EXISTENCE AND UNIQUENESS IN THE FINITE ELASTOSTATIC DIRICHLET PROBLEM

A Dissertation

Presented to

the Faculty of the Department of Physics

University of Houston

In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

by

John F. Pierce

December 1973

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ABSTRACT

A qualitative model for the finite elastostatic Dirichlet problem is presented. The principal feature is that the solution space is a differentiable manifold as opposed to a topological vector space. The nature of the solution manifold reflects the imposed boundary condition, the body topology, and varies with them. The model permits one to utilize contemporary mathematical methods to resolve existence and uniqueness questions.

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EXISTENCE AND UNIQUENESS IN THE FINITE ELASTOSTATIC DIRICHLET PROBLEM

I. INTRODUCTION

THE ULTIMATE PURPOSE OF THE WORK

This thesis attempts to build a mathematical model for the finite elastostatic Dirichlet boundary value problem (or place boundary value problem) which will allow one to phrase more accurately and answer in greater detail the following "qualitative" questions:

(1) Existence in Elastostatics: Given a nonlinear material body, a body force field, and a specific place boundary condition, does there exist at least one configuration of the material body satisfying the boundary condition, and equilibrating the given body force?

(2) Regularity in Elastostatics: Given a configuration satisfying the requirements of question (1), how smooth is it? Can one be assured that if the given place boundary condition is sufficiently regular, and the body force and the stress-strain relation vary smoothly from point to point over the body, the configuration equilibrating the body force and satisfying the boundary condition will be at least as smooth?

(3) Global and Local Uniqueness in Elastostatics: Given one configuration satisfying the requirements of question (1), is it possible to find a second configuration satisfying the place boundary condition and equilibrating the body force? Can the second configuration be gained from the first configuration by a slight perturbation? Is it gained from the first configuration by a finite deformation?

The model will allow one to investigate these questions more effectively by transforming the finite elastostatic Dirichlet problem from its real analysis setting as a classical partial differential equation with boundary conditions to a geometric/topological setting as a differentiable mapping between manifolds. By so doing, the mathematical methods of global analysis, differential topology, and differential geometry can be utilized in the investigation.

INSTANCES OF NONUNIQUE BEHAVIOR IN ELASTIC SYSTEMS

Existence, uniqueness, and regularity questions are always a part of any physical theory where the motion of the system is represented by differential equations. However, for the "usual" linear differential equations encountered by the physicist in his study of classical elasticity, investigating the uniqueness and regularity questions yields results which are more or less what is intuitively expected: if a solution exists, it is unique, and as smooth as the body, the boundary conditions, the forces, and the differential equation itself permits. Moreover, the existence of a solution is demonstrated in a straightforward way: one solves the equation. In contrast, classical elastic systems whose linear differential equations yield results counter to what is expected were regarded as physically untenable for many years. As a result, the study of the existence and uniqueness questions for a classical elastic system came to be regarded as a rather academic, formal exercise, which for all practical purposes was devoid of physical content.

But when one examines normally classical elastic systems near critical behavior situations, or when one examines continuum mechanical systems undergoing finite deformations, questions of existence, uniqueness, and regularity gain more than a passing interest. For instance, in the process of bursting, a balloon can be pictured as a classical elastic isotropic body, but one whose modulii lie outside the "physically reasonable" range of values established by classical criteria.[1] Also, an elastic column undergoing finite deformation may buckle, in which case it admits a unique elastostatic solution for one set of place boundary conditions, and two distinct elastostatic solutions for another set.[2] Such nonunique behavior is uncomfortably foreign to physicists schooled in the usual linear models.

Factors Which Influence Uniqueness in Linear Elasticity

In linear elasticity, two factors reveal themselves as affecting the uniqueness conclusions for a given solid:

the stress-strain relation for the material comprising the solid, and the shape (topology) of the body itself. For example, by Kirchoff's theorem, if the elastic coefficients of the material satisfy a certain definiteness condition, or in particular, if the moduli for an isotropic material lie in a particular range of values, one is assured that the linear elastic place boundary value problem admits at most one equilibrating solution. Moreover, the uniqueness result holds for all place boundary conditions, all body forces, and for all shapes of the body. Provocatively, when the elastic coefficients fail to satisfy the Kirchoff condition, one does not automatically obtain nonunique results. For instance,[3] for an isotropic material one finds that one geometric shape for the body leads to nonunique equilibrating solutions only for certain values of the moduli outside the range established by Kirchoff; a second geometric shape leads to nonuniqueness at a different set of values for the moduli; a third geometric shape continues to display unique equilibrating configurations for all values of the moduli. In linear elasticity theory there is currently no general way to anticipate how the geometric shape of the body will affect the uniqueness conclusions when the elastic coefficients for the material fail to satisfy the definiteness conditions. One must simply resolve the question for each geometric shape on an individual basis.

Additional Factors in the Nonlinear Elasticity Theory Which Affect Uniqueness Conclusions

In finite deformation theory a third factor reveals itself as affecting the uniqueness conclusions: the boundary condition. In the linear elasticity theory, for a given body shape and material, the conclusion one draws as to the uniqueness or nonuniqueness of the solution to the Dirichlet boundary value problem for one particular place boundary condition continues to hold for all place boundary conditions. This universal property ceases to hold when one passes to nonlinear elastic materials, or the finite deformation theory.[4] At present, even when one is given the topology of the body, and the (nonlinear) elastic response of the material comprising it, there is no general method by which one can anticipate which boundary conditions admit unique solutions and which do not. Here again, one must investigate each boundary condition separately.

The Objective this Thesis Pursues

How can one begin to incorprate the boundary conditions and the body topology factors into his study of existence and uniqueness in a unified way? It is this author's contention that the mathematical means by which one can model nonlinear continuum mechanical systems in a way sensitive to all three factors affecting the existence, uniqueness, and regularity questions have only recently become available. Moreover, the mathematical tools which are capable of resolving the questions are only now being forged as theorems in the Algebraic Topology, the Differential Topology, and the Differential Geometry. The task undertaken in this thesis is to construct a geometric/topological model for nonlinear continuum mechanical systems which can exploit the contemporary mathematical tools as they become available.

THE VALUE OF A GEOMETRIC/TOPOLOGICAL MODEL

The Model is the Continuum Mechanical Counterpart to Poincare's Qualitative Model for a Point Mechanical Dynamic System

Why would one desire a model which is more of a topological nature, as opposed to one utilizing the more familiar elements of the real and complex analysis? The reason lies in the "qualitative" or global nature of the questions under investigation. One can appreciate the meaning of this statement by examining a parallel development in the mathematical model for point particle mechanical systems.

Usually, a point particle mechanical system is represented by a system of nonlinear ordinary differential equations. The solution to the system is not always available in closed form. Rather, the best one can sometimes do is gain a formal, open expansion for the solution whose radius of convergence (domain of validity) is limited. H. Poincare noticed that the information one desired to gain about a nonlinear system could be grouped into two classes. One class involved questions about the general nature and characteristics of the physical behavior admitted by the system. For instance, does the system admit periodic behavior, or equilibrium points; or, is a particular behavior admitted by the system sustained under perturbation? Such information is called "qualitative" or "global" information. The second class involved more specific solution questions: given the existence of a solution having some desired property (like periodicity) what precisely is its morphology?

To answer these questions Poincare found it advantageous to examine the nonlinear mechanical system from two points of view.[5] The first point of view pictures the equations of motion in geometric terms as specifying a field of vectors on a suitably chosen phase space. The second point of view realizes the equations of motion using elements of the real analysis as ordinary differential equations. Both points of view, the geometric model, and the analysis model, are legitimate ways of picturing the same dynamic system. One finds that the two points of view complement each other: questions which are difficult to examine from one point of view yield to the other point of view. In particular, Poincare's investigations indicated that questions of a qualitative nature, like existence, uniqueness, local uniqueness, regularity, and stability of motion could be viewed more easily and with greater insight from the geometric point of view. Questions concerning the morphology of solutions could be more successfully resolved in the analysis model.

In the Continuum Mechanical Case, the Qualitative Model has Distinct Practical Value

Although the geometric viewpoint allowed Poincare to view the qualitative questions of a point mechanical system with greater clarity, the geometric and topological tools available were insufficient to allow him to actually resolve very many. Indeed, the tools capable of resolving such questions did not begin to appear until the middle of the twentieth century. The first significant achievement of the geometric model after the work of Poincare came in the 1960's with the proof of the existence and orbital stability of the Poincare orbits of the second kind in the restricted three-body problem.[6]

Regrettably, during the interim between Poincare's efforts and today, numerical techniques have advanced sufficiently that the brilliance of the geometric breakthrough is dulled considerably from a practical point of view. With the advent of high speed computers the applied mathematician finds that, as far as orbital analysis is concerned, applying improved numerical integration techniques to the real analysis model with randomly varying initial conditions resolves the qualitative questions with sufficient efficiency and at a reasonable cost. The necessity of constructing an entirely new model for the same phenomenon is abnegated. Sadly, one must conclude that, from an applied point of view, the breakthrough in the geometric model for point particle systems germinated a decade too late to bear noteworthy fruit.

When one goes to continuum systems, however, the situation is reversed. Attempts to numerically integrate the three-dimensional partial differential equations have been stymied: few materials can be modeled with sufficient simplicity so that integration procedures can be applied; moreover, the cost of the computer time necessary to integrate even the simplest of equations is prohibitive. Hence, fertile ground is available for the germination of a qualitative model for continuum mechanical systems.

The Qualitative Model Complements the Analytical Methods

One can anticipate how results from the qualitative model can complement investigations in the real analysis continuum model. If one can be assured that a given continuum mechanical system admits a motion having the desired physical characteristics (for instance, bounded, or periodic), if one can be assured that the solution is stable with respect to perturbations in the initial conditions, boundary conditions, and the equation of motion, and if one can even be given the initial conditions generating the desired motion, then numerically integrating the three-dimensional partial differential equations to gain the morphology of the particular motion, though expensive, becomes a cost effective procedure.

QUALITATIVE MODELS ARE RECENT IN ORIGIN AND DIFFICULT TO CONSTRUCT

The Space of Possible Solutions is Necessarily Infinite-Dimensional

Several problems obstruct the generalization of the geometric model for point particle systems to continuum mechanical systems. The most prominent question is what is the "configuration space" for a continuum system? For a single point particle, the configuration space is obvious: a three-dimensional Euclidean space. For two point particles, if one imposes impenetrability, the configuration space is roughly a six-dimensional region lying in Euclidean six-space. As one goes to a continuum system, one must expect the dimension of the configuration space to grow accordingly; in short, to be some sort of infinite-dimensional space. What is the explicit specification of this space? What is its geometric and topological structure? Can one even meaningfully use these terms? Moreover, even if one can construct a configuration space, how do the equations of motion manifest themselves upon it? How does one incorporate the boundary conditions? Finally, with such a model, how does one draw even one conclusion from it?

The Mathematical Tools for Developing and Exploiting a Qualitative Model are Quite Recent

The possibility of constructing a geometric model for a continuum mechanical system is quite recent. Erecting a reasonable configuration space was severely hampered until the advent of the infinite-dimensional differentiable manifold theory in the early 1960's.[7] The first significant breakthrough in a global geometric model for linear elliptic differential equations dates from the middle 1960's.[8] The topological and geometric foundation for a global nonlinear analysis was only first set forth in a complete form in 1968.[9] Finally, the mathematical methods capable of utilizing this structure to resolve the questions of interest are only now evolving. In short, the elements are only now available for developing a qualitative model for a continuum mechanical system.

CURRENTLY EXISTING QUALITATIVE MODELS

What is a Geometric Model in Linear Elasticity Theory, and How is it Used?

In chpater two some currently existing geometric and analysis models for nonlinear continuum systems are presented.

Generally speaking, they are outgrowths of the models which were successful in resolving existence and uniqueness questions for linear elastic systems. The success of the linear system models rests upon the ability to view the linear differential equations of motion as a linear mapping between two topological vector spaces, a "solution" space and a "data" space. One can easily see how this point of view arises. If ϕ , θ are two solutions to a linear differential equation, their sum $\phi + \theta$, and any multiple $\lambda\phi$ are also solutions to the same equation. Hence, to speak of a "space" of possible solutions to a linear differential equation is to speak of a set of functions which form a vector space. Moreover, if one adopts some sense of the "nearness" of one function to another, the space of possible solutions becomes a topological vector space. Likewise the inhomogeneous terms of the equation are a set of functions which can form a topological vector space, called the "data" space in the model. Finally, the linear differential equation itself, when viewed as acting on these function spaces, becomes an operator associating with each function of the solution vector space, a function in the data vector space. Since the differential equation makes the association in a linear manner, it may be viewed as a linear operator. Depending upon how the topologies are chosen the linear operator may take nearby solution functions into nearby data functions, in which case the operator is called a continuous linear operator.

When cast in this manner the questions of the existence of a solution to the linear elastic system may be rephrased in terms of the properties of the continuous linear operator: when does the continuous linear operator map the solution vector space onto, as opposed to into, the data vector space? The question of uniqueness may be rephrased as: when is the continuous linear operator one-to-one? These questions can be answered by exploiting theorems from the mathematical theory of continuous linear operators. Hence, the questions of existence, uniqueness, and regularity for linear elastic systems can be answered when the proper topological vector spaces are found to model the space of solutions and the space of data, and the proper linear operator is constructed to model the linear elastostatic differential equations.

The Main Features of the Currently Existing Nonlinear Geometric Models

The qualitative models for nonlinear elastic systems currently in the literature are modifications of the models for linear elastic systems. Whereas the solution space and the data space are still assumed to be topological vector spaces, the elastostatic equation is pictured as a nonlinear operator relating the two spaces. By utilizing the linear structure of the solution and data spaces, some conditions can be found which if satisfied by the nonlinear operator insure the existence and regularity of solution. These conditions, however, appear as _ad hoc_ elements in the elasticity theory and are without a really firm physical basis. For this reason, these models have met with limited acceptance.

Objections are Raised to these Models

More specific objections can be raised to modeling the finite elastostatic systems in terms of nonlinear operator between topological vector spaces. Three objections appear in chapter four of the thesis. They question whether one can model the space of solutions for a finite elastostatic system as a topological vector space at all. The first objection is that the usual vector space models for the space of solutions possess elements which can not correspond to physical configurations of the body. The second objection is that if one adopts any topological vector space model for the space of solutions, one precludes *a priori* alternatives of possible behavior for the body. The third objection is that a topological vector space is insensitive to changes in boundary conditions and the shape (topology) of the body. These objections will be developed in turn.

The usual models for the space of solutions arise in the following way. Let the body be viewed as occupying a region Ω of physical space R^3. If the body is deformed into a new configuration, the body points which were initially in the region Ω will be taken into another region of R^3. The deformation can be represented mathematically by a function δ which takes points of Ω into points in R^3, a vector-valued function on Ω. To reflect the fact that the body is not torn apart during the deformation, mathematical conditions are placed upon the function: it is continuous (C^0), continuous through first derivatives (C^1), and so on. If one wishes to model all possible configurations of the body it suffices to view them as elements of the set of all C^k functions from Ω into R^3, denoted $C^k(\Omega\ ,\ R^3)$. All possible configurations of the body Ω in R^3 have a representative in this set. Moreover, the set is a vector space, albeit infinite dimensional. Finally, by introducing a notion of distance between functions, it becomes a topological vector space. The topological vector

$C^k(\Omega, R^3)$ serves as the initial choice for the space of solutions in the current models for the finite elastostatic problem. More advanced choices are built from it by adding more elements, and introducing more exotic topologies.

The First Objection

Although the topological vector space $C^k(\Omega, R^3)$ contains all possible configurations of the body Ω in R^3, the first objection raised in the thesis is that it also contains elements which cannot correspond to postures physically attainable by the body. In chapter three, a function in $C^k(\Omega, R^3)$ is cited which physically would correspond to a deformation in which a region of the body collapses upon itself. Hence, using the entire vector space $C^k(\Omega, R^3)$, or one of its generalizations, as the space of solutions is physically untenable, and the models incorporating such a solution space must be reviewed to determine if they used any of the non-physical elements of the space in drawing their conclusions.

The Second Objection

In addition to objecting to the specific use of the topological vector space $C^k(\Omega, R^3)$ or its generalization as the model for the space of solutions to the nonlinear elastostatic problem, one can object to the use of any topological vector space. The second objection raised in the thesis addresses this point: if one adopts a topological linear space for the space of solutions, one precludes *a-priori* alternatives of possible behavior for the material body. An example is given in chapter four for a material body which possesses two equilibrating configurations for the same boundary condition, and body force. If one assumes that the space of solutions is a topological vector space it follows that the two equilibrating configurations can be continuously deformed from one into the other without violating the boundary condition. Such a conclusion precludes *a-priori* the alternative that the two configurations can be deformed into one another only by violating the boundary condition at some state in the deformation. This latter alternative may be easily visualized.

The Third Objection

The third objection to a topological vector space is that its mathematical structure is too simple to reflect how intimately the existence and uniqueness conclusions in a finite elastostatic problem depend upon the particular boundary condition under consideration and the shape (topology) of the body itself. When one models the solution space for a finite elastostatic problem as a topological vector space, the solution spaces for various particular boundary conditions appear as subspaces which are simply translates of each other. They all have the same mathematical structure. Hence, simply by looking at the solution subspaces one cannot anticipate that for one boundary condition the system might exhibit unique behavior, while for another it might exhibit nonunique behavior. Likewise if one drills a hole in the experimental sample, or otherwise alters the topology of the material body, one cannot see it registered as any alteration in the mathematical structure of the solution space.

THE NONLINEAR QUALITATIVE MODEL PROPOSED IN THIS WORK

(1) The Solution Manifold

In light of these objections an alternative formulation for the space of possible solutions for a finite elastostatic problem is presented in chapter four. Rather than adopt *a-priori* a topological linear space, the thesis firstly identifies which mathematical functions can represent meaningful configurations for the material body, and subsequently determines what geometric structure the set of admissible functions can possess.

Two types of solution space models are constructed. The first one includes all functions which can represent configurations for the material body, and subject to no other condition. This set represents all possible solutions to the finite elastostatic problem subject to no boundary conditions. For this reason it is called the solution set for the finite elastostatic free boundary problem, or the free boundary solution set. The second solution set is excised from the first by selecting only those configurations which carry the boundary of the material body into a given prescribed shape or place in the physical space. This subcollection of configurations constitutes the set of all possible solutions to the finite elastostatic problem subject to a given Dirichlet boundary condition. For this reason this subset is called the solution set for a finite elastostatic Dirichlet boundary condition. For simplicity this subset is called the solution set for a finite elastostatic Dirichlet boundary value problem. It remains then to determine what geometric structure these sets can possess.

The Solution Set for the Free Boundary Problem is a Differentiable Manifold

What one finds in the case of the free boundary solution set is that the functions representing configurations of the material body constitute a subset of the topological vector space $C^k(\Omega\ \ ,R^3)$, namely the subset of embeddings of Ω into R^3. Under suitable conditions, the subset of embeddings possesses the structure of a C^∞ differentiable manifold which lies as an open submanifold or "open domain" in $C^k(\Omega\ \ ,R^3)$. When viewed with this geometric structure the set of possible solutions to the free boundary problem will be called the free boundary solution manifold.

Complications Arise in Modeling Place Boundary Conditions

The investigation of the solution set of a particular Dirichlet boundary condition is somewhat more involved. The first complication one must encounter is the question of whether one can even realize a given Dirichlet boundary condition at all. If one arbitrarily chooses a place boundary condition for a material, it does not follow automatically that there exists even one configuration of the body which has the proper degree of smoothness required for the problem, and which permits the boundary of the body to satisfy the given place boundary condition. For example, one can impose postures for a boundary of a body which can be fulfilled only by mathematical functions which would represent postures where the interior of the body would collapse into itself, or which would represent postures where the body would develop a "kink" or a "crease". The first instance would have to be ruled out as a non-physical situation, while the second would have to be dismissed by virtue of the inadequacy of the mathematical tools used. Until recently, one's only recourse was to assume a-priori that if the boundary condition was "sufficiently smooth" then one could be assured there exists at least one configuration of the material body satisfying the given place boundary condition. Indeed, in most models for the finite elastostatic Dirichlet problem currently in the literature, this assumption is one of the axioms of the model. In the model presented here, however, one can go a step further in investigating this question. By utilizing some recent results from the global analysis and algebraic topology, one can begin to analyze how the smoothness characteristics of the place boundary condition and the topology of the material body itself determine whether a given place boundary condition can be realized by at least one physical configuration of the body.

A second complication arises when one considers the question of just how the set of functions satisfying a given boundary condition is to be selected. There are at least two mathematical ways in which one can group these functions. Interestingly enough, they correspond physically to the situations in which one maintains the given Dirichlet boundary condition by simply supporting the boundary, and by rigidly supporting the boundary, respectively.[10]

Two Solution Manifolds are Proposed for the Dirichlet Boundary Condition

What one finds for the Dirichlet boundary condition case is that the set of functions representing configurations satisfying the given boundary condition which corresponds physically to a simple support not only lies as a subset of the free boundary solution set, but also as a "surface" or closed submanifold in the free boundary solution manifold. Moreover, if one further constrains the boundary condition by requiring that it be maintained more rigidly, (an overdetermined situation), one gains an even finer structure for the Dirichlet solution set. The Dirichlet solution manifold corresponding to the simply supported boundary condition itself becomes viewed as composed of "surfaces" or closed submanifolds. Each submanifold corresponds to a particular way of rigidly supporting the given boundary condition. In either case, however, the geometric structure which one may endow upon the Dirichlet solution set is well defined.

The Topology of the Solution Manifolds Reflects Alternatives of Mechanical Behavior for the Body

The models for the free boundary and Dirichlet solution manifolds introduced in chapter four overcome the three objections raised to the topological vector space models. They have features which make them very attractive candidates. By virtue of being a differential manifold as opposed to a topological vector space the topology of the free boundary solution manifold may now be more complex than that of previous models. In particular, the model introduced here may not be simply connected. It may have "holes" in it, a condition not possible for a topological vector space. Consequently, a submanifold representing the solution manifold for a particular Dirichlet boundary condition may or may not be connected, depending upon how it intersects the holes. The connectedness or non-connectedness of a Dirichlet solution manifold has physical significance in that it indicates whether or not two configurations satisfying the same Dirichlet boundary conditions can be deformed into one another with or without violating the given boundary condition. If the Dirichlet solution manifold is connected, i.e. it consists of one piece, or component, then any two configurations satisfying the same Dirichlet boundary condition can be continuously deformed into one another without violating the boundary condition. However, if the manifold consists of more than one piece, or component, then there are configurations satisfying the given Dirichlet boundary condition which can be deformed into one another only by violating the boundary condition. This latter alternative is precluded if one assumes that the space of solutions is a topological vector space, as it is always simply connected.

The Topology of a Dirichlet Solution Manifold can Vary from Boundary Condition to Boundary Condition

Significantly, the connectedness or non-connectedness of a Dirichlet solution manifold can vary from Dirichlet boundary condition to Dirichlet boundary condition. Thus, alternatives of mechanical behavior can vary from boundary condition to boundary condition. This result is in contrast to the "universal" result one gains with the linear models or their generalizations, where the solution vector spaces for the various Dirichlet boundary conditions are identical in mathematical structure.

The Topology of the Solution Manifold Depends Upon the Topology of the Body

Finally, one is not at a loss in discerning the number of components comprising the solution manifold for a Dirichlet problem. The thesis indicates how contemporary mathematical methods (in particular, the Obstruction theory) can be utilized to resolve the question. One finds that the number of components in the solution manifold is influenced by the topology of the material body itself, or the shape of the experimental sample. Roughly speaking, if one drills a hole in the experimental specimen, one can alter the number of components in the solution manifold, and thereby one can alter the alternatives of behavior possible for a Dirichlet boundary condition.

(2) The Dynamic Elements

The Data Space is a Linear Vector Space

In chapter five the dynamic elements of the finite elastostatic model are investigated. The usual model for the data space of body force density fields is a topological vector space of functions. This model is also adopted in this thesis. The primary motivation for retaining the topological vector space structure for the data space is the fact that body force density fields superimpose: their combined effects are additive. One can represent a body force density field as a vector-valued function defined over the region Ω occupied by the body which has a suitable degree of smoothness (i.e. C^0, C^1, etc.). For modeling the data space of body force density fields, then, the topological vector space of C^k vector-valued functions defined over Ω, $C^k(\Omega, R^3)$, for a suitable choice of k, is a quite legitimate candidate.

The Finite Elastostatic Operator Prescribes the Links Between the Solution Manifolds and the Data Manifolds

The remainder of chapter five deals with the construction of the non-linear operator which models the finite elastostatic differential equations and links the above solution manifolds and data spaces. If the material body is a materially uniform, simple, elastic body of grade one, a particular kind of non-linear material body, the finite elastostatic differential equations for it determine a nonlinear operator linking the solution manifolds and data spaces which is a differential operator of order two. The "differential operator of order two" nature of the correspondence means that the operator associates with configurations continuous through k^{th} order derivatives over Ω, body force density fields continuous only through $(k-2)^{th}$ order derivatives. Hence, the specific way in which the non-linear operator links the solution manifolds to the data spaces is by taking C^k configurations into C^{k-2} body force density fields. If one were to consider material bodies other than the one specified, for instance, a material body of grade ℓ, non-elastic bodies, materials with facing memory, or non-simple bodies, the non-linear operator modeling the finite elastostatic differential equation would not be a simple differential operator of order two. Rather, it would be a more complicated integro-differential operator. Consequently, the operator would link the solution manifolds and data spaces in a manner entirely different from that for the simple, grade one, elastic material body. One can now appreciate how models proliferate as one considers finite, non-linear continuum problems, in contrast to the single, universal, linear-differential-operator-of-order-two model which completely characterized infinitesimal, classical linear elasticity theory.

For the materially uniform, simple, elastic material body of grade one, the combination of free boundary or Dirichlet solution manifolds, body force density field (data) spaces, and non-linear differential operator developed above serves as the models which will be used in this thesis for the finite elastostatic free boundary and Dirichlet problems defined over the classical functions. The fourth chapter ends with a comparison of these models with ones which currently exist in the literature, in order to point out contrasts.

THE MODEL IS ADAPTED TO ACCOMODATE CONTEMPORARY MATHEMATICAL TOOLS

Why an Adaptation is Required

By adapting the models for the finite elastostatic free boundary and Dirichlet problems completed in Chapter four, one has replaced the hybrid models involving non-linear differential operators linking topological vector spaces by models in which the solution manifolds possess no linear structure. By doing so, however, one loses not only the results which were gained from the previous models by the use of the linear structure on the solution space, but also the tools, theorems, and techniques which made these results possible. If the new models introduced here are to have a utilitarian values, one must find alternative mathematical methods to replace those which have been rendered inapplicable. These methods and tools are only now evolving in the fields of infinite dimensional geometry, and algebraic topology. In order to be in a position to exploit these methods, however, one must make one additional improvement in the models for the finite elastostatic problems developed here: they must be extended from models over the classical C^k functions to functions having a more general form of continuity and differentiability.

The Extension in the Case of the Earlier Linear Models

The technique of extending models for differential equations constructed over classical C^k functions to models constructed over more general function spaces has its origin in the "method of weak solutions" for linear elastic systems. The motivation for the technique is that sometimes it is easier to answer questions about a linear differential equation by viewing it in its "integrated form", as opposed to its differential equation form. The method of solution by Green's function is such an example. In such a method, however, one must extend one's solution and data spaces to include functions which are not continuous or differentiable in the usual limit process sense, but which can be regarded as continuous, or differentiable when viewed under the integral sign. The procedure, generally speaking is to choose the generalized function spaces for the space of solutions and the space of data in such a way as to yield as easily as possible criteria for the existence and uniqueness of solution to the linear differential equation. One then seeks to prove "regularity theorems" which state that if the given data is in fact a classical C^k function then the generalized function solution corresponding to it is also a C^k function. Hence, the three areas of investigation which grow out of the method of weak solution to linear differential equations are: to establish existence and uniqueness criteria for the differential equation when viewed in the generalized function setting, and then seek to pull back the results to the classical function setting by regularity theorems.

The Extension of Currently Existing Non-Linear Models

For the models of a non-linear elastostatic problem in which topological vector spaces serve as the space of solutions and the space of data, the extension to a generalized function setting is relatively straightforward. The space of solutions and the space of data extend to the same generalized function spaces that arise in the linear differential equation theory: The space of square integrable (L^2) functions over the body Ω, or some closed linear subspace of it, like the Holder spaces or the Sobolev spaces. The extension of the non-linear operator to an operator linking the generalized function spaces is a little more involved than in the linear case. However, one can establish by extensive norm calculations that for the Holder or Sobolev spaces the non-linear operator can be extended to the generalized function spaces, and the extension is as continuous as the original non linear operator linking the solution and data spaces built over the classical functions.

The Extension of the Free Boundary Solution Manifolds Proposed Here is More Complicated

The extension of the model introduced in the thesis to the generalized function is not at all straightforward.

In fact, the mathematical tools necessary to construct the extension did not exist until the work of R. Palais and S. Smale in the late 1960's.[11] The principal difficulty in extending the model lies in the fact that the classical function solution sets is not a topological space, but rather a differentiable manifold. In chapter six the free boundary solution manifold is extended to the generalized functions. The extension, when it exists, has the structure of a C^{∞} differentiable manifold, and lies as an open submanifold or "open domain" in a generalized function topological vector space. The conditions sufficient to permit the extension, and the manifold structure endowed upon the extension are provided by theorems from the mathematical theory of global non linear analysis. In addition, the method of extension may be applied to many different classes of generalized functions, including the Holder and Sobolev spaces. Hence, in extending the free boundary solution manifold, one is not limited to a single class of generalized functions, as is the case in the currently existing models. Rather, one may choose that class which is most advantageous for the problem of interest.

The Extension of the Data Space

The space of body force density fields may also be extended to the generalized functions. As it is a topological vector space its extension is rather straightforward, and the resultant generalized function space parallels the equivalent elements found in the other models.

The Extension of the Finite Elastostatic Operator is Achieved by Theorem, as Opposed to Norm Calculation

The remainder of chapter six is devoted to the extension of the non linear operator representing the finite elastostatic differential equation to the generalized function manifolds representing the extended free boundary solution manifolds and data spaces. The technique for extending the operator used in this thesis differs significantly from the extension technique used in other models. First of all, the existence of the extension, and its continuity properties (i.e., whether or not the extended operator is continuous, C^1, etc.) are gained by theorem as opposed to extensive norm calculations. Hence, the details of the mathematics do not cloud the important features of the extension. Moreover, when the extension is accomplished by theorem as opposed to norm calculation, one can extend the operator to several different classes of generalized functions simultaneously. If one relies on norm calculations one must treat each class of generalized function extension separately, as the norms are distinct.

Chapter six ends with an explicit display of the finite elastostatic free boundary value problem modeled over a particular class of generalized functions, the Sobolev spaces. In this extended form the model is sufficiently abstracted so as to be able to utilize the methods of infinite-dimensional differential geometry and algebraic topology to answer questions of existence and uniqueness. This is the level in the formulation of the model for the free boundary problem where one can begin to recoup the existence and uniqueness statements that were rendered inapplicable when the linear structure of the solution space was lost, and begin to develop others.

The Extension of the Dirichlet Boundary Solution Manifold

In chapter seven the model for the finite elastostatic Dirichlet problem constructed over the classical functions is likewise extended. The models for simply supported and rigidly supported boundary conditions are treated separately. The extension parallels the development of the previous chapter. Once again, one gains a model which is sufficiently abstracted so that the original questions of existence, uniqueness, and regularity for the non-linear elastic system can now be formulated in terms amenable to analysis by contemporary geometric and topological methods. Moreover, one can anticipate a means for more deeply analyzing non-linear systems which exhibit locally nonunique behavior. The models show that in some cases a body can exhibit locally nonunique behavior under simply supported boundary conditions but locally unique behavior when the boundary conditions are more rigidly supported.

HOW THE MODEL CAN BE UTILIZED IS ANTICIPATED

The remainder of the thesis is devoted to indicating specifically how the mathematical methods of infinite dimensional geometry and topology can be utilized. As the mathematical tools are rather new, the thesis concentrates on indicating how they may be used, as opposed to developing particular results. Two methods are discussed in detail: the Morse Theory, and the Lusternik-Schnirelman Theory.

If one restricts his attention to hyperelastic materials, (materials which possess a strain-energy function), and the body forces are conservative, then the partial differential equations for both the free boundary and Dirichlet problem may be represented in the analytical model as Euler-Lagrange equations of a variational integral. When the variational integral is viewed in the qualitative model, it specifies a function defined over infinite-dimensional solution manifold. The configurations in the solution manifold which are the critical points of the function are the solutions of the Euler-Lagrange equations; hence they are the equilibrating configurations for the elastostatic problem. The Morse theory permits one to determine how many critical points the function has. The theory indicates that the number of critical points of the function depends upon the nature of the function itself (hence, the material response), and the topology of the solution manifold (hence, the boundary condition, and the topology of the sample). Thus, the topology of the solution manifold becomes directly related to the mechanical behavior of the body. The richness of the topology, and its variation with boundary condition and body shape provide a wealth of provocative subjects for future study.

The Lusternik-Schnirelman theory is an attractive tool for extracting information from the finite elastostatic models. Existence and uniqueness questions are, once again, related to the topology of the solution manifold and the nature of the finite elastostatic operator. The results, however, are not extensive as with the Morse theory. Even in non-variational cases directions for investigations are becoming apparent. References are given which addresses this point.

In short a model for the finite elastostatic Dirichlet problem is constructed in this thesis. It has the attractive feature that its solution manifold is not topologically trivial, and can vary in a predictable way with boundary condition and body shape. Moreover, it can serve as a vehicle by which heretofore pure, contemporary mathematical methods may be brought to bear on fundamental questions in non-linear continuum mechanics. It awaits exploitation. The areas suggested here provide some directions for future study.

II. THE DISTINCTION BETWEEN ANALYTICAL AND QUALITATIVE MODELS

What comprises a geometric or qualitative model for an elastic system? How does the approach for answering existence and uniqueness questions using it differ from the approach using analytical methods? In this chapter, examples of some analytical results for existence and uniqueness questions for elastic bodies are presented. One finds that they usually deal with very specific situations: the response must depend linearly upon the infinitesimal strain, the material must be isotropic, the moduli must lie in a certain range. If one wishes to pursue the questions in a more general point of view one must adopt another method. The qualitative models provide such an alternative. To understand what comprises a qualitative model and how it is used, two models for the elastostatic Dirichlet problem currently in the literature are examined in detail.

RESULTS FROM PURELY ANALYTICAL METHODS

Some existence and uniqueness statements for the elastostatic Dirichlet problem arise from analytical methods. The most familiar results reside in linear elasticity theory.[12]

Foremost among these is a strong version of Kirchoff's result

Theorem (Kirchoff) Let B be a linearly elastic solid occupying a bounded region in space, and possessing elastic coefficients $C_{ijkl}(x)$ satisfying the Cauchy symmetry condition

$$C_{ijkl} = C_{ijlk} = C_{jikl}$$

(a) If the elastic coefficients satisfy a positive definiteness condition

$$C_{ijkl}(x) \quad \xi_{ij}(x) \; \xi_{kl}(x) > 0, \quad \text{for all } x \in B,$$

and for all symmetric tensor fields $\xi_{ij}(x)$ on B, then the linear elastostatic mixed boundary value problem has at most one classical solution.

(b) If the material is isotropic

$$C_{ijkl}(x) = \lambda(x) \; \delta_{ij}\delta_{kl} + \mu(x)(\delta_{ij}\delta_{kl} - \delta_{il}\delta_{jk})$$

for $\lambda(x)$, $\mu(x)$ the Lamé and shear moduli, respectively, then the positive definiteness condition holds if and only if, for each $x \in B$

$$\mu(3\lambda + 2\mu) > 0$$

or equivalently

$$\mu(x) \neq 0, \text{ and } \sigma(x) \in (-1,1/2)$$

for $\sigma(x)$ the Poisson ratio defined by

$$\frac{2\mu(x)\sigma(x)}{(1-2\sigma(x))} = \lambda(x)$$

The theorem is established by determining what conditions are sufficient to insure that the only solution to the homogeneous displacement boundary value problem is the trivial solution.

A second example of a uniqueness theorem in linear elasticity theory arises when one considers homogeneous materials. Again one constrains the elastic coefficients, but in a way not as strong as the positive definiteness condition.

Theorem. If B is a homogeneous body

(a) and the elastic coefficients satisfy the strong ellipticity condition

$$C_{ijkl}\alpha_i\alpha_k\beta_j\beta_l \geq a\alpha_i\alpha_i\beta_j\beta_j \text{ , } \quad a>0 \quad \text{a constant}$$

then there exists at most one (weak) solution to the linear elastostatic Dirichlet problem.

(b) If the material is isotropic, the strong ellipticity condition is satisfied if and only if

$$\mu > 0 \quad \text{and} \quad \lambda + 2\mu > 0$$

$$\text{or } \mu > 0, \quad \text{and} \quad \sigma \varepsilon (-\infty, 1/2) \cup (1, \infty)$$

Examples of Nonunique Behavior in Linear Elasticity

Instances of nonunique behavior in linear elasticity allow one to evaluate how necessary conditions set forth in the uniqueness theorems are. Knops and Payne[13] present a varied collection of such counter examples to unique behavior. Among them are:

(1) An ellipsiod of homogeneous isotropic material with boundary given by

$$x_1^2 + a^2 x_2^2 + b^2 x_3^2 - 1 = 0$$

For homogeneous body force and displacement boundary conditions, non-trivial solutions are possible of the form

$$U_i = \Omega P_i,$$

for P_i a polynomial of non-negative degree, if and only if the modulii satisfy the condition

$$\mu(\lambda + 2\mu) < 0.$$

This solution may be added to any other solution to the linear elastostatic equation for a given body force and boundary condition to produce a second solution.

Notice that the condition cited in the example fails to satisfy the strong ellipticity criterion. Thus the strong ellipticity criterion is necessary in the following sense: there exists a class of (ideal) materials which do not satisfy the strong ellipticity condition, and a body shape for which nonunique behavior is revealed.

(2) The homogeneous isotropic elastic sphere with boundary

$$x_1^2 + x_2^2 + x_3^2 - 1 = 0$$

exhibits nonunique behavior if and only if Poisson's ratio has value

$$\sigma = 1$$

$$\text{or} \quad \sigma_n = \tfrac{1}{2}(1-3n)(1-2n)^{-1} \qquad n = 1,2,\ldots$$

Notice that the values of Poisson's ratio lie outside the uniqueness range cited in the strong ellipticity condition; however, for the body shape given, only particular values for the ratio lead to nonunique behavior. All other values yield unique behavior.

(3) The inhomogeneous isotropic material in the shape of a sphere with a cavity

$$n\pi \leq r \leq (n + m)\pi$$

with elastic moduli satisfying the conditions

$$\lambda + 2\mu = \frac{1}{r^2}, \quad \mu = \frac{1}{4}\frac{3}{r^2} + \ln \frac{(n+m+1)r}{r}$$

admits the nontrivial displacement solution

$$(U_r, U_\theta, U_\phi) = (\sin r\,, 0, 0)$$

to the homogeneous problem. Thus it exhibits nonunique behavior.

Notice in this example that the moduli in this example satisfy the strong ellipticity condition at every point. Hence, the strong ellipticity condition is not sufficient to insure uniqueness in the case of inhomogeneous materials.

Comments on the Nature of the Existence Theorems

Two important points follow from these examples. Firstly, while the conditions advanced in the theorems are sufficient to achieve unique behavior, their necessity is qualified. Even when the elastic moduli fall outside the range of values cited, it is still possible to achieve unique behavior. Secondly, the uniqueness theorems involve conditions on only the elastic coefficients of the material. They are independent of the body shape (topology). Yet in the counter examples, the body shape plays an integral part.

Uniqueness Theorems for Nonlinear Systems are even More Complicated

When one uses analytical methods to establish conditions sufficient to guarantee unique behavior for a system undergoing finite deformations, one finds that the restrictions placed upon the elastic response are even more specialized and complex than in the linear case. Counter-examples to nonunique behavior involve specifying the boundary conditions, as well as the body shape and the material response. Moreover, one must distinguish between local nonunique behavior and global nonunique behavior. In particular, it is possible for a body to exhibit unique behavior with respect to small deformations about any arbitrary state of strain, yet still exhibit nonunique behavior under finite deformation.

The Qualitative Point of View

Must the investigation of existence and uniqueness questions be so fractured? Can one not adopt an approach which would enable him to picture all factors contributing to the existence and uniqueness conclusions in a unified way? Such a point of view would require an approach different from the analytical methods. Qualitative models provide such an alternative setting. From this new perspective, existence and uniqueness conclusions would follow as a consequence of a relationship between the material itself, the boundary condition under consideration, and the body topology, as opposed to the categorical imposition of "necessary" and sufficient conditions on some one factor. For this reason, the goal of this thesis is to establish an adequate, general setting for the finite elastostatic Dirichlet problem which manifests the qualitative point of view.

EXISTING NONLINEAR QUALITATIVE MODELS

There are models for the finite elastostatic Dirichlet problem in the literature which manifest the geometric/topological point of view. Examples are the models advanced by W. Van Buren,[14] and I. Beju.[15]

Although these models are available in the literature, it is instructive to examine them in detail before going further. One can then see what has already been accomplished, what particular elements of the models are noteworthy, and what particular elements appear vulnerable. The information will be valuable in indicating where the model developed in this work ought to coincide with the models which currently exist, and where it ought to depart. Moreover, the investigation will also provide a clue as to how one can use the model once it has been established.

VAN BUREN'S MODEL

The model of W. Van Buren is an outgrowth of the works of F. John and F. Stoppelli.[16] The works of the latter authors permit one to examine local existence and uniqueness questions for the finite elastostatic place and traction boundary value problems in the vicinity of a natural state by viewing a corresponding classical linear infinitesimal elasticity boundary value problem. Van Buren's generalization allows one the opportunity to make local uniqueness and existence statements in the vicinity of states of finite strain, or non-natural states. The statements are gained by judiciously exploiting a geometric model for the finite elastostatic place boundary value problem. Van Buren's model is erected by making sufficient assumptions about the nature of the possible solutions, the possible loads, and the differential equation governing the elastostatic problem so that the problem may be viewed geometrically as a nonlinear differential operator linking Banach spaces. Once this is done, Van Buren utilizes the inverse function theorem for Banach spaces to give a local existence and uniqueness theorem for a body in a given equilibrated strained state. For completeness of exposition, the notation and basic definitions employed by Van Buren are summarized in his article.

Van Buren's Analytical Model

For Van Buren, the analytical representation for the finite elastostatic place boundary value problem consists of the classical partial differential equation

$$\mathrm{Div}_{\underline{X}}(\hat{S}(F(\underline{X}), \underline{X}) + \underset{\sim}{b}_a(\underset{\sim}{X}) = \underset{\sim}{0} \qquad \underline{X} \in B, \qquad \text{(II.1.)}$$

subject to the place boundary condition

$$\underset{\sim}{u}(\underset{\sim}{Z}) = \underset{\sim}{u}_a(\underset{\sim}{Z}) \qquad \underline{Z} \in \partial B. \qquad \text{(II.2)}$$

Here, B is the region in R^3 occupied by the body in the reference configuration, ∂B is its boundary, $\hat{S}(F,\underset{\sim}{X})$ is the first Piola-Kirchoff stress tensor field relative to the given reference configuration, $\underset{\sim}{u}(\underset{\sim}{X})$ is the displacement vector field relating the deformed configuration to the reference configuration, $F(\underset{\sim}{X})$ is the deformation gradient tensor field relative to the reference configuration, $\underset{\sim}{b}_a$ is the given body force density per unit mass in the reference configuration, and $\underset{\sim}{u}_a$ is the given displacement boundary vector field corresponding to the place boundary condition. Notice that the reference configuration need not be a natural state.

Van Buren's analytical model for the problem of infinitesimal displacement superimposed upon a given state of strain consists of the partial differential equation

$$\mathrm{Div}_{\underset{\sim}{X}}([A(\underset{\sim}{X})][\nabla \underset{\sim}{u}(\underset{\sim}{X})]) + \underset{\sim}{b}^*_a(\underset{\sim}{X}) = 0, \quad \underline{X} \in B, \qquad \text{(II.3)}$$

and the boundary condition (II.2). Here, the reference configuration is chosen to be the given strained state, $\underset{\sim}{b}^*_a(\underset{\sim}{X})$ is the excess body force density vector field, representing how much the given body force density exceeds that necessary to equilibrate the reference configuration, and $A(\underset{\sim}{X})$ is called the elasticity tensor field for the material body for the given reference configuration, and is the partial gradient of the first Piola-Kirchoff tensor field with respect to the first variable F,

$$A(\underset{\sim}{X}) = \nabla_1 \hat{S}(F,\underset{\sim}{X})\Big|_{(1,\underset{\sim}{X})}$$

Van Buren now proceeds to develop a geometric model for the finite and infinitesimal problems by realizing them as differential operator equations linking two Banach spaces, a "solution space", and a "load space".

Conditions Van Buren Imposes on his Qualitative Model

Local existence and uniqueness statements about the finite elastostatic operator are intimately linked with the invertability of the infinitesimal operator in the geometric model. The critical point in erecting Van Buren's model and utilizing it lies in the proper matching of the solution space, the load space, and the differential operators in order to guarantee invertability. To insure proper matching from the point of view of differentiability Van Buren imposes three sets of conditions on the possible solutions, loads, and the differential operators.

Hypothesis 1. The region $\mathcal{B}$ occupied by the body in the reference configuration

a) is an open connected set whose closure is compact.

b) The boundary $\partial\mathcal{B}$ of $\mathcal{B}$ is $C^{2+\alpha}$ for some fixed α, $0 < \alpha < 1$.

Hypothesis 2. The response function $\hat{S}$ and its partial gradients

$$\nabla_1\hat{S},\quad \nabla_2\hat{S},\quad \nabla_1^{(2)}\hat{S},\quad \nabla_1\nabla_2\hat{S},\quad \nabla_2^{(2)}\hat{S},\quad \nabla_1^{(3)}\hat{S},\quad \nabla_1^{(2)}\nabla_2\hat{S},$$

$$\nabla_1\nabla_2^{(2)}\hat{S},\quad \nabla_2^{(3)}\hat{S}$$

are bounded and uniformly continuous on the domain $N_\gamma \times \beta$ where

$$N_\gamma = \{F\varepsilon L(R^3,R^3) \ : \ |F-1|<\gamma\}$$

Hypothesis 3. The following vector fields satisfy the conditions:

a) b_a is Holder continuous on B

b) u^a_a and its first two tangential gradients are Holder continuous of ∂B.

Some Comments on the Conditions

It is instructive to examine in detail the attractiveness of these axioms, since it will be imperative to adopt axioms which will insure the compatability of the elements of the model which will be developed in this work.

The first hypothesis is most natural. As Van Buren points out, Hypothesis 1(b) insures that (1) the exterior unit normal and its tangential gradients are Holder continuous on the boundary of the body, (2) a function which is Holder continuous on the body may be extended continuously to the boundary of the body, and (3) a function with a bounded continuous gradient on the body is Holder continuous on it.

The compactness condition imposed by Hypothesis 1 (a) upon the body plus boundary allows one to speak of the set of possible solutions as a "space". The compactness insures that the set of all continuous bounded displacement vector fields is a linear space capable of supporting a Banach space structure. Moreover, the connectivity of the body, a physically justifiable assumption, greatly simplifies the mathematical structure of the model. It is relied upon heavily when one characterizes a configuration as a vector-valued function which satisfies a (local) impenetrability condition. This point is made explicit in chapter IV.

The attractiveness of Hypothesis 2 rests upon the facts that (1) it is sufficient to insure that the order of partial differentiation of the Piola-Kirchoff stress tensor field is immaterial, and (2) along with Hypothesis 1(b), it insures that the second order partial derivatives of $\hat{S}$, and thereby the coefficients of the infinitesimal elasticity operator are Holder continuous. Holder continuity of the coefficients of the infinitesimal elasticity operator is desirable, because there are differential equations whose coefficients are continuous, but not Holder continuous, and which, even under the most favorable circumstances do not admit solutions with the desired degree of differentiability. (This fact is succinctly revealed in chapter IV). Hence, lack of Holder continuity in the coefficients of the infinitesimal elasticity operator can potentially jeopardize its invertibility, an essential requirement for establishing local uniqueness and existence theorems.

Van Buren also reveals the attractiveness of Hypothesis 3. Without the requirement of Holder continuity on the load, one can not be assured that the solution to the infinitesimal elasticity equation, if it exists, will have continuous derivatives up to the order of the equation, and be, thereby, a solution in the classical sense. Such a situation would potentially jeopardize the invertibility of the infinitesimal elasticity operator. Conversely, the Holder continuity requirements placed upon the possible solutions, coupled with Hypothesis 2 insures that for any possible displacement, the load equilibrating it (or equivalently, linked to it by the finite or infinitesmial elasticity operator) is itself Holder continuous.

Van Buren's Qualitative Model

Van Buren's geometric model may now be erected. As the load space, Van Buren chooses the collection

$$\mathbb{B} = \{(\underset{\sim}{b},\underset{\sim}{w}) : \underset{\sim}{b} \text{ and } \underset{\sim}{w} \text{ satisfy Hypothesis 3}\}$$

of pairs consisting of possible body force density fields $\underset{\sim}{b}$ and place boundary condition fields $\underset{\sim}{w}$. The set $\mathbb{B}$ is a linear space. It may be given the structure of a Banach space if one imposes the norm

$$\begin{aligned} \|(b,w)\| = {} & \sup_{\underset{\sim}{X}\varepsilon\mathcal{B}} |\underset{\sim}{b}(\underset{\sim}{X})| + C_{\underset{\sim}{b}}(\mathcal{B}) + \\ & \sup_{\underset{\sim}{Z}\varepsilon\partial\mathcal{B}} \{|\underset{\sim}{w}(\underset{\sim}{Z})| + |\text{grad } \underset{\sim}{w}(\underset{\sim}{Z})| + \\ & |\text{grad}^{(2)} \underset{\sim}{w}(\underset{\sim}{Z})|\} + C_{\text{grad}^{(2)}\underset{\sim}{w}}(\mathcal{B}), \end{aligned}$$

where grad $\underset{\sim}{w}$ and $\text{grad}^{(2)}\underset{\sim}{w}$ represent the first and second gradients of $\underset{\sim}{w}$, and $C_{\underset{\sim}{b}}(\mathcal{B})$ and $C_{\text{grad}^{(2)}\underset{\sim}{w}}(\mathcal{B})$ are suitably chosen constants.

As the solution space, Van Buren chooses the collection of displacement vector fields

$$\mathbb{D} = \{\underset{\sim}{u} : \underset{\sim}{u} \text{ is } C^{2+\alpha} \text{ on } \mathcal{B}\}$$

where $C^{2+\alpha}$ means that the displacement vector field and its first two derivatives are continuous in the Holder sense with exponent α. This set is also a linear space which can be given the structure of a Banach space by choosing the norm

$$\|\underset{\sim}{u}\| = \sup_{\underset{\sim}{X}\varepsilon B} \{|\underset{\sim}{u}(\underset{\sim}{X})| + |\text{grad } \underset{\sim}{u}(\underset{\sim}{X})| + |\text{grad}^{(2)}\underset{\sim}{u}(\underset{\sim}{X})|\}$$

$$+ \; C_{\text{grad}^{(2)}\underset{\sim}{u}}(\mathcal{B})$$

Here the first and second gradients of the vector field $\underset{\sim}{u}$ are as indicated, and $C_{\text{grad}(2)\underset{\sim}{u}}(\mathcal{B})$ is a suitably chosen constant. For purposes of setting the finite elastostatic operator it is convenient to introduce open subsets in $\mathbb{D}$ which represent neighborhoods of displacement vector fields about the given reference configuration whose associated deformation gradients do not deviate greatly from the identity. For γ a real number, $\gamma > 0$, let $\mathbb{N}_\gamma$ denote the subset

$$\mathbb{N}_\gamma = \{\underset{\sim}{u}\,\varepsilon\,\mathbb{D} \; : \; |\text{grad } \underset{\sim}{u}| < \gamma\}.$$

One may now realize the finite and infinitesimal elastostatic problems as differential operators linking the solution space $\mathbb{D}$, and the load space $\mathbb{B}$. Acting on the solution space $\mathbb{D}$ and the subset $\mathbb{N}_\gamma$, assuming γ is properly chosen, one may define the infinitesimal and finite body force operators $\bar{a}$ and $\bar{b}$ by, for $\underset{\sim}{v} \varepsilon D$ and $\underset{\sim}{u} \varepsilon N_\gamma$,

$$\bar{a}(\underset{\sim}{v})\Big|_{\underset{\sim}{X}} = -\mathrm{Div}_{\underset{\sim}{X}}([A(\underset{\sim}{X})](\mathrm{grad}\ \underset{\sim}{v}(\underset{\sim}{X}))$$

$$\bar{b}(\underset{\sim}{u})_{\underset{\sim}{X}} = -\mathrm{Div}_{\underset{\sim}{X}}(\hat{S}(1 + \mathrm{grad}\ \underset{\sim}{u}(\underset{\sim}{X}), \underset{\sim}{X}))$$

for $\underset{\sim}{X} \varepsilon B$. Moreover, one may define the boundary displacement operator $\bar{w}$ acting on D by, for $\underset{\sim}{v} \varepsilon D$ and $\underset{\sim}{Z} \varepsilon \partial B$

$$\bar{w}(\underset{\sim}{v})_{\underset{\sim}{Z}} = \underset{\sim}{v}(\underset{\sim}{Z})$$

One may now construct two operators which realize the finite and infinitesimal elastostatic equations as mappings defined on $\mathbb{N}_\gamma$ and $\mathbb{D}$, respectively. Moreover, if one imposes the conditions of Hypothesis 2, one is then assured that the range of the operators is the load space $\mathbb{B}$. Hence, the proper matching of the solution space, the load space, and the differential operators from the point of view of differentiability allows one to depict the operators having domains of definition $\mathbb{N}_\gamma$ and $\mathbb{D}$, respectively, and range $\mathbb{B}$:

$$\Phi \ : \ N_\gamma \longrightarrow B \qquad \text{(II.4)}$$
$$\underset{\sim}{u} \qquad \Phi(\underset{\sim}{u}) = (\overline{b}(\underset{\sim}{u}), \overline{w}(\underset{\sim}{u}))$$

$$\Lambda \ : \ D \longrightarrow B \qquad \text{(II.5)}$$
$$\underset{\sim}{v} \qquad \Lambda(\underset{\sim}{v}) = (\overline{a}(\underset{\sim}{v}), \overline{w}(\underset{\sim}{v}))$$

Finally, if one denotes a given body force density field and boundary displacement field and their excess over the reference load as the elements of $\mathbb{B}$

$$\underset{\sim}{\ell}_a = (\underset{\sim}{b}_a, \underset{\sim}{u}_a) \quad \text{and} \quad \underset{\sim}{\ell}_a^* = (\underset{\sim}{b}_a^*, \underset{\sim}{u}_a^*),$$

respectively, then the place boundary value problems for the finite and infinitesimal cases, relative to the given reference configuration, may be written as the operator equations

$$\Phi(\underset{\sim}{u}) = \underset{\sim}{\ell}_a$$

and

$$\Lambda(\underset{\sim}{v}) = \underset{\sim}{\ell}_a^*$$

These equations justify regarding Φ and Λ as the finite and infinitesimal elastostatic operators for the place boundary value problem.

