
Step 2. Solve problem (4.3.23) to find search direction ΔL, then go to Step 3.
Step 3. Use Algorithm 3 to perform a line search to find a suitable step size;
Step 4. Update the liability matrix via $L^+ = L + \beta\Delta L$.

4.3.4 An Integrated Approach

We first present an integrated algorithm that combines several update schemes described in previous sub-sections.

Algorithm 5 The Integrated Algorithm

```
Inputs: α, L^{KL} [2]
Output: L^{WC}
if |I_1| = 1 then
    update the liability matrix via update scheme I;
end if
while |I_1| > 1 do
    if the optimal objective function value of problem (4.3.14) is positive then
        update the liability matrix via update scheme II;
    else
        if the optimal value of (4.3.23) is negative then
            update the liability matrix via update scheme III;
        end if
        Stop;
    end if
end while
```

Next we explore the properties of the sequence from Algorithm 5.

Theorem 4.3.5. Let L^{WC} be the liability matrix obtained from Algorithm 5. Then, the sequence generated from Algorithm 5 converges to a stationary point of (4.3.2). Particularly, if Assumption 4.3.1 holds at the final solution and the optimal solution to problem (4.3.23) is unique and trivial with $\Delta L^* = 0$, then the solution provided by the integrated algorithm is locally optimal.

Proof. Let (L^{WC}, λ^{WC}) denote the solution provided by Algorithm 5. To prove this theorem, we consider the following three cases: Case (i): the system is solvent. Case (ii): there is only one default node in the system. Case (iii): there are more than one default node in the system. In the first case, as discussed in Proposition 4.3.2, Algorithm 5 will fail to improve the optimal value of (4.2.2). Now we consider case (ii) where $|\mathcal{I}_1| = 1$. In this case, as it is shown in Theorem 4.3.1, Algorithm 5 will fail if for all solvent nodes (j) problem (4.3.6) is infeasible or it has non-positive optimal value. For the third case, Algorithm 5 will fail to improve the optimal value of (4.2.2) when the objective function value at the optimal solutions of both problems (4.3.14) and (4.3.23) are zero. In this case, we have $\mathcal{D}_{\Delta L^*} f(L) = 0$. Since ΔL^* is the optimal solution of (4.3.23), for every feasible solution

ΔL, it holds $\mathcal{D}_{\Delta L} f(L) \geq 0$. This shows that (L^{WC}, λ^{WC}) is a stationary point of (4.3.2).

Note that if Assumption 4.3.1 holds at the final solution and the optimal solution to problem (4.3.23) is unique with $\Delta L^* = 0$, then for every nontrivial $\Delta L \neq 0 \in \mathcal{U}_L$, it holds $\mathcal{D}_{\Delta L} f(L) > 0$, which further implies that (L^{WC}, λ^{WC}) is locally optimal.

Next we explore the network structure of the liability matrix at the solution provided by Algorithm 5. For such a purpose, we need to construct some graph $(G = (V, E))$ corresponding to the liability matrix identified by Algorithm 5. One way to do that is as follows. We first cast all the non-zero elements of the liability matrix, i.e., $V = \{(i,j) : l_{ij} > 0\}$. Now we describe an algorithm to construct the edges in the induced graph.

Algorithm 6 Constructing the Induced Graph

```
Input: V
Output: E : the set of edges
for i ≠ j = 1 to |V| do
    if v_i and v_j are from the same row or the same column then
        add (v_i, v_j) to E;
    end if
end for
```

We note that the induced graph G is different from the financial network. Since $\Delta L \in \mathcal{U}_L$, we are interested in the extreme points of $\mathcal{U}_L$ such that each extreme point represents a Δ-loop defined below.

Definition 4.3.2. A Δ-loop in $\mathcal{U}_L$ is a matrix such that each of its nontrivial row (or column) contains exactly two nonzero elements, one with value $+\Delta$ and another with $-\Delta$.

On the other hand, for every ΔL defined by (4.3.5), there exists some positive integer K satisfying the following relation:

$$\Delta L = \sum_{k=1}^{K} \Delta L^k,$$

where ΔL^k is a Δ-loop for every $k = 1, \ldots, K$. Therefore, one can conclude that if the objective function value at the optimal solution of problem (4.3.23) is negative, then there must exist a Δ-loop (denoted by $\Delta \tilde{L}$) satisfying

$$\mathcal{D}_{\Delta\tilde{L}} f(L) < 0.$$

From Theorem 4.3.5 we immediately obtain the following result:

Corollary 4.3.1. Suppose that induced graph G is constructed using Algorithm 6 based on the liability matrix provided by Algorithm 5. If there exists no cycles in G, then the solution provided by Algorithm 5 is locally optimal.

Next we characterize some structure of the liability matrix under which the associated graph contains no cycles. One such example is the so-called tridiagonal structure specified as

$$l_{ij} \begin{cases} > 0 & \text{if } |i-j| \leq 1; \\ = 0 & \text{Otherwise.} \end{cases} \tag{4.3.25}$$

It is worthwhile mentioning that in our recent work [52], we show that there exists a significant domino effect of bankruptcy in the system if the system is dominated by a monopoly node and the liability matrix has a tridiagonal structure. We also point out that the identified worst-case liability matrix in Example 4.2.1 does not have a tridiagonal structure, however its corresponding graph does not have any cycle and thus the solution provided by Algorithm 5 is locally optimal. One can further show that if any cycle in the constructed induced graph (G) does not include any vertex associated with some element in sub-matrix L_{11}, then the solution provided by Algorithm 5 is still locally optimal. There may exist other network structures under which the solution provided by Algorithm 5 is locally optimal, we leave such exploration to interested readers.

We note that the identified network structure in this paper is different from [1], where they show that complete network is the most stable one for small shock and is the least stable one for large shock.

4.4 Identifying the Most Stable Network Structure

In this section, we consider the issue of identifying the most stable structure of the financial system under which the total repayment of the financial institutions is maximal. For this, we introduce the following optimization model (BCLO):

$$\begin{aligned} \max_{\Delta L \in \mathcal{U}_L} \quad \max_{x} \quad & p^T x \\ s.t. \quad & (P - L^T - \Delta L^T)x \le \alpha; \\ & x \le e. \end{aligned} \tag{4.4.1}$$

Next we develop two update schemes to update both x and ΔL alternatively. Like in Section 3, we consider the following two cases:

$$\text{(i)} : |\mathcal{I}_1| \le 1; \qquad \text{(ii)} : |\mathcal{I}_1| > 1.$$

Note that in case (i), there is at most one default node in the system. In such a case, one can show that for any $\Delta L \in \mathcal{U}_L$, the objective function value at the optimal solution of (4.2.2) can not be improved. This shows that the current network structure is the best case in terms of the stability of the system.

Next, we consider case (ii) where there are multiple default nodes in the system. In this case, we develop two update schemes II$'$, and III$'$ to update the liability matrix and the repayment vector (x) alternatively. Let x^* be the optimal primal solutions of problem (4.2.2) with the liability matrix L. In order to find a scheme to update the liability matrix ($L^+ = L + \Delta L$), we solve the following problem:

$$\begin{aligned} \max_{\Delta L} \quad & e_{\mathcal{I}_1}^T \Delta L^T x^* \\ s.t. \quad & \Delta L^T x^* \ge (P - L^T)x^* - \alpha; \\ & \Delta L \in \mathcal{U}_L. \end{aligned} \tag{4.4.2}$$

Correspondingly, we obtain the new update scheme II$'$ where we replace problem (4.3.14) in Scheme II by problem (4.4.2).

Next we use the linear approximation approach and line search to improve the optimal objective value of (4.2.2) when update scheme II$'$ does not work. In this case, when directional derivative ($\mathcal{D}_{\Delta L} f(L)$) has positive value, we can choose the step size such that the optimal value of (4.2.2) strictly increases. Based on this, we change optimization model (4.3.23) as follows:

$$\begin{aligned} \max_{\Delta L} \quad & (\alpha^T - e^T(P-L))_{\mathcal{I}_1}(P-L)^{-1}_{\mathcal{I}_{11}} \Delta L_{\mathcal{I}_{11}} (P-L)^{-1}_{\mathcal{I}_{11}} p_{\mathcal{I}_1} \\ s.t. \quad & \Delta L \in \mathcal{U}_L. \\ & ||\Delta L||_1 \leq 2 \min\{l_{ij} > 0, \forall i, j \in \mathcal{I}\}. \end{aligned} \tag{4.4.3}$$

Correspondingly, we obtain a new update scheme III$'$ where we replace problem (4.3.23) in Scheme III by problem (4.4.3). If the optimal value of (4.4.3) is positive, then we adopt the line search procedure to find a step size to update the liability matrix ($L^+ = L + \beta \Delta L$) to ensure that the optimal objective function of problem (4.2.2) can be improved after the update of the liability matrix. Now we are ready to described the modified integrated algorithm for problem (4.4.1). Our next result establishes the convergence of the sequence

Algorithm 7 The Modified Integrated Algorithm

Inputs: α, L^{KL}
Output: L^{BC}
while $|\mathcal{I}_1| > 1$ **do**
    if the optimal objective function value of problem (4.4.2) is positive **then**
        update the liability matrix via update scheme II$'$;
    else
        if the optimal objective function value of problem (4.4.3) is negative **then**
            update the liability matrix via update scheme III$'$;
        end if
        stop;
    end if
end while

generated by Algorithm 7.

Theorem 4.4.1. Let L^{BC} be the liability matrix obtained from Algorithm 7. Then, the sequence generated from Algorithm 7 converges to a stationary point of (4.4.3). Particularly, if Assumption 4.3.1 holds at the final solution and the optimal solution to problem (4.4.3) is unique and trivial with $\Delta L^* = 0$, then the solution provided by Algorithm 7 is locally optimal.

The proof of the above theorem follows a similar vein as that of Theorem 4.4.1, thus omitted here.

4.5 Numerical Results

In this section, we report some numerical results to assess the resilience/vulnerability of a financial network under the identified worst(best)-case structure. The section consists of two parts. In the first subsection, we study the contagious effect of failure under three network structures, i.e., the identified best-case, worst-case, and the one estimated based on the KL-divergence. In the second subsection, we compare the systemic loss generated in the financial network with the above-mentioned three structures under different scenarios of the asset vector.

