GLOBAL EXISTENCE OF SOLUTIONS TO REACTION-DIFFUSION SYSTEMS WITH MASS TRANSPORT TYPE BOUNDARY CONDITIONS

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Global Existence of Solutions to Reaction-Diffusion Systems with Mass Transport Type Boundary Conditions.

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ABSTRACT

We consider coupled reaction-diffusion models, where some components react and diffuse on the boundary of a region, while other components diffuse in the interior and react with those on the boundary through mass transport. We proved if vector fields are locally Lipschitz functions and satisfy quasi-positivity conditions, and if initial data are component-wise bounded and non-negative then there exists $T_{\max} > 0$ such that our model has component-wise non-negative solution with $T = T_{\max}$. Our criterion for determining local existence of the solution involves derivation of a priori estimates, as well as regularity of the solution, and the use of a fixed point theorem. Moreover, if vector fields satisfy certain conditions outlined in the dissertation, then there exists solution for all time $t > 0$. Classical potential theory and estimates for linear initial boundary value problems are used to prove local well-posedness and global existence. This type of system arises in mathematical models for cell processes.

Contents

Chapter 1

Introduction and Motivation

Systems of semilinear parabolic partial differential equations result from a general conservation law which states that the rate of change of the amount of constituents in a domain is equal to the rate of flow of the constituents across the boundary into the domain, plus the amount of the constituent created in the domain. They arise naturally in the modeling of a variety of biological and chemical processes. In those settings, they are also referred to as reaction-diffusion systems. The idea that reaction-diffusion phenomena is essential to the growth of living organisms seems quite intuitive. Indeed, it would be rather hard to envision how any organism could grow and operate without moving its constituents around and using them in various bio-chemical reactions [23]. For example, bacterial cytokinesis is one of the processes which involves reaction-diffusion systems. During the bacterial cytokinesis process, a proteinaceous contractile ring assembles in the middle of the cell. The ring tethers to the membrane and contracts to form daughter cells; that is, the "cell divides". One mechanism to center the ring involves the pole-to-pole oscillation of proteins Min C, Min D, and Min E. Oscillations cause the average concentration of Min C, an inhibitor of the ring assembly, to be lowest at the midcell and highest near the poles [36]. This centering mechanism, relating molecular-level interactions to supra-molecular ring positioning (the interactions between spatial oscillation of Min proteins and FtsZ reactions) can be modelled as a system of semilinear parabolic equations. The model is developed within the context of a cylindrical cell consisting of 2 subsystems; one involving Min oscillations and the other involving FtsZ reactions. The Min subsystem consists of ATP-bound cytosolic MinD(D_{cyt}^{ATP}), ADP-bound cytosolic MinD(D_{cyt}^{ADP}), membrane-bound MinD(D_{mem}^{ATP}), cytosolic MinE(E), and membrane bound MinD:MinE complex($E : D_{mem}^{ATP}$). These Min proteins react with certain reaction rates that are illustrated in Table 1.1. Our study involves analysis of the system involving the Min proteins. The evolution of the Min concentrations is described by a reaction-diffusion system of the form

$$\begin{aligned}
u_t &= D\Delta u + H(u) && x \in \Omega, \quad 0 < t < T \\
v_t &= \tilde{D}\Delta_M v + F(u,v) && x \in M, \quad 0 < t < T \\
D\frac{\partial u}{\partial \eta} &= G(u,v) && x \in M, \quad 0 < t < T \\
u &= u_0 && x \in \Omega, \quad t = 0 \\
v &= v_0 && x \in M, \quad t = 0
\end{aligned} \tag{1.0.1}$$

where Ω is a bounded domain in $\mathbb{R}^3$ with smooth boundary M, Δ and Δ_M denote the Laplace and Laplace Beltrami operators, η is the unit outward normal vector to Ω at points on M,

Table 1.1: Reactions and Reaction Rates

Chemicals	Reactions	Reaction Rates
Min D	$D_{cyt}^{ADP} \xrightarrow{k_{exc}} D_{cyt}^{ATP}$	$R_{exc} = k_{exc}[D_{cyc}^{ADP}]$
Min D	$D_{cyt}^{ATP} \xrightarrow{k_{Dcyt}} D_{mem}^{ATP}$	$R_{Dcyt} = k_{Dcyt}[D_{cyc}^{ATP}]$
	$D_{cyt}^{ATP} \xrightarrow{k_{Dmem}[D_{mem}^{ATP}]} D_{mem}^{ATP}$	$R_{Dmem} = k_{Dmem}[D_{mem}^{ATP}][D_{cyc}^{ATP}]$
Min E	$E + D_{mem}^{ATP} \xrightarrow{k_{Ecyt}} E : D_{mem}^{ATP}$	$R_{Ecyt} = k_{Ecyt}[E][D_{mem}^{ATP}]$
	$E + D_{mem}^{ATP} \xrightarrow{k_{Emem}[E:D_{mem}^{ATP}]^2} E : D_{mem}^{ATP}$	$R_{Emem} = k_{Emem}[D_{mem}^{ATP}][E][E : D_{mem}^{ATP}]^2$
Min E	$E : D_{mem}^{ATP} \xrightarrow{k_{exp}} E + D_{cyt}^{ADP}$	$R_{exp} = k_{exp}[E : D_{mem}^{ATP}]$

$$u = \begin{pmatrix} [D_{cyt}^{ATP}] \\ [D_{cyt}^{ADP}] \\ [E_{cyt}] \end{pmatrix}, v = \begin{pmatrix} [D_{mem}^{ATP}] \\ [E : D_{mem}^{ATP}] \end{pmatrix}$$

$$\tilde{D} = \begin{pmatrix} \sigma_{Dmem} & 0 \\ 0 & \sigma_{E:Dmem} \end{pmatrix}, \quad D = \begin{pmatrix} \sigma_{Dcyt} & 0 & 0 \\ 0 & \sigma_{ADyct} & 0 \\ 0 & 0 & \sigma_{Ecyt} \end{pmatrix}$$

$$F = \begin{pmatrix} R_{Dcyt} + R_{Dmem} - R_{Ecyt} - R_{Emem} \\ -R_{exp} + R_{Ecyt} + R_{Emem} \end{pmatrix}, \ G = \begin{pmatrix} -R_{Dcyt} - R_{Dmem} \\ R_{exp} \\ R_{exp} - R_{Ecyt} - R_{Emem} \end{pmatrix},$$

$$H = \begin{pmatrix} R_{exc} \\ -R_{exc} \\ 0 \end{pmatrix},$$

and expressions of the form K_α and σ_β are positive constants. In (1.0.1), Ω represents the cell and M represents its membrane. There are some components that are bound to a membrane, and other components that move freely in the cytoplasm. Also, the components on the membrane and cytoplasm react together on the boundary.

System (1.0.1) is somewhat similar to two-component systems where both of the unknowns react and diffuse inside Ω, with various homogeneous boundary conditions and nonnegative initial data. In that setting, global well-possedness and uniform boundedness has been studied by many researchers Alikakos [3], Masuda [29], Hollis Martin Pierre [19], and many others [33].
Consider

$$\begin{aligned}
u_t &= d\Delta u + F(u,v) && x \in \Omega, \quad 0 < t < T \\
v_t &= \tilde{d}\Delta v + G(u,v) && x \in \Omega, \quad 0 < t < T \\
d\frac{\partial u}{\partial \eta} &= \tilde{d}\frac{\partial v}{\partial \eta} = 0 && x \in M, \quad 0 < t < T \\
u &= u_0 && x \in \Omega, \quad t = 0 \\
v &= v_0 && x \in \Omega, \quad t = 0
\end{aligned} \tag{1.0.2}$$

For $d, \tilde{d} > 0$, and a vector field having the form

$$F(u,v) = -uh(v)$$
$$G(u,v) = uh(v)$$

where h is a given non-negative function, and u and v satisfy various homogeneous boundary conditions, Alikakos [3] proved that solutions of (1.0.2) exist globaly provided that $h(z) \leq Kz^{\frac{n+2}{n}-\epsilon} + L$ for some non-negative constants K and L, and Masuda [29] proved global existence and uniform boundedness on $\Omega \times (0,\infty)$ for $h(z) \leq Kz^r + L$ with r arbitrary, as well as convergence as $t \to \infty$. Haraux and Youkana [15] were able to obtain the result for $h(z) \leq Ke^{\sqrt{z}} + L$, and similar results were obtained in [5] and [18].

In [19], Hollis, Martin and Pierre proved global existence and uniform boundedness of solution for a class of reaction-diffusion systems involving two unknowns. They assumed that the unknown u is uniformly bounded,

1. There exists $\gamma \geq 1$, $L_0 \geq 0$ such that $|G(\zeta,\mu)| \leq L_0(1+\mu+\zeta)^\gamma$ for all $\zeta, \mu \geq 0$.

2. There exists $\delta_0 \geq 0$ such that $F(\zeta,\mu)+G(\zeta,\mu) \leq \delta_0(1+\mu+\zeta)$ for all $\zeta, \mu \geq 0$.

Then they employed duality arguments to obtain L_p estimates for v for arbitrary $p \in (1,\infty)$, which together with polynomial growth in the reactions term implies an L_∞ bound for v. Later, Hollis and Morgan [20] showed that if blow-up occurs, then this blow up must occur for both components at the same point in $\overline{\Omega}$. Morgan [30],[31] extended the results of Hollis *et al.* to handle arbitrary m component systems under the assumption of quasipositivity, polynomial growth of reaction terms, and an "intermediate sums" condition. Pierre [33] gives an excellent survey of these results.

This gives rise to a fundamental mathematical question concerning global existence for (1.0.1). Namely, what conditions on F and G will guarantee that (1.0.1) has global solutions, and how are these conditions related to the results listed in Pierre [33] . The focus of this dissertation is to give a partial answer to this question. We observed that in order to obtain global well posedness of the solution to system (1.0.1), it was sufficient to analyze the system

$$\begin{aligned}
u_t &= d\Delta u && x \in \Omega, \quad 0 < t < T \\
v_t &= \tilde{d}\Delta_M v + F(u,v) && x \in M, \quad 0 < t < T \\
d\frac{\partial u}{\partial \eta} &= G(u,v) && x \in M, \quad 0 < t < T \\
u &= u_0 && x \in \Omega, \quad t = 0 \\
v &= v_0 && x \in M, \quad t = 0
\end{aligned} \tag{1.0.3}$$

where $d, \tilde{d} > 0$, F, $G : \mathbb{R}^2 \to \mathbb{R}$, and u_0 and v_0 are smooth functions that satisfy the compatibility condition

$$d\frac{\partial u_0}{\partial \eta} = G(u_0, v_0) \quad \text{on } M.$$

From a physical standpoint, it is natural to ask under what conditions the solutions of (1.0.3) are non-negative, and the total mass is either conserved or reduced. It is also important to ask whether these conditions arise in the problems similar to the above mentioned cell biology system. Conditions that are similar in spirit to Lightbourne [28] and Hollis, Martin, and Pierre [19], [33] result in non-negative solutions for system (1.0.3). More precisely, we show (1.0.3) has non-negative solutions for all choices of non-negative initial data u_0 and v_0 if and only if F and G are quasi-positive. That is, $F(\nu, 0) \geq 0$ and $G(0, \nu) \geq 0$ whenever $\nu \in [0, \infty)$. Conservation or reduction of mass translates to requiring

$$\int_\Omega u(x,t)dx + \int_M v(\zeta,t)d\zeta \leq \int_\Omega u(x,0)dx + \int_M v(\zeta,0)d\zeta \tag{1.0.4}$$

for all $t \geq 0$. Integration leads to

$$\begin{aligned} \frac{d}{dt}\left(\int_\Omega u + \int_M v\right) &= d\int_\Omega \Delta u + \tilde{d}\int_M \Delta v + \int_M F(u,v) \\ &= \int_M (G(u,v) + F(u,v)) \end{aligned}$$

Clearly, for (1.0.4) to hold, we need

$$\int_0^T \int_M (G(u,v) + F(u,v)) \leq 0$$

Therefore we expect to need an assumption such as

$$G(u,v) + F(u,v) \leq 0 \quad \forall u, v \in \mathbb{R}_+$$

More generally, for (1.0.3) we assume there exists $\alpha > 0$ such that

$$F(\zeta,\nu) + G(\zeta,\nu) \leq \alpha(\zeta + \nu + 1) \quad \text{ for all } \quad \nu \geq 0, \zeta \geq 0 \tag{1.0.5}$$

Assumption (1.0.5) and Lemma 6.1.12 (which is proved in Chapter 6), generalizes mass conservation by implying that total mass, $\int_\Omega u(x,t)\ dx + \int_M v(\zeta,t)\ d\zeta$, grows at most exponentially in time t.

Now we return to our goal of proving global wellpossedness of solutions to (1.0.3). The natural conditions, quasipositivity and conservation of mass, are not sufficient to obtain global existence in (1.0.2) (see [34]), and we suspect the same will be true for (1.0.3). In order to show solutions do not blow up in finite time, we need stronger conditions. First we obtain L_p bounds for all $p > 1$ and then obtain L_∞ bounds. To obtain L_p bounds for all $1 < p < \infty$, we impose a condition similar to Morgan's intermediate sums [30] and [31]. Namely, that there exists $K_g > 0$ such that

$$G(\zeta,\nu) \leq K_g(\zeta+\nu+1) \quad \text{for all} \quad \nu \geq 0,\ \zeta \geq 0 \tag{1.0.6}$$

Our technique of obtaining L_p estimates of the solution is similar in spirit to that of Morgan [30], but in the setting of the structure of the system (1.0.3). To obtain L_∞ bounds, we adopt a natural assumption of polynomial growth, which has been considered in the context of chemical and biological modeling (see Horn and Jackson [21]). That is there exists $l \in \mathbb{N}$ and $K_f > 0$ such that

$$F(\zeta,\nu) \leq K_f(\zeta+\nu+1)^l \quad \text{for all} \quad \nu \geq 0,\ \zeta \geq 0$$

We also verify and make use of a remark of Brown [6], that if the Neumann data

γ lies in $L_p(M)$ for $p > n+1$, then the solution to

$$
\begin{aligned}
\varphi_t &= d\Delta\varphi && x \in \Omega, \quad 0 < t < T \\
d\frac{\partial \varphi}{\partial \eta} &= \gamma && x \in M, \quad 0 < t < T \\
\varphi &= 0 && x \in \Omega, \quad t = 0
\end{aligned}
\tag{1.0.7}
$$

is Hölder continuous on $\overline{\Omega} \times (0,T)$. We provide the proof of this result in chapter 4 for completeness of our arguments. As a result, L_p estimates for $1 < p < \infty$ imply L_∞ estimates for solutions of (1.0.3).

Our criterion for determining local existence of the solution to (1.0.3) involves derivation of a priori estimates, as well as regularity of the solution, and the use of a fixed point theorem. In our case, given a Cauchy problem on the manifold M,

$$
\begin{aligned}
\Psi_t &= \tilde{d}\Delta_M \Psi + f && (\xi, t) \in M \times (0,T) \\
\Psi\big|_{t=0} &= \Psi_0 && \xi \in M
\end{aligned}
\tag{1.0.8}
$$

we prove the following result.

Theorem 1.0.1. *If* $1 < p < \infty$ *and* $T > 0$*, then there exists* $\hat{C}_{p,T} > 0$ *such that whenever* $\Psi_0 \in W_p^{2-\frac{2}{p}}(M)$ *and* $f \in L_p(M_T)$*, there exists a unique solution* Ψ*, solving* (1.0.8)*, and*

$$
\|\Psi\|_{p,M_T}^{(2)} \leq \hat{C}_{p,T}(\|f\|_{p,M_T} + \|\Psi_0\|_{p,M}^{(2-\frac{2}{p})})
$$

Tracing through the work in this direction, we found that Polden in [35] and [22] and J.J Sharples [24] showed there exists a unique weak solution in $W_2^{2,1}(M\times(0,T))$ of (1.0.8), provided the initial data belongs to $W_2^1(M)$ and $f \in L_2(M \times (0,T))$.