The triples of Banach spaces and mappings between them given by relations (II.4) and (II.5) constitute Van Buren's geometric models for the finite and infinitisimal elastostatic place boundary value problems. They complement the analytical models given by equations (II.1), (II.2), and (II.3). By means of this model one may now proceed to exploit theorems in the infinite dimensional Banach space theory to gain existence and uniqueness information.

How the Local Uniqueness Problem Manifests Itself

In particular, Van Buren's model allows one to investigate solutions to the finite elastostatic problem in the neighborhood of a given equilibrated configuration. One can establish that the finite elastostatic place boundary value problem (II.1) and (II.2) has the (analytic) property of local existence and uniqueness of solution in the vicinity of the given configuration if one can establish the (geometric) property that the operator Φ in (II.4) has an inverse on some neighborhoods of the vector fields $\underset{\sim}{u} = \underset{\sim}{0}$ in $\mathbb{N}_\gamma$, and $\Phi(\underset{\sim}{0}) = \underset{\sim}{\ell}_o$ in $\mathbb{B}$. For, if Φ admits such an inverse, then for any load $\underset{\sim}{\ell}_a$ sufficiently close to the reference load $\underset{\sim}{\ell}_o$, one is assured that there exists one and only one displacement field $\underset{\sim}{u}$ near $\underset{\sim}{0}$ which is equilibrated by it. Notice that such a local uniqueness property does not imply a global uniqueness property: there may exist displacements $\underset{\sim}{u}$ distant from $\underset{\sim}{0}$ equilibrated by the same given load.

In order to establish sufficient conditions for the local existence and uniqueness property, Van Buren exploits the inverse function theorem for Banach spaces. If one can show that the finite elasticity operator Φ is continuously differentiable on $\mathbb{N}_\gamma$ and its Frechet differential $\delta\Phi(\underset{\sim}{0})$ at the zero displacement field $\underset{\sim}{0}$ is a linear hemeomorphism of $\mathbb{D}$ onto $\mathbb{B}$ (hence, the critical requirement that the spaces be properly matched), then the inverse function theorem insures that there is a neighborhood $\mathbb{E}$ of $\underset{\sim}{0}$ in N_γ such that the restriction of Φ to $\mathbb{E}$ is an invertible mapping of $\mathbb{E}$ onto a neighborhood $\Phi(\mathbb{E})$ of the reference load $\Phi(\underset{\sim}{0}) = \underset{\sim}{\ell}_o$ in $\mathbb{B}$.

Van Buren's Result for Local Uniqueness

Van Buren's accomplishment for the place boundary value problem consists of showing that if one chooses the spaces $\mathbb{D}$ and $\mathbb{B}$ to satisfy the hypotheses given above, and if one bestows upon them the norms as stated, then by a series of rather intricate norm calculations one can verify that Φ is continuously differentiable on N_γ, when γ is suitably chosen, and that the Frechet differential $\delta\Phi(\underset{\sim}{0})$ at $\underset{\sim}{0}$ is the infinitesimal elasticity operator Λ.

> Theorem: (Van Buren) Φ is of class C^1 on N_γ and its Frechet differential at $0 \varepsilon N_\gamma$ coincides with the infinitesimal elasticity operator Λ:
>
> $$\delta\Phi(0) = \Lambda.$$

Even with the proper match of operators and spaces provided by Hypothesis 1 through Hypothesis 3, and the above result, the invertibility of Van Buren's infinitesimal elasticity operator does not immediately follow. For this reason, Van Buren is forced to impose an additional hypothesis:

> Hypothesis 4: The body, reference configuration, and the elasticities
>
> $$A(\underset{\sim}{X}) = \text{grad}_1 \hat{S}(1,\underset{\sim}{X}) \ , \ X \epsilon B$$
>
> are such that the infinitesimal place boundary value problem
>
> $$\Lambda(\underset{\sim}{u}) = \underset{\sim}{\ell}{}_a^*$$
>
> defined by (II.5) has, for each $\underset{\sim}{\ell}{}_a^*$ belonging to B, a unique solution $\underset{\sim}{u}$ belonging to D.

If one imposes such a condition, one gains the following theorem on the existence and local uniqueness of solution to the finite elastostatic place boundary value problem as an analytical translation of the geometric result obtained from the application of the inverse function theorem.

> Theorem: Let Hypothesis 1 through 4 be satisfied. Then there are positive numbers Ψ_1 and ρ_1 such that for all data $(\underset{\sim}{b}_a, \underset{\sim}{u}_a)$ in B satisfying
>
> $$\left|\left|(\underset{\sim}{b}_a, \underset{\sim}{u}_a) - (\underset{\sim}{b}_o, \underset{\sim}{0})\right|\right| \leq \Psi_1$$
>
> the finite elastostatic place boundary value problem
>
> $$\mathrm{Div}_{\underset{\sim}{X}} \hat{S}(1 + \mathrm{grad}\ \underset{\sim}{u}(\underset{\sim}{X}), \underset{\sim}{X}) + \underset{\sim}{b}_a(\underset{\sim}{X}) = \underset{\sim}{0}, \quad \underset{\sim}{X} \varepsilon B$$
>
> $$\underset{\sim}{u}(\underset{\sim}{Z}) - \underset{\sim}{u}_a(\underset{\sim}{Z}) = \underset{\sim}{0}, \quad Z \varepsilon \partial B$$
>
> has in the space $\mathbb{D}$ one and only one solution $\underset{\sim}{u}$ for which
>
> $$\left|\left|\underset{\sim}{u}\right|\right| \leq \rho_1 .$$

Comments on the Invertibility Hypothesis

The attractiveness of Hypothesis 4 is questionable. Two comments indicate where it is vulnerable.

(1) When the reference configuration κ is a natural configuration the infinitesimal elasticity operator

is precisely the classical linear elasticity operator. For this particular case Van Buren's results reduce to those originally gained by F. John.[17] Under this circumstance, one may then physically justify imposing additional conditions which insure the invertibility of the infinitesimal elasticity operator based upon results from the theory of the classical linear elasticity place boundary value problem. For instance, with experimental justification one may impose one of the classical elasticity inequalities, or by requiring real wave speeds in linear elastic materials, one may impose a strong ellipticity condition, or finally, one may impose the Coleman-Noll condition, or one of its generalizations. Any of these conditions insure the invertibility of the classical linear elasticity operator, and thereby insure Van Buren's Hypothesis 4.

(2) When the reference configuration κ is not a natural configuration, then the infinitesimal elasticity operator is not the classical linear elasticity operator which would be associated with the configuration.[18] As a result, one has much less physical justification for imposing upon the infinitesimal elasticity operator the conditions one imposed when κ was a natural configuration. Hence, for this situation, though mathematically sufficient, Hypothesis 4 appears much more <u>ad hoc</u> than the previous three.

In short, Van Buren's model provides one means for gaining local existence and uniqueness information about an elastostatic system about a given reference state of strain. Exploitation of the implicit function theorem, however, does not give rise to information regarding finite deformations from the reference state. As F. John has pointed out,[19] a material body may satisfy a local uniqueness condition like that of Van Buren's for every conceivable reference configuration, and yet for a given place boundary condition possess two equilibrating configurations. (A "global" non-uniqueness of solution to the finite elastostatic place boundary value problem).

THE MODEL OF I. BEJU

If local existence and uniqueness conclusions do not lead immediately to conclusions concerning finite deformations, how does one gain such information? What additional mathematical tools are available for exploitation?

One approach for gaining global information is provided by the model of I. Beju.[20] Generally speaking, Beju's geometric model for the finite elastostatic place boundary value problem is similar in structure to

Van Buren's, in that it realizes the problem as a nonlinear mapping between Banach, (in fact Hilbert) spaces. However, by limiting his attention to materials whose response functions are derivable from strain energy functions, (hyperelastic materials), Beju is able to go further than Van Buren in gaining global information. By imposing sufficient conditions to insure that the finite elastostatic operator is monotone, Beju is able to utilize the mathematical theory of monotone operators to yield a global uniqueness statement for the finite place boundary value problem. By imposing slightly stronger conditions he is able to apply "maximum-minimum" theorems of the variational theory to the energy integral constructed from the strain energy function to gain existence statements.

The basic concepts and mathematical tools used by Beju in constructing his model and using it are summarized in his article. The salient features are set forth below.

Monotone Operator Tools Which Beju Uses

The primary mathematical tool utilized by Beju to gain global uniqueness information from his model is the following abstract mathematical theorem, which Beju attributes to A. Lagenbach.

Theorem II.1.[21] (Lagenbach) Let Ω be a bounded region of R^n with boundary $\partial\Omega$. Let $H(\Omega)$ be any Hilbert space of vector-valued functions on Ω. Assume that the boundary of Ω is sufficiently regular to insure the validity of the Green's-Stokes Theorem. Let P be a non inear operator,

$$P : D(P) \longrightarrow H(\Omega), \quad D(P) \subset H(\Omega)$$

and for $f \in H(\Omega)$, consider the nonlinear equation

$$P(u) = f \tag{II.6}$$

subject to the set of linear homogeneous boundary conditions

$$\{L_i u = 0, \quad i - 1,2,\ldots,p\} \tag{II.7}$$

Let $D_o(P) = \{u \varepsilon D(P) : u \text{ satisfies (II.7)}\}$.

Assume

(1) $D_o(P)$ and $D(P)$ are linear sets, and $D_o(P)$ is dense in $H(\Omega)$,

(2) for all $u \varepsilon D(P)$, $h \varepsilon D(P)$, P has a linear Gateaux differential and $(DP)(h)$ is a continuous mapping of u in every two dimensional hyperplane containing u,

(3) $P(0) = 0$

(4) for all $u \varepsilon D(P)$, $h,g, \varepsilon D_o(P)$

$$<(DP)(u)h,g> = <(DP)(u)g,h> \text{ (a symmetry condition)}$$

(5) for all $u \varepsilon D(P)$, $h \varepsilon D_o(P)$, $h \neq 0$

$$<(DP)(u)h,h> > 0 \text{ (a positive definiteness condition)}$$

then

(a) if there exists a solution $u_o \varepsilon D_o(P)$ to (II.6), it is unique, and on $D_o(P)$ it minimizes the functional

$$\begin{aligned} F : D_o(P) &\longrightarrow R \\ u &\longrightarrow F(u) = J(u) - \langle f,u\rangle \end{aligned} \qquad \text{(II.8)}$$

for

$$\begin{aligned} J : D(P) &\longrightarrow R \\ u &\longrightarrow J(u) = \int_0^1 \langle P(tu),u\rangle dt \end{aligned} \qquad \text{(II.9)}$$

(b) Conversely, if an element $u \varepsilon D_o(P)$ minimizes the functional (II.8) it is a solution of (II.6).

Comments on the Theorem

Theorem II.1.† is a global uniqueness theorem. A moment's reflection gives one insight into how the hypotheses are utilized in establishing the conclusions. Four of the assumptions are particularly critical: the linear and dense nature of $D_o(P)$, the existence and special continuity properties of the linear Gateaux differential of P, the symmetry property of the differential, and the positive definiteness property of the differential. They will be examined in turn.

The linearity of the domain of definition $D(P)$ of the operator P and the existence and continuity of the

†Theorem II.1 should be distinguished from Equation II.1. This procedure for eenoting equations, theorems, etc., holds throughout the paper.

linear Gateaux differential allow one to relate the action of the operator at two points to an integral expression involving its derivative:

for $x, x_0 \varepsilon D(P)$

$$P(x) - P(x_0) = \int_0^1 [(DP)(x_0 + t(X - x_0))](x - x_0)\, dt.$$

The symmetry property of the differential relative to the Hilbert space inner product, allows one to relate variation of the integral function F with an integral expression involving the Gateaux differential (DP). The positive definiteness property allows one to establish that a solution $u_0 \varepsilon D_0(P)$ to the equation (II.6) minimizes the integral function F on $D_0(P)$.

Moreover, positive definiteness property of (DP) insures that the operator P is a monotone operator on $D_0(P)$. This property allows one to establish the uniqueness of solution $u_0 \varepsilon D_0(P)$, if it exists.

One can now begin to appreciate the diverse mathematical elements entering into a global uniqueness theorem: a "minimax" principle for the variation of an integral expression, and a result from the theory of monotone operators.

If one strengthens the positive definiteness property one begins to gain existence information.

Theorem II.2.[22] If condition (5) of Theorem II.1 is changed into a stronger one

$$<(DP)(u)h,h> \geq c||h||^2, \quad u\varepsilon D(P), \quad h\varepsilon D_0(P), \; c>0, \text{ a constant,}$$

then

(a) the functional (II.8) is bounded below on $D_0(P)$,

(b) moreover, it is strictly convex on $D_0(P)$, that is, for $u,v \varepsilon D_0(P)$, $u \neq v$, and for $t\varepsilon(0,1)$,

$$F(tu+(1-t)v) \leq tF(u)+(1-t)F(v)$$

(c) Any minimizing sequence of the functional is convergent in $H(\Omega)$.

Definition II.1. A generalized solution to equation (II.6) is defined to be the limit of a minimizing sequence for the functional (II.8).

Theorem II.3. The generalized solution to (II.6) is unique.

The proof of Theorem II.2 is non-constructive, in the sense that the desired solution is not explicitly constructed in the proof. After intricate computations one establishes that some minimizing sequence must exist; its specific nature is not determined.

One can, however, gain some limitations on where in $H(\Omega)$ the solution lies from additional theorems which can be found in Lagenbach's paper.

How is Beju able to utilize Lagenbach's abstract results to gain information about the finite elastostatic problem? He accomplishes the feat by carefully erecting a geometric model for certain finite elastostatic problems which fulfill the assumptions of Theorem II.1. The first elements which are formulated from the analytical model are the data and solution spaces and the finite elastostatic operator.

Beju's Analytical Model

Beju's analytical model for the finite elastostatic place boundary value problem is gained in the following way. Let Ω represent the region in R^3 occupied by the body in the reference configuration. Let the boundary $\partial\Omega$ be sufficiently regular to guarantee that the Stokes-Green theorems hold. For a given deformation $\underset{\sim}{\chi}$ of the body from the reference configuration,

$$x = \underset{\sim}{\chi}(X) \ , \qquad \underset{\sim}{X}\varepsilon\Omega$$

let the deformation gradient field $F(\underset{\sim}{X})$ be given by

$$\underline{F}(\underset{\sim}{X}) = \text{Grad } \underset{\sim}{X}(\underset{\sim}{X})$$

Let the constitutive relation for the first Piola-Kirchoff tensor field be given by

$$T_R = h(\underline{F}, \underset{\sim}{X})$$

Let $\underset{\sim}{b}(\underset{\sim}{X})$ represent the given density of external body forces per unit mass. Then the differential equation for finite elastostatics becomes

$$\text{Div } h(\underset{=}{F},\underset{\sim}{X}) + \rho_R \underset{\sim}{b}(\underset{\sim}{X}) = 0$$

where ρ_R is the mass density in the reference configuration. The boundary condition of place is specified by imposing a requirement that the boundary of the body assume a given shape

$$\underset{\sim}{\chi}(\underset{\sim}{X}) = \underset{\sim}{\chi}_0(\underset{\sim}{X}), \qquad \underset{\sim}{X} \varepsilon \partial\Omega$$

For convenience of presentation, consider the case where the body is homogeneous, and the reference configuration is a homogeneous reference configuration. The response function becomes a function of the deformation gradient only,

$$h = h(\underset{=}{F}).$$

To formulate the model it is convenient to characterize the deformation in terms of a displacement vector field $\underset{\sim}{u}(\underset{\sim}{X})$,

$$\underset{\sim}{u}(\underset{\sim}{X}) = \underset{\sim}{\chi}(\underset{\sim}{X}) - \underset{\sim}{X}.$$

The differential equation for finite elastostatics and the boundary condition of place may then be expressed in terms of the displacement vector field as

$$\mathrm{Div}\, h + \rho_R \underset{\sim}{b} = \underset{\sim}{0}, \quad \text{for } \underset{\sim}{X} \varepsilon \Omega \qquad \text{(II.10)}$$

$$\underset{\sim}{u}(\underset{\sim}{X}) = \underset{\sim}{a}(\underset{\sim}{X}) \quad \text{for } \underset{\sim}{X} \varepsilon \partial\Omega \qquad \text{(II.11)}$$

where $\quad h = h(\underline{F})$,

$\qquad \underline{F}(\underset{\sim}{X}) = 1 + \underline{H}(\underset{\sim}{X})$

and $\quad \underline{H}(\underset{\sim}{X}) = \mathrm{Grad}\, \underset{\sim}{u}(\underset{\sim}{X})$.

The Homogeneous Boundary Condition Formulation

Equations (II.10) and (II.11) constitute Beju's analytical model for the finite elastostatic place boundrry value problem. It is a differential equation with inhomogeneous boundary conditions. In view of Langenbach's results, the first step in Beju's construction is to transform the boundary value problem into one with homogeneous boundary conditions. This step is accomplished by introducing a known, but arbitrary vector field $\underset{\sim}{v}$ defined over the reference configuration plus boundary $\overline{\Omega}$ which satisfies the given place boundary conditions:

$$\underset{\sim}{v}(\underset{\sim}{X}) = \underset{\sim}{a}(\underset{\sim}{X}), \quad X \varepsilon \partial\Omega.$$

Beju now takes as the unknown the vector field $\underset{\sim}{w}$ defined by

$$\underset{\sim}{w}(\underset{\sim}{X}) = \underset{\sim}{u}(\underset{\sim}{X}) - \underset{\sim}{v}(\underset{\sim}{X}), \qquad \underset{\sim}{X}\varepsilon\overline{\Omega}$$

One may now formulate the finite elastostatic boundary value problem as a homogeneous value problem in terms of the vector field $\underset{\sim}{w}$, and develop a finite elastostatic operator. Define an operator A which takes vector fields $\underset{\sim}{u}$ into vector fields by

$$A(\underset{\sim}{u}) = -(1/\rho_R)\ \mathrm{Div}\ h(1 + \underline{H}) - \underset{\sim}{b} \tag{II.12}$$

The finite elastostatic differential equation becomes

$$A(\underset{\sim}{u}) = A(\underset{\sim}{v} + \underset{\sim}{w}) = 0$$

Introduce a second operator $E_{\underset{\sim}{v}}$ which takes vector fields into vector fields by

$$E_{\underset{\sim}{v}}(\underset{\sim}{w}) = A(\underset{\sim}{v} + \underset{\sim}{w}) - A(\underset{\sim}{v}).$$

If one defines the vector field $\underset{\sim}{f}$ by

$$\underset{\sim}{f} = -A(\underset{\sim}{v}),$$

then the finite elastostatic boundary value problem of place (II.10) and (II.11) is transformed into a nonlinear boundary value problem with homogeneous place boundary conditions

$$[E_{\underset{\sim}{v}}(\underset{\sim}{w})](\underset{\sim}{X}) = \underset{\sim}{f}(\underset{\sim}{X}) \qquad \underset{\sim}{X}\varepsilon\Omega \tag{II.13}$$

$$\underset{\sim}{w}(\underset{\sim}{X}) = \underset{\sim}{0} \qquad \underset{\sim}{X}\varepsilon\partial\Omega \tag{II.14}$$

The operator E_v will serve as the finite elastostatic operator associated with the place boundary value problem. Notice that it depends upon the choice of the fixed vector field $\underset{\sim}{v}$. The subscript on the operator emphasizes this point. Comments about the dependence upon $\underset{\sim}{v}$ are reserved until later.

Beju's Geometric Model

Equations (II.13) and (II.14) serve as Beju's analytical model for the finite elastostatic place boundary value problem in homogeneous boundary formulation. A geometric/topological model may now be developed by extending the finite elastostatic operator from acting on vector fields with classical differentiability properties, to vector fields which are differentiable in a more generalized sense. Beju's data space is constructed from $L_2(\Omega)$, the Hilbert space of vector-valued functions which are square integrable over Ω. The inner product on the space is chosen to be

$$\langle u,v\rangle = \int_{\Omega} \rho_R u_i v^i d\Omega$$

As the space of possible generalized solutions, $W^2(\Omega)$, Beju chooses the subset of $L_2(\Omega)$ which belongs to $C^2(\Omega)$ and satisfy the requirement (II.14). An element of $W^2(\Omega)$ thereby satisfies the homogeneous place boundary condition and is continuously differentiable through second order in the classic sense. As Beju points out, it is possible to show that $W^2(\Omega)$ is a linear dense subset of $L_2(\Omega)$. Notice, however, that the set is not complete in the Cauchy sense.

Finally, Beju's geometric model for the finite elastostatic place boundary value problem is gained by considering the finite elastostatic operator of equation (II.13) extended to the generalized solution space. As $\underset{\sim}{E_v}$ is classically a second order operator, it can be shown by theorem that the extension of

$$\underset{\sim}{E_v} : W^2(\Omega) \longrightarrow L_2(\Omega) \qquad \text{(II.15)}$$

Hence, Beju's solution space, data space, and finite elastostatic operator are "properly matched" in a manner similar to Van Buren's model (II.15) is Beju's geometric model for the problem.

How Beju Utilizes His Model

Like Van Buren's model, Beju's geometric model consists of a nonlinear mapping between two topological vector spaces. It differs from Van Buren's model in that the data space is a Hilbert space, as opposed to simply a Banach space. Secondly, the place boundary condition is incorporated as an algebraic constraint which helps define the solution space $W^2(\Omega)$, as opposed to being incorporated into the data space. Finally, for Beju, the finite elastostatic differential operator need not be continuously differentiable in the sense of Van Buren. This is why Beju investigates the Gateaux differentiability of the operator as opposed to the Frechet differentiability. Gateaux differentiability is slightly more general.

It is in how the model is utilized that Beju's conclusions go beyond those of Van Buren. By a series of lemmas Beju is able to determine sufficient conditions on the elastostatic operator in order that it satisfy the hypotheses of the Langenbach theorems. When interpreted physically, these conditions delimit the class of material bodies to which the conclusions of the Lagenbach theorems apply.

The first lemma indicates what conditions must be placed upon the finite elastostatic operator and the body force to insure that it have a linear Gateaux differential with the continuity requirements needed by the Langenbach theorem. These conditions fulfill the role played by Hypothesis 2 in Van Buren's model.

> Lemma II.1.[23] If the response function h and the body force density field $\underset{\sim}{b}$ have continuous derivatives of second and first order, respectively, then
>
> a) the operator $E_{\underset{\sim}{v}}$ has a linear Gateaux differential on $W^2(\Omega)$
>
> b) The differential can be represented explicitly: for $\underset{\sim}{w}$, $\underset{\sim}{g} \in W^2(\Omega)$
>
> $$[(DE_{v_i})(\underset{\sim}{w})]\underset{\sim}{g} = -(1/\rho_R)[\Lambda_{ik}^{\alpha\beta} g^k{}_{,\beta}]_{,\alpha} - b_{i,k} g^k$$
>
> $$\text{for} \quad \Lambda_{ik}^{\alpha\beta} = \frac{\partial h_i^\alpha}{\partial X^k{}_{,\beta}} = \frac{\partial h_i^\alpha}{\partial F_\beta^k}$$
>
> c) For a given $\underset{\sim}{g}$, $[(DE_{\underset{\sim}{v}})(\underset{\sim}{w})]\underset{\sim}{g}$ is a continuous mapping of $\underset{\sim}{w}$ in every hyperplane which contains the point $\underset{\sim}{w}$.
>
> d) $E_{\underset{\sim}{v}}(\underset{\sim}{0}) = \underset{\sim}{0}$.

Hyperelastic Materials with Certain Material Symmetry Satisfy the Criterion of Langenbach's Theorems

Next, Beju shows that if one restricts attention to hyperelastic materials experiencing conservative body forces, the Gateaux differential of the finite elastostatic operator fulfills the symmetry property demanded by the Langenbach theorem.

Lemma II.2.[24] If the body is hyperelastic and if the body forces $\underset{\sim}{b}$ are conservative:

$$h = \rho_R \sigma_F$$

$$h^i_\alpha = \rho_R \frac{\partial \sigma}{\partial x^k}{}_{,\alpha}$$

$$\underset{\sim}{b} = -\text{Grad } V$$

$$v_i = -V_{,i}$$

then the operator $\underset{\sim}{E}_v$ has the following symmetry property

$$<(D\underset{\sim}{E}_v)(\underset{\sim}{w})\underset{\sim}{g},\underset{\sim}{\ell}> = <(D\underset{\sim}{E}_v)(\underset{\sim}{w})\underset{\sim}{\ell},\underset{\sim}{g}> \quad \text{for } \underset{\sim}{w},\ \underset{\sim}{\ell},\ \underset{\sim}{g}\varepsilon W^2(\Omega),$$

where

$$<(D\underset{\sim}{E}_v)(\underset{\sim}{w})\underset{\sim}{g},\underset{\sim}{\ell}> = \int_\Omega A^{\alpha\beta}_{ik}\, g^i{}_{,\alpha}\, \ell^k{}_{,\beta} d\Omega + \int_\Omega V_{,ik}\ell^i g^k \, dm$$

The third and fourth lemmas provide mathematical conditions which insure that the Gateaux differential of $\underset{\sim}{E}_v$ satisfies the positive definite properties of Theorem II.1 or Theorem II.2,

Lemma II.3.[25] If the hypothesis of the previous lemma are satisfied and if for all $\underset{\sim}{w},\underset{\sim}{g} \varepsilon W^2(\Omega)$, $\underset{\sim}{g} \neq 0$,

$$\int_{\Omega} [V_{,ik}\, g^i g^k + \frac{\partial^2 \sigma}{\partial X^i{}_{,\alpha} \partial X^k{}_{,\beta}}\, g^i{}_{,\alpha} g^k{}_{,\beta}]\, dm \geq 0 \qquad \text{(II.16)}$$

then the linear Gateaux differential of $E_{\underset{\sim}{V}}$ is positive:

$$<(DE_{\underset{\sim}{V}})(\underset{\sim}{w})\underset{\sim}{g},\underset{\sim}{g}> \; > 0$$

Lemma II.4.[26] If the hypothesis of the previous lemma are satisfied and if condition (II.16) is strengthened to

$$\int_{\Omega} [V_{,ik}\, g^i g^k + \frac{\partial^2 \sigma}{\partial X^i{}_{,\alpha} \partial X^k{}_{,\beta}}\, g^i{}_{,\alpha}\, g^k{}_{,\beta}]\, dm \geq c \int_{\Omega} g_i g^i\, dm \qquad \text{(II.17)}$$

then the linear Gateaux differential of E_V is positive definite

$$<(DE_V)(\underset{\sim}{w})\underset{\sim}{g},\underset{\sim}{g}> \; \geq c\, \|\underset{\sim}{g}\|^2_{L_2(\Omega)} \quad , \quad c>0, \text{ a constant.}$$

As Beju points out, the mathematical conditions (II.16) and (II.17) have a physically justifiable basis. They are related to the condition imposed by Coleman and Noll to insure the static stability of a configuration of a hyperelastic material under conservative body forces. Notice also, that the last two conditions are global conditions, (integrated conditions), as opposed to local conditions which must be satisfied point by point.

A Global Uniqueness Result in the Weak Problem

The above lemmas now allow Beju to gain global existence and uniqueness statements for finite elastostatics from his model. Lemmas II.1 through II.4 insure that the conditions of Langenbach's Theorem II.1 are satisfied. Applying the theorem, Beju gains the following uniqueness theorem for some hyperelastic materials.

Theorem II.4.[27] If

(1) the material body is hyperelastic,
(2) the body forces are conservative,
(3) the constitutive equation satisfies equation (II.16),

then

(a) given $\underset{\sim}{f} \in L_2(\Omega)$, if a solution $\underset{\sim}{w}_0 \in W^2(\Omega)$ of the equation

$$E_{\underset{\sim}{v}_0}(\underset{\sim}{w}) = \underset{\sim}{f} \tag{II.18}$$

exists, it is unique and attains on $W^2(\Omega)$ the minimum of the functional

$$F : W^2(\Omega) \longrightarrow R$$

$$\underset{\sim}{w} \longrightarrow F(\underset{\sim}{w}) = J(\underset{\sim}{w}) + \langle \underset{\sim}{f}, \underset{\sim}{w} \rangle \tag{II.19}$$

where $J(\underset{\sim}{w})$ is defined in Theorem II.1

(b) Conversely, if an element $\underset{\sim}{w}_0 \in W^2(\Omega)$ attains on $W^2(\Omega)$ the minimum of the functional (II.19), then it is a generalized solution of (II.18).

Uniqueness in the Classical Problem

The uniqueness theorem guarantees the global uniqueness of solution of equation (II.18) in Beju's model. However, Beju notes, it does not immediately insure the uniqueness of the solution to the classical inhomogeneous boundary value problem (II.10) and (II.11). One must establish the role played by the vector field $\underset{\sim}{v}_0$ used to convert the problem to a homogeneous boundary condition. Given two distinct vector fields $\underset{\sim}{v}_0$ and $\underset{\sim}{v}_{00}$ which represent the inhomogeneous boundary condition, the Theorem II.4 insures the uniqueness of solutions $\underset{\sim}{w}_0$ and $\underset{\sim}{w}_{00}$ to equation (II.18) for the operators $E_{\underset{\sim}{v}_0}$ and $E_{\underset{\sim}{v}_{00}}$, respectively. However, it does not insure that the two solutions to the inhomogeneous problem (II.10) and (II.11)

$$\underset{\sim}{u}_0 = \underset{\sim}{v}_0 + \underset{\sim}{w}_0$$

$$\underset{\sim}{u}_{00} = \underset{\sim}{v}_{00} + \underset{\sim}{w}_{00}$$

are identical. A critical element in understanding Beju's conclusions rests in comprehending under what conditions the two classical solutions are unique.

> Lemma II.5.[28] Given Beju's model (II.15), if a solution $\underset{\sim}{u} \in C^2(\Omega)$ exists, to the inhomogeneous problem (II.10) and (II.11), it is unique.

The proof of the lemma requires that one establish that the operator A defined by equation (II.12) is strictly monotone on the domain

$$D(A) = \{\underset{\sim}{u} \varepsilon C^2(\Omega) : \underset{\sim}{u}(\underset{\sim}{X}) = \underset{\sim}{a}(\underset{\sim}{X}),\ \underset{\sim}{X} \varepsilon \partial\Omega\}.$$

Beju gains this result by showing that the domain $D(A)$ is convex, and that the condition (II.16) insures that the directional derivative

$$\frac{d}{dt} \langle A(\underset{\sim}{u} + t\underset{\sim}{g}), \underset{\sim}{g} \rangle \Big|_{t=0}$$

is positive for all $\underset{\sim}{u}\ \varepsilon\ D(A)$ and $\underset{\sim}{g}$ such that $\underset{\sim}{g}\big|_{\partial\Omega} = \underset{\sim}{0}$.

An Existence Result

Finally, an existence theorem is gained if one imposes the stronger condition (II.17):

Theorem II.5.[29] If the hypotheses of Theorem II.4 are satisfied and the constitutive relation satisfies (II.17), then

(a) The functional (II.19) is bounded below on $W^2(\Omega)$.

(b) The functional (II.19) is strictly convex on $W^2(\Omega)$.

(c) Any minimizing sequence of (II.19) is convergent in $L_2(\Omega)$, and its limit is a generalized solution of (II.18)

(d) The generalized solution is unique.

The proof of the theorem is non-constructive; however, one can gain some limitations as to where in $L_2(\Omega)$ the generalized solution may lie. One is referred to Beju's paper for this result.

Some Comments on Beju's Model

Several features in Beju's model are worthy of note for one who wishes to build a more general geometric model for the finite elastostatic place boundary value problem. The main reason why Beju's conclusions go beyond those of Van Buren's, at least in one direction, is that at the proper time Beju releases himself from the full generality of his geometric model. Like Van Buren's model, Beju's geometric model (II.16) holds for all material bodies with the proper degree of smoothness in body and response. But whereas Van Buren attempts to draw conclusions utilizing the universal model, Beju restricts his attention to the subclass of smoothly responding materials which are hyperelastic and satisfy a condition like the Coleman-Noll condition for finite stability.

Such restriction is in keeping with the spirit of nonlinear investigations. When one investigates phenomena in terms of a linear model, for instance, infinitesimal deformations, one expects to draw conclusions of a universal nature applicable to all materials. But as one investigates more finite deformations one expects the particular characteristics of materials to manifest themselves. Out of the common, universal linear behavior sprout many different subclasses of nonlinear behavior, which become more numerous as the deformations become more extreme. Thus, in order to utilize any future model for the finite elastostatic place boundary value problem to any profitable degree, one must expect to restrict one's attention, in turn, to particular subclasses of materials. The particular mathematical tools available influence which subclasses will be considered. For instance, in Beju's model, it was the Langenbach symmetry condition which dictated the restriction to hyperelastic materials.

The specific class of materials chosen by Beju, the hyperelastic materials, is an especially propitious one, in that the finite elastostatic operator may be viewed as derivable from a variational principle. In the Algebraic and Differential Topology there is a wealth of untapped resources which may be applied to operators of the variational type. The Langenbach theorems are but one example of the utilization of a part of this resource: a minimax principle coupled with a theory of monotone operators.

POINTS OF DEPARTURE FOR FUTURE MODELS

The models of Van Buren and Beju also contain elements which are promising points of departure for future models. Attention will be focused upon four particular points: the role played by the mathematical structure imposed upon the model in drawing existence and uniqueness conclusions, how the models reflect changes in the boundary conditions and the body shape, which elements of the models are constrained in order to gain existence and uniqueness conclusions, and which mathematical methods are used to gleen information from the models. These points of departure will be examined in turn.

How many of the existence and uniqueness conclusions gained in the models presented above depend upon the particular mathematical structure imposed? In Van Buren's model, the Holder space structure is relied upon heavily in drawing conclusions. For example, the theorem relating the Frechet derivative of the finite elasticity operator and the infinitesimal elasticity operator follows only after intricate calculations in the norms previously specified. Is the relationship independent of the particular norms imposed? If not, one would be faced with the unenviable task of physically motivating the choice of norms.

A second difficulty arising from Van Buren's reliance upon a particular Holder space structure is that many results which are available from the classical linear elasticity theory cannot be immediately implemented. They are gained by imposing a Sobolev space structure on the solution and load spaces, as opposed to a Holder space structure. To incorporate them into Van Buren's model one is faced with the task of adopting a suitable Sobolev norm for Van Buren's model, and performing a myriad of intricate norm calculations to determine if Van Buren's original conclusions withstand the revision.

A profitable alternative to Van Buren's approach would be to cast the geometric model in terms of several different function space settings simultaneously. In chapter three, mathematical methods are presented which give circumstances under which one can accomplish this reformulation. One may then determine which conclusions hold for all settings. Moreover, one gains these conclusions by theorem, as opposed to intricate norm calculations. Hence, a means is available by which one can choose a particularly convenient function space setting for analyzing a particular aspect of the boundary value problem, draw conclusions, and then carry the conclusions over to another function space setting to analyze some other aspect of the problem.

The second point of departure is concerned with how conditions are imposed upon the model to achieve existence and uniqueness conclusions. In Van Buren's and Beju's models conclusions follow from conditions placed upon the elastostatic operator and its derivative, or the response of the material comprising the body. The conclusions drawn in both models are independent of the shape of the body and the boundary condition imposed. An attractive alternative would be a model in which all three factors can affect the conclusions. This perspective leads to the next point of departure: in constructing a model sensitive to all three factors, ought one to regard the solution space as a linear space?

In the linear infinitesimal theory a linear solution space is quite natural; however, as one generalizes to a finite nonlinear model, uncomfortable features arise. From Van Buren's and Beju's models one sees that if the space of all possible solutions is a linear space, then the subsets representing different place boundary conditions are all alike topologically. They are but translates of the zero boundary displacement space. Consider a problem where one place boundary condition admits a unique equilibrating configuration for a given load, while a second one admits more than one solution for the same load. In both models one cannot expect to anticipate the discrepancy by simply looking for topological differences between the two solution subspaces. One is forced to solve the problem before the information becomes available. In other words, the solution spaces in either model are insensitive to the dependence of the uniqueness question upon the boundary condition imposed.

Could the dependence be incorporated into a topological distinction between various place boundary solution subspaces? The question is made more provocative when one recalls that in Beju's proof of the uniqueness of the classical solution of the finite elastostatic problem, a critical element was the convex nature of the domain D(A), a topological property. This particular alternative will be investigated more thoroughly in chapter four.

Another question of interest is what happens to the uniqueness and existence conclusions if one drills a hole in the material body? If one changes the topology of the material body it is not obvious that one can anticipate how Van Buren's model or Beju's model will be altered. It would be most satisfying if a future model would allow one the luxury of accommodating such changes in the material body. This point will also be examined in chapter four.

Finally, in addition to the basic structure of the geometric model, Beju's model and Van Buren's model point the way to more refined methods for gleaning information from future models. For instance, Beju's use of Langenbach's theorems gives one a grasp on situations where behavior is uniquely determined; however, it does not give one any insight into non-unique situations. There are, however, mathematical tools which may be applied to variational problems which give some insight into the extent of nonuniqueness of solution. The Morse theory is an example. Even the non-variational case can be investigated to some extent using newer functional analysis methods.

Thus in examining the models of Van Buren and Beju one is introduced to what constitutes a geometric model for the elastostatic place boundary value problem, what features are particularly valuable for any model, and where one might begin to improve on existing models. With this foundation one may now confidently break ground on a new model.

III. THE MODERN SETTING FOR GLOBAL ANALYSIS

How to Read this Chapter

One can improve upon the existing qualitative models for continuum mechanics by employing some mathematical tools which are quite recent in origin, and which, heretofore, were regarded as "pure" in nature. Recent advances made in setting the foundation for the theory of global analysis yields the capacity to revise the current models with an economy of effort.

One may see this point most clearly by example. In the last chapter one found that the solution space could be improved markedly if one could replace its topological vector space structure with an alternative structure which was more sensitive to change in the boundary conditions and the body shape. Such alternative structures were not available even as late as a decade ago. However, with the advances in the differential geometry and algebraic topology, and the development of a geometric/topological setting for the theory of differential equations, which have been made in the last ten years, alternative structures are now becoming available. In particular, subsequent chapters will show that the solution space can possess a well-defined structure as a differentiable, but infinite-dimensional, manifold.

The problem with utilizing the new mathematical techniques in that they are not common knowledge among physicists. Therefore, for the convenience of the reader, this chapter presents a summary of the mathematical setting which will be utilized to cast the qualitative model developed in this thesis. As one examines this chapter, one should keep in mind several features which the contemporary geometric/topological setting for the theory of differential equations exhibit. These features reflect comments made at the end of the previous chapter. They are:

(1) that the solution and data spaces may be differentiable manifolds, as opposed to simply topological vector spaces;

(2) that the abstract setting is "categorical" in nature; hence, one may simultaneously cast a differential equation in several function space settings simultaneously, as opposed to being limited to one setting at a time;

(3) that one may deduce properties of the elements of the setting, like a linearization of a differential operator, by theorem, as opposed to resorting to intricate norm calculation.

The chapter is divided into two parts. The first part provides a description of how the theory of differential equations evolved to its contemporary abstract setting. The non-mathematical description provides one with an overview of the setting. The second part sets forth the particular mathematical elements of the setting. It is recommended that one simply scan this latter part on the first reading, and return to it as specific elements are called upon.

THE EVOLUTION OF MODERN GLOBAL ANALYSIS

The Use of Function Spaces to Study Linear Partial Differential Equations

Recall that the theory of linear partial differential equations in n-variables gained new impetus when the "method of weak solutions" was investigated in a rigorous manner.[30] The result of the investigation was a program for considering linear partial differential equations which consisted of three steps. Firstly, a collection or "chain" of function spaces, which were well-defined infinite dimensional topological vector spaces, were specified. These spaces generalized the set of weak solutions in the classic partial differential equation theory. Examples of function spaces used frequently were the C^k spaces, the Holder spaces, the Lipschitz spaces, and the Sobolev spaces. Secondly, a set of embedding theorems relating function spaces of the chain was developed; for example, the Rellich and Sobolev embedding theorems. Finally, the linear partial differential equation was generalized to a continuous linear map between spaces of the chain, extending the classic notion of the linear partial differential equation in the "weak" or integrated form. In this setting, the questions of the existence, uniqueness, and regularity of solution to the linear partial differential equation could be investigated using the theory of linear operators on Banach spaces. The theory of Elliptic Differential Operators in n-variables[31] is a good example of the success of such an approach.

Linear Partial Differential Equations in Non Euclidean Manifolds

In recent years the above program has been "globalized". With the development of the vector bundle theory it has become possible to "piece together" an n-variable theory to investigate linear partial differential equations on mathematical manifolds which can only locally be identified with an n-dimensional Euclidean space. In this setting, the functions which were investigated in the n-variable setting are now generalized to sections of a vector bundle. The chains of function spaces which served as the solution and data spaces in the n-variable theory are now replaced by chains of spaces of sections. Each section spaces is a Banach Space. The differential equation again is manifested as a linear mapping between Banach spaces. Hence, over non Euclidean manifolds the setting for a linear partial differential equation is similar in structure to the function space setting of the n-variable theory. The major difference is that the analytical expressions one usually encounters for differential equations now are regarded as local coordinate representations of a global, intrinsic statement.

Some statements about the existence and uniqueness of Elliptic partial differential operators on manifolds have been achieved using this setting.[32] Such results are particularly of value to Continuum Mechanics when one investigates material bodies which can not be mathematically modeled as an n-dimensional Euclidean space, or for which such a model is inconvenient. Inhomogeneous bodies serve as an example.

The Abstract Setting for Global Linear Analysis

With the advent of the Category theory, it became possible to abstract the program for linear differential equations to a theory independent of the choice of function spaces serving as the data and solution spaces. As a result, the formulation of existence and uniqueness questions become much clearer, because the complicating factors which arise solely from the particular choice of function space setting can be removed. It is this abstract formulation of the original program for the study of linear equations which is called the theory of global linear analysis.

The Abstract Setting for Global, Nonlinear Analysis

Quite recently, through the effort of S. Smale[33] and R. Palais,[34] a foundation for a theory of global nonlinear analysis has been achieved by extending or "piecing together" the theory of global linear analysis. In the theory, a chain of infinite-dimensional manifolds, each having a well-defined differentiable structure, is achieved. These manifolds are modeled on a suitable Banach function space; hence, locally, the chain of manifolds resemble the chain of elements of a global linear theory. Secondly, embedding theorems are obtained which relate the manifolds of the chains, analogous to the imbedding theorems for the global linear theory. Thirdly, the nonlinear partial differential equation gains representation as a differentiable mapping between manifolds of the chain. The derivative of the mapping, viewed as a linear mapping between the Banach function spaces modeling the manifolds of the nonlinear theory, may be related to the classic "linearization" of the nonlinear differential operator. Fourthly, the boundary condition associated with a Dirichlet problem manifests itself as a constraint which selects a subset of points of the solution manifold associated with the "free boundary" problem. A remarkable achievement is that this subset of points has a well-defined structure as a differentiable submanifold of the free boundary solution manifold. The Dirichlet problem may then be viewed as the study of the nonlinear differential operator restricted to the submanifold of possible solutions satisfying the boundary conditions.

Finally, whereas the theory of linear operators could be used to investigate questions in the global linear analysis, the entire weight of the Differential Geometry, the Differential Topology, and the Algebraic Topology may be brought to bear on questions in nonlinear analysis, when formulated in the above terms. One element of the Differential Topology, the Inverse Mapping Theorem, has been utilized in previous works. Yet it is the still untapped resources which hold the most promise for exploitation. More will be said about these possibilities at the conclusion of this work.

ELEMENTS OF THE GLOBAL LINEAR ANALYSIS

Having summarized the current setting for the theory of global analysis, the particular elements of the theory necessary for the reformulation of finite elastostatics may now be presented. The elements of the global linear theory will be presented first. How they extend to the nonlinear theory will then be summarized, using the results of Palais cited above.

The Banach Space-Valued Section Functor

The initial element of the global linear analysis is the capability of associating with any vector bundle over a given mathematical manifold a set of Banach spaces which can serve as solution and data spaces for linear equations. Exploiting the Category theory, this task can be most effectively accomplished by the specification of a Banach space-valued section functor from the category of vector bundles over the given manifold and vector bundle morphisms into the category of Banach spaces and continuous linear maps.

Definition III.1.[35] (Banach space-valued section functor). Denote by $\mathcal{M}$ a covariant functor from the category of C^∞ vector bundles over a finite-dimensional, compact, C^∞ manifold, M, (possibly with boundary), and vector bundle morphisms into the category of Banach spaces and continuous linear maps:

a) $\mathcal{M}$ is a function which associates with a C^∞ vector bundle ξ over M a complete normable topological vector space $\mathcal{M}(\xi)$ of sections of ξ which includes $C^\infty(\xi)$ and which is a subspace of the vector space $S(\xi)$ of all sections of ξ.

b) $\mathcal{M}$ associates with every C^∞ vector bundle morphism f of C^∞ vector bundles ξ, η over M,

$$f: \xi \longrightarrow \eta_1$$

a continuous linear map $\mathcal{M}(f)$ of the image section spaces

$$\mathcal{M}(f): \mathcal{M}(\xi) \longrightarrow \mathcal{M}(\eta).$$

Term $\mathcal{M}$ a Banach space-valued section functor.

An example of such a functor is C^0.

Given a functor $\mathcal{M}$ it is convenient to generate from it a set of derivative functors defined as follows. Recall that for ξ a C^∞ vector bundle over M, $J^k(\xi)$ denotes the bundle of k-jets of sections of ξ, also a C^∞ vector bundle over M.

Definition III.2.[36] (Derivative Functors). Given a Banach space-valued section functor $\mathcal{M}$, for every C^∞ vector bundle ξ over the compact C^∞ n-manifold M, let

a) $$\mathcal{M}_{(k)}(\xi) = \{s \varepsilon C^k(\xi) : j_k s \varepsilon \mathcal{M}(J^k(\xi))\},$$

a vector space. Topologize $\mathcal{M}_{(k)}(\xi)$ by the requirement that the map

$$j_k: \mathcal{M}_{(k)}(\xi) \longrightarrow \mathcal{M}(J^k(\xi))$$

be an into homeomorphism. $\mathcal{M}_{(k)}(\xi)$ is then a normable topological space.

b) Define $M_k(\xi)$ to be the completion of $M_{(k)}(\xi)$ in the above topology, so that j_k extends to a continuous linear isomorphism of $M_k(\xi)$ onto a closed subspace of $M(J^k(\xi))$.

An example of a collection of derivative functors would be the functors C^k, derived from the functor C^0 as $(C^0)_k$.

The properties desired for such functors may be conveniently expressed in terms of four axioms stated in Palais' text. More will be said about them later.

A Differential Operator is Represented as a Linear Mapping of Banach Spaces

Introducing the section functor is valuable, in that it allows the notion of a linear differential operator to be conveniently expressed and viewed as a linear mapping on well defined vector spaces:

Definition III.3.[37] (Linear Differential Operator) A linear differential operator D of order k (with C^∞ coefficients) taking sections of ξ into sections of η is a linear continuous mapping.

$$D: \quad C^\infty(\xi) \longrightarrow C^\infty(\eta)$$

which factors through the k^{th} order jet bundle of sections of ξ. That is to say, there exists a C^∞ vector bundle morphism

$$f: \quad J^k(\xi) \longrightarrow \eta$$

such that $D = f_* \cdot j_k$:

$$\begin{tikzcd} C^{\infty}(\xi) \arrow[rr, "D"] \arrow[dr, "j_k"'] & & C^{\infty}(\xi) \\ & C^{\infty}(J^k(\xi)) \arrow[ur, "f_*"'] & \end{tikzcd}$$

Denote the set of $k^{\underline{th}}$ order linear differential operators from ξ into η by $\mathrm{Diff}_k(\xi,\eta)$.

> Theorem III.1.[38] (The action of linear differential operators on section functor spaces.) If M is a Banach space-valued section functor satisfying axioms (B§1) through (B§4) in Palais' work , and if D is a linear differential operator of order k from ξ into η, then
>
> $$D: C^{\infty}(\xi) \longrightarrow C^{\infty}(\eta)$$
>
> extends to a set of continuous linear mappings of the section spaces $\{M_r(\xi)\}$ and $\{M_r(\eta)\}$ as
>
> $$D_r = M_r(D): M_{r+k}(\xi) \longrightarrow M_r(\eta),$$
>
> $r = 0,1,2,\ldots.$ When there is no confusion, the extended operator D_r will be denoted simply by D.

It is with the last theorem that the connection can be seen between the Banach section functor setting and the modern theory of linear partial differential equations. For if M is taken to be the Holder functor C^{α}, or the Sobolev functor L_0^p (in particular $H^0 = L_0^2$), assuming for the moment that these functors satisfy the four axioms, then a system of linear partial differential equations may be viewed as defining a linear mapping between the well-defined vector spaces. The questions of existence, uniqueness, and regularity of solution for given data may then be investigated in this setting.

ELEMENTS OF THE GLOBAL NONLINEAR ANALYSIS

Axiom B§5: The Banach Manifold-Valued Section Functor

The recent significant advance in the theory of analysis of interest here, is the fact that if the functor $\mathcal{M}$ satisfies one additional axiom, the above setting for global linear analysis extends to a setting for a global nonlinear analysis.

Axiom III.1.[39] (B§5). If ξ is a vector bundle over a compact, C^∞, n-dimensional manifold M, and $\mathcal{M}$ is a Banach space-valued section functor,

a) then $\mathcal{M}(\xi) \subseteq C^0(\xi)$, and the inclusion map is linear and continuous.

b) Moreover, if η is a second vector bundle over M, and

$$f: \xi \longrightarrow \eta$$

is a C^∞ fiber bundle morphism, then

$$f_*: C^0(\xi) \longrightarrow C^0(\eta)$$

restricts to a continuous map (albeit not linear)

$$\mathcal{M}(f) : \mathcal{M}(\xi) \longrightarrow \mathcal{M}(\eta).$$

Lemma III.1.[40] If $\mathcal{M}$ satisfies axioms (B§1) through (B§5), then $\mathcal{M}_k$ satisfies the axioms (B§1) through (B§5).

Theorem III.2.[41] (The Banach manifold-valued section functor.) If $\mathcal{M}$ satisfies (B§1) through (B§5), then $\mathcal{M}$ extends to a covariant functor from the category of C^∞ fiber bundles and C^∞ fiber bundle morphisms over compact C^∞ n-manifolds into the category of Banach manifolds and C^∞ manifold maps. That is to say:

a) if E is a C^∞ fiber bundle over M, $\mathcal{M}(E)$ possesses a well defined structure as a C^∞ differentiable manifold modeled on a Banach space, and

b) if f is a C^∞ fiber bundle morphism from the fiber bundle E_1, into a fiber bundle E_2, then $\mathcal{M}(f)$ is a C^∞ mapping of Banach manifolds

$$\mathcal{M}(f):\ \mathcal{M}(E_1) \longrightarrow \mathcal{M}(E_2)$$
$$s \qquad\qquad\qquad \mathcal{M}(f)s = f_* s.$$

A Nonlinear Differential Operator is Represented As a Differentiable Mapping Between Manifolds

In this setting, nonlinear differential operators become well defined as manifold mappings:

Definition III.4.[42] (Nonlinear differential operator). Given fiber bundles E_1 and E_2, a nonlinear differential operator of order k, from E_1 into E_2 is a mapping D of C^∞ sections

$$D:\ C^\infty(E_1) \longrightarrow C^\infty(E_2)$$

which factors through the k^{th} order jet bundle of sections of E_1, a well-defined fiber bundle over M. That is to say, there exists a C^∞ fiber bundle morphism

$$f: \; J^k(E_1) \longrightarrow E_2$$

such that $D = f_* \cdot j_k$:

$$\begin{tikzcd}
C^\infty(E_1) \arrow[rr, "D"] \arrow[dr, "j_k"'] & & C^\infty(E_2) \\
& C^\infty(J^k(E_1)) \arrow[ur, "f_*"'] &
\end{tikzcd}$$

Denote the set of $k^{\underline{th}}$ order differential operators from E_1 into E_2 by $Df_k(E_1,E_2)$.

Theorem III.3.[43] Let $\mathcal{M}$ satisfy (B§1) through (B§5), and for E_1, E_2, C^∞ fiber bundles over compact, C^∞ n-dimensional manifold M, let $D \in Df_k(E_1,E_2)$. Then D extends to a C^∞ map of manifolds, also denoted by D,

$$D: \; \mathcal{M}_{k+r}(E_1) \longrightarrow \mathcal{M}_r(E_2),$$

$r = 0,1,2,\ldots$.

Notice that the ability to extend nonlinear differential operators to mappings of manifolds is a property of the functor $\mathcal{M}$, as opposed to the particular fiber bundles.

The Derivative of a Nonlinear Operator Mapping Extends the Classic Notion of the Linearization of the Operator

Theorem III.3 insures that the extended differential operator is continuous when viewed as a nonlinear mapping of manifolds, and its derivative exists and is continuous. The derivative of the mapping may be explicitly determined and related to an extension of the classic notion of the linearization of the operator. Moreover, the relation is independent of the particular functor $\mathcal{M}$ under consideration, so long as it satisfies the axioms.

Theorem III.4.[44] (Tangent space to $\mathcal{M}(E_1)$). Let E_1 be a C^∞ fiber bundle over M. Let $\mathcal{M}$ be a Banach manifold valued section functor satisfying (B§1) through (B§5).

a) If $s \varepsilon C^\infty(E_1)$ then the tangent space to the manifold $\mathcal{M}(E_1)$ at s may be identified canonically with a Banach space $\mathcal{M}(T_s(E_1))$,

$$T(\mathcal{M}(E_1))_s = \mathcal{M}(T_s(E_1)).$$

Here, $T_s(E_1)$ is a vector bundle over derived from E_1 and s. Hence, $\mathcal{M}(T_s(E_1))$ is a well-defined Banach space.

b) If

$$f: \; E_1 \longrightarrow E_2$$

is a C^∞ fiber bundle morphism, and if

$$\mathcal{M}(f): \quad \mathcal{M}(E_1) \longrightarrow \mathcal{M}(E_2)$$

the induced manifold map, then the differential of $\mathcal{M}(f)$ at s,

$$d\mathcal{M}(f)_s: \; T(\mathcal{M}(E_1))_s = \mathcal{M}(T_s(E_1)) \to T(\mathcal{M}(E_2))_{f\cdot s} = \mathcal{M}(T_{f\cdot s}(E_2))$$

is given by

$$d(M(f))_s = M(\delta_s f).$$

where $\delta_s f$ is a C^∞ vector bundle morphism

$$\delta_s f: \quad T_s(E_1) \longrightarrow T_{f\cdot s}(E_2)$$

defined in Palais' work. (Essentially, $\delta_s f$ is the vertical differential of f along s).