4.5.1 Contagious Effect of Failure

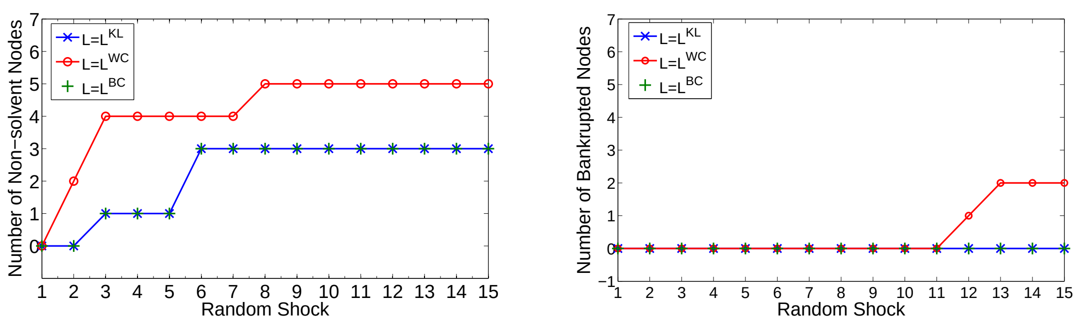
We first use an example to evaluate the contagious risk under the identified worst-case network structure, the best-case network structure and the one estimated based on the KL-divergence. For this, we consider the same data matrix as in Example 4.2.1. The asset vector is $\alpha^T = (160, 0, 0, 0, 1000, 1700, 1570, 1670)^T$ with $e^T\alpha = 6100$. Since the system is well-balanced, all the nodes are solvent. In this case, based on the results in Theorem 4.3.2 the stability of the system does not depend on its network structure.

To identify the least stable network structure in such a circumstance, we consider some specific scenario where a small negative shock is received by one of the three nodes (nodes 2,3 and 4) whose asset has zero value and a positive shock is received by one of the solvent nodes. We also assume that the total asset in the system remains invariant. Under this setting, we have fifteen different scenarios. Then, we use Algorithm 5 to find the worst-case structure. We observe that the identified worst-case structures are the same for all these fifteen scenarios which is given in Table 4.2. We also identify the best-case structure via Algorithm 7 for all fifteen scenarios. It also gives us the same network structure as shown in Table 4.3.

Table 4.3: The identified best-case liability matrix.

Node	1	2	3	4	5	6	7	8
1	0	5713.3	8922.9	10088.2	7057.2	6144.2	1189.7	2109.6
2	2495.8	0	10571.7	11965.3	8297.3	7167.4	1034.9	2076.6
3	2902.7	10469.5	0	31710.1	15096.8	11846.2	1274.3	5043.4
4	2990.6	11895.7	31715.5	0	18076.3	13655.7	1306.4	5879.8
5	2705.6	8243.7	15222.3	18216.6	0	9128.5	1158.7	3152.5
6	2616.3	7110.6	11910.6	13539.8	9300.3	0	1078.0	2386.4
7	6441.8	91.5	0	0	0	0	0	508.7
8	21072.2	84.8	0	0	0	0	0	0

To assess the effect of the network structure on the stability of the system, we solve problem (4.2.2) with randomly generated shocks s under three different structures. In our experiments, we use MATLAB function NORMRAND to generate fifteen random shocks satisfying $s_i \sim \mathcal{N}(0, \sigma^2), \sigma_i = 0.25\bar{p}_i \forall i = 1, \ldots, n,$. Figure 4.1(a) gives the number of default nodes and the number of bankrupted nodes in these scenarios.



((a)) The number of insolvent nodes in the system, when $L = L^{KL}$, $L = L^{WC}$, and $L = L^{BC}$.

((b)) The number of bankrupted nodes in the system, when $L = L^{KL}$, $L = L^{WC}$, and $L = L^{BC}$.

Figure 4.1: The horizontal axis shows fifteen different scenarios where a random shock triggered the system.

From Figure 4.1(a), one can see that the number of insolvent nodes under the identified worst-case structure is larger than that number under two other network structures. We have also observed bankruptcy in the system under the worst-case network structure for four scenarios. However, under the best-case structure and the KL case, we do not have any bankruptcy in the system. This implies that the system under the worst-case network structure is the least stable one.

To examine the contagious effect in the system with three different structures, we list the optimal solutions of (4.3.12) for three shocks in Table 4.4. As one can see, the shadow price under the best-case network structure is the smallest, while the shadow price under the worst-case network structure is the largest. This is consistent with our theoretical conclusion that the identified best-case structure is more vulnerable (than any other network structure) to market shocks.

Table 4.4: The shadow price when the financial system is subjected to shock vectors s under three different network structure (L^{KL}, L^{WC}, L^{BC}).

Optimal dual solution when $L = L^{KL}$

λ_1^*	λ_2^*	λ_3^*	λ_4^*	λ_5^*	λ_6^*	λ_7^*	λ_8^*	
0.00	1.18	1.23	1.22	0.00	0.00	0.00	0.00	$\alpha = \alpha^1$
2.54	2.53	2.42	2.39	0.00	0.00	0.00	0.00	$\alpha = \alpha^2$
2.54	2.53	2.42	2.39	0.00	0.00	0.00	0.00	$\alpha = \alpha^3$

Optimal dual solution when $L = L^{WC}$

λ_1^*	λ_2^*	λ_3^*	λ_4^*	λ_5^*	λ_6^*	λ_7^*	λ_8^*	
0.00	0.66	4.30	3.76	0.00	0.00	0.00	0.00	$\alpha = \alpha^1$
47.80	46.80	11.22	9.07	0.00	0.00	0.00	0.00	$\alpha = \alpha^2$
47.80	46.80	11.22	9.07	0.00	0.00	0.00	0.00	$\alpha = \alpha^3$

Optimal dual solution when $L = L^{BC}$

λ_1^*	λ_2^*	λ_3^*	λ_4^*	λ_5^*	λ_6^*	λ_7^*	λ_8^*	
0.00	0.00	1.00	0.00	0.00	0.00	0.00	0.00	$\alpha = \alpha^1$
2.38	2.32	2.31	2.26	0.00	0.00	0.00	0.00	$\alpha = \alpha^2$
2.38	2.32	2.31	2.26	0.00	0.00	0.00	0.00	$\alpha = \alpha^3$

To better characterize the vulnerability of the system under three different structures, we next consider a specific scenario corresponding to the asset vector ($\alpha = \alpha^{12}$). We estimate the maximum amount of negative shock ($s^-_{KL}, s^-_{WC}, s^-_{BC}$) that a node can survive without being bankrupted (Conclusion (i) in Theorem 4.2.1), the minimum amount of positive shock (or asset gain) ($s^+_{KL}, s^+_{WC}, s^+_{BC}$) via (Conclusion (ii) in Theorem 4.2.1) that a default node needs to become solvent. Note that, if the underling node i is solvent, then s^+_i refers to the maximal negative shock under which the solvent node can remain solvent. The results are summarized in Table 4.5.

Table 4.5: The maximum amount of negative shock and the minimum amount of positive shock estimated when $\alpha = \alpha^{12}$

	KL		WC		BC	
	s^-_{KL}	s^+_{KL}	s^-_{WC}	s^+_{WC}	s^-_{BC}	s^+_{BC}
1	-6100	1840	-132.9	1840	-6100	1840
2	-6100	448	-236.0	1840	-6100	456.31
3	-6100	732.06	-5015.4	1840	-6100	693.14
4	-6100	788.09	-5141.9	1840	-6100	744.63
5	-6100	-176.6	-6100	840	-6100	-182.03
6	-6100	-1040	-6100	-860	-6100	-1027.79
7	-6100	-2483.2	-6100	-2570	-6100	-2471.50
8	-6100	-2400.2	-6100	-2670	-6100	-2418.67

From Table 4.5, one can see that under the identified worst-case structure, the maximum amount of negative shock that financial institutions can survive is the least, while the minimum amount of positive shock (upper bound) for default institutions to become solvent under the identified worst-case structure is the largest.

4.5.2 The Systemic Loss in the Financial System

In this subsection, we evaluate the systemic losses generated in the system under on the identified best-case, worst-case structure and the one estimated based on the KL-divergence . For this, the system consisting of the banking sectors in eight European countries for December 2009 is considered. We consider the same data matrix and asset vector as in [21] (see Table 1 and Table 2 in [21]). In this example when we have $\alpha = \alpha(0)$ all the nodes in the system are solvent, and therefore we have $|\mathcal{I}_1| = 0$. In this case, from Theorem 4.3.2, we can conclude that by updating the liability matrix the stability of the system remains the same. Here we use the same procedure as described in the previous subsection to generate ten random shocks. Correspondingly, we denote the resulting asset vector for the generated scenarios by $\alpha(n), n = 1, \ldots, 10$. Next we solve problem (4.2.2) under the asset vector $(\alpha(n))$, and then use Algorithm 5 and 7 to find the worst-case and best-case liability matrix, respectively. The results show that at every scenario, the number of insolvent nodes under the worst-case structure is larger than the number under the most stable network structure. For every scenario, we also compute the total loss in the system under different network structure. The results are summarized in Figure 4.2.

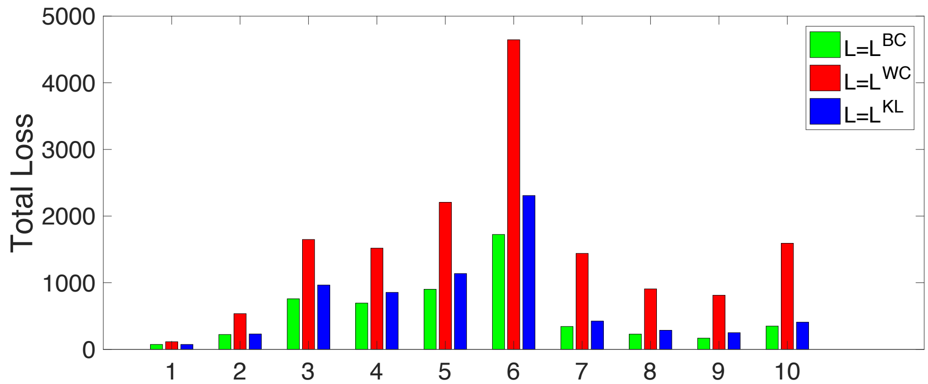


Figure 4.2: Systemic losses in the system.

As one can see from Figure 4.2 that for every scenario, the network under the least stable structure (L^{WC}) has the maximal systemic loss, while the network with the most stable structure has the minimal systemic loss.