Note that the results of Amann [4] can be used to guarantee the local well posedness of (1.0.3) subject to appropriate conditions on initial data and the functions F and G. But semigroup theory does not provide the explicit estimates that are needed in our this setting. Moreover, the semigroup approach (see [4], [32]) requires the features of fractional spaces and their intermediate spaces which make the analysis much harder. Our approach keeps the analysis on comparetively simpler L_p spaces.

We note that many results obtained in the non-manifold setting also hold in our setting. However, there is a difference. For example, our results do not apply to the system

$$\begin{aligned}
u_t &= \Delta u && x \in \Omega, \quad 0 < t < T \\
v_t &= \tilde{d}\Delta_M v - u^a v^b && x \in M, \quad 0 < t < T \\
\frac{\partial u}{\partial \eta} &= u^a v^b && x \in M, \quad 0 < t < T \\
u &= u_0 && x \in \Omega, \quad t = 0 \\
v &= v_0 && x \in M, \quad t = 0
\end{aligned}$$

The results [19] apply in the non-manifold setting, but in our setting it is still an open question.

This dissertation is organized as follows. In chapter 2, we provide the statement of the main results, notations, and definitions used in the further work. Since the understanding of our problem depends mainly on the understanding of the Cauchy problem on a manifold, in chapter 3 we give the extension of results by Ladyzenskaya on a manifold. Chapter 4 discusses the detailed proof of the remark in [6] made by Brown. Both of these chapters are helpful in establishing local, global wellpossedness and uniform estimates of the solution of our problem. In chapter 5 and 6, the proofs of the main results on local existence and global existence are presented. In chapter 7, we apply our results to the 5 component model example, which also provides the motivation of working on this problem.

Chapter 2

Notation and Definitions

This section defines the notations that make for the lucid comprehension of the subject.

Throughout, we assume Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary, $M = \partial\Omega$ belonging to the class of $C^{2+\mu}$ with $\mu > 0$ such that Ω lies locally on one side of M. We note that M is said to be a $C^{2+\mu}(\mu > 0)$ manifold together with the C^1 Riemannian metric g if only if

- For each point $\xi \in M$ there exists a pair (V_ξ, ϕ_ξ) consisting of an open set V_ξ of M containing ξ and a C^2 diffeomorphism $\phi_\xi : U \to V_\xi$, from open subset U of $\mathbb{R}^{n-1}$ containing the origin onto V_ξ.

- A Riemannian metric on a differentiable manifold M is an inner product g_ξ on the tangent space $T_\xi(M)$ at each point ξ that varies smoothly from point to point in the sense that if X and Y are vector fields on M, then $\xi \mapsto g_\xi(X(\xi), Y(\xi))$ is a C^1 function for all $X, Y \in T_\xi(M)$. The family g_ξ of inner products is called a Riemannian metric (tensor). g_ξ is determined by a positive definite, symmetric (C^1) matrix $(g_{ij}(\xi)) = (g_\xi(\partial_{x^i}, \partial_{x^j}))$, where $\{\partial_{x^i} = \frac{\partial}{\partial x^i}\}_{i=1}^{n-1}$ is a basis of $T_\xi(M)$. See [39], [42] for more information.

Note that the metric g is defined on the entire manifold M, whereas $g_{i,j}(\xi)$ are defined only in a coordinate chart (V_ξ, ϕ_ξ).

$\mathbb{R}_+$ is the set of all non-negative real numbers.

$\overline{\Omega}$ is the closure of Ω, so that $\overline{\Omega} = \Omega \cup M$.

Ω_T is the cylinder $\Omega \times (0, T)$, which contains points (x, t) with $x \in \Omega$ and $t \in (0, T)$.

M_T is the cylinder $M \times (0, T)$, which contains points (ξ, t) with $\xi \in M$ and $t \in (0, T)$.

η is the outward unit normal to Ω on M, and η_j are direction cosines of the outward unit normal η.

∇ and $\nabla\cdot = \sum_{k=1}^N \frac{\partial}{\partial x_k}$ are the gradient and divergence operators, respectively. Note that $\frac{\partial}{\partial \eta} = \eta \cdot \nabla$ and that the Laplacian operator on Ω is given by $\Delta = \nabla \cdot \nabla$.

Throughout, m, k, n, i, and j are positive integers, and D and $\tilde{D}$ are $k \times k$ and $m \times m$ diagonal matrices with positive diagonal entries $\{d_i\}_{1 \le i \le k}$ and $\{\tilde{d}_j\}_{1 \le j \le m}$ respectively. Also, $d_{min} = \min\{d_i : 1 \le i \le k\}$.

Laplace Beltrami operator: (M, g) is a Riemannian manifold together with C^1 Riemannian metric, with $g_{i,j}(\xi)$ being defined in a coordinate chart (V_ξ, ϕ_ξ) and $g^{ij}(\xi)$ are entries of the inverse matrix, $(g_{i,j}(\xi))^{-1}$. The Laplacian of $\tilde{u} = u \circ \phi$ on $U \subseteq \mathbb{R}^{n-1}$, in local coordinates, is given by

$$\Delta_M u = \frac{1}{\sqrt{det\ g}} \partial_j (g^{ij} \sqrt{det\ g}\ \partial_i \tilde{u})$$

Δ_M is known as the Laplace Beltrami operator. For more details, see Rosenberg [39] and Taylor [42]. It is worth mentioning that although it seems that the above expression depends on local coordinates, it does not change with the local coordinates, and hence it is well defined.

2.1 Basic Function Spaces

Let $\mathcal{B}$ be a bounded domain on $\mathbb{R}^n$. We will define all function spaces on $\mathcal{B}$ and $\mathcal{B}_T = \mathcal{B} \times (0,T)$. $L_q(\mathcal{B})$ is the Banach space consisting of all measurable functions on $\mathcal{B}$ that are $q^{th}(q \geq 1)$ power summable on $\mathcal{B}$. The norm is defined as

$$\|u\|_{q,\mathcal{B}} = \left(\int_{\mathcal{B}} |u(x)|^q dx\right)^{\frac{1}{q}}$$

Also,

$$\|u\|_{\infty,\mathcal{B}} = ess\sup\{|u(x)| : x \in \Omega\}$$

Measurability and summability are to be understood everywhere in the sense of Lebesgue. The elements of $L_q(\mathcal{B})$ are equivalence classes of functions on $\mathcal{B}$. Also, if $p, q \in [1,\infty]$ with $p < q$ then $L_q(\mathcal{B}) \subset L_p(\mathcal{B})$ and there exists $C > 0$ so that $\|u\|_{p,\mathcal{B}} \leq C\|u\|_{q,\mathcal{B}}$ for all $u \in L_q(\mathcal{B})$. Indeed, if $p \in [1,\infty)$, $p \leq q = pr < \infty$, and $\frac{1}{r} + \frac{1}{r'} = 1$ then by Hölder's inequality,

$$\|u\|_{p,\mathcal{B}} \leq \left(\left(\int_{\mathcal{B}} |u|^{pr}\right)^{\frac{1}{r}} \left(\int_{\mathcal{B}} 1\right)^{\frac{1}{r'}}\right)^{\frac{1}{p}} = |\mathcal{B}|^{\frac{q-1}{pq}} \|u\|_{q,\mathcal{B}}$$

That is $C = |\mathcal{B}|^{\frac{q-1}{pq}}$. This inequality is obvious in the case where q is infinite. Also note that with these norms, Hölder's inequality is expressed as

$$\|uv\|_{1,\mathcal{B}} \leq \|u\|_{p,\mathcal{B}}\|v\|_{q,\mathcal{B}}$$

for $u \in L_p(\mathcal{B}), v \in L_q(\mathcal{B})$, and $\frac{1}{p} + \frac{1}{q} = 1$.

If $p \geq 1$, then $W_p^2(\mathcal{B})$ is the Sobolev space of functions $u : \mathcal{B} \to \mathbb{R}$ with generalized derivatives, $\partial_x^s u$ (in the sense of distributions) where $s \leq 2$ and each of the derivatives belongs to $L_p(\mathcal{B})$. The norm in this space is

$$\|u\|_{p,\mathcal{B}}^{(2)} = \sum_{r=0}^{2} \|\partial_x^s u\|_{p,\mathcal{B}}$$

Here $s = (s_1, s_2,...,s_n), |s| = s_1 + s_2 + .. + s_n$, and $\partial_x^s = \partial_1^{s_1}\partial_2^{s_2}...\partial_n^{s_n}$ where $\partial_i = \frac{\partial}{\partial x_i}$.

Similarly, $W_p^{2,1}(\mathcal{B}_T)$ is the Sobolev space of functions $u : \mathcal{B}_T \to \mathbb{R}$ with generalized derivatives, $\partial_x^s \partial_t^r u$ (in the sense of distributions) where $2r + s \leq 2$ and each of the derivatives belongs to $L_p(\mathcal{B}_T)$. The norm in this space is

$$\|u\|_{p,\mathcal{B}_T}^{(2)} = \sum_{2r+s=0}^{2} \|\partial_x^s \partial_t^r u\|_{p,\mathcal{B}_T}$$

Here $s = (s_1, s_2,...,s_n), |s| = s_1 + s_2 + .. + s_n$, and $\partial_x^s = \partial_1^{s_1}\partial_2^{s_2}...\partial_n^{s_n}$ where $\partial_i = \frac{\partial}{\partial x_i}$.

In addition to $W_p^{2,1}(\mathcal{B}_T)$, we will encounter two spaces with different ratios of upper indices. $W_2^{1,0}(\mathcal{B}_T)$ is the Hilbert space with scalar product

$$(u,\ v)_{W_2^{1,0}(\mathcal{B}_T)} = \int_{\mathcal{B}_T} (uv + u_{x_k} v_{x_k}) dx\ dt$$

and $W_2^{1,1}(\mathcal{B}_T)$ is the Hilbert space with scalar product

$$(u,\ v)_{W_2^{1,1}(\mathcal{B}_T)} = \int_{\mathcal{B}_T} (uv + u_{x_k} v_{x_k} + u_t v_t) dx\ dt$$

$V_2(\mathcal{B}_T)$ is the Banach space consisting of all elements of $W_2^{1,0}(\mathcal{B}_T)$ having a finite norm

$$|u|_{V_2(\mathcal{B}_T)} = vrai \max_{0 \le t \le T} \|u(x,t)\|_{2,\mathcal{B}} + \|u_x\|_{2,\mathcal{B}_T}$$

where here and below

$$\|u_x\|_{2,\mathcal{B}_T} = \left(\int_{\mathcal{B}_T} u_x^2\ dx\ dt \right)^{\frac{1}{2}}$$

$V_2^{1,0}(\mathcal{B}_T)$ is the Banach space consisting of all elements of $V_2(\mathcal{B}_T)$ that are continuous in t with respect to the $L_2(\mathcal{B})$ norm, and having a finite norm

$$|u|_{V_2^{1,0}(\mathcal{B}_T)} = \max_{0 \le t \le T} \|u(x,t)\|_{2,\mathcal{B}} + \|u_x\|_{2,\mathcal{B}_T}$$

The continuity in t of a function $u(\cdot, t)$ in the norm of $L_2(\mathcal{B})$ means that

$$\|u(\cdot, t + \Delta t) - u(\cdot, t)\|_{2,\mathcal{B}} \to 0$$

as $\Delta t \longrightarrow 0$. The space $V_2^{1,0}(\mathcal{B}_T)$ is obtained by completing the set $W_2^{1,1}(\mathcal{B}_T)$ in the norm of $V_2(\mathcal{B}_T)$.

$V_2^{1,\frac{1}{2}}(\mathcal{B}_T)$ is the subset of those elements $u \in V_2^{1,0}(\mathcal{B}_T)$ for which

$$\int_0^{T-h} \int_{\mathcal{B}} h^{-1} [u(x,t+h) - u(x,t)]^2\ dx\ dt \longrightarrow 0 \quad \text{as } h \to 0$$

We will also introduce $W_p^l(\mathcal{B})$, where l is not an integer, because initial data will be taken from these spaces.

The space $W_p^l(\mathcal{B})$ with non-integral l, is a Banach space consisting of elements of $W_p^{[l]}$ ($[l]$ is the largest integer less than l) with the finite norm

$$\|u\|_{p,\mathcal{B}}^{(l)} = \langle u\rangle_{p,\mathcal{B}}^{(l)} + \|u\|_{p,\mathcal{B}}^{([l])}$$

where

$$\|u\|_{p,\mathcal{B}}^{([l])} = \sum_{s=0}^{[l]} \|\partial_x^s u\|_{p,\mathcal{B}}$$

and

$$\langle u\rangle_{p,\mathcal{B}}^{(l)} = \sum_{s=[l]} \left(\int_{\mathcal{B}} dx \int_{\mathcal{B}} |\partial_x^s u(x) - \partial_y^s u(y)|^p . \frac{dy}{|x-y|^{n+p(l-[l])}} \right)^{\frac{1}{p}}$$

$W_p^{l,\frac{l}{2}}(\partial\mathcal{B}_T)$ spaces with non-integral l also plays an important role in the study of the boundary value problems with nonhomogeneous boundary conditions, especially in the proof of exact estimates for there solutions. It is a Banach space when $p \geq 1$, which is defined by means of parametrization of the surface $\partial\mathcal{B}$. Let $\partial\mathcal{B}$ be covered by sets $\partial\mathcal{B}_1, \partial\mathcal{B}_2, ..., \partial\mathcal{B}_k, ...$ such that $\cup_k \partial\mathcal{B}_k = \partial\mathcal{B}$, and for each point $x \in \partial\mathcal{B}$ there exists k such that $x \in \partial\mathcal{B}_k$, and the distance from x to $\partial\mathcal{B} \setminus \partial\mathcal{B}_k$ exceeds a certain fixed positive number, δ independent of x. Further, every $\partial\mathcal{B}_k$ intersects only a finite number of other $\partial\mathcal{B}_j$, not exceeding some number m_k, and is mapped onto some canonical domain σ of $n-1$ dimensional Euclidean space. In other words, for $x \in \partial\mathcal{B}_k$, there exists (V_k, ϕ_k) consisting of an open set V_k of M containing $x \in \partial\mathcal{B}_k$ and a C^2 diffeomorphism $\phi_k : U \to V_k$, from open subset U of $\mathbb{R}^{n-1}$ containing the origin onto V_k. Let $u_k(z,t) = u(\phi_k(z), t)$ for all

$(z,t) \in U \times (0,T)$. The space $W_p^{l,\frac{l}{2}}(\partial\mathcal{B}_T)$ is defined as a set of functions with finite norm

$$\|u\|^{(l)}_{p,\partial\mathcal{B}_T} = \left(\sum_k \left(\|u_k(z,t)\|^{(l)}_{p,\sigma_T} \right)^p \right)^{\frac{1}{p}}$$

where

$$\|u_k(z,t)\|^{(l)}_{p,U_T} = \sum_{0 \le 2r+s < l} \|\partial_t^r \partial_x^s u_k\|_{p,U_T} + \sum_{2r+s=[l]} \langle \partial_t^r \partial_x^s u_k \rangle^{(l-[l])}_{p,x,U_T} + \sum_{0 < l-2r-s < 2} \langle \partial_t^r \partial_x^s u_k \rangle^{(\frac{l-2r-s}{2})}_{p,t,U_T}$$

and for $0 < \alpha < 1$

$$\langle v \rangle^{(\alpha)}_{p,x,U_T} = \left(\int_0^T dt \int_U dx \int_U |v(x,t) - v(y,t)|^p . \frac{dy}{|x-y|^{n-1+\alpha p}} \right)^{\frac{1}{p}}$$

$$\langle v \rangle^{(\alpha)}_{p,t,U_T} = \left(\int_U dx \int_0^T dt \int_0^T |v(x,t) - v(x,t')|^p . \frac{dt'}{|t-t'|^{1+\alpha p}} \right)^{\frac{1}{p}}$$

The need of $W_p^{l,\frac{l}{2}}(\partial\mathcal{B}_T)$ spaces is connected to the fact that the differential properties of the boundary values of the function from classes $W_p^{2,1}(\mathcal{B}_T)$ and of certain of its derivatives, $\partial_x^s \partial_t^r$, can be exactly described in terms of the spaces $W_p^{l,\frac{l}{2}}(\partial\mathcal{B}_T)$, where $l = 2 - 2r - s - \frac{1}{p}$.