Corollary III.1. (Derivative of an extended differential operator). If M satisfies (B§1) through (B§5), and if D

$$D = f_* \, j_k : \quad C^\infty(E_1) \longrightarrow C^\infty(E_2)$$

is a nonlinear differential operator of order k, which extends to a C^∞ manifold map

$$D: \quad M_k(E_1) \longrightarrow M(E_2),$$

a) then for $s \varepsilon C^\infty(E_1)$, the derivative of D at s is given by

$$dD_s: \quad T(M_k(E_1))_s = M_k(T_s(E_1)) \longrightarrow M(T_{Ds}(E_2))$$

$$\sigma \longrightarrow [M(\delta_{j_k s} f)](j_k \sigma)$$

b) More generally, if $\bar{s} \varepsilon M_k(E_1)$ and if ξ_1 and ξ_2 are vector bundle neighborhoods of s and Ds in E_1 and E_2, respectively, then

$$dD_{\bar{s}}: \quad T(M_k(E_1))_{\bar{s}} = M_k(\xi_1) \longrightarrow T(M(E_2))_{D\bar{s}} = M(\xi_2)$$

$$\sigma \qquad [M(\delta j_k \bar{s} f)](j_k \sigma).$$

Theorem III.5.[45] (The classic linearization in vector bundle terms). Let $D \varepsilon Df_k(E_1, E_2)$,

$$D = f_* . j_k .$$

a) If $s \varepsilon C^\infty(E_1)$, then

$$\delta j_k s f: \quad T_{j_k s}(J^r(E_1)) = J^r(T_s(E_1)) \longrightarrow T_{Ds}(E_2)$$

is a C^∞ vector bundle morphism.

b) $\delta_{j_k s} f$ thus determines a linear differential operator of order k

$$\Lambda(D)_s \in \mathrm{Diff}_k(T_s(E_1), T_{Ds}(E_2)),$$

$$\Lambda(D)_s = (\delta_{j_k s} f)_* \cdot j_k : \quad C^\infty(T_s(E_1)) \to C^\infty(T_{D_s}(E_2))$$

Term it the linearization of D at s.

c) If ξ is a vector bundle neighborhood of s in E_1 (so that $T_s(E_1) = \xi$), then for $\sigma \varepsilon C^\infty(\xi)$,

$$\Lambda(D)_s(\sigma)(x) = d/dt \, \{D(s+t\sigma)(x)\}\Big|_{t=0}$$

Hence, $\Lambda(D)_s$ is a global extension of the classic linearization of a differential operator.

Corollary III.2.[46] If D is a nonlinear differential operator of order k, and M satisfies axioms (B§1) through (B§5), then for $s \varepsilon C^\infty(E_1)$,

$$dD_s: \quad T(M_{k+r}(E_1))_s = M_{k+r}(T_s(E_1)) \longrightarrow$$

$$T(M_r(E_2))_{Ds} = M_r(T_{Ds}(E_2)),$$

$r = 0, 1, 2, \ldots$, is an extension of the classic linearization

$$\Lambda(D)_s: \quad C^\infty(T_s(E_1)) \longrightarrow C^\infty(T_{Ds}(E_2))$$

to the chain of Banach spaces determined by M.

A representation of the linearization of D at s in local coordinates is developed in Palais' work. It reduces to the classic linearization of a nonlinear differential operator in n-variables about a given function.

The Linearization is Prescribed by Theorem, as Opposed to Computation

The important point to realize is that the lineari-zation at s of a given differential operator is independent of the particular choice of section functor M used in the formulation. If one has calculated it once, no matter what the setting (Holder, Sobolev, etc.), then one has calculated the linearization for all settings. Moreover, the extension of the linearization to the various functor spaces is provided by the derivative of the extended operator. This derivative will vary from functor to functor; however, one is assured that it exists, and in fact, a prescription for it is given by Corollary III.1. No further intricate norm calculations are required.

Casting the differential operator and its linearization in global terms allows one to examine properties of the operator from a global point of view. In particular, the symbol of a differential operator is a global object, and from it one can meaningfully define a nonlinear e-liptic differential operator. As these definitions are rather intricate, and their presentation at this stage might detract from the main purpose, one is referred to

Palais' text. Reference will be made to them, however, once the model for continuum mechanics has been erected, and one begins to utilize it.

HOW A DIRICHLET BOUNDARY CONDITION MAY BE SET

In setting the boundary conditions of place, knowledge of a particular submanifold of the Banach manifold $\mathcal{M}(E)$ is valuable:

Definition III.5.[47] (A setting for the place boundary condition). Let $\mathcal{M}$ satisfy axioms (B§1) through (B§5), and let $f \in \mathcal{M}(E)$. Define the subset $\mathcal{M}_{\partial f}(E)$ of $\mathcal{M}(E)$ to be the closure in $\mathcal{M}(E)$ of the set of all sections $g \in \mathcal{M}(E)$ such that for some neighborhood U of ∂M (U depending on g),

$$f|_U = g|_U .$$

Theorem III.6.[48] (A characterization of $\mathcal{M}_{\partial f}(E)$). If $\mathcal{M}$ satisfies axioms (B§1) through (B§5), then for $f \in \mathcal{M}(E)$,

a) $\mathcal{M}_{\partial f}(E)$ is a closed C^∞ submanifold of $\mathcal{M}(E)$ and the injection is C^∞.

b) In fact, if $s_o \in \mathcal{M}_{\partial f}(E)$, and ξ is a vector bundle neighborhood of s_o in E, then

$$\mathcal{M}(\xi) \cap \mathcal{M}_{\partial f}(E) = s_o + \overset{o}{\mathcal{M}}(\xi)$$

where $\mathcal{M}^o(\xi)$ is the closed linear subspace of $\mathcal{M}(\xi)$ obtained by taking the closure in $\mathcal{M}(\xi)$ of

$$C_o(\xi) = \{s \in C^\infty(\xi) : \text{support of } s \text{ is disjoint from } \partial M\}.$$

c) In particular, if $E = \xi$, a vector bundle,

(1) $M_{\partial f}(\xi) = s_o + M^o(\xi)$, for $s_o \in M_{\partial f}(\xi)$,

(2) If $M = L_k^2 = H^k$, then $(L_k^2)_{\partial f}(\xi)$ is a Hilbert manifold (a closed submanifold of the Hilbert space $L_k^2(\xi)$).

Corollary III.3. (Independence of the submanifold from the particular choice of f). If $g \in M_{\partial f}(E)$, then

$$M_{\partial f}(E) = M_{\partial g}(E).$$

Notice that the setting for the place boundary condition set forth here is available for all functors M satisfying the axioms. Moreover, some of the basic properties of the submanifold representing a place boundary condition (for instance, the model space being $M^o(\xi)$) may be expressed in $M^o(\xi)$ a general way in terms of the functor M. Hence, one is not immediately forced into examining features of the particular functors.

THE FUNCTION SPACES REGULARLY USED IN CONTINUUM MECHANICS EXTEND TO A NONLINEAR ANALYSIS SETTING

Finally, and most importantly, the function spaces of current interest in Continuum Mechanics may be viewed as functors satisfying the axioms, including BS5. Hence, they extend to a global nonlinear setting.

Definition III.6.[49] Let ξ be a C^∞ vector bundle over a C^∞, compact, n-manifold M with a Riemannian structure $\langle , \rangle_{\xi_x}$, $x \in M$.

a) (1) (The C^o functor). Let $C^o(\xi)$ be the complete normable topological vector space with respect to the norm

$$||s|| = \sup_{x \in M} \langle s(x), s(x) \rangle_{\xi_x}^{1/2}.$$

(2) $(C^o)_k = C^k$, with the usual "C^k topology".

b) (1) (The Holder functor, C^α). For $0 < \alpha < 1$, define $C^\alpha(\xi)$ to be the complete normable topological vector space of sections s of ξ which satisfy a global Holder condition of order α, which is gained by piecing together the following local requirement: the local coordinate representation of a section is bounded to order α in the Holder sense. That is to say, for any chart (U, θ_o) of M and local representation s_{θ_o} of s, then there exists a constant $K_U > 0$ such that

$$|s_{\theta_o}(x) - s_{\theta_o}(y)| < K_U \; ||x-y||$$

for all x, y $\theta_o(U)$. A norm for s can be the least upper bound of the collection $\{K_U\}$, which exists, since M is compact.

(2) Denote $(C^\alpha)_k$ by $C^{k+\alpha}$.

c) The Lipschitz functor C^{1-}). If α is set to 1, denote the resulting functor by C^{1-}. Likewise, set $(C^{1-})_k = C^{k+1-}$.

d) (1) (The Sobolev functor L^p). Choosing a strictly positive smooth measure μ on M, and a Riemannian structure $\langle\ \rangle_{\xi_x}$ on ξ, let $L^p(\xi)$ be the normable complete topolotical vector space of all Borel measureable sections s of ξ, such that

$$||s||_{L^p(\xi)} = \left(\int_M \langle s(x), s(x) \rangle_{\xi_x}^{p/2} \, d\mu_x \right)^{1/p} < \infty.$$

(2) Set $(L^p)_k = L^p_k$.

(3) (The Sobolev functor H^k). In particular, L^2_k is a Hilbert space-valued section functor, usually denoted H^k.

Theorem III.7.[50] The axioms (B§1) through (B§5) are satisfied by

a) C^k for all $k = 0,1,2, \ldots$

b) $C^{k+\alpha}$ for all $k=0,1,2, \ldots,$ and $0 < \alpha < 1$.

c) C^{k+1-} for all $k=0,1,2,\ldots$

d) L_k^p if $k>n/p$, where n=dimension M.

Thus if it is possible to view the elastostatic field equations for a material body as a non-linear differential operator taking sections of a bundle into sections of a second bundle, the above theorems will immediately:

(1) show under what conditions the elastostatic field equations extend to the desired function spaces,

(2) show under what conditions the extended operator is continuous, and in fact C^ω, without resorting to intricate norm calculations, and no matter which function space is chosen,

(3) explicitly specify what the derivative of the operator is, again without extended norm calculations.

The model for the finite elastostatic boundary value problem of place will now be initiated. One may begin by showing how the configurations and the kinematic state of a material body may be viewed as sections of suitable bundles, and under what conditions the elastostatic field equations may be viewed as determining a differential operator.

IV. FORMULATING THE KINEMATIC ELEMENTS; THE MATHEMATICAL MODEL FOR THE SPACE OF CONFIGURATIONS

The first step in creating a qualitative model for the finite elastostatic Dirichlet problem is to select a mathematical model for the solution space. The solution set consists of all the possible configurations of a material body undergoing finite deformations. In this chapter this set will be given the structure of an infinite-dimensional differentiable manifold. Its topology can be quite complicated. Specification of a Dirichlet boundary condition singles out a particular closed submanifold lying in the configuration manifold. The topology of the solution manifold is sensitive to changes in the boundary conditions, and the shape (topology) of the material body itself.

THE CRITERION FOR ASSIGNING A MATHEMATICAL STRUCTURE TO THE SOLUTION SPACE

How does one mathematically represent a configuration of a material body undergoing a finite deformation? One may adopt the definition of a configuration set forth by Truesdell, Noll, and Wang.[51] Let E denote Euclidean three-space For convenience, fix the origin and a rectilinear

Cartesian coordinate system, so that E may be viewed as the thee-dimensional real number space R^3 with the trivial connection. Let B be a C^ℓ material body. A configuration of B in E may be represented as a differentiable mapping of B into R^3 which is an embedding:

Definition IV.1[52] A configuration ϕ of B in E is a C^ℓ mapping

$$\phi : B \longrightarrow R^3$$

which is open, and if $p \neq q$ in B, then

$$\phi(p) \neq \phi(q).$$

Let $C^\ell(B,R^3)$ denote the set of all C^ℓ mappings of B into R^3. Let $Emb^\ell(B,R^3)$ denote the subset of these mappings which represent configurations. What topological and geometric structure can this subset support?

The criterion for assigning a geometric and topological structure to the subset of configurations which is adopted in this thesis is that the mathematical structure neither preclude alternatives of mechanical behavior which are physically conceivable for the body, nor permit alternatives which are not. To appreciate how strong this criterion is, a mathematical structure which the criterion precludes for the space of configurations will now be examined.

THE SOLUTION SPACE IS NOT A TOPOLOGICAL VECTOR SPACE

The mathematical model for the set of configurations for a material body subject to finite deformations is not a topological linear space. Three reasons are given for this assertion. The first reason is that the linear spaces generally used in the literature to model the set of configurations are too large. They contain elements which cannot correspond to configurations. The second reason is that the structure of any topological linear space is too simple to accommodate the variety of behavior a material body can physically display under finite deformations. The very topology of such a space excludes alternatives of mechanical behavior. Thirdly, a topological vector space is not sensitive enough to accommodate the variations in behavior which occur when one changes the boundary condition, or alters the shape of the specimen. One may expand on these ideas in turn.

Usual Topological Vector Spaces are too Large

Which topological linear spaces are used to model the set of configurations and why are they too large? An example best illustrates how such a space emerges as a candidate for the model. Let a material body be viewed in a reference configuration as a solid block given by body coordinates (X^1, X^2, X^3) subject to the conditions

$$-1 \leq X^i \leq 1 \qquad i = 1,2,3.$$

Let the boundary of the body in the reference configuration be specified by the planes $X^1 = \pm 1$, $X^2 = \pm 1$, $X^3 = \pm 1$. Assume the corners of the boundary are suitably rounded to prevent any mathematical difficulty. Denote the body viewed in this configuration as B. Attempts to model all possible deformations of the body from the reference configuration begin by characterizing the deformations in terms of a displacement vector field $\underset{\sim}{u}$ over B, so that a given deformation $\chi^i(X^1X^2X^3)$ may be specified as

$$\chi^i(X^1X^2X^3) = X^i + u^i(X^1X^2X^3) \quad i = 1,2,3.$$

Let $C^\ell(B,R^3)$ denote the set of all vector fields over B having a suitable degree of differentiability, ℓ. $C^\ell(B,R^3)$ is a linear space of vector-valued functions which may be given a topology. The resulting topological linear space $C^\ell(B,R^3)$, or some generalization of it, for instance the Sobolev space $H^\ell(B,R^3)$ or the Holder space with exponent α, $C^{\ell+\alpha}(B,R^3)$, is then taken to be the model for the space of configurations. By a model, one means that the finite elastostatic differential equations are cast in terms of an operator equation from the space modeling the set of configurations into a space modeling the data. In general, the operator is not linear. When the problem is abstracted in this way, questions about the existence and uniqueness of a solution to the operator equation for given data may be investigated using the techniques of nonlinear operator theory.

The difficulty with choosing $C^{\ell}(B,R^3)$ or some generalization of it as the model for the set of configurations for a material body undergoing finite deformations is that the space is too large. It contains elements representing postures which the material body cannot physically attain. For instance, consider the displacement vector field

$$\underset{\sim}{u}(X^1,X^2,X^3) = (4/3(X^1)^3 - 4/3\, X^1,\ 0,\ 0)$$

The mapping is an element of $C^{\ell}(B,R^3)$ for any ℓ. Under this displacement from the reference configuration, the boundary of the body would remain fixed; however, points interior to the body would "fold" upon one another, as can be discerned by graphing the function. The condition violates the constraint of material impenetrability. Since $\underset{\sim}{u}$ cannot represent a possible configuration for the material body, it would not lie in the domain of definition of the finite elastostatic operator. Hence if one abstracts the finite elastostatic problem into the operator formalism, one must consider the domain of definition of the finite elastostatic operator to be some subset of $C^{\ell}(B,R^3)$, as opposed to the entire linear space.

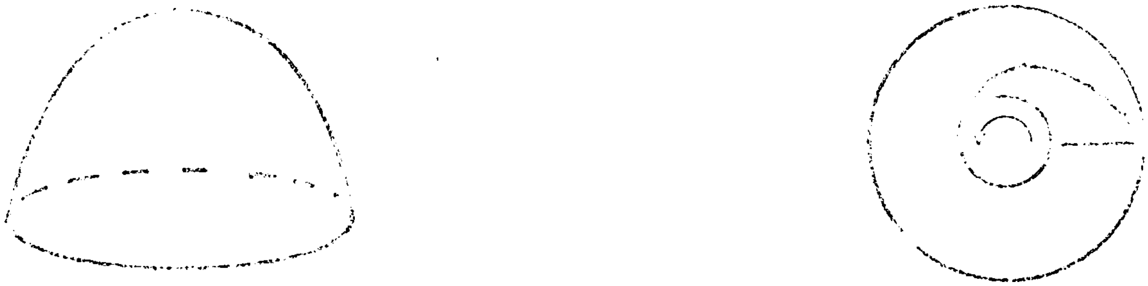
Topological Vector Spaces Preclude Alternatives of Mechanical Behavior

If $C^{\ell}(B,R^3)$, or some generalization of it, is too large a linear space, could not the model for the set of configurations be some other topological linear space? Again, the answer is in the negative. The topology of such a space is too simple to adequately represent some of the alternatives of mechanical behavior available for a material body. The following alternative which the topology precludes illustrates this point.

In finite deformation theory a two dimensional hemispherical shell made of suitable material can exhibit two equilibrating configurations to the zero boundary displacement problem: its "natural" configuration and its inverted configuration. The two equilibrating configurations differ from each other by a finite deformation. The question arises: is it possible to deform from one of the equilibrating configurations to the other without violating the zero displacement boundary condition at any stage of the deformation? For the case of the two dimensional shell, it is quite possible, as illustrated in Figure IV.1.

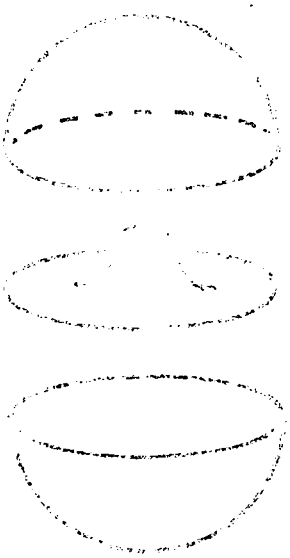
However, there are situations where it is impossible to deform between two equilibrating configurations satisfying the same boundary conditions without violating the boundary condition at some stage of the deformation. An example similar to one set forth by F. John illustrates this second alternative of behavior. Consider a ball which is composed of a homogeneous, isotropic material, and which possesses a spherical cavity. Assume it is possible to rotate the inner boundary through a straight angle about some axis in a positive sense. The resulting configuration equilibrates a particular Dirichlet boundary value problem. The same boundary value problem can be equilibrated by a second configuration which is gained from the reference state by rotating the inner boundary through a straight angle about the same axis, but in a negative sense. Notice that for this situation one may deform the body from the first equilibrating configuration to the other; however, at some stage of the deformation one must release the boundary condition. This alternative situation is also illustrated in Figure IV.1.

Could a topological linear space model admit the two alternatives? No. For, assume the model for the



The First Equilibrating
Configuration

The Second Equilibrating
Configuration



Deforming from One
into the Other

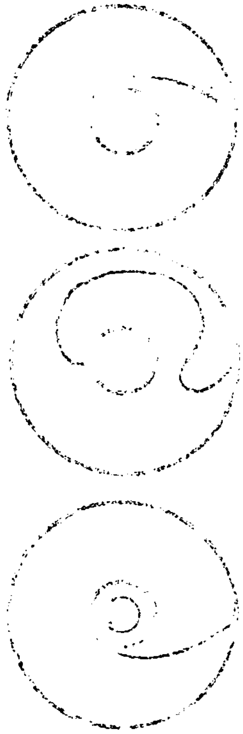


FIGURE IV. 1.

Two Situations Wherein One May or May Not be able to Deform between two Equilibrating Configurations Without Violating the Dirichlet Boundary Condition.

space of configurations is topological linear space. Let the set of configurations which satisfy the same boundary conditions be modeled as a linear subspace. Two distinct configurations satisfying the same boundary conditions would correspond to two distinct points in the same linear subspace. To say that the two configurations can be deformed into each other without violating the boundary conditions is to say that the two distinct points corresponding to the configurations may be joined by a curve which lies entirely in the linear subspace. Topologically speaking, the two points are said to lie in the same component of the linear subspace. To say that the two configurations can not be deformed into each other without violating the boundary conditions is to say that the two points may not be joined by a curve lying entirely in the linear subspace modeling the given boundary conditions. The two points may be connected by a curve; however, the curve must depart from the linear subspace at some point. Topologically speaking, the two points would lie in different components of the linear subspace modeling the given boundary conditions.

But, a topological linear space cannot model the latter alternative. A topological linear space is simply connected. Hence, the space itself, or any linear subspace, possesses but one component. The choice of a topological linear space of any sort as the mathematical model for the space of configuration automatically removes from consideration the second alternative of behavior mentioned above.

Thus, if one wishes to maintain as many alternatives of mechanical behavior as is possible in one's mathematical model one must consider spaces having more complex topologies than that afforded by the topological linear space. Moreover, the topology of the mathematical model must depend upon the geometry of the material body, and the choice of place boundary condition, if the alternatives are to vary from body to body, and from boundary condition to boundary condition.

Topological Vector Spaces are Insensitive to Changes in the Boundary Conditions and the Body Topology

When one admits the possibility that the topology of the configuration space affects the alternatives of behavior which it models, then the topological vector space exhibits some additional features which make it an undesirable candidate. A topological vector space model is insensitive to changes which occur when one varies the Dirichlet boundary conditions. As seen in the models of Van Buren and Deju, if one models the configurations in terms of a topological linear space, the subsets representing various Dirichlet boundary conditions are but different translates of the same linear subspace. But translates of a topological linear subspace are identical topologically. Hence, if one alternative of behavior is precluded by the topology of subspace modeling one boundary condition, it is precluded for all boundary conditions. Thus alternatives of mechanical behavior can not vary from boundary condition to boundary condition. In order to admit the possibility that alternatives of behavior which hold for one boundary condition need not hold for another, one must consider mathematical models whose topology is more complex than that of a topological linear space.

Finally, if one models the configuration space as a vector space, and one drills a hole in the material body, or otherwise alters its shape, one can not register this alteration in the topological structure of the space. If one desires a model which can adjust to alterations in the sample one must turn to more complex mathematical models.

Some Consequence if the Solution Space is not a Topological Vector Space

If one takes the model for the space of configurations to be anything except a topological linear space, two difficulties arise. Firstly, if the configuration space does not have the structure of a topological linear space, what precisely is its structure? Secondly, if the set of configurations cannot be modeled as a topological linear space, it becomes very difficult to apply the existence and uniqueness theorems which have been used in previous models. The theorems, particularly those arising from the theory of monotone operators, rely heavily upon the simple topology of a topological linear space. If the set of configurations has a structure which is too complex topologically, the theorems become inapplicable. One is then faced with the problem of finding mathematical tools to replace those which have been rendered useless.

Alternatives to a Topological Vector Space

Redoubtable as the previously mentioned difficulties are, mathematical tools are available with which to overcome them. In chapter three the abstract setting for a nonlinear partial differential equation was extended beyond that of a nonlinear operator on a topological vector space. More intricate solution spaces may be utilized. Moreover, advances in nonlinear functional analysis through the introduction of some elements from the Algebraic Topology and the Differential Topology have provided new methods for resolving questions of existence and uniqueness for nonlinear problems cast in the setting of Palais. As might be anticipated, the qualitative model which will be developed here fits well into such a setting.

THE FREE BOUNDARY SOLUTION SPACE IS A DIFFERENTIABLE MANIFOLD

How the Manifold Structure for the Space of Configurations will be Developed

The mathematical structure for the set of configurations will now be presented. It is shown that the set admits the structure of an infinite-dimensional differentiable manifold which lies as an open submanifold of a Banach space. Charts for the manifold may be explicitly displayed, and their use may be related to the usual procedure of characterizing the set of configurations in terms of "vector displacement fields" relative to a reference configuration. Since it lies as an open submanifold of a Banach space, the global structure of the configuration manifold is not that of a topological vector space; however, locally it resembles one. The model in general possesses a quite complicated topology.

One obtains the atlas for the manifold of $C^{\hat{k}}$ configurations subject to no boundary conditions by using the mathematical tools of chapter three. The development is not simple and straightforward. For the comfort of the reader only the results of the various steps of the development, and the mathematical elements which give them meaning are presented in the chapter proper. The proofs of [illegible] results, as well as most of the mathematical work, [illegible] rendered in appendices. The inquisitive reader [illegible], encouraged to make use of them to gain clarity and greater understanding of what is being asserted. As before, paragraph headings will be used to guide the reader through the various stages of the development.

Several steps are involved in capturing the structure of the manifold of C^ℓ configurations.[13] Firstly, the set of C^ℓ configurations is identified with a subset of the C^ℓ sections of a vector bundle. This subset is then identified as the intersection of two other subsets. The C^ℓ immersive sections, $\mathrm{Imm}^\ell(\eta)$, and the C^ℓ injective sections, $\mathrm{Inj}^\ell(\eta)$, of the vector bundle. The C^ℓ immersive sections $\mathrm{Imm}^\ell(\eta)$ are identified with the set of all $C^{\ell-1}$ sections of a particular fiber bundle; hence, by the methods of chapter three, $\mathrm{Imm}^\ell(\eta)$ inherits a natural structure as an infinite-dimensional differentiable manifold. Moreover, this manifold is shown to lie as an open submanifold in the Banach space of C^ℓ sections of η.

The C^ℓ injective sections $\mathrm{Inj}^\ell(\eta)$ are shown to lie in a second open submanifold of the Banach space of C^ℓ sections of η, the C^ℓ sections which correspond to degree one mappings of B into R^3, $\mathrm{Deg}_1{}^\ell(\eta)$. Finally, as the intersection of two open sets in the Banach space

$C^{\ell}(\eta)$, the set of C^{ℓ} configurations, $\mathrm{Emb}^{\ell}(\eta)$, is an open set in a Banach space; hence, it possesses the structure of an infinite-dimensional open submanifold.

The topology of the free boundary manifold of configurations $\mathrm{Emb}^{\ell}(\eta)$ is best understood when one examines the submanifolds representing particular Dirichlet boundary conditions. When one chooses a given configuration for the boundary of the body one finds that the set of C^{ℓ} configurations satisfying the boundary condition constitute a subset of the free boundary manifold $\mathrm{Emb}^{\ell}(\eta)$. The subset may be given the structure of a closed submanifold in $\mathrm{Emb}^{\ell}(\eta)$. Once again, charts in the manifold atlas may be displayed. Moreover, one may examine in surprising detail how the topology of the Dirichlet configuration manifolds may vary with boundary condition, and with the topology of the specimen itself. One may then appreciate the overall attractiveness of the model for the configuration space which is presented here:

(1) The elements of the space are configurations and only configurations.

(2) The space has a well defined geometric and topological structure, and there are techniques by which one can investigate them.

(3) The manifold is sensitive enough to reflect topologically changes in boundary condition and the shape (topology) of the specimen. Moreover, techniques are available to determine these changes.

The Main Result for the Free Boundary Configuration Manifold

The purpose of the development which follows is to give meaning to and prove the following assertion about the geometric structure which the set of C^{ℓ} configurations may possess.

Theorem IV.1.[54] Let B be a C^{∞}, compact, connected, oriented material body with boundary ∂B. Let η represent the vector bundle of all possible positions which a material point p may take in R^3, taken over all points of B:

$$\pi_{\eta} : \eta = B \times R^3 \longrightarrow B$$

a) A configuration Ψ of B in R^3 may be represented as a section s_{ψ} of the bundle η, given by

$$s_{\psi} : B \longrightarrow \eta = B \times R^3$$
$$p \longrightarrow s_{\psi}(p) = (p, \psi(p)).$$

Under the representation the set of all C^{ℓ} configurations subject to no boundary conditions constitutes a subset, denoted $Emb^{\ell}(\eta)$, of the set of all C^{ℓ} sections of the vector bundle η, denoted $C^{\ell}(\eta)$:

$$Emb^{\ell}(\eta) \subset C^{\ell}(\eta).$$

b) The set of C^{ℓ} configurations $Emb^{\ell}(\eta)$ has the structure of an infinite-dimensional differentiable manifold which may be viewed as the intersection of two manifolds,

$$Emb^{\ell}(\eta) = Imm^{\ell}(\eta) \cap Deg_1^{\ell}(\eta),$$

and which lies as in open submanifold of the Banach space $C^{\ell}(\eta)$. The model space for the manifold is the Banach space $C^{\ell}(\eta)$.

c) For any configuration s_χ one may display explicitly manifold charts about s_χ: one may specify an open neighborhood U_{s_χ} of s_χ in $\mathrm{Emb}^\ell(\eta)$ and a diffeomorphism Σ of the neighborhood onto an open neighborhood in $C^\ell(\eta)$

$$\Sigma : U_{s_\chi} \longrightarrow \Sigma(U_{s_\chi}) \qquad C^\ell(\eta)$$

which takes s_χ into the zero section in $C^\ell(\eta)$, and which is C^∞ compatible with other intersecting charts.

d) In particular, about each configuration s_χ there is a chart (U, Σ_ε)

$$\Sigma_\varepsilon \; U_{s_\chi} \longrightarrow \Sigma_\varepsilon(U_{s_\chi}) \subset C^\ell(\eta)$$

$$t_\Psi \longrightarrow \Sigma_\varepsilon(t_\Psi)$$

where

$$\Sigma_\varepsilon(t_\Psi) : B \longrightarrow \eta = B \times R^3$$

$$p \longrightarrow (\Sigma_\varepsilon(t))(p) = (p, u_\chi(p))$$

and $u_\chi(p) = \Psi(p) - \chi(p)$

is the "displacement vector field" characterizing the configuration Ψ when χ is used as a reference configuration.

The Set of Configurations is a Subset of a Set of Sections of a Vector Bundle

To describe the manifold structure of the set of configurations, it is mathematically convenient to view a configuration as a section of a vector bundle. The transition is a simple process. A configuration is a mapping of the body into physical space,

$$\begin{aligned} \phi : B &\longrightarrow R^3 \\ p &\longrightarrow \phi(p) \end{aligned}$$

Associate with ϕ a mapping s_ϕ from B into a product space of the body B and the physical space R^3:

$$\begin{aligned} s_\phi : B &\longrightarrow B \times R^3 \equiv \eta \\ p &\longrightarrow s_\phi(p) = (p, \phi(p)). \end{aligned}$$

If one views the product space η together with the body manifold B,

$$\begin{aligned} \pi_\eta : \eta = B \times R^3 &\longrightarrow B \\ (p,r) &\longrightarrow p \end{aligned}$$

one may regard η as attaching to each body point p of B the space $\{p\} \times B \equiv \eta_p$, which represents all the possible ways that the single body point can lie in physical space. The segment η_p is called the fiber of η at p. The product space η thereby constitutes a "bundling together" of all possible ways each body point of B may lie in physical space. η is called a bundle space. Since the fiber over each point is a vector space, η is called a vector bundle space. The triple consisting of the vector bundle space η, the body manifold B, and the projective link between them π_η, is called a vector bundle. The structural details of the vector bundle π_η are rendered in Appendix IV,1.

In the vector bundle context the mapping s_ϕ associates with each body point p of B a particular position in physical space. Notice the composition of s_ϕ with the projective link π_η is the identity map on B

$$\pi_\eta \cdot s_\phi : B \longrightarrow B$$
$$p \longrightarrow p.$$

Mappings of B into η which have this property are called sections of the vector bundle π_η. The set of all sections of π_η which are continuously differentiable through order ℓ, (C^ℓ), form a vector space, denoted $C^\ell(\eta)$. The vector space may be given a Banach space structure.

By the association established above, the set of all C^ℓ configurations of B in physical space, $Emb^\ell(B,R^3)$, may be identified with a subset of C^ℓ sections of η. Denote this subset by $Emb^\ell(\eta)$. One then has the correspondence

$$Emb^\ell(B,R^3) \longleftrightarrow Emb^\ell(\eta).$$
$$\phi \longleftrightarrow s_\phi$$

Not all sections of π_η correspond to configurations. Thus, the set $Emb^\ell(\eta)$ lies as a proper subset of the space of all C^ℓ sections,

$$Emb^\ell(\eta) \subset C^\ell(\eta).$$

To emphasize this fact, C^ℓ sections of π_η which correspond to C^{ℓ} configurations will be called C^{ℓ} configuration sections. The set $\mathrm{Emb}^{\ell}(\eta)$ will be called the set of C^{ℓ} configuration sections, or, on occasion, the set of C^{ℓ} embedding sections of π_{η}.

The Configuration Sections are the Injective Immersive Sections

One may now begin to capture the manifold structure for the set of C^{ℓ} configurations by determining the manifold structure for the set $\mathrm{Emb}^{\ell}(\eta)$ of C^{ℓ} configuration sections. In order for a C^{ℓ} section of π_{η} to be a configuration section the mapping

$$\begin{aligned} s_{\phi} : B &\longrightarrow \eta = B \times R^{3} \\ p &\longrightarrow s_{\phi}(p) = (p, \phi(p)) \end{aligned}$$

must satisfy a condition of impenetrability. The condition can be interpreted mathematically as two constraints:

(1) that there be no "internal collapse", a sufficient condition for which is the requirement that on the interior and boundary of the body, the determinant of the derivative of the configuration be non-zero,

$$\det[\phi_{*}(p)] \neq 0$$

$p \varepsilon B$;

(2) that there is no boundary penetration. That is to say, the mapping is one-to-one, or injective over the entire body manifold with boundary.

The first requirement implies that the configuration is a C^{ℓ} immersion of the manifold, which one denotes by

$$s_{\phi} \in \mathrm{Imm}^{\ell}(\eta) \subset C^{\ell}(\eta).$$

The second requirement implies that ϕ is a C^{ℓ} injection of the body manifold, denoted

$$s_{\phi} \in \mathrm{Inj}^{\ell}(\eta) \subset C^{\ell}(\eta)$$

If the body B is modeled as a compact manifold, (that is, it occupies a closed, bounded region in physical space), by adapting the work of R. Abraham[55] one may show that the set of C^{ℓ} configuration sections is the intersection of the two sets of sections determined by the impenetrability criterion:

$$\mathrm{Emb}^{\ell}(\eta) = \mathrm{Imm}^{\ell}(\eta) \bigcap \mathrm{Inj}^{\ell}(\eta).$$

One may then determine the manifold structure of the set of C^{ℓ} configuration sections from the manifold structure of the intersecting section spaces.

The Immersive Sections are Sections of a Fiber Bundle

One obtains the manifold structure for the immersive section by firstly establishing that they constitute a set of $C^{\ell-1}$ sections of a fiber bundle. The manifold structure for the set then follows by applying the methods of chapter three.

The fiber bundle wherein the immersive sections lie is a subbundle of the first jet bundle extension of the position vector bundle η. The jet bundle is defined as as follows:

Definition IV.2[56] Let B be a C^∞ material body which is compact. Let η be the vector bundle of positions. Define the vector bundle $J^1(\eta)$,

$$\pi_{J^1(\eta)} : J^1(\eta) \longrightarrow B$$

in the following way:

a) The set $J^1(\eta)$:

(1) For t, $s \varepsilon C^\ell(\eta)$, $\ell > 1$, and for $p \epsilon B$, define the equivalence relation

$t \underset{p}{\sim} s \Longleftrightarrow$ for (α, α_o, U) a vector bundle chart for η about p, and $\alpha_o(p) = x$,

$p_\alpha t(x) = p_\alpha s(x)$

$Dp_\alpha t(x) = Dp_\alpha s(x)$.

Denote by $j_1 s_p$ the equivalence class

$$j_1 a_p = \{t \varepsilon C^\ell(\eta) : t \underset{p}{\sim} s\}.$$

Term $j_1 s_p$ the first jet extension of s at p.

(2) Denote by $J^1(\eta)_p$ the set

$$J^1(\eta)_p = \{j_1 s_p : s \varepsilon C^\ell(\eta)\}.$$

Term $J^1(\eta)_p$ the fiber to the first jet bundle of ${}^p\eta$ at p.

(i) As a set, note that $J^1(\eta)_p$ may be put into one-to-one correspondence with the set

$$R^3 \times L(R^3, R^3)$$

(ii) $J^1(\eta)p$ may be given a vector space structure by defining

$$j_1 s_p + j_1 t_p \equiv j_1(s+t)_p \quad , \quad s,t \in C^\ell(\eta)$$

$$\lambda j_1 s_p \equiv j_1(\lambda s)_p \quad , \quad \lambda \in R, \ s \in C^\ell(\eta)$$

(3) Denote by $J^1(\eta)$ the bundle of sets taken over all points p of the body

$$J^1(\eta) = \bigcup_{p \in B} J^1(\eta)p.$$

(4) Define $\pi_{J^1(\eta)}$ to be the projective mapping

$$\pi_{J^1(\eta)} : J^1(\eta) \longrightarrow B$$
$$j_1 s_p \longrightarrow p.$$

The triple may be given a vector bundle structure. Details of the structure, and in particular, the vector bundle atlas, is presented in the referred work.

Lemma IV.1. $J^1(\eta)$ is a $C^{\ell-1}$ vector bundle with standard fiber

$$R^3 \times L(R^3 R^3)$$

$J^1(\eta)$ is called the first set bundle of sections of η.

The bundle of positions η and the first jet bundle $J^1(\eta)$ are related. For $\ell > 0$, the C^ℓ sections of η

lie in a nice way in the $C^{\ell-1}$ sections of $J^1(\eta)$:

> Lemma IV.2.[57] Let η be the bundle of positions of a C^∞, compact material body B, and let $J^1(\eta)$ denote the first jet bundle of sections of η. Then the first jet extension map
>
> $$j_1 : C^\ell(\eta) \longrightarrow C^{\ell-1}(J^1(\eta))$$
> $$s \longrightarrow j_1 s$$
>
> is a linear map taking the vector space of C^ℓ sections of η into the vector space of $C^{\ell-1}$ sections of $J^1(\eta)$. Moreover, when $C^\ell(\eta)$ and $C^{\ell-1}$ $(J^1(\eta))$ are viewed as Banach spaces, j_1 is a homeomorphism of $C^\ell(\eta)$ onto a closed linear submanifold of $C^{\ell-1}(J^1(\eta))$.

The fiber bundle in which the immersive sections lie may now be excised from the first jet bundle of η. Recall that the standard fiber for $J^1(\eta)$ is the vector space $R^3 \times L(R^3,R^3)$. If one views the elements of $L(R^3,R^3)$ as three-by-three matrices, one may single out the subset of nonsingular matrices,

$$GL(3,R) = \{g \varepsilon L(R^3,R^3) : \det g \neq 0\} \quad L(R^3,R^3).$$

If one bestows upon $L(R^3,R^3)$ the structure of a nine-dimensional Euclidean space, the subset $GL(3,R)$ inherits the structure of an open differentiable submanifold, which is nine-dimensional, and has two components. Roughly speaking, $GL(3,R)$ may be viewed as an open, somewhat complicated region of a Euclidean space upon which functions may be defined and differentiations may be carried out. Notice, however, that GL(3,R) does not inherit a vector space structure.

Consequently, if one views $R^3 \times L(R^3,R^3)$ as a twelve dimensional Euclidean space, the subset $R^3 \times GL(3,R)$ constitutes a twelve-dimensional, differentiable, open submanifold which is not a vector subspace.

One can construct a fiber bundle whose standard fiber is $R^3 \times GL(3,R)$. It lies as an open fiber subbundle in $J^1(\eta)$.

Proposition IV.1.[58] Let B be a compact, C^∞, material body. Let η be its bundle of positions, and let

$$\pi_{J^1(\eta)} : J^1(\eta) \longrightarrow B$$

be the first jet bundle of sections of η. Define by $W^{(3)}$ the following subset of $J^1(\eta)$:

$$W^{(3)} = \{j_1 s_p \in J^1(\eta) : \text{relative to some vector bundle chart of } J^1(\eta),\ j_1 s_p \text{ lies in } R^3 \times GL(3,R)\}.$$

Then

a) the set $W^{(3)}$ is independent of the particular charts used in its specification,

b) $W^{(3)}$ is an open subset in $J^1(\eta)$,

c) $W^{(3)}$ respects the fibration of $J^1(\eta)$; hence,

$$\pi_{W(3)} \equiv \pi_{J^1(\eta)}\big|_{W(3)} : W^{(3)} \longrightarrow B$$

is an open fiber subbundle in $J^1(\eta)$.

The proposition is proven in the reference. As one might intuitively suspect, $W^{(3)}$ may be related to the bundle of local configurations over B, and sections of $W^{(3)}$ may be related to local configuration fields. As shown in the reference, a section of the fiber bundle $W^{(3)}$ corresponds to the specification of simultaneously a configuration of B in R^3, and a local configuration field over B.

One may now identify the C^ℓ immersive sections with a subset of $C^{\ell-1}$ sections of $W^{(3)}$.

> Lemma IV.3.[59] Let B be a compact, C^∞ material body, let η be its bundle of positions, and $J^1(\eta)$ be the first jet bundle of extensions. Let
>
> $$\mathrm{Imm}^\ell(\eta) \subset C^\ell(\eta)$$
>
> denote the subset of C^ℓ sections of η which are immersive. Then $\mathrm{Imm}^\ell(\eta)$ may be identified as the set
>
> $$\mathrm{Imm}^\ell(\eta) = j_1{}^{-1}\{j_1(C^\ell(\eta)) \cap C^{\ell-1}(W^{(3)})\}$$
>
> That is to say, the C^ℓ immersive sections are those $C^{\ell-1}$ sections of the fiber bundle $W^{(3)}$ which are integrable.

The lemma follows from the definition of $W^{(3)}$, and the one-to-one nature of the first jet extension map j_1 introduced in Lemma IV.1.

When the immersive sections are viewed in this way, the geometric structure for $\mathrm{Imm}^\ell(\eta)$ follows immediately, using the methods of chapter three.

The Space of Immersive Sections is a Differentiable Manifold

One discovers the geometric structure which $\mathrm{Imm}^{\ell}(\eta)$ supports by examining the spaces which comprise it in Lemma IV.3. The critical step is the identification of the geometric structure supported by the $C^{\ell-1}$ sections of $W^{(3)}$.

> Proposition IV.2. Given the hypothesis of the previous lemma then
>
> $$C^{\ell-1}(W^{(3)})$$
>
> is an infinite-dimensional differentiable manifold which lies as an open submanifold in the Banach space $C^{\ell-1}(J^1(\eta))$

The proposition follows immediately. By Proposition III.7, "$C^{\ell-1}$" itself may be viewed as a Banach space-valued section functor which satisfies the axioms for a global nonlinear extension. Thus, as $W^{(3)}$ is a fiber bundle over B, the set of $C^{\ell-1}$ sections, $C^{\ell-1}(W^{(3)})$ supports the structure of an infinite dimensional differentiable manifold, given by Proposition III.2. Moreover, since $W^{(3)}$ lies as an open subbundle of the vector bundle $J^1(\eta)$, the differentiable manifold $C^{\ell-1}(W^{(3)})$ lies as an open submanifold in the Banach space $C^{\ell-1}(J^1(\eta))$.

Hence, without any further effort, the structure of $C^{\ell-1}(W^{(3)})$ is completely specified. In fact, a prescription exists for displaying its manifold charts, if one desires them.

The structure of the second component space comprising $Imm^{\ell}(\eta)$ follows in an equally straightforward way. By Lemma IV.2, the first jet extension map is a linear homeomorphism onto a closed linear subspace; hence, $j_1(C^{\ell}(\eta))$ is a closed Banach subspace of $C^{\ell}(J^1(\eta))$.

Combining these results, one gains the structure for $Imm^{\ell}(\eta)$. As $C^{\ell-1}(W^{(3)})$ is an open submanifold in $C^{\ell-1}(J^1(\eta))$, the intersection

$$j_1(C^{\ell}(\eta)) \cap C^{\ell-1}(W^{(3)})$$

is a submanifold of the Banach subspace $j_1(C^{\ell}(\eta))$ which is open in the relative topology. Since j_1 is a homeomorphism, or one-to-one and open both ways, the preimage of the intersection

$$j_1^{-1}\{j_1(C^{\ell}(\eta)) \cap C^{\ell-1}(W^{(3)})\} \equiv Imm^{\ell}(\eta)$$

is an open submanifold in $C^{\ell}(\eta)$. Therefore, the C^{ℓ} immersive sections $Imm^{\ell}(\eta)$ support the structure of an open submanifold in the Banach space $C^{\ell}(\eta)$.

The Injective Sections Lie in an Open Set in $C^{\ell}(\eta)$

In determining the manifold structure on the set of injections $Inj^{\ell}(\eta)$, a problem arises. The C^{ℓ}

injecting sections do not constitute an open subset in the Banach space $C^{\ell}(\eta)$ of C^{ℓ} sections of η. An example given in Appendix IV.2 establishes this point. The injecting sections, however, may be viewed as elements of a larger subset which is open in $C^{\ell}(\eta)$, and which may be used to characterize the C^{ℓ} configuration sections. The subset is the collection of degree one configuration sections of π_{η}. They are best defined in terms of the degree one mappings from B into R^3.

Definition IV.3.[60] Let B be a compact region in R^3, $\partial B \neq \phi$, and let

$$\Psi : B \longrightarrow R^3$$

be a differentiable mapping of B into R^3. Let $r \varepsilon R^3$, and for $p \varepsilon f^{-1}(r)$, let

$$\mathrm{sgn}(J(f)(p))$$

denote the sign of the Jacobian of the mapping at p. Define the degree of f at r relative to B to be the integer

$$\deg(f,r,B) = \sum_{p \varepsilon f^{-1}(r)} \mathrm{sgn}(J(f)(p))$$

If $f^{-1}(r) = \phi$, take $\deg(f,r,B) = 0$

For a more rigorous definition of the degree of a mapping, especially for the definition of the degree of f when r is a critical value, f is continuous, but not differentiable, or when B is a compact manifold with boundary, as opposed to a region of R^3, one is referred to the Schwartz text.[61] The properties of the degree of a map given in the reference will be used extensively.

One may easily carry over the notion of degree one mappings of B into R^3 to sections of the position bundle η. Recall one has a bijective correspondence between a mapping ψ of B into R^3 and a section s_ψ of π_η given by

$$s_\psi : B \longrightarrow \eta = B \times R^3$$

$$p \longrightarrow (p,\psi(p))$$

Hence, one may speak of the degree of the section s_ψ at a point r relative to B in terms of the degree of the associated mapping.

From the notion of the degree of a section three facts follow. If B is a connected manifold one may define unambiguously a subset of the space of sections $C^\ell(\eta)$ whose elements are of degree one on B. Moreover, the subset is open in $C^\ell(\eta)$, and has the structure of an open Banach submanifold. Finally, those degree one mappings which are also immersions are precisely the injective immersions, or the embeddings. For convenience, these facts are formalized as a theorem:

Theorem IV.2. Let B be a C^ℓ compact, connected manifold with boundary B and interior ∂B. Assume B is orientable and oriented.

(a) The collection

$\mathrm{Deg}_1^{\ell}(\eta) = \{s_\psi \varepsilon C^{\ell}(\eta): \deg(\psi, \psi(p), B) = 1,$ for some, hence all $p\varepsilon B\}$

is a well-defined subset of sections in $C^{\ell}(\eta)$.

(b) $\mathrm{Deg}_1^{\ell}(\eta)$ is an open subset of $C^{\ell}(\eta)$, and has a structure of an open Banach submanifold in $C^{\ell}(\eta)$.

(c) (1) The intersection

$\mathrm{Imm}^{\ell}(\eta) \cap \mathrm{Deg}_1^{\ell}(\eta)$

is open in $C^{\ell}(\eta)$.

(2) Set theoretically,

$\mathrm{Imm}^{\ell}(\eta) \cap \mathrm{Deg}_1^{\ell}(\eta) = \mathrm{Imm}^{\ell}(\eta) \cap \mathrm{Inj}^{\ell}(\eta)$.

The Space of Embedding Sections is a Differentiable Manifold

The manifold structure for the space of configuration sections $\mathrm{Emb}^{\ell}(\eta)$ now follows. As the intersection of two open manifolds in $C^{\ell}(\eta)$, $\mathrm{Emb}^{\ell}(\eta)$ has the structure of an infinite-dimensional differentiable manifold which lies as an open submanifold of the Banach space $C^{\ell}(\eta)$. Moreover, by using the bundle exponential map introduced in chapter three manifold charts for $\mathrm{Emb}^{\ell}(\eta)$ may be displayed. In particular, the manifold charts built off the trivial bundle exponential map may be related to the usual representation of the manifold of configurations in terms of a set of "displacement vector fields" relative to a reference configuration. In order to coalesce these ideas, they are formalized as a proposition, the details of which may be found

Theorem IV.3.[62] Let B be a C^ℓ compact, connected, oriented material body with boundary ∂B.

a) The set of configurations $Emb^\ell(\eta)$ has the structure of the infinite dimensional differentiable manifold

$$Emb^\ell(\eta) = Imm^\ell(\eta) \cap Deg_1^\ell(\eta)$$

which lies as an open Banach submanifold of the Banach space $C^\ell(\eta)$ of all C^ℓ sections of the position bundle η. The model space for the manifold is the Banach space $C^\ell(\eta)$.

b) Let Exp be a bundle exponential map for π_η. (For a definition, see the reference Let s_ψ be an arbitrary configuration. A manifold chart for $Emb^\ell(\eta)$ at s_ψ can be built from Exp: there is an open neighborhood $U_{s_\psi,Exp}$ of s_ψ in $Emb^\ell(\eta)$ and a mapping $\Sigma_{s_\psi,Exp}$ of the neighborhood onto an open neighborhood in $C^\ell(\eta)$,

$$\Sigma_{s_\chi,Exp}: \tilde{U}_{s_\chi,Exp} \longrightarrow \Sigma_{s_\chi,Exp}(\tilde{U}_{s_\chi,Exp}) \quad C^\ell(\eta)$$

$$t \qquad\qquad [[s_\chi{}^*(Exp)]_*]^{-1}t$$

which takes s_χ to the zero section in $C^\ell(\eta)$, and is C^∞ compatible with intersecting charts.

c) For the particular case where Exp is the trivial bundle exponential map E,

$$
\begin{array}{ccc}
E : TF\eta \approx \eta \times_B \eta & \longrightarrow & \eta \\
(b,c) & & b+c
\end{array}
$$

the manifold chart at s_χ associated with E is globally diffeomorphic onto an open subset $D_{s_\chi, E}$ of $C^\ell(\eta)$:

$$
\begin{array}{ccc}
\Sigma_{s_\chi, E} : \mathrm{Emb}^\ell(\eta) & \longrightarrow & s_{\chi, E}(\mathrm{Emb}^\ell(\eta)) = \\
 & & D_{s_\chi, E} \quad C^\ell(\eta) \\
t & & t - s = s_{t, s_\chi}
\end{array}
$$

(1) Let

$$u : B \longrightarrow R^3$$

denote the C^ℓ function such that

$$
\begin{array}{ccc}
s_{t,s_\chi} : B & \longrightarrow & \eta \\
p & & s_{t,s_\chi}(p) = (p, u(p)).
\end{array}
$$

Then u is the "displacement vector field" characterizing the configuration t when χ is used as the reference configuration.

(2) The manifold chart $(\Sigma_{s_\chi, E}, U_{s_\chi, E})$ extends to a global characterization corresponding to the usual representation found in the literature for configurations in terms of a subset of displacement vector fields relative to a reference configuration. (Note that the pair $(\Sigma_{s_\chi, E}, \mathrm{Emb}^\ell(\eta))$ need not be a manifold chart for $\mathrm{Emb}^\ell(\eta)$).

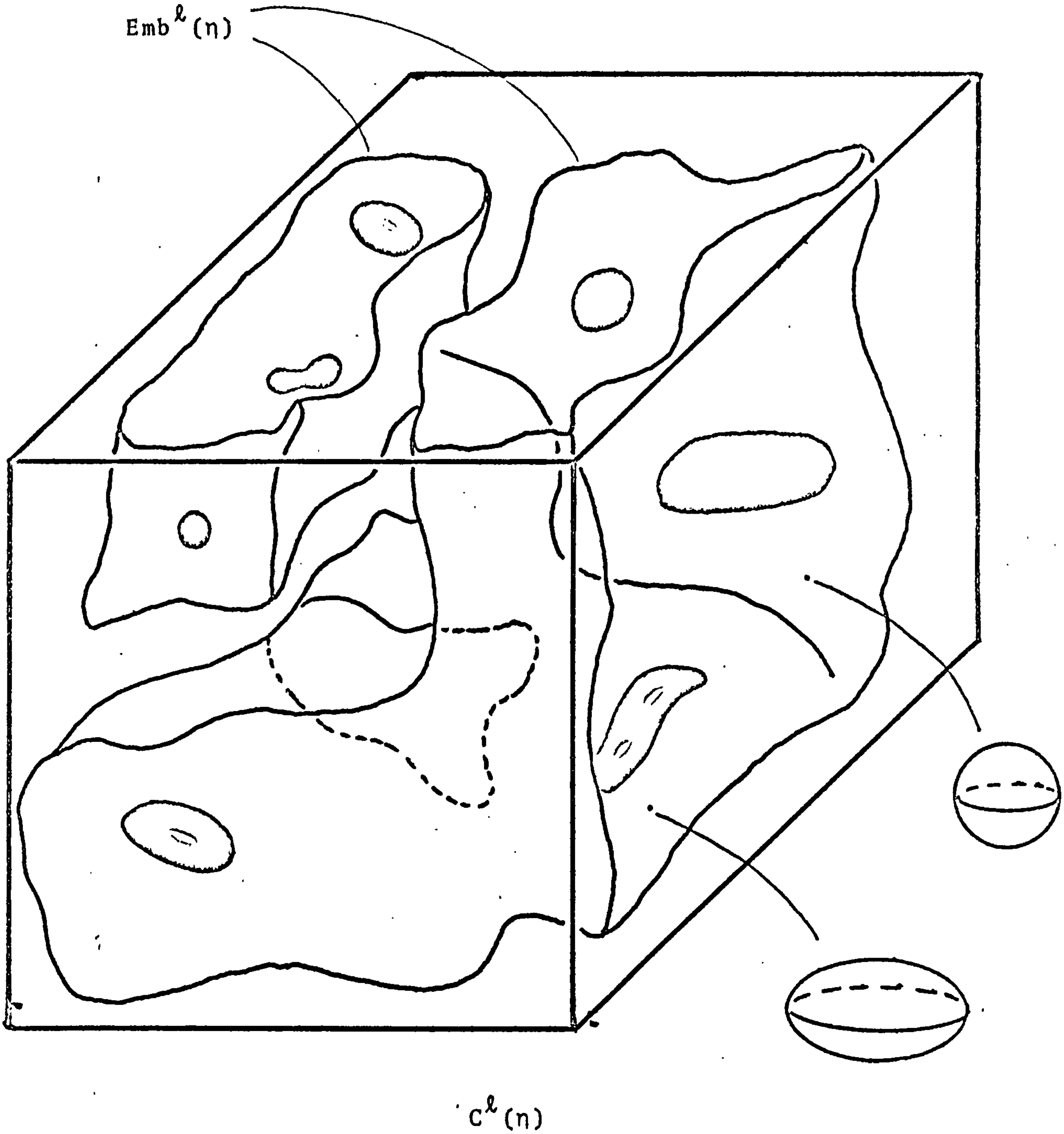


FIGURE IV. 2.