4.6 Conclusion

In this paper, we study the issue of assessing systemic risk in a financial network. For this, we introduce two bi-level linear optimization models to identify the least stable and the most stable network structure, and develop two integrated approaches to solve these new optimization models respectively. Numerical experiment demonstrates that the contagious effect of failure is the most significant under the identified least stable network, while the total loss in the system is minimal under the identified most stable network.

Several issues are of interests for future research. For example, the integrated algorithm developed in this paper can only find a local optimal solution to the underlying optimization model. It will be interesting to design new effective global algorithms for the model, and characterize the network structure at the global solution and investigate the domino effect of bankruptcy in such a network. We also note that after the financial crisis in 2007-2008, new regulations have been implemented/enforced for the financial market. It will be interesting to incorporate the new regulations in the optimization model for risk assessment and develop new resolution techniques accordingly.

Chapter 5

Stabilizing Financial Networks via Merging

5.1 Introduction

A typical financial network is comprised of multiple financial institutions that interact with each other through borrowing and lending or some indirect interconnections through the market by holding similar shares or portfolios. The tight linkages within the financial networks has a two-fold effect. On the one hand, it improves the trading performance within the network and helps to diversify the risk. On the other hand, it also creates a channel through which the failure of some institution can quickly spill over to the entire system. This is usually referred as the so-called systemic risk. The Asian banking crisis in the late 1990s, and the more recent financial crisis in 2007-2008 are two pieces of the evidence of this disaster.

The catastrophic disaster of the above-mentioned financial crisis has caught a growing attention from different researchers and a large literature has been established in the study of systemic risk assessment, the contagion effect in the financial network, and the policies and strategies to mitigate the risk. In a seminal paper, Eisenberg and Noe [30] introduce the basic clearing agent model (E-N) to assess the systemic risk in financial networks. They propose a clearing algorithm, called Fictitious Default Algorithm to solve the basic clearing agent model. They also establishes the existence and uniqueness of the clearing payment vector. Elsinger et al. [33, 35] study the impact of market shocks on the stability of a financial system. Several different variants of the E-N model are proposed in the literature. These include Rogers and Veraart [65] and Glasserman and Young [44], where the authors extend the E-N model by taking the bankruptcy and liquidation costs into account. Several researchers study the stability and resilience of financial networks via exploring the impact of market shock and network structures on the stability of the financial system. Works in this direction include Acemoglu et al. [1], Glasserman and Young [44], and Khabazian and Peng [52].

Various policies/strategies have been proposed by different researchers to stabilize the financial system. For example, Pokuta et al. [64] extend the E-N model to identify optimal bail-out strategies to minimize the loss in the financial system under certain constraints on bail-out budget. Rogers and Veraart [65] discuss how to form a rescue consortium to help the failing banks in the system. They extend the E-N model by incorporating the liquidation cost for both the outside asset and inter-banks asset. They show that if there is no liquidation costs, then there is no incentive for solvent banks to rescue the insolvent banks in the system. Kallio and Khabazian [51] propose cooperative private bail-outs to stabilize the financial system. Bernard et al. [14] consider three intervention policies, bail-outs, bail-ins and subsidized bail-ins to stabilize a financial system. They analyze the impact of shock size, the size of the recovery rate, and the level of inter-bank connectivity.

In this paper, we propose a new mitigation strategy based on merging and acquisitions to stabilize a financial system. Here, the merger refers to the case when two financial institutions agree to merge into a single financial institution, and acquisition is the case where a financial institution buys all the shares of another institution which may be later merged with the buying institution. Subsequently, a merger may refer to an acquisition as well.

A critical issue in merging is how to measure the *merger gain*. In this work, we propose to quantify the *merger gain* of a merging pair in terms of the gap between the total loss of the merging pair (or a single bank when the bank is unmerged) before and after the merging. To compute the merger gain, we introduce a two-step procedure. We first consider the extended E-N model, which take the liquidation and bankruptcy cost into account similar to the models used in [65] and [44]. By solving the extended E-N model, we obtain the loss for every bank in the network. Moreover, we can divide the banks into two sets, the set of *'fundamentally insolvent banks'* with negative equity (book value) and its complement set, the set of *solvent banks.* Now let us consider a matching matrix ξ whose elements have values 0 or 1, where $\xi_{ij} = 1$ indicates an insolvent bank i will merge with another solvent bank j. For a given matching matrix, we obtain a new merger network consisting of the merged pairs corresponding to the nonzero elements of ξ, and some banks that do not involve in merging. The liabilities in the new merger network can be computed in a similar vein as the consortium in [65]. Like in [65], we also define the so-called *merging incentive* for a solvent bank to merge with another insolvent bank as the change in the equity of the solvent bank after the merging. Then, we resolve the extended E-N model for the merger network to compute the loss for all the merger pairs in the network. The total merger gain is computed by summing up the difference between the loss of a merger pair (or a single bank if it is unmerged) before and after the merging. Let us define the total merger gain as a function $g(\xi)$ of the matching matrix ξ. The major task of this work is to identify the matching matrix ξ^* whose merger gain attains the maximum.

Mathematically speaking, the problem of finding the optimal matching matrix (ξ^*) can be casted as a bi-level optimization where the subproblem involves solving the extended E-N model to evaluate $g(\xi)$ for a given matching matrix ξ. However, since the function $g(\xi)$ does not have an explicit form and is not continuously differentiable in general, solving such a bi-level optimization problem is rather challenging. As a remedy for such an issue, we consider a *subsidized merger* where the social planner (SP) provides some bail-out money b_{ij} to cover part of the liabilities of the insolvent bank and merging cost. We assume that with the subsidies from the social planner, all the banks in the new merger network are solvent. Such an assumption is reasonable if we restrict us to the network of systemically important banks, as the SP would like to rescue all these important banks. Note that for a given merging pair (i, j), the minimum subsidy it needs to become solvent and has incentive to rescue another insolvent bank can be computed explicitly and thus the merger gain can also be estimated easily. This allows us to develop a goal programming approach to identify the optimal matching matrix that minimizes the total subsidy and maximizes the merging gain, respectively. We explore conditions under which one can achieve the maximal gain with the minimal bail-out cost simultaneously. When these two goals can not be realized simultaneously, we present a new integer linear optimization model to maximize the merging gain under constraint on the bail-out budget, and develop a Lagrangian search method for it. We compare the public bail-out cost of the strategy proposed in our work with that of several other strategies in the literature using major European banks and a scenario linked to the adverse economic scenario used in 2016 EU-wide stress testing. Our numerical results demonstrate that our subsidized merger policy can significantly reduce the cost of other policies based on the public bail-out.

The major contributions of our work are as follows. First we extend the E-N clearing model for financial networks by taking into account bankruptcy and liquidation costs, and leverage ratio requirement. Second we propose a new way to estimate the merging gain of a merging pair in the merging process. Under the assumption that all the banks after merging become solvent, we give an explicit formulae to compute the merging gain for all the merging pairs in the network. Third, we introduce a goal programming approach to maximize the total merging gain and minimize the bail-out cost based on the estimated merging gains. We also explore the relationship between the optimal solutions of the two optimization models for the maximal merging gain and the minimal bail-out cost, respectively, and show that under certain conditions, the optimal solution of these two models can be obtained by solving only a single optimization model. Forth, we introduce a new integer linear optimization model (ILP) to manage the trade-off between the merging gain and the bail-out cost, and develop an effective Lagrangian search method for it.

Our approach is different from Khallio and Khabazian [51] and Bernard et al. [14] where they consider two-level problems where the SP first makes certain choices and the banks choose thereafter. In [51] the SP chooses the threat tax level applicable in case the cooperative private bail-outs fail, and the banks either choose to collaborate and agree on bail-out payments or not. In [14], the SP first proposes the amount of subsidies to the insolvent banks and suggests the way that solvent and insolvent banks should be matched.

Second, the banks individually can accept or reject SP's proposal. Finaly, depends on the outcome of the second stage, the SP may face additional choices. Bernard et al. [14] consider such three stage game in extensive form and the unique sub-game perfect equilibrium is considered as a solution of the model.

The paper is organized as follows. Section 5.2 introduces the clearing equilibrium and instruments for clearing payments. Section 5.3 introduces two optimization models to maximize the merging gain and minimize the bail-out cost respectively. An integer linear optimization model is introduced to maximize the merging gain subject to constraints on the bail-out cost and an effective algorithm for it is developed. Section 5.4 evaluates the performance of the proposed models using a network of major European banks. Section 5.5 concludes the paper by discussing future research topics.

5.2 The Clearing Agent Model

We first describe the linear optimization model introduced in [30]. Consider a financial network with n banks (represented by n nodes) interconnected to each other. A clearing agent is in charge of the process of settling the liabilities among these nodes. The ability of one node to settle its obligations depends on the repayment of other nodes to this node and also its own asset. Let $L \in \Re^{n\times n}$ be the interbank liability matrix where l_{ij} is the liability of node i toward node j. Since each nominal claim is nonnegative and no node has a nominal claim against itself, we have $l_{ij} \geq 0$ and $l_{ii} = 0$, $\forall i, j = 1, ..., n$. Let α be the exogenous asset and κ be the outside liability. The total liability of node i is equal to $p_i + \kappa_i$ where $p_i = \sum_{j=1}^{n} l_{ij}$, and the total receivables of bank i from other banks in the system is equal to $r_i = \sum_j l_{ji}$. In the balance sheet, the equity of bank i satisfies:

$$e_i = \alpha_i + r_i - p_i - \kappa_i.$$

We assume that the outside liabilities have seniority and they are cleared first, thus the available asset of bank i for the clearing can be obtained as $\hat{\alpha} = \alpha - \kappa$. Based on this, the payment made by node i to node j, i.e., $x_i l_{ij}$ is obtained by solving the following problem:

$$\begin{aligned} \min_{x} \quad & p^T(\vec{1} - x) \; and \\ s.t. \quad & (P - L^T)x \leq \hat{\alpha}; \\ & 0 \leq x \leq \vec{1}, \end{aligned} \tag{5.2.1}$$

where $\vec{1}$ is the all ones vector. Based on (5.2.1), the leverage ratio for node i is defined by

$$\gamma_i = \frac{[\hat{\alpha} - (P - L^T)x]_i}{(1 - x_i)p_i}.$$

We note that after financial crisis in 2007-2008, Basel III imposed a minimum leverage ratio requirement at 3% Basel. In July 2013, the U.S. Federal Reserve announced that the minimum leverage ratio would be 6% for eight systemically important financial institutions Basel III. By incorporating the minimum ratio requirement (denoted by γ) into model (5.2.1), we drive the following extended model:

$$\begin{aligned} \min_{x} \quad & p^T(\vec{1} - x) \; and \\ s.t. \quad & [(P - L^T)x]_i \leq \hat{\alpha}_i - \gamma(1 - x_i)p_i, \quad \forall i = \{1, \ldots, n\}; \\ & 0 \leq x \leq \vec{1}, \end{aligned} \tag{5.2.2}$$

where $0 < \gamma < 1$. We remark that the above model is very close to model by Glasserman and Young [44], indicating that Glasserman and Young's model implicitly imposes some minimum leverage ratio requirement.