$C^{\alpha,\frac{\alpha}{2}}(\overline{\mathcal{B}_T})$ are Hölder spaces, where $0 < \alpha < 1$. It is a Banach space of continuous functions $u(x,t)$ with the finite norm

$$|u|^{(\alpha)}_{\overline{\mathcal{B}}_T} = \sup_{(x,t) \in \mathcal{B}_T} |u(x,t)| + [u]^{(\alpha)}_{x,\mathcal{B}_T} + [u]^{(\frac{\alpha}{2})}_{t,\mathcal{B}_T}$$

where

$$[u]^{(\alpha)}_{x,\overline{\mathcal{B}}_T} = \sup_{\substack{(x,t),(x',t)\in\mathcal{B}_T \\ x\neq x'}} \frac{|u(x,t)-u(x',t)|}{|x-x'|^{\alpha}}$$

and

$$[u]^{(\frac{\alpha}{2})}_{t,\overline{\mathcal{B}}_T} = \sup_{\substack{(x,t),(x,t')\in\mathcal{B}_T \\ t\neq t'}} \frac{|u(x,t)-u(x,t')|}{|t-t'|^{\frac{\alpha}{2}}}$$

Notation: $C^{\frac{\alpha}{2}}(\overline{\mathcal{B}}_T)$ stands for $C^{\frac{\alpha}{2},\frac{\alpha}{2}}(\overline{\mathcal{B}}_T)$.

$C(\mathcal{B}_T,\mathbb{R}^n)$ is the set of all continuous functions $u:\mathcal{B}_T\to\mathbb{R}^n$.

$C^{1,0}(\mathcal{B}_T,\mathbb{R}^n)$ is the set of all continuous functions $u:\mathcal{B}_T\to\mathbb{R}^n$ having continuous derivatives u_x in $\mathcal{B}_T$.

$C^{2,1}(\mathcal{B}_T,\mathbb{R}^n)$ is the set of all continuous functions $u:\mathcal{B}_T\to\mathbb{R}^n$ having continuous derivatives u_x, u_{xx} and u_t in $\mathcal{B}_T$.

Note that similar definition can be given for $\overline{\mathcal{B}}_T$.

2.2 Hölder and Sobolev Spaces on Manifolds

Let M be a compact Riemannian manifold with metric g. For $p\geq 1$, define the Lebesgue space $L_p(M)$ to be the set of locally integrable funcions u on M for which the norm

$$\|u\|_{L_p} = \left(\int_M |u|^p dV_g\right)^{\frac{1}{p}}$$

is finite. Here dV_g is the volume form of the metric g. Suppose that $p,q\geq 1$ and that $\frac{1}{p}+\frac{1}{q}=1$. If $u\in L_p(M)$ and $v\in L_q(M)$, then $uv\in L_1(M)$, and

$$\|uv\|_{L_1} \leq \|u\|_{L_p}\|v\|_{L_q}$$

This is Hölder inequality. One can easily define the Sobolev spaces $W_p^k(M)$, following what is done in the more traditional Euclidean context. For instance, when $k > 0$ and $p > 1$, one may define the Sobolev space $W_p^k(M)$ as follows: for $u \in C^\infty(M)$, we let

$$\|u\|_{W_p^k(M)} = \left(\sum_{j=0}^{k} \int_M |\nabla^j u|^p dV_g \right)^{\frac{1}{p}}$$

We then define $W_p^k(M)$ as the completion of $C^\infty(M)$ with respect to $\|.\|_{W_p^k(M)}$. Also, as for bounded open subsets of the Euclidean space, the Sobolev embedding theorem (continuous embeddings) and the Rellich-Kondrakov theorem (compact embeddings) [16],[42] do hold.

In further arguments we need a result stating that when $p > n$, the order of differentiability or the order of integrability is so large that the Sobolev space can be embedded in Hölder spaces, and it will stated later. More developments on Sobolev spaces, Sobolev inequalities, and the notion of best constants can be found in Druet-Hebey [8] and [9].

Following standard notation, we let $C^k(M)$ be the space of k times continuously differentiable functions on M. If $u \in C^k(M)$, then

$$\|u\|_{C^k(M)} = \sum_{i=0}^{k} \|\nabla^i u\|_\infty$$

is finite. The norm $\|.\|_{C^k(M)}$ induces a Banach space structure on the space $C^k(M)$. Let $d_g(x, y)$ be the distance between x and $y \in M$ calculated using g, and let $\alpha \in (0, 1)$. Then a function u on M is said to be Hölder continuous with exponent

α if

$$[u]_\alpha = \sup_{x \neq y \in M} \frac{|u(x) - u(y)|}{d_g(x,y)^\alpha}$$

is finite. Any Hölder continuous function u is continuous. The vector space $C^{0,\alpha}(M)$ is the set of continuous bounded functions on M which are Hölder continuous with exponent α, and the norm on $C^{0,\alpha}(M)$ is

$$\|u\|_{C^{0,\alpha}(M)} = \|u\|_{\infty,M} + [u]_\alpha$$

is finite. Concerning the spaces $C^{k,\alpha}(M)$ and $C^{k,\alpha}_B(M)$, where $k \geq 1$ is an integer and $\alpha \in (0,1)$, a possible definition is the following: a function $u : M \to \mathbb{R}$ is in $C^{k,\alpha}(M)$ if and only if it is in $C^k(M)$, and given a system of charts on M, the coordinates of the tensor $\nabla^k u$ are in $C^{0,\alpha}$ when read via a chart. This definition is naturally independent of the choice of a C^∞ system of charts. See more detail in the Handbook of Global Analysis [25].

Chapter 3

The Cauchy Problem on a Compact Manifold

Let M be a compact Riemannian manifold without boundary with metric g. Consider the system

$$\begin{aligned} \Psi_t &= \tilde{d}\Delta_M \Psi + f & (\xi, t) &\in M_T \\ \Psi\big|_{t=0} &= \Psi_0 & \xi &\in M \end{aligned} \tag{3.0.1}$$

where $\tilde{d} > 0$, $f \in L_p(M_T)$ and $\Psi_0 \in W_p^{(2-\frac{2}{p})}(M)$. Searching the literature, we surprisingly could not find the $W_p^{2,1}$ estimates for the solution to linear parabolic problems in this setting. Tracing through the work in this direction, we found that Polden in his PhD thesis [22] and [35], and J.J Sharples [24] give a result in the setting where $p = 2$. Sharples also corrected a minor error in Polden's work. Using their $W_2^{2,1}(M_T)$ estimate, we obtain $W_p^{2,1}(M_T)$ apriori estimates for solutions of (3.0.1) for all $p > 1$. For smooth functions $f, g : M \times [0, \infty) \to \mathbb{R}$, Polden considered weighted inner products:

$$\langle f, g\rangle_{LL_a} = \int_0^\infty e^{-2at}\langle f(\cdot,t), g(\cdot,t)\rangle_{L^2(M)}dt$$

$$\langle f, g\rangle_{LW_a^1} = \int_0^\infty e^{-2at}\langle f(\cdot,t), g(\cdot,t)\rangle_{W_2^1(M)}dt$$

$$\langle f, g\rangle_{LW_a^2} = \int_0^\infty e^{-2at}\langle f(\cdot,t), g(\cdot,t)\rangle_{W_2^2(M)}dt$$

$$\langle f, g\rangle_{WW_a} = \langle f(\cdot,t), g(\cdot,t)\rangle_{LW_a^1} + \langle D_t f, D_t g\rangle_{LL_a}$$

Where LL_a, LW_a and WW_a are the Hilbert spaces formed by the completion of $C^\infty(M \times [0,\infty))$ in the corresponding norms and WW_a^0 is the completion of $C_c^\infty(M \times [0,\infty))$ in WW_a. See [24] for the proof of the following result.

Theorem 3.0.1 (Polden, J.J Sharples)**.** *Suppose Ψ_0 lies in $W_2^1(M)$ and $f \in LL_a(M \times [0,\infty)))$. Then for sufficiently large a, the system* (3.0.1) *has a unique weak solution in WW_a^0.*

Furthermore using apriori estimates in [24], they showed that the solution belongs to $W_2^{2,1}(M \times [0,\infty))$.

Theorem 3.0.2. *Let $\Psi \in WW_a$ be solution of* (3.0.1) *with $\Psi_0 \in W_2^1(M)$ and $f \in L_p(M_T)$. Then $\Psi \in LW_a^2$, and there exists $C > 0$ independent of Ψ_0 and f such that*

$$\|\Psi\|^2_{LW_a^2} \le C(\|\Psi_0\|^2_{W_2^1(M)} + \|f\|^2_{LL_a})$$

Proof. See Lemma 4.3 in [24]. □

Corollary 3.0.3. *Let $0 < T < \infty$. Suppose $\Psi_0 \in W_2^1(M)$ and $f \in L_2(M_T)$. If $a = 0$, there exists a unique weak solution to (3.0.1) in $W_2^{2,1}(M_T)$, and there exists $C > 0$ independent of Ψ_0 and f such that*

$$\|\Psi\|^2_{W_2^{2,1}(M_T)} \leq C(\|\Psi_0\|^2_{W_2^1(M)} + \|f\|^2_{L_2(M_T)})$$

Remark 3.0.4. Theorem 1 and Theorem 2 also hold on a C^2 manifold.

We will use this $W_2^{2,1}(M_T)$ result to derive $W_p^{2,1}(M_T)$ a priori estimates for solutions to (3.0.1) for all $p > 1$. To obtain these estimates, we transform the Cauchy problem defined locally on M to a bounded domain on $\mathbb{R}^{n-1}$ where $n-1$ is the dimension of the manifold, and obtain the estimates over this bounded domain. Then we pull the resulting estimates back to the manifold. Repeating this process over every neighborhood on the manifold, and using compactness of the manifold, we get estimates over the entire manifold.

The following results will help us obtain a priori estimates for the Cauchy problem on M and prove the existence of solutions in $W_p^{2,1}(M_T)$. Lemmas 3.0.5, 3.0.8 and 3.0.12 can be found on page 341 in [27], page 49 in [26], and [16] respectively.

Let $\mathcal{B}$ be a smooth bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\mathcal{B}$ belonging to the class $C^{2+\mu}$ with $\mu > 0$ such that $\mathcal{B}$ lies locally on one side of the boundary, $\partial\mathcal{B}$. Let $T > 0$ and $p > 1$. Suppose $\Theta \in L_p(\mathcal{B}_T)$, $w_0 \in W_p^{(2-\frac{2}{p})}(\mathcal{B})$ and $\gamma \in W_p^{2-\frac{1}{p},1-\frac{1}{2p}}(\partial\mathcal{B}_T)$. Also, in further arguments the coefficient functions $a_{i,j}$ are symmetric, uniformly continuous on $\overline{\mathcal{B}}_T$, and satisfy the uniform ellipticity condition, that for some $\lambda > 0$

$$\sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i\xi_j \geq \lambda\xi^2 \text{ for all } (x,t) \in \mathcal{B}_T \text{ and for all } \xi \in \mathbb{R}^n$$

Consider the problem

$$\begin{aligned} \frac{\partial w}{\partial t} - \sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial^2 w}{\partial x_i \partial x_j} + \sum_{i=1}^{n} a_i(x,t)\frac{\partial w}{\partial x_i} &= \Theta(x,t) \qquad (x,t) \in \mathcal{B}_T \\ w &= \gamma(x,t) \qquad (x,t) \in \partial\mathcal{B}_T \\ w\big|_{t=0} &= w_0(x) \qquad x \in \mathcal{B} \end{aligned} \tag{3.0.2}$$

Lemma 3.0.5. *Let $p > 1$. Suppose that the coefficients a_{ij} and a_i are uniformly continuous on $\mathcal{B}_T$, $\Theta \in L_p(\mathcal{B}_T)$, $w_0 \in W_p^{(2-\frac{2}{p})}(\mathcal{B})$ and $\gamma \in W_p^{2-\frac{1}{p},1-\frac{1}{2p}}(\partial\mathcal{B}_T)$ with $p \neq \frac{3}{2}$. Then (3.0.2) has a unique solution $w \in W_p^{2,1}(\mathcal{B}_T)$, satisfying in the case $p > \frac{3}{2}$, the compatibility condition of zero order, $w_0|_{\partial\mathcal{B}} = \gamma|_{t=0}$. Furthermore, there exists $C_T > 0$ independent of Θ, w_0 and γ such that*

$$\|w\|_{p,\mathcal{B}_T}^{(2)} \leq C_T(\|\Theta\|_{p,\mathcal{B}_T} + \|w_0\|_{p,\mathcal{B}}^{(2-\frac{2}{p})} + \|\gamma\|_{p,\partial\mathcal{B}_T}^{(2-\frac{1}{p},1-\frac{1}{2p})})$$

Proof. See Theorem 9.1 in Chapter 4 of Ladyzenskaya [27] $\square$

Lemma 3.0.6. *Let $q \geq p$, $2-2r-s-\left(\frac{1}{p}-\frac{1}{q}\right)(n+2) \geq 0$ and $0 < \delta \leq \min\{d; \sqrt{T}\}$. Then there exists c_3, c_4 depending on r, s, n, p and Ω such that*

$$\|D_t^r D_x^s u\|_{q,\mathcal{B}_T} \leq c_3\delta^{2-2r-s-\left(\frac{1}{p}-\frac{1}{q}\right)(n+2)}\|u\|_{p,\mathcal{B}_T}^{(2)} + c_4\delta^{-(2r+s+\left(\frac{1}{p}-\frac{1}{q}\right)(n+2))}\|u\|_{p,\mathcal{B}_T}$$

for all $u \in W_p^{2,1}(\mathcal{B}_T)$. Moreover, if $2-2r-s-\frac{(n+2)}{p} > 0$, Then for $0 \leq \alpha <$

$2-2r-s-\frac{(n+2)}{p}$ *there exists constants* c_5, c_6 *depending on* r, s, n, p *and* Ω *such that*

$$|D_t^r D_x^s u|_{\mathcal{B}_T}^{(\alpha)} \leq c_5 \delta^{2-2r-s-\frac{n+2}{p}-\alpha} \|u\|_{p,\mathcal{B}_T}^{(2)} + c_6 \delta^{-(2r+s+\frac{(n+2)}{p}+\alpha)} \|u\|_{p,\mathcal{B}_T}$$

for all $u \in W_p^{2,1}(\mathcal{B}_{\mathcal{T}})$.

Proof. See Lemma 3.3 in Chapter 2 of [27] □

Corollary 3.0.7. *Suppose the conditions of Lemma* 3.0.5 *are fulfilled for* $p > \frac{n+2}{2}$. *Then there exists* $\hat{c} > 0$ *depending on* n, p *and* Ω *such that the solution of a problem* (3.0.2) *is a Hölder continuous function in* x *and* t *with*

$$|w|_{\mathcal{B}_T}^{(2-\frac{n+2}{p})} \leq \hat{c} \|w\|_{p,\mathcal{B}_T}^{(2)}$$

Lemma 3.0.8. *Let* $1 < p < \infty$ *and* $\epsilon > 0$. *If* $p < n$ *then* $W_p^1(\mathcal{B})$ *imbeds continuously into* $L_q(\mathcal{B})$ *for* $p \leq q \leq p^* = \frac{np}{n-p}$. *Furthermore, there exists* $C_\epsilon > 0$ *such that*

$$\|v\|_{L_q(\mathcal{B})}^p \leq \epsilon \|v_x\|_{L_p(\mathcal{B})}^p + C_\epsilon \|v\|_{L_1(\mathcal{B})}^p$$

for all $v \in W_p^1(\mathcal{B})$. *The imbedding constants for the imbeddings above depend only on* n, p, q *and* Ω.