A Visualization of How $Emb^{\ell}(\eta)$ may lie as an Open Submanifold in $C^{\ell}(\eta)$.

THE DIRICHLET SOLUTION SPACE IS A CLOSED SUBMANIFOLD OF THE FREE BOUNDARY SOLUTION SPACE

One feature remains outside the model for the kinematic elements; the ability to model a given place boundary condition. Until now, the model for the space of configurations has been a "free boundary" one. A model for the space of configurations satisfying a given place boundary condition will be excised from the free boundary model. Once again, the model developed here will differ from those usually found in the literature in that it is a manifold, as opposed to a subspace of a topological linear space. As a consequence, one will gain some distinct features not found in the usual models; the topology of the manifold of configurations satisfying a given place boundary condition can vary with the place boundary condition, and can vary with the topology of the body itself. Hence, the model presented here has the significant capacity of, for instance permitting some alternatives of mechanical behavior for one boundary condition, while denying them for another.

The Model for the Dirichlet Solution Space for the Linear Elastic Case

How does one incorporate a Dirichlet boundary condition into the formulation of a solution space? One may modify the mathematical procedure used to specify the Dirichlet solution space for an infinitesimal elastic problem.

In the linear theory the usual model for the free boundary space of configurations is the topological linear space $C^{\ell}(\mathcal{B},R^3)$ of C^{ℓ} displacement vector fields defined over some region $\mathcal{B}$ of R^3 which serves as a reference configuration for the body. One excises the usual model for the space of configurations satisfying a given place boundary condition, by introducing the boundary operator $|_{\partial\mathcal{B}}$. This operator assigns to each C^{ℓ} displacement vector field g defined on the entire region $\mathcal{B}$ the C^{ℓ} restriction $g|_{\partial\mathcal{B}}$ of the field to the boundary $\partial\mathcal{B}$ of the body. The operator is linear; hence, it may be viewed as a linear mapping of the Banach space $C^{\ell}(\mathcal{B},R^3)$ into the linear space $C^{\ell}(\partial\mathcal{B},R^3)$

$$
|_{\partial\mathcal{B}} \;:\; \begin{array}{ccc} C^{\ell}(\mathcal{B},R^3) & \longrightarrow & C^{\ell}(\partial\mathcal{B},R^3) \\ g & & g|_{\partial\mathcal{B}} \end{array}
$$

Moreover, the linear operator is continuous. Finally, either by assumption or theorem one establishes the fact that if the boundary is of a suitable degree of smoothness, say ℓ, then the boundary operator is surjective. Hence, given any place boundary condition g with a suitable degree of smoothness ℓ, and viewed as an element of $C^{\ell}(\partial\mathcal{B},R^3)$, one is assured that there exists at least one C^{ℓ} displacement vector field g defined over the entire body which assumes the given boundary condition,

$$g|_{\partial\mathcal{B}} = g.$$

The preimage of g under the boundary operator,

$$[|_{\partial\mathcal{B}}]^{-1}(g) = \{g\epsilon C^{\ell}(\mathcal{B},R^3) : g|_{\partial\mathcal{B}} = g\}.$$

constitutes the usual model for the set of configurations satisfying the given boundary condition in the linear theory. The surjectivity of the boundary operator insures that the set is not empty. The continuity of the operator insures that the set is a closed subset of the topological linear space $C^{\ell}(\mathcal{B},R^3)$. The linearity of the operator insures that the set is a translation of a closed linear subspace in $C^{\ell}(\mathcal{B},R^3)$, the subspace modeling the zero displacement boundary condition. That is to say, if $g\epsilon[|_{\partial\mathcal{B}}]^{-1}(g)$, then

$$[|_{\partial\mathcal{B}}]^{-1}(g) = g + [|_{\partial\mathcal{B}}]^{-1}(0),$$

where the sum is in the sense of $C^{\ell}(\mathcal{B},R^3)$, and

$$[|_{\partial\mathcal{B}}]^{-1}(0) = \{h\epsilon C^{\ell}(\mathcal{B},R^3) : h|_{\partial\mathcal{B}}=0\} = \ker(|_{\partial\mathcal{B}}) \equiv [C^{\ell}(\mathcal{B},R^3)]_{\circ}$$

is a closed linear subspace of $C^{\ell}(\mathcal{B},R^3)$. One denotes the usual model for the set of C^{ℓ} configurations satisfying the boundary condition g by $[C^{\ell}(\mathcal{B},R^3)]_g$,

$$[C^{\ell}(\mathcal{B},R^3)]_g = \{|_{\partial\mathcal{B}}\}^{-1}(g).$$

How One Incorporates the Boundary Condition into a Nonlinear Model

One may excise from the free boundary finite elastostatic C^{ℓ} configuration manifold $\mathrm{Emb}^{\ell}(\eta)$ a model for the finite elastostatic Dirichlet configuration manifold by introducing a bundle counterpart to the boundary operator.

Let B be a material body with a non-void boundary ∂B. Let η denote its vector bundle of positions,

$$\pi_{\eta}: \eta = B\times R^3 \longrightarrow B$$

All possible positions of the boundary points of B may be represented by a second vector bundle $\eta_{|\partial B}$ which is the restriction of the position bundle to the boundary:

$$\pi_{\eta|\partial\mathcal{B}}: \eta|_{\partial\mathcal{B}} = \partial B\times R^3 \longrightarrow \partial B$$

A C^{ℓ} section of this bundle specifies a position in R^3 for each point of the boundary of the body which is C^{ℓ} smooth, Hence one may specify a C^{ℓ} smooth Dirichlet boundary condition for B by specifying a C^{ℓ} section of the bundle $\eta|_{\partial B}$.

The boundary operator may now be viewed as an operator which takes C^{ℓ} sections of the position bundle η into C^{ℓ} sections of the restricted vector bundle $\eta|_{\partial B}$:

$$|_{\partial B} : C^{\ell}(\eta) \longrightarrow C^{\ell}(\eta|_{\partial B})$$
$$s \longrightarrow s|_{\partial B}$$

When the section spaces are viewed with their Banach structure, the boundary operator is linear, continuous, and surjective if $\ell > 0$.

The Dirichlet Configuration Manifold Which Models Simple Support

Lying as an open submanifold in $C^{\ell}(\eta)$ is the finite elastostatic free boundary C^{ℓ} configuration manifold $\mathrm{Emb}^{\ell}(\eta)$. One may consider the boundary operator restricted to this manifold:

$$|_{\partial B}|_{\mathrm{Emb}^{\ell}(\eta)} : \mathrm{Emb}^{\ell}(\eta) \longrightarrow C^{\ell}(\eta|_{\partial B}).$$

Since $\mathrm{Emb}^{\ell}(\eta)$ is an open set in $C^{\ell}(\eta)$, under the restriction the operator remains a C^{∞} mapping; however, it is no longer linear. Moreover, its restriction to $\mathrm{Emb}^{\ell}(\eta)$ is a submersion, or a local surjection of the free boundary

C^{ℓ} configuration manifold.

One may now excise the various Dirichlet configuration manifolds. Let g be a C^{ℓ} section of the restricted position bundle $\eta_{|\partial B}$ which represents a possible Dirichlet boundary condition. Let $[Emb^{\ell}(\eta)]_g$ denote the set of C^{ℓ} configurations whose image under the restricted boundary restriction operator is the section g:

$$[Emb^{\ell}(\eta)]_g = [|_{|\partial B}|_{Emb^{\ell}(\eta)}]^{-1}(g).$$

The elements of the set are C^{ℓ} configurations associate with interior of the body very different postures, yet as one approaches the boundary of the body, they all coalesce into the same boundary condition g. From a physical point of view, the set mathematically represents all C^{ℓ} smooth postures for the body which are conceivable while one maintains the boundary condition in a simple supporting manner. In Figure IV.3, one visualizes some configuration sections which would lie in such a set, in the case where B is one-dimensional.

The Geometric Structure of the Manifold

The continuous and submersive properties of the restricted boundary operator insure that the preimage of g

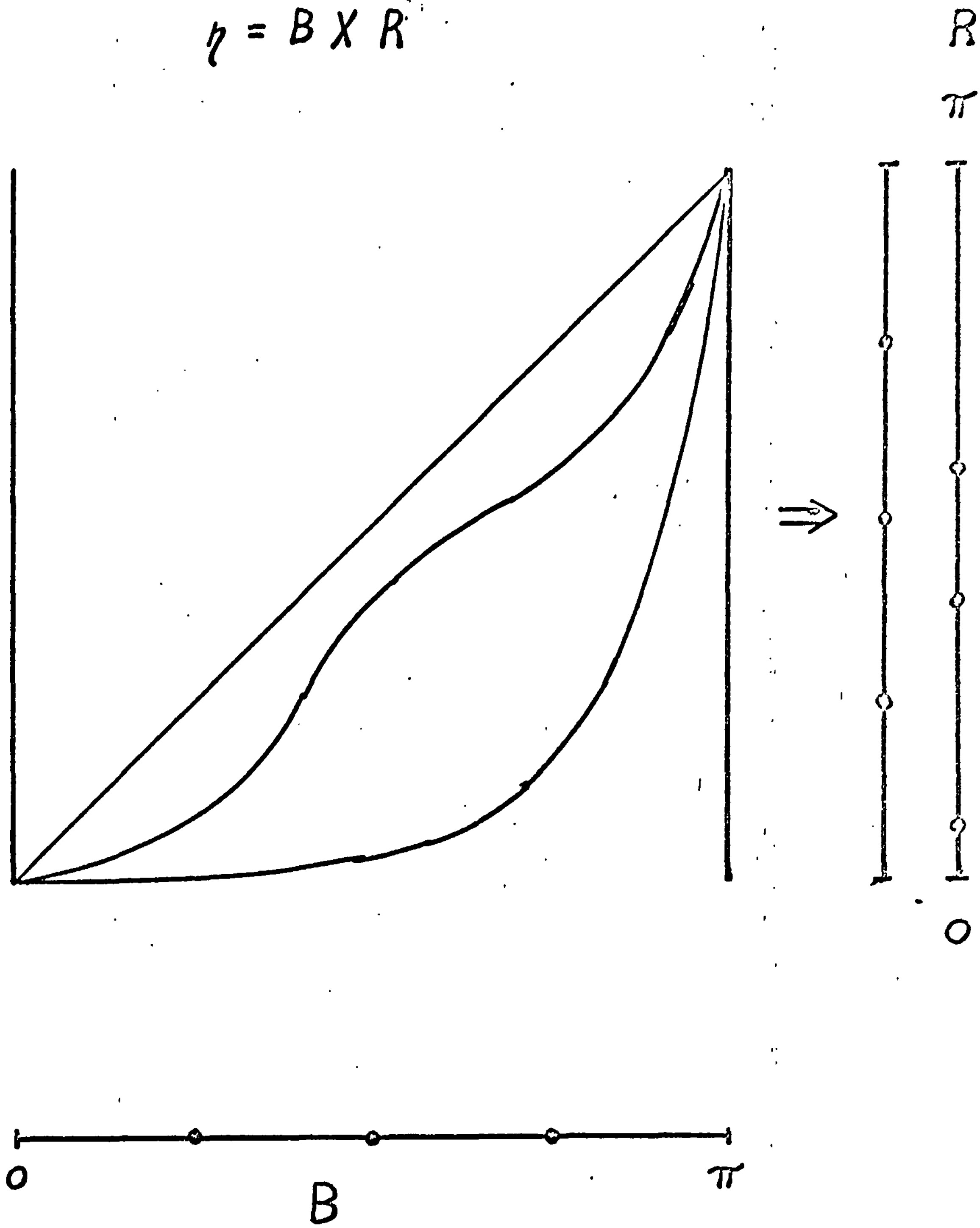


FIGURE IV. 3.

Several Elements of $[\mathrm{Emb}^{\ell}(\eta)]_g$ for $\eta = [0,\pi] \times R$, $B = [0,\pi]$, and $\partial B = \{0,\pi\}$.

under the mapping is a closed subset of the free boundary manifold $Emb^{\ell}(\eta)$, lies as a closed submanifold, and is specifically the intersection of the preimage of g under the unrestricted operator and the free boundary manifold:

$$[Emb^{\ell}(\eta)]_g \equiv [|_{\partial B}|_{Emb^{\ell}(\eta)}]^{-1}(g) = (|_{\partial B})^{-1}(g) \bigcap Emb^{\ell}(\eta)$$

$$= [C^{\ell}(\eta)]_g \bigcap Emb^{\ell}(\eta).$$

As in the free boundary case, one may explicitly detail the geometric structure of $[Emb^{\ell}(\eta)]_g$ and display some of its manifold charts. The following corollary summarizes these aspects:

Corollary IV.1. Let B be a C^{∞} material body with non-void boundary ∂B. Let η be its bundle of positions and let $|_{\partial B}$,

$$|_{\partial B} : C^{\ell}(\eta) \longrightarrow C^{\ell}(\eta|_{\partial B})$$

denote the boundary restriction operator. Let $g \varepsilon C^{\ell}(\eta|_{\partial B})$ represent a Dirichlet boundary condition.

a) If not null,

$$[Emb^{\ell}(\eta)]_g \equiv [|\,|_{\partial B}|_{Emb^{\ell}(\eta)}]^{-1}(g),$$

is a C^{∞} differentiable manifold whose model space is

$[C^{\ell}(\eta)]_{\circ}$, a closed Banach subspace of $C^{\ell}(\eta)$.

b) $[Emb^{\ell}(\eta)]_g = (|_{\partial B})^{-1}(g) \cap Emb^{\ell}(\eta)$, and lies as a closed submanifold of the C^{ℓ} free boundary manifold.

c) As a closed submanifold of $Emb^{\ell}(\eta)$, the manifold atlas for $[Emb^{\ell}(\eta)]_g$ follows by a suitable restriction of the atlas of the free boundary manifold. In particular, for $g \varepsilon [Emb^{\ell}(\eta)]_g$, there is a neighborhood U and a diffeomorphism Σ_g given by

$$\Sigma_g : U \longrightarrow \Sigma_g(U) \quad [C^{\ell}(\eta)]_{\circ},$$
$$\bar{g} \longrightarrow \Sigma_g(\bar{g}) = \bar{g} - g$$

where the difference is in the sense of $[C^{\ell}(\eta)]_{\circ}$

Part c) of the corollary allows one to compare this model for the Dirichlet configuration manifold with that of I. Beju presented in chapter two. $[C^{\ell}(\eta)]_{\circ}$ is the linear space used by Beju to model the Dirichlet problem. Thus, locally about any given configuration s satisfying the boundary condition, the Dirichlet configuration manifold presented here and Beju's model coincide. However, as one considers finite deformations from the reference configuration, the two models differ. In particular, the model developed here may be quite complicated topologically.

One can envision how the various Dirichlet configuration manifolds may inherit such a complex structure from the free boundary manifold using finite dimensional illustrations. Figure IV.4. illustrates, in finite dimensional terms, this phenomenon.

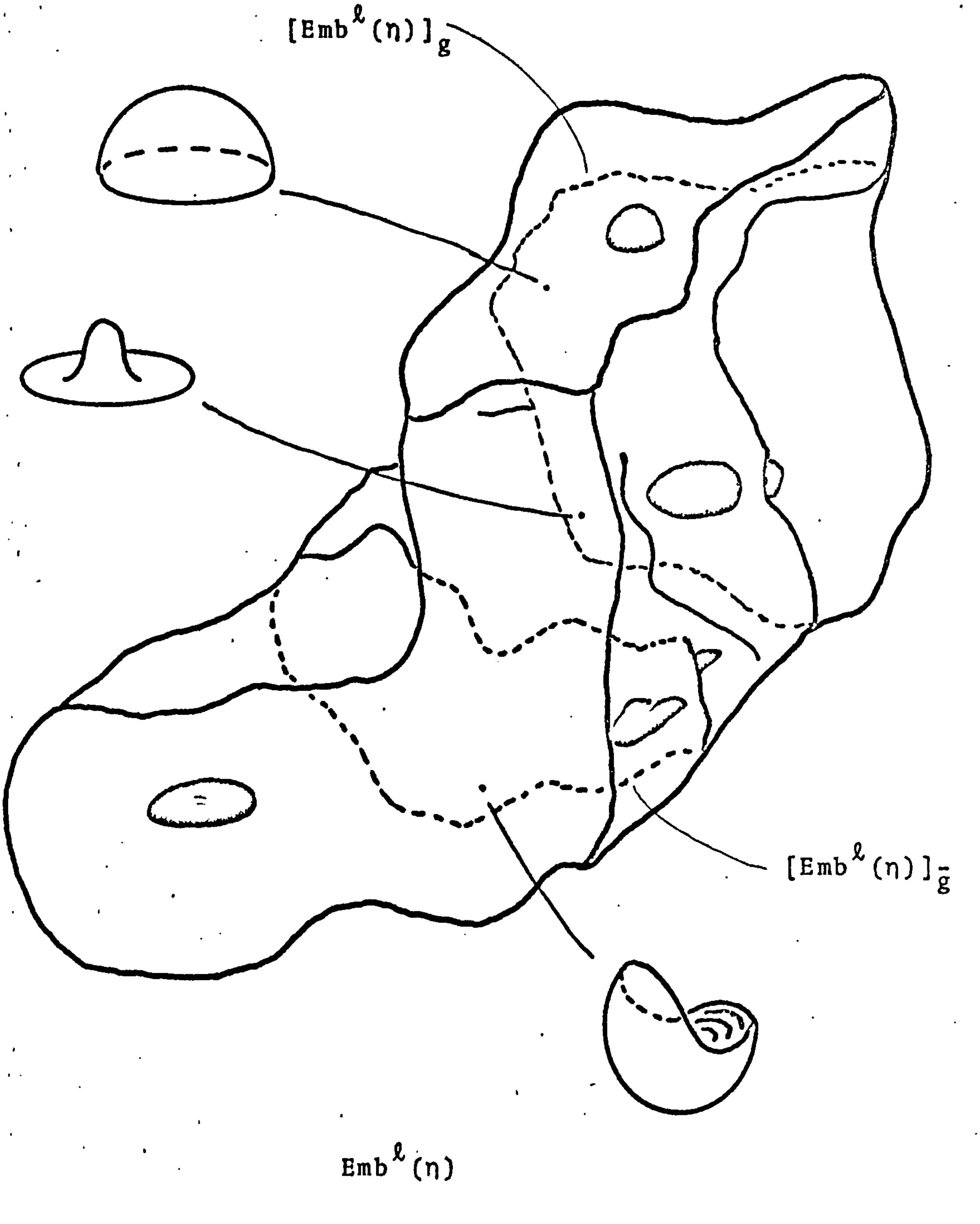


FIGURE IV. 4.

A Visualization of How the Dirichlet Configuration Manifolds may lie in the Free Boundary Configuration Manifold.

Different Dirichlet Configuration Manifolds Need Not Be Homeomorphic

The manifolds $[\mathrm{Emb}^{\ell}(\eta)]_g$ for $g \in C^{\ell}(\eta_{|\partial B})$ have one distinct feature which make them most attractive candidates for the models for the Dirichlet configuration manifolds. They need not be topologically identical, or homeomorphic.

> Proposition IV.3. Let g and $\bar{g}$ be sections of $\eta_{|\partial B}$ which represent different Dirichlet boundary conditions. Let
>
> $$[\mathrm{Emb}^{\ell}(\eta)]_g \quad \text{and} \quad [\mathrm{Emb}^{\ell}(\eta)]_{\bar{g}}$$
>
> denote the corresponding Dirichlet C^{ℓ} configuration manifolds. Then
>
> a) the topology of each manifold depends upon the boundary condition, and the topology of the body itself.
>
> b) the two manifolds need not be identical topologically.

It is the non-homeomorphic property of the Dirichlet configuration manifolds developed here which stands in direct contrast to the models which exist in the literature, and which gives the model developed here the capacity to reflect changes in the boundary condition and the body shape.

One may intuitively visualize how the topology of the Dirichlet configuration manifold can change from boundary condition to boundary condition using Figure IV.4. The topology of a Dirichlet manifold depends intimately upon how the subset of configurations representing the given boundary condition intersects the "holes" in the free boundary manifold. Subsets representing different boundary conditions may intersect different numbers of holes. In such a case, the manifolds would differ topologically, and thereby be non-homeomorphic.

HOW ONE GAINS THE TOPOLOGY FOR THE DIRICHLET CONFIGURATION MANIFOLDS

The Dirichlet Configuration Manifold is Identified with the Dirichlet Immersive Section Manifold

There are in fact mathematical methods available by which one can discern information about the topology of the Dirichlet configuration manifolds, and witness the dependence upon the boundary condition and body shape. One is able to employ these methods in the special case of a connected body, where one can identify $[Emb^{\ell}(\eta)]_g$ with the single manifold $[Imm^{\ell}(\eta)]_g$, as opposed to the intersection of two manifolds

$$[\mathrm{Emb}^{\ell}(\eta)]_g = [\mathrm{Imm}^{\ell}(\eta)]_g \bigcap [\mathrm{Deg}_1^{\ell}(\eta)]_g .$$

The first manifold of the intersection conveys a requirement which a configuration must satisfy on a local, or point-to-point basis: a determinant at each point must be non-zero. The second manifold conveys a global requirement, or a requirement which involves all points of the body simultaneously: no two points of the body must be taken into the same point in R^3 by the configuration. The following theorem indicates certain conditions under which the knowledge that the global requirement is satisfied by the boundary condition insures that the requirement is satisfied over the interior of the body as well by all configurations which satisfy the boundary condition.

Theorem IV.4 [63] (Interior penetration) Let B be a C^{ℓ} material body with boundary ∂B which is compact, connected, and for convenience, oriented. Let $s_{\psi} \varepsilon \mathrm{Emb}^{\ell}(\eta)$ be a given configuration serving as reference. Let

$$s_{\phi} \varepsilon \mathrm{Imm}^{\ell}(\eta),$$

and since B is oriented, take the sign of the Jacobian of ϕ to be positive at all points in the interior of B,

$$\mathrm{sgn}\, J(\phi_*)(p) > 0 \quad \text{for } p \;\varepsilon\; \mathring{B}.$$

If the boundary value of s_{ϕ} is identical to that of the reference configuration,

$$s_{\phi}|_{\partial B} = s_{\psi}|_{\partial B},$$

then s_ϕ is a configuration section of η,

$$s_\phi \varepsilon \mathrm{Emb}^\ell(\eta).$$

proof: see Appendix IV.3.

By the theorem, if one is assured that the boundary condition can be realized by a C^ℓ configuration, then the local requirement manifested by the manifold $[\mathrm{Imm}^\ell(\eta)]_g$ completely determines the Dirichlet C^ℓ configuration manifold. One may thereby characterize $[\mathrm{Emb}^\ell(\eta)]_g$ in terms of the single manifold $[\mathrm{Imm}^\ell(\eta)]_g$:

Corollary IV.2. Let B be a C^ℓ material body which is compact, connected, and, for convenience, oriented. Let the boundary of the body be denoted ∂B. Let

$$g : \partial B \longrightarrow \eta|_{\partial B}$$

be a given place boundary condition section, and suppose

$$[\mathrm{Emb}^\ell(\eta)]_g \neq 0.$$

Then

$$[\mathrm{Emb}^\ell(\eta)]_g = [\mathrm{Imm}^\ell(\eta)]_g,$$

where

$$[\mathrm{Imm}^\ell(\eta)]_g = \{f \varepsilon \mathrm{Imm}^\ell(\eta) : f|_{\partial B} = g\}$$

is a closed submanifold of the free boundary manifold of immersions $\mathrm{Imm}^\ell(\eta)$.

proof: see Appendix IV.4.

The Obstruction Theory is Employed to Gain Information About the Topology of the Dirichlet Immersive Section Manifold

One is now in a position to utilize the Obstruction Theory to gain information about the topology of the Dirichlet configuration manifold. Recall that a C^{ℓ} immersive section may be viewed as a $C^{\ell-1}$ section of the fiber bundle $W^{(3)}$. To examine those immersive sections which satisfies a given boundary condition g, one is led to consider the following diagram, which is a common setting in the mathematical Obstruction Theory:[64]

$$\begin{tikzcd}
\partial B \arrow[r, "j_1 g"] \arrow[d, "i"'] & W^{(3)} \arrow[d, "w^{(3)}"] \\
B \arrow[r, "1_B"'] \arrow[ur, dashed, "(?)"] & B
\end{tikzcd}$$

Two questions investigated in the Obstruction Theory in terms of the diagram are: (1) does there exist at least one extension of $j_1 g$ to all of B which commutes the diagram, (as indicated by the diagonal line), and (2) if more than one exists, are any two homotopic relative to the diagram? When one interprets these questions in terms of the Dirichlet configuration manifold, one sees that they provide information about its existence and topology.

An answer to the first question indicates whether or not a Dirichlet configuration manifold corresponding to the boundary condition g exists. If one is informed that there does not exist a diagonal mapping which commutes the diagram, then one is assured that the set of immersive sections of modeling the boundary condition g is null. Consequently, a negative answer to the first question implies that the section g cannot specify a Dirichlet boundary condition. Conversely, if the Dirichlet C^{ℓ} configuration manifold $[\mathrm{Emb}^{\ell}(\eta)]_g$ for the given boundary condition is not null, then a diagonal mapping for the diagram exists, and the answer to the first question is in the affirmative.

An answer to the second question gives one information about the number of components comprising the Dirichlet configuration manifold. Assume that the manifold $[\mathrm{Emb}^{\ell}(\eta)]_g$ is not null. Then the first jet extension of each section in the manifold is a diagonal map for the Obstruction Theory diagram. To say that any two of them are homotopic relative to the diagram is to say one can continuously deform from one to the other without violating the boundary condition. The two sections would, thereby, lie in the same component of the Dirichlet configuration manifold. Hence, if one finds that all diagonal maps of the Obstruction diagram are homotopic one may conclude that all sections in the Dirichlet configuration manifold under consideration are connected to each other, or the manifold has but one component. Accordingly, the number of distinct homotopy classes of diagonal mappings of the Obstruction diagram indicates the maximum number of components comprising $[\mathrm{Emb}^{\ell}(\eta)]_g$.

The Topology of the Dirichlet Configuration Manifolds can Vary with Boundary Condition and Body Shape

One is now in a position to apply the mathematical results which are available in the Obstruction theory to gain information about the topology of the Dirichlet configuration manifolds. In particular, two elementary observations indicate immediately how the topology of a Dirichlet manifold can vary as one changes the boundary condition, and as one changes the topology of the body itself.

One may compare two different Dirichlet configuration manifolds, say $[\mathrm{Emb}^{\ell}(\eta)]_g$ and $[\mathrm{Emb}^{\ell}(\eta)]_{\bar{g}}$, using the Observation theory by examining the two diagrams

$$\begin{tikzcd} \partial B \arrow[r, "j_1 g"] \arrow[d, "i"'] & W^{(3)} \arrow[d, "\pi_{W^{(3)}}"] \\ B \arrow[r, "1_B"] & B \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \partial B \arrow[r, "j_1 \bar{g}"] \arrow[d, "i"'] & W^{(3)} \arrow[d, "\pi_{W^{(3)}}"] \\ B \arrow[r, "1_B"] & B \end{tikzcd}$$

Since the boundary conditions g and $\bar{g}$ are distinct, one finds that the conclusions drawn from the first diagram need not be the same as the conclusions drawn from the second. For example, the number of homotopically distinct diagonal maps in the first diagram need not be the same as the number of homotopically distinct diagonal maps in the second. When rephased in terms of the configuration manifolds, such a conclusion would imply that the number of components of $[\mathrm{Emb}^{\ell}(\eta)]_g$ differs from the number of components of $[\mathrm{Emb}^{\ell}(\eta)]_{\bar{g}}$. Hence, different Dirichlet configuration manifolds can possess different topologies, and thereby admit different alternatives of mechanical behavior. Significantly, there are results in the Obstruction theory which allow one to ascertain when two diagrams yield the same conclusions and when they do not.[65]

One may also ascertain how the change in the body topology affects the topology of the Dirichlet configuration manifold. Speaking in general terms, the principal mathematical elements which one utilizes in the Obstruction theory to answer the two central questions about a diagram, which were presented above, are the cohomology groups of the base space, relative to the boundary,

$$\{H^n(\mathcal{B}, \partial B, \pi)\},$$

where π is a suitable ring of coefficients. In the situation considered here, the base space is the material body manifold itself. If one alters the cohomology groups, one alters the conclusions drawn about the Obstruction diagram.

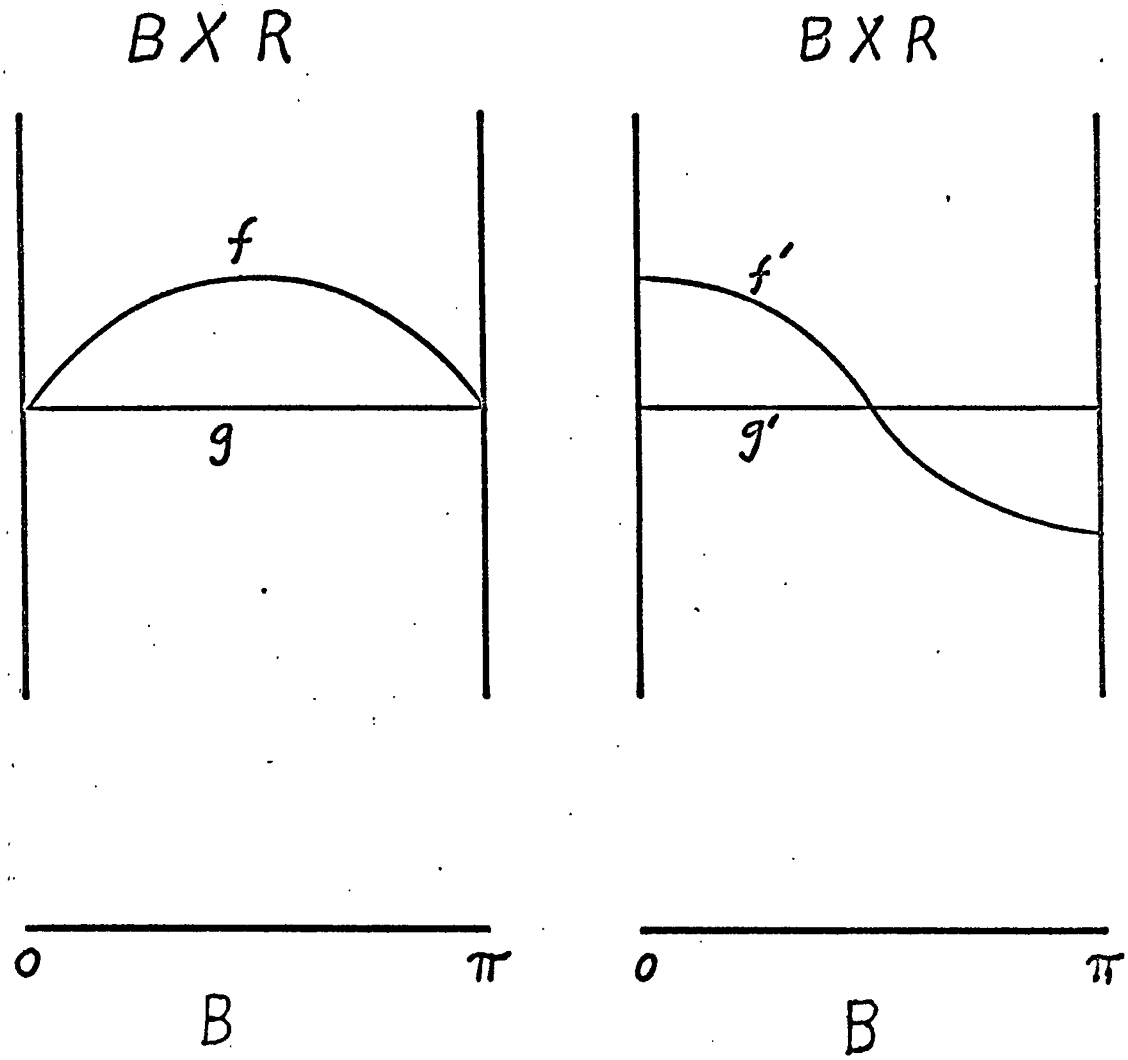
In particular, an Obstruction theory result relates the number of generators of the cohomology groups and the number of homotopically distinct classes of diagonal maps of an Obstruction diagram, under suitable circumstances. If one rephases the result in terms of the Dirichlet configuration manifold, one may relate the number of generators of the cohomology groups of the material body manifold, relative to the boundary, and the number of components of the particular Dirichlet configuration manifold under consideration. Hence, if one alters the topology of the material body itself, by drilling a hole in it, for instance, one can alter considerably the relative cohomology groups, and in turn, the topology of the Dirichlet configuration manifold.

Thus the Dirichlet configuration manifold model presented here has the most attractive feature of being sensitive to changes in the given boundary condition and body topology. Moreover, once one has achieved this setting, one is in a position to apply the full force of the Obstruction theory to resolve specific continuum mechanical problems. This approach promises to be a most fruitful direction of inquiry. A comment to this end is reserved for the last chapter.

THE DIRICHLET CONFIGURATION MANIFOLD WHICH MODELS RIGID SUPPORT

Before concluding the chapter it is worth mentioning that one can devise other models for the Dirichlet configuration manifold which are as sensitive to varying conditions as the one presented above, and which have an even finer structure. One particularly attractive model is the one suggested in chapter three. The configurations in the manifold are required not only to satisfy the boundary condition, but also to come off the boundary in a designated manner.

The motivation for the alternative manifold is the observation that two configurations for a material body may model the same boundary condition but may differ markedly even very close to the boundary. A visualization of the situation is suggested in Figure IV.5, in which two



$g(x) = 0$
$f(x) = \sin(x)$ $x \varepsilon [0,\pi]$

FIGURE IV. 5.

Two Functions on $[0,\pi]$ Modeling the same Dirichlet Boundary Conditions Which are Not Coincident on any Neighborhood of the Boundary in the C^1 sense.

real-valued functions defined on $[o,\pi]$ model the same zero boundary condition, but are not close in the C^{ℓ} sense in any neighborhood of the boundary.

One formally specifies the subset of configurations modeling a given boundary condition and particular way of coming off the boundary by means of the following lemma and definition:

Lemma IV.4[66] (The set $C^k_{\partial g}(\eta)$) Let B be a material body which is compact, connected, orientable, and oriented. Let ∂B denote the boundary of B. Let η be the vector of bundle of positions of B in R^3. Let $C^{\ell}(\eta)$ denote the Banach space of all C^{ℓ} sections of η. Let $g \varepsilon C^{\ell}(\eta)$. Then the set

$$(C^k)_{\partial g}(\eta) \equiv \text{the closure in } C^k(\eta) \text{ of } \left\{ \begin{array}{l} s\varepsilon C^k(\eta): \text{ an open neighborhood} \\ U_s \text{ of } \partial B \text{ on which } s_{|U_s} = g_{|U_s} \end{array} \right\}$$

a) is a closed set in $C^k(\eta)$

b) is a translate of a closed linear subspace Namely, for

$$\bar{g} \varepsilon (C^k)_{\partial g}(\eta) \quad \text{arbitrary,}$$

$$(C^k)_{\partial g}(\eta) = \bar{g} + (C^k)_o(\eta)$$

where $(C^k)_o(\eta)$ = the closure in $C^k(\eta)$ of

$$\left\{ \begin{array}{l} s\varepsilon C^k(\eta): \text{ support of } s \text{ is disjoint} \\ \text{from } \partial B \end{array} \right\}$$

Definition IV.3. Let $Emb^{\ell}(\eta)$ be the free boundary C^{ℓ} configuration manifold. Let be a C^{ℓ} configuration section for B. Define the set $(Emb^{\ell})_{\partial g}(\eta)$ by

$$(Emb^{\ell})_{\partial g}(\eta) = (C^{\ell})_{\partial g}(\eta) \cap Emb^{\ell}(\eta).$$

Notice that although the behavior of the configuration sections in the set are severely limited near the boundary, it is unrestrained in the interior. The situation is illustrated in Figure IV.6. The elements of the set may be regarded from a physical point of view as modeling configurations for the material body for which the Dirichlet boundary condition is maintained in a more constrained, rigidly supported manner, as opposed to a simply supported manner.[67]

As one may anticipate, the new sets have the structure of a differentiable manifold, lies as a closed submanifold of the free boundary configuration manifold, and also lie as closed submanifolds of the previously defined Dirichlet configuration manifolds. Moreover, each previously defined Dirichlet configuration manifold may be viewed as "partitioned" by the newer manifolds, yielding a structure which is finer than its original structure:

Theorem IV.5 Let g be a Dirichlet boundary condition, and $[Emb^{\ell}(\eta)]_g$ the corresponding Dirichlet configuration manifold. For each $g \in [Emb^{\ell}(\eta)]_g$, let

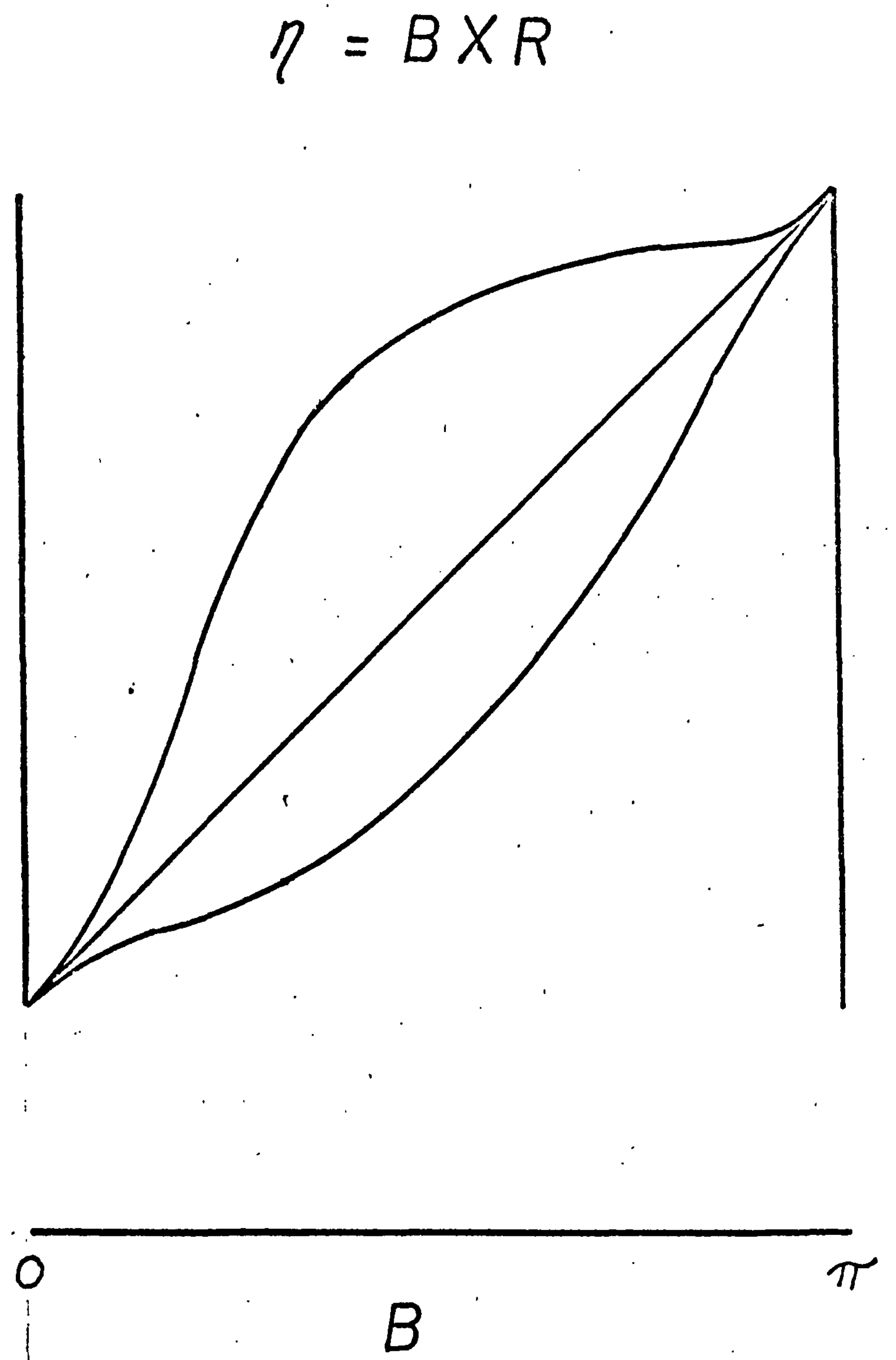


FIGURE IV. 6.

Several Elements of $(C^{\ell})_{\partial g}(\eta)$, for $\eta = [0,\pi] \times R$, $B = [0,\pi]$, $\partial B = \{0,\pi\}$.

$$(\mathrm{Emb}^{\ell})_{\partial g}(\eta)$$

be the Dirichlet configuration manifold defined above. Then

a) $(\mathrm{Emb}^{\ell})_{\partial g}(\eta)$ is differentiable manifold whose model is the Banach space $C_o^{\ell}(\eta)$.

b) $(\mathrm{Emb}^{\ell})_{\partial g}(\eta)$ lies as a closed submanifold of the free boundary manifold $\mathrm{Emb}^{\ell}(\eta)$.

c) Moreover, if $g|_{\partial B} = g$, then $(\mathrm{Emb}^{\ell})_{\partial g}(\eta)$ lies as a closed submanifold of the Dirichlet configuration submanifold $[\mathrm{Emb}^{\ell}(\eta)]_g$.

d) Set theoretically,

$$[\mathrm{Emb}^{\ell}(\eta)]_g = \bigcup_{g \in [\mathrm{Emb}^{\ell}(\eta)]_g} \{(\mathrm{Emb}^{\ell})_{\partial g}(\eta)\},$$

a disjoint union.

e) The topology induced upon the set $[\mathrm{Emb}^{\ell}(\eta)]_g$ when it is viewed as a disjoint union of closed submanifolds is finer than its manifold topology.

Proof: see Theorem III.6.

As will become evident in chapter seven, the finer structure will be extremely valuable in allowing one to study local nonuniqueness problems.

A SUMMARY

To recapitulate, in this chapter the free boundary configuration set has been modeled as an infinite dimensional differentiable manifold. The set of configurations satisfying a given place boundary condition have been modeled in two ways, depending on whether one maintains the boundary condition by simple or rigid support. The manifold structure and topology of these models have been developed somewhat and compared to the usual topological linear space model. The models developed here are shown to be more versatile than the linear space models in that their topology may vary in accordance with the topology of the body and the boundary condition under consideration. The variations in the topology in the models reflects variations in the alternatives of possible physical behavior available to the body. One may now turn one's attention to the formulation of the dynamic elements of the theory.

V. A FORMULATION OF THE DYNAMIC ELEMENTS AND THE FINITE ELASTOSTATIC OPERATOR

In this chapter the dynamic elements of the finite elastostatic theory are incorporated into the geometric model using the tools available from the global nonlinear analysis. The incorporation is accomplished in a two step process. First of all, the classical dynamic elements must be presented in such a way that one can apply the global analysis techniques of chapter three. This requirement leads one to cast the elements in terms of a fiber bundle formulation. Once the dynamic elements are set, the global analysis theorems may be applied, and the geometric model follows straightforwardly.

One achieves the following results in this chapter. The set of body force density fields are shown to constitute a Banach space of sections of a vector bundle. The stress tensor fields can also be cast into a bundle formulation. From them, one can evolve a finite elastostatic operator which serves as the geometric representative for the elastostatic equations. The operator links the free boundary or Dirichlet configuration manifolds with the body force density Banach spaces. The particular response of the material determines the particular characteristics of the elastostatic operator link. With the specification of the configuration manifolds, the body force density spaces, and the finite elastostatic operator linking them, one completes the qualitative model for the finite elastostatic free boundary and Dirichlet problems.

Inevitably, the development in this chapter taxes the reader considerably. The frustration must be compounded further when the reader discovers, in contrast to the dramatic departure seen in the solution manifolds, that the results obtained for the dynamic elements do not differ significantly from previous models. In truth, the chapter plays a supporting role in the overall work. However, the role is an essential one, and one must maintain the rigor if results are to be warranted. Consequently, the casual reader is forwarned to take the chapter with a considerable grain of salt, and the involved reader is encouraged to review the immense detail of the development at his leisure.

MATHEMATICAL PRELIMINARIES

The Vector Bundles Used to Formulate the Dynamic Elements are Introduced

The dynamic elements of a finite elastostatic theory, such as the traction vector field, the body force density field, and the stress tensor field, may be given a bundle formulation. In the classical formulation the traction vector field and the body force field are defined over a region of Euclidean space as opposed to the body itself. Indeed, if the configuration of the body changes, the region of Euclidean space over which the fields are defined changes. This situation is most inconvenient for setting the mathematical problem. Hence, new bundles are introduced which allow these fields to be viewed as different vector fields over the same base space. They are the vertical tangent bundle, and its pullback relative to a given configuration.

The Vertical Tangent Bundle

One introduces the vertical tangent bundle in order to view fields defined over a region of physical space as fields defined over the abstract body itself. Let B be a material body, and let $\eta = B \times R$ be its bundle of positions. At each point (p,r) in η one may identify a closed subspace of the tangent space $T\eta_{(p,r)}$ to η at (p,r): the vector subspace which is tangent to the fiber η_p,

$$T(\eta_p)_{(p,r)} \subset T\eta_{(p,r)}.$$

Figure V.1 displays the subspace. It is called the space of vectors tangent to the fiber of η at (p,r), or the

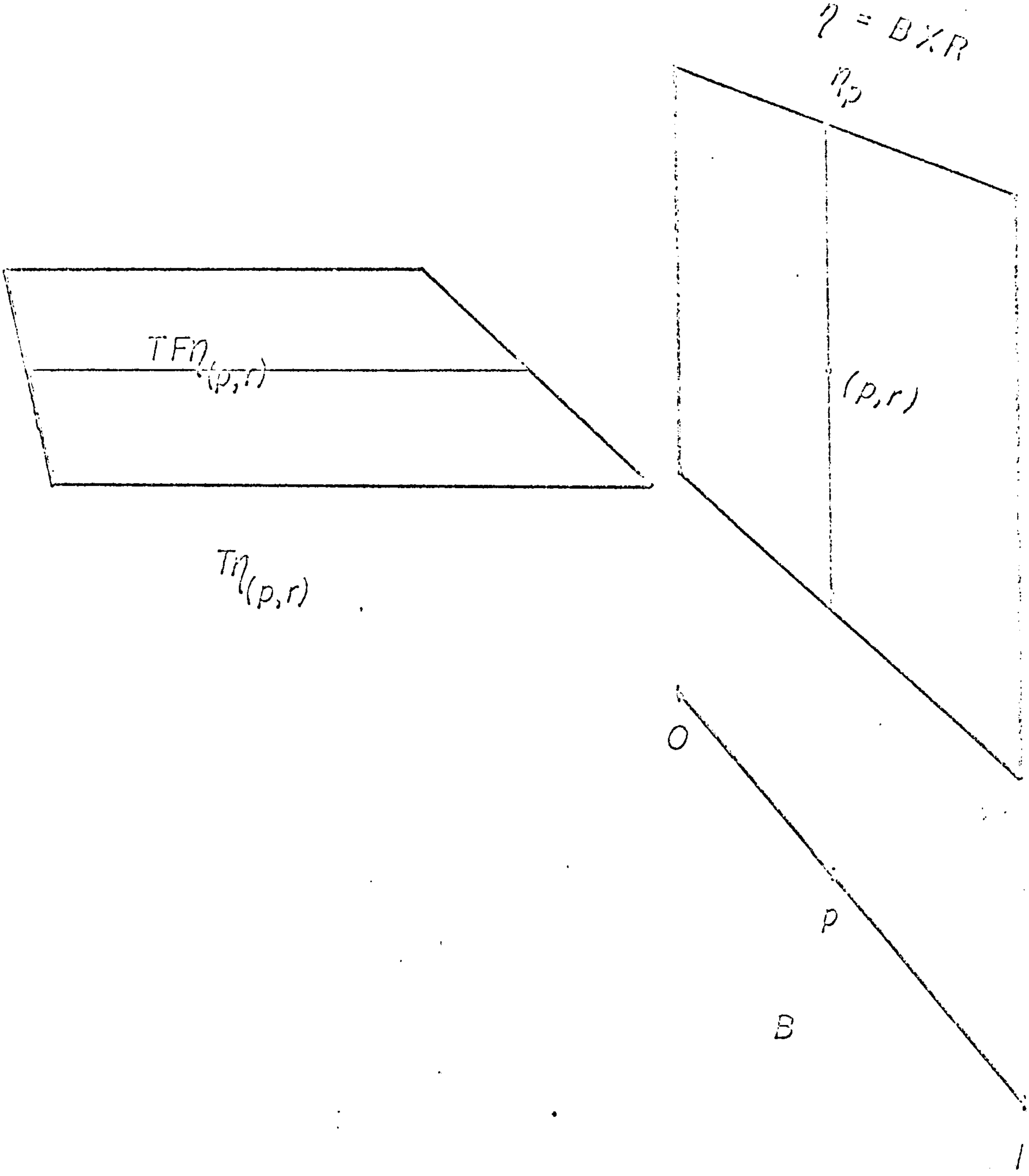


FIGURE V.1.

The Tangent Space to η at (p,r), and the Vertical Subspace $TF\eta_{(p,r)}$, for $\eta = B \times R$, and $B = [0,1]$.

space of vertical vectors (to η) at (p,r). The vertical subspaces may be bundled together to form a vector bundle over η, the vertical tangent bundle over η. Its standard fiber is R^3. It is a closed subbundle of the tangent bundle $T\eta$, and is denoted $TF\eta$.[68] The subbundle relationship is indicated schematically in the following diagram

$$\begin{tikzcd} TF\eta = \bigcup\limits_{(p,r)\in\eta} T(\eta_p)_{(p,r)} \arrow[r] \arrow[d] & T\eta \arrow[d] \\ \eta \arrow[r] & \eta \end{tikzcd}$$

Note in particular, that there is a canonical identification of the fibers of $TF\eta$ and the fibers of η: that is, if $W_p \in \eta$, W_p determines isomorphism of the fibers[69]

$$W_p : TF\eta_{W_p} = T(\eta_p)_{W_p} \longrightarrow \eta_p .$$

The Pullback of the Vertical Tangent Bundle

The vertical tangent bundle allows fields defined over regions of R^3 occupied by the body to be characterized by fields defined on the body B itself, provided one is given a reference configuration. Given a configuration ψ of B in R^3, or equivalently, given a configuration s_ψ of η, one may view the body B as a submanifold of the bundle space η, namely the closed submanifold $s_\psi(B)$. The restriction of the vertical tangent bundle to this submanifold of η, $TF\eta|_{s_\psi(B)}$, has the structure of a vector bundle over B. Mathematically, it is the vertical tangent bundle "pulled back" by s_ψ: $s_\psi{}^*(TF\eta)$. The bundle is denoted $T_{s_\psi}\eta$. The vector bundle which makes the diagram

$$\begin{tikzcd} s_\psi{}^*(TF\eta) = T_{s_\psi}\eta \arrow[r, "s_\psi^*(\tau_\eta)"] \arrow[d, "\tau_\eta{}^*(s_\psi)"'] & TF\eta \arrow[d, dashed, "\tau_\eta"] \\ B \arrow[r, "s_\psi"] & \eta \end{tikzcd}$$

commute. The details of the bundle are given in the footnoted reference.[70] In particular, note that the bundles $T_{s_\psi}\eta$ and η are isomorphic, through the isomorphism mentioned above.

The two bundles introduced here allow the dynamic elements of the elastostatic theory to be placed in a mathematical setting to which the global analysis machinery may be applied. The elements which will be placed in the setting are: the body force density field, the traction vector density field, and the stress tensor field.

BODY FORCE DENSITY FIELDS ARE SECTIONS OF THE VERTICAL TANGENT BUNDLE[71]

Recall that the body force density field is defined classically as follows: if χ is a configuration of the body B in the physical space R^3, then the specification of a body force density field on $\chi(B)$ is the specification at each point x of $\chi(B)$ of a body force density vector $\underset{\sim}{b}(x)$, viewed as an element of R^3. Thus the body force density field is an R^3- valued function on $\chi(B)$. To set the field in the bundle theory, recall that at each place x in $\chi(B)$, the tangent space to R^3 at x, $TR^3{}_x$, may be identified with R^3. Hence, a body force density field may be regarded as a specification at each place x in $\chi(B)$ of an element of the tangent space to R^3 at x. Let s_χ be the section of the bundle of positions η associated with the configuration χ. From the isomorphism between the tangent space TR^3_x at the place x in $\chi(B)$ and the vertical tangent space $TF\eta_{s_\chi(p)}$ at $s_\chi(p)$, one may view the body force density field, $\underset{\sim}{b}$ as specifying at each configuration section point $s_\chi(p)$ an element $\underset{\sim}{b}$, of the vertical tangent space $TF\eta_{s_\chi(p)}$. Hence, the body force density field may be viewed as a section of the restricted vertical tangent bundle $TF\eta|_{s_\chi(B)}$. Finally, pulling back the vertical tangent bundle by s_χ, one may express the body force density field as a section of the bundle $T_{s_\chi}\eta$. One denotes this fact by writing $\underset{\sim}{b} \in \Gamma(T_{s_\chi}\eta)$. In this way one transforms a body force density field from a field defined over a region of R^3 to a section of the vector bundle $T_{s_\chi}\eta$ over the body B associated with the configuration section s_χ.