Next, we extend model (5.2.2) by taking the loss factor (β) into account as in [65]. We consider that when institution i defaults in the system, only a fraction $(1 - \beta)$ of its asset, i.e., $(1 - \beta)\hat{\alpha}_i$ is available for the clearing. Based on this, we drive the following extended model:

$$\min_{x} \quad p^T(\vec{1} - x) \; and \tag{5.2.3}$$

$$\begin{aligned} s.t. \quad & [(P - L^T)x]_i \leq \hat{\alpha}_i - \gamma(1 - x_i)p_i - \beta\hat{\alpha}_i\mathbb{I}_{x_i<1}, \quad \forall i = \{1, \ldots, n\}; \\ & 0 \leq x \leq \vec{1}, \end{aligned}$$

where $\mathbb{I}_{x_i<1}$ is a binary indicator variable taking value 1 if $x_i < 1$ and value 0 otherwise.

Note that problem (5.2.2) is a mixed integer linear optimization problem. We next propose an iterative approach to solve problem (5.2.2). Let $\mathcal{I}$ denote the set of default nodes. We can rewrite problem (5.2.3) as the following problem:

$$\begin{aligned} \min_x \quad & p^T(\vec{1} - x) \; and \\ s.t. \quad & [(P - L^T)x]_i \leq \hat{\alpha}_i - \gamma(1 - x_i)p_i - \beta\hat{\alpha}_i\mathbb{I}_{i\in\mathcal{I}}, \quad \forall i = \{1, \ldots, n\}; \\ & 0 \leq x \leq \vec{1}, \end{aligned} \tag{5.2.4}$$

where $\mathbb{I}_{i\in\mathcal{I}}$ is a binary indicator variable taking value 1 if $i \in \mathcal{I}$ and value 0 otherwise. Note that if $\mathcal{I}$ is empty, then problem (5.2.4) reduces to problem (5.2.2). Now we are ready to describe the iterative procedure for problem (5.2.3).

Algorithm 8

```
Inputs: Liability matrix L, asset vector α̂, loss factor β and leverage ratio γ.
Set k = 0, I^k = ∅;
Find the optimal solution (denoted by x^(k)) to problem (5.2.4) with index set I^k;
Update k = k + 1, I^k = {i : x_i^(k−1) < 1};
while I^k ≠ I^(k−1) do
    Solve problem (5.2.4) with index set I^k;
    Set k = k + 1, I^k = {i : x_i^(k−1) < 1};
end while
Outputs: I* = I^k, x* = x^(k).
```

From the above procedure, one can obtain the following relation:

$$\mathcal{I}^k \subset \mathcal{I}^{k+1} \subset \{1, 2, \cdots, n\}, \quad \forall k = 1, 2, \cdots.$$

It follows immediately

Theorem 5.2.1. The optimal solution of (5.2.3) can be located by Algorithm 8 within at most n-iterations.

Given the optimal payment share vector (x^*) obtained from Algorithm 8, we can find the equity of node i after clearing (denoted by e_i^-) as

$$e_i^- = \hat{\alpha}_i + \sum_{j \neq i} l_{ji} x_j^* - p_i x_i^* - \gamma(1 - x_i^*) p_i - \beta \hat{\alpha}_i \mathbb{I}_{i \in \mathcal{I}}, \tag{5.2.5}$$

Using the optimal solution obtained from Algorithm 8, we can introduce the following index sets:

$$\mathcal{N} = \{1, \ldots, n\}, \quad \mathcal{I}^* = \{j : e_j^- = 0\}, \quad \mathcal{J}^* = \mathcal{N} - \mathcal{I}^*,$$

The set $\mathcal{I}^*$ consists of defaulting banks, and the set $\mathcal{J}^*$ consists of solvent banks.

5.3 Subsidized merging policies

In this section, we introduce two optimization models for matching solvent banks $\mathcal{J}^*$ and insolvent banks $\mathcal{I}^*$ to determine merging pairs. The section consists of two subsections. In the first subsection, we consider the merger gain to identify the merging banks. In the second subsection, we take into account the incentive of the solvent bank to help the insolvent one to justify the merger pair.

We first discuss how to estimate the merging gain. We assume that the merger pair (i, j) is associated with a merging cost denoted by c_{ij}. If solvent node i and insolvent node j are merged together, then cost of size c_{ij} occurs. Now we describe how to find the new asset vector and liability matrix after the merger of two banks in the system as follows.

Definition 5.3.1. Suppose banks $i \in \mathcal{J}^*$ and $j \in \mathcal{I}^*$ are merged into new bank denoted by ij. In this case, the new vector α^+ of outside assets is as follows: Given $k \neq i$ and $k \neq j$, we have $\alpha_k^+ = \hat{\alpha}_k$ and $\alpha_{ij}^+ = \hat{\alpha}_i + \hat{\alpha}_j - c_{ij}$. For the revised liability matrix L^+, we have $l_{ij,k}^+ = l_{ik} + l_{jk}$, $l_{k,ij}^+ = l_{ki} + l_{kj}$ and $l_{ij,ij}^+ = 0$, while the liabilities among non-merging banks remain unchanged and the liabilities between banks i and j are canceled.

5.3.1 Maximizing the merger gain and minimizing the bail-out cost

In this subsection, we study the case that the merger pair will benefit from the merger. Let consider the merger pair $(i,j) \in \hat{\mathcal{N}}$ such that $i = j$ or $i \in \mathcal{J}^*$ and $j \in \mathcal{I}^*$. The total loss for both nodes before the merging is $\ell_i + \ell_j = \sum_{k \in \mathcal{I}^*} l_{ki}(1 - x_k^*) - e_j^-$. Under the assumption that the merger pairs will become solvent after the merging, the total loss of the merger pair (i,j) can be obtained, i.e., $\ell_{ij}^+ = c_{ij}$. Based on this, we define the *subsidized merger gain* as the gap between the total loss of the merger pair before and after the merging, i.e.,

$$g_{ij} = \sum_{k \in \mathcal{I}^*} l_{ki}(1 - x_k^*) - e_j^- - c_{ij}. \tag{5.3.1}$$

Here we assume that if an insolvent bank j remains unmerged, then it will bankrupt and thus its equity value (e_j^-) equals zero.

We next describe a matching model. For all $(i,j) \in \hat{\mathcal{N}}$, let a binary variable ξ_{ij} be 1 if bank i is matched with bank j and 0 otherwise. Each bank is matched with another bank or with itself. If $\xi_{ii} = 1$, then i stays unmerged. Hence, we require

$$\sum_{i \in \mathcal{J}^*} \xi_{ij} = 1 \ \ \forall\, j \in \mathcal{I}^*, \ \textit{and} \tag{5.3.2}$$

$$\xi_{ii} + \sum_{j \in \mathcal{I}^*} \xi_{ij} = 1 \ \ \forall\, i \in \mathcal{J}^*. \tag{5.3.3}$$

Relaxing the integrality requirements for ξ_{ij} in (5.3.2)–(5.3.3) and maximizing the total merger gain leads to the linear programming problem of finding ξ_{ij}, for $(i,j) \in \hat{\mathcal{N}}$, to

$$\max \ \{ \sum_{(i,j) \in \hat{\mathcal{N}}} g_{ij}\xi_{ij} \mid (5.3.2),\ (5.3.3),\ \text{and } \xi_{ij} \geq 0 \ \forall\ (i,j) \in \hat{\mathcal{N}} \ \}. \tag{5.3.4}$$

Note that in certain cases, the subsidized merger gain g_{ij} may be negative due to the merging cost (c_{ij}) and thus the two banks may not be interested in merging. To encourage merging, the SP can provide some amount of bail-out b_{ij} to cover part of the liabilities of insolvent banks and merging cost. The minimal amount of bail-out to ensure a nonnegative merger gain can be computed as follows:

$$b_{ij} = \max\{0, -g_{ij}\}. \tag{5.3.5}$$

Correspondingly, we call $b_{ij} + g_{ij}$ *the subsidized merger gain.* We note that if $g_{ij} < 0$, both i and j would be better off without merger ij. If $g_{ij} > 0$, merger is possible but the subdivision of the excess g_{ij} among i and j is subject to negotiation. Based on this bail-out policy, we propose to solve the following optimization model to find the matching matrix:

$$\min \ \{ \sum_{(i,j)\in\hat{\mathcal{N}}} b_{ij}\xi_{ij} \mid (5.3.2),\ (5.3.3),\ \text{and } \xi_{ij} \geq 0 \ \forall \ (i,j) \in \hat{\mathcal{N}} \ \}. \tag{5.3.6}$$

Since both problems (5.3.4) and (5.3.6) have the same constraint set but with different objective functions, we can cast them as a typical goal programming. Note that from (5.3.5), we can easily verify that $b_{ij} = -g_{ij}$ if $g_{ij} \leq 0, \forall i, j \in \hat{\mathcal{N}}$. It follows immediately:

Theorem 5.3.1. Let ξ^g be the optimal solution to problem (5.3.4). Then the following conclusions hold:

(i) If $g_{ij} \leq 0, \forall (i,j) \in \hat{\mathcal{N}}$, then ξ^g is also the optimal solution to problem (5.3.6);

(ii) If $g_{ij} \geq 0$ for every meaningful matching pair (satisfying $\xi^g_{ij} = 1$), then ξ^g is also the optimal solution to problem (5.3.6).