Proof. See page 49 of Ladyzenskaya [26] □

The following result seems to be well known, but in the absence of a good references we include the proof.

Lemma 3.0.9. *Let $\epsilon > 0$, $1 < p < \infty$ and $v \in W_p^2(\mathcal{B})$. Then there exists $C_\epsilon > 0$ such that*

$$\|v_x\|_{p,\mathcal{B}} \leq \epsilon \|v_{xx}\|_{p,\mathcal{B}} + C_\epsilon \|v\|_{p,\mathcal{B}}$$

for all $v \in W_p^2(\mathcal{B})$.

Proof. By way of contraction, suppose there exist $\epsilon > 0$, and a sequence $v_m \in W^{2,p}(\mathcal{B})$ such that,

$$\|v_{m_x}\|_{p,\mathcal{B}} > \epsilon \|v_{m_{xx}}\|_{p,\mathcal{B}} + m\|v_m\|_{p,\mathcal{B}} \tag{3.0.3}$$

for all $m \in \mathbb{N}$. Let $k_m = \|v_{m_x}\|_{p,\mathcal{B}}$ and $w_m = \frac{1}{k_m} v_m$. Dividing inequality (3.0.3) by k_m and using homogeneity, we have

$$1 = \|w_{m_x}\|_{p,\mathcal{B}} > \epsilon \|w_{m_{xx}}\|_{p,\mathcal{B}} + m\|w_m\|_{p,\mathcal{B}} \tag{3.0.4}$$

From (3.0.4) $\|w_{m_{xx}}\|_{p,\mathcal{B}} < \frac{1}{\epsilon}$, $\|w_{m_x}\|_{p,\mathcal{B}} = 1$ and $\|w_m\|_{p,\mathcal{B}} < \frac{1}{m}$. This implies $\{w_m\}$ is a bounded sequence in $W^{2,p}(\mathcal{B})$. Now, $W^{2,p}(\mathcal{B})$ embedds compactly into $W^{1,p}(\mathcal{B})$, so there exists a subsequence $\{w_{m_k}\}$ and $w \in W^{1,p}(B)$ such that, $w_{m_k} \to w$. Again, from (3.0.4), $\|w_{m_k}\|_{p,\mathcal{B}} < \frac{1}{m}$ so, as $m \to \infty$ $w_{m_k} \to 0$ in $L_p(\mathcal{B})$. Therefore, $w = 0$ with $\|w\|_{W^{1,p}} = 1$, which is not possible. Hence, there exists $C_\epsilon > 0$ such that

$$\|v_x\|_{p,\mathcal{B}} \leq \epsilon \|v_{xx}\|_{p,\mathcal{B}} + C_\epsilon \|v\|_{p,\mathcal{B}}$$

for all $v \in W_p^2(\mathcal{B})$. $\square$

Lemma 3.0.10. *Suppose $f \in L_2(M_T)$ and $\Psi_0 \in W_2^1(M)$. Then the unique solu-*

tion Ψ *of* (3.0.1) *satisfies the estimate*

$$\|\Psi\|_{1,M} \quad \leq \quad \|f\|_{1,M_T} + \|\Psi_0\|_{1,M}$$

Proof. Write $f = f^+ - f^-$ and $\Psi_0 = \Psi_0^+ - \Psi_0^-$. Since $\Psi_0 \in W_2^1(M)$, $\Psi_0^+, \Psi_0^- \in W_2^1(M)$. Consider the systems

$$\begin{aligned} \Psi_{1t} &= \tilde{d}\Delta_M \Psi_1 + f^+ & \qquad (\xi, t) \in M_T \\ \Psi_1\big|_{t=0} &= \Psi_0^+ & \xi \in M \end{aligned}$$

and

$$\begin{aligned} \Psi_{2t} &= \tilde{d}\Delta_M \Psi_2 + f^- & \qquad (\xi, t) \in M_T \\ \Psi_2\big|_{t=0} &= \Psi_0^- & \xi \in M \end{aligned}$$

Note that the solutions Ψ_1 and Ψ_2 are unique from Theorem 3.0.1. Using Theorem 3.0.1 and linearity, $\Psi = \Psi_1 - \Psi_2$ solves (3.0.1). Since f^+ and f^- are non-negative functions, maximum priniciples imply Ψ_1 and Ψ_2 are non-negative. Integrating both the systems over M_T, we get

$$\int_M \Psi_1 \leq \int_{M_T} f^+ + \int_M \Psi_0^+$$

and

$$\int_M \Psi_2 \leq \int_{M_T} f^- + \int_M \Psi_0^-$$

As a result,

$$\int_M |\Psi| \leq \int_M \Psi_1 + \int_M \Psi_2 \leq \int_{M_T} (f^+ + f^-) + \int_M (\Psi_0^+ + \Psi_0^-)$$
$$= \|f\|_{1,M_T} + \|\Psi_0\|_{1,M}$$

□

Lemma 3.0.11. *Let $p > n$ and $\alpha \in (0,1)$ such that $\alpha < 1 - \frac{n}{p}$. Then $W_p^1(\mathcal{B})$ embedds compactly in $C^{0,\alpha}(\overline{\mathcal{B}})$.*

Proof. See Adams [2] □

Lemma 3.0.12. *Let $(\mathfrak{D}, g)$ be a compact Riemannian manifold of dimension greator than or equal to 1. Let $p > n$. Then the embedding $W_p^1(\mathfrak{D}) \subset C^{0,\alpha}(\mathfrak{D})$ is compact for all $0 < \alpha < 1 - \frac{n}{p}$.*

Proof. See Emmanuel [16] □

3.1 $W_p^{2,1}$ Estimates on Manifolds

Let $\mathcal{F}$ be a subset of $\mathbb{R}_+$ with following property:

$p > 1$ belongs to $\mathcal{F}$ if and only if there exists $\hat{C}_{p,T} > 0$ such that whenever $\Psi_0 \in W_p^{2-\frac{2}{p}}(M)$ and $f \in L_p(M_T)$, then there exists $\Psi \in W_p^{2,1}(M_T)$, such that Ψ solves (3.0.1) and

$$\|\Psi\|_{p,M_T}^{(2)} \leq \hat{C}_{p,T}(\|f\|_{p,M_T} + \|\Psi_0\|_{p,M}^{(2-\frac{2}{p})})$$

Note: From Theorem 3.0.1 and 3.0.2, $2 \in \mathcal{F}$.

Lemma 3.1.1. *$[2, \infty) \subset \mathcal{F}$.*

Proof. We will show that if $p \in \mathcal{F}$ then $[p, p + \frac{1}{n-1}] \subset \mathcal{F}$. To this end, let $p \in \mathcal{F}$ and $q \in [p, p+\frac{1}{n-1}]$ such that $\Psi_0 \in W_q^{2-\frac{2}{q}}(M)$ and $f \in L_q(M_T)$. Then $f \in L_p(M_T)$ and $\Psi_0 \in W_p^{2-\frac{2}{p}}(M)$. Since $p \in \mathcal{F}$, there exists $\hat{C}_{p,T} > 0$ independent of Ψ_0 and f, and $\Psi \in W_p^{2,1}(M_T)$ solving (3.0.1) such that

$$\|\Psi\|_{p,M_T}^{(2)} \leq \hat{C}_{p,T}(\|f\|_{p,M_T} + \|\Psi_0\|_{p,M}^{(2-\frac{2}{p})}) \tag{3.1.5}$$

Let $R > 0$ and $B(0, R)$ be an open ball in $\mathbb{R}^{n-1}$ containing the origin. Now, M is a C^2 manifold. Therefore, for each point $\xi \in M$ there exists a pair (V_ξ, ϕ_ξ) of an open set V_ξ containing ξ of M and a C^2 diffeomorphism $\phi_\xi : B(0, R) \xrightarrow{\text{onto}} V_\xi$. Let $\Phi = \Psi \circ \phi_\xi$, $\tilde{f} = f \circ \phi_\xi$, $\Phi_0 = \Psi_0 \circ \phi_\xi$ and using the Laplace Beltrami operator (defined before), (3.0.1) takes the form

$$\Phi_t = \frac{\tilde{d}}{\sqrt{det\ g}}\partial_j(g^{ij}\sqrt{det\ g}\ \partial_i\Phi) + \tilde{f}(x,t) \qquad x \in B(0,R), \quad 0 < t < T$$

$$\Phi = \Phi_0 \qquad\qquad x \in B(0,R), \quad t=0 \tag{3.1.6}$$

That is, in a bounded region $B(0,R)\times(0,T)$ of the Euclidean space, we have

$$\mathcal{L}(\Phi) = \Phi_t - \sum_{i,j=1}^{n-1} a_{ij}(x,t)\Phi_{x_i x_j} + \sum_{i=1}^{n-1} a_i(x,t)\Phi_{x_i} = \tilde{f}(x,t) \tag{3.1.7}$$

$$\Phi\big|_{t=0} = \Phi_0(x) \tag{3.1.8}$$

where,

$$a_{ij}(x,t) = \tilde{d}\; g^{ij}(x,t)$$

$$a_i(x,t) = \frac{-\tilde{d}}{\sqrt{det\ g}}\partial_j(g^{ij}\sqrt{det\ g})$$

Note $\Psi \in W_p^{2,1}(M_T)$ implies $\Phi \in W_p^{2,1}(B(0,R)\times(0,T))$. Take $0 < 2r < R$ and define a cut off function $\psi \in C_0^\infty(\mathbb{R}^{n-1},[0,1])$ such that,

$$\psi(x) = \begin{cases} 1 & \forall x \in B(0,r) \\ 0 & \forall x \in \mathbb{R}^{n-1}\backslash B(0,2r) \end{cases} \tag{3.1.9}$$

In $Q = B(0,2r)$, $Q_T = B(0,2r)\times(0,T)$ and $S_T = \partial B(0,r)\times(0,T)$, $w = \psi\Phi$ satisfies the equation

$$\frac{\partial w}{\partial t} - \sum_{i,j=1}^{n-1} a_{ij}(x,t)\frac{\partial^2 w}{\partial x_i \partial x_j} + \sum_{i=1}^{n-1} a_i(x,t)\frac{\partial w}{\partial x_i} = \theta(x,t) \qquad (x,t)\in Q_T$$

$$w = 0 \qquad\qquad (x,t) \in S_T$$

$$w\big|_{t=0} = \psi\Phi_0(x) \quad t=0, x\in Q$$

where,

$$\theta(x,t) = \tilde{f}\psi - 2\sum_{i=1}^{n-1} a_{ij}(x,t)\frac{\partial \Phi}{\partial x_i}\frac{\partial \psi}{\partial x_j} - \Phi\sum_{i,j=1}^{n-1} a_{ij}(x,t)\frac{\partial^2 \psi}{\partial x_i \partial x_j} + \Phi\sum_{i=1}^{n-1} a_i(x,t)\frac{\partial \psi}{\partial x_i}$$

Since $\psi \in C_0^\infty(\mathbb{R}^{n-1},[0,1])$ and $\Phi \in W_p^{2,1}(B(0,R)\times(0,T))$, therefore $\theta - \tilde{f}\psi \in W_p^{1,1}(Q_T)$.

Case 1. Suppose $p < n$. From Lemma 3.0.8, $\theta \in L_{\min\{q,\ p+\frac{p^2}{n-p}\}}(Q_T)$. In particular since $p + \frac{1}{n-1} < p + \frac{p^2}{n-p}$, $\theta \in L_q(Q_T)$. As a result

$$\begin{aligned} \|\theta\|_{q,Q_T} &\leq \|\tilde{f}\psi\|_{q,Q_T} + C_1\|\Phi\|_{q,Q_T} + C_2\|\Phi_x\|_{q,Q_T} \\ &\leq \|\tilde{f}\psi\|_{q,Q_T} + C_1\|\Phi\|_{q,Q_T} + C_2\|\Phi_x\|_{p,Q_T}^{(1)} \end{aligned}$$

where $C_1, C_2 > 0$ are independent of f. Now in order to estimate $\|\Phi_x\|_{p,Q_T}^{(1)}$, apply the change of variable

$$\|\Phi_x\|_{p,Q_T}^{(1)} = \|\Psi_x|\det((\phi_\xi^{-1})')|\|_{p,(\phi_\xi(Q))_T}^{(1)}$$

and using (3.1.5), we get

$$\|\theta\|_{q,Q_T} \leq \|\tilde{f}\psi\|_{q,Q_T} + C_1\|\Phi\|_{q,Q_T} + C_{2p,T}(\|f\|_{p,M_T} + \|\Psi_0\|_{p,M}^{(2-\frac{2}{p})})$$

where $C_{2p,T} > 0$ is independent of f and Ψ_0. At this point, we need an estimate on $\|\Phi\|_{q,Q_T}$. Again $\|\Phi\|_{q,Q_T} = \|\Psi|\det(({\phi_\xi}^{-1})')|\|_{q,(\phi_\xi(Q))_T}$ and from Lemma 3.0.8,

$$\|\Psi|\det(({\phi_\xi}^{-1})')|\|_{q,(\phi_\xi(Q))_T} \leq \|\Psi|\det(({\phi_\xi}^{-1})')|\|_{p,(\phi_\xi(Q))_T}^{(1)}$$

Thus

$$\|\theta\|_{q,Q_T} \leq K_{p,T}(\|f\|_{p,M_T} + \|\Psi_0\|_{p,M}^{(2-\frac{2}{p})}) \tag{3.1.10}$$

where $K_{p,T} > 0$ is independent of f and Ψ_0.

Since $g_{i,j}$ are the C^1 functions on the compact manifold M, $a_{i,j}(x,t)$ and $a_i(x,t)$ satisfies the hypothesis (bounded continuous function in $\overline{Q_T}$) of Lemma 3.0.5. Therefore using Lemma 3.0.5 (for $q > \frac{3}{2}$),

$$\|w\|_{q,Q_T}^{(2)} \leq C_{q,T}(\|\theta\|_{q,Q_T} + \|\psi\Phi_0\|_{q,Q}^{(2-\frac{2}{q})}) \tag{3.1.11}$$

where $C_{q,T} > 0$ is independent of θ and $\psi\Phi_0$. Combining (3.1.11) and (3.1.10) we get,

$$\begin{aligned}\|w\|_{q,Q_T}^{(2)} &\leq C_{2r,T}(\|\theta\|_{q,Q_T} + \|\psi\Phi_0\|_{q,Q}^{(2-\frac{2}{q})}) \\ &\leq \tilde{K}_{p,T}(\|f\|_{p,M_T} + \|\Psi_0\|_{p,M}^{(2-\frac{2}{p})} + \|\psi\Phi_0\|_{q,Q}^{(2-\frac{2}{q})})\end{aligned}$$

where $\tilde{K}_{p,T} > 0$ is independent of f, θ and $\psi\Phi_0$. Note that $w = \Phi$ on $W_T = B(0,r) \times (0,T)$. Thus

$$\|\Phi\|_{q,W_T}^{(2)} \leq \tilde{K}_{p,T}(\|f\|_{p,M_T} + \|\Psi_0\|_{p,M}^{(2-\frac{2}{p})} + \|\psi\Phi_0\|_{q,Q}^{(2-\frac{2}{q})}) \tag{3.1.12}$$

Observe (3.1.12) is over $B(0,r) \times (0,T) \subset \mathbb{R}^{n-1} \times \mathbb{R}_+$. To get the estimate back on the manifold, apply the change of variable, $\|\Phi\|_{q,W_T}^{(2)} = \|\Psi|\det((\phi^{-1})')|\|_{q,\phi(W_T)}^{(2)}$.