THE CAUCHY STRESS TENSOR FIELD IS GIVEN A BUNDLE FORMULATION

For a Given Configuration the Cauchy Stress Tensor Field is a Section of a Linear Map Bundle

The Cauchy stress tensor field associated with the body B in the configuration χ by the material comprising it is classically viewed[72] as a field of linear mappings of R^3 defined over the region $\chi(B)$ of physical space:

$$T : \chi(B) \longrightarrow L(R^3; R^3).$$

It may also be formulated as a section of a vector bundle defined over B. First of all, by the identification of R^3 with the tangent space to R^3 at the place x, the Cauchy stress tensor at x may be regarded as a linear mapping of tangent spaces

$$T_x \in L(R^3, R^3) = L(TR^3{}_x, TR^3{}_x).$$

Let $p \in B$ be the material point occupying the place x in the configuration χ, and let s_χ be the section of associated with χ. If $I_{s_\chi(p)}$ denotes the isomorphism of the fibers TR^3_x and $T_{s_\chi}\eta_p$ provided by s_χ, T_x may be viewed as a linear mapping of fibers of $T_{s_\chi}\eta$:

$$\tilde{T}(s_\chi(p)) = I^{-1}_{s_\chi}(p) \cdot T_{\chi(p)} \cdot I_{s_\chi}(p) \in L(T_{s_\chi}\eta_p, T_{s_\chi}\eta_p).$$

$L(T_{s_\chi}\eta_p, T_{s_\chi}\eta_p)$ is the fiber at p of a bundle of linear maps $L(T_{s_\chi}\eta, T_{s_\chi}\eta)$ over B. Allowing the point $p \in B$ to vary, one achieves a mapping

$$T(s_\chi(-)) : B \longrightarrow L(T_{s_\chi}\eta, T_{s_\chi}\eta).$$

In fact, this mapping is a cross section of the linear map bundle. One denotes this fact by writing

$$\tilde{T}(s_\chi(-)) \in \Gamma\,(L(T_{s_\chi}\eta, T_{s_\chi}\eta).$$

Hence, for a given present configuration section s_χ, specification of the Cauchy stress tensor field is equivalent to the specification of a cross section of a vector bundle. In fact, the differentiability of the Cauchy stress tensor field may be considered in terms of the differentiability of the cross section.

THE FIRST PIOLA-KIRCHOFF STRESS TENSOR FIELD IN BUNDLE FORMULATION

Why the Field is Introduced

At this stage in the formulation the first Piola-Kirchoff stress tensor field may be conveniently presented. While its introduction is not essential in order to formulate the Dirichlet problem, it is necessary when one formulates the Neumann, or traction boundary value problem. By formulating the Dirichlet model in terms of the first Piola-Kirchoff stress tensor field initially, the author is anticipating a future effort in which the model is extended to encompass the traction boundary value problem.

The First Piola-Kirchoff Stress Tensor Field is a Section of a Vector Bundle

Recall that the first Piola-Kirchoff stress tensor field is a mixed tensor field defined over a reference configuration $\kappa(B)$ for the material body which characterizes the stress of the body in some deformed configuration $\phi(B)$.[73] The bundle formulation of the tensor field proceeds in a manner similar to that of the Cauchy stress tensor field. By identifying the tangent bundle to the physical space R^3 with the vertical tangent bundle to the present configuration section s_χ, and by pulling back to a bundle over the body B by the reference configuration section s_κ, one can identify the first Piola-Kirchoff tensor at $p \in B$ as a linear mapping of the fiber of the vector bundle over B associated with the reference configuration section s_κ:

$$T_{s_\kappa}(s_\chi, p) : T_{s_\kappa}\eta p \longrightarrow T_{s_\kappa}\eta p,$$

$$T_{s_\kappa}(s_\chi, p) \in L(T_{s_\kappa}\eta\, p, T_{s_\kappa}\eta p).$$

Allowing the point p to vary, $T_{s_\kappa}(s_\chi, -)$ may be regarded as a cross section of a bundle of linear maps, denoted

$$T_{s_\kappa}(s_\chi, -) \in \Gamma(L(T_{s_\kappa}\eta, T_{s_\kappa}\eta))$$

The Piola-Kirchoff Stress Operator is Developed

A pivotal point in the development is now reached. With the bundle formulation one gains the ability to view the Piola-Kirchoff stress tensor field $T_{s_\kappa}(s_\chi, -)$ in the configuration s_χ as arising from an operator correspondence.

One may easily see how this point of view arises. Let the present configuration of the material body change, say to s_ϕ, and let the new Cauchy stress tensor field be denoted $T(s_\phi(-))$. As before, the Cauchy stress tensor field generates a new first Piola-Kirchoff stress tensor field over the reference configuration κ. The new field may be viewed as a section of the bundle of linear maps from the reference configuration bundle $T_{s_\kappa}\eta$ into the reference configuration bundle $T_{s_\kappa}\eta$

$$T_{s_\kappa}(s_\phi,-) \in L(T_{s_\kappa}\eta, T_{s_\kappa}\eta).$$

The new Piola-Kirchoff stress tensor field is a section of the same vector bundle as the old one. Hence, in this formulation, a change in the present configuration of the body manifests itself as a change in section of the fixed linear map bundle $L(T_{s_\kappa}\eta, T_{s_\kappa}\eta)$. Consequently, one may assert that there exists an operator correspondence $\hat{T}_{s_\kappa}$ which associates the sections of η which correspond to configurations, sections of the linear map bundle $L(T_{s_\kappa}\eta, T_{s_\kappa}\eta)$. Symbolically, one may write

$$\hat{T}_{s_\kappa} : \{\text{set of configuration sections}\} \longrightarrow \Gamma(L(T_{s_\kappa}\eta, T_{s_\kappa}\eta))$$

$$s_\chi \longrightarrow \hat{T}_{s_\kappa}(s_\chi) = T_{s_\kappa}(s_\chi,-).$$

The correspondence will be called the Piola-Kirchoff stress tensor operator, and will serve as the basic element with which one incorporates the elastostatic field equations into the geometric model.

In order to set the elastostatic field equations in terms of the first Piola-Kirchoff stress operator, it is convenient to make one more mathematical identification. If a connection is specified on the bundle of configurations η, and if the reference configuration section s_κ is such that its first jet extension is invertible in the jet bundle sense,[74] then the covariant derivative of s_κ, $\nabla_\eta s_\kappa$, determines a vector bundle isomorphism between the tangent bundle body B, and the bundle $T_{s_\kappa}\eta$. Hence, a section of $T_{s_\kappa}\eta$ is uniquely identifiable with a vector field over B.[75] With this identification, the first Piola-Kirchoff tensor field relative to the reference configuration section s_κ for a presént configuration section s_χ, $T_{s_\kappa}(s_\chi,-)$, may be identified with a section of the bundle of linear maps from the tangent bundle over B into the bundle of configuration η,

$$T_{s}(s_\chi)(-) = T_{s_\kappa}(s_\chi,-)\cdot\nabla_\eta s_\kappa(-) \in \Gamma(L(TB,T_{s_\kappa}\eta)).$$

Thus, the Piola-Kirchoff stress operator may be viewed as a correspondence T_{s_κ} which associates with configuration sections of the body B sections of the vector bundle $L(TB,T_{s_\kappa}\eta)$,

$$T_{s_\kappa} : \{\text{set of configuration sections}\} \longrightarrow \Gamma(L(TB, T_{s_\kappa}\eta))$$

$$s_\phi \longrightarrow T_{s_\kappa}(s_\phi)$$

The elastostatic operator will be developed from this operator.

THE MATERIAL RESPONSE SPECIFIES THE PIOLA-KIRCHOFF STRESS OPERATOR

Once one is assured of the existence of a Piola-Kirchoff stress operator, one may examine ways of specifying it. The stress operator is specified in analytical terms by mathematical constitutive relation which characterizes the particular response properties of the material comprising the body. By employing some elements of the global analysis one may see how the material response of the body manifests itself in the geometric model. One finds that it determines the specific way in which the Piola-Kirchoff stress operator "links" the various configuration manifolds to their corresponding spaces of stress tensor fields.

A Smoothly Responding Material Specifies the Piola-Kirchoff Operator as a Mapping Defined on the Configuration Manifolds

How can the geometric model reflect the property that the material comprising the material body is smoothly responding? To say that a material body B has a "smooth response" means that a smooth present configuration of B results in a smooth Piola-Kirchoff stress tensor field.[76] In mathematical terms this means that a configuration section s_χ of the bundle η if C^∞ gives rise to a Piola-Kirchoff stress field $T_{s_\kappa}(s_\chi)$, which when viewed as a section of the vector bundle $L(TB, T_{s_\kappa}\eta)$, is also C^∞. Hence, for the purposes here, if the body B has "smooth response", the Piola-Kirchoff stress operator T_{s_κ} takes the elements of the C^∞ manifold of configurations, $\mathrm{Emb}^\infty(\eta)$, into the space of C^∞ sections of $L(TB, T_{s_\kappa}\eta)$, or

$$T_{s_\kappa} : \mathrm{Emb}^\infty(\eta) \subset C^\infty(\eta) \longrightarrow C^\infty(L(TB, T_{s_\kappa}\eta)).$$

The Piola-Kirchoff Operator is an 1^{st} Order Differential Operator When the Body is Elastic of Degree 1

To say that the material body B is simple elastic of degree 1 is to say that at each point the Piola-Kirchoff stress tensor depends only upon the first derivative of the present configuration.[77] Any other configuration identical to the present configuration up to order 1 at a particular place generates the same stress. This condition may be expressed in global analysis terms as stating that the

Piola-Kirchoff stress correspondence $\mathbb{T}_{s_K}$ is determined by a constitutive relation which depends only upon the 1^{st} order jet of the configuration section at each point.

From the definition in chapter three, mappings which take smooth sections of a bundle into smooth sections of another bundle and factor through the jet section map, or a covariant derivative map, are called differential operators (with C^∞ coefficients). Thus, one may express the conditions imposed upon the Piola-Kirchoff stress operator by the material response in the language of global analysis in the following way:

> Definition V.1. Let B be a material body, η the position bundle of B in R^3, and TB the tangent bundle to B. To say that B is a smooth elastic material body simple of grade 1 is to say that the first Piola-Kirchoff stress tensor operator relative to any reference configuration is determined by the response function associated with the material body which is a differential operator of order one:
>
> a) That is to say $\mathbb{T}_{s_K}$ takes the manifold of smooth configuration sections, an open submanifold of the smooth sections of the vector bundle η, into the smooth sections of a linear map space:
>
> $$\mathbb{T}_{s_K} : \mathrm{Emb}^\infty(\eta) \subset C^\infty(\eta) \longrightarrow C^\infty(L(TB, T_{s_K}\eta)),$$
>
> and,
>
> b) the constitutive relation for $\mathbb{T}_{s_K}$ factors through the manifold of C^∞ local configuration sections $W^{(3)}$ by means of a fiber bundle morphism;
>
> $$\mathbb{T}_{s_K} = (h_{s_K})_* \cdot j_1,$$

where

$$h_{s_K} : W^{(3)} \longrightarrow L(TB, T_{s_K}\eta)$$

is a fiber bundle morphism, and

$$(h_{s_K})_* : C^\infty(W^{(3)}) \longrightarrow C^\infty(L(TB, T_{s_K}\eta))$$

is the induced mapping on the section manifolds. One has the diagram

$$\begin{tikzcd}
\mathrm{Emb}^\infty(\eta) \arrow[r, phantom, "\subset"] \arrow[dr, "j_1|_{\mathrm{Emb}^\infty(\eta)}"'] & C^\infty(\eta) \arrow[rr, "\mathbf{T}_{s_K}"] \arrow[dr, "j_1"] & & C^\infty(L(TB, T_{s_K}\eta)) \\
& C^\infty(W^{(3)}) \arrow[r, phantom, "\subset"] \arrow[urr, "(h_{s_K})_*"'] & C^\infty(J^1(\eta)) &
\end{tikzcd}$$

To say that the stress depends only upon the gradient of the deformation is to say that the Piola-Kirchoff constitutive relation factors through the manifold of local configuration sections of grade one in a special way: the factoring is accomplished by the covariant derivative on η:

$$\mathbf{T}_{s_K} = (h_{s_K} \cdot i)_* \cdot \nabla_\eta ,$$

for $\quad \nabla_\eta : \mathrm{Emb}^\infty(\eta) \longrightarrow C^\infty(L^{(3)}(TB, T_{s_K}\eta))$

$\qquad\quad i \;\; : L^{(3)}(TB, \eta) \longrightarrow W^{(3)},$

and one has the diagram

$$\begin{tikzcd}
\mathrm{Emb}^\infty(\eta) \arrow[r, phantom, "\subset"] \arrow[d, "\nabla_\eta"'] & C^\infty(\eta) \arrow[r, "\mathbf{T}_{s_K}"] \arrow[dl, "\nabla_\eta"] & C^\infty(L(TB, T_{s_K}\eta)) \\
C^\infty(L^{(3)}(TB\eta)) \arrow[r, phantom, "\subset"] & C^\infty(L(TB\eta)). \arrow[ur, "(h_{s_K} \cdot i)_*"'] &
\end{tikzcd}$$

The bundles are as defined in chapter three.

One may also specify a material response of higher order in similar terms.

Complex as these diagrams may appear their raison d'etre is quite simple. They show in a precise, succinct way how the constitutive relation for the material manifests itself in the geometric model as specifying the link between the (kinematic) manifold of configurations and the (dynamic) space of stress tensor fields. Indeed, if the material body were not simple of degree one, but rather simple of degree k, or composed of a material with fading memory, then the specification of the Piola-Kirchoff stress operator link would be radically different. The operator could no longer be viewed as factoring through the manifold of local configuration sections; rather, it would factor through a higher order bundle, or a series of bundles.[78] The situation is characterized in global analysis terms when the operator is characterized as a differential operator of higher order, or a more general integro-differential operator. The question will be examined further after the geometric model is presented.

A Local Representation for the Material Response

Formulating the stress operator in terms like the diagrams above is excellent for investigating its global properties, and in particular, existence and uniqueness questions. This advantage will be seen shortly. However, in order to gain some insight into the diagrams, and more importantly, in order to utilize results from Continuum Mechanics in the geometric model, one must have the ability to represent the stress operator in terms of the classical analysis. At this stage, then, it is instructive to show what the stress operator looks like in terms of a coordinate specification. By so doing, one will discover how the classical representation for the response function relative to a given reference local configuration field determines the fiber bundle morphism h_γ which occurs in the global diagram.

If one chooses a coordinate chart (α_o, U) on the body B with coordinate functions X, and trivializations $((\alpha_o x \gamma_o), \alpha_o, U)$ on η, with coordinate functions (X, x), for x the classical "place" coordinate functions for R^3, and using the induced charts on $J^1(\eta)$, TB, and $L(TB, T_{s_K}\eta)$, any differential operator of order one from η into $L(TB, T_{s_K}\eta)$,

$$f_* \cdot j_1 : C^\infty(\eta) \longrightarrow C^\infty(L(TB, T_{s_\kappa}\eta))$$

may be represented as an operator on the polynomial space built on R^3 into the space of linear maps of R^3 into itself:

$$[(\alpha_o x\gamma_o)\otimes T^*\alpha_o]\cdot f\cdot[J'(\alpha_o x\gamma_o)]^{-1} : \alpha_o(U) \times R^3 \times L(R^3,R^3) \to \alpha_o(U) \times L(R^3,R^3)$$

$$(X, x, F) \longrightarrow (X, f(X, x, F))$$

In particular, if the differential operator factors through the covariant derivative, as is the case with the Piola-Kirchoff stress tensor operator, the representation is somewhat simplified: for

$$F = f_* \cdot \nabla_\eta,$$

implies that the local representation of the operator is

$$[(\alpha_o x\gamma_o)\otimes T^*\alpha_o]f[(\alpha_o x\gamma_o)\otimes T^*\alpha_o]^{-1} : \alpha_o(U) \times L(R^3,R^3) \to \alpha_o(U) \times L(R^3,R^3)$$

$$(X, F) \qquad\qquad (X, f(X, F)).$$

For the Piola-Kirchoff stress tensor operator relative to the reference configuration s_κ,

$$T_{s_\kappa} = (h_{s_\kappa}\cdot i)_*\cdot\nabla_\eta$$

the local representation is of the following form

$$[(\alpha_0 \times \gamma_0)\otimes T^*\alpha_0]\, h_{S_K}\, [(\alpha_0 \times \gamma_0)\otimes T^*\alpha_0]^{-1} : \alpha_0(U) \times L(R^3,R^3) \to \alpha_0(U) \times L(R^3,R^3)$$

$$(X\,,\,F) \qquad\qquad (X, \mathring{h}_{S_K}(X,\,F))$$

The tensor-valued function $\mathring{h}_{S_K}(X,F)$ is the classic representation of the Piola-Kirchoff response function relative to the choice of charts. One gains the usual interpretation of $\mathring{h}_{S_K}$ as the representation of the elastic Piola-Kirchoff response function relative to a field of reference local configurations[79] upon realizing that, given a connection on η, the admissible charts on $L(TB,\eta)$ are in bijective correspondence with the fields of the first jet bundle of sections of η which are invertible at each point. These invertible fields may be indentified with fields in the bundle of frames of B, which in turn are identified with the fields of reference local configurations.

THE GEOMETRIC MODEL FOR THE FINITE ELASTOSTATIC PROBLEM

The finite elastostatic field equations may now be set into the geometric model through the introduction of a finite elastostatic operator. The operator specifies a link between the manifold of C^∞ configuration sections and the space of smooth body force density sections. The methods of chapter three then allow one to extend the link to configuration sections which have lesser degrees of smoothness.

The Finite Elastostatic Operator

The bundle formulation of the divergence operation allows one to develop the finite elastostatic operator. In the theory of vector bundles,[80] the divergence operation "div" may be viewed as a linear mapping which takes the smooth sections of the linear map bundle $L(TB,\eta)$ into smooth sections of the position bundle η,

$$\mathrm{div} : C^{\infty}(L(TB,\eta)) \longrightarrow C^{\infty}(\eta).$$

From this operator and the bundle isomorphism between η and $T_{S_K}\eta$, one obtains the bundle counterpart to the divergence operator over the reference configuration. It is a linear mapping "Div" which takes smooth sections of the linear map bundle $L(TB,T_{S_K}\eta)$ into smooth sections of the bundle $T_{S_K}\eta$,

$$\mathrm{Div} : C^{\infty}(L(TB,T_{S_K}\eta)) \longrightarrow C^{\infty}(T_{S_K}\eta).$$

Moreover, in the language of the global analysis of chapter three, it is a linear differential operator of order one.[81]

Consequently if a Piola-Kirchoff stress tensor field $T_{s_\kappa}(s_\chi)$ is viewed in the bundle formulation as a smooth section of the above linear map bundle, its divergence is a well-defined smooth section of the bundle $T_{s_\kappa}\eta$,

$$\mathrm{Div}(T_{s_\kappa}(s_\chi)) \in C^\infty(T_{s_\kappa}\eta)$$

Thus, if the material comprising the body has a smooth response, one may define an operator which links the manifold of C^∞ configuration sections with smooth vector fields over the body by

$$\begin{aligned} \mathrm{Div}\ T_{s_\kappa}:\ \mathrm{Emb}^\infty(\eta) &\longrightarrow C^\infty(T_{s_\kappa}\eta) \\ s_\chi &\longrightarrow \mathrm{Div}(T_{s_\kappa}(s_\chi)). \end{aligned}$$

This operator serves as the finite elastostatic operator for the geometric model developed here.

The specifications placed upon the Piola-Kirchoff stress operator by the material response may be carried over to the finite elastostatic operator. In particular, if the material body is simple, elastic of degree one, then the finite elastostatic operator may be characterized in global analysis terms as a particular kind of differential operator of order two,

$$\mathrm{Div}\ T_{s_\kappa} = \mathrm{Div}\cdot(h_{s_\kappa})_*\cdot\nabla_\eta : \mathrm{Emb}^\infty(\eta) \longrightarrow C^\infty(T_{s_\kappa}\eta)$$

The factoring may be viewed in terms of the diagram

$$
\begin{tikzcd}
\mathrm{Emb}^{\infty}(\eta) \arrow[dd, "\nabla_{\eta}"] & \subset & C^{\infty}(\eta) \arrow[rr, dashed, "\mathrm{Div}\,\mathbb{T}_{s_K}"] \arrow[dd, "\nabla_{\eta}"] & & C^{\infty}(T_{s_K}\eta) \\
& & & & C^{\infty}(L(TB, T_{s_K}\eta)) \arrow[u, "\mathrm{Div}"'] \\
C^{\infty}(L^{(3)}(TB,\eta)) \arrow[urrrr, "(h_{s_K})_*"'] & \subset & C^{\infty}(L(TB,\eta)) & &
\end{tikzcd}
$$

More generally, if the material body is simple, elastic of degree k, or a body with fading memory, then the finite elastostatic operator would be characterized as a differential operator of higher order, or as a more complex integro-differential operator.

The Finite Elasostatic Model over the Classical Functions

Once the nature of the finite elastostatic operator is specified in global analysis terms, the results of chapter three may be employed to construct a geometric model for the finite elastostatic problem over the classical functions. The model for the simple elastic body of degree one will now be presented.

Let $\{\mathrm{Emb}^{\ell}(\eta);\ \ell = 0,1,2,\ldots\}$ and $\{C^{\ell}(T_{s_K}\eta);\ \ell=0,1,2\ldots\}$ denote the families of configuration section manifolds and body force density spaces of varying degrees of differentiability.

The geometric model over the classical functions is completed when one specifies which configuration manifolds are linked with which body force density spaces by the finite elastostatic operator. From Van Buren's model one sees that the proper choice of compatibility condition on the orders of differentiability is critical to establishing the link. Indeed, if one is interested in questions of existence, one must be extremely careful not to mismatch the solution and data spaces, if one is not to include unnecessarily data for which no solution exists. In fact, the matching problem is so significant that it constitutes one of Van Buren's assumptions for his model.

If one employs the global analysis setting of chapter three to establish the link, the compatibility is automatically incorporated into the model in the extension of the finite elastostatic operator. For, the classically differentiable sections of η are characterized by the Banach space-valued section functor C°. The functor satisfies the axioms for a global nonlinear setting. Consequently, if the material body is simple, elastic of grade one, its finite elastostatic operator

$$\mathrm{Div}\ T_{S_K}\colon \mathrm{Emb}^{\infty}(\eta) \longrightarrow C^{\infty}(T_{S_K}\eta),$$

a differential operator of order two, extends to a C^{∞} nonlinear mapping defined over the manifolds of C^{ℓ}

configurations

$$\mathrm{Div}\ T_{S_K}: \mathrm{Emb}^{\ell+2}(\eta) \longrightarrow C^{\ell}(T_{S_K}\eta) \quad \ell=0,1,2,\ldots$$

A prescription for the extended operator is given by theorem, as opposed to intricate norm calculation, its differentiability is assured, and the compatibility domain and range is assured.

For the simple, elastic body of degree one, the triples

$$\mathrm{Div}\ T_{S_K}: \mathrm{Emb}^{\ell+2}(\eta) \longrightarrow C^{\ell}(T_{S_K}\eta) \quad \ell=0,1,2,\ldots,$$

consisting of a configuration section manifold, a body force density space, and a finite elastostatic operator linking them constitute the geometric model for the finite elastostatic free boundary problem, built over the classical functions. Roughly speaking, the model consists of two manifolds with a nonlinear differentiable mapping linking them. The model may be pictured in terms of finite-dimensional elements as in Figure V.2. Notice that the particular way in which the configuration manifolds are linked to the body force spaces by the finite elastostatic operator depends upon the material response. In the simple, elastic, degree one case, it is the differential-operator-of-order-two nature of the finite elastostatic operator which dictates that the $C^{\ell+2}$

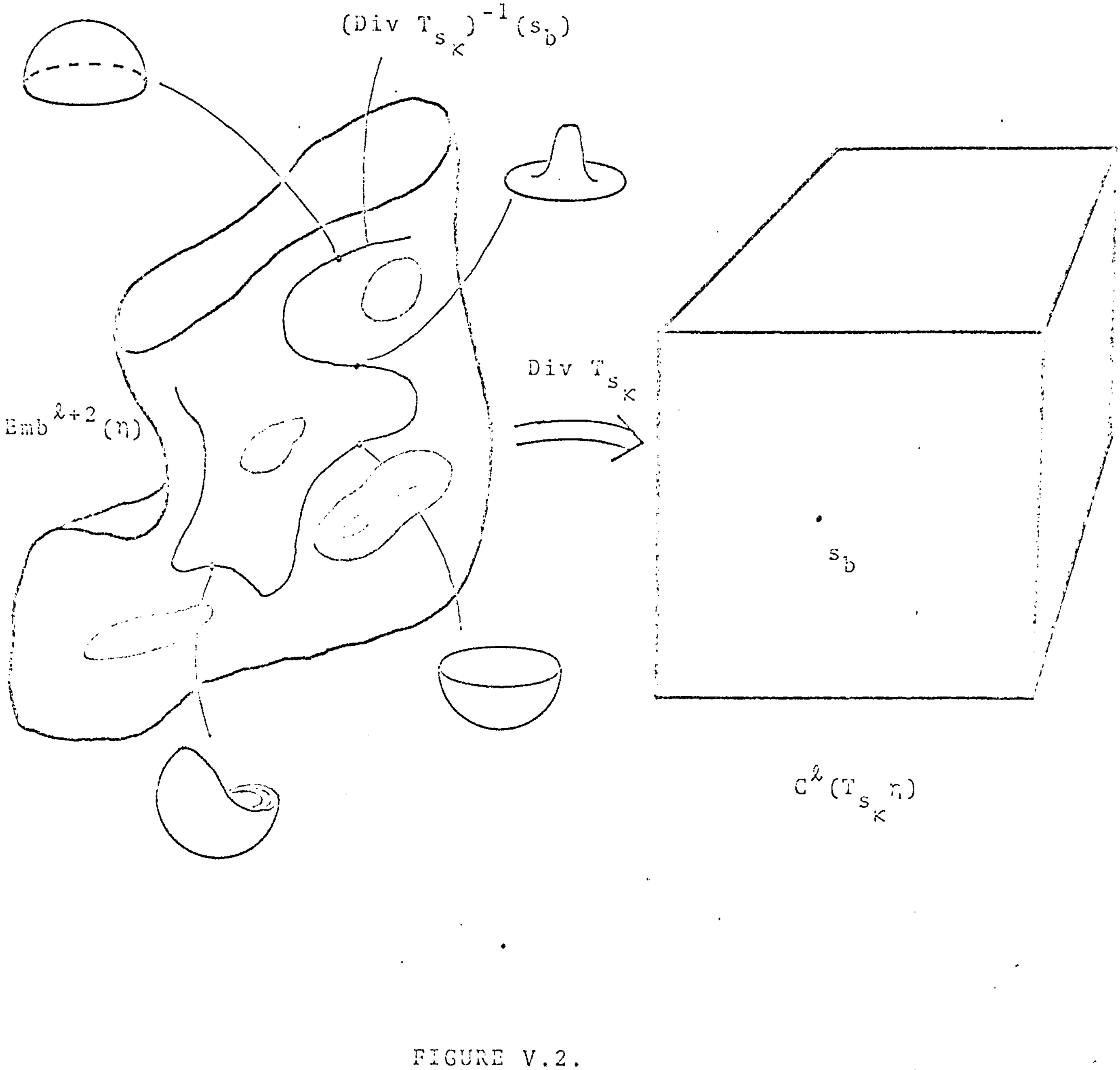


FIGURE V.2.

A Depiction of the Geometric Model for the Free Boundary Problem using Finite Dimensional Elements.

configurations are equilibrated by C^{ℓ} body forces. If the material response of the body were different, for example, elastic of a higher degree, the extension of the finite elastostatic operator would have linked spaces of entirely different degrees of differentiability.

Using the results of chapter four one may excise from the free boundary model a geometric model for the finite elastostatic Dirichlet problem. Given a C^{ℓ} place boundary condition $g \in C^{\ell}(\eta_{|\partial B})$ the manifold of C^{ℓ} configurations modeling the boundary condition in a simple supporting manner

$$[\mathrm{Emb}^{\ell}(\eta)]_g$$

lies as a closed submanifold of the free boundary manifold. As the extended finite elastostatic operator is C^{∞} over the free boundary manifolds, its restriction to the closed submanifold remains continuously differentiable. One thereby gains the family of triples

$$\mathrm{Div}\ \mathbb{T}_{s_{\kappa}}{}_{[\mathrm{Emb}^{\ell}(\eta)]_g} : [\mathrm{Emb}^{\ell+2}(\eta)]_g \longrightarrow C^{\ell}(T_{s_{\kappa}}\eta),\ \ell=0,1,2,\ldots$$

This family serves as the geometric model for the finite elastostatic Dirichlet problem. It is illustrated in Figure V.3.

One may envision how the qualitative questions manifest themselves in the geometric model. The existence

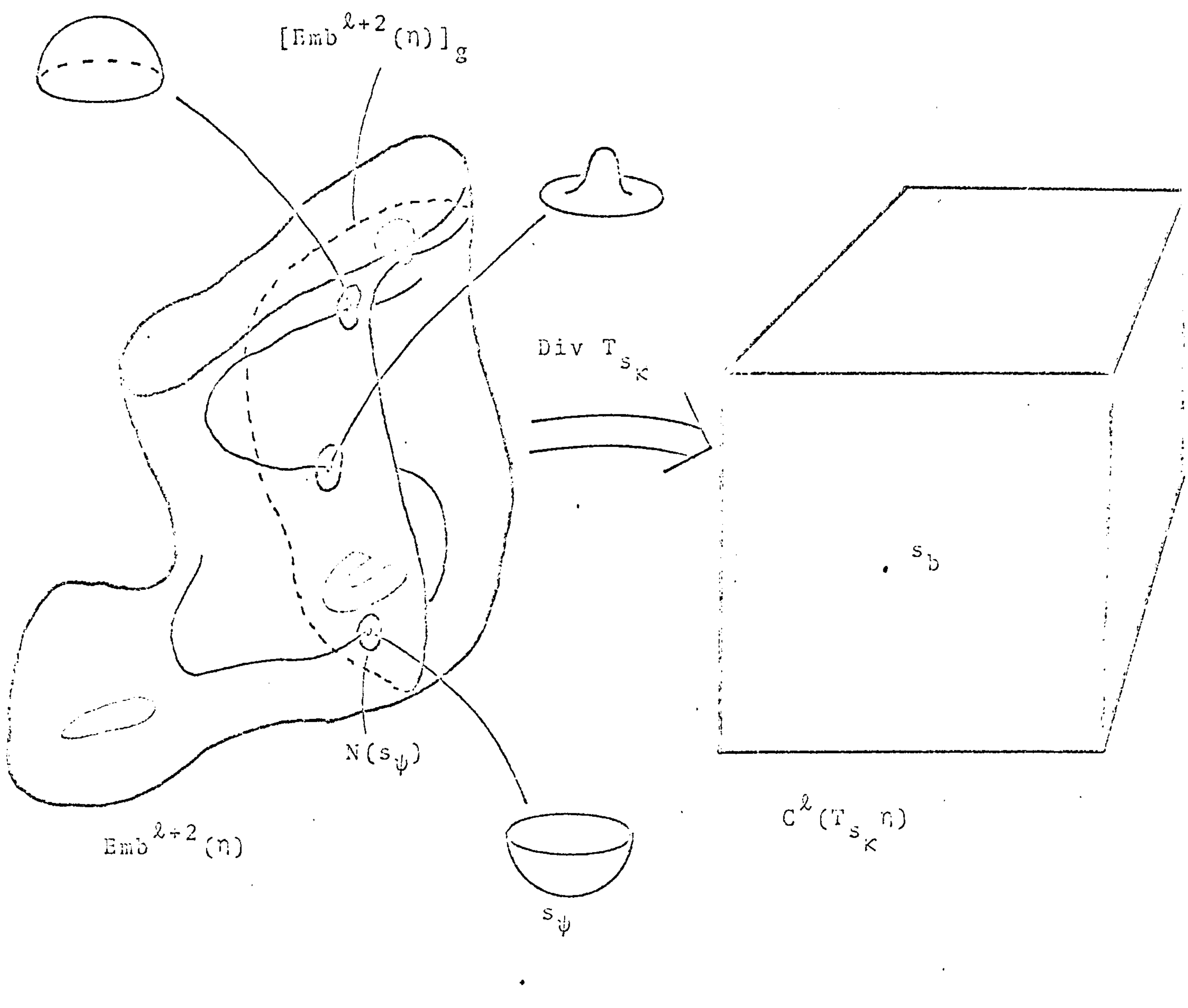


FIGURE V.3.

Visualization of the Geometric Model for the Finite Elastostatic Dirichlet Problem using Finite Dimensional Elements.

question for the finite elastostatic Dirichlet problem relative to the body force density s_b asks if it is possible to find a configuration section s satisfying the boundary condition g and equilibrating s_b. In terms of the geometric model, the body force density section s_b may be viewed as a point in the section space $C^\ell(T_{s_K}\eta)$. The existence question then becomes, does s_b have a preimage under the finite elastostatic operator $\mathrm{Div}\ Ts_K$ in the closed submanifold $[\mathrm{Emb}^{\ell+2}(\eta)]_g$? The global uniqueness question may be phrased as, does there exist more than one preimage of s_b in $[\mathrm{Emb}^{\ell+2}(\eta)]_g$? The local uniqueness question inquires whether or not it is possible to separate configuration sections in $[\mathrm{Emb}^{\ell+2}(\eta)]_g$ equilibrating s_b by neighborhoods. An example of these situations is depicted in Figure V.3.

One quickly appreciates the topological and geometric nature of the reformulated questions. One can easily anticipate how the geometric and topological theory of differentiable mappings on manifolds can be employed to resolve them.

A COMPARISON WITH BEJU'S GEOMETRIC MODEL

It is instructive to compare the geometric model presented here with those existing in the literature. A comparison will be made with the model of I. Beju. One finds, pleasantly enough, that the finite elastostatic operators are comparable, and that the critical dissimilarities in the models stem from the basic differences in the way the configuration manifolds are pictured.

Beju's representation for the finite elastostatic operator is as a global operator defined over the body B whose argument is a displacement vector field. One may gain such a representation from the bundle finite elastostatic operator developed above. The procedure is a particular instance of choosing a chart for the manifold of configurations $Emb^{\ell}(\eta)$. Recall from Chapter four charts on $Emb^{\ell}(\eta)$ about any section can be explicitly displaying using the bundle exponential map. In particular, if s_{κ} is a reference configuration and Exp is a bundle exponential map on $TF\eta$, there is an open set $\nu_{Exp,s_{\kappa}}$ about s_{κ} in the manifold $Emb^{\ell}(\eta)$, an open neighborhood of vector bundle sections $\omega_{Exp,s_{\kappa}}$ about the zero section in the Banach space $C^{\ell}(\eta)$, and a diffeomorphism $\Omega_{Exp,s_{\kappa}}$ linking them

$$\Omega_{Exp,s_{\kappa}} : \nu_{Exp,s_{\kappa}} \quad Emb^{\ell}(\eta) \longrightarrow \omega_{Exp,s_{\kappa}} \subset C^{\ell}(\eta)$$

which takes the configuration s_{κ} into the zero section in $\omega_{Exp,s_{\kappa}}$. The pair $(\nu_{Exp,s_{\kappa}}, \Omega_{Exp,\, s_{\kappa}})$ is a manifold chart for $\mathrm{Emb}^{\ell}(\eta)$ about s_{κ}, which may be viewed as associating with each section s nearby s_{κ} a vector field of η near the zero section.

In particular, if one chooses the trivial exponential map, E, on $TF\eta$ the map $\Omega_{E,s_{\kappa}}$ can be globalized, and one may parametrize the entire manifold $\mathrm{Emb}^{\ell}(\eta)$ in terms of "displacement vector fields" relative to s_{κ}:

$$\Omega_{E,s\ ,r} : \mathrm{Emb}^{\ell}(\eta) \longrightarrow \Omega_{E,s_{\kappa},r}(\mathrm{Emb}^{\ell}(\eta)) \equiv D_{E,s_{\kappa},r} \subset C^{\ell}(\eta),$$

$$s_{\phi} \longmapsto u$$

where $s_{\kappa} + u = s_{\phi}$, and the sum is in the sense of $C^{\ell}(\eta)$. $\Omega_{E,s_{\kappa}}$ is the classic representation of the C^{ℓ} configurations relative to s_{κ} as C^{ℓ} displacement vector fields. Recall, that although this parametrization exists, $(\mathrm{Emb}^{\ell}(\eta), \Omega_{E,s_{\kappa},r})$ need not be a manifold chart at s_{κ}, since $\mathrm{Emb}^{\ell}(\eta)$ may not be contractible to s_{κ}.

If one represents the finite elastostatic operator introduced above in terms of the parametrization, one achieves an operator whose argument is a displacement vector field

$$\mathrm{Div}\, T_{s_{\kappa}} \Omega^{-1}_{\ ,s_{\kappa}} : D_{E,s_{\kappa}} \quad C^{\ell+2}(\eta) \longrightarrow C^{\ell}(\eta)$$

$$u \longmapsto \mathrm{Div}\, T_{s_{\kappa}}(s_{\kappa} + u)$$

The operator so gained in Beju's free boundary operator A, minus the body force part. The critical difference between the two operators lie in their domains of definition: the operator gained here is defined only over an open submanifold of $C^{\ell+2}(\Omega)$, as opposed to the topological linear space.

The principal difference between the model presented here and the Beju's model lies with the Dirichlet problem. In Beju's model, the submanifolds representing different Dirichlet solution spaces are diffeomorphic. Hence, Beju reduces all Dirichlet problem models to a single geometric model by introducing a homogeneous boundary condition space, and a suitably modified elastostatic operator E. In the model presented here, the different Dirichlet configuration manifolds are topologically distinct, in general. Hence, it is impossible to reduce all finite elastostatic Dirichlet problem models to a single homogeneous model. Comments have already been made about the significance of this difference in the previous chapter.

VI. THE GEOMETRIC MODEL FOR THE FREE BOUNDARY PROBLEM BUILT OVER THE SOBOLEV SPACES

WHY THE GEOMETRIC MODEL MUST BE MODIFIED

In the previous chapter the geometric models for the finite elastostatic free boundary and Dirichlet problems for the simple, elastic material body were completed. One now begins to examine the question of how one gleans information from them.

Previously Used Mathematical Methods are Useless

The question cannot be quickly dispatched. For when one replaces the hybrid geometric models involving topological vector spaces by a model whose solution manifold possesses no linear structure, one loses not only many of the hybrid model results, but also the very use of the tools, theorems, and techniques which made them possible. In order to utilize the models developed in the previous chapter, one must discover alternative mathematical methods to replace those rendered useless.

The New Methods Require More General Settings

The alternative methods and tools are only now evolving as abstract theorems in infinite-dimensional geometry theory and algebraic topology. Some of the tools, like the implicit function theorem used by Van Buren, can be applied immediately to the geometric model built over the classical C^k functions to obtain existence and uniqueness conclusions. However, most of the tools cannot be applied effectively to the geometric model as it now stands. In order to accommodate them, one must modify the models. For example, to employ the Morse Theory, or the Lusternik-Schnirelman Theory effectively, it is most profitable to extend the models to ones built over functions having a more general form of continuity and differentiability. This situation parallels that in Beju's model, where, in order to effectively apply Langenbach's results, Beju generalized his data space to the Hilbert space of square integrable fields.

The Holder functions and the Sobolev functions mentioned in chapter three are particularly attractive candidates to use in the extension. Indeed, since much work on the existence and uniqueness problems for the linear infinitesimal elastostatic problem has been done using the Sobolev function spaces, it is of value to cast the nonlinear problem in the same setting.

The Regularity Question Arises

But how does one gain existence or uniqueness conclusions for "concrete", classically C^k configurations for a material body from a generalized function setting? The procedure one develops consists of two steps. Firstly, one chooses a generalized setting for the geometric model which yields as easily as possible conclusions for the existence and uniqueness questions for the finite elastostatic problem. One then attempts to pull back the results to the classical function setting by proving theorems which state that if the given data are in fact C^k differentiable, then the generalized function solution is in fact C^ℓ differentiable. Theorems which accomplish this purpose are called "regularity theorems," and the investigation of which conclusions can be pulled back and which cannot is called "the regularity question". Hence, one witnesses the three fundamental areas of investigation in a geometric model: existence, uniqueness, and regularity.

What the Development Accomplishes

In short, this chapter casts the geometric model for the free boundary in a more general mathematical setting which will allow one to more effectively apply contemporary mathematical methods to draw physical conclusions about existence and uniqueness questions. The model one obtains has features similar to the classical function model. It sets the problem as a differentiable mapping between manifolds. The specification of the configuration manifold, data space, and finite elastostatic operator is given by Definition VI.1, Theorem VI.1, and Theorem VI.2, which constitute the main results of the chapter. The generalized configuration manifolds have the same homotopy type as the C^k manifolds; hence, alternatives of mechanical behavior are not gained or lost by the mathematical generalization. However, the topology and geometry of the generalized configuration manifolds are more sophisticated than that of its classical function counterparts. The more sophisticated structure permits the more effective use of the mathematical tools. In chapter seven, the Dirichlet model will be similarly generalized.

In this chapter and the next, the reader will encounter a development which is more mathematically than physically motivated. However, as illustrated in Beju's model, the development is a necessary step if one is to glean physical conclusions from the model. The relevance of the development will manifest itself in chapter eight, where one actually perceives how to glean information from the model.

PROBLEMS ONE ENCOUNTERS IN EXTENDING THE FREE BOUNDARY MODEL

The Extension of the Infinitesimal Elasticity Model Relies Heavily Upon Its Linear Structure

How does one go about extending the free boundary model to a generalized function setting? One may examine the extension of the infinitesimal elasticity model for clues.

The extension of the linear elastostatic place boundary value problem to a generalized function space setting is an intricate, but straightforward process.[82] The solution spaces and data spaces are completed to form two families of Banach spaces of generalized functions. The families are linked by the extension of the infinitesimal elasticity operator, whose properties are usually gained by a series of intricate calculations.

The Problems in Extending the Finite Elasticity Model

Two problem areas arise in extending the finite elastostatic model: the generalization of the C^ℓ configuration manifolds, and the extension of the elastostatic operator.

The problem one encounters with extending the configuration manifolds rests in their lack of linear structure. The extension of the linear solution spaces $C^\ell(\eta)$ for the infinitesimal elastostatic problem rely heavily upon their linear structure. The manifold of C^ℓ configurations $\mathrm{Emb}^\ell(\eta)$, which lies only as a subset of $C^\ell(\eta)$, and possesses no linear structure, cannot be extended in the same manner. Rather one must pursue the extension more carefully.

A parallel problem arises in the extension of the finite elastostatic operator. If the operator were defined over the entire Banach space $C^\ell(\eta)$, its extension to a generalized function setting would be given immediately by theorem. However, its domain of definition consists only of the open set $\mathrm{Emb}^\ell(\eta)$. Hence, the extension is not a trivial matter.

Although the nonlinear problem does not extend to the generalized function space setting with the speed and dispatch that the linear problem does, one is not relegated to a hopeless position. The tools provided by the global nonlinear analysis provided in chapter three still allow one to set the configuration manifolds, and extend the finite elastostatic operator in a precise manner. Both the Holder functions and the Sobolev functions may be viewed in terms of Banach space-valued section functors: the Holder functors of exponent α, $C^{k+\alpha}$, and the Sobolev functors $L^2_k = H^k$. By Theorem III. 7 , both functors satisfy the axioms for a global nonlinear setting. Thus the tools of the global nonlinear analysis may be employed to assist in the extension of the configuration manifolds and the elastostatic operator, and to insure the proper matching of the solution manifolds and the data spaces. Most importantly, one may accomplish the extensions by use of theorems, as opposed to extensive norm calculations.

THE EXTENSION OF THE DATA SPACES

Since the body force density section spaces $C^r(T_{S_K}\eta)$ built over the classical functions maintain their linear structure, their extension to the Sobolev spaces follows immediately by theorem from the methods of chapter three. The Sobolev functor H° is a Banach (in fact, Hilbert) space-valued section functor which satisfies the axioms B§1 through B§4 of the global linear analysis. Hence by theorem, one is assured that the classical force density section spaces $C^r(T_{s_K}\eta)$ may be extended in a continuous way to a chain $\{H^r(T_{s_K}(\eta))\}$ of Banach spaces of force density sections with more generalized differentiability and continuity properties. For the convenience of the reader the extension is summarized as a corollary, which one may establish from the work of chapter three:

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Corollary VI.1. Let B be a smooth, compact, material body. Let $\eta = B \times R^3$ be its bundle of positions, s_K be a reference configuration section for B, and let $C^r(T_{s_K}\eta)$ denote the space of C^r body force density field relative to s_K.

Then

a) for each integer r, the Sobolev space $H^r(T_{s_K}\eta)$ of sections of $T_{s_K}\eta$ which are square integrable to order r over B is a well defined Hilbert space,

b) the classical body force density section spaces $C^r(T_{s_K}\eta)$ inject in a continuous, linear manner into the corresponding Sobolev space:

$$i: C^r(T_{s_K}\eta) \longrightarrow H^r(T_{s_K}\eta) \quad r=0,1,2,3,\ldots,$$

c) one has the following embedding theorems which indicate the relation between the classically differentiable sections and the generalized sections: for k, r, into

(1) $i_{kr}: H^k(\eta) \longrightarrow H^r(\eta)$, for $k>r>0$,

$$(2)\ i_{r\ell}: H^r(\eta) \longrightarrow C^\ell(\eta), \text{ for } r>\tfrac{3}{2}+\ell,\ \ell\geq 0.$$

The family of spaces $\{H^r(\eta)\}$ serves as the geometric model for the body force density sections built over the Sobolev function spaces.

HOW THE FREE BOUNDARY CONFIGURATION MANIFOLD EXTENDS TO THE SOBOLEV SPACES

The manifolds of C^ℓ configurations may be extended to open submanifolds of the Sobolev spaces $H^\ell(\eta)$. The definition of the classical $\mathrm{Emb}^\ell(\eta)$ manifolds themselves provide the clue for constructing the extensions. Moreover, the fact that one may employ the methods of chapter three permits one to construct the extensions without being inundated with a plethora of norm calculations. The essential elements of the extension are contained in the following definition and theorem.

Definition VI.1. Let B be a smooth, compact, connected material body which is orientable and oriented. Let η be the vector bundle of positions of B in R^3. Let $W^{(3)}$ be the open fiber subbundle of non-degenerate one jets in $J^1(\eta)$. Let $\mathrm{Deg}_1^\circ(\eta)$ denote the open submanifold of continuous sections of η which are of degree one.

a) Let

$$j_1: H^r(\eta) \longrightarrow H^{r-1}(J^1(\eta))$$

denote the continuous linear homeomorphism onto a closed subspace which extends the one jet extension map from the classical (C^r) section spaces. For $r \geq 3$, define

$$\operatorname{Imm} H^r(\eta) = j_1^{-1}(H^{r-1}(W^{(3)}))$$

Term $\operatorname{Imm} H^r(\eta)$ the H^r-generalized immersive sections of η.

b) For $r \geq 2$, let

$$i_{ro} : H^r(\eta) \longrightarrow C^\circ(\eta)$$

be the continuous, linear inclusions specified in Corollary VI.1. Define

$$\operatorname{Deg}_1 H^r(\eta) = i_{ro}^{-1}(\operatorname{Deg}_1^\circ(\eta)),$$

and term it the H^r-generalized degree one sections of η.

c) For $r \geq 3$, define

$$\operatorname{Emb} H^r(\eta) = \operatorname{Imm} H^r(\eta) \cap \operatorname{Deg}_1 H^r(\eta)$$

to be the H^r-generalized embedding sections of η, or the H^r configuration sections.

Theorem VI.1. Under the hypotheses of Definition VI.1,

$$\operatorname{Emb} H^r(\eta) \subseteq H^r(\eta)$$

is an open C^∞ submanifold having the following properties which reflect the fact that it is an "admissible" extension of $\operatorname{Emb}^r(\eta)$:

a) The continuous linear inclusion

$$i: C^r(\eta) \longrightarrow H^r(\eta)$$

restricts to a C^∞ inclusion

$$i: \operatorname{Emb}^r(\eta) \longrightarrow \operatorname{Emb} H^r(\eta),$$

for $r \geq 3$.

b) For $r \geq 3$, $\ell \geq 0$ integers satisfying the relation $r \geq 3/2 + \ell$, then the continuous, linear inclusion

$$i_{r\ell} : H^r(\eta) \longrightarrow C^{\ell}(\eta)$$

(1) restricts to a C^{∞} inclusion

$$i_{r\ell} : \mathrm{Emb}H^r(\eta) \longrightarrow \mathrm{Emb}^{\ell}(\eta).$$

Hence, elements of $\mathrm{Emb}H^r(\eta)$ are embeddings of B into R^3 in some classical (C^{ℓ}) sense.

(2) Moreover,

$$i^{-1}_{r\ell}(\mathrm{Emb}^{\ell}(\eta)) = \{s \varepsilon H^r(\eta) : i_{r\ell}(s) \varepsilon \mathrm{Emb}^{\ell}(\eta)\} = \mathrm{Emb}H^r(\eta).$$

That is to say, all elements of $H^r(\eta)$, which are embeddings in any classical (C^{ℓ}) sense are in $\mathrm{Emb}H^r(\eta)$.

c) For $k \geq r \geq 3$ integers, the linear, continuous inclusion

$$i_{kr} : H^k(\eta) \longrightarrow H^r(\eta)$$

(1) restricts to a C^{∞} inclusion

$$i_{kr} : \mathrm{Emb}H^k(\eta) \longrightarrow \mathrm{Emb}H^r(\eta).$$

(2) Moreover,

$$i^{-1}_{kr}(\mathrm{Emb}H^r(\eta)) = \{s \varepsilon H^k(\eta) : i_k(s) \ \varepsilon \ \mathrm{Emb}H^r(\eta)\} = \mathrm{Emb}H^k(\eta).$$

That is to say, all elements of $H^k(\eta)$ which are H^r_k embeddings for some r, $3 \leq r \leq k$, are H^k embeddings.

d) $\mathrm{Emb}H^{\infty}(\eta) = \mathrm{Emb}^{\infty}(\eta)$.

The proof of the theorem follows straightforwardly from the work of Palais.[84] Moreover some insight into the topology of the extended configuration manifolds can be gained from the following theorem relating the homotopy type of $EmbH^k(\eta)$ and $Emb^\circ(\eta)$:

Theorem VI.2.[85] For $k \geq 3$, the inclusion map

$$EmbH^k(\eta) \longrightarrow Emb^\circ(\eta)$$

is a homotopy equivalence.

The particular details of the extended manifolds $EmbH^k(\eta)$ have not been included in this work, as they are not particularly germane to the development of the model. Rather, it suffices to say that within the chain of Sobolev spaces over η, $\{H^k(\eta)\}$, there exist open submanifolds which generalize the C^r domains of definition of the finite elastostatic operator.

THE EXTENSION OF THE FINITE ELASTOSTATIC OPERATOR

It remains to see how the finite elastostatic operator extends. If traditional methods were employed to resolve this question, one would now be faced with a profusion of norm calculations. However, using the global nonlinear analysis methods set forth in chapter three, one may now establish by theorem, as opposed to calculation, those domains to which the finite elastostatic operator extends, and the degree of differentiability of the extended operator over these generalized domains.

For a smooth materially uniform elastic material body simple of degree one, the finite Piola-Kirchoff elastostatic operator relative to a reference section s_K is a differential operator of order two defined over the C^∞ configuration sections. That is to say,

$$\mathrm{Div}\, T_{s_K} : \mathrm{Emb}^\infty(\eta) \subseteq C^\infty(\eta) \longrightarrow C^\infty(T_{s_K}\eta),$$

and

$$\mathrm{Div}\, T_{s_K} = \mathrm{Div}(h_{s_K})_* \nabla_\eta \equiv (H_{s_K})_* j_2$$

satisfies the diagram

$$\begin{tikzcd}
\mathrm{Emb}^\infty(\eta) \subset C^\infty(\eta) \arrow[rr] \arrow[d, dash, "j_2"] \arrow[dr, dash, "j_2"] & & C^\infty(T_{s_K}\eta), \arrow[dl, dash, "(H_{s_K})_*"'] \\
C^\infty((W_2^{(3)})) \subset & C^\infty(J^2(\eta)) &
\end{tikzcd}$$

for

$$H_{s_K} : (W_2^{(3)}) \longrightarrow T_{s_K}\eta$$

a C^∞ fiber bundle morphism on $W_2^{(3)}$, an open fiber subbundle of the vector bundle $J^2(\eta)$. Taking n=3, p=2, and choosing $k=2 > 3/2$, Theorem III.2 and Theorem III.7 assure that the functor $L_2^2 = H^2$ satisfies the axioms B§2 and B§5. Hence, one is assured that the fiber bundle morphism H_{s_K} extends to a C^∞ mapping of the chain of C^∞ manifold $H^{2+s}(W_2^{(3)})$ of generalized sections:

$$H^{2+s}(H_{s_K}) : H^{2+s}(W_2^{(3)}) \longrightarrow H^{2+s}(T_{s_K}\eta), \; s=0,1,2,...$$

The extension of the j_2 map is also well behaved when restricted to open submanifolds:

Lemma VI.1. The continuous linear map

$$j_2 : H^{2+s+2}(\eta) \longrightarrow H^{2+s}(J^2(\eta))$$

restricts to a C^∞ inclusion

$$j_2 : \mathrm{Emb}H^{4+s}(\eta) \longrightarrow H^{2+s}(W_2^{(3)}),$$

for $s=0,1,2,...$

Combining the results, (and, indeed, without any further calculation), one is assured that the finite elastostatic operator

$$\mathrm{Div}\, T_{s_K} : \mathrm{Emb}^\infty(\eta) \longrightarrow C^\infty(T_{s_K}\eta)$$

extends to a nonlinear mapping of manifolds

$$H^{2+s}(H_{s_K})\cdot j_2 : \mathrm{Emb}H^{2+s+2}(\eta) \longrightarrow H^{2+s}(T_{s_K}\eta)$$

for $s=0,1,2,...,$ which is C^∞. Thus, one is assured that the generalized operator is continuous, and has continuous Frechet derivative at every point in the domain of definition.

THE EXTENDED GEOMETRIC MODEL FOR THE FREE BOUNDARY PROBLEM

Viewing the finite elastostatic operator simply as a nonlinear differential operator of order two, the above equation asserts that the most general data over the Sobolev spaces consistent with the C^{∞} extension of the finite elastostatic operator is that lying in $H^2(T_{s_K}\eta)$, the space of sections whose derivatives through order two are Lebesque square integrable over the body B:

$$\mathrm{Div}\ T_{s_K} : \mathrm{Emb}H^4(\eta) \longrightarrow H^2(T_{s_K}\eta).$$

For an arbitrary second order operator, this is as far as the global analysis permits the operator to be extended. However, since the finite elastostatic operator is a particular type of differentiable operator, a divergence operator, one may proceed at least one step further in the extension. Since a divergence operator of order two is a composition of a linear differential operator of order one and a nonlinear operator of order one, and since a linear operator extends continuously to all $H^k(\eta)$ spaces, $k \geq 0$, the finite elastostatic operator may be extended to a C^{∞} map from the generalized H^3 configuration manifold into data which is in $H^1(\eta)$:

Theorem VI.2.[86] Given a finite elastostatic operator

$$\mathrm{Div}\, T_{s_K} = \mathrm{Div}(h_{s_K})_* \nabla_\eta : \mathrm{Emb}^\infty(\eta) \longrightarrow C^\infty(T_{s_K}),$$

a divergence operator of order two, then the operator extends to a C^∞ map

$$\mathrm{Div}\, T_{s_K} : \mathrm{EmbH}^{2+s+1}(\eta) \longrightarrow H^{2+s-1}(T_{s_K}\eta),\ s=0,1,2,\ldots \qquad \text{(VI-1)}$$

In particular, the most general data for the C^∞ extended operator lies in $H^1(T_{s_K}\eta)$

$$\mathrm{Div}\, T_{s_K} : \mathrm{EmbH}^3(\eta) \longrightarrow H^1(T_{s_K}\eta)$$

The proof of the theorem is given in the reference.