It should be pointed out that in general, problem (5.3.4) and problem (5.3.6) may have different optimal solutions. In such cases, we are more interested in finding a solution that minimizes the total amount of bail-out subject to constraint that the total amount of gain is also close to the optimal value of (5.3.4). For this, we propose the following integer linear optimization problem:

$$\min \ \{ \sum_{(i,j)\in\hat{\mathcal{N}}} b_{ij}\xi_{ij} \ |\Delta \geq g^* - \sum_{ij} g_{ij}\xi_{ij},\ (5.3.3),\ (5.3.2),\ \text{and } \xi_{ij} \geq 0 \ \forall \ (i,j) \in \hat{\mathcal{N}} \ \} \tag{5.3.7}$$

Here ξ^g is the optimal solution of (5.3.4), g^* is the objective function value at the optimal solution of problem (5.3.4) ($g^* = \sum_{ij} g_{ij}\xi^g_{ij}$), and $\Delta > 0$ is a parameter indicating how close is the merging gain at the optimal solution of problem (5.3.7) to g^*. However, due to the new constraint on merging gain, problem (5.3.7) is nontrivial hard to solve. Next we develop a Lagrangian search method for problem (5.3.7). Let us consider the following parameterized Lagrangian problem:

$$\min \ \{ \ (1-\lambda)\sum_{ij} b_{ij}\xi_{ij} + \lambda(g^* - \sum_{ij} g_{ij}\xi_{ij}) \mid (5.3.3), \ (5.3.2), \ \text{and } \xi_{ij} \geq 0 \ \forall \ (i,j) \in \hat{\mathcal{N}} \ \} \tag{5.3.8}$$

where $\lambda \in [0,1]$ is a parameter. For each fixed $\lambda \in [0,1]$, problem (5.3.8) reduces to a standard matching problem that can be solved effectively by many existing optimization solvers. For a fixed Lagrangian multiplier λ, let $\xi(\lambda)$ denote the optimal solution of (5.3.8) and define $f_b(\lambda) = \sum_{ij} b_{ij}\xi(\lambda)_{ij}$, $f_g(\lambda) = g^* - \sum_{ij} g_{ij}\xi(\lambda)_{ij}$. We have

Theorem 5.3.2. The function $f_g(\lambda)$ is decreasing in terms of λ while the function $f_b(\lambda)$ is increasing in terms of λ.

Proof. Without loss of generality, we consider two different parameters $\lambda^1, \lambda^2 \in (0,1)$. Let $\xi(\lambda^1)$ and $\xi(\lambda^2)$ denote the optimal solution of (5.3.8) when $\lambda = \lambda^1$ and $\lambda = \lambda^2$ respectively. By the optimality of $\xi(\lambda^1)$ and $\xi(\lambda^2)$, we have that:

$$\begin{aligned}
\sum_{ij} b_{ij}\xi(\lambda^1)_{ij} + \frac{\lambda^1}{1-\lambda^1} f_g(\lambda^1) &\leq \sum_{ij} b_{ij}\xi(\lambda^2)ij + \frac{\lambda^1}{1-\lambda^1} f_g(\lambda^2); \\
\sum_{ij} b_{ij}\xi(\lambda^2)_{ij} + \frac{\lambda^2}{1-\lambda^2} f_g(\lambda^2) &\leq \sum_{ij} b_{ij}\xi(\lambda^1)_{ij} + \frac{\lambda^2}{1-\lambda 62} f_g(\lambda^1).
\end{aligned}$$

Adding the above two inequalities, we obtain:

$$\frac{\lambda^1}{1-\lambda^1} f_g(\lambda^1) + \frac{\lambda^2}{1-\lambda^2} f_g(\lambda^2) \leq \frac{\lambda^1}{1-\lambda^1} f_g(\lambda^2) + \frac{\lambda^2}{1-\lambda^2} f_g(\lambda^1),$$

which further yields

$$(\lambda^1 - \lambda^2)(f_g(\lambda^1) - f_g(\lambda^2)) \leq 0.$$

Therefore, $f_g(\lambda)$ is decreasing in terms of λ. The proof for $f_b(\lambda)$ follows similarly.

Based on Theorem 5.3.2, the optimal solution to problem (5.3.7) can be obtained via solving the following problem

$$\begin{aligned} \min \quad & \lambda \\ s.t. \quad & f_g(\lambda) \leq \Delta. \end{aligned} \tag{5.3.9}$$

One can easily see that when $\lambda = 0$, problem (5.3.8) reduces to problem (5.3.6) and we thus have $f_g(0) > \Delta$. If $\lambda = 1$, problem (5.3.8) reduces to problem (5.3.4) and we have $f_g(1) = 0 < \Delta$. By using Theorem 5.3.2 and the above observation, we present the following line search algorithm to solve problem (5.3.7).

Algorithm 9 A Binary Line Search Algorithm

Inputs: Δ, g^*, and a stop criteria ϵ.
Set $l = 0, u = 1$;
while $u - l > \epsilon$ **do**
    set $\lambda = (l + u)/2$;
    Solve problem (5.3.8);
    if $f_g(\lambda) \leq \Delta$ **then**
        Set $u = \lambda$,
    else
        Set $l = \lambda$,
    end if
end while
Output: $\lambda = (u + l)/2$.

5.3.2 Maximizing the merging incentive

In this subsection, we consider the case where the solvent banks will always benefit from the merger. Note that if $g_{ij} \geq 0$, then the insolvent bank j will always benefit from merging. However, for the solvent bank i, the high merging cost may make it unwilling to merge. To see this, let us consider the merging incentive (similar to the rescue incentive in [65]). Let

e_{ij}^{+} denote the equity of the merger after merging, i.e., $e_{ij}^{+} = e_i^{+} + e_j^{+}$. We define the merging incentive by

$$\delta_i = e_{ij}^{+} - e_i^{-} - c_{ij}. \tag{5.3.10}$$

From (5.3.1) we have that $\delta_i \leq g_{ij}$, if bank j is fundamentally default. This implies that even when the merging gain is positive, the merging incentive is negative and thus there is no incentive for bank i to rescue the insolvent bank j. To encourage bank i to merge with bank j, the minimal bail-out (b_{ij}^{inc}) must satisfy the following condition

$$e_{ij}^{+} - e_i^{-} \geq c_{ij} - b_{ij}^{inc}, \quad \text{or} \quad b_{ij}^{inc} = \max\{0, -\delta_i\}; \quad i \in \mathcal{I}^*, j \in \mathcal{J}^*. \tag{5.3.11}$$

Condition (5.3.11) ensures that the equity of the solvent node i will not decrease after the merging, and thus there is an incentive for bank i to merge with bank j. Correspondingly, we call $b_{ij} + \delta_i$ the *subsidized merging incentive.*

Next, we consider a variant of problem (5.3.4) that incorporates the merging incentive condition (5.3.11) as

$$\max \ \{ \ \sum_i \delta_i \xi_{ij} \mid (5.3.2),\ (5.3.3),\ \text{and}\ \xi_{ij} \geq 0\ \forall\ (i,j) \in \hat{\mathcal{N}} \ \}. \tag{5.3.12}$$

In this model, all the solvent banks have incentives to merge with some insolvent bank j.

Based on merging incentive condition, we can rewrite problem (5.3.6) as

$$\min \ \{ \sum_{(i,j)\in\hat{\mathcal{N}}} b_{ij}^{inc} \xi_{ij} \mid (5.3.2),\ (5.3.3),\ \text{and}\ \xi_{ij} \geq 0\ \forall\ (i,j) \in \hat{\mathcal{N}} \ \}. \tag{5.3.13}$$

Here b_{ij}^{inc} is defined by (5.3.11).

Problem (5.3.12) and (5.3.13) have the same feasible set but different objective function, and therefore we can cast them as a typical goal programming. Note that from (5.3.11), we can easily verify that $b_{ij}^{inc} = -\delta_{ij}$ if $\delta_{ij} \leq 0, \forall i, j \in \hat{\mathcal{N}}$. It follows immediately:

Theorem 5.3.3. Let ξ^δ be the optimal solution of problem (5.3.12). Then the following conclusions hold:

(i) If $\delta_{ij} \leq 0, \forall (i,j) \in \hat{\mathcal{N}}$, then ξ^δ is also the optimal solution to problem (5.3.13);

(ii) If $\delta_{ij} \geq 0$ for every meaningful matching pair (satisfying $\xi^\delta_{ij} = 1$), then ξ^δ is also the optimal solution to problem (5.3.13).

We note that in general problem (5.3.13), and (5.3.12) may have different optimal solutions. In such cases, we use the similar idea as in the previous subsection, and we propose to solve the following integer linear optimization model:

$$\min\{\sum_{(i,j)\in\hat{\mathcal{N}}} b^{inc}_{ij}\xi_{ij} \mid \Delta \geq \delta^* - \sum_{ij}\delta_{ij}\xi_{ij},\ (5.3.3),\ (5.3.2),\ \text{and } \xi_{ij} \geq 0\ \forall\ (i,j) \in \hat{\mathcal{N}}\}. \tag{5.3.14}$$

Here ξ^δ is the optimal solution of (5.3.11) and δ^* is the objective function value at the optimal solution of (5.3.11) ($\delta^* = \sum_{ij}\delta_{ij}\xi^\delta_{ij}$). $\Delta > 0$ is a parameter indicating how close is the merging incentive at the optimal solution of problem (5.3.11) to δ^*. As we mentioned, this problem is hard to solve. Therefore, we next develop a Lagrangian search method for problem (5.3.14). Let us consider the following parameterized Lagrangian problem:

$$\min\{(1-\lambda)\sum_{ij} b^{inc}_{ij}\xi_{ij} + \lambda(\delta^* - \sum_{ij}\delta_{ij}\xi_{ij}) \mid (5.3.3),\ (5.3.2),\ \text{and } \xi_{ij} \geq 0\ \forall\ (i,j) \in \hat{\mathcal{N}}\}. \tag{5.3.15}$$

Here $\lambda \in [0,1]$ is a parameter. Similarly, we can use the binary search algorithm to find the best value for λ.

5.4 A case study on European banks

In this section, similar to [51], we use banks listed by the EBA as Global Systemically Important Institutions (G-SIIs) to demonstrate our approach. The EBA data base provides the following data for 36 G-SIIs: total exposures, intra-financial system assets and intra-financial system liabilities, which we use as total assets, receivables from other banks and liabilities to other banks, respectively. For the equity (net value) of each bank we use Tier 1 capital[1] which is obtained from the EBA data base or from banks' annual reports of 2015. Consequently, we obtain the data for all the 36 banks in the data base. Like in [51], we also use 22 of these banks for the demonstration. The complete network structure is obtained from the information criterion by [70].