Thus

$$\|\Psi\|^{(2)}_{q,\phi(W_T)} \leq \tilde{K}_{p,T,\xi}(\|f\|_{p,M_T} + \|\Psi_0\|^{(2-\frac{2}{p})}_{p,M} + \|\Psi_0\|^{(2-\frac{2}{q})}_{q,\phi(Q)}) \tag{3.1.13}$$

where $\tilde{K}_{p,T,\xi} = \tilde{K}_{p,T}\left(\dfrac{\max\limits_{\xi\in\phi(Q)} |\det((\phi^{-1})')|}{\min\limits_{\xi\in\phi(Q_T)} |\det((\phi^{-1})')|}\right)$.

So far, an estimate in one open neighborhood of some point $\xi \in M$ is obtained. As one varies the point ξ on M, there exists corresponding open neighborhoods V_ξ and a smooth diffemorphisms $\phi_\xi : B(0,r) \longrightarrow V_\xi$, which results in different $\tilde{K}_{p,T,\xi}$ for every V_ξ. Consider an open cover of M such that $M = \bigcup_{\xi\in M} V_\xi$. Since M is compact, there exists $\{\xi_1, \xi_2, ..., \xi_N\}$ such that $M \subset \bigcup_{\substack{\xi_j\in M\\ 1\leq j\leq N}} V_{\xi_j}$ and $\tilde{K}_{p,T,\xi_j}$ corrresponding to each V_{ξ_j}. Let, $\hat{C}_{p,T} = \sum_{j=1}^N \tilde{K}_{p,T,\xi_j}$. Inequality (3.1.13) implies

$$\|\Psi\|^{(2)}_{q,M_T} \leq \hat{C}_{p,T}(\|f\|_{q,M_T} + \|\Psi_0\|^{(2-\frac{2}{q})}_{q,M})$$

Thus $[p, p+\frac{1}{n-1}] \subset \mathcal{F}$.

Case 2. Suppose $p \geq n$. By Lemma 3.0.8 and Theorem 4.12 in [2], $\theta \in L_q(Q_T)$, for all $q \in [p,\infty)$, and proceeding similarly to Case 1, we get

$$\|\Psi\|^{(2)}_{q,M_T} \leq \hat{C}_T(\|f\|_{q,M_T} + \|\Psi_0\|^{(2-\frac{2}{q})}_{q,M})$$

where $\hat{C}_{p,T} > 0$ is independent of f, θ and $\psi\Phi_0$. Hence $[2,\infty) \subset \mathcal{F}$.

□

Theorem 3.1.2. $\mathcal{F} = (1, \infty)$.

Proof. From Lemma 3.1.1, it remains to show that $(1,2) \subset \mathcal{F}$. Let $1 < p < 2$, $f \in L_p(M_T)$ and $\Psi_0 \in W^{2-\frac{2}{p}}(M)$. Since $C^\infty(M_T)$ is dense in $L_p(M_T)$, there exists a sequence of functions $\{f_k\}$ and $\{\Psi_{0k}\}$ in $C^\infty(\overline{M}_T)$ such that f_k converges to f in $L_p(M_T)$ and Ψ_{0k} converges to Ψ_0 in $W_p^{2-\frac{2}{p}}(M)$. Consider a sequence of Ψ_k such that,

$$
\begin{aligned}
&\Psi_{k_t} = \tilde{d}\Delta_M \Psi_k + f_k && \xi \in M, \quad 0 < t < T \\
&\Psi_k = \Psi_{0k} && \xi \in M, \quad t = 0
\end{aligned}
\qquad (3.1.14)
$$

Now we transform system (3.1.14) over a bounded region in $\mathbb{R}^{n-1}$. Similar to the proof of Lemma 3.1.1. Corresponding to each k, let $\tilde{f}_k = f_k \circ \phi_\xi$, $\Phi_{0k} = \Psi_{0k} \circ \phi_\xi$ and using the Laplace Beltrami operator (defined before), (3.1.14) on $B(0,R) \subset U$ takes the form

$$
\begin{aligned}
&\Phi_{kt} = \frac{\tilde{d}}{\sqrt{det\ g}} \partial_j (g^{ij} \sqrt{det\ g}\ \partial_i \Phi_k) + \tilde{f}_k(x,t) && x \in B(0,R), \quad 0 < t < T \quad (3.1.15) \\
&\Phi_k = \Phi_{0k} && x \in B(0,R), \quad t = 0
\end{aligned}
$$

Consequently, in a bounded region $B(0,R) \times (0,T)$ of the Euclidean space, we consider (3.1.15) in the nondivergence form, $\mathcal{L}$ defined in (3.1.7) for each Φ_k with $\tilde{f}$ replace by $\tilde{f}_k$ and Φ_0 by Φ_{0k}. Taking $2r < R$ and using a cut off function $\psi \in C_0^\infty(\mathbb{R}^{n-1}, [0,1])$ defined in (3.1.9), the following Dirichlet homogeneous boundary value problem is obtained. In $Q = B(0,2r)$, $Q_T = B(0,2r) \times (0,T)$ and $S_T =$

$\partial B(0,r) \times (0,T)$, $w_k = \psi\Phi_k$ satisfies the equation

$$\frac{\partial w_k}{\partial t} - \sum_{i,j=1}^{n-1} a_{ij}(x,t)\frac{\partial^2 w_k}{\partial x_i \partial x_j} + \sum_{i=1}^{n-1} a_i(x,t)\frac{\partial w_k}{\partial x_i} = \theta_k(x,t) \qquad (x,t) \in Q_T$$

$$w_k = 0 \qquad (x,t) \in S_T$$

$$w_k\Big|_{t=0} = \psi\Phi_{0k}(x) \quad t = 0, \forall x \in Q$$

where,

$$\theta_k(x,t) = \tilde{f}_k\psi - 2\sum_{i=1}^{n-1} a_{ij}(x,t)\frac{\partial \Phi_k}{\partial x_i}\frac{\partial \psi}{\partial x_j} - \Phi_k \sum_{i,j=1}^{n-1} a_{ij}(x,t)\frac{\partial^2 \psi}{\partial x_i \partial x_j} + \Phi_k \sum_{i=1}^{n-1} a_i(x,t)\frac{\partial \psi}{\partial x_i}$$

Note f_k and Ψ_{0k} are smooth functions. Therefore from Lemma 3.1.1, there exists a solution. $\Phi_k \in W_q^{2,1}(Q_T)$ for all $q \geq 2$. Thus $\theta_k \in L_q(Q_T)$ for all $q \geq 2$. Recall $\psi \in C_0^\infty(\mathbb{R}^{n-1}, [0,1])$. Using Lemma 3.0.9 for $\epsilon > 0$ there exists $c_\epsilon > 0$ such that

$$\begin{aligned} \|\theta_k\|_{p,Q_T} &\leq \|\tilde{f}_k\psi\|_{p,Q_T} + M_1\|\Phi_k\|_{p,Q_T} + M_2\|\Phi_{kx}\|_{p,Q_T} \\ &\leq \|\tilde{f}_k\|_{p,Q_T} + M_1\|\Phi_k\|_{p,Q_T} \\ &\quad + M_2(\epsilon\|\Phi_{kxx}\|_{p,Q_T} + c_\epsilon\|\Phi_k\|_{p,Q_T}) \end{aligned} \tag{3.1.16}$$

Here $M_1, M_2 > 0$ are independent of f and Ψ_0. At this point we need an estimate for $\|\Phi_k\|_{p,Q_T}$. From Lemma 3.0.8 for $1 < p \leq n < q$ there exists $C_\epsilon > 0$ such that

$$\|\Phi_k\|^p_{L_{\frac{pq}{q-p}}(Q_T)} \leq \epsilon(\|\Phi_{kx}\|^p_{p,Q_T} + \|\Phi_{kt}\|^p_{p,Q_T}) + C_\epsilon\|\Phi_k\|^p_{1,Q_T}$$

Since $p < \frac{pq}{q-p}$, from Hölder's inequality, ϵ and C_ϵ get scaled to $\tilde{\epsilon} > 0$ and $C_{\tilde{\epsilon}} > 0$, and

$$\begin{aligned}\|\Phi_k\|_{p,Q_T} \leq \tilde{\epsilon}(\|\Phi_{kt}\|_{p,Q_T} + \|\Phi_{kx}\|_{p,Q_T}) \\ + C_{\tilde{\epsilon}}\|\Phi_k\|_{1,Q_T}\end{aligned} \tag{3.1.17}$$

From (3.1.16) and (3.1.17),

$$\begin{aligned}\|\theta_k\|_{p,Q_T} \leq (M_1 + M_2 c_\epsilon)(\tilde{\epsilon}(\|\Phi_{kt}\|_{p,Q_T} + \|\Phi_{kx}\|_{p,Q_T}) + C_{\tilde{\epsilon}}\|\Phi_k\|_{1,Q_T}) \\ + \|\tilde{f}_k\|_{p,Q_T} + M_2\epsilon\|\Phi_{kxx}\|_{p,Q_T}\end{aligned}$$

Recall $g_{i,j}$ are C^1 functions on the compact manifold M. Therefore $a_{i,j}(x,t)$ and $a_i(x,t)$ satisfy the hypothesis (bounded continuous function in $\overline{Q_T}$) of Lemma 3.0.5. Using Lemma 3.0.5 for $p \neq \frac{3}{2}$,

$$\|w_k\|_{p,Q_T}^{(2)} \leq C_{p,T}(\|\theta_k\|_{p,Q_T} + \|\psi\Phi_{0k}\|_{p,Q}^{(2-\frac{2}{p})}) \tag{3.1.18}$$

where $C_{p,T}$ is independent of θ and $\psi\Phi_0$. Combining (3.1.16) and (3.1.18), we get

$$\begin{aligned}\|w_k\|_{p,Q_T}^{(2)} &\leq C_{p,T}(\|\theta_k\|_{p,Q_T} + \|\psi\Phi_{0k}\|_{p,Q}^{(2-\frac{2}{p})}) \\ &\leq C_{p,T}\{\|\tilde{f}_k\|_{p,Q_T} + M_2\epsilon\|\Phi_{kxx}\|_{p,Q_T} \\ &\quad + (M_1 + M_2 c_\epsilon)(\tilde{\epsilon}(\|\Phi_{kt}\|_{p,Q_T} + \|\Phi_{kx}\|_{p,Q_T}) + C_{\tilde{\epsilon}}\|\Phi_k\|_{1,Q_T}) \\ &\quad + \|\psi\Phi_{0k}\|_{p,Q}^{(2-\frac{2}{p})}\}\end{aligned}$$

Note that $w_k = \Phi_k$ on $W_T = B(0,r) \times (0,T)$. Thus

$$\begin{aligned}\|\Phi_k\|^{(2)}_{p,W_T} \leq C_{p,T}\{&\|\tilde{f}_k\|_{p,Q_T} + M_2\epsilon\|\Phi_{kxx}\|_{p,Q_T}\\ &+ (M_1 + M_2c_\epsilon)(\tilde{\epsilon}(\|\Phi_{kt}\|_{p,Q_T} + \|\Phi_{kx}\|_{p,Q_T}) + C_{\tilde{\epsilon}}\|\Phi_k\|_{1,Q_T})\\ &+ \|\psi\Phi_{0k}\|^{(2-\frac{2}{p})}_{p,Q}\}\end{aligned} \tag{3.1.19}$$

Observe (3.1.19) is over $B(0,r) \times (0,T) \subset \mathbb{R}^{n-1} \times \mathbb{R}_+$. To get an estimate on the manifold, apply the change of variable, $\|\Phi_k\|^{(2)}_{p,W_T} = \|\Psi_k|\det((\phi^{-1})')|\|^{(2)}_{p,\phi(W_T)}$. This gives

$$\begin{aligned}\|\Psi_k\|^{(2)}_{p,\phi(W_T)} \leq \tilde{C}_{p,\xi,T}\{&\|f_k\|_{p,\phi(Q_T)} + M_2\epsilon\|\Psi_{kxx}\|_{p,\phi(Q_T)}\\ &+ (M_1 + M_2c_\epsilon)(\tilde{\epsilon}(\|\Psi_{kt}\|_{p,\phi(Q_T)} + \|\Psi_{kx}\|_{p,\phi(Q_T)}) + C_{\tilde{\epsilon}}\|\Psi_k\|_{1,(\phi(Q_T))})\\ &+ \|\Psi_{0k}\|^{(2-\frac{2}{p})}_{p,\phi(Q)}\}\end{aligned} \tag{3.1.20}$$

where $\tilde{C}_{p,\xi,T} = \hat{C}_{p,T}\left(\dfrac{\max\limits_{\xi\in\phi(Q_T)} |\det((\phi^{-1})')|}{\min\limits_{\xi\in\phi(Q_T)} |\det((\phi^{-1})')|}\right)$.

So far an estimate in one open neighborhood of some point $\xi \in M$ is obtained. As one varies the point ξ on M, there exist corresponding open neighborhoods V_ξ and a smooth diffemorphisms $\phi_\xi : B(0,r) \longrightarrow V_\xi$, which result in different $\tilde{C}_{p,\xi,T}$ for every V_ξ. Consider an open cover of M such that $M = \bigcup_{\xi\in M} V_\xi$. Since M is compact, there exists $\{\xi_1, \xi_2, ..., \xi_N\}$ such that $M \subset \bigcup_{\substack{\xi_j\in M\\ 1\leq j\leq N}} V_{\xi_j}$ and $\tilde{C}_{p,\xi_j,T}$ corrresponding to each V_{ξ_j}. Let $\hat{C}_{p,T} = \sum_{j=1}^N \tilde{C}_{p,\xi_j,T}$. Inequality (3.1.20) implies

$$\begin{aligned}\|\Psi_k\|^{(2)}_{p,M_T} \leq \hat{C}_{p,T}\,\{&\|f_k\|_{p,M_T} + M_2\epsilon\|\Psi_{kxx}\|_{p,M_T} + \|\Psi_{0k}\|^{(2-\frac{2}{p})}_{p,M}\\ &+ (M_1 + M_2c_\epsilon)(\tilde{\epsilon}(\|\Psi_{kt}\|_{p,M_T} + \|\Psi_{kx}\|_{p,M_T}) + C_{\tilde{\epsilon}}\|\Psi_k\|_{1,M_T})\Big\}\end{aligned} \tag{3.1.21}$$

Also from Lemma 3.0.10,

$$\|\Psi_k\|_{1,M} \leq \|f_k\|_{1,M_T} + \|\Psi_{0k}\|_{1,M}$$

Implies, $\|\Psi_k\|_{1,M_T} \leq \|f_k\|_{1,M_T} + \|\Psi_{0k}\|_{1,M}$. Now, choose $\epsilon > 0$ such that,

$$\max\{\hat{C}_{p,T}M_2\epsilon, \quad \hat{C}_{p,T}\tilde{\epsilon}(M_1 + M_2c_\epsilon)\} < \frac{1}{2}$$

For this choice of ϵ, (3.1.21) gives us the $W_p^{2,1}$ estimates of the Cauchy problem on manifold.

$$\|\Psi_k\|_{p,M_T}^{(2)} \leq \hat{C}_{p,T}(\|f_k\|_{p,M_T} + C_\epsilon(\|f_k\|_{1,M_T} + \|\Psi_{0k}\|_{1,M}) + \|\Psi_{0k}\|_{p,M}^{(2-\frac{2}{p})})$$

$$\|\Psi_k\|_{p,M_T}^{(2)} \leq \hat{K}_{p,T}(\|f_k\|_{p,M_T} + \|\Psi_{0k}\|_{p,M}^{(2-\frac{2}{p})}) \tag{3.1.22}$$

where $\hat{K}_{p,T} > 0$ is independent of f_k and Ψ_{0k}. It remains to show that the sequence $\{\Psi_k\}$ converges to Ψ in $W_p^{2,1}(M_T)$, and Ψ solves (3.0.1). From linearity and (3.1.22), if $m, l \in \mathbb{N}$ then $\Psi_m - \Psi_l$ satisfies

$$\begin{aligned}
(\Psi_m - \Psi_l)_t &= \tilde{d}\Delta_M(\Psi_m - \Psi_l) + f_m - f_l & \xi \in M, \quad 0 < t < T \\
\Psi_m - \Psi_l &= \Psi_{0m} - \Psi_{0l} & \xi \in M, \quad t = 0
\end{aligned}$$

with

$$\|\Psi_m - \Psi_l\|_{p,M_T}^{(2)} \leq \hat{K}_{p,T}(\|f_m - f_l\|_{p,M_T} + \|\Psi_{0m} - \Psi_{0l}\|_{p,M}^{(2-\frac{2}{q})})$$

This implies $\{\Psi_k\}$ is a Cauchy sequence in $W_p^{2,1}(M_T)$. Let $k \to \infty$ then f_k converges to f in $L_p(M_T)$, Ψ_{0k} converges to Ψ_0 in $W_p^{2-\frac{2}{p}}(M)$, and Ψ_k converges to $\Psi \in W_p^{2,1}(M_T)$ in the $W_p^{2,1}$ norm. Thus Ψ solves (3.0.1) and (3.1.22) implies

$$\|\Psi\|_{p,M_T}^{(2)} \leq \hat{K}_{p,T}(\|f\|_{p,M_T} + \|\Psi_0\|_{p,M}^{(2-\frac{2}{p})})$$

Hence $\mathcal{F} = (1, \infty)$. $\square$

Chapter 4

The Linear Neumann Boundary Value Problem

In the past, local solvability of nonlinear quasilinear parabolic systems has been proved using both classical and semigroup theory approaches (see Amann [4], Weidemaier [43], and Acquistapace and Terreni [1]). Giaquinta and Modica [14] used classical techniques based on a priori estimates, without assuming any growth condition on functions to prove local solvability. For more details see Giaquinta and Modica [14]. We are interested in Hölder estimates of a solution to a linear Neumann boundary value problem, provided Neumann data lies in $L_p(M \times (0,T))$ for p sufficiently large. These Hölder estimates will be used in the next section to prove local existence for the coupled reaction diffusion system (1.0.1).