The collection of configuration manifolds, data spaces and finite elastostatic operator link set forth in equation (VI.1) constitutes the geometric model for the finite elastostatic free boundary problem built over the Sobolev spaces. The model possesses four features which are sufficient significant, so as to warrant the attention of the reader. Firstly, it is a geometrical topological model. As in the classical function setting, the final result for the free boundary geometric model built over the Sobolev sections is a family of differentiable manifolds linked by well prescribed differentiable mappings. One may therefore view existence and uniqueness questions from a geometric and topological perspective. The perspective will be examined in chapter eight.

The second feature is that, unlike Van Buren's model, the model presented is a <u>global</u> model. If one restricts the model to a small region about given configuration, the model presented here reduces to the classical infinitesimal elasticity model built over the Sobolev functions. The model, hence, represents a particular "piecing together" of infinitesimal elasticity models. How the patching process is carried out is determined by the topology of the material body itself, and the material response.

Thirdly, many of the properties of the model are specified by theorem, as opposed to extensive calculation. Not least among them are that the finite elastostatic operator is differentiable, and that, in fact, a prescription for its derivative is set forth in chapter three.

Finally, one should appreciate how the proper matching of solution and data spaces follows automatically by theorem when one can employ the global analysis tools. In particular, one can appreciate how tenuous is the position of postulating that the generalized problem is "suitably set" in terms of spaces which are given *a priori*.

In summary, setting the free boundary problem in finite elastostatics is more involved than its counterpart in the infinitesimal case. When the finite elastostatic operator may be extended, it is defined only over a portion of an entire Sobolev space. The topology of its domain of definition is thus more complicated than that of the

Hilbert space in which it lies. Complex as this situation is, however, theorems are available from the global analysis which permit the generalized problem to be set with surprising precision, and with a minimum amount of information. One may now look forward to the utilization of the model, which will be discussed in chapter eight.

VII. THE GENERALIZED SETTING FOR THE FINITE ELASTOSTATIC DIRICHLET PROBLEM

THE MAIN RESULTS

The Generalized Dirichlet Configuration Manifold for a Simply Supported Boundary Condition

The tools provided by the global nonlinear analysis permit one to excise from the free boundary model a geometric model for the finite elastostatic Dirichlet problem built over the generalized function spaces. Two geometric models for the Dirichlet problem are constructed. They correspond to whether one simply supports or rigidly supports the Dirichlet boundary condition.

The first part of the chapter concentrates upon the simple support model. One shows that the classical (C^k) configuration manifold for a simply supported Dirichlet boundary condition, which was introduced in chapter four, has a generalization to the Sobolev function spaces. The generalized manifold has a well-defined structure; moreover, it lies as a closed submanifold of the generalized free boundary configuration manifold introduced in chapter six. The topology of the Dirichlet configuration manifolds may be quite complex; in addition, two configuration manifolds for different Dirichlet boundary conditions need not be diffeomorphic. The details of the manifold structure, and the relationship to the free boundary configuration manifold are presented in Theorem VII.3, and Theorem VII.4.

How the extended finite elastostatic operator behaves upon restriction to the Dirichlet submanifolds is then examined. Using the tools available from chapter three, one may show, by theorem, as opposed to norm calculation, that the generalized finite elastostatic operator maintains its differentiability properties when restricted to the submanifolds. Thus, one achieves a geometric model for the fintie elastostatic Dirichlet problem built over the Sobolev generalized functions. It may be viewed as a differentiable mapping linking infinite dimensional manifolds in a nonlinear way. The section closes with some comments on how existence, uniqueness, and regularity questions reveal themselves as topological and geometric questions in the model.

The Generalized Dirichlet Manifold for a Rigidly Supported Boundary Condition

The extended geometric model for the Dirichlet problem for which the boundary conditions are maintained by rigid support is constructed in the second part of the chapter. For this case, elements of a particular configuration manifold not only satisfy the given Dirichlet boundary condition, but also represent a particular way of "coming off the boundary." In Theorem VII.4, one shows that the generalized configuration manifold for a rigidly supported boundary condition lies as a closed submanifold of the generalized configuration manifold for the simply supported boundary condition. When one restricts the finite elastostatic operator to the new submanifold, one finds that, once again, its differentiability properties are maintained. Hence, one may construct a second geometric model for the Dirichlet problem, one which models a rigidly supported boundary condition.

One Model Refines the Other

In the last part of the chapter, one investigates the relationship between the two geometric models for the Dirichlet problem. By Theorem VII.4, one finds that one may view the Dirichlet configuration manifold for a given simply supported boundary condition as the disjoint union of those configuration manifolds which are associated with the various ways of rigidly supporting the same boundary condition. By so doing, one may bestow upon the Dirichlet configuration manifold representing a simple support, a geometric and topological structure which is finer than that which it was shown to possess in the first part of the chapter. The section ends with some comments which indicate that, with a finer structure on the solution manifold, one may probe deeper into questions on the uniqueness and local uniqueness of solutions to Dirichlet problems.

THE GENERALIZED SETTING FOR THE PLACE BOUNDARY OPERATOR

In Chapter four the specification of the boundary condition of place was given a bundle formulation through the introduction of the boundary restriction operator. In the bundle formulation, the specification of a boundary condition of place amounts to the specification of a cross section g of the bundle of positions restricted to the boundary of the body, $\eta|_{\partial B}$. The assignment of a boundary condition is accomplished through the introduction of the boundary operator $|_{\partial B}$, which is a linear differential operator which takes C^∞ sections of the bundle of positions η into C^∞ sections of the boundary restricted bundle $\eta|_{\partial B}$:

$$|_{\partial B} : C^{\infty}(\eta) \longrightarrow C^{\infty}(\eta_{|\partial B})$$

$$\mathcal{g} \qquad\qquad \mathcal{g}|_{\partial B}$$

Two questions now arise which must be resolved before one can consider the generalized Dirichlet models. Firstly, how does the boundary operator extend to the Sobolev spaces? Secondly, if one is given a boundary condition g, when can one be assured that there exists at least one configuration section of a suitable generalized differentiability class, say $\mathcal{g} \varepsilon \mathrm{EmbH}^{k}(\eta)$, which when restricted to the boundary of the body coincides with the boundary condition:

$$\mathcal{g}|_{\partial B} = g?$$

The extension of the boundary operator to the Sobolev spaces is not a trivial result. In general, it is impossible to find any section of the bundle of positions which can model the given boundary condition without some condition being placed upon its differentiability. It becomes necessary, therefore, to investigate what conditions on the boundary configuration section are sufficient to guarantee at least one extension with the proper degree of differentiability. Until recently, the only available alternative was to conjecture that certain reasonable assumptions on the boundary conditions were sufficient.[87]

However, with the advent of the linear global analysis, sufficient conditions have been deduced by theorem. Hormander,[88] Atiyah,[89] and others have shown that if one allows the Sobolev spaces to be defined for non-integer orders, the boundary restriction operator extends to a continuous linear mapping of the Sobolev space of order $k \geq 1$ into the Sobolev space of order $k-1/2$:

$$|_{\partial B} : H^k(\eta) \longrightarrow H^{k-1/2}(\eta|_{\partial B})$$

The significant result for the purposes here, however, is that the linear global analysis allows one to prove that the continuous linear extension is in fact surjective.

> Theorem VII.1.[90] Let η be a vector bundle over a compact manifold M with C^∞ boundary ∂M. Then for $k-1/2 \geq 0$, the linear continuous mapping
>
> $$|_{\partial M} : H^k(\eta) \longrightarrow H^{k-1/2}(\eta|_{\partial M})$$
>
> admits a linear continuous section, and hence is surjective.

The conditions sufficient for modeling the boundary conditions for a differential operator whose domain of definition is an entire Sobolev space follows immediately from this theorem. Given such a differential operator of order r, whose free boundary model is

$$D : H^k(\eta) \longrightarrow H^{k-r}(\eta),$$

and given a place boundary condition $g\epsilon(\eta|_{\partial B})$, if the derivatives of on the boundary ∂B, (the "tangential derivaties" locally), are Lebesque square integral on ∂B, at least through order k-1/2, the one is assured that there exists at least one mapping of B into R^3 of suitable differentiability $(\mathcal{g}\epsilon H^k(\eta))$ which "models" the boundary condition, in the sense that

$$\mathcal{g}|_{\partial B} = g.$$

In particular, for the models of the finite elastostatic problem found in the literature, where the domain of definition of the finite elastostatic operator is an entire Sobolev space, the usual axiom of requiring the tangential derivatives of the given boundary condition to have the same order of differentiability as the solution is now seen to be more than adequate.

When one adopts a more rigorous model for the finite elastostatic problem, one in which the generalized configumanifold of the finite elastostatic is <u>not</u> an entire Sobolev space, then the previous simple existence theorem for modeling boundary conditions becomes narrowed. One can achieve the following modification:

Theorem VII.2. Let B be a materially uniform body, simple of grade one, compact, connected, orientable, and oriented. Let ∂B denote the boundary of the body. Let

$$\eta = B \times R^3$$

denote the bundle of positions, and let

$$\eta|_{\partial B} = \partial B \times R^3$$

denote the bundle restricted to the boundary of the body.

a) Then for $k \geq 3$, k integer, the boundary restriction map extends to a C^∞ submersion

$$|_{\partial B} : \mathrm{Emb}H^k(\eta) \longrightarrow \mathrm{Emb}H^{k-1/2}(\eta|_{\partial B})$$

Hence, (1) if $g \varepsilon \mathrm{Emb}H^{k-1/2}(\eta|_{\partial B})$ has a preimage in $\mathrm{Emb}H^k(\eta)$, every $\bar{g}$ near g has a preimage in $\mathrm{Emb}H^k(\eta)$

(2) If $\mathcal{g} \varepsilon \mathrm{Emb}H^k(\eta)$ is such that $\mathcal{g}|_{\partial B} = g$, then every $\bar{\mathcal{g}}$ nearby it in $\mathrm{Emb}H^k(\eta)$ has a boundary restriction nearby g.

b) However, for

$$|_{\partial B}^{-1}(\mathrm{Emb}H^{k-1/2}(\eta|_{\partial B})) \equiv \{s \varepsilon H^k(\eta) \;:\; s|_{\partial B} \varepsilon \mathrm{Emb}H^{k-1/2}(\eta|_{\partial B})\}$$

then

$$\mathrm{Emb}H^k(\eta) \subset |_{\partial B}^{-1}(\mathrm{Emb}H^{k-1/2}(\eta|_{\partial B})),$$

and the containment is proper.

What the proposition asserts is that although simple differentiability conditions on the boundary value g are sufficient to guarantee the existence of some mapping of B into R^3 which models g and has the proper degree of differentiability, (i.e., it lies in $H^k(\eta)$), one cannot be assured that the mapping is in fact a configuration (i.e., it lies in $\mathrm{EmbH}^k(\eta))$. One can only be assured that if one finds a boundary condition g which can be modeled by a configuration, then boundary values $\overline{g}$ nearby g can be modeled by configurations. It appears that if one wishes to impose additional conditions to insure the modeling of the boundary value by configurations, these conditions will be imposed upon the body itself, as opposed to just the boundary conditions. It is conjectured that the additional conditions concern the topology of the material body. At present, however, the problem of the additional conditions is left as an open question.

THE EXTENDED GEOMETRIC MODEL CORRESPONDING TO SIMPLY SUPPORTED BOUNDARY CONDITIONS

The Generalized Dirichlet Configuration Manifold Corresponding to Simply Supported Boundary Conditions

In spite of the difficulties one encounters in modeling a given boundary condition g in terms of a configuration, if one can be assured that g can be modeled by a configuration with the proper degree of differentiability, one can deduce a structure for the set of all configurations satisfying the given boundary condition. It is a closed

C^∞ submanifold of the free boundary solution manifold:

Theorem VII.3. Let B be a connected, C^∞, compact, materially uniform body simple of grade one which is orientable and oriented. Let the free boundary finite elastostatic problem be given, relative to a reference configuration section, by

$$\mathrm{Div}\, T_{s_K} : \mathrm{EmbH}^k(\eta) \longrightarrow H^{k-1/2}(\eta)$$

Let $g \in H^{k-1/2}(\eta|_{\partial B})$ be a given Dirichlet boundary condition. Then for

$$[\mathrm{EmbH}^k(\eta)]_g = \{s \in \mathrm{EmbH}^k(\eta) : s|_{\partial B} = g\}$$

either

(1) $[\mathrm{EmbH}^k(\eta)]_g = \emptyset$
or (2) $[\mathrm{EmbH}^k(\eta)]_g$ is a closed C^∞ submanifold of $\mathrm{EmbH}^k(\eta)$.

Moreover,

$$[\mathrm{EmbH}^k(\eta)]_g = |_{\partial B}^{-1}(g) \cap \mathrm{EmbH}^k(\eta)$$

for

$$|_{B}^{-1}(g) = \{s \in H^k(\eta) : s|_{\partial B} = g\}.$$

The proof parallels that given for C^k case in chapter four. Such parallel results emphasize the "categorical" nature of the global nonlinear analysis setting, as mentioned in chapter three.

From the proposition one sees that one may slice out of the free boundary solution manifold for a finite elastostatic problem for a given material body submanifolds, $[\mathrm{EmbH}^k(\eta)]_g$, which can serve as solution manifolds for the finite elastostatic problem subject to the given place boundary condition. Since $[\mathrm{EmbH}^k(\eta)]_g$ is a submanifold as opposed to a topological vector space, its topology may be quite complicated. For two different boundary conditions g and $\overline{g}$ the Dirichlet solution manifolds $[\mathrm{EmbH}^k(\eta)]_g$ and $[\mathrm{EmbH}^k(\eta)]_{\overline{g}}$ need not be diffeomorphic. The topology of the Dirichlet manifolds, and thereby alternatives of mechanical behavior, may vary with the boundary condition and the topology of the body. In short, one may obtain results which parallel those obtained for the C^k case in chapter four. One may thus picture the generalized Dirichlet manifold using finite dimensional elements as shown in Figure VII.1.

The Geometric Model for the Dirichlet Problem Built Over the Sobolev Spaces

One may erect the geometric model for the finite elastostatic Dirichlet problem over the Sobolev spaces by examining how the extended finite elastostatic operator developed in chapter six behaves under restriction to the Dirichlet configuration manifolds corresponding to a simply supported boundary condition. For a connected, compact material body simple of degree one, Theorem VI.2 indicates that the finite elastostatic operator extends to a nonlinear, but C^∞ mapping between the generalized free boundary configuration manifolds and the Sobolev data spaces:

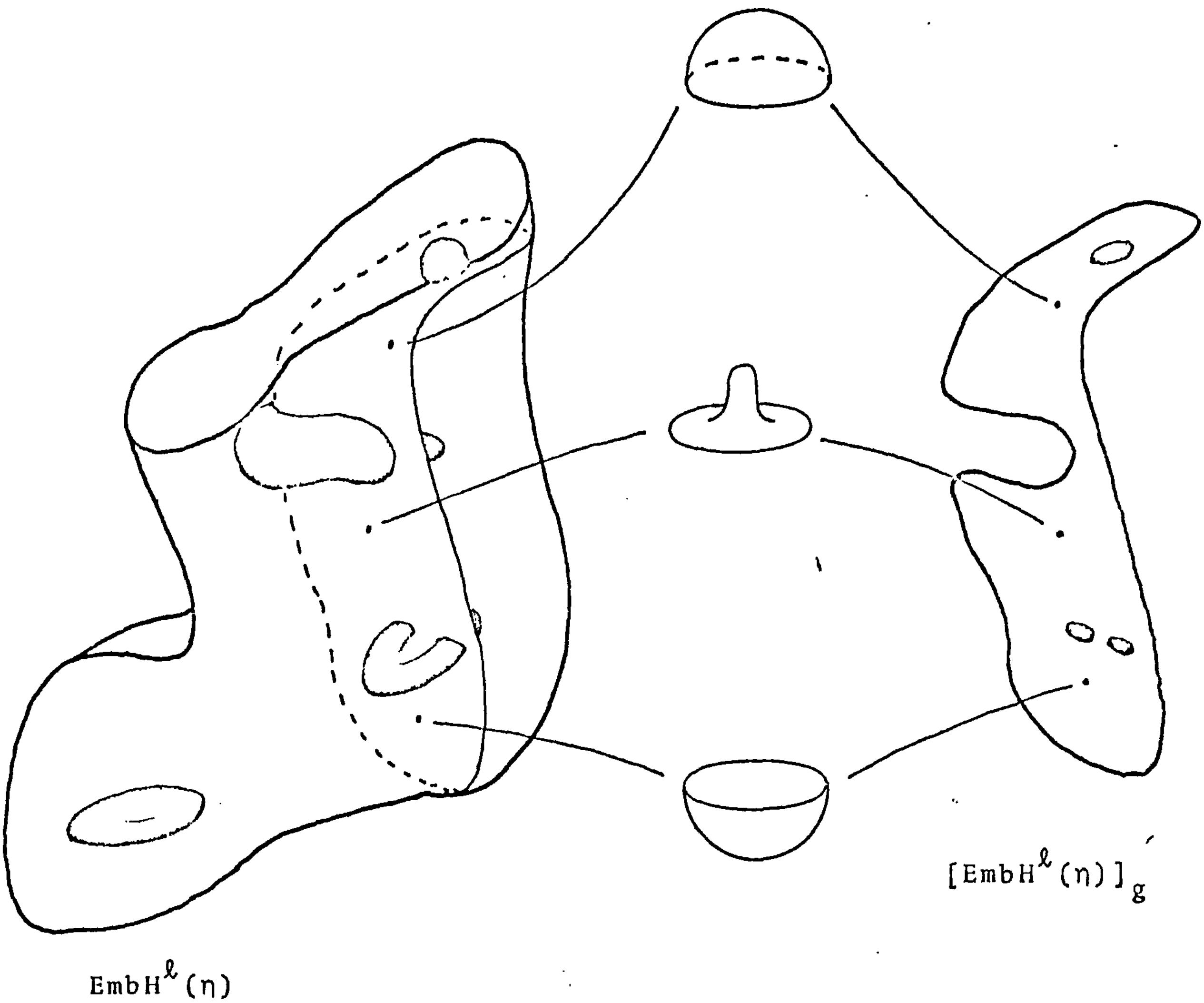


FIGURE VII. 1.

A Visualization of the Generalized Dirichlet Configuration Submanifold Using Finite Dimensional Elements.

$$\mathrm{Div}\, T_{s_K} : \mathrm{EmbH}^{2+\ell}(\eta) \longrightarrow H^{\ell}(T_{s_K}\eta) \quad \ell = 1,2,\ldots .$$

If one specifies a boundary condition g, the restriction of the extended finite elastostatic operator to the generalized Dirichlet configuration manifolds $[\mathrm{EmbH}^k(\eta)]_g$ determined by g determines a mapping

$$[\mathrm{Div}T_{s_K}]\Big|_{[\mathrm{EmbH}^{2+\ell}(\eta)]_g} : [\mathrm{EmbH}^{2+\ell}(\eta)]_g \longrightarrow H^{\ell}(T_{s_K}\eta), \quad \ell=1,2,\ldots$$

Since $[\mathrm{EmbH}^{2+\ell}(\eta)]_g$ is a closed submanifold of the free boundary solution manifold, the finite elastostatic operator maintains its differentiability properties upon restriction. Hence, one is assured that the restricted operator is a non-linear, C^∞ mapping between differentiable manifolds.

The triples, consisting of the restricted finite elastostatic operator, the differentiable manifolds of possible solutions satisfying the given place boundary conditions, and the spaces of data constitute the generalized geometric model for the finite elastostatic Dirichlet problem built over the Sobolev spaces.

One can anticipate how questions of existence, uniqueness, regularity, and local uniqueness can be viewed in terms of this geometric model. The question of the existence of a solution for the given data and place boundary condition manifests itself as the question of whether there exists a point on the differentiable manifold of solutions satisfying the boundary conditions which is mapped by the finite elastostatic operator into the given point in the data space. The question of uniqueness of solution manifests itself as the question of how many points on the solution manifold are mapped into the given data point. Regularity reveals itself as the following question: if the data point is an element of a "more regular" (C^r) subspace of the data Sobolev space, will the point which maps into it under the finite elastostatic operator be simultaneously an element of the solution manifold and an element of a "more regular" ($C^{r+\ell}$) subspace of the free boundary solution Sobolev space? Finally, the question of local uniqueness reveals itself as the question: if a solution is given to a Dirichlet problem for given data and boundary conditions, do there exist neighborhoods of the solution and data points upon which the finite elastostatic operator is one-to-one and C^∞ both ways? The important fact to notice is that the questions of existence, uniqueness, regularity, and local uniqueness, which were originally viewed in terms of the analytical model, have now been transformed into topological and geometric questions in the qualitative model.

How Existence and Uniqueness Questions Appear in the Geometric Model

One can now appreciate the fact that by causing the finite elastostatic Dirichlet problem in the above form all the tools of the Algebraic Topology, the Differential Topology, and the Differential Geometry may be directly employed to probe the questions of existence, uniqueness, and regularity of solution. In particular, the Inverse Mapping theorem, which was used by VanBuren to investigate questions of local uniqueness, represents but one tool in the vast resource. Other tools which can be used will be considered in chapters eight and nine.

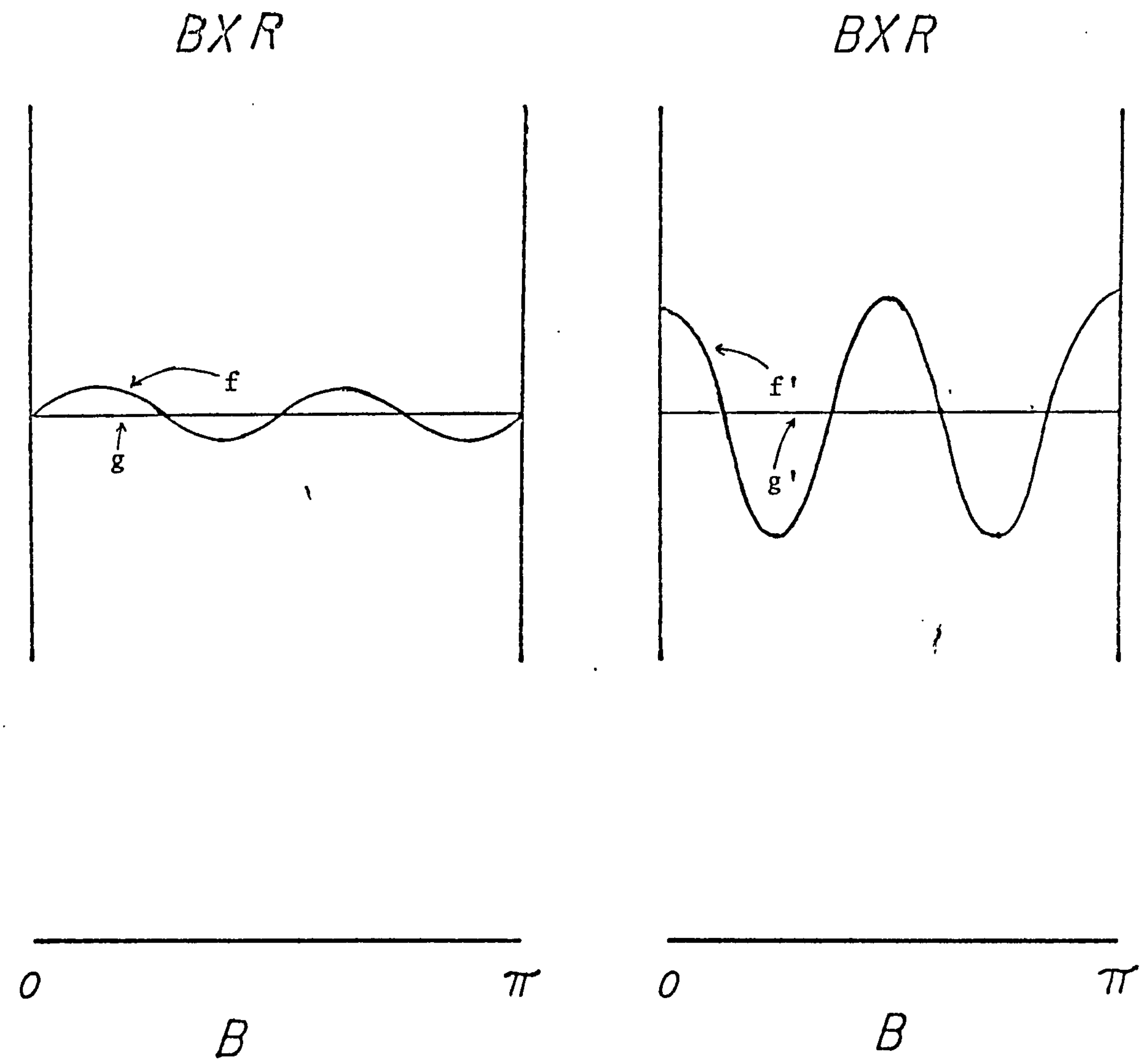
THE EXTENDED GEOMETRIC MODEL CORRESPONDING TO RIGIDLY SUPPORTED BOUNDARY CONDITIONS

The Generalized Dirichlet Configuration Manifold Corresponding to Rigidly Supported Boundary Conditions

Attractive as the above model for the Dirichlet problem is, there are questions of local uniqueness which it cannot resolve because the structure imposed upon the manifold of configurations is still too simple. It is of value to see how the global nonlinear analysis provides an alternative way of viewing the solution manifold which provides some additional structure. Roughly speaking, the global analysis results allow the configuration manifold for a given boundary condition to be viewed as partitioned into a family of mutually disjoint closed submanifolds. Each closed submanifold represents a "judiciously overdetermined" set of possible configurations for the Dirichlet problem which, not only model the given boundary condition, but also represent a particular way of coming off the boundary. Physically speaking, the situation corresponds to a circumstance where the boundary condition is maintained not in a simple supporting manner, but in a more constraining rigid supporting manner.[91]

One recognizes the finer structure by realizing that although two configurations in the above defined Dirichlet manifold may model the same boundary condition, they may differ markedly even very close to the boundary. The two sections may come off the common boundary condition in very different ways. A visualization of this situation is suggested in Figure VII.2, in which two real-valued functions defined on $[0,\pi]$ model the same zero boundary condition, but do not maintain the closeness in the H^1 sense in any neighborhood of the boundary.

For many questions concerning the given Dirichlet problem, it is inconsequential whether one chooses one or



$$f(x) = A_o \sin 4x$$

$$g(x) = 0$$

$$x \in [0,\pi]$$

FIGURE VII. 2.

Two Functions Defined on $[0,\pi]$ Which are Not Close in the H^1 Sense.

the other of the two sections to model the given boundary condition. It is sufficient simply that the boundary condition can be modeled. However, there are some questions where the specific choice can make a marked difference. This consequence is particularly true in situations where the original solution manifold for the given Dirichlet problem has many components.

If there are many ways to model the same boundary condition in terms of configuration sections coming off the boundary in different ways, is it possible to subclassify the configurations into collections which represent this feature? If so, do these subcollections possess a geometric structure? The answers to both questions are in the affirmative. From Palais' work, one has the following definition of the subcollections, and theorems about their geometric structure.

Lemma VII.1.[92] (The set $(H^k)_{\partial g}(\eta)$). Let B be a material body, which is compact, connected, orientable, and oriented. Let B denote the boundary of B. Let $\eta = B \times R^3$ be the vector bundle of positions of B in R^3. Let $H^k(\eta)$ denote the Hilbert space of all sections of η which are continuous in the H^k sense, and let $g \in H^k(\eta)$. Then the set

$(H^k)_{\partial g}(\eta)$ = the closure in $H^k(\eta)$ of $s \in C^k(\eta)$: an open neighborhood U_s of ∂B on which $s|_{U_s} = g|_{U_s}$

(a) is a closed set in $H^k(\eta)$;
(b) is a translate of a closed linear subspace of $H^k(\eta)$.

Namely, for

$$\bar{g}\varepsilon(H^k)_{\partial g}(\eta) \quad \text{arbitrary,}$$

$$(H^k)_{\partial g}(\eta) = \bar{g} + (H^k)_o(\eta),$$

where

$$(H^k)_o(\eta) = \text{the closure in } H^k(\eta) \text{ of } \begin{array}{l} s\varepsilon C^k(\eta): \\ \text{support of } s \text{ is} \\ \text{disjoint from } \partial B. \end{array}$$

and the sum is in the $H^k(\eta)$ sense.

The elements of $(H^k)_{\partial g}(\eta)$ are those sections which not only model the same boundary condition as g, (i.e. $\bar{g}|_{\partial B} = g|_{\partial B}$), but also come off the boundary in the same way as g (i.e. $\bar{g}|U_{\bar{g}} = g|U_{\bar{g}}$). Notice that although the behavior of sections in this set are severely limited near the boundary of B, they are unrestrained in the interior. The situation is visualized in Figure VII.3.

Using the space introduced in Lemma VII.1 as the basic building block, one may construct the Dirichlet configurations manifolds corresponding to rigidly supported boundary conditions and investigate their properties.

Definition VII.2. Given the conditions of the previous lemma, let $\mathrm{Emb}H^k(\eta)$ denote the manifold of H^k configuations of B in R^3. Let g be an H^k configuration,

$$g\,\varepsilon\mathrm{Emb}H^k(\eta).$$

Define the set

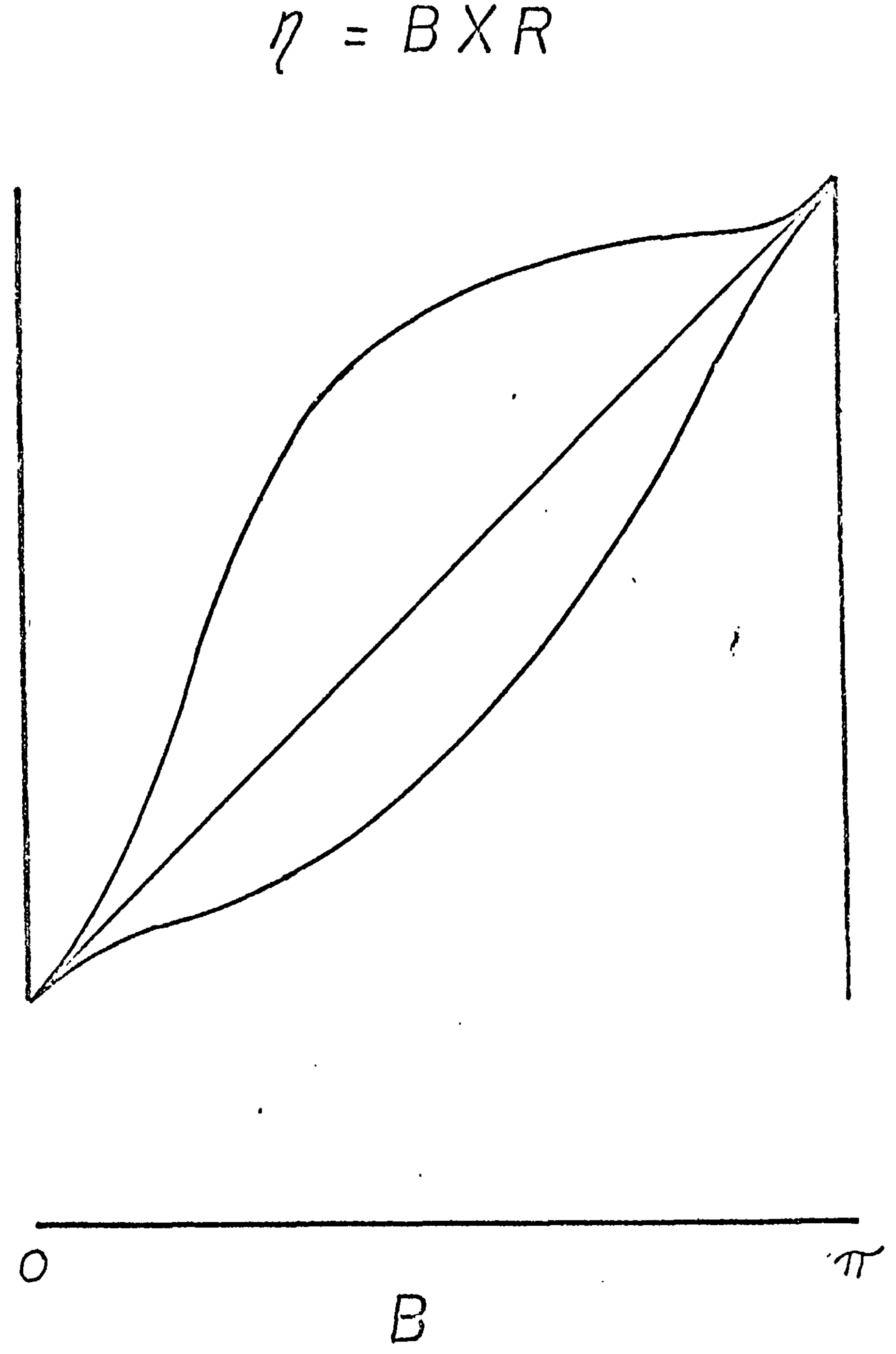


FIGURE VII. 3.

Several Elements of $(H^{\ell})_{\partial g}(\eta)$ for $\eta = [o,\pi] \times R$, $B = [0,\pi]$ and $\partial B = \{0,\pi\}$.

$$(\mathrm{EmbH}^k)_{\partial \mathcal{g}}(\eta) = (H^k)_{\partial \mathcal{g}}(\eta) \cap \mathrm{EmbH}^k(\eta)$$

Theorem VII.4. Given the conditions of the previous definition, let

$$\mathcal{g}|_{\partial B} = g,$$

and let $[\mathrm{EmbH}^k(\eta)]_g$ denote the Dirichlet solution manifold for the boundary condition g which was previously defined. Then

(a)

$$(\mathrm{EmbH}^k)_{\partial \mathcal{g}}(\eta)$$

is a closed C^∞ submanifold in $\mathrm{EmbH}^k(\eta)$ which is also closed relative to the Dirichlet solution manifold

$$[\mathrm{EmbH}^k(\eta)]_g$$

(b) (1) If $\overline{\mathcal{g}} \varepsilon (\mathrm{EmbH}^k)_{\partial \mathcal{g}}(\eta)$, then

$$(\mathrm{EmbH}^k)_{\partial \mathcal{g}}(\eta) = (\mathrm{EmbH}^k)_{\partial \mathcal{g}}(\eta).$$

That is to say, the specification of the manifold is independent of the particular element used to characterize it.

(2) If $\overline{g}$ $(\mathrm{EmbH}^k)_{\partial g}(\eta)$, then

$$(\mathrm{EmbH}^k)_{\partial \overline{g}}(\eta) \cap (\mathrm{EmbH}^k)_{\partial \mathcal{g}}(\eta) = \emptyset$$

That is to say, manifolds modeling different ways of coming off the same boundary condition are disjoint.

(c) Set theoretically,

$$[\mathrm{EmbH}^k(\eta)]_g = \bigcup_{\mathcal{g} \varepsilon [\mathrm{EmbH}^k(\eta)]_g} (\mathrm{EmbH}^k)_{\partial \mathcal{g}}(\eta) \quad .$$

(d) The topology induced upon the set $[\mathrm{EmbH}^k(\eta)]_g$ when it is viewed as a disjoint union of closed submanifolds is finer than its manifold topology.

The proof of the theorem parallels that of Theorem IV.5.

As a consequence of the above theorem the previously defined Dirichlet solution manifold may be viewed as partitioned into a disjoint set of closed submanifolds. A visualization in terms of finite dimensional figures of how the partitioning can occur is offered in Figure VII.4. By a "finer topology" one means that open sets in the manifold topology for $[\mathrm{EmbH}^k(\eta)]_g$ may be constructed from open sets in the "disjoint union" topology for $[\mathrm{EmbH}^k(\eta)]_g$, but not vice-versa. The finer structure allows one to differentiate between points in the Dirichlet solution set $[\mathrm{EmbH}^k(\eta)]_g$ to a degree that would otherwise not be possible. For instance, points which could not be separated in the usual structure could be separated in the finer structure. This property will be most important in resolving local uniqueness questions. The reader is again referred to Figure VII.5 for a visualization of this assertion.

The Generalized Geometric Model for the Dirichlet Problem Corresponding to a Rigidly Supported Boundary Condition

One can anticipate how the geometric model for the Dirichlet problem corresponding to a rigidly supported boundary condition can be posed in terms of the Sobolev spaces.

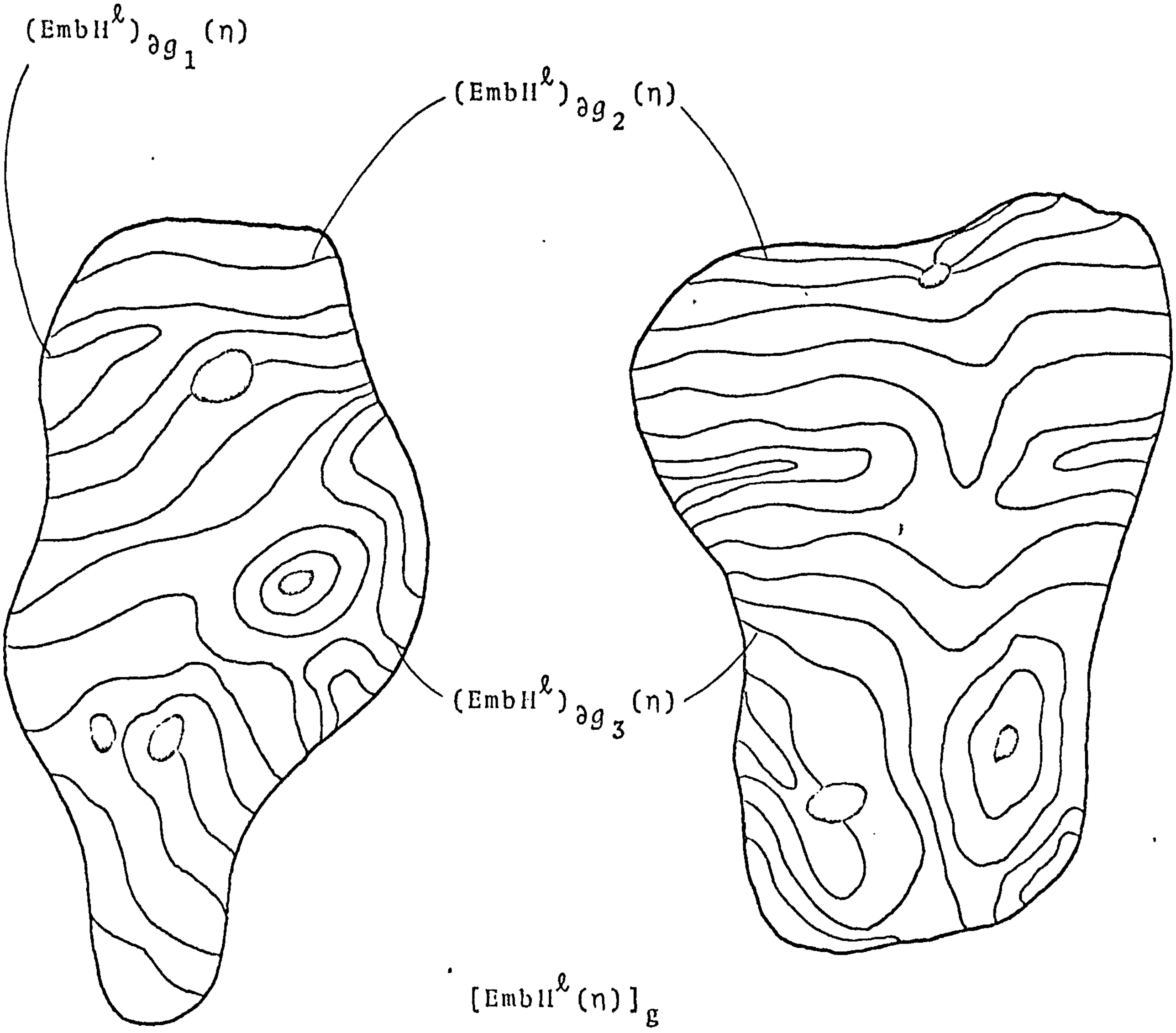
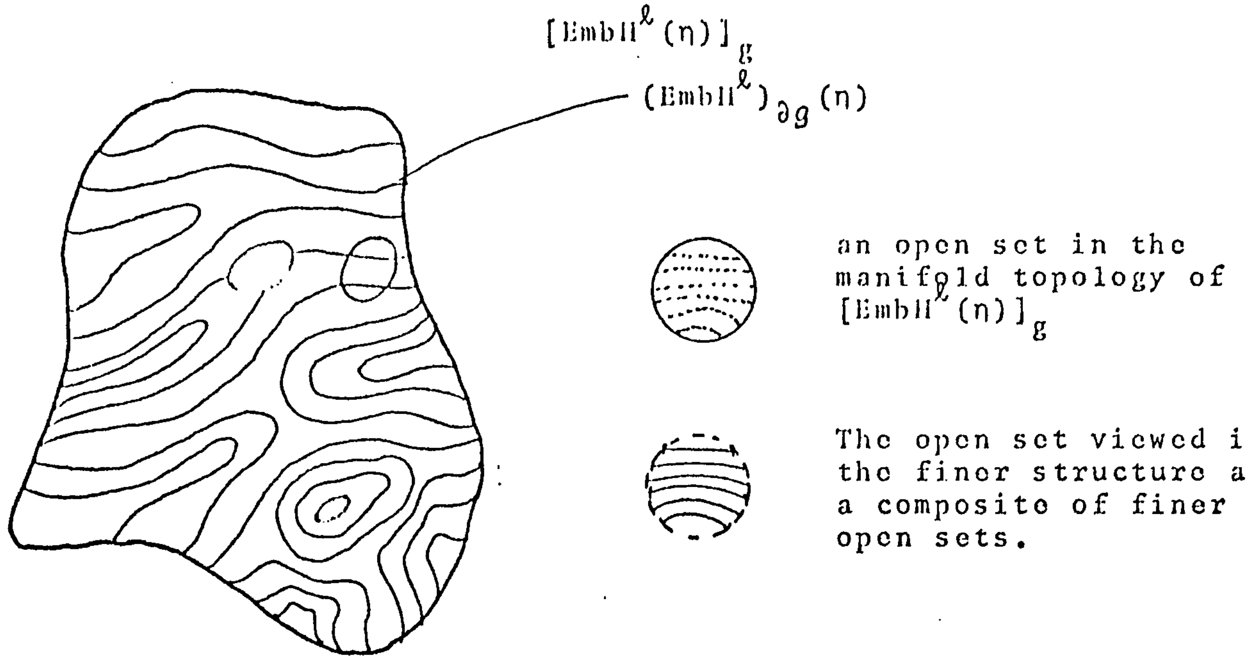
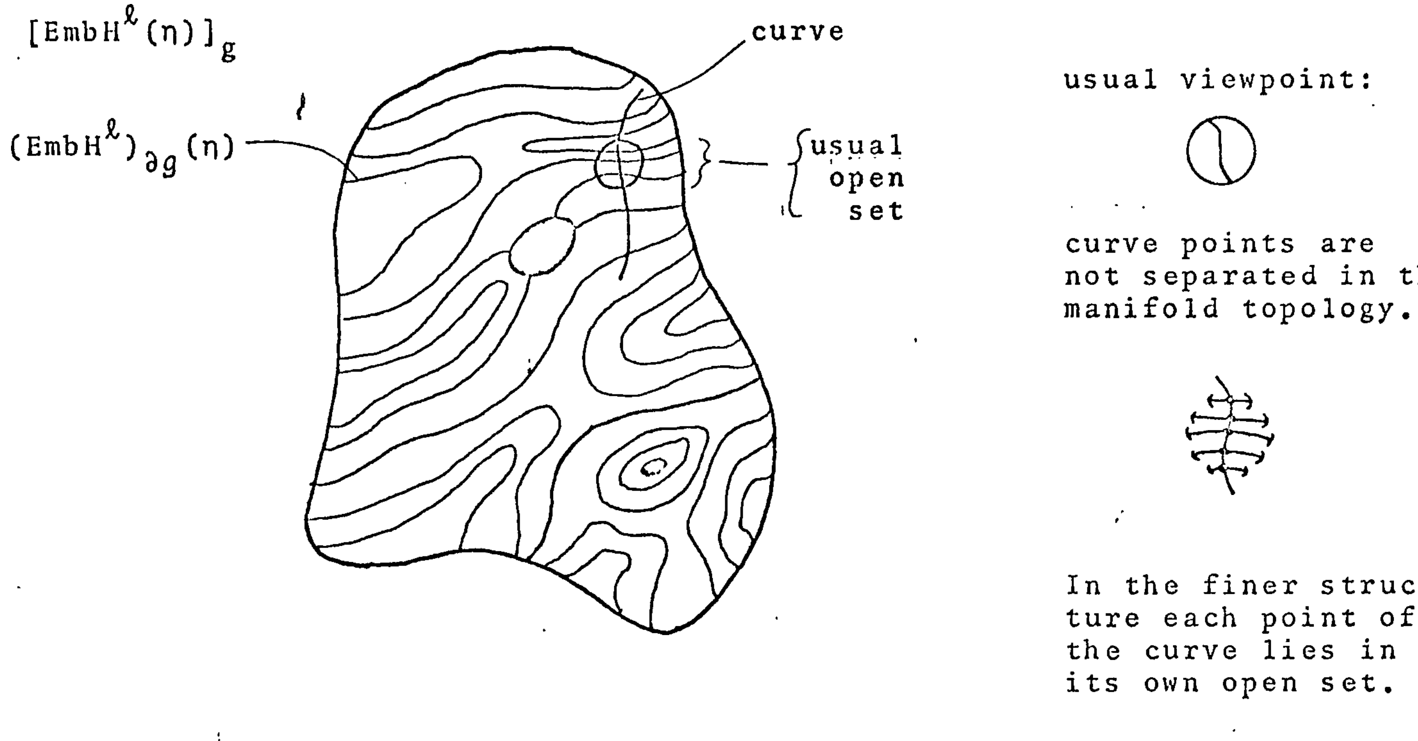


FIGURE VII. 4.

The Dirichlet Solution Manifold $[\mathrm{EmbH}^{\ell}(\eta)]_g$ for the Boundary Condition g now Viewed as a Disjoint Union of Submanifolds, $(\mathrm{EmbH}^{\ell})_{\partial g}(\eta)$.



a) Open Sets in [EmbH$^\ell$(η)]$_g$ from the Manifold Topology and the Finer Disjoint Union Topology.



b) How points in the set [EmbH$^\ell$(η)]$_g$ which are not separated in the usual manifold topology can be separated in the finer disjoint union topology.

FIGURE VII. 5.

Theorem VII.5. Let $k \geq 3$, and let $[\mathrm{Emb}^k(\eta)]_g$ be the Dirichlet configuration manifold for the simply supported boundary condition g. Let $\mathcal{G}\varepsilon[\mathrm{EmbH}^k(\eta)]_g$. Then

$$\mathrm{DivT}_{s_\kappa}|_{(\mathrm{EmbH}^{2+\ell})_{\partial\mathcal{G}}(\eta)} : (\mathrm{EmbH}^{2+\ell})_{\partial\mathcal{G}}(\eta) \longrightarrow H^\ell(T_{s_\kappa}\eta),$$

$$\ell = 1,2,\ldots$$

is a nonlinear C^∞ mapping from the smooth, differentiable manifold $(\mathrm{EmbH}_\ell^{3+\ell})_{\partial\mathcal{G}}(\eta)$ into the topological vector space $H^{2+\ell}(T_{s_\kappa}\eta)$.

The theorem is established when one shows that under restriction, the finite elastostatic operator maintains its smoothness properties. By Theorem VII.4d), the disjoint union topology of $[\mathrm{EmbH}^k(\eta)]_g$ is finer than its manifold topology. Consequently, any property the finite elastostatic operator has on $[\mathrm{EmbH}^k(\eta)]_g$ with respect to its usual manifold topology, in particular continuity or smoothness, it possesses over $(\mathrm{EmbH}^k)_{\partial\mathcal{G}}(\eta)$ in the finer topology. Thus, if the finite elastostatic Dirichlet problem can be modeled as a smooth, nonlinear differential operator linking the Dirichlet configuration manifold corresponding to a simply supported boundary condition with the generalized data space, its restriction to the finer submanifolds corresponding to particular ways of rigidly supporting the boundary condition is also smooth. Therefore, the triple specified by Theorem VII.5 may be taken as the geometric model for the generalized finite elastostatic Dirichlet problem with boundary condition g modeled near the boundary by the configuration g.

THE TWO GEOMETRIC MODELS ARE RELATED

How the Finer Structure can be Used to Examine Local Uniqueness Questions

One can now appreciate how a finer, but more complicated as the one for the Dirichlet problem under rigid support can be valuable in dissecting problems which are locally nonunique (degenerate) when viewed in terms of the model for the Dirichlet problem under a simply supported boundary condition. Consider a situation where a finite elastostatic Dirichlet problem corresponding to a given simply supported boundary condition does not have a locally unique solution. That is to say, a slight perturbation of a given equilibrating configuration while keeping the boundary configuration fixed in a simply supported manner results in a new equilibrium configuration. In order to gain some insight into the nature of the nonuniqueness, one might inquire if it is possible to impose additional constraint upon the boundary of the body which would eliminate the local nonuniqueness.

Figure VII.5 b) for the finer structure of the rigidly supported Dirichlet problem provides a visual clue as to how to constrain the material body physically in order to possibly remove the degeneracy. If a solution to a given Dirichlet problem with simply supported boundary condition is locally nonunique, the solutions to the problem manifest themselves geometrically in the configuration manifold $[\mathrm{EmbH}^k(\eta)]_g$ as a curve, like the one pictured in Figure VII.5 b). The various points on the curve, or equilibrating configurations, cannot be separated using the the manifold topology of $[\mathrm{EmbH}^k(\eta)]_g$. However, if one imposes the finer structure of the rigidly supported model, also pictured in Figure VII.5 b), one may be able to separate points of the curve of equilibrating solutions. If a particular rigidly supported Dirichlet configuration manifold contains but one point of the curve, one is then assured that the corresponding rigidly supported Dirichlet problem would have a locally unique solution. For instance, one might find that the configuration manifold $(\mathrm{EmbH}^k)_{\partial g}(\eta)$ is transversal to the curve, while $(\mathrm{EmbH}^k)_{\partial \bar{g}}(\eta)$ is not. One would then know that the local nonuniqueness of the equilibrating configuration in the simply supported problem is of such a nature, that in order to resolve the degenerate situation, one would have to (1) rigidly support the boundary, as opposed to just simply supporting it, and

(2) rigidly support it in a particular way, (as dictated by g as opposed to $\bar{g}$).[93] An example of this circumstance is suggested in Chapter eight.

A SUMMARY

In summary, it has been shown that the Dirichlet problem on finite elastostatics may be given an extended geometrical formulation over the Sovolev function spaces as a nonlinear, differentiable mapping between infinite dimensional differentiable manifolds. Two ways of formulating the geometric model have been presented, one corresponding to a simple support of the boundary condition, the other corresponding to a rigid support. The relationship between the two models have been examined, and the value of the finer structure provided by the latter model in analyzing local uniqueness questions has been anticipated.

The casting of the mathematical model for the finite elastostatic Free Boundary and Dirichlet problems is now as complete as is necessary for this thesis. One may now begin to concentrate on how one can exploit the contemporary mathematical methods and tools now becoming available in the Algebraic Topology, Differential Geometry, and Differential Topology to glean information from the models, and actually resolve questions of existence and uniqueness.

VIII. GAINING LOCAL INFORMATION FROM THE MODEL

In chapters six and seven, the geometric models for the finite elastostatic free boundary and Dirichlet problems built over the Sobolev function spaces were completed. The erection of the models represents the principal effort of the thesis. In the remaining two chapters of this work one considers how one gains information from these models.

As mentioned in the previous chapters, when one releases the topological vector space structure on the solution manifold one loses many of the mathematical tools and methods previous models had been able to exploit to answer local and global questions. What alternative methods replace them? How does one employ the new methods? Which conclusions carry over to the new models, and which are altered? One cannot hope to answer these questions in this work. However, one will find in this chapter and the next, that new mathematical methods are emerging which will replace those rendered inapplicable. Moreover, one will gain an insight into how to employ them to gain existence and uniqueness information. Finally, one can anticipate that many of the local existence and uniqueness conclusions may be carried over and even augmented, while the global conclusions are severely altered.

In this chapter one examines how one gains local information from the models. The first part of this chapter concentrates upon the development of the geometric model for the free boundary and Dirichlet infinitesimal deformation problems. The finite elastostatic models are linearized using the methods of chapter three. Upon restriction to small deformations the finite elastostatic model presented here reduces to the well known linear infinitesimal elasticity models, which have proven so successful. Two points are worth noting about the derivation of the infinitesimal models. The linearization may be accomplished (1) by theorem, as opposed to computation; and (2) simultaneously for several settings of the elastostatic problem (Holder, Sobolev, etc.).

From the linearized models one obtaines local uniqueness results which parallel those developed in the previous models. In addition, the geometric nature of the models allows one to exploit differential topological tools to dwell deeper into the local existence questions, and augment the previously drawn conclusions. As an example, ways of modifying Van Buren's invertibility hypothesis (Hypothesis 4) are examined through the introduction of the index of a nonlinear Fredholm operator. A mathematical result of particular relevance to this investigation is the infinite dimensional version of Sard's theorem proposed by Smale. As a consequence, several questions about Fredholm mappings under current mathematical consideration, and heretofore regarded as abstract in nature, now become quite relevant.

Finally, other tools for investigating local uniqueness questions are mentioned. Principal among these is the algebraic degree of a nonlinear mapping. A sequence of Dirichlet elastostatic problems is envisioned in which the branching of a family of secondary equilibrating configurations from a given one is anticipated by means of a change in the degree of the finite elastostatic operator. All in all, one gains some appreciation of the wealth of untapped resources which the models make available.

As the mathematical tools with which one explores the models are only now emerging, the comments made in this chapter are necessarily general. They are meant to indicate how the models introduced here permit one to formulate local uniqueness and existence questions in a geometric manner to examine how one uses the abstract tools to explore the questions, employing specific examples when available, and finally to anticipate which questions show promise, and are resolvable.

THE GEOMETRIC MODEL FOR THE FREE BOUNDARY AND DIRICHLET INFINITESIMAL DEFORMATION PROGLEMS

One can derive from the geometric models for the finite elastostatic free boundary and Dirichlet problems models for the infinitesimal deformation problem about a given initial configuration. The derivation of the linearized model is quite geometric in spirit.