Table 5.1: Bank data for an adverse economic scenario. α_i = outside assets in the adverse scenario (bn); r_i = intra-network assets (bn); κ_i = outside liabilities (bn); p_i = intra-network liabilities (bn); e_i = equity before clearing (bn);

i	*bank*	α_i	κ_i	p_i	r_i	e_i
1	B_1	352.95	384.95	55.81	90.95	3.14
2	B_3	1106.46	1117.90	210.81	232.12	9.87
3	B_5	691.63	634.37	93.80	72.96	36.42
4	B_7	1673.93	1579.08	197.30	187.01	84.56
5	B_8	1033.68	916.45	127.63	93.18	82.78
6	B_9	371.27	368.14	134.09	127.02	-3.93
7	B_{10}	1189.41	1153.42	171.23	174.74	39.51
8	B_{11}	510.46	522.08	72.26	115.02	31.15
9	B_{12}	375.32	398.05	20.61	58.27	13.75
10	B_{13}	1167.22	1127.87	219.49	237.52	57.38
11	B_{18}	2305.71	2169.08	260.19	225.94	102.38
12	B_{19}	931.39	912.22	131.32	148.36	36.21
13	B_{20}	476.49	477.85	70.71	127.36	55.28
14	B_{22}	300.47	289.94	22.46	23.60	11.67
15	B_{24}	818.65	832.34	84.30	69.21	-28.78
16	B_{27}	465.32	495.47	47.93	98.49	20.41
17	B_{29}	642.64	607.80	46.97	27.32	15.19
18	B_{30}	718.34	749.24	167.16	155.84	-42.23
19	B_{31}	1233.11	1159.70	148.26	95.92	21.07
20	B_{32}	274.47	253.20	41.21	33.38	13.43
21	B_{33}	1009.79	1012.04	174.65	151.30	-25.60
22	B_{36}	752.16	717.82	202.33	155.00	-12.99
total		18400.87	17879.02	2700.50	2700.50	520.67

We note that using the available data for the outside assets no banks default. To have some default banks in the system, we randomize the outside assets by following a log-normal distribution as in [51]. Table **??** shows the resulting outside assets α_i, outside liabilities κ_i, total liabilities p_i, intra-network assets r_i and equity levels before clearing e_i, assuming that total liabilities p_i and intra-network assets r_i remain at the base case levels at the end of 2015. Under such a setting, we have 5 fundamentally defaulting banks (banks $6, 15, 18, 21$ and 22 with negative equity ($e_i < 0$)) and 17 solvent banks with positive equity ($e_i > 0$).

[1] Tier 1 capital is the core measure of a bank's financial strength from a regulator's point of view. It is composed primarily of common stock and disclosed reserves (or retained earnings).

Based on this, we can calculate the minimal bail-out cost to ensure all the banks in the system are solvent as

$$b^* = -\sum_{i:e_i<0} e_i = 113.53.$$

For convenience, we call it the cost of public bail-out.

5.4.1 Clearing without merging

Next, we use Algorithm 8 to find the clearing payment ratio x^* under generated random outside asset α_i, when $\gamma = 0.03$ and for five different value of $\beta = 0.1, 0.2, 0.3, 0.4, 0.5$. We assume that the all the banks clear their outside liabilities first. Based on this, the available asset of each back for the clearing can be obtained as $\alpha_i - \kappa_i$. As one can see from Table **??**, in all five different cases, we have 7 default banks including 5 fundamentally defaulting banks with $e_i < 0$ and 2 contagiously defaulting banks i (banks 1 and 2) with $e_i > 0$ and $x_i^* < 1$.

5.4.2 Subsidized merging cases

Given lack of data for the merger cost c_{ij}, for demonstration we use $c_{ij} = \epsilon_{ij}\sqrt{(\alpha_i + \alpha_j)}$, where parameters ϵ_{ij} were drawn independently from the uniform distribution ($\epsilon_{ij} \sim \mathcal{U}(-1, 1)$). Then we solve problem (5.3.4) to find the best matching matrix ξ^g. In this subsection, we only consider the case that $\beta = 0.1$ for illustration. The results for the case that $\beta = 0.2, 0.3, 0.4, 0.5$ are given in Appendix B.1.

Table **??** shows the merger gain obtained from (5.3.1) among all the possible merger pairs for the case that $\beta = 0.1$. As one can see from Table **??**, the merger gains for all the merging pairs at the optimal solution are positive, and therefore the minimal amount of bail-out in this case is zero. This illustrates that the merging strategy can significantly reduce the contagion risk.

Next, we estimate the merging incentive (δ) for a solvent bank (i) when merged with a default bank (j) as listed in Table **??**. As one can see from Table **??**, for insolvent banks 15, 18, 21, and 22, the merging incentives for all the solvent banks are negative, which indicates

Table 5.2: The clearing solutions based on Algorithm 8 with $\gamma = 0.03$. x_i^* = share of liabilities repaid; ℓ_i = loss of bank i without merging (bn).

	$\beta = 0.1$		$\beta = 0.2$		$\beta = 0.3$		$\beta = 0.4$		$\beta = 0.5$	
i	x_i^*	ℓ_i	x_i^*	ℓ_i	x_i^*	ℓ_i	x_i^*	ℓ_i	x_i^*	ℓ_i
1	0.89	-3.14	0.82	-3.14	0.75	-3.14	0.68	-3.14	0.61	-3.14
2	0.96	-9.87	0.95	-9.87	0.93	-9.87	0.92	-9.87	0.91	-9.87
3	1.00	4.91	1.00	5.47	1.00	6.02	1.00	6.57	1.00	7.13
4	1.00	13.14	1.00	14.63	1.00	16.11	1.00	17.59	1.00	19.08
5	1.00	6.36	1.00	7.07	1.00	7.79	1.00	8.51	1.00	9.23
6	0.90	3.93	0.89	3.93	0.88	3.93	0.87	3.93	0.86	3.93
7	1.00	12.15	1.00	13.52	1.00	14.89	1.00	16.27	1.00	17.63
8	1.00	7.69	1.00	8.55	1.00	9.42	1.00	10.29	1.00	11.16
9	1.00	3.82	1.00	4.25	1.00	4.68	1.00	5.11	1.00	5.54
10	1.00	16.88	1.00	18.78	1.00	20.69	1.00	22.59	1.00	24.48
11	1.00	16.33	1.00	18.17	1.00	20.02	1.00	21.84	1.00	23.69
12	1.00	10.15	1.00	11.29	1.00	12.44	1.00	13.58	1.00	14.73
13	1.00	8.51	1.00	9.47	1.00	10.43	1.00	11.39	1.00	12.34
14	1.00	1.55	1.00	1.72	1.00	1.90	1.00	2.07	1.00	2.25
15	0.58	28.78	0.56	28.78	0.54	28.78	0.51	28.78	0.49	28.78
16	1.00	6.52	1.00	7.26	1.00	8.00	1.00	8.73	1.00	9.47
17	1.00	1.81	1.00	2.01	1.00	2.21	1.00	2.42	1.00	2.62
18	0.67	42.22	0.65	42.22	0.62	42.22	0.59	42.22	0.57	42.22
19	1.00	6.60	1.00	7.34	1.00	8.09	1.00	8.83	1.00	9.57
20	1.00	2.21	1.00	2.45	1.00	2.70	1.00	2.95	1.00	3.20
21	0.79	25.59	0.79	25.59	0.78	25.59	0.77	25.59	0.76	25.59
22	0.86	12.99	0.85	12.99	0.82	12.99	0.80	12.99	0.77	12.99
Loss		219.12		232.50		245.90		259.26		272.62

that no solvent banks are willing to merge with those insolvent banks unless some subsidies are provided from the social planner. We also solve problem (5.3.12) to find the best matching matrix ξ^δ. Since in model (5.3.12), all the insolvent banks must be merged with one of the solvent banks in the system, and therefore, at the optimal solution of (5.3.12), we obtain a negative total incentive for the identified matching pairs. This implies that such a merging plan is not attractive for the solvent banks. To ensure all the involved solvent banks have nonnegative incentives, we solve problem (5.3.13) to obtain the merging pairs with minimal bail-out cost.

Table 5.3: The optimal solution of model (5.3.4).

i \ j	g_{ij} 1	2	6	15	18	21	22
3	-0.44	2.70	4.10	-1.95	-4.40	-1.58	0.70
4	6.88	-0.97	**10.00**	9.88	6.62	-5.84	-0.88
5	-12.07	1.89	-4.22	4.63	-4.53	0.05	2.63
7	8.53	5.34	4.63	7.27	10.60	9.80	**9.69**
8	7.69	7.69	7.69	**7.69**	7.69	7.69	-0.46
9	3.82	3.82	3.82	3.82	3.82	3.82	2.43
10	**16.74**	6.99	5.11	12.44	14.35	15.12	8.57
11	2.36	**14.74**	8.16	-5.11	1.59	8.23	-1.81
12	10.15	9.36	6.61	9.53	**10.15**	2.49	-3.33
13	8.51	8.51	8.35	8.51	8.51	**8.51**	6.14
14	1.55	1.55	-3.92	1.55	1.55	-4.15	-2.34
16	6.52	6.52	6.52	6.52	6.52	6.52	3.70
17	-1.14	0.54	-8.99	-6.33	-0.43	-8.00	1.58
19	-5.86	-1.18	-7.13	-1.28	0.47	2.12	0.74
20	2.21	0.37	-7.41	2.11	2.21	0.11	-3.51

Table 5.5: The optimal solution of model (5.3.13)

i \ j	b_{ij}^{inc} 1	2	6	15	18	21	22
3	0.00	**0.00**	0.00	30.72	46.62	27.18	12.28
4	0.00	0.00	0.00	**18.91**	35.62	31.45	13.89
5	8.95	0.00	8.16	24.16	46.77	25.57	10.38
7	0.00	0.00	0.00	21.50	31.64	15.80	**3.30**
8	0.00	0.00	0.00	21.08	34.54	17.91	13.45
9	0.00	0.00	1.29	26.14	39.59	22.96	11.74
10	0.00	0.00	0.00	16.33	27.88	**10.47**	4.42
11	0.00	0.00	0.00	33.90	40.66	17.38	14.82
12	0.00	0.00	0.00	19.25	**32.09**	23.11	16.33
13	0.00	0.00	0.00	20.27	33.73	17.10	6.86
14	0.00	0.00	7.85	27.23	40.68	29.75	15.33
16	0.00	0.00	**0.00**	22.26	35.71	19.08	9.29
17	0.00	0.00	12.92	35.11	42.66	33.59	11.41
19	2.73	0.00	11.07	30.06	41.76	23.49	12.25
20	**0.00**	0.00	11.34	26.67	40.02	25.48	16.49

Table **??** shows the resulting minimal amount of bail-out (b^{inc}). As one can see from Table **??** and **??**, the optimal matching pair obtained from two models (5.3.12) and (5.3.13) are different. As a compromise, we then solve problem (5.3.15) to find a solution that minimizes the total amount of bail-out subject to constraint on the total amount of incentives. In our example, we use $\Delta = 0.5$. The optimal matching pairs for all four models when $\beta = 0.1$ are given in Table **??**.