As earlier let Ω be a bounded region with smooth boundary $M = \partial\Omega$ (say, belong to the class of $C^{2+\mu}(\mu > 0)$ such that Ω lies locally on one side of M). Here η_j are direction cosines of the outward unit normal to M, $d > 0$ and $\frac{\partial \varphi}{\partial \eta} =$

$\sum_{j=1}^{n} \frac{\partial \varphi}{\partial x_j} \eta_j(x)$. Consider the linear parabolic system,

$$\begin{aligned} \varphi_t &= d\Delta\varphi + \theta && x \in \Omega, \quad 0 < t < \hat{T} \\ d\frac{\partial \varphi}{\partial \eta} &= \gamma && x \in M, \quad 0 < t < \hat{T} \\ \varphi &= \varphi_0 && x \in \Omega, \quad t = 0 \end{aligned} \tag{4.0.1}$$

and the subsystem,

$$\begin{aligned} \varphi_t &= d\Delta\varphi && x \in \Omega, \quad 0 < t < \hat{T} \\ d\frac{\partial \varphi}{\partial \eta} &= \gamma && x \in M, \quad 0 < t < \hat{T} \\ \varphi &= 0 && x \in \Omega, \quad t = 0 \end{aligned} \tag{4.0.2}$$

If $\gamma \in W_p^{1-\frac{1}{p},\frac{1}{2}-\frac{1}{2p}}(M \times (0,\hat{T}))$, then from Theorem 9.1, chapter 4 in [27] there exists a unique solution to (4.0.1) in $W_p^{2,1}(\Omega \times (0,\hat{T}))$. More precisely,

Lemma 4.0.3. *Let $p > 1$. Suppose that $\theta \in L_p(\Omega \times (0,\hat{T}))$, $\varphi_0 \in W_p^{(2-\frac{2}{p})}(\Omega)$ and $\gamma \in W_p^{1-\frac{1}{p},\frac{1}{2}-\frac{1}{2p}}(M \times (0,\hat{T}))$ with $p \neq 3$. In addition, when $p > 3$ assume*

$$d\frac{\partial \varphi_0}{\partial \eta} = \gamma(x,0) \quad \text{on } M$$

Then (4.0.1) has a unique solution $\varphi \in W_p^{2,1}(\Omega \times (0,\hat{T}))$ and there exists C independent of θ, φ_0 and γ such that

$$\|\varphi\|_{p,(\Omega\times(0,\hat{T}))}^{(2)} \leq C(\|\theta\|_{p,(\Omega\times(0,\hat{T}))} + \|\varphi_0\|_{p,\Omega}^{(2-\frac{2}{p})} + \|\gamma\|_{p,(\partial\Omega\times(0,\hat{T}))}^{(1-\frac{1}{p},\frac{1}{2}-\frac{1}{2p})})$$

Proof. See Ladyzenskaya [27] $\square$

Definition 4.0.4. φ is said to be a weak solution of system (4.0.1) from $V_2^{1,\frac{1}{2}}(\Omega_{\hat{T}})$ if and only if

$$-\int_0^{\hat{T}}\int_\Omega \varphi\nu_t - \int_0^{\hat{T}}\int_{\partial\Omega} d\,\nu\frac{\partial\varphi}{\partial\eta} + \int_0^{\hat{T}}\int_\Omega d\,\nabla\nu.\nabla\varphi - \int_0^{\hat{T}}\int_\Omega \theta\nu$$
$$= -\int_\Omega \nu(x,\hat{T})\varphi(x,\hat{T}) + \int_\Omega \nu(x,0)\varphi(x,0)$$

for any $\nu \in W_2^{1,1}(\Omega_{\hat{T}})$ that is equal to zero for $t = \hat{T}$.

We also need a notion of solution of (4.0.2) which was first introduced in the study of Dirichlet and Neumann problems for the Laplace operator in a bounded C^1 domain by E. B. Fabes, M. Jodeit JR and N. M Rivier [11]. They used Albert Calderon's result in [7] on L^p continuity of Cauchy integral operators for C^1 curves. Further in [12], Fabes and Riviere constructed solutions to the initial Neumann problem for the heat equation in a cylindrical domain, $D\times(0,T)$, with D a bounded C^1 domain of $\mathbb{R}^n$ satisfying the zero initial condition in the form of a single layer heat potential, when densities belong to $L_p(\partial D \times (0,T))$, $1 < p < \infty$. We will consider the solution to (4.0.2) in the sense of one which is constructed in [12]. Define

$$J(g(p,t)) = \lim_{\epsilon\to 0}\int_0^{t-\epsilon}\int_M \frac{\langle p-y,\eta_p\rangle}{(t-s)^{\frac{n}{2}+1}}\exp\left(-\frac{|p-y|^2}{4(t-s)}\right) g(s,y)\; dy\; ds$$

for a.e $p \in M$ (for a smooth manifold it is true for all p), where

$$g(p,t) = -2[-c_nI + J]^{-1}\gamma(p,t)$$

and η_p is unit inward normal to M at p. For $Q \in M$, $(x,t) \in \Omega_T$ and $t > s$, consider

$$W(t-s,x,Q) = \frac{\exp\left(\frac{-|x-Q|^2}{4(t-s)}\right)}{(t-s)^{\frac{n}{2}}} \text{ and } g(Q,t) = -2[-c_n I + J]^{-1}\gamma(Q,t)$$

where c_n is given in [12]. In [12], corresponding properties for the integral operator J which will be associated to the restriction of the normal derivative of the single layer potential were obtained. For completeness we state the result in the following proposition.

Proposition 4.0.5. *Assume Ω is a C^1 domain. For $\epsilon > 0$ set*

$$J_\epsilon(g(p,t)) = \int_0^{t-\epsilon}\int_M \frac{\langle p-y,\eta_p\rangle}{(t-s)^{\frac{n}{2}+1}} \exp\left(-\frac{|p-y|^2}{4(t-s)}\right) g(s,y)\, dy\, ds$$

Then

1. *For every $1 < p < \infty$ there exists $C_p > 0$ such that $\sup_{\epsilon>0}|J_\epsilon f(p,t)| = \tilde{J}(f)(p,t)$ satisfies*

$$\|\tilde{J}f\|_{L_p(M\times(0,\hat{T}))} \le C_p\|f\|_{L_p(M\times(0,\hat{T}))} \text{ for all } f \in L_p(M\times(0,\hat{T}))$$

2. *$\lim_{\epsilon\to 0^+} J_\epsilon f = Jf$ exists in $L_p(M\times(0,\hat{T}))$ and pointwise for almost every $(p,t) \in (M\times(0,\hat{T}))$ provided $f \in L_p(M\times(0,\hat{T})), 1 < p < \infty$.*

3. *$c_n I + J$ is invertible on $L_p(M\times(0,\hat{T}))$ for each $c_n \neq 0$ and $1 < p < \infty$.*

Proof. See [12]. □

Definition 4.0.6. φ is said to be a (classical) solution of system (4.0.2), with $d = 1$ if and only if $\varphi(x,t) = \int_0^t \int_M W(t-s,x,y)g(s,y)\ dy\ ds$ for all $(x,t) \in \mathbb{R}^{n+1} \setminus M$ satisfies the properties:

- $\partial_t \varphi - \Delta\varphi = 0$ for all $(x,t) \in \mathbb{R}^{n+1} \setminus M$
- $\langle \nabla_x \varphi(x,t), \eta_p \rangle \to \gamma(p,t)$ pointwise for almost every $(p,t) \in M \times (0,\hat{T})$ as $x \to$ p, $\langle x-p, \eta_p\rangle > \alpha|x-p|$, for some positive constant α.

4.1 Hölder Estimates

Brown made a remark in [6], that if Neumann data lies in $L_p(M \times (0,\hat{T}))$ for p sufficiently large then the solution to problem (4.0.2) is Hölder continuous. Here we will make use of potential theory to give the proof of this remark and then use this result to get Hölder estimates of (4.0.1).

Lemma 4.1.1. *Let $p > n+1$. Suppose (x,T), $(y,\tau) \in \Omega_{\hat{T}}$, $g \in L_p(M \times (0,\hat{T}))$ and $\mathcal{R}^c = \{(Q,s) \in M \times (0,\tau) : |x-Q| + |T-s|^{\frac{1}{2}} < 2(|x-y| + |T-\tau|^{\frac{1}{2}})\}$. Then for $0 < \epsilon < 1$, there exists $K_1 > 0$ independent of g such that,*

$$\int_{\mathcal{R}^c} |(W(T-s,x,Q) - W(\tau-s,y,Q))g(s,Q)|\ dQ\ ds$$
$$\leq K_1 \left(|x-y| + |T-\tau|^{\frac{1}{2}}\right)^{\frac{\epsilon(p-1)}{p}} \parallel g \parallel_{p,M\times[0,\tau]}.$$

Proof.

$$\int_{\mathcal{R}^c} |(W(T-s,x,Q) - W(\tau-s,y,Q))g(s,Q)|\ dQ\ ds$$

$$= \int_{\mathcal{R}^c} \left| \frac{\exp\left(\frac{-|y-Q|^2}{2C(\tau-s)}\right)}{(\tau-s)^{\frac{n}{2}}} - \frac{\exp\left(\frac{-|x-Q|^2}{2C(T-s)}\right)}{(T-s)^{\frac{n}{2}}} \right| |g(Q,s)|\ dQ\ ds$$

Define $A = \{Q \in M : |x-Q| < 2|x-y| + |T-\tau|^{\frac{1}{2}}\}$. Since $|T-\tau| < |T-s|$, $\mathcal{R}^c \subset A \times (0,\tau)$.

Therefore,

$$\begin{aligned}
&\int_{\mathcal{R}^c} \left| \frac{\exp\left(\frac{-|y-Q|^2}{2C(\tau-s)}\right)}{(\tau-s)^{\frac{n}{2}}} - \frac{\exp\left(\frac{-|x-Q|^2}{2C(T-s)}\right)}{(T-s)^{\frac{n}{2}}} \right| |g(Q,s)|\ dQ\ ds \\
&\leq \int_0^\tau \int_A \left| \frac{\exp\left(\frac{-|y-Q|^2}{2C(\tau-s)}\right)}{(\tau-s)^{\frac{n}{2}}} - \frac{\exp\left(\frac{-|x-Q|^2}{2C(T-s)}\right)}{(T-s)^{\frac{n}{2}}} \right| |g(Q,s)|\ dQ\ ds \\
&\leq \left(\int_0^\tau \int_A \frac{\exp\left(\frac{-p'|y-Q|^2}{2C(\tau-s)}\right)}{(\tau-s)^{\frac{np'}{2}}}\ dQ\ ds \right)^{\frac{1}{p'}} \| g \|_{p,A\times[0,\tau]} \\
&\quad + \left(\int_0^\tau \int_A \frac{\exp\left(\frac{-p'|x-Q|^2}{2C(T-s)}\right)}{(T-s)^{\frac{np'}{2}}}\ dQ\ ds \right)^{\frac{1}{p'}} \| g \|_{p,A\times[0,\tau]}
\end{aligned}$$

Using the property $w^N \cdot \exp(-w) \leq c \cdot N$ for N$=\frac{n-1-\epsilon}{2}$ and for some $c > 0$, gives

$$\begin{aligned}
&\int_{\mathcal{R}^c} \left| \frac{\exp\left(\frac{-|y-Q|^2}{2C(\tau-s)}\right)}{(\tau-s)^{\frac{n}{2}}} - \frac{\exp\left(\frac{-|x-Q|^2}{2C(T-s)}\right)}{(T-s)^{\frac{n}{2}}} \right| |g(Q,s)|\ dQ\ ds \\
&\leq C_1 \left(\int_0^\tau (\tau-s)^{\frac{n-1-\epsilon-np'}{2}} ds \int_A \frac{1}{|y-Q|^{n-1-\epsilon}}\ dQ \right)^{\frac{1}{p'}} . \| g \|_{p,A\times[0,\tau]} \\
&\quad + C_2 \left(\int_0^\tau (T-s)^{\frac{n-1-\epsilon-np'}{2}} ds \int_A \frac{1}{|x-Q|^{n-1-\epsilon}}\ dQ \right)^{\frac{1}{p'}} . \| g \|_{p,A\times[0,\tau]}
\end{aligned}$$

Let $\rho_y = |y-Q|$, $\rho_x = |x-Q|$. Notice that in set A, $0 < \rho_x < 2|x-y| + |T-\tau|^{\frac{1}{2}}$ and $0 < \rho_y < |x-y| + \rho_x < 3|x-y| + |T-\tau|^{\frac{1}{2}}$. Therefore,

$$\begin{aligned}
C_1 & \left(\int_0^\tau (\tau - s)^{\frac{n-1-\epsilon-np'}{2}} ds \int_A \frac{1}{|y-Q|^{n-1-\epsilon}} dQ \right)^{\frac{1}{p'}} . \parallel g \parallel_{p,A\times[0,\tau]} \\
& + C_2 \left(\int_0^\tau (T - s)^{\frac{n-1-\epsilon-np'}{2}} ds \int_A \frac{1}{|x-Q|^{n-1-\epsilon}} dQ \right)^{\frac{1}{p'}} . \parallel g \parallel_{p,A\times[0,\tau]} \\
\leq \tilde{C}_1 & \left(\int_0^\tau (\tau - s)^{\frac{n-1-\epsilon-np'}{2}} ds \int_0^{3|x-y|+|T-\tau|^{\frac{1}{2}}} \rho_y^{\epsilon-1} d\rho_y \right)^{\frac{1}{p'}} . \parallel g \parallel_{p,A\times[0,\tau]} \\
& + \tilde{C}_2 \left(\int_0^\tau (T - s)^{\frac{n-1-\epsilon-np'}{2}} ds \int_0^{2|x-y|+|T-\tau|^{\frac{1}{2}}} \rho_x^{\epsilon-1} d\rho_x \right)^{\frac{1}{p'}} . \parallel g \parallel_{p,A\times[0,\tau]} \\
\leq & \frac{\tilde{C}_1}{\epsilon^{\frac{1}{p'}}} (\tau)^{\frac{n+1-\epsilon-np'}{2p'}} . \left(3|x-y| + |T-\tau|^{\frac{1}{2}} \right)^{\frac{\epsilon}{p'}} . \parallel g \parallel_{p,A\times[0,\tau]} \\
& + \frac{\tilde{C}_2}{\epsilon^{\frac{1}{p'}}} \left(T^{\frac{n+1-\epsilon-np'}{2}} - (T-\tau)^{\frac{n+1-\epsilon-np'}{2}} \right)^{\frac{1}{p'}} \left(2|x-y| + |T-\tau|^{\frac{1}{2}} \right)^{\frac{\epsilon}{p'}} \parallel g \parallel_{p,A\times[0,\tau]}
\end{aligned}$$

provided $p' < \frac{n+1-\epsilon}{n}$. Thus,

$$\begin{aligned}
\int_{\mathcal{R}^c} & |(W(T-s,x,Q) - W(\tau-s,y,Q))g(s,Q)| \; dQ \; ds \\
& \leq K_1 \left(|x-y| + |T-\tau|^{\frac{1}{2}} \right)^{\frac{\epsilon}{p'}} \parallel g \parallel_{p,M\times[0,\tau]}.
\end{aligned}$$

□

The proof of the following Lemma makes use of Brown's corollary of his Theorem 3.1 in [6], and also gives a detailed explanation of his remark.