The Geometric Models for the Infinitesimal Problems are Derived from the Finite Elastostatic Models

For a compact, connected, oriented simple material body of degree one, the geometric models for the finite elastostatic free boundary problem, and the simply supported and rigidly supported Dirichlet problems are, respectively:

$$\mathrm{Div}\, T_{s_\kappa} : \mathrm{Emb}H^{\ell+2}(\eta) \longrightarrow H^{\ell}(T_{s_\kappa}\eta),\ \ell = 1,2,\ldots,$$

$$\mathrm{Div}\, T_{s_\kappa}\big|_{[\mathrm{Emb}H^{\ell+2}(\eta)]_g} : [\mathrm{Emb}H^{\ell+2}(\eta)]_g \longrightarrow H^{\ell}(T_{s_\kappa}\eta), \ell=1,2,\ldots$$

$$\mathrm{Div}\, T_{s_\kappa}\big|_{(\mathrm{Emb}H^{\ell+2})_{\partial g}(\eta)} : (\mathrm{Emb}H^{\ell+2})_{\partial g}(\eta) \longrightarrow H^{\ell}(T_{s_\kappa}\eta), \ell=1,2,\ldots$$

As established in Theorem VI.2, Theorem VII.2, and Theorem VII.4, the finite elastostatic operators are differentiable mappings. At a given $H^{\ell+2}$ differentiable configuration, say s, the derivative of the operator is a continuous, linear mapping of the tangent space to the $H^{\ell+2}$ configuration manifold at s into the tangent space to the body force density space at Div $T_{s_\kappa}(s)$. Symbolically, one may write

$$d(\mathrm{Div}T_{s_\kappa})(s):\ T(\mathrm{Emb}H^{\ell+2}(\eta))_s \longrightarrow T(H^{\ell}(T_{s_\kappa}(\eta))_{\mathrm{Div}T_{s_\kappa}(s)},$$

$$d(\mathrm{Div}T_{s_\kappa}|_{[\mathrm{Emb}H^{\ell+2}(\eta)]_g})(s):\ T([\mathrm{Emb}H^{\ell+2}(\eta)]_g)_s \longrightarrow$$

$$T(H^{\ell}(T_{s_\kappa}(\eta)))_{\mathrm{Div}T_{s_\kappa}(s)},$$

$$d(\mathrm{Div}T_{s_\kappa}|_{(\mathrm{Emb}H^{\ell+2})_{\partial g}(\eta)})(s):\ T((\mathrm{Emb}H^{\ell+2})_{\partial g}(\eta))_s \longrightarrow$$

$$T(H^{\ell}(T_{s_\kappa}(\eta)))_{\mathrm{Div}T_{s_\kappa}(s)},$$

and one may visualize the derived models geometrically in terms of finite dimensional elements as shown in Figure VIII.1.

The Derived Models are the $H^{\ell+2}$ Extension of the Classic Linearized Models

One may relate the derived models to the classical linearization of the finite elastostatic models about the

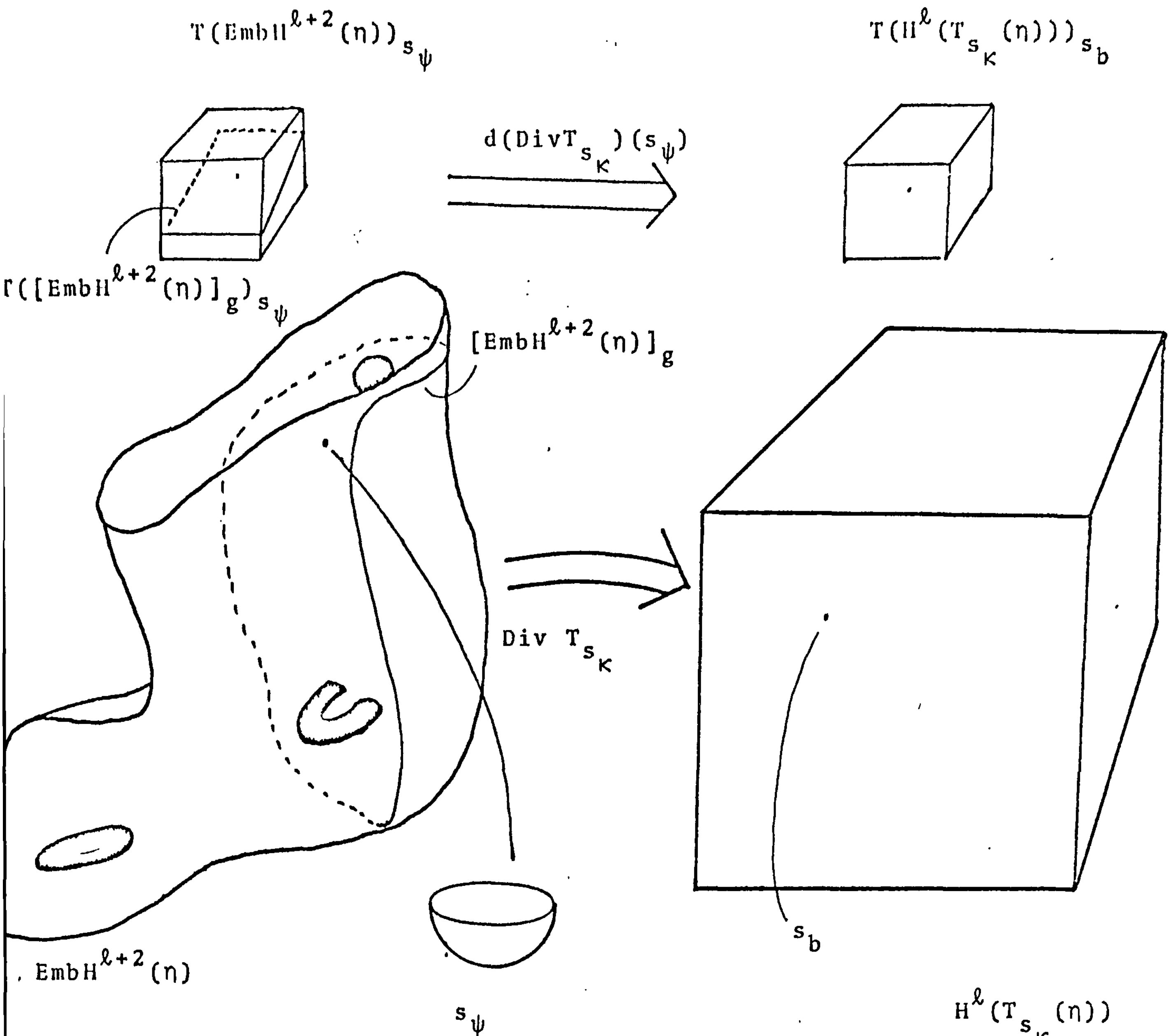


FIGURE VIII. 1.

A Visualization of the Finite Elastostatic Free Boundary and Simply Supported Dirichlet Models Using Finite Dimensional Elements.

given initial configuration. The nonlinear analysis tools presented in the latter part of chapter three permit one to establish by theorem what the tangent spaces in the derived models look like, and what the derivative of the finite elastostatic operator is. For convenience of the development the initial $H^{\ell+2}$ configuration is taken to be smoothly differentiable, say s_ψ.

As the generalized data space $H^\ell(T_{s_\kappa}\eta)$ is a Hilbert space, its tangent space at any point is easy to specify. It is but a copy of the Hilbert space itself, and the identification is a natural one:

$$T(H^\ell(T_{s_\kappa}\eta))_{\mathrm{Div}T_{s_\kappa}(s_\psi)} = H^\ell(T_{s_\kappa}\eta).$$

The tangent space to the configuration manifold at s_ψ is obtained in an equally straightforward manner. By Theorem VI.2, Theorem VII.2, and Theorem VII.4, the model spaces for the free boundary, the simply supported Dirichlet, and the rigidly supported Dirichlet configuration manifolds are, respectively, $H^{\ell+2}(\eta)$, $[H^{\ell+2}(\eta)]_o$, and $(H^{\ell+2})_{\partial o}(\eta)$, where "o" is the zero section in η. By Theorem III.1, the tangent spaces at s_ψ are isomorphic with the model spaces. Hence, one has the identifications

$$T(\mathrm{Emb}H^{\ell+2}(\eta))_{s_\psi} \approx H^{\ell+2}(\eta)$$
$$T([\mathrm{Emb}H^{\ell+2}(\eta)]_g)_{s_\psi} \approx [H^{\ell+2}(\eta)]_o$$
$$T((\mathrm{Emb}H^{\ell+2})_{\partial g}(\eta))_{s_\psi} \quad (H^{\ell+2})_{\partial o}(\eta),$$

although the identifications are not canonical.

Finally the derivatives of the finite elastostatic operators are prescribed by Theorem III.4. If $\Lambda(\mathrm{Div}T_{s_\kappa})s_\psi$ represents the classic linearization of the finite elastostatic free boundary operator at s_ψ, (see Corollary III.1),

$$\Lambda(\mathrm{Div}T_{s_\kappa})_{s_\psi} : C^\infty(\eta) \longrightarrow C^\infty(T_{s_\kappa}\eta),$$

a linear differential operator of order two, then the derivative of the linearized operator extended to the $H^{\ell+2}$ function spaces is the $H^{\ell+2}$ extension of the linearization,

$$d(\mathrm{Div}T_{s_\kappa})(s_\psi) = H^{\ell+2}(\Lambda(\mathrm{Div}T_{s_\kappa})_{s_\psi}) : T(\mathrm{Emb}H^{\ell+2}(\eta)) \approx H^{\ell+2}(\eta) \to H^{\ell}(\eta), \quad \ell = 1,2,\ldots .$$

Applying the theorems in a similar manner to the Dirichlet problems, one determines that the derivatives of the finite elastostatic Dirichlet operators at s_ψ are the Sobolev extensions of their classic linearizations:

$$d(\mathrm{Div}T_{s_\kappa}|_{[\mathrm{Emb}H^{\ell+2}(\eta)]_g})_{s_\psi} = H^{\ell+2}(\Lambda(\mathrm{Div}T_{s_\kappa}|_{[\mathrm{Emb}H^{\ell+2}(\eta)]_g})_{s_\psi})$$

$$T([\mathrm{Emb}H^{\ell+2}(\eta)]_g)_{s_\psi} \approx [H^{\ell+2}(\eta)]_0 \longrightarrow H^{\ell}(\eta)$$

$$d(\mathrm{DivT}_{s_\kappa}|_{(\mathrm{EmbH}^{\ell+2})_{\partial g}(\eta)})(s_\psi) = H^{\ell+2}(\Lambda\mathrm{DivT}_{s_\kappa}|_{(\mathrm{EmbH}^{\ell+2})_{\partial g}(\eta)})_{s_\psi}):$$

$$T((\mathrm{EmbH}^{\ell+2})_{\partial g}(\eta))_{s_\psi} \approx (H^{\ell+2})_{\partial o}(\eta) \longrightarrow H^{\ell}(\eta)$$

The Derived Models are the Geometric Models for the Infinitesimal Deformation Problem at the Initial Configuration

Viewed in this manner, the differentiation of the finite elastostatic models at s_ψ reduce to the geometric models for the infinitesimal elastostatic free boundary and Dirichlet problems about the initial configuration s_ψ, built over the Sobolev function spaces.[94] The identification may be made more transparent if one takes the reference configuration s_κ to be the initial configuration, and if one represents the derived model in terms of a local coordinate representation about it. For example, let $\mathrm{EmbH}^{\ell+2}(\eta)$ be the solution manifold for the finite elastostatic free boundary problem, and let s_κ denote both the reference configuration, and the configuration about which the linearization is formulated. Let the finite elastostatic operator be specified in terms of a response function as Definition V.1)

$$\mathrm{Div}\ T_{s_\kappa} = \mathrm{Div}\cdot(H_{s_\kappa})_*\cdot\nabla_\eta .$$

For any point $p \in B$, let $(\gamma_0 x1, \gamma_0, U)$ be a vector bundle chart on η. Let the parametric representation of the finite elastostatic operator about p relative to the induced charts by (see Chapter IV)

$$\partial_\alpha (h_{s_K}{}^{i}_{\alpha}(X,F)),$$

for $h_{s_K}{}^{i}_{\alpha}$ the local coordinate representation of the response function, relative to the induced charts

$$(T_{s_K}\gamma \otimes T\gamma_0)\, H_{s_K} \cdot (T_{s_K} \otimes T\gamma_0)^{-1} : \gamma_0(U) x L(R^3 R^3) \to L(R^3 R^3)$$

$$X \quad F \to (h_{s_K})^{i}_{\alpha}(X,F) \ .$$

By theorem, the classical linearization of the operator has the parametric representation

$$\partial_\alpha (\partial_{F^{\ell}_{\beta}} h^{i}_{\alpha}(X,F)\big|_{F=1}) \equiv \partial_\alpha (A^{i\beta}_{\alpha\ell}(X)) \ : \ C^\infty(\eta) \to C^\infty(\eta)$$

The $H^{\ell+2}$ extension of the linearized operator is the infinitesimal elastostatic free boundary operator for the infinitesimal deformation problem about the initial configuration s_K, built over the Sobolev function spaces.[95] Thus for small deformations about a given configuration, the finite elastostatic model presented in this work reduces to the linear elastostatic model for small deformations superimposed upon a given strain. One thereby gains the reassuring result that, for local questions, the conclusions drawn by the currently existing infinitesimal theories are sustained by the model developed here.

The Models for the Dirichlet Infinitesimal Deformation Problems are Restrictions of the Free Boundary Infinitesimal Deformation Model

As one might intuit, the geometric models for the rigidly supported and simply supported Dirichlet infinitesimal deformation problems at s_ψ are not independent of each other, or of the free boundary infinitesimal deformation model. One may establish that the manifest themselves as restrictions of the linearized free boundary problem to various subspaces of the tangent space at s_ψ. Figure VIII.1, anticipates this idea.

The simply supported Dirichlet configuration manifold lies as a closed submanifold of the free boundary one. Its tangent space at s_K may then be viewed as a linear subspace of the free boundary tangent space at s_K:

$$T([\mathrm{Emb}H^{\ell+2}(\eta)]_g)_{s_\psi} \subset T(\mathrm{Emb}H^{\ell+2}(\eta))_{s_\psi} \quad \ell = 1,2,\ldots,$$

or equivalently, in terms of model spaces for the manifolds

$$[H^{\ell+2}(\eta)]_o \subset H^{\ell+2}(\eta) \quad \ell = 1,2,\ldots,$$

Moreover, the submanifold nature of the Dirichlet configuration manifold insures that it "splits" locally relative to the free boundary manifold. Consequently, the derivative of the Dirichlet finite elastostatic operator may be viewed as a "partial derivative" of the free boundary operator, or more properly, as the restriction of the derivative of the free boundary operator to the tangent subspace to the Dirichlet submanifold. The model for the simply supported Dirichlet infinitesimal problem at s_ψ thus may be displayed as

$$d(\mathrm{Div}T_{s_\kappa}|_{[\mathrm{Emb}H^{\ell+2}(\eta)]_g})(s_\psi) = d(\mathrm{Div}T_{s_\kappa})(s_\psi)\Big|_{T([\mathrm{Emb}H^{\ell+2}(\eta)]_g)_{s_\psi}} :$$

$$T([\mathrm{Emb}H^{\ell+2}(\eta)]_g)_{s_\psi} \to T(H^{\ell}(T_{s_\kappa}\eta))_{\mathrm{Div}T_{s_\kappa}(s_\psi)}$$

In a like manner, the model for the rigidly supported Dirichlet infinitesimal problem at s_ψ, whose solution manifold lies as a closed submanifold of the simply supported one, may be cast into the form

$$d(\mathrm{Div}T_{s_\kappa}\big|_{(\mathrm{Emb}H^{\ell+2})_{\partial g}(\eta)})(s_\psi) = d(\mathrm{Div}T_{s_\kappa})(s_\psi)\Big|_{T((\mathrm{Emb}H^{\ell+2})_{\partial g}(\eta))_{s_\psi}} :$$

$$T((\mathrm{Emb}H^{\ell+2})_{\partial g}(\eta))_{s_\psi} \to T(H^{\ell}(T_{s_\kappa}\eta))_{\mathrm{Div}T_{s_\kappa}(s_\psi)}$$

Although the relationship between the free boundary and Dirichlet models developed here seems an obvious one, it is by no means trivial. As will be seen presently, one gains a valuable tool for examining local uniqueness questions when one can "judiciously overdetermine" a system by imposing stronger and stronger boundary conditions.

The Infinitesimal Deformation Models are the Sobolev Counterparts to Van Buren's Infinitesimal Deformation Model

One may also develop a relationship between the linearized models developed here and the infinitesimal models developed by Van Buren. The infinitesimal elastostatic operators developed above closely parallel those of Van Buren, with the only difference being that the Sobolev functor is used here, as opposed to the Holder functor. Most particularly, notice that the derivative of the finite elastostatic free boundary operator introduced here, and

Van Buren's infinitesimal free boundary operator are respectively, the Sobolev and Holder extensions of the same classical linear operator, $\Lambda(\mathrm{Div}T_{s_K})s_K$. This parallel is precisely what is required by Theorem III.4. By the theorem, Van Buren's free boundary infinitesimal deformation model would result if one were to extend the geometric model for the finite elastostatic free boundary problem introduced in chapter five to the Holder function spaces, as opposed to the Sobolev spaces, using the methods of chapter three. Naturally, the finite elastostatic free boundary model which would be so obtained would still differ from Van Buren's finite deformation.

A Comment on How the Local Models were Derived

Two points are worth noting about the derivation of the infinitesimal deformation models presented here. Firstly, the linearization of the models are gained by theorem as opposed to intricate norm calculations. By using the tools of the nonlinear analysis, one may remove much of the computational aspects of the linearization procedure, which tend to confuse the local problem rather than clarify it. Secondly, the linearization process presented here is "categorical" in nature. That is to say, the theorems by which one linearizes the model may be applied without modification to models, cast over the Sobolev function spaces, the Holder function spaces, or any other function space setting whose associated functor satisfies the axioms introduced in chapter three. Hence, one may begin to see most clearly which aspects of the linearized problem are independent of the particular function setting chosen, and which are not. Such insight is invaluable for understanding the models and using them effectively.

HOW THE LOCAL UNIQUENESS QUESTIONS MANIFEST THEMSELVES

The Meaning of Local Uniqueness and Local Existence Assertions

It is advisable to standardize what is meant by a local uniqueness or local existence assertion in elastostatics. Let s_b be the body force density equilibrated by the configuration $s\psi$ for a given elastostatic problem. To say that the finite elastostatic operator is locally unique about $s\psi$ means physically that if one perturbs the equilibrium configuration $s\psi$ by a suitably small deviation while maintaining the body force s_b, and any Dirichlet boundary conditions which may be imposed by the problem, then the resulting condition of the body is not an equilibrium state, but a dynamic one. In this sense, the configuration s_ψ is the only one in its immediate vicinity which equilibrates the body force density s_b. Hence, it is the only one in its immediate neighborhood which is associated with s_b by the finite elastostatic operator. Notice that if one releases the restriction on the size of the perturbation it is possible to achieve a second equilibrating configuration for s_b. However, it will be related to s_ψ by a finite deformation. Also notice that the local uniqueness assertion involves the boundary conditions imposed. It is conceivable that for the free boundary problem the finite elastostatic operator is not locally unique at s_ψ, but for one of the Dirichlet problems the operator is locally unique.

If a configuration s_ψ is equilibrated by a body force section s_b in a given elastostatic problem, and if one perturbs the body force section by a small deviation, while maintaining any boundary condition which might be imposed, then one is assured that there is at most one configuration nearby the initial one which will restore equilibrium. The prohibition "at most" is necessary because there may exist no configuration nearby the original one which restores equilibrium. To assert that every body force section nearby s_b is equilibrated by a configuration, and that the prohibition "at most" may be removed, is to assert that the finite elastostatic operator possesses a local existence property at s_b. Notice that, once again the boundary condition enters into the formulation of the assertion. It is conceivable that for a given material the finite elastostatic operators for the Dirichlet problems exhibit no local existence property, while the free boundary operator does. Notice also that the local existence question has an elastodynamic manifestation as inquiring which acceleration fields may be induced upon the body by perturbing the equilibrating configuration while maintaining the boundary condition and initial body force. To fully comprehend this point of view one must be familiar with the relationship between the elastostatic and elastodynamic problems.[96]

A Geometric Representation of Local Existence and Local Uniqueness Assertions: Van Buren's Approach

How may one geometrically represent a local existence or uniqueness property for a finite elastostatic operator? As a beginning, one may adopt the specification introduced by Van Buren, in which the properties find expression in the immersive and submersive characteristics of the finite elastostatic operator.

Recall[97] that a mapping between two manifolds

$$f: X \longrightarrow Y$$
$$p \qquad f(p)$$

is an immersion at a point p of X if the mapping takes a neighborhood of p in X onto a neighborhood of a submanifold of Y about $f(p)$ in a diffeomorphic way. The mapping is a submersion at p if there is a neighborhood of p in X which is taken onto a neighborhood of Y about $f(p)$ by the mapping. When one applies these concepts to the geometric models for the finite elastostatic problems, one finds that the (analytical) assertion that a given equilibrating configuration is locally unique is satisfied by the (geometric) assertion that the finite elastostatic operator is immersive at the given configuration. Similarly, if one is given a body force density, and a configuration equilibrating it, the (analytical) assertion that the finite elastostatic operator possesses an equilibrating configuration for all body force densities gained by a small perturbation from the given one is satisfied by the (geometric) assertion that the operator is a submersion about the given equilibrating configuration.

The Sobolev Counterpart to Van Buren's Result

A criteria for determining when the finite elastostatic operator exhibits immersive or submersive characteristics follows if one invokes the inverse mapping theorem.[98] The theorem relates these characteristics of the operator to properties of the derivative of the operator. When restated in the context of the finite elastostatic operator the local uniqueness and existence criteria is given by the following theorem

Theorem VIII.1. Let

$$\mathrm{DivT}_{s_K} : \mathrm{EmbH}^{\ell+2}(\eta) \longrightarrow H^{\ell}(T_{s_K}\eta),\ \ell = 1,2,\ldots$$

be a finite elastostatic free boundary problem for a smooth, materially uniform, simple, connected body. Let s_ψ be a configuration equilibrated by a body force density s_b. Then

a) a sufficient condition that s_ψ be a locally unique equilibrating configuration for s_b is that the finite elastostatic operator is an immersion at s_ψ, for which a necessary and sufficient condition is that the derivative map

$$d(\mathrm{DivT}_{s_K})_{s_\psi} : T(\mathrm{EmbH}^{\ell+2}(\eta))_{s_\psi} \longrightarrow H^{\ell}(T_{s_K}(\eta))$$

is injective (one-to-one), and its range splits.

b) a sufficient condition that the finite elastostatic operator possesses a local existence property at s_b is that it be a submersion at s_ψ, for which a necessary and sufficient condition is that the derivative map

$$d(\mathrm{DivT}_{s_K})_{s_\psi} : T(\mathrm{EmbH}^{\ell+2}(\eta))_{s_\psi} \longrightarrow H^{\ell}(\eta)$$

is surjective (onto), and its kernel splits;

c) a sufficient condition that for each body force in some $H^{\ell}(\eta)$ neighborhood of s_b there exists a unique equilibrating configuration in some neighborhood of s_ψ is that the $H^{\ell+2}$ extension of the linearized elastostatic operator at s_ψ

$$d(\mathrm{Div}T_{s_\kappa})_{s_\psi} : T(\mathrm{Emb}H^{\ell+2}(\eta))_{s_\psi} \longrightarrow H^{\ell}(\eta)$$

be a one-to-one and onto linear mapping.

The theorem represents the counterpart to Van Buren's results for the geometric models built over the Sobolev function spaces. It indicates that one, indeed, can gain local information about an elastostatic system from the models presented here, and that the principal local existence and uniqueness result carries over to the Sobolev setting.

AUGMENTING VAN BUREN'S RESULTS: THE SMALE-SARD THEOREM

Some Difficulties with Using Van Buren's Theorem

As mentioned previously, it is difficult to use Van Buren's result or its Sobolev space counterpart. Firstly, the theorem provides a criteria which, if satisfied assures that the finite elastostatic operator possesses local existence and uniqueness properties. The theorem does not, however, insure that criteria are satisfied. To overcome the difficulty Van Buren adopted his fourth axiom, the invertibility hypothesis. The vulnerable aspects of this axiom were examined in chapter two.

A second difficulty is the restrictive nature of the geometric representation of local existence and uniqueness properties. To equate these properties with the submersive and immersive character of the finite elastostatic operator is too stringent. For example, the theorem would not allow one to attribute local existence and uniqueness properties to the one-dimensional nonlinear mapping.

$$f(x) = x^3$$

about the origin. In general, there are many elliptic differential operators which are locally unique, yet fail to satisfy the criteria of the theorem.

To examine if Van Buren's local existence and uniqueness results can be augmented, and if less stringent conditions can be imposed upon the elastostatic operator, one may employ some additional mathematical tools. One promising opportunity lies in a generalization of the Sard theorem to infinite dimensional manifolds made by S. Smale.[99] After some preliminaries, the results from his work will be summarized, and applied.

The Kernel and Cokernel Spaces for the Linearized Elastostatic Operator

In characterizing the immersive and submersive properties of a finite elastostatic operator, two subspaces of the tangent spaces play a pivitol role: the kernel of the derivative operator, and the cokernel, or complement to the image of the derivative operator. The kernel of the derivative of the finite elastostatic operator is the linear subspace of the tangent space $T(\mathrm{EmbH}^{\ell+2}(\eta))_{s_\psi}$ which consists of those elements which are taken into the zero element of $H^{\ell}(\eta)$:

$$\ker\{d(\mathrm{DivT}_{s_\kappa})_{s_\psi})\} = \{u\varepsilon T(\mathrm{EmbH}^{\ell+2}(\eta))_{s_\psi} : [d(\mathrm{DivT}_{s_\kappa})_{s_\psi}]\ u=0\} ,$$
$$\subset T(\mathrm{EmbH}^{\ell+2}(\eta))_{s_\psi}.$$

Since $H^{\ell}(\eta)$ is a Hilbert space, the cokernel of the derivative operator may be defined as the subspace in $H^{\ell}(\eta)$ which is the orthogonal complement to the image of the derivative operator under the inner product,

$$\mathrm{coker}[d(\mathrm{DivT}_{s_\kappa})_{s_\psi}] = [\mathrm{Im}(d(\mathrm{DivT}_{s_\kappa})_{s_\psi}]^{\perp\ H^{\ell}(\eta)} .$$

The dimensions of the kernel and cokernel spaces can characterize when the finite elasostatic operator exhibits local uniqueness or local existence properties. In order for the finite elastostatic operator to be immersive at s_ψ its derivative must possess a zero dimensional kernel. To be submersive at s_ψ, its derivative must possess a zero dimensional cokernel. The local existence and uniqueness result gained in Theorem VIII.1 rests upon the condition that the dimensions of the kernel and cokernel spaces of the derivative operator be simultaneous zero, a rather restrictive condition.

The Index of a Fredholm Mapping

The important point to notice about the relationship between the kernel and cokernel spaces and the immersive and submersive properties is that the certain algebraic quantities associated with the operator (the dimensions of the kernel and cokernel spaces) characterize certain geometric/topological properties of it (immersion and submersion). The augmentation of Van Buren's results provided by the Smale-Sard theorem rests upon the relationship between another algebraic quantity, the index of a mapping, and the geometric and topological properties of its image and preimage.

In the theory of finite dimensional linear spaces, if one is given a linear operator between two vector spaces

$$L : V \longrightarrow W,$$

then the index of the operator is the integer which is the difference of the dimensions of the kernel and cokernel spaces of the operator:

$$\text{index } L = \dim \ker L - \dim \operatorname{coker} L$$

Roughly speaking, the index measures the size of the kernel of the operator relative to the cokernel.

When one passes to the theory of infinite dimensional linear spaces, the index is defined only for a subclass of linear operators, the Fredholm operators:

Definition VIII.1.[100] A (linear) Fredholm operator is a continuous linear mapping between Banach spaces

$$L : E_1 \longrightarrow E_2$$

which has the properties:

a) $\dim \operatorname{Ker} L < \infty$
b) Image L is closed
c) $\dim \operatorname{Coker} L < \infty$

For L Fredholm, define the index of L to be the integer

$$\text{index } L = \dim \operatorname{Ker} L - \dim \operatorname{Coker} L$$

The definition allows one to specify a particular class of mappings between infinite dimensional manifolds, the nonlinear Fredholm mappings.

Definition VIII.2.[101] Let

$$f : M \longrightarrow N$$

be a differentiable mapping between Banach (infinite dimensional) manifolds.

a) f is a Fredholm map if at each point $x \in M$ the derivative map

$$Df(x) : T_x M \longrightarrow T_{f(x)} N$$

is a linear Fredholm operator

b) If f is a Fredholm mapping, on each component of M define the index of f to be the index of Df(x) at some, hence all, x in the component.

When one views the definition in terms of the geometric model for the finite elastostatic free boundary problem, and Theorem VIII.1, one sees that finite elastostatic operators which satisfy the Sobolev counterpart to Van Buren's invertibility hypothesis are particular examples of Fredholm mappings, namely those f such that

$$\dim \operatorname{Ker} Df(x) = \dim \operatorname{Coker} Df(x) = 0$$

Thus the class of finite elastostatic operators which are Fredholm mappings augment Van Buren's class.

Thé Smale-Sard Theorem

In finite dimensional theory Sard's theorem gives one information about the extent to which a function between two manifolds fails to be submersive. If

$$f : M \longrightarrow N$$

is a differentiable mapping between finite dimensional manifolds, one terms $x \varepsilon M$ a regular point if the derivative map

$$Df(x) : T_x M \longrightarrow T_{f(x)}$$

is surjective, and a singular point if it is not. The image under f of a singular point is called a critical value; all other points in the range of f are called regular values. Sard's theorem relates the extent to which the range space consists of critical values to the dimension of the spaces involved, and the differentiability of the map.

Theorem VIII.2 (Sard).[102] Let U be an open subset of R^p, and let

$$f : U \longrightarrow R^q$$

be a C^s map, where $s > \max(p-q, 0)$. Then the set of critical values in R^q has measure zero.

Smale's work extends Sard's theorem to a particular class of mappings between infinite dimensional manifolds, the nonlinear Fredholm mappings.

Theorem VIII.3.[103] Let

$$f : M \longrightarrow N$$

be a C^q Fredholm map with

$$q > \max(\text{ index } f, 0)$$

(on each component). Then the regular values of f are "almost all" of N; that is to say, except for a set of the first Baire category.

Of particular interest are the following corollaries to the theorem.

Corollary VIII.2.[104] If

$$f : M \longrightarrow N$$

is a C^q Fredholm map, $q>(\text{index } f, 0)$, then for almost all $y \in N$, its preimage $f^{-1}(y)$ is a submanifold of M whose dimension is equal to index f, or is empty.

Corollary VIII.3.[105] If

$$f : M \longrightarrow N$$

is a Fredholm map of negative index, its image contains no interior points.

Local Uniqueness Results for the Free Boundary Problem

When one applies the Smale-Sard theorem to the geometric models for the free boundary and Dirichlet elastostatic problems, one achieves results which greatly enhance Van Buren's work, and generate some rather provocative questions which appear quite promising and, most importantly, resolvable. The first corollary permits one to relate the index of a Fredholm elastostatic operator to (1) the existence of equilibrating configurations for a given body force, and (2) the extent of local nonuniqueness. The second corollary gives one a nonexistence theorem.

Proposition VIII.1 (Free Boundary Version) Let

$$\mathrm{Div}\; T_{s_\kappa} : \mathrm{EmbH}^{\ell+2}(\eta) \longrightarrow H^{\ell}(T_{s_\kappa}\eta)$$

represent a free boundary finite elastostatic problem. Suppose on each component of $\mathrm{EmbH}^{\ell+2}(\eta)$ the finite elastostatic operator is a Fredholm map. Then for almost all body force density sections $s_b \epsilon H^{\ell}(T_{s_\kappa}\eta)$, the set of configurations equilibrating s_b in each component of $\mathrm{EmbH}^{\ell+2}(\eta)$ is a submanifold of (finite) dimension index DivT_{s_κ}, or is empty.

Proposition VIII.2. (Free Boundary Version) Under the hypotheses of the previous proposition, if the index of the finite elastostatic operator is negative on any component, then almost all body force densities are incapable of being equilibrated by configurations lying in that component.

The theorems indicate that the more informative quantity with which to study local uniqueness properties of the finite elastostatic operator is the index of the map, as opposed to the separate dimensions of the kernel and cokernel spaces. The index is constant on each component of the solution manifold, whereas the dimensions of the kernel space, for example, can change abruptly from configuration to configuration. If the index of the operator is greater than zero on any component, then the finite elastostatic operator is locally nonunique about most equilibrating configurations in that component. Finally, the actual value of the index gives one a measure of the extent of local nonuniqueness.

Local Uniqueness Results for the Dirichlet Problems

One may also formulate the propositions for the Dirichlet settings and obtain local existence and uniqueness results. One of Smale's own applications of the generalized Sard theorem is a local uniqueness theorem for the nonlinear, elliptic Dirichlet problem of second order built over the Holder function spaces.

Corollary VIII.3.[106] (Holder Space Version) Let Ω be a bounded region in R^n with smooth boundary $\partial\Omega$.

(1) Let

$$\Phi : C^{s+2}(\overline{\Omega}) \longrightarrow \underset{\Phi(u)}{C^{s}(\overline{\Omega})} \qquad s \geq 0$$

be a nonlinear partial differential equation of second order defined in terms of the map

$$F : J^2(\overline{\Omega}) \longrightarrow R$$

by $\Phi(u)(x) = F(j_2u)(x) = F(x,u(x),Du(x),D^2u(x))$

(2) For $C^{\alpha}(\overline{\Omega})$ the space of functions on $\overline{\Omega}$ which are Holder continuous of order α, let

$$f_o \varepsilon C^{2+\alpha}(\overline{\Omega})$$

and define

$$C^{2+\alpha}_{f_o}(\overline{\Omega}) = \{f \varepsilon C^{2+\alpha}(\overline{\Omega}) : f|_{\partial\Omega} = f_o|_{\partial\Omega}\}$$

Then:

(a) If F is a (strongly) elliptic operator then the induced map

$$\Phi : C^{2+\alpha}_{f_o}(\overline{\Omega}) \longrightarrow C^{\alpha}(\overline{\Omega})$$

is a Fredholm map of index zero.

(b) For almost all $g \epsilon C^{\alpha}(\overline{\Omega})$ the set of u $C^{2+\alpha}_{f_o}(\overline{\Omega})$ such that

$$\Phi(u)(x) = g(x)$$

is discrete.

The corollary enhances the local uniqueness results of Van Buren by allowing one to release somewhat the stringent requirement that the derivative operator be invertible at each configuration. Moreover, the second assertion of the corollary indicates clearly how the local uniqueness concousions do not necessarily give rise to global unique ones.

Similar theorems may be formulated for the geometric models for the Dirichlet settings presented here. For full generality, they are formulated in a manner paralleling Corollary VIII.2.

Proposition VIII.3. (Simple support version) Let

$$g : \partial B \text{ —— } \eta|_{\partial B}$$

specify a boundary configuration which models a Dirichlet boundary condition. Let

$$\mathrm{Div}T_{s_K}|_{[\mathrm{Emb}H^{\ell+2}(\eta)]_g} : [\mathrm{Emb}H^{\ell+2}(\eta)]_g \text{ —— } H^{\ell}(T_{s_K}\eta)$$

specify a simply supported Dirichlet problem. Suppose on each component of the configuration manifold the Dirichlet finite elastostatic operator is a Fredholm map.

a) Then for almost all body force density sections $s_b \varepsilon H^{\ell}(\eta)$, the set of configurations in each component of $[\mathrm{Emb}H^{\ell+2}(\eta)]_g$ equilibrating it and satisfying the boundary condition is a submanifold of (finite) dimension index $(\mathrm{Div}T_{s_\kappa}|_{[\mathrm{Emb}H^{\ell+2}(\eta)]_g})$,

or is empty.

b) In particular, if the index is zero on any component, then the set of equilibrating configurations which satisfy the boundary condition is discrete.

Proposition VIII.4 (Rigid Support Version) Let

$$g : B \longrightarrow \eta$$

designate a particular configuration which models the Dirichlet condition g, and let

$$\mathrm{Div}T_{s_\kappa}|_{(\mathrm{Emb}H^{\ell+2})_{\partial g}(\eta)} : (\mathrm{Emb}H^{\ell+2})_{\partial g}(\eta) \longrightarrow H^{\ell}(T_{s_\kappa}\eta)$$

specify a rigidly supported Dirichlet problem. Suppose on each component of the configuration manifold the finite elastostatic operator is a Fredholm map.

a) Then for almost all body force density sections $s_b \varepsilon H^{\ell}(\eta)$ the preimage under the finite elastostatic operator in each component of the configuration manifold is a submanifold of (finite) dimension index $(\mathrm{Div}T_{s_\kappa}|_{(\mathrm{Emb}H^{\ell+2})_{\partial g}(\eta)})$,

or is empty.

b) In particular, if the index is zero on any component, then the preimage is a discrete set.

How the Local Uniqueness Conclusions of the Various Models are Related

The local uniqueness conclusions drawn from the free boundary and Dirichlet models are not independent of each other. Since the finite elastostatic operators for the Dirichlet problems arise as restrictions of the free boundary operator, one gains the following results linking the indices of the operators.

> Lemma VIII.1. Let $\mathcal{g}\varepsilon\mathrm{Emb}H^{\ell+2}(\eta)$ be an $H^{\ell+2}$ configuration, and
>
> $$\mathcal{g}|_{\partial B} \equiv g.$$
>
> Then the indices of the finite elastostatic operators associated with the rigidly supported Dirichlet problem, the simply supported Dirichlet problem, and the free boundary problem, when defined, are ordered by the relation
>
> $$\text{index } (\mathrm{Div}T_{s_K}|_{(\mathrm{Emb}H^{\ell+2})_{\partial\mathcal{g}}(\eta)}) \leq$$
>
> $$\text{index } (\mathrm{Div}T_{s_K}|_{[\mathrm{Emb}H^{\ell+2}(\eta)]_g}) \leq$$
>
> $$\text{index } \mathrm{Div}T_{s_K}.$$

As one imposes stronger and stronger boundary conditions, the index of the elastostatic operator decreases.

A provocative avenue of inquiry opens as to the role played by the boundary condition in affecting local existence and uniqueness conclusions. One may examine elastostatic problems which are locally nonunique in the free boundary setting, but locally unique in the Dirichlet boundary setting. Little work has been done on this aspect of the existence and uniqueness problem. The principal efforts in the literature are directed towards determining what conditions can be imposed upon the response function for the finite elastostatic operator to insure local uniqueness in any, hence all Dirichlet problems. Neither the manner in which the place boundary condition is imposed, nor the the possibility of local uniqueness holding for one boundary condition, but not another usually enter into the treatment.

How Three Locally Nonunique Situations Manifest Themselves

Using the primitive tool provided by the relationship among the indices of the three finite elastostatic operators one may begin to examine how local uniqueness conclusions may vary with boundary conditions. For example, one may discern between three types of locally nonunique situations associated with the finite elastostatic problem.

Let a simple elastic material body be in a configuration $s\psi$ equilibrated by a body force s_b. The first type of local nonuniqueness results when the free boundary finite elastostatic operator is not locally unique at $s\psi$, but a simply supported Dirichlet finite elastostatic operator is. In terms of the geometry of the configuration manifolds, the situation may be visualized in the following way. To say that the free boundary operator is not locally unique at $s\psi$ is to say that associated with the body force s_b are many configurations in the free boundary configuration manifold, of which $s\psi$ is but one. The index of the free boundary finite elastostatic operator would be greater than zero; the preimage of s_b would be a manifold of dimension index DivT_{s_κ}. Geometrically, it would be impossible to separate $s\psi$ from other equilibrating configurations using open sets of the free boundary configuration manifold $\mathrm{EmbH}^{\ell+2}(\eta)$.

However, if one restricts to a suitable simply supported Dirichlet configuration submanifold $[\mathrm{EmbH}^{\ell+2}(\eta)]_g$ containing $s\psi$, local uniqueness would be characterized by the zero index of the associated Dirichlet finite elastostatic operator. Geometrically, one could separate $s\psi$ from other equilibrating configurations using open sets of $[\mathrm{EmbH}^{\ell+2}(\eta)]_g$. A visualization of this nonunique situation is given in Figure VIII.2.a), using finite dimensional elements.

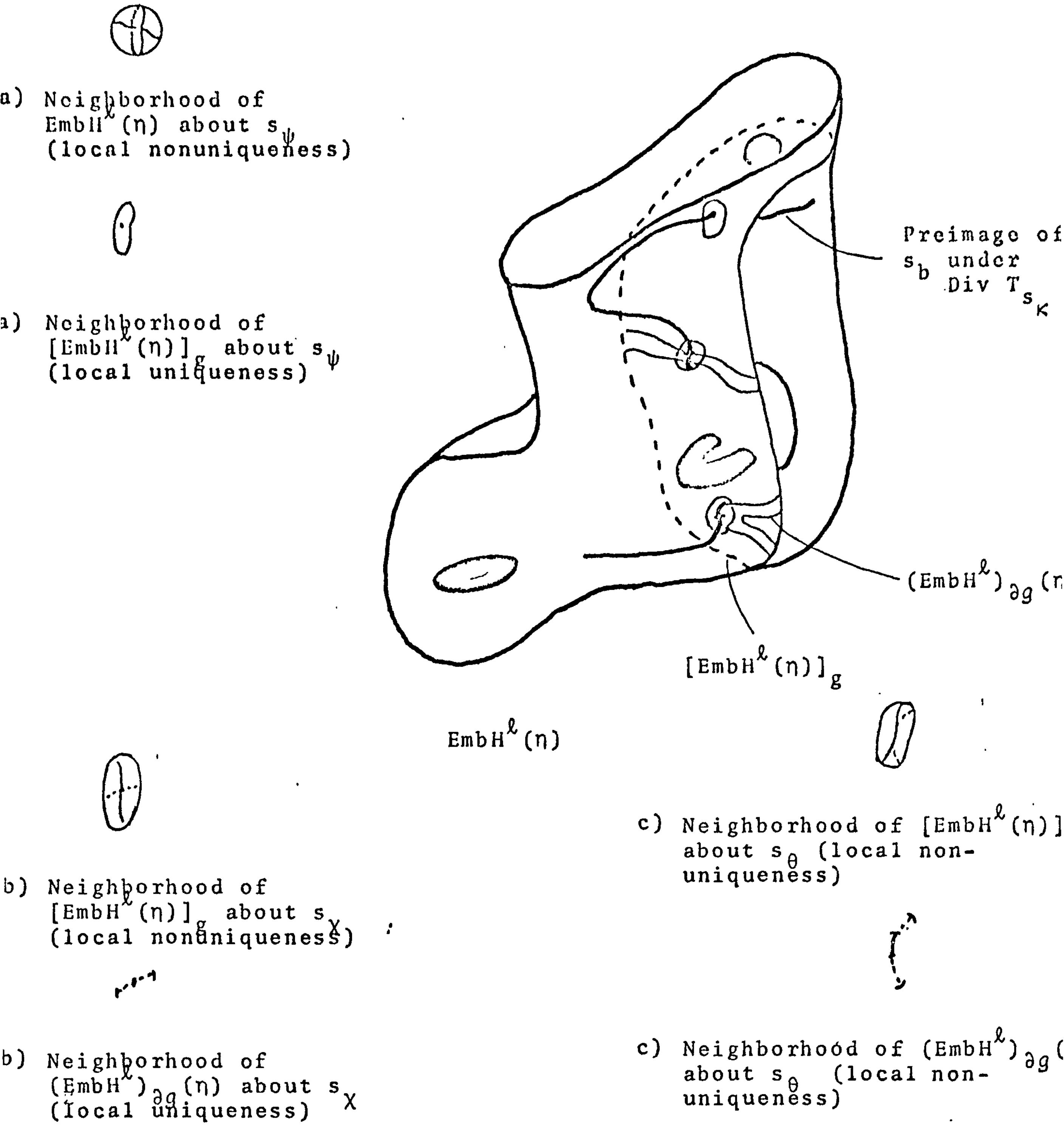


FIGURE VIII. 2.

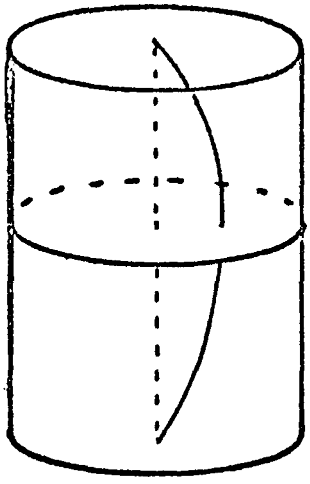
Geometrically Visualizing Three Situations of Local Nonuniqueness.

One is well acquainted with physical situations corresponding to this type of local nonuniqueness. For example, if a homogeneous isotropic hyperelastic solid subject to no boundary conditions possesses an equilibrating configuration $s\psi$ for a zero body force density field, then any rigid body motion from the configuration will result in a new equilibrating one. If the free boundary operator is Fredholm, its index would be six, indicating the extent of local nonuniqueness at $s\psi$. However, if $s\psi$ is a natural state, then the imposition of a simply supported place boundary condition suffices to insure that $s\psi$ is locally unique.[107]

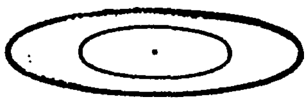
A second type of local nonuniqueness results when both the free boundary and simply supported Dirichlet finite elastostatic operators are locally nonunique at s_χ, but the rigidly supported Dirichlet problem is locally unique. If the elastostatic operators are Fredholm, the situation would reveal itself as one where the indices of the free boundary and simply supported Dirichlet finite elastostatic operators, though ordered, are both positive, while the index of the rigidly supported Dirichlet operator is zero. Geometrically, the situation is one where s_χ may not be separated from other equilibrating configurations using open sets in $[\mathrm{EmbH}^{\ell+2}(\eta)]_g$, but may be separated from them using open sets in $(\mathrm{EmbH}^{\ell+2})_{\partial g}(\eta)$. A visualization of this instance is given in Figure VIII.2.b).

A physical situation corresponding to this type of nonuniqueness can be conjectured.[108] Consider a homogeneous, isotropic cylinder subject to a Dirichlet boundary condition. Suppose the boundary condition is so chosen, and the response of the material is such that the cylinder admits an interior buckling as depicted in Figure VIII.3.a). Assume that the buckling may be axially symmetric. Then for any cross-sectional plane of the cylinder the locus of possible buckling of the centerline of the cylinder would be a circle centered about the axis. Given any buckled configuration s_ψ satisfying the boundary condition and a zero body force, any perturbation of the buckled point around the circle would result in a new equilibrium configuration, as depicted in Figure VIII.3.b). As the propagation of the buckled point could be made as small as desired, the simply supported Dirichlet problem would exhibit local nonuniqueness at s_ψ.

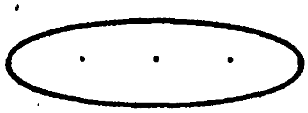
However, if one further constrains the boundary condition by requiring that the equilibrating configuration come off the boundary in a definite, nonisotropic way, by rigidly supporting the boundary condition, for example,



a) An Instance of the Buckling is Envisioned.



b) A Plane Showing the Locus of the Possible Buckling.



c) The Locus of the Buckling Under Suitably Chosen Rigid Support.

FIGURE VIII. 3.

An Instance of Nonuniqueness in the Simply Supported Dirichlet Boundary Condition.

then it may be possible to reduce the equilibrating buckled configurations in the plane section to antipodal points of the circle, as visualized in Figure VIII.3.c). As such equilibrating configurations are related by a finite deformation, the equilibrating configurations are locally unique in the rigid support setting.

The third type of local nonuniqueness occurs when the finite elastostatic operator has a nontrivial index in all three settings. A geometric visualization of the configuration manifold for this situation is given in Figure VIII.2.c). A physical situation corresponding to it arises from the previous example. If one chooses a rigidly supported model for the boundary condition where the way in which the configuration comes off the boundary is itself required to be isotropic, then one again loses the local uniqueness property.

As cursory as these examples are, they do emphasize the fact that for the nonlinear models presented here, local uniqueness conclusions drawn for one boundary condition need not hold for another.

They indicate how one can dissect a locally unique elastostatic situation by judiciously imposing stronger and stronger boundary conditions. Moreover, the index is revealed as a simple algebraic indicator for characterizing when one attains a locally unique situation, or the extent of non-uniqueness. As more mathematical information becomes available on the computation of the index of a mapping, how it depends upon the topology of the domain of definition, and how it behaves under the restriction of the mapping, a more complete understanding of how one can analyze nonunique finite elastostatic problems will emerge.

Questions Outstanding in the Application of the Smale-Sard Theorem

Several promising areas of investigation arise when one views the Smale-Sard theorem in the context of the elastostatic problem. The first deals with the availability of studying finite elastostatic operators which are Fredholm. A second deals with determining when the operator is Fredholm. A third seeks ways to characterize local existence and uniqueness properties of Fredholm maps.

One can formulate questions in these areas which appear resolvable in the immediate future. Firstly, how restrictive is the precondition of Fredholmness on the finite elastostatic operator? Secondly, if one knows that the derivative of a finite elastostatic operator is a Fredholm operator at one configuration, under what conditions can one view the finite operator as itself a Fredholm mapping? Thirdly, if one is given a finite elastostatic operator with the Fredholm property, can one determine its index without direct knowledge of its kernel and cokernel spaces?

Surprisingly, these questions currently are the focus of much mathematical activity. However, they are disguised in terms of abstract investigations. For example, the first two questions are subjects of interest in the inquiry into whether, in the class of all mappings between Banach manifolds, characterize those which are Fredholm, and those which can be approximated by Fredholm mappings. The reader is referred to the literature to examine the extent to which abstract results have been obtained, but have yet to be applied.[109, 110]

Resolution of the third question has been advanced significantly by the identification of the index of the mapping, the mapping itself, and the algebraic topological properties of the solution manifold.[111, 112] Although the specific results are too complex to present here, it deserves mention that results like these point once again to the intimate relationship between the operator, the topology of the configuration manifold, and the existence and uniqueness properties.

OTHER MATHEMATICAL TOOLS

Local Uniqueness and the Degree of a Mapping

The application of the Smale-Sard theorem represents but one example of the powerful mathematical resources which become available with the introduction of the geometric models for the finite elastostatic free boundary and Dirichlet models. Another prominent tool is the algebraic degree of a nonlinear mapping.

In chapter four one found that one may associate with a mapping between finite dimensional manifolds

$$f : U \subset R^n \longrightarrow R^n$$

an algebraic quantity, $\deg(f, q_o, U)$, which characterized, roughly speaking the number of solutions $p_o \in U$ which satisfy the equation

$$f(p) = q_o.$$

For more precision, the reader is referred to Definition IV.3.

The definition of the degree of a nonlinear mapping between infinite dimensional manifolds is under current mathematical investigation. The degree of a Fredholm mapping of a Banach space was defined by Smale in 1965.[113] When the Fredholm map has index zero, the degree of the map is an integer whose definition parallels the finite dimensional definition. When the index of the map is non-zero, (a locally nonunique situation), the degree is no longer an integer; rather, it is a more abstract, but equally comprehensible quantity. Much work has been done on generalizing the definition to Fredholm mappings between manifolds, and consequences, like an infinite dimensional version of the Fredholm Alternative, and a rank theorem.[114]

Notably, the theory of the algebraic degree of a nonlinear mapping between topological vector spaces has received much attention in some special areas of hydrodynamics and elastodynamics, when the dynamic operator is a particular type of compact perturbation of the identity.[115] Some investigations of the onset of secondary flows as one varies the Reynolds number have been made, in which the onset is modeled as a bifurcation phenomenon in a nonlinear eigenvalue problem.[116] Also, problems in the buckling of shells which arise in the Elastica theory have been

117,118

similarly modeled. In each case, the hydrodynamic or elastodynamic operator is strongly elliptic, (hence, Fredholm of index zero), for each value of the parameter. By intricate computation, usually, one establishes that for some values of the parameter, the degree of the operator has one integer value, and for other values of the parameter the degree has a second value. By theorem or computation one may then determine the value of the parameter at which the discontinuity in the degree occurs. It is at these values of the parameter that new motions branch off from previously known ones. The chief feature of the nonlinear eigenvalue problem, in contrast to the linear one, is that the new motions continue to exist and grow as the parameter increases.

A Local Uniqueness Problem in Elastostatics is Viewed in terms of the Degree

One may envision a family of Dirichlet elastostatic problems which display an onset of secondary, locally unique equilibrating solutions. One may view them in terms of the geometric models presented here as bifurcation problems, to which the algebraic degree theory may be applied.

Consider in greater detail the example proposed by F. John mentioned in chapter four. It exhibits a body displaying locally unique, but globally nonunique equilibrating configurations for a given Dirichlet problem. The body consists of an infinite cylinder with concentric cylindrical boundaries of radius $R_1 < R_2$, respectively. If the material comprising the body were isotropic, and if the body admits an equilibrating configuration in which the inner cylinder is rotated through a straight angle, then the body admits at least two locally unique equilibrating configurations to that particular Dirichlet boundary value problem. The situation is pictured in Figure IV.1.

Now suppose that the material comprising the body is such that the inner cylinder may be rotated through one straight angle, but not through two. One would then have a situation where the elastostatic Dirichlet problems in which the inner cylinder is rotated through an angle θ in an interval (θ_1, θ_2) would possess two locally unique equilibrating configurations, while those Dirichlet problems in which the inner cylinder were rotated through an angle in the interval $[0, \theta_1)$ or $(\theta_2, 2\pi]$ would possess a unique equilibrating configuration. The situation is visualized in Figure VIII.4.

How would such a problem appear in the geometric models presented here, and how would one characterize the behavior? The boundary conditions would determine a one-parameter family of simply supported Dirichlet configuration manifolds,

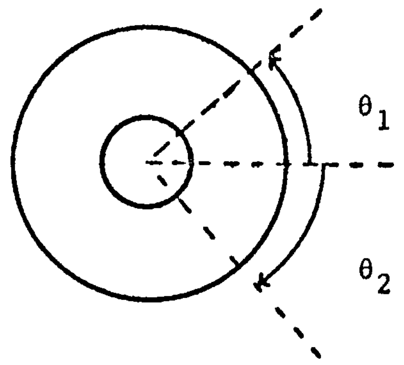
$$\{[\mathrm{Emb}H^{\ell+2}(\eta)]_{g(\theta)},\ \theta\varepsilon[0,2\pi)\},$$

lying in the free boundary manifold. Assume that for each Dirichlet boundary condition the finite elastostatic operator is Fredholm. Then the locally unique character of the equilibrating configurations would be reflected in the index of the elastostatic operator. For each value of θ, the corresponding Dirichlet finite elastostatic operator would be zero, and the preimage of the body force density would be a zero dimensional manifold. Allowing the parameter to vary continuously, one generates a one dimensional submanifold which may be quite complicated topologically.