Table 5.4: The optimal solution of model (5.3.12)

i \ j	δ_{ij}						
	1	2	6	15	18	21	22
3	2.70	12.57	0.17	-30.72	-46.62	-27.18	-12.28
4	10.00	8.88	**6.06**	-18.91	-35.62	-31.45	-13.89
5	-8.95	11.74	-8.16	-24.16	-46.77	-25.57	-10.38
7	11.67	15.21	0.70	-21.50	-31.64	-15.80	**-3.30**
8	10.83	17.56	3.76	**-21.08**	-34.54	-17.91	-13.45
9	5.77	12.50	-1.29	-26.14	-39.59	-22.96	-11.74
10	**19.88**	16.86	1.18	-16.33	-27.88	-10.47	-4.42
11	5.48	**24.59**	4.21	-33.90	-40.66	-17.38	-14.82
12	13.28	19.22	2.68	-19.25	**-32.09**	-23.11	-16.33
13	11.64	18.37	4.42	-20.27	-33.73	**-17.10**	-6.86
14	4.69	11.42	-7.85	-27.23	-40.68	-29.75	-15.33
16	9.66	16.39	2.59	-22.26	-35.71	-19.08	-9.29
17	2.00	10.40	-12.92	-35.11	-42.66	-33.59	-11.41
19	-2.73	8.69	-11.07	-30.06	-41.76	-23.49	-12.25
20	5.34	10.24	-11.34	-26.67	-40.02	-25.48	-16.49

Table 5.6: The merging pairs obtained from four different problems when $\beta = 0.1$

Model	Total gain	Total incentive	Bail-out with incentive	Bail-out without incentive	Merging Pairs
Model (5.3.4)	77.50	-23.04	73.57	0.00	(4,6),(7,22),(8,15),(10,1),(11,2),(12,18),(13,21)
Model (5.3.12)	77.50	-23.04	73.57	0.00	(4,6),(7,22),(8,15),(10,1),(11,2),(12,18),(13,21)
Model (5.3.13)	56.26	-44.27	64.77	0.00	(3,2),(4,15),(7,22),(10,21),(12,18),(16,6),(20,1)
Model (5.3.15)	77 .38	-23.17	71.40	0.00	(4,15),(7,22),(8,6),(10,1),(11,2),(12,18),(13,21)

As one can see from Table **??**, in this case the optimal solution from model (5.3.4) and (5.3.13) are the same. The solution from model (5.3.4) achieves the maximal merger gain with the maximal bail-out cost, while the solution from model (5.3.13) has the minimal bail-out cost and the minimal (negative) merger gain. The solution from model (5.3.15) reaches a good balance between the merger gain and the bail-out cost. We also would like to point out that the bail-out cost in all the above four cases is smaller than the cost of the public bail-out. Particularly, if we do not take the incentives for the solvent banks into account in the model such as in model (5.3.4), then the corresponding bail-out cost can be very small.

5.5 Concluding discussion

In this paper, we have developed various models to identify optimal policies for the subsidized mergers to stabilize the financial network. For this, we first extend the Glassermann-Young model to estimate the merger gain for any given merging pair in the network. Based on the computed merger gains, we develop a goal programming approach to find the matching pairs with the maximal gain and minimal bail-out cost respectively. We identify conditions under which the maximal gain and the minimal bail-out can be obtained simultaneously. We also develop similar models to maximize the incentives for the solvent banks involved in merging, and minimize the bail-out used to subsidize the solvent banks, and discuss how to manage the trade-off between the incentive and the bail-out cost. We test the proposed models on a network of major European banks and a scenario linked to the adverse economic scenario used in 2016 EU-wide stress testing. Our numerical experiments demonstrate that the proposed subsidized merging can significantly reduce the bail-out cost compared with the cost of the public bail-out.

Several issues are of interests for future research. The first issue is how the proposed merging strategy can be implemented in practice. One possible way is that the SP first provides some recommendations (without considering the merging incentive) for the banks to merge. Then the banks can make their choices whether to merge with another bank or not. If they do not accept the recommendations, the SP will provide some subsidies by taking into account the merging incentive or they provide public bail-out, and then the banks can choose among these options. We also point out that a similar approach is proposed in [14]. The second issue is that the merging may lead to the so-called “too big too fail" risk . Further study is needed to investigate how to avoid such a risk.

Chapter 6

Future work

In this chapter, we explain future studies in order to extend and improve the current results of this dissertation.

6.1 Studying the Systemic Risk Assessment by Taking into Account the Market Impact on the Asset Liquidation

For the case that a bank has to liquidate some of its assets to pay its liabilities, there is usually a loss caused in the liquidation process. Note that in such a case, the bank itself may be solvent after the liquidation. However, it still incurs a liquidation loss. Moreover, based on works in asset liquidation (see Chen et al. [23] and the references therein), the liquidation loss is usually nonlinear in terms of the amount of liquidated assets. This is very different from the literature in systemic risk study [44, 65] where linear functions are adopted to measure the loss, which implies the market impact on the liquidation has been neglected. More study is needed to address such an issue.

6.2 Studying How to Avoid "Too Big to Fail" Risk

The merging policy that we proposed in Section 5 may lead to the so-called "too big to fail" risk. It will be interesting to investigate how to avoid such a risk.

6.3 Analyzing the Vulnerability of Financial Network with Cross-holding

We can extend the results in Chapter 3 and 4 by considering a network of banks which are linked with each other by financial obligations and cross holdings. Cross holding is a situation where a financial institution owns shares or stock in another institution which leads to a well-known problem of inflating book values. We can do some analysis to see how the presence of cross-holding will impact the size of risk exposure and what is the best mitigation strategy to reduce such negative effect.

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Appendix A

Supplement for the Results in Section 4

A.1 Proof of Theorem 4.3.4

We first present a technical result.

Lemma A.1.1. Let $\hat{\lambda}^i$ be the optimal solution of (4.3.17). Then for every $j \in \mathcal{I}_1$, $\hat{\lambda}^i_j$ is also the optimal solution of the following linear optimization problem:

$$\begin{aligned} \max_{\lambda_{\mathcal{I}_1}} \quad & \lambda_j \\ s.t. \quad & (P-L)_{\mathcal{I}_1}\lambda_{\mathcal{I}_1} \leq p_i e_i, \quad \lambda_{\mathcal{I}_1} \geq 0; \end{aligned} \tag{A.1.1}$$

Proof. From (4.3.18) we can conclude that $\hat{\lambda}^i_{\mathcal{I}_1}$ is a solution of the following linear equation system.

$$(P-L)_{\mathcal{I}_1}\hat{\lambda}^i_{\mathcal{I}_1} = p_i e_i.$$

Now, let us consider the following optimization problem:

$$\max_{\lambda_{\mathcal{I}_1}} \quad \sum_{j \in \mathcal{I}_1} \lambda_j, and \tag{A.1.2}$$

$$s.t. \quad (P-L)_{\mathcal{I}_1}\lambda_{\mathcal{I}_1} \leq p_i e_i; \tag{A.1.3}$$

$$\lambda_{\mathcal{I}_1} \geq 0.$$

We next show that at the optimal solution of (A.1.2) constraint (A.1.3) is active for all $i \in \mathcal{I}_1$. Suppose to the contrary that at the optimal solution of (A.1.2), constraint (A.1.3) is not active for all $i \in \mathcal{I}_1$. Here, we have two cases:

(i) Constraint (A.1.3) is not active for $i \in \mathcal{I}_1$;

(ii) Constraint (A.1.3) is not active for $j \neq i \in \mathcal{I}_1$.

We first consider the first case. Thus, we have

$$[(P-L)_{\mathcal{I}_1}\hat{\lambda}^i_{\mathcal{I}_1}]_i < p_i.$$

From (4.3.17) we can see that for arbitrary $\epsilon > 0$, it holds

$$\epsilon(\alpha^T - e^T(P-L))\hat{\lambda}^i_{\mathcal{I}_1} < 0, \quad \epsilon(P-L)\hat{\lambda}^i_{\mathcal{I}_1} \leq \epsilon p_i e_i.$$

By choosing a sufficiently small ϵ satisfying

$$\epsilon[(P-L)\hat{\lambda}^i_{\mathcal{I}_1}]_i \leq \epsilon p_i \leq (1+\epsilon)[\Delta L^* \hat{\lambda}^i_{\mathcal{I}_1}]_i,$$

we have that

$$[(P-L^+)(1+\epsilon)\hat{\lambda}^i_{\mathcal{I}_1}]_i \leq p_i.$$

Therefore, $(1+\epsilon)\hat{\lambda}^i_{\mathcal{I}_1}$ is a feasible solution for problem (4.3.17) with liability matrix L^+. One can see the fact that $\sum_{j\in\mathcal{I}_1}(1+\epsilon)\hat{\lambda}^i_j > \sum_{j\in\mathcal{I}_1}\hat{\lambda}^i_j$ contradicts to the assumption that $\hat{\lambda}^i$ is the optimal solution of (A.1.2). Next we consider case (ii). In this case, we have

$$[(P-L)_{\mathcal{I}_1}\hat{\lambda}^i_{\mathcal{I}_1}]_j < 0.$$

We can choose a vector $\hat{\lambda}^{\epsilon_i}_{\mathcal{I}_1}$ whose elements are defined by

$$\hat{\lambda}^{\epsilon_i}_k = \hat{\lambda}^i_k, \quad \forall k \neq j \in \mathcal{I}_1, \hat{\lambda}^{\epsilon_i}_j = \hat{\lambda}^i_j + \epsilon,$$

where ϵ is sufficiently small to ensure that

$$[(P-L)_{\mathcal{I}_1}\hat{\lambda}^{\epsilon_i}_{\mathcal{I}_1}]_j \leq 0.$$

Since $\hat{\lambda}^{\epsilon_i}_{\mathcal{I}_1} > \hat{\lambda}^{i}_{\mathcal{I}_1}$, we have

$$[(P-L)_{\mathcal{I}_1}\hat{\lambda}^{\epsilon_i}_{\mathcal{I}_1}]_k - (P-L)_{\mathcal{I}_1}\hat{\lambda}^{i}_{\mathcal{I}_1}]_k \leq 0, \forall k \neq j.$$

From the above relations we can conclude that $\hat{\lambda}^{\epsilon_i}_{\mathcal{I}_1}$ is a feasible solution for problem (4.3.17). One can see the fact that $\sum_{j\in\mathcal{I}_1}\hat{\lambda}^{\epsilon_i}_j > \sum_{j\in\mathcal{I}_1}\hat{\lambda}^{i}_j$ contradicts to the assumption that $\hat{\lambda}^i$ is the optimal solution of (A.1.2). Therefore, at the optimal solution of (A.1.2) constraint (A.1.3) is active for all $j \in \mathcal{I}_1$.