Lemma 4.1.2. *Let $p > n+1$. Suppose (x,T),$(y,\tau) \in \Omega_{\hat{T}}$, $g \in L_p(M \times (0,\hat{T}))$ and $\mathcal{R} = \{(Q,s) \in M \times (0,\tau) : 2(|x-y| + |T-\tau|^{\frac{1}{2}}) < |x-Q| + |T-s|^{\frac{1}{2}}\}$. Then* *for $a < 1 - \frac{n+1}{p}$ there exists $K_2 > 0$ independent of g such that,*

$$\int_{\mathcal{R}} |(W(T-s,x,Q) - W(\tau-s,y,Q))g(s,Q)|\ dQ\ ds$$
$$\leq K_2 \left(|x-y| + |T-\tau|^{\frac{1}{2}}\right)^a \parallel g \parallel_{p,M\times[0,\tau]} .$$

Proof.

$$\begin{aligned}
&|\int_{\mathcal{R}}(W(T-s,x,Q) - W(\tau-s,y,Q))g(s,Q)\ dQ\ ds| \\
&\leq \int_{\mathcal{R}} C\left(\frac{|T-\tau|^{\frac{1}{2}} + |x-y|}{|T-s|^{\frac{1}{2}} + |x-Q|}\right)(1+(T-s)^{\frac{-n}{2}})\exp\left(\frac{-|x-Q|^2}{2C(T-s)}\right)|g(Q,s)|\ dQ\ ds \\
&\leq D_1\left(\frac{1}{2}\right)^{1-a}\int_{\mathcal{R}}\left(\frac{|T-\tau|^{\frac{1}{2}} + |x-y|}{|T-s|^{\frac{1}{2}} + |x-Q|}\right)^a \frac{\exp\left(\frac{-|x-Q|^2}{2C(T-s)}\right)}{(T-s)^{\frac{n}{2}}}|g(Q,s)|\ dQ\ ds \\
&\leq \tilde{D}_1\int_{\mathcal{R}}\frac{1}{|x-Q|^a}\frac{\exp\left(\frac{-|x-Q|^2}{2C(T-s)}\right)}{(T-s)^{\frac{n}{2}}}|g(Q,s)|\ dQ\ ds
\end{aligned}$$

where $D_1 = C(T^{\frac{n}{2}}+1)$ and $\tilde{D}_1 = D_1\left(\frac{1}{2}\right)^{1-a}\left(|T-\tau|^{\frac{1}{2}} + |x-y|\right)^a$. Applying Hölder's inequality and $w^N \cdot \exp(-w) \leq c \cdot N$ for N$=\frac{n-1-\epsilon-ap'}{2}$ and some $c > 0$,

$$\begin{aligned}
&\tilde{D}_1\left(\int_{\mathcal{R}}\frac{1}{|x-Q|^{ap'}}\frac{\exp\left(\frac{-p'|x-Q|^2}{2C(T-s)}\right)}{(T-s)^{\frac{np'}{2}}}\ dQ\ ds\right)^{\frac{1}{p'}} \parallel g \parallel_{p,\mathcal{R}} \\
&\leq \tilde{D}_1\left(\int_0^\tau\int_M\frac{(2C(T-s))^{\frac{n-1-\epsilon-ap'}{2}}}{(T-s)^{\frac{np'}{2}}}\frac{1}{|x-Q|^{n-1-\epsilon}}\ dQ\ ds\right)^{\frac{1}{p'}} \parallel g(Q,s) \parallel_{p,M\times[0,\tau]} \\
&\leq \tilde{D}_1\left(\int_0^\tau (C(T-s))^{\frac{n-1-\epsilon-ap'-np'}{2}}\ ds.\int_M\frac{1}{|x-Q|^{n-1-\epsilon}}\ \ dQ\right)^{\frac{1}{p'}} \parallel g \parallel_{p,M\times[0,\tau]}
\end{aligned}$$

Now again let $\rho_x = |x-Q|$. Then by change of variable,

$$\tilde{D}_1 \left(\int_0^\tau (C(T-s))^{\frac{n-1-\epsilon-ap'-np'}{2}} ds. \int_M \frac{1}{|x-Q|^{n-1-\epsilon}} dQ \right)^{\frac{1}{p'}} \parallel g \parallel_{p,M\times[0,\tau]}$$
$$\leq \beta_1 \tilde{D}_1 \left((C(T))^{\frac{n-1-\epsilon-ap'-np'}{2}+1}. \int_\alpha^{\alpha_M} \frac{\rho_x^{n-2}}{\rho_x^{n-1-\epsilon}} d\rho_x \right)^{\frac{1}{p'}} \parallel g \parallel_{p,M\times[0,\tau]}$$

provided,

$$\frac{n+1-\epsilon-ap'-np'}{2p'} > 0; \quad \text{i.e.} \quad p' < \frac{n+1-\epsilon}{n+a}.$$

Notice, if $a+\epsilon<1$, then $p > \frac{n+1-\epsilon}{1-(a+\epsilon)} > n+1$. Hence, for $a < 1 - \frac{n+1}{p} - \epsilon p'$

$$\int_{\mathcal{R}} |(W(T-s,x,Q) - W(\tau-s,y,Q))g(s,Q)| \; dQ \; ds$$
$$\leq K_2 \left(|x-y| + |T-\tau|^{\frac{1}{2}} \right)^a \parallel g \parallel_{p,M\times[0,\tau]} .$$

□

Lemma 4.1.3. *Let $p > n+1$. Suppose (x,T),$(y,\tau) \in \Omega_{\hat{T}}$ and $g \in L_p(M \times (0,\hat{T}))$. Then for $\epsilon > 0$ there exists $K_3 > 0$, independent of g such that,*

$$\int_\tau^T \int_M \frac{\exp\left(\frac{-|x-Q|^2}{2C(T-s)}\right)}{(T-s)^{\frac{n}{2}}} |g(Q,s)| \; dQ \; ds \leq K_3 (T-\tau)^{\frac{n+1-\epsilon-np'}{2p'}} \parallel g \parallel_{p,M\times[\tau,T]}$$

Proof. Again making use of the property, $w^N \cdot \exp(-w) \leq c \cdot N$ for N$=\frac{n-1-\epsilon}{2}$ and for some $c > 0$,

$$\int_\tau^T \int_M \frac{\exp\left(\frac{-|x-Q|^2}{2C(T-s)}\right)}{(T-s)^{\frac{n}{2}}} |g(Q,s)| \; dQ \; ds$$

$$\leq C_3 \int_\tau^T \int_M \frac{\tilde{C}(T-s)^{\frac{n-1-\epsilon}{2}}}{(T-s)^{\frac{n}{2}}} . \frac{1}{|x-Q|^{n-1-\epsilon}} |g(Q,s)| \; dQ \; ds$$
$$\leq C_3 \left(\int_\tau^T (C(T-s))^{\frac{n-1-\epsilon-np'}{2}} ds. \int_M \frac{1}{|x-Q|^{n-1-\epsilon}} dQ \right)^{\frac{1}{p'}} \parallel g \parallel_{p,M\times[\tau,T]}$$

Similarly, by the change of variable

$$C_3 \left(\int_\tau^T (C(T-s))^{\frac{n-1-\epsilon-np'}{2}} ds. \int_M \frac{1}{|x-Q|^{n-1-\epsilon}} dQ \right)^{\frac{1}{p'}} \parallel g \parallel_{p,M\times[\tau,T]}$$
$$\leq \tilde{C}_3 \left(|(-C(T-\tau))^{\frac{n-1-\epsilon-np'}{2}+1}|. \int_\alpha^{\alpha_M} \frac{\rho_x^{n-2}}{\rho_x^{n-1-\epsilon}} d\rho_x \right)^{\frac{1}{p'}} \parallel g \parallel_{p,M\times[\tau,T]}$$
$$\leq K_3 (T-\tau)^{\frac{n+1-\epsilon-np'}{2p'}} \parallel g \parallel_{p,M\times[\tau,T]}$$

□

Proposition 4.1.4. *Suppose that $\gamma \in L_p(M \times (0,\hat{T}))$ for $p > n+1$ then the classical solution of* (4.0.2) *is Hölder continuous on $\overline{\Omega} \times (0,\hat{T})$ with Hölder exponent $0 < a < 1 - \frac{n+1}{p}$ and there exists $\tilde{K}_p > 0$, independent of γ such that*

$$|\varphi(x,T) - \varphi(y,\tau)| \leq \tilde{K}_p \Big(|T-\tau|^{\frac{1}{2}} + |x-y| \Big)^a \parallel \gamma \parallel_{p,M\times(0,\hat{T})} .$$

Proof. We first prove this proposition for $d = 1$. Let $\hat{\Omega}$ be a closed and bounded subset of Ω. Using a "cut off" function, system (4.0.2) can be converted into a zero boundary value Dirichlet problem and from Ladyzenskaja, Theorem 9.1 in [27], solution of the later system belongs to $W_p^{2,1}(\hat{\Omega} \times (0,T)$. Now $W_p^{2,1}(\hat{\Omega} \times (0,T)$ embedds continuously into the space of Hölder continuous functions (see [27]). As a result we have Hölder continuity of the solution to (4.0.2) in the interior of Ω. We want to extend this behaviour to the boundary.

Pick points (x,T), $(y,\tau) \in Q_{\hat{T}}$ and consider a set $\mathcal{R} = \{(Q,s) \in M \times (0,\tau) : 2(|x-y| + |T-\tau|^{\frac{1}{2}}) < |x-Q| + |T-s|^{\frac{1}{2}}\}$. We know from Fabes and Riviere [12] that the solution of (4.0.2) is given by

$$\varphi(x,T) = \int_0^T \int_M W(T-s,x,Q) g(s,Q)\ dQ\ ds$$

where $W(T-s,x,Q) = \frac{\exp\left(\frac{-|x-Q|^2}{2C(T-s)}\right)}{(T-s)^{\frac{n}{2}}}$, $g(p,t) = -2[-cI+J]^{-1}\gamma(p,t)$ and

$$J(g(p,t)) = lim_{\epsilon \to 0} \int_0^{t-\epsilon} \int_M \frac{\partial W(t-s,p,y)}{\partial \eta_p} g(s,y)\ dy\ ds$$

for a.e $p \in M$ (for smooth manifold it is true for all p), η_p being the unit inward normal at p.

$$\begin{aligned}
|\varphi(x,T) - \varphi(y,\tau)| = &| \int_0^\tau \int_M (W(T-s,x,Q) - W(\tau-s,y,Q)) g(s,Q)\ dQ\ ds \\
&+ \int_\tau^T \int_M W(T-s,x,Q) g(s,Q)\ dQ\ ds| \\
\leq &| \int_{\mathcal{R}^c} (W(T-s,x,Q) - W(\tau-s,y,Q)) g(s,Q)\ dQ\ ds| \\
&+ | \int_{\mathcal{R}} (W(T-s,x,Q) - W(\tau-s,y,Q)) g(s,Q)\ dQ\ ds| \\
&+ \int_\tau^T \int_M C(1+(T-s)^{\frac{-n}{2}}) \exp\left(\frac{-|x-Q|^2}{2C(T-s)}\right) |g(Q,s)|\ dQ\ ds
\end{aligned}$$

Now using Lemma 4.1.1 for $\epsilon > 0$ such that $\frac{\epsilon(p-1)}{p} < a$, where $a < 1 - \frac{n+1}{p}$ (which is possible because $\frac{ap}{p-1} < 1 - \frac{n}{p-1} < 1$ for $p > 1$), and Lemma 4.1.2 and 4.1.3,

$$|\varphi(x,T) - \varphi(y,\tau)| \leq K_1 \left(|x-y| + |T-\tau|^{\frac{1}{2}}\right)^{\frac{\epsilon(p-1)}{p}} \parallel g \parallel_{p, M\times(0,\tau)}$$

$$+ K_2\Big(|T-\tau|^{\frac{1}{2}} + |x-y|\Big)^a \parallel g \parallel_{p,M\times(0,\tau)}$$
$$+ K_3(T-\tau)^{\frac{n+1-\epsilon-np'}{2p'}} \parallel g \parallel_{p,M\times(\tau,T)}$$

So,

$$|\varphi(x,T)-\varphi(y,\tau)| \leq \tilde{K}_p\Big(|T-\tau|^{\frac{1}{2}} + |x-y|\Big)^a \parallel g \parallel_{p,M\times(0,\hat{T})}$$

Thus the proposition is proved for the case $d=1$. The general case reduces to this one, since (4.0.2), as is known , can be written as a heat equation. For this purpose it is necessary to carry out a simple linear transformation of the coordinates. If one introduces a new variable

$$z = \frac{x}{\sqrt{d}}$$

then (4.0.2) becomes

$$\begin{aligned} \varphi_t &= \Delta\varphi && z\in\Omega, \quad 0<t<\hat{T} \\ \frac{\partial\varphi}{\partial\eta} &= \frac{\gamma}{\sqrt{d}} && z\in M, \quad 0<t<\hat{T} \\ \varphi &= 0 && z\in\Omega, \quad t=0 \end{aligned}$$

Hence, φ satisfying (4.0.2) is Hölder continuous with

$$\|\varphi\|^{(a)}_{p,\Omega\times(0,\hat{T})} \leq C_1(\hat{T})\|\gamma\|_{p,M\times(0,\hat{T})}$$

□

Now we combine Hölder estimates and Ladyzenskaja Theorem 9.1(chapter 4) in [27] to get the existence of a Hölder continuous solution to system (4.0.1) for any finite time $T > 0$.