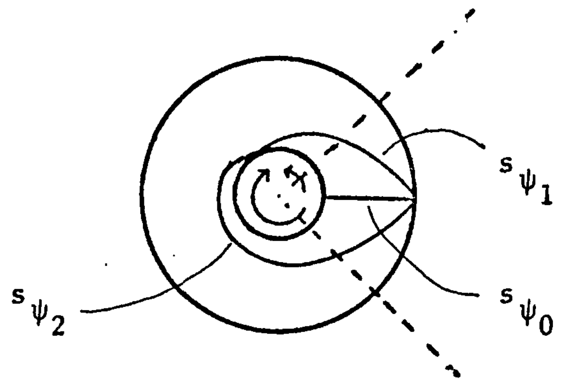
The onset of the second equilibrating configuration would be signaled by the degree of the elastostatic operator. For some values of the parameter the degree of the associated Dirichlet finite elastostatic operator would be unity, while for other values it would deviate from unity. The parameter values at which the discontinuities in the degree occurs would mark the boundary values at which the second equilibrating configuration appears.

Some Outstanding Questions About the Application of the Degree

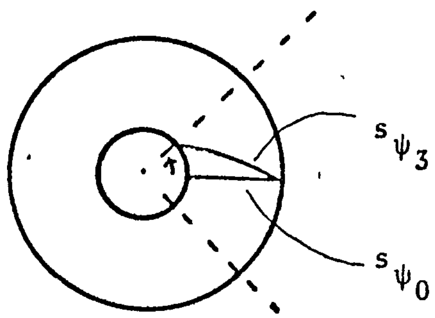
The above conjecture reveals several very important questions concerning the application of degree theory to



a) The Critical Angles at Which the Number of Dirichlet Equilibrating Configurations Change.



b) Two Equilibrating Configurations for a Boundary Condition within the Nonunique Region.



c) The Equilibrating Configuration for a Boundary Condition Outside the Nonunique Domain.

FIGURE VIII. 4.

A Situation Where the Number of Equilibrating Configurations Vary with the Boundary Condition.

elastostatic problems, and some provocative areas for further inquiry. The primary question is how precisely can one compute the degree of the finite elastostatic operator, and how does one determine when and where it changes? As mentioned previously, much work has been done on this question for Fredholm mappings which are particular types of compact perturbations of the identity.[119] If the elastostatic operators have the particular form referenced, the eigenvalues of the operator linearized about a trivial solution constitute the values of the parameter at which bifurcation occurs. Moreover, an examination of the higher order terms of the operators indicate at which of these parameter values a discontinuity of the degree in fact occurs; hence, where branching occurs, and how many branches form.

Current mathematical investigations center about characterizing the degree of a Fredholm mapping which links infinite dimensional manifolds, as opposed to topological vector spaces, and in particular maps which do not possess the form alluded to above.[120] Ways of determining the degree from algebraic topological information also appear imminent.[121] Finally, the higher order degrees, which arise when the index of the finite elastostatic operator has non zero index (hence, is not strongly elliptic) remains a provocative, yet completely untapped resource.[122]

A CONCLUDING REMARK

Embryonic as these developments are, they show what promise resides in the geometric examination of local uniqueness and existence questions. The above examples permit the reader the opportunity to experience, at least in general terms, how the geometric models developed here provide the vehicle by which heretofore abstract mathematical results are rendered quite relevant. The avenues of inquiry which follow, only a few of which were mentioned here, promise a more complete local uniqueness theory of elastostatics, and stand as fruitful vistas for future work resulting from this thesis.

IX. GAINING GLOBAL INFORMATION FROM THE MODELS

Can one gain global information from the models for the elastostatic problems presented in this work? If so, will the conclusions one draws from them differ significantly from the results gained from previous models? In this chapter, one is introduced to the mathematical methods which are becoming available for application to the new models, and which can replace those methods rendered inapplicable in chapter four.

The examples permit one to anticipate significant departures in the approach to investigating existence and global uniqueness questions. With the previous models the burden of the existence and uniqueness conclusions rested primarily upon the material response. The methods introduced in this chapter permit one to incorporate an intimate dependence upon the boundary conditions, and the body topology as well. The chief feature of the application of the methods is the importance which is placed upon the nontrivial topological nature of the manifold of configurations.

The Methods Which Will be Considered in this Chapter

The examples presented in the chapter study the existence and uniqueness questions for bodies composed of hyperelastic materials. Two methods of analysis are considered for application: the Lusternik-Schnirelman theory, and the Morse theory.

When a body is composed of material possessing a hyperelastic response the geometric model permits one to view the existence and uniqueness questions as a problem of the existence and number of critical points of a function defined over the configuration manifold. One identifies the elastostatic Dirichlet equations as the Euler-Lagrange equations of an action integral. The action integral determines a function which is defined over the Dirichlet configuration manifold, and the equilibrating configurations are critical points of the function. The existence of a critical point for the function insures the existence of an equilibrating configuration for the Dirichlet problem, and the number of critical points characterizes the global nonuniqueness of the problem.

When the hyperelastic Dirichlet problem is viewed in this manner, the abstract methods of the critical point theory may be engaged to resolve existence and global uniqueness questions. The Lusternik-Schnirelman theory and the Morse theory represent but two of these methods.

One does not go deeply into the application of these methods in this chapter. Rather the purpose they serve is to illustrate how one can gain global information from the geometric models presented in this work, and how the information one gains can differ significantly from the information gained in previous models.

Comments in this chapter are limited to hyperelastic materials. However, some mathematical methods are becoming available for resolving existence questions for bodies composed of nonhyperelastic materials.[123] An example is the generalization of the Leray-Schauder degree.[124] Consideration of these methods will be reserved for future study.

Why the Global Conclusions will Differ from Previous Models

The application of critical point theories to study existence and uniqueness questions in the abstract Dirichlet problem, though rather new, is not novel.[125] The methods have even been applied to some previous models of elastic systems.[126] What makes their application to the models presented in this work distinct is the topological complexity of the models themselves. The significant feature of the methods which are illustrated is the intimate relationship they establish between the algebraic topological properties of the solution space and the existence and uniqueness conclusions. In previous models the solution space has been trivial from an algebraic topological point of view; hence, existence and uniqueness conclusions have depended primarily upon the properties of the material response. In the models presented here, the solution space is a manifold, as opposed to a topological space. Its topology can be quite complex, and the manifolds for different Dirichlet boundary conditions need not be homeomorphic. Consequently, when one applies the critical point methods the variety of possible existence and global uniqueness conclusions one draws from them is greatly enhanced. They come to depend as intimately upon the choice of boundary condition and the specimen topology, as upon the nature of the material response.

With the completion of the illustrations one ends the general study of the geometric models. The models have been shown to be well defined, capable of providing information about finite elastostatic systems, and most importantly, potentially able to generate results quite different from previous models. One may now turn to the study of particular elastostatic problems in terms of the models. In this spirit, the chapter ends with some specific questions which can be resolved, and whose resolution will be a concrete contribution to the theory of existence and uniqueness in finite elasticity.

CRITICAL POINT THEORIES AND THE GLOBAL CONCLUSIONS FOR HYPERELASTIC MATERIALS

An Equilibrating Configuration is a Critical Point of a Function

When a continuum mechanical body is composed of material exhibiting a hyperelastic response, one may geometrically view the equilibrating configurations for a Dirichlet elastostatic problem as critical points of a function defined on the Dirichlet configuration manifold. The function arises as the action integral determined by the strain energy function associated with the material.[127]

For convenience of the development, let the body force density be zero. Relative to the reference configuration section, one may associate with the hyperelastic material a strain energy function whose derivative generates the Piola-Kirchoff stress tensor field.[128] In terms of the free boundary model, one may view it geometrically as a real-valued function defined over the classically differentiable configurations. In the notation of chapter five, one may write the function as

$$\Sigma_{s_K} : \mathrm{Emb}^k(\eta) \longrightarrow \mathrm{BxR}$$

If the material response is smooth, the function takes C^∞ configurations into C^∞ functions over the body. In general, the function depends upon the higher order derivatives of the configuration section. For example, for a simple elastic body, Σ_{s_K} is a nonlinear differential operator of order one

$$\Sigma_{s_K} = (\sigma_{s_K})_* \cdot \nabla_\eta ,$$

where

$$\sigma_{s_K} : L^{(3)}(TB, \eta) \longrightarrow B \times R$$

represents the usual specification of the strain energy function as a morphism over the bundle of local configurations.

One may associate with Σ_{s_K} an action integral. Geometrically, the integral defines a function on the manifolds of classically differentiable configurations. The function is C^∞ if the material response is smooth. One may denote the function as

$$J^{\Sigma_{s_K}} : \mathrm{Emb}^k(\eta) \longrightarrow R$$

$$s_\psi \qquad \int_B \Sigma_{s_K}(s_\psi)(p)\, d\mu_{s_K}(p) \quad .$$

$d\mu s_K$ denotes the volume element on B determined by the reference configuration. If one takes into account the axioms of chapter four, 129 the action integral function extends to a C^∞ function on the free boundary Sobolev configuration manifolds,

$$J^{\Sigma s_\kappa} : \mathrm{EmbH}^k(\eta) \longrightarrow R, \; k \geq 3$$

Upon restriction, the action integral function determines a C^∞ function on the Dirichlet configuration manifolds, for example,

$$J^{\Sigma s_\kappa}\Big|_{(\mathrm{EmbH}^k)_{\partial g}(\eta)} : (\mathrm{EmbH}^k)_{\partial g}(\eta) \longrightarrow R$$

The derivative of the action integral function is given by theorem from chapter four. Choosing s_ψ a smooth configuration for convenience,

$$dJ^{\Sigma s_\kappa}(s_\psi) : \; T((\mathrm{EmbH}^k)_{\partial g}(\eta))_{s_\psi} \approx H^k_o(\eta) \longrightarrow R$$

by $$\sigma \longrightarrow [dJ^{\Sigma s_\kappa}(s_\psi)]\sigma = \int [\Lambda(J^{\Sigma s_\kappa})_{s_\psi}]\sigma(p)\,d\mu_{s_\kappa}(p)$$

As one might expect, the derived operator $dJ^{\Sigma s_\kappa}$ is the Euler-Lagrange operator in integrated form.[130] The equilibrating configurations for the elastostatic Dirichlet problem are precisely those which render the derived operator trivial,

$$dJ^{\Sigma s_\kappa}(s_\psi) = 0 \in T^*((\mathrm{EmbH}^k)_{\partial g}(\eta))_{s_\psi} \quad .$$

Hence, they constitute the critical points of the action integral function.

One thus achieves a representation for the generalized elastostatic Dirichlet problem as a function on the extended Dirichlet configuration manifold. Its critical points are the equilibrating configurations for the Dirichlet problem. The existence of critical points insures the existence of elastostatic configurations for the body. If the critical points are isolated, then the static configurations are locally unique. The number of critical points characterizes how globally nonunique the elastostatic problem is.

HOW ONE ENGAGES THE LUSTERNIK-SCHNIRELMAN THEORY

The Idea of the Lusternik-Schnirelman Theory

How many critical points may an action integral function possess on a Dirichlet solution manifold, and what are their character? In order to resolve the question, one may engage the various "abstract" critical point theories which have emerged in the modern mathematical literature. Two theories will be discussed in this chapter. The first theory was developed by Lusternik and

Schnirelman in 1934.[131] It has recently been extended to Banach manifolds of arbitrary (infinite) dimension.[132]

The Lusternik-Schnirelman theory allows one to determine the critical values of the function, and to associate with each critical value an integer which characterizes the topological nature of the corresponding critical point set. In general, if one judiciously chooses a class of subsets of the manifold, determines the maximum value of the function on each set of the class, then minimizes the maximum values for all sets in the class, one gains a critical value for the function.[133] The Lusternik-Schnirelman theory provides a particular choice for the class of subsets over which to carry out the minimax procedure. One finds that the choice intimately involves certain topological features of the subsets of the manifold. The features may be characterized by an algebraic topological invariant, the (L-S) category of the set.

The (L-S) Category of a Set

To define the invariant quantity, one begins by identifying when a subset of a topological space is "trivial", or contractable.

Definition IX.1.[134] A closet set A of a topological space X is contractable over X if the injection of A into X is homotopic over X to a constant map. That is to say, there is a continuous map

$$H : A \times I \longrightarrow X$$

having the properties

$$H(-,0) = 1_A : A \longrightarrow A \subset X,$$

$$H(-,1) : A \longrightarrow \{P\} \in X$$

Not all subsets of a topological space are contractable. One introduces an algebraic quantity which characterizes the extent to which a subset is not contractable. The quantity is the (L-S) category of the set.

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Definition IX.2. The (L-S) category of a closed set A in X, denoted $\text{cat}_X(A)$, is the least integer n such that A can be covered by n closed subsets of X each of which are contractable. Denote the category of X in X by cat(X).

Lemma IX.1. If A is contractable in X then

$$\text{cat}_X(A) = 1.$$

The (L-S) category is an algebraic indicator of the topological complexity of a space. If a space is trivial topologically, its (L-S) category is quite low. For example, $\text{Cat}(R^n) = 1$ reflects the contractability of Euclidean space. $\text{Cat}(S^n) = 2$ indicates that S^n, though still rather simple topologically, is more complex than R^n. As one might intuit, the (L-S) category of a space is closely related to its homology and cohomology. The following theorem illustrates how knowledge of the cohomology of a space provides information about its category.

Proposition IX.1.[136] Let X be an arcwise connected metric space. Let $H^{k_i}(X,F)$ denote the k_ith cohomology group of X with coefficients in a field F. Let

$$\mathrm{cuplong}(X)$$

denote the largest integer n such that for $\Gamma_i \varepsilon H^{k_i}(X,F)$, $k_i > 0$, $1 \leq i \leq n$, the cup product

$$\Gamma_1 \vee \ldots \vee \Gamma_n \neq 0$$

Then

$$\mathrm{cat}(X) \geq \mathrm{cuplong}(X) + 1.$$

(L-S) Category and Critical Points

In what follows consideration is limited to manifolds modeled on a Hilbert space which possess a Riemann structure. The limitation is imposed for convenience; however, it is consistent with the previous work. For example, $\mathrm{EmbH}^k(\eta)$ is a Hilbert manifold, and inherits a Riemannian structure as an open submanifold of the Hilbert space $H^k(\eta)$.

Definition IX.3.[137] Let M be a Riemannian manifold modeled on a Hilbert space. Let f be a smooth function on M. Let ∇f be the gradient associated with its derivative by the Riemann structure. For c a real number, define the set of critical points of f at level c by

$$K_c = \{p \varepsilon M : f(p) = c, \quad \nabla f(p) = 0\}.$$

Which values of f may be critical values? An attractive collection of candidate values arises when one minimizes the maximum values the function f may take over sets of M of suitable category.

Definition IX.4. 138 For k an integer, let $\Gamma_k(M)$ denote the set of all subsets of M of (L-S) category $\geq k$. Define

$$c_k(f) = \inf_{A \in \Gamma_k(M)} \{\sup_{p \in A} f(p)\}.$$

If $\Gamma_k(M) = \emptyset$, take $c_k(f) = \infty$.

The fundamental results of the Lusternik Schnirelman theory are that under suitable conditions the values $\{c_m(f)\}$ are critical values for f, and moreover, the (L-S) category of the critical point set $K_{c_m}(f)$ possesses a lower bound which may be determined. Consequently, one gains information about how nontrivial the critical point set of a function is topologically.

The Palais-Smale Condition (C)

From one's experience with non-Fredholm operators in the previous chapter, one may correctly intuit that not all functions defined over arbitrary infinite-dimensional manifolds will admit a Lusternik-Schnirelman theory. In 1965, R. Palais extended the theory to a class of infinite dimensional manifolds and functions defined on them which can include the elastostatic models presented here.[139] Let M denote an infinite dimensional manifold modeled on a separable Hilbert space. Assume that M possesses a C^n Riemann metric whose geodesics may be extended indefinitely, (that is, M is a complete Riemannian manifold). Consider those smooth functions f defined on M which satisfy the following condition (C).

> Condition IX.1.[140] (Condition (C)). If S is a subset of M on which $|f|$ is bounded, but on which $||\nabla f||$ is not bounded away from zero, then there is a critical point of f in the closure of S.

For example, the condition is satisfied if f is a proper map. In particular, if M is compact (hence, finite dimensional), any smooth function satisfies the condition.

The Main Results of the Lusternik-Schnirelman Theory

Under the Palais-Smale condition (C), one achieves the following theorem which identifies the critical values of f and determines the lower bounds on the (L-S) category of the critical point sets.

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Theorem IX.1. Let (M,f) satisfy condition (C). Let $\{c_m(f)\}$ and K_c be as defined previously. Let K be the entire critical point set of f on M. Then

(a) $c_1(f) = \text{imf } \{f(p) : p \varepsilon M\}$, and $c_1(f) = \text{Min } \{f(p) : p \varepsilon M\}$ if f is bounded below.

(b) For each integer $n \geq 1$,

$$c_n(f) \leq c_{n+1}(f).$$

(c) If $-\infty < c_k(f) < \infty$, then $c_k(f)$ is a critical value for f

(d) If some $c_k(f) = \infty$, then f is unbounded on K, and K is infinite. In fact,

$$c_m(f) \leq \sup \{f(p) : p \varepsilon K\}.$$

for all m.

(e) If $0<m<n \leq \text{Cat } (M)$, and $-\infty < c \equiv c_m(f) = c_n(f) < \infty$, then

$$\text{Cat}_M(K_c) \geq n-m+1.$$

From the Lusternik-Schnirelman theorem one may gain information about the existence and number of critical points of a function defined on the manifold. The following corollary illustrates the type of abstract existence and global uniqueness theorem one gains.

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Corollary IX.1. Let (M,f) satisfy condition (C), and let f be bounded below.

(a) Then f assumes a minimum on each component of M.

(b) In particular, on each component M_o of M, there are at least as many critical points of f as the Lusternik-Schnirelman category $\text{Cat}_M(M_o)$ for that component.

The Significance of the Lusternik-Schnirelman Theory for Elastostatics

The Lusternik-Schnirelman critical point theory provides one method for resolving existence and global uniqueness questions for the finite elastostatic Dirichlet problem for a hyperelastic material. The manifold M is taken to be the manifold of possible configurations satisfying the given Dirichlet boundary condition. The function f is taken to be the action integral function. The critical points of the function are the equilibrating configurations for the elastostatic problem. The conclusions on the existence of critical points for the function and the lower bounds on their number informs one about the existence and the extent of global nonuniqueness of the solutions to the finite elastostatic Dirichlet problem.

This value of the Lusternik-Schnirelman theory has been recognized by architects of previous continuum mechanical models. Melvyn Berger has shown that if one models the buckling problem for a two-dimensional elastic body by means of the Elastica theory as a nonlinear eigenvalue problem, the application of the Lusternik-Schnirelman theory successfully resolves some existence and global uniqueness questions.[143] In fact, he foresees the Lusternik-Schnirelman theory as producing the nonlinear generalization of the Sturm-Liouville theory for the eigenvalue problem.[144]

The Significance of the Lusternik-Schnirelman Results for the Models Developed Here

If one applies the Lusternik-Schnirelman theory to the models for the elastostatic problem developed in this work, will the existence and global uniqueness conclusions differ significantly from those of previous models? Corollary IX.1 indicates that they will differ quite significantly. The basis for the assertion is the intimate involvement of the topology of the solution manifold with the number of critical points which the action integral function can sustain. For the models presented here, the solution manifolds can be highly nontrivial topologically. Hence, the quantity $Cat_M(M_o)$ may differ quite markedly from unity. For the previous models, where the solution manifolds are topological vector spaces, the quantity is precisely unity.

It is instructive to examine this point in detail. For a given boundary condition, the Dirichlet configuration manifolds for the models presented here may have many components, each of which are of a different, nontrivial topological character. A nontrivial topological character signals a large category. Hence, $Cat_M(M_o)$ can vary for each component, and, most importantly, differ from unity.

Corollary IX.1 then indicates that the number of critical points which the action integral function possesses may vary from component to component, and may have a lower bound which is greater than unity. Moreover, since the Dirichlet configuration manifolds need not be homeomorphic in these models, the existence and uniqueness conclusions will change as one varies the boundary condition, or alters the specimen topology. In other words, the conclusions on the existence and global uniqueness of equilibrating configurations from the models presented here would be rich and varied.

Contrast these results with those which one would gain from previous models. The Dirichlet solution manifolds are affine subspaces of a Banach space. They are all diffeomorphic to a closed linear subspace. A subspace of a Banach space is contractable. Thus, the category of all Dirichlet solution manifolds would be unity. Consequently, Corollary IX.1 could indicate the existence of equilibrating configurations for these models, but could convey no information on the global uniqueness question. Moreover, since all Dirichlet configuration manifolds are diffeomorphic in these models, the Lusternik-Schnirelman theory could not anticipate how the number of critical points of the action integral might change as one varied the boundary condition.

In short, the Lusternik-Schnirelman theory provides a concrete indication of how the existence and number of equilibrating configurations of a Dirichlet elastostatic problem can depend as intimately upon the topology of the configuration manifold as upon conditions imposed upon the elastostatic operator itself. From the nontrivial topological character of the Dirichlet configuration manifolds of the models presented here, one gains information on the global uniqueness questions which is unavailable to previous topological linear space models. Finally, the manifestation of the variations in the topology of the configuration manifolds as one changes the boundary condition, or alters the specimen topology, as changes in the least number of equilibrating configurations is most provocative. It heralds a deeper insight into the confounding problem of the interdependence of the solution of a nonlinear problem and the shape of its boundary and domain.

THE MORSE THEORY AND THE HYPERELASTIC MODELS

The application of the Morse theory to the finite elastostatic models for the hyperelastic material body illustrates the intimate relationship between the existence and number of critical points of the action integral function and the topology of the configuration manifold even more dramatically than the Lusternik-Schnirelman theory. When applicable, the Morse theory allows one to utilize more topological information from the configuration manifold to make stronger statements about the number of critical points. Moreover, one can discern something of the nature of the critical points: are they absolute minima, absolute maxima, or saddle points?

The Idea of the Morse Theory

The relationship between the presence and nature of critical points for a function and the topological structure of the manifold on which it is defined can be much more subtle and intimate than the Lusternik-Schnirelman theory suggests. A simple, but dramatic example of the relationship is provided by the height function on a two-dimensional torus.[145] This example is reproduced as Figure IX.1. Notice that the onset of a critical point of the height function is signaled by fundamental changes in the algebraic topological structure of the function's preimage. Moreover, near the critical points, one may choose a coordinate system for which the function is represented as a quadratic expression.[146] The number of negative signs in the quadratic is called the index of the critical point. It characterizes the nature

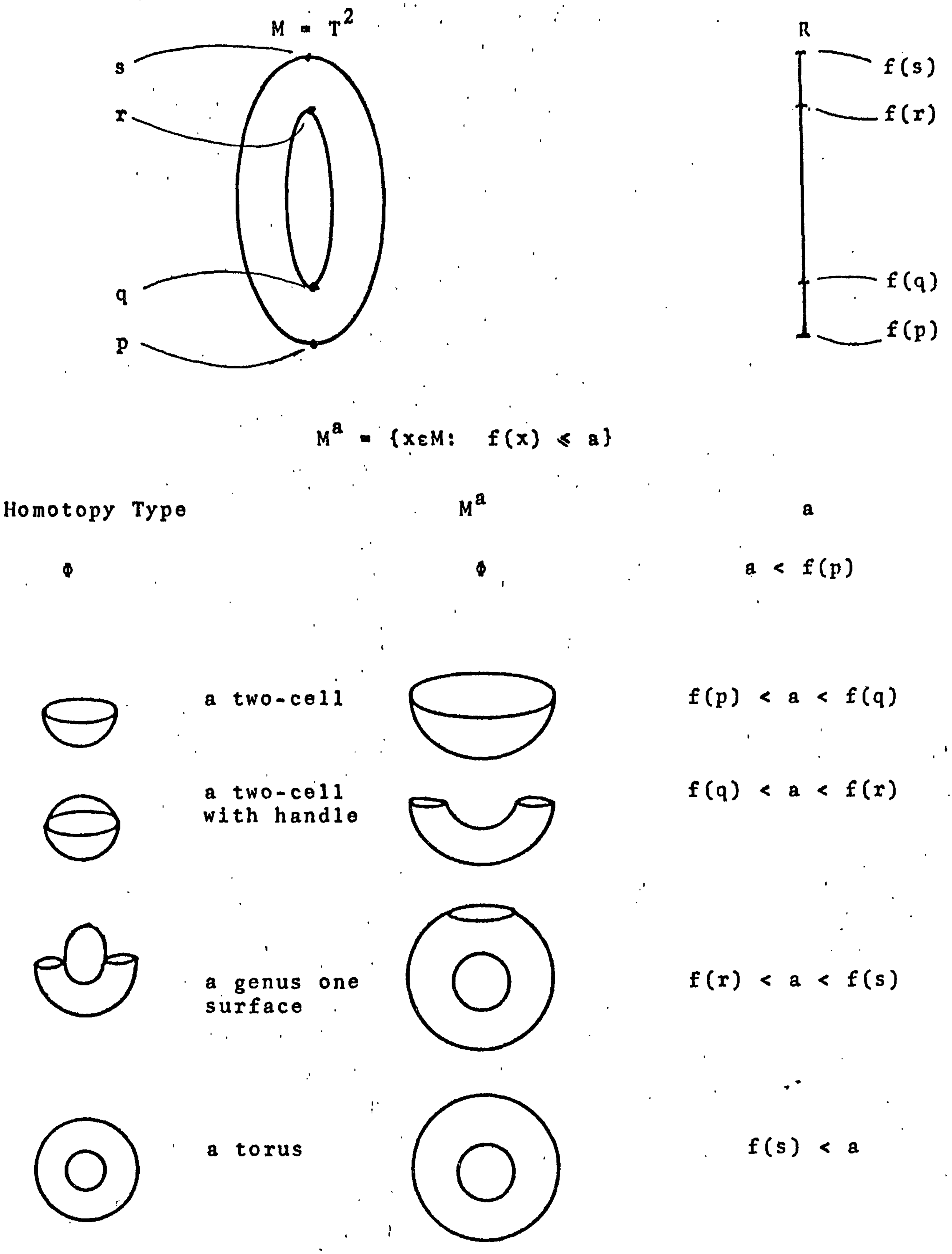

FIGURE IX. 1.

Milnor's Example Which Illustrates the Correlation Between the Critical Points of f and the Homotopy Type of M^a.

of the critical point: maxima, minima, saddlepoint. The index varies with the critical point and is also related to the algebraic topological character of the function's preimage. In short, the existence, nature and number of critical points of a function is intimately related to the algebraic topological structure of its domain of definition.

The Basic Elements of the Morse Theory

The fundamental results of the Morse theory for finite dimensional manifolds are available from a number of sources. For the convenience of the reader the pertinent definitions and theorems are set forth here. The central element of the theory is the nondegenerate critical point of a function. Its nature is characterized by the index of the function at that point.

Definition IX.5.[147] Let f be a smooth, real-valued function defined on an n-manifold M.

(a) A point $p \varepsilon M$ is a critical point of f if the tangent map

$$f_{x_p} : TM_p \longrightarrow R$$

is zero. Relative to a local coordinate system (x^i) about p, the requirement implies

$$\left.\frac{\partial f}{\partial x^1}\right|_{x(p)} = \left.\frac{\partial f}{\partial x^2}\right|_{x(p)} = \ldots = \left.\frac{\partial f}{\partial x^n}\right|_{x(p)} = 0,$$

where f is the representation of f relative to the coordinate system.

(b) A critical point p is nondegenerate if relative to any local coordinate system (x^i) about p the matrix

$$\left(\frac{\partial^2 f}{\partial x^i \partial x^j} \bigg|_{x(p)} \right)$$

is nonsingular. The matrix is the coordinate representation of the Hessian of f at p, the symmetric bilinear form on TMp given by

$$H_p = f_{**p} : TM_p \times TM_p \longrightarrow R.$$

(c) If p is a nondegenerate critical point of f, the index of f at p is defined to be the dimension of the maximal subspace of TM_p on which the Hessian of f is negative definite.

Lemma IX.2. (Morse).[148] Let p be a nondegenerate critical point of f. Then there exists a local coordinate neighborhood U of p, and a local coordinate system (y^i) such that $y(p) = 0$, and in U

$$f = f(0)-(y^1)^2-(y^2)^2-\ldots-(y^\lambda)^2+(y^{\lambda+1})^2+\ldots+(y^n)^2,$$

where λ is the index of f at p.

The critical points of f are those points at which f attains an extremal value. Notice that the extremal value is an absolute minimum for f if the index of the nondegenerate critical point is zero, and an absolute maximum if the index is $n = \dim M$. For index values between these extremes the critical point is a saddle point.[149]

The fundamental theorems relating the appearance of critical points of a function with a change in the algebraic topological structure of its preimage now follow. For $s \in R$ a possible value for the function f, define the preimage of f in M up through the value s as the set

$$M^s \equiv f^{-1}(]-\infty,s]) = \{p \in M : f(p) \leq s\}.$$

It follows straightforwardly that M^s is a closed submanifold of M, possibly empty.[150] The first fundamental theorem asserts that a necessary condition for the absence of a critical value for a function in a given interval $[a,b]$ of values is the geometric equivalence, and in fact homotopy equivalence of the two preimages M^a and M^b.

> Theorem IX.2.[151] Let f be a real valued, smooth function defined on an n-manifold. Let $a<b$ in R, and let the set $f^{-1}([ab])$ be compact and contain no critical points of f. Then
>
> (a) M^a is diffeomorphic to M^b
>
> (b) M^a is a deformation retract of M^b, and the inclusion map
>
> $$i : M^a \longrightarrow M^b$$
>
> is a homotopy equivalence.

The second theorem asserts that the previous homotopy equivalence condition is also sufficient to insure against the appearance of new critical points. In fact, it dictates how the two preimage manifolds must differ homotopically if additional critical points are to appear.

Theorem IX.3.[152] Let

$$f : M \longrightarrow R$$

be a smooth, real valued function on an n-manifold M. Let p be a nondegenerate critical point of f with index λ. Let $f(p) = c$, and that for some $\varepsilon > 0$,

$$f^{-1}([c-\varepsilon,\ c+\varepsilon])$$

is compact and contains no other critical points of f besides p. Then for sufficiently small ε, $M^{c+\varepsilon}$ with a λ cell attached (as a "handle").

The latter theorem is more commonly recognized in its original form as the Morse inequality theorem.

Theorem IX.4.[153] Let $B_i(M)$ denote the i^{th} Betti number of the manifold M:

$$B_i(M) = \dim H_i(M,R),$$

for $H_i(M,R)$ the ith homology group of M with coefficients in R, a real vector space. If C_λ denotes the number of critical points of index λ on a compact manifold M, then

$$B_\lambda(M) \leq C_\lambda \quad \text{for each} \quad \lambda,$$

$$\sum_{m=0}^{k} (-1)^{k-m} B_m(M) \leq \sum_{m=0}^{k} (-1)^{k-m} C_m,$$

and

$$\sum_{m=0}^{\infty} (-1)^m B_m(M) = \sum_{m=0}^{\infty} (-1)^m C_m.$$

These results of the Morse theory explicitly show how the existence, number, and character of the critical points of a function relate intimately to algebraic topological character of the manifold on which it is defined. Roughly speaking, the more intricate the manifold structure the larger the variety of critical points available.

The Infinite Dimensional Generalization

In order to utilize the Morse theory in the geometric models developed here one must generalize the above results to infinite dimensional manifolds. As with the Lusternik-Schnirelman theory not all functions defined over arbitrary infinite-dimensional manifolds can support a Morse theory.

In 1964 Palais and Smale showed that one could extend the Morse theory to infinite dimensional manifolds modeled on a separable Hilbert space, and which possessed a complete C^2 Riemannian structure, if the function f satisfied the condition (C) set forth in Condition IX.1. Under these conditions one achieves the following theorem on the existence of minimum points for f.

> Theorem IX.5.[154] If (M,f) satisfy condition (C) and if f is bounded below on a component M_0 of M, then f assumes its greatest lower bound on M_0.
>
> (a) If ∇f represents the gradient field of f, and for $p \varepsilon M_0$ if $\phi_t(p)$ represents the flow through p associated with ∇f, then
>
> (1) $\phi_t(p)$ is defined for all positive t and has a critical point as a limit point as $t \to \infty$.

(2) If all critical points of f are nondegenerate then in fact

$$\lim_{t\to\infty} \phi_t(p)$$

exists.

(b) If f is bounded below on all of M then f assumes its greatest lower bound on M provided the critical point set of f has no interior, and in particular, if the critical points of f are nondegenerate.

One also gains the following infinite dimensional version of the Morse inequality theorem.

Theorem IX.6.[155] Let (M,f) satisfy condition (C) and assume that the critical points of f are nondegenerate. Let $a,b\varepsilon R$ be regular values for f. Let

$$f^a = \{p\varepsilon M : f(p) \leq a\}$$

$$f^{a,b} = f^{-1}([a,b])$$

Let $R_i^{a,b}$ be the i^{th} Betti number of (f^b,f^a) with coefficients in a field and let $C_i^{a,b}$ be the number of critical points of index i in $f^{a,b}$. Then

$$\sum_{i=0}^{m} (-1)^{m-i} R_i^{a,b} \leq \sum_{i=1}^{m} (-1)^{m-i} C_i^{a,b}$$

$$R_i^{a,b} \leq C_i^{a,b}$$

$$\sum_{i=0}^{\infty} (-1)^i R_i^{a,b} = \sum_{i=0}^{\infty} (-1)^i C_i^{a,b}$$

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Corollary IX.2. Let f be bounded below on M and let R_i denote the i^{th} Betti number of M with coefficients in a field. Let C_i be the number of critical points of f having index i. Then

$$R_i \leq C_i ,$$

and if each C_i is finite, $i=0,1,2,\ldots,m$, then

$$\sum_{i=0}^{m} (-1)^{m-i} R_i \leq \sum_{i=0}^{m} (-1)^{m-i} C_i .$$

An Example Showing How One Uses the Morse Theory

One can now appreciate the value of the infinite dimensional Morse Theory for resolving infinite dimensional existence and uniqueness questions. As mentioned previously, applying the Morse theory in this manner, though relatively new, is not novel. The example provided by Smale and Palais illustrates well how one transforms an abstract nonlinear Dirichlet problem into a Morse theory question, and how one uses the topology of the solution manifold, and knowledge of some critical points to deduce information about other ones.

For the convenience of presentation the example is set forth in propositional form. The first proposition casts the abstract Dirichlet problem in a geometric setting, The second one imposes some conditions sufficient for the extension of the model to an infinite dimensional Hilbert space setting. While the conditions differ from the axiom conditions imposed by the approach used in this work, the two approaches are consistent. The third proposition models the Dirichlet boundary condition solution space as a complete Riemannian manifold. Finally, the fourth proposition suggests conditions which insure the satisfaction of condition (C). Consequently, the abstract Dirichlet problem is transformed into a Morse theory question. For details about a Morse theory question. The references are indicated if one wishes greater details about the example.

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Proposition IX.2. Let M be a compact, smooth differentiable manifold, let ξ be a finite dimensional vector bundle over M, and let be a smooth measure on M. Let

$$F: J^r(\xi) \longrightarrow R_M$$

be a smooth mapping. Define the integral functi

$$\begin{aligned} J: C^r(\xi) &\longrightarrow R \\ f & \int_M F(j_k f)\,d\mu. \end{aligned}$$

If $r \geq k$, then J is a C^∞ function on $C^r(\xi)$.

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Proposition IX.3. Let

$$F: J^k(\xi) \longrightarrow R_M$$

satisfy the following growth conditions; relative to a local coordinate system (x,p) on $J^k(\xi)$

$$F(x,p) \leq C_1 \; ||p||^2 + C_2$$

$$F_{pp}(x,p)(\beta,\beta) \leq C_3 \; ||\beta||^2,$$

$$\beta \varepsilon E \times L(V,E) \times \ldots \times L_s^k(V,E)$$

Then the C^∞ function J on $C^k(\xi)$ extends to a C^∞ function on $H^k(\xi)$.

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Proposition IX.4. For $f_o \varepsilon C^k(\xi)$, let $C_o^k(\xi)$ be the affine subspace of maps $f \varepsilon C^k(\xi)$ such that

$$j_{k-1}f = j_{k-1}f_o$$

on ∂M. Let $H_o^k(\xi)$ denote the closure of $C_o^k(\xi)$ in $H^k(\xi)$. Then $H_o^k(\xi)$ is a complete Riemannian manifold.

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Proposition IX.5. If in addition, on each local coordinate system (x,p) in $J^k(\xi)$, F satisfies the following conditions

$$C_4||p^k||^2 - C_5 \leq \int_U F(x,p)dx, \quad p^k \varepsilon L_s^k(V,E)$$

$$C_6||\beta||^2 \leq F_{p^k p^k}(x,p) \; (\beta,\beta),$$

then the restricted function

$$J_o = J|_{H_o^i(\xi)} \; : H_o^k(\xi) \longrightarrow R$$

satisfies condition (C). Hence, J_o has a minimum on $H_o^k(\xi)$, and if the critical points are non-degenerate, the Morse theory is valid on $(H_o^k(\xi), J_o)$.

How does the Morse theory dictate the number and nature of the critical points? One invokes one's knowledge of the topological character of the solution space $H_o^k(\xi)$ and the Morse inequalities to resolve these questions. To illustrate, two observations by Palais and Smale follow.

Observation IX.1.[161] Let Proposition IX.5. hold. Then J_o has at least one critical point of index zero, and thus, J_o attains a minimum on $H_o^k(\xi)$

Proof:

(1) $H_o^k(\xi)$ is a real topological vector space, hence contractable.

(2) From classic algebraic topological results $H_o^k(\xi)$ contractable implies $R_i(H_o^k(\xi))=\begin{cases}1 & i=0\\ 0 & i\neq 0\end{cases}$

(3) If C_o is the number of critical points of index zero, Corollary IX.2 implies

$$1 \leq C_o.$$

(4) Finally, a critical point of index zero is a minimum.

Observation IX.2.[162] If J_o admits two local non-degenerate minima on $H_o^k(\xi)$, then J_o admits at least one other critical point.

Proof:

(1) Let $C_o = 2$. By Observation IX.1, and Corollayy IX.2,

$$-1 \leq -C_o + C_1$$

or

$$C_1 \geq 1.$$

Thus, there exists at least one critical point of index one.

The example thus illustrates how intimately the number and variety of critical points depends upon the topological nature of the solution space as well as the conditions placed upon the integrand.

The Morse Theory and the Elastostatic Models

How the global conclusions drawn from the elastostatic models presented in this work can differ significantly from conclusions drawn from previous models becomes even more apparent in the light of the previous example. When the algebraic topological character of the solution space is trivial, as is the case with previous elastostatic models, and in the previous example, the information about the critical points available from the Morse inequalities is quite limited. However, as the Betti numbers of the solution space become nontrivial the number and variety of critical points enlarge, and the information available from the Morse inequalities increases. Palais and Smale anticipate this possibility in their remark,

> Presumably the theorem of this section extends to subbundles η of ξ under suitable conditions of F and 2k>dim M. Then usually the homology of $H_0^k(\eta)$ will be highly nontrivial and the existence theory will imply much more.[163]

As indicated in chapter four, the configuration manifolds for the finite elastostatic problem are, roughly speaking, sections of a fiber subbundle of a vector bundle. Indeed, they have a highly nontrivial algebraic topological character, in general. Moreover, this character may vary with the boundary condition, and can be altered if the topology of the experimental specimen is altered. Hence if the Morse theory can be applied to the elastostatic model for a hyperelastic material body one may expect conclusions for the global nonuniqueness of equilibrating configurations which would vary from boundary condition to boundary condition, and with the topology of the specimen. Such results stand in contrast to conclusions one can draw using the Morse theory on previous models, as the Dirichlet boundary manifolds are all alike topologically.

OUTSTANDING QUESTIONS ON THE APPLICATION OF THE CRITICAL POINT THEORIES

Several outstanding questions now become quite pertinent for the application of the critical point theory to the models presented here. Some appear immediately resolvable. Their resolution will provide concrete examples of the conclusions anticipated in the previous sections.

What is the Cohomology of the Configuration Manifolds Presented Here? In Particular, What are their Betti Numbers?

The application of the Lusternik-Schnirelman theory to elastostatics can provide information about the global uniqueness problem when (1) the number of components of the Dirichlet configuration manifold is greater than one, and (2) when some component has an (L-S) category greater than one. Proposition IX.1 indicates that knowledge of the cohomology of the configuration manifold permits one to determine lower bounds on the (L-S) category of its components. Moreover, the zeroeth Betti number counts the number of components of the manifold.

At present, the zeroeth Betti number appears obtainable for the Dirichlet configuration manifold associated with particular specimen topologies. In chapter four one observed how the obstruction theory could be used to determine the number of components of the configuration manifold. According to the theory the number depends upon the elements of the cohomology classes of the material body, relative to its boundary

$$\{H^*(B,\partial B)\}.$$

Two avenues of investigation follow from this question and appear quite promising. First of all, the relative cohomology classes are available for some specimen shapes which are physically important: the solid ball, the solid finite cylinder with a hole, the infinitely long solid cylinder with a hole, and the ball with a cavity. If the conclusions for the zeroeth Betti number drawn from the obstruction theory differ among any of these cases, then the application of either the Lusternik-Schnirelman theory or the Morse theory will yield a concrete instance where alteration of the specimen topology alters the number and nature of equilibrating configurations.

Secondly, one may probe deeper into the global nonuniqueness question and the possibility of interior buckling. Cohomology and Homotopy information about the Dirichlet configuration manifolds is most helpful in answering the following questions. Are there two or more equilibrating configurations for the given boundary condition? Can one deform from one to the other without violating the boundary condition? If the (L-S) category of any component is greater than one, the answer is immediate. Alternatively, one may rephrase the second question as, "Are the two equilibrating configurations homotopically distinct extensions of the boundary conditions?", and apply the obstruction theory. Once again, the cohomology groups of the material body, relative to its boundary, governs the answer. The question may be considered for the particular specimen topologies mentioned above.

How Does a Functional Analysis Condition like Condition (C) Interface with the Physical Theory of Elastostatics?

The condition (C) of Palais and Smale is sufficient to insure the application of the critical point theories mentioned here. Is it a physically meaningful condition to impose? What alternatives are available? In 1968 Palais indicated analytical conditions which would guarantee that an integral function would be bounded below and satisfy Condition (C).[164] The form of the condition sufficiently parallels the generalized Coleman-Noll condition and, to a lesser extent, Beju's conditions, as to warrant a thorough investigation. Conditions like GCN heretofore have been difficult to comprehend because their exact purpose is somewhat vague. Perhaps they might be more advantageously viewed in the geometric setting as opposed to the analytical one. For example, one might find that conditions motivated by a desire to insure global uniqueness, or even stability might be too strong. Condition (C) reflects this possibility in that it permits many critical points (nonuniqueness) of higher and higher index (stability).

In the Elastodynamic Model, Does the Index of the Critical Point Provide Information About the Stability of Equilibrating Configurations?

One may extend the geometric model for nondissipative, hyperelastic material bodies experiencing conservative body forces to an elastodynamic model. J. Marsden first suggested the possibility in 1970,[165] although, once again, the configuration manifold was a topological vector space. The basic feature of the model is that the evolution of the body is portrayed by a flow on the configuration manifold which is governed by a set of Hamilton's equations derivable from the action integral function. Under the equivalence of the Hamiltonian and Lagrangean representations for this situation the equilibrium points of the Hamiltonian flow are the critical points of the gradient of the action integral function. Hence, the elastostatic configurations are the equilibrium points for the elastodynamic problem for this case. One may now ask, are these equilibrium points stable, unstable, or saddlepoint stable?

Until the mid 1960's no effective way was available for characterizing the stability of equilibrium points of a flow beyond Poincare's theory of separatrices for two-dimensional flows. In the middle 1960's Poincare's concept of separatrix was generalized to larger finite-dimensional dynamic flows through the introduction of the stable manifold, the unstable manifold, and the center manifold.[166] Several stability theorems on the dimension of these manifolds followed.[167] A current subject of investigation in Topological Dynamics is the extension of these theorems to infinite dimensional flows.[168]

If the equilibrium points of an elastodynamic flow are nondegenerate, can the index of the critical points provide information about their stability? The index of a critical point carries information about its stability. For example, a configuration minimizing the action integral function has index zero. A higher order index indicates that the critical point is a minimum point relative to variations in some directions, but a maximum relative to variations in other directions. One may conjecture that the index carries information about the dimension of the stable, unstable, and center manifolds at the equilibrium point. Conversely, perhaps information about these manifolds would permit one to deduce the index of the critical point.

What Technical Difficulties Does One Encounter in Applying the Critical Point Theories?: Possible Future Models

Attractive as the possible consequences of the critical point theories are, one must first carefully establish when they may legitimately be applied. One difficulty which arises is when are the critical points of the action integral function nondegenerate and when are they degenerate?

Melvyn Burger indicates situations where the critical points of a continuum mechanical model built from Elastica theory exhibits degenerate critical points.[169] For example, a locally nonunique equilibrating configuration, or an equilibrating configuration at which a bifurcation occurs relative to some parameter would be degenerate critical points. For this situation, the Morse theory is inapplicable, while the Lusternik-Schnirelman theory may hold.

The principal difficulty with the application of the critical point theories is the requirement that the configuration spaces be complete Riemannian manifolds. The H^k generalized configuration spaces are Riemannian manifolds; however, their completeness is not immediate. They are not closed submanifolds in $H^k(\eta)$. Roughly speaking, an evolving elastodynamic system can "run off the manifold" by kinking, tearing or collapsing. To render the configuration manifolds geodesically complete, one must augment them. Perhaps the augmentation requires simply the incorporation of the boundaries of the manifolds; perhaps more is required. Is the algebraic topological character of the manifolds altered radically by the augmentation? Since the augmenting elements would not correspond to configurations as have been previously defined, what can one say about them? Such questions point towards geometric models for continuum mechanical systems which would be more encompassing than the ones presented here.

A CONCLUDING REMARK

In summary, the chapter illustrates that one can gain global information from the geometric models presented here, and how the nature of the information and the manner in which one acquires it can differ significantly from previous models.

In concluding the work, one must remark on the value of the approach taken here for other physical theories. The models which were constructed and the mathematical techniques which were considered can equally well be attempted for any other nonlinear classical field theory. What makes finite elastostatics particularly attractive is the preciseness and fidelity of its mathematical model. One knows exactly which mathematical functions are configurations and which are not. It is precisely those functions which are excluded from the set of configurations which give the manifold its topological richness. At present, few other field theories can boast of such a well defined mathematical model.

Moreover, if one were to adopt the spirit of the approach used here to other nonlinear field theories, the results could be most beneficial. The days when "nonlinear" meant "not linear" are gone. One has advanced to questions and phenomena which do not follow from "suitable linearizations". One has progressed to systems where a solution is the last gem, given reluctantly, from a treasurehouse of information. If the specifics of the work presented here are not immediately relevant to the reader, perhaps the promise its spirit holds, and the new directions of inquiry it shows plausible will prove inspiring.

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APPENDIX IV.1

THE VECTOR BUNDLE STRUCTURE OF π_η

In this appendix one demonstrates that π_η satisfies the mathematical criteria for a vector bundle. In addition, one shows that the choice of a vector bundle chart corresponds to the choice of a neighborhood reference configuration. One identifies sections of the bundle with configurations of the body in physical space. Finally, one relates the representation of a section with respect to a vector bundle chart to the representation of a given configuration as a relative deformation from a reference configuration.

From Abraham and Robin, Transversal Mappings and Flows, Chapter I, a triple (S, B, π) must satisfy the following criteria in order to be a C^k vector bundle.

(1) S and B are C^k manifolds,

(2) $\pi: S \to B$ is an open, C^k, surjective map.

(3) For $p \varepsilon B$, the fiber over p is in bijective correspondence to a Banach space E:

$$\pi^{-1}(p) \approx E.$$

(4) For (α_o, U) a chart on B, there is a mapping

$$\alpha : \pi^{-1}(U) \longrightarrow \alpha_o(U) X E$$

(a) one-to-one, and C^k,

(b) its inverse is C^k,

(c) when restricted to a fiber,

$$\alpha|_p : \pi^{-1}(p) \longrightarrow \alpha_o(p) X E,$$

it is linear,

(d) The diagram

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\alpha} & \alpha_o(U) \times E \\ \pi \downarrow & & \downarrow \\ U & \xrightarrow{\alpha_o} & \alpha_o(U) \end{array}$$

commutes.

(α, α_o, U) is called a bundle chart for π.

(5) If $(\alpha, \alpha_o\ V)$ is a second bundle chart for π, and $p\ U^o V$, the coordinate descriptions are related by a transition function $\Psi_{\beta\alpha}$ given by

$$\beta \cdot \alpha^{-1}: \alpha_o(U \cap V) \times E \longrightarrow \beta_o(U\ V) \times E$$

$$(x,\ e) \qquad (\beta_o \alpha_o^{-1}(x),\ [\Psi_{\beta\alpha}(x)]e).$$

One requires that, for each x, $\Psi_{\beta\alpha}(x)$ be linear,

$$\Psi_{\beta\alpha}: \alpha_o(U \cap V) \longrightarrow L(E,E),$$

and $\Psi_{\beta\alpha}$ be C^k in x.

For the triple $(B \times R^3,\ B, \pi_r)$ specified on pages 128-129, one has a natural way of constructing bundle charts. If B is a C^k material body, for each point p of B there is a neighborhood U of p and at least one C^k configuration α_o of U in R^3,

$$\alpha_o : U \longrightarrow \alpha_o(U) \subset R^3.$$

The triple $(\alpha_o \times 1, \alpha_o, U)$, specified by the diagram

$$\begin{tikzcd}
\underset{(p,r)}{\pi_\eta^{-1}(U)} \arrow[r, "\alpha_o X1"] \arrow[d] & \underset{(\alpha_o(p),r)}{\alpha_o(U) \times R^3} \arrow[d, no head] \\
\underset{p}{U} \arrow[r] & \underset{\alpha_o(p)}{\alpha_o(U)}
\end{tikzcd}$$

satisfies the criterion for a bundle chart at p. Classically speaking, α_o corresponds to a reference configuration for U in R^3.

If β_o is a second C^k configuration of U in R^3, $(\beta_o \times 1, \beta_o, U)$ is also a bundle chart about p. The transition function relating the two coordinate descriptions is quite simple:

$$[(\beta_o X1)\cdot(\alpha_o X1)^{-1}](x,r) = (\beta_o \alpha_o^{-1}(x), r),$$

or

$$\Psi_{\beta\alpha}(x) = 1_{R^3}$$

for $x \varepsilon \alpha_o(U)$. The transition function satisfies the last criterion. The collection of all such bundle charts constitutes a C^k atlas for π_η.

A section of π_η is a mapping

$$s: B \longrightarrow \eta$$

such that

$$\pi_\eta \cdot s = 1_B.$$

Such a mapping may be written as

$$\begin{aligned} s : B &\longrightarrow B \times R^3 \\ p &\quad (p, s(p)). \end{aligned}$$

One may thereby identify configurations s of B in R^3 with sections of π_η. Not all sections of π_η correspond to configurations, however.

If $(\alpha_o \times 1, \alpha_o, U)$ is a bundle chart, onemmay represent the section s relative to the chart as

$$\begin{aligned} p_\alpha s : \alpha_o(U) &\longrightarrow R^3 \\ x &\quad s \cdot \alpha_o^{-1}(x), \end{aligned}$$

by

$$\begin{aligned} (\alpha_o \times 1) \cdot s \cdot \alpha_o^{-1} : \alpha_o(U) &\longrightarrow \alpha_o(U) \times R^3 \\ x &\quad (x, p_\alpha s(x)) \end{aligned}$$

$p_\alpha s$ is called the principal part of the section relative to the bundle chart. Classically speaking, it represents the configuration s as the relative deformation $p_\alpha s$ from the reference configuration $\alpha_o(U)$.

APPENDIX IV.2

$\mathrm{Inj}^k(\eta)$ IS NOT AN OPEN SET IN $C^k(\eta)$.

In order to establish the proposition for $\eta = B \times R^3$, it suffices to show that the set of all C^k injections of B into R^3 is not open in the set of all C^k maps of B into R^3. For convenience, choose a body B and a reference configuration for which the body appears as a cube with coordinates (X^1, X^2, X^3) $-1 \leq X^i \leq +1$, $i=1,2,3$. Consider a relative deformation

$$\lambda : (X^1, X^2, X^3) \longrightarrow ((X^1)^3, X^2, X^3).$$

The configuration it represents is injective. The sequence of deformations

$$\lambda_\alpha : (X^1, X^2, X^3) \longrightarrow (\alpha(X^1)^3 + (1-\alpha)X^1, X^2, X^3),$$

$0 \leq \alpha \leq 1$, require that some portion of the body must collapse upon itself. The C^k distance is given by

$$||\lambda_\alpha - \lambda||_{C^k} = \sup_{-1 \leq X^i \leq 1} ||(\lambda_\alpha - \lambda)(X^1, X^2, X^3)|| + \sup_{-1 \leq X^i \leq 1} ||D(\lambda_\alpha - \lambda)(X^1, X^2, X^3)|| +$$

$$\ldots + \sup_{-1 \leq X^i \leq 1} ||D^k(\lambda_\alpha - \lambda)(X^1, X^2, X^3)|| .$$

When one evaluates the sum one finds that given $\varepsilon > 0$ there is at least one λ_α in a ball of radius ε about λ. Thus, one cannot find a neighborhood of λ which does not contain at least one non-injective map. Consequently, the set of C^k injective mapping is not open in the set of all C^k mappings.

APPENDIX IV.3

A PROOF OF THEOREM IV.4

Theorem IV.4 follows as a consequence of the dependence of the degree of a mapping upon the boundary. In Schwartz Nonlinear Functional Analysis, p 72, one finds the following property for a degree.

Dependence only on boundary value: if

$$\phi|_{\partial D} = \Psi|_{\partial D}, \text{ and } p \notin \phi(\partial D) = \Psi(\partial D), \text{ then}$$

$$\deg(p,\phi,D) = \deg(p,\Psi,D).$$

If one takes Ψ in Theorem IV.4 as a reference configuration, $D = \Psi(B)$, and $\partial D = \Psi(\partial B)$, then by Appendix IV.1, ϕ may be viewed as a relative deformation

$$\begin{aligned} \phi_\Psi : D &\longrightarrow R^3 \\ x &\quad\ \ \phi\cdot\Psi^{-1}(x) \end{aligned}$$

Obviously

$$\Psi_\Psi = 1_D$$

Since

$$\phi_\Psi|_{\partial D} = \Psi_\Psi|_{\partial D}$$

for any point $x \in D$, $x \notin D$,

$$\deg(x, \phi_\Psi, D) = \deg(x, \Psi_\Psi, D) = 1$$

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