Let $\bar{\lambda}_{\mathcal{I}_1}$ be a vector whose i-element $\bar{\lambda}_i$ be the objective function value at the optimal solution of problem (A.1.1). Let $\hat{\lambda}^i$ be the optimal solution of (A.1.1) and thus, we have $\bar{\lambda}_i = \hat{\lambda}^i_i$. Then we have:

$$p_i\bar{\lambda}_i - \sum_{j\neq i} l_{ij}\hat{\lambda}^i_j = p_i\hat{\lambda}^i_i - \sum_{j\neq i} l_{ij}\hat{\lambda}^i_i \leq p_i,$$

which implies that

$$p_i\bar{\lambda}_i - \sum_{j\neq i} l_{ij}\bar{\lambda}_j \leq p_i.$$

This shows that $\bar{\lambda}_{\mathcal{I}_1}$ is the unique solution to problem (A.1.1). This completes the proof of the lemma.

Next we prove Theorem 4.3.1. We start with the first conclusion. Let us consider decomposed problem (A.1.1). In this case, from feasibility condition (4.3.20) we have that

$$(P - L^+)\hat{\lambda}^i(L) \leq p_i e_i,$$

which implies that $\hat{\lambda}^i(L)$ is the feasible solution of (A.1.1) with liability matrix L^+. It follows immediately

$$\hat{\lambda}^i(L^+) \geq \hat{\lambda}^i(L), \quad \forall i \in \mathcal{I},$$

where $\hat{\lambda}^i(L)$ and $\hat{\lambda}^i(L^+)$ denote the optimal solution of (4.3.17) with liability matrix L and L^+, respectively. Since $e^T_{\mathcal{I}_1}\Delta L^*\lambda^* > 0$, we can conclude that $\exists i \in \mathcal{I}_1$ such that $[\Delta L^*\lambda^*]_i > 0$.

Based on this, following the similar vein as in the proof of Lemma A.1.1, we can conclude that $\exists i \in \mathcal{I}_1$ such that the following holds.

$$\hat{\lambda}^i_j(L^+) > \hat{\lambda}^i_j(L), \quad j \in \mathcal{I}_1. \tag{A.1.4}$$

Since $\lambda^* = \sum_{i\in\mathcal{I}} \hat{\lambda}^i$, we have that

$$\lambda^*_j(L^+) > \lambda^*_j(L), \quad j \in \mathcal{I}_1.$$

Next we consider the second conclusion. Since $\hat{\lambda}^i(L)$ is the feasible solution of (4.3.17), it follows immediately that

$$(\alpha^T - e^T(P-L))\hat{\lambda}^i(L^+) + p_i \le (\alpha^T - e^T(P-L))\hat{\lambda}^i(L) + p_i, \quad \forall i \in \mathcal{I}.$$

Based on duality theorem Bazaraa, we have that

$$x^*_i(L^+) \le x^*_i(L), \quad \forall i \in \mathcal{I}.$$

Now it suffices to show that strict inequality holds for insolvent nodes. Similarly, since we have $e^T_{\mathcal{I}_1}\Delta L^*\lambda^* > 0$, we can conclude that $\exists i \in \mathcal{I}_1$ such that $[\Delta L^*\lambda^*]_i > 0$. Therefore, inequality (A.1.4) holds. Based on this, and from Lemma A.1.1 we have

$$(\alpha^T - e^T(P-L))\hat{\lambda}^i(L^+) + p_i < (\alpha^T - e^T(P-L))\hat{\lambda}^i(L) + p_i. \tag{A.1.5}$$

Thus, based on duality theorem Bazaraa we can conclude that $x^*_i(L^+) < x^*_i(L)$. This finishes the proof of the theorem.

A.2 Estimating the Directional Derivative of f

In this section, we provide more details on how to estimate the directional derivative $(\mathcal{D}_{\Delta L}f(L))$. We follow a similar vein as in [37] where they estimate the directional derivative of a clearing payment vector with respect to a perturbation matrix. Let us consider the equality constraint as

$$(P - L - \beta \Delta L)_{\mathcal{I}_{11}} \lambda_{\mathcal{I}_1}(L^+) = p_{\mathcal{I}_1},$$

where $\lambda_{\mathcal{I}_1}(L^+)$ is the optimal solution of (4.3.12) when the liability matrix is L^+ and $\beta \in (0, 1]$. Since $(P - L)_{\mathcal{I}_{11}}$ is nonsingular, we can multiply each side of the equality by $(P - L)^{-1}_{\mathcal{I}_{11}}$ and obtain $\lambda_{\mathcal{I}_1}(L^+)$ as

$$\lambda_{\mathcal{I}_1}(L^+) = \beta (P - L)^{-1}_{\mathcal{I}_{11}} \Delta L_{\mathcal{I}_{11}} \lambda_{\mathcal{I}_1}(L^+) + (P - L)^{-1}_{\mathcal{I}_{11}} p_{\mathcal{I}_1}. \tag{A.2.1}$$

Based on this, we have

$$\lambda_{\mathcal{I}_1}(L) = (P - L)^{-1}_{\mathcal{I}_{11}} p_{\mathcal{I}_1}. \tag{A.2.2}$$

From the feasibility condition we have that $e^T \Delta L = 0$. Thus,

$$(\alpha^T - e^T(P - L^+))_{\mathcal{I}_1} = (\alpha^T - e^T(P - L))_{\mathcal{I}_1}. \tag{A.2.3}$$

From (A.2.1), (A.2.2), and (A.2.3) we can obtain the following.

$$\begin{aligned} \mathcal{D}_{\Delta L} f(L) \quad &= (\alpha^T - e^T(P - L))_{\mathcal{I}_1} (P - L)^{-1}_{\mathcal{I}_{11}} \Delta L_{\mathcal{I}_{11}} \lambda_{\mathcal{I}_1}(L); \qquad \text{(A.2.4)} \\ &= (\alpha^T - e^T(P - L))_{\mathcal{I}_1} (P - L)^{-1}_{\mathcal{I}_{11}} \Delta L_{\mathcal{I}_{11}} (P - L)^{-1}_{\mathcal{I}_{11}} p_{\mathcal{I}_1}. \end{aligned}$$

Appendix B

Supplement for Numerical Experiment in Section 5

B.1 Numerical Results for Section 5

Table B.1: The merging pairs obtained from four different problems when $\beta = 0.2$

Model	Total gain	Total incentive	Bail-out with incentive	Bail-out without incentive	Merging Pairs
Model (5.3.4)	82.81	-17.68	59.70	0.00	(4,15),(7,18),(8,6),(10,1),(11,21),(12,22),(13,2)
Model (5.3.12)	82.81	-17.68	59.70	0.00	(4,15),(7,18),(8,6),(10,1),(11,21),(12,22),(13,2)
Model (5.3.13)	62.75	-37.74	56.32	0.00	(3,6),(4,2),(7,15),(10,18),(11,21),(12,22),(20,1)
Model (5.3.15)	81.98	-18.51	56.46	0.00	(4,15),(7,2),(8,6),(10,18),(11,21),(12,22),(13,1)

Table B.2: The merging pairs obtained from four different problems when $\beta = 0.3$

Model	Total gain	Total incentive	Bail-out with incentive	Bail-out without incentive	Merging Pairs
Model (5.3.4)	100.55	0.05	50.88	0.00	(4,15),(7,22),(8,6),(10,18),(11,1),(12,2),(13,21)
Model (5.3.12)	100.55	0.05	50.88	0.00	(4,15),(7,22),(8,6),(10,18),(11,1),(12,2),(13,21)
Model (5.3.13)	66.49	-34.03	48.32	2.88	(3,6),(4,15),(7,22),(10,21),(11.18),(14,1),(20,2)
Model (5.3.15)	94.13	-6.37	48.32	0.00	(4,15),(7,22),(8,6),(10,21),(11,18),(12,1),(13,2)

Table B.3: The merging pairs obtained from four different problems when $\beta = 0.4$

Model	Total gain	Total incentive	Bail-out with incentive	Bail-out without incentive	Merging Pairs
Model (5.3.4)	107.02	6.48	57.17	0.00	(4,21),(7,18),(8,2),(10,6),(11,1),(12,22),(13,15)
Model (5.3.12)	107.02	6.48	57.17	0.00	(4,21),(7,18),(8,2),(10,6),(11,1),(12,22),(13,15)
Model (5.3.13)	75.39	-25.17	51.15	0.00	(3,1),(4,21),(7,18),(8,6),(10,15),(13,22),(20,2)
Model (5.3.15)	102.30	1.76	51.19	0.00	(4,21),(7,6),(8,2),(10,18),(11,1),(12,22),(13,15)

Table B.4: The merging pairs obtained from four different problems when $\beta = 0.5$

Model	Total gain	Total incentive	Bail-out with incentive	Bail-out without incentive	Merging Pairs
Model (5.3.4)	115.16	14.65	54.17	0.00	(4,15),(7,21),(8,6),(10,22),(11,2),(12,1),(13,18)
Model (5.3.12)	115.16	14.65	54.17	0.00	(4,15),(7,21),(8,6),(10,22),(11,2),(12,1),(13,18)
Model (5.3.13)	68.98	-31.55	45.43	9.08	(3,6),(4,22),(5,2),(7,21),(10,15),(12,18),(16,1)
Model (5.3.15)	112.50	11.99	45.61	0.00	(7,22),(7,21),(8,6),(10,18),(11,2),(12,1),(13,15)