Theorem 4.1.5. *Let $p > 1$. Suppose $\theta \in L_p(\Omega \times (0, \hat{T}))$, $\gamma \in L_p(M \times (0, \hat{T}))$ and $\varphi_0 \in W^{2-\frac{2}{p}}(\Omega)$ with $p \neq 3$. Then for $p = 2$ system (4.0.1) has a unique weak solution in $V_2^{1,\frac{1}{2}}(\Omega \times (0, \hat{T}))$. In addition, when $p > 3$ assume*

$$d\frac{\partial \varphi_0}{\partial \eta} = \gamma(x, 0) \quad \text{on } M$$

Then for $p > n+1$ there exists $0 < \beta < 1$ such that $\beta < 1 - \frac{n+1}{p}$, and $C(\hat{T}, p)$ independent of θ, γ and φ_0 such that

$$|\varphi|^{(\beta)}_{\Omega \times (0,\hat{T})} \leq C(\hat{T}, p)(\|\theta\|_{p,\Omega \times (0,\hat{T})} + \|\gamma\|_{p,M \times (0,\hat{T})} + \|\varphi_0\|^{(2-\frac{2}{p})}_{p,\Omega})$$

Proof. Chapter 4, Theorem 5.1 in [27] implies system (4.0.1) has a unique weak solution. In order to get Hölder estimates, we break (4.0.1) into two sub systems,

$$\begin{aligned}
\varphi_{2t} &= d\Delta\varphi_2 + \theta & & x \in \Omega, \quad 0 < t < \hat{T} \\
d\frac{\partial \varphi_2}{\partial \eta} &= 0 & & x \in M, \quad 0 < t < \hat{T} \\
\varphi_2 &= \varphi_0 & & x \in \Omega, \quad t = 0
\end{aligned} \tag{4.1.3}$$

$$\begin{aligned}
\varphi_{1t} &= d\Delta\varphi_1 & & x \in \Omega, \quad 0 < t < \hat{T} \\
d\frac{\partial \varphi_1}{\partial \eta} &= \gamma & & x \in M, \quad 0 < t < \hat{T} \\
\varphi_1 &= 0 & & x \in \Omega, \quad t = 0
\end{aligned} \tag{4.1.4}$$

Again from Theorem 9.1 in [27], we know there exists a unique solution of (4.1.3) in $W_p^{2,1}(\Omega \times (0,\hat{T}))$, and $C_1(\hat{T},p) > 0$ independent of θ and φ_0 such that

$$\|\varphi_2\|^{(2)}_{p,\Omega\times(0,\hat{T})} \leq C_1(\hat{T},p)(\|\theta\|_{p,\Omega\times(0,T)} + \|\varphi_0\|^{(2-\frac{2}{p})}_{p,\Omega})$$

Using proposition (4.1.4), there exists $\varphi_1 \in C^{\beta,\frac{\beta}{2}}(\overline{\Omega} \times (0,T))$, satisfying (4.1.4) with

$$|\varphi_1|^{(\beta)}_{\Omega\times(0,T)} \leq C_2(\hat{T},p)\|\gamma\|_{p,M\times(0,\hat{T})}$$

By linearity, $\varphi = \varphi_1 + \varphi_2$ solves (4.0.1). Moreover, since for sufficiently large p, $W_p^{2,1}(\Omega \times (0,\hat{T}))$ embedds continuously into the space of Hölder continuous functions, there exists $\tilde{C}(\hat{T},p) > 0$ independent of θ, γ and φ_0 such that

$$|\varphi|^{(\beta)}_{\Omega\times(0,\hat{T})} \leq \tilde{C}(\hat{T},p)(\|\theta\|_{p,\Omega\times(0,\hat{T})} + \|\gamma\|_{p,M\times(0,\hat{T})} + \|\varphi_0\|^{(2-\frac{2}{p})}_{p,\Omega}) \tag{4.1.5}$$

Remark 4.1.6. Obviously, sup norm estimates follow immediately from the Hölder estimates, but sup norm estimates can also be obtained by using the weak formulation only. We need Hölder estimates to prove our local existence result.

□

Chapter 5

Local Existence, Uniqueness, and Non-negativity

Recall, m, k, n, i and j are positive integers, D and $\tilde{D}$ are $k \times k$ and $m \times m$ diagonal matrices with positive entries $\{d_i\}_{1 \le i \le k}$ and $\{\tilde{d}_j\}_{1 \le j \le m}$ respectively. Also, $d_{min} = \min\{d_i : 1 \le i \le k\}$. η is the unit outward normal (from Ω) to M at each of its points; $\dfrac{\partial u}{\partial \eta} = \displaystyle\sum_{i,j=1}^{n} \frac{\partial u}{\partial x_i} \eta_j$, where η_j is the j^{th} component of η. The primary concern of this work is the system

$$
\begin{aligned}
u_t &= D\Delta u + H(u) & x \in \Omega, \quad & 0 < t < T \\
v_t &= \tilde{D}\Delta_M v + F(u, v) & x \in M, \quad & 0 < t < T \\
D\frac{\partial u}{\partial \eta} &= G(u, v) & x \in M, \quad & 0 < t < T \\
u &= u_0 & x \in \Omega, \quad & t = 0 \\
v &= v_0 & x \in M, \quad & t = 0
\end{aligned}
\tag{5.0.1}
$$

where $F = (F_i) : \mathbb{R}^k \times \mathbb{R}^m \to \mathbb{R}^m, G = (G_j) : \mathbb{R}^k \times \mathbb{R}^m \to \mathbb{R}^k$ and $H = (H_j) : \mathbb{R}^k \to \mathbb{R}^k$, and $u_0 = (u_{0j}) \in W_p^{2-\frac{2}{p}}(\Omega)$, $v_0 = (v_{0i}) \in W_p^{2-\frac{2}{p}}(M)$ with $p > n$. Also, for $p > 3$, u_0 and v_0 satisfy compatibility condition

$$D\frac{\partial u_0}{\partial \eta} = G(u_0, v_0) \quad \text{on } M.$$

Remark 5.0.2. For $p > n$, u_0 and v_0 are Hölder continuous functions on $\overline{\Omega}$ and M respectively (See [2], [10]).

Definition 5.0.6. A function (u, v) is a solution of (5.0.1) if and only if

$$u \in C(\overline{\Omega} \times [0, T), \mathbb{R}^k) \cap C^{1,0}(\overline{\Omega} \times (0, T), \mathbb{R}^k) \cap C^{2,1}(\Omega \times (0, T), \mathbb{R}^k)$$

and

$$v \in C(M \times [0, T), \mathbb{R}^m) \cap C^{2,1}(M \times (0, T), \mathbb{R}^m)$$

such that (u, v) satisfies (5.0.1). Moreover, if $T = \infty$, the solution is said to be a global solution.

Definition 5.0.7. A function (u, v) defined for $0 \leq t < b$ is a maximal solution of (5.0.1) if and only if (u, v) solves (5.0.1) with $T = b$, and if $d > b$ and $(\tilde{u}, \tilde{v})$ solves (5.0.1) for $T = d$ then there exists $0 < c < b$ such that $(u(\cdot, c), v(\cdot, c)) \neq (\tilde{u}(\cdot, c), \tilde{v}(\cdot, c))$.

Definition 5.0.8. F, G and H are quasi-positive if and only if $F_i(\zeta, \eta) \geq 0$ whenever $\eta \in [0, \infty)^m$ and $\zeta \in [0, \infty)^k$ with $\eta_i = 0$ for $i = 1, ..., m$, and $G_j(\zeta, \eta) \geq 0$, $H_j(\zeta) \geq 0$ whenever $\eta \in [0, \infty)^m$ $\zeta \in [0, \infty)^k$ with $\zeta_j = 0$, for $j = 1, ..., k$.

Theorem 5.0.9. *Suppose F, G and H are Lipschitz and componentwise uniformly bounded. Then* (5.0.1) *has a unique global solution.*

Proof. Let $T > 0$, Set

$$X = \{(u, v) \in C(\overline{\Omega} \times [0, T]) \times C(M \times [0, T]) : u(x, 0) = 0, \forall\ x \in \overline{\Omega}, v(x, 0) = 0, \forall\ x \in M\}$$

Note $(X, \|\cdot\|_\infty)$ is a Banach space. Also, for simplicity of notations in remainder of the proof, we assume $u_0, v_0 = 0$, and compatibility condition holds. Fix $(\tilde{u}, \tilde{v}) \in X$, and consider

$$\begin{aligned}
u_t &= D\Delta u + H(\tilde{u}) && x \in \Omega, \quad 0 < t < T \\
v_t &= \tilde{D}\Delta_M v + F(\tilde{u}, \tilde{v}) && x \in M, \quad 0 < t < T \\
D\frac{\partial u}{\partial \eta} &= G(\tilde{u}, \tilde{v}) && x \in M, \quad 0 < t < T \\
u &= u_0 && x \in \Omega, \quad t = 0 \\
v &= v_0 && x \in M, \quad t = 0
\end{aligned} \tag{5.0.3}$$

From Theorems 3.1.2 and 4.1.5, system (5.0.3) possesses a unique weak solution (u, v). Furthermore, from embeddings, $(u, v) \in C(\overline{\Omega} \times [0, T]) \times C(M \times [0, T])$. Now define a map,

$$S : X \to X$$

$$\text{given by } S(\tilde{u}, \tilde{v}) = (u, v)$$

We will see that S is continuous and compact, with respect to the natural norm on X.

Let $(\tilde{u},\tilde{v})$ and $(\tilde{U},\tilde{V}) \in X$, and define $(u,v) = S(\tilde{u},\tilde{v})$ and $(U,V) = S(\tilde{U},\tilde{V})$. Using linearity, $(u-U, v-V)$ solves

$$\begin{aligned}
u_t - U_t &= D\Delta(u-U) + H(\tilde{u}) - H(\tilde{U}) && x\in\Omega, \quad 0<t<T\\
v_t - V_t &= \tilde{D}\Delta_M(v-V) + F(\tilde{u},\tilde{v}) - F(\tilde{U},\tilde{V}) && x\in M, \quad 0<t<T\\
D\frac{\partial(u-U)}{\partial\eta} &= G(\tilde{u},\tilde{v}) - G(\tilde{U},\tilde{V}) && x\in M, \quad 0<t<T\\
u-U &= 0 && x\in\Omega, \quad t=0\\
v-V &= 0 && x\in M, \quad t=0
\end{aligned}$$

From Theorem 4.1.5, if $p > n+1$ there exists K independent of $H, G, F, \tilde{u}, \tilde{v}, \tilde{U}$ and $\tilde{V}$ such that

$$\begin{aligned}
\|u-U\|_{\infty,\Omega\times(0,T)} + \|v-V\|_{\infty,M\times(0,T)} \le K(&\|F(\tilde{u},\tilde{v}) - F(\tilde{U},\tilde{V})\|_{p,M\times(0,T)}\\
&+ \|G(\tilde{u},\tilde{v}) - G(\tilde{U},\tilde{V})\|_{p,M\times(0,T)}\\
&+ \|H(\tilde{u}) - H(\tilde{U})\|_{p,\Omega\times(0,T)})
\end{aligned}$$

Using the boundedness of Ω and M, there exists $\tilde{K} > 0$ such that

$$\begin{aligned}
\|u-U\|_{\infty,\Omega\times(0,T)} + \|v-V\|_{\infty,M\times(0,T)} \le \tilde{K}(&\|F(\tilde{u},\tilde{v}) - F(\tilde{U},\tilde{V})\|_{\infty,M\times(0,T)}\\
&+ \|G(\tilde{u},\tilde{v}) - G(\tilde{U},\tilde{V})\|_{\infty,M\times(0,T)}\\
&+ \|H(\tilde{u}) - H(\tilde{U})\|_{\infty,\Omega\times(0,T)})
\end{aligned}$$

Since, F, G, H are Lipschitz functions there exists $\tilde{M} > 0$ such that

$$\|u - U\|_{\infty,\Omega\times(0,T)} + \|v - V\|_{\infty,M\times(0,T)} \leq \tilde{M}(\|\tilde{u} - \tilde{U}\|_{\infty,\overline{\Omega}\times(0,T)} + \|\tilde{v} - \tilde{V})\|_{\infty,M\times(0,T)})$$

Therefore S is continuous. Moreover, for $p > n+1$, from Theorem 4.1.5, 3.1.2, and Lemma 3.0.12 there exists $\hat{C}(T,p) > 0$, independent of $F(\tilde{u},\tilde{v}), G(\tilde{u},\tilde{v}), H(\tilde{u}), u_0$ and v_0 such that for all $\alpha < 1 - \frac{n}{p}$, $\beta < 1 - \frac{n+1}{p}$,

$$\begin{aligned}|u|^{(\beta)}_{\Omega\times(0,T)} + |v|^{(\alpha)}_{M\times(0,T)} \leq \hat{C}(T,p)(&\|H(\tilde{u})\|_{p,\Omega\times(0,T)} + \|G(\tilde{u},\tilde{v})\|_{p,M\times(0,T)} \\ &+ \|F(\tilde{u},\tilde{v})\|_{p,M\times(0,T)} + \|v_0\|^{(2-\frac{2}{p})}_{p,M} + \|u_0\|^{(2-\frac{2}{p})}_{p,\Omega})\end{aligned}$$

Note H, F and G are uniformly bounded which implies S is compact.

Now we show S has a fixed point. To this end, we show that the set A=$\{(u,v) \in C(\overline{\Omega}\times[0,T]) \times C(M\times[0,T]) : (u,v) = \lambda S(u,v)$ for some $0 < \lambda \leq 1\}$ is bounded in $C(\overline{\Omega}\times[0,T]) \times C(M\times[0,T])$. Let $(u,v) \in$ A. Then there exists $0 < \lambda \leq 1$ such that $(u,v) = \lambda S(u,v)$. That is, (u,v) solves

$$\begin{aligned}
u_t &= D\Delta u + \lambda H(u) && x \in \Omega, \quad 0 < t < T \\
v_t &= \tilde{D}\Delta_M v + \lambda F(u,v) && x \in M, \quad 0 < t < T \\
D\frac{\partial u}{\partial \eta} &= \lambda G(u,v) && x \in M, \quad 0 < t < T \\
u &= u_0 && x \in \Omega, \quad t = 0 \\
v &= v_0 && x \in M, \quad t = 0
\end{aligned}$$

From Theorem 4.1.5 and again using uniform boundedness of H, F and G, there exists $N > 0$ such that $\|(u,v)\|_\infty \leq N$, with N independent of λ, u and v. Hence boundedness of the set is accomplished. Thus, applying Schaefer's theorem (see [10]), we conclude S has a fixed point (u,v) which in turn solves (5.0.1). Further bootstrapping the regularity of (u,v), we obtained a solution in the sense of (5.0.6).

Now we show the solution of (5.0.1) is unique. Suppose $(u,v), (\hat{u},\hat{v})$ solve (5.0.1). Then, $(u-\hat{u}, v-\hat{v})$ satisfies

$$\begin{aligned}
u_t - \hat{u}_t &= D\Delta(u-\hat{u}) + H(u) - H(\hat{u}) && x \in \Omega, \quad t > 0\\
v_t - \hat{v}_t &= \tilde{D}\Delta_M(v-\hat{v}) + F(u,v) - F(\hat{u},\hat{v}) && x \in M, \quad t > 0\\
D\frac{\partial(u-\hat{u})}{\partial\eta} &= G(u,v) - G(\hat{u},\hat{v}) && x \in M, \quad t > 0\\
u - \hat{u} &= 0 && x \in \Omega, \quad t = 0\\
v - \hat{v} &= 0 && x \in M, \quad t = 0
\end{aligned}$$

Taking the dot product of the $v_t - \hat{v}_t$ equation with $(v-\hat{v})$, and the $u_t - \hat{u}_t$ equation with $(u-\hat{u})$, and integrating over M and Ω respectively, yields

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|v-\hat{v}\|_{2,M}^2 + \|u-\hat{u}\|_{2,\Omega}^2) + D\|\nabla(u-\hat{u})\|_{2,\Omega}^2\\
&\qquad \leq \|v-\hat{v}\|_{2,M}\|F(u,v) - F(\hat{u},\hat{v})\|_{2,M} + \|u-\hat{u}\|_{2,\Omega}\|H(u) - H(\hat{u})\|_{2,\Omega}\\
&\qquad\quad + \|u-\hat{u}\|_{2,M}\|G(u,v) - G(\hat{u},\hat{v})\|_{2,M}\\
&\qquad \leq K\|v-\hat{v}\|_{2,M}\left(\|u-\hat{u}\|_{2,M} + \|v-\hat{v}\|_{2,M}\right)\\
&\qquad\quad + K\|u-\hat{u}\|_{2,M}\left(\|u-\hat{u}\|_{2,M} + \|v-\hat{v}\|_{2,M}\right) + +K\|u-\hat{u}\|_{2,\Omega}^2\\
&\qquad \leq K(\|v-\hat{v}\|_{2,M}^2 + \|u-\hat{u}\|_{2,M}^2)\\
&\qquad\quad + 2K\|u-\hat{u}\|_{2,M}\|v-\hat{v}\|_{2,M} + K\|u-\hat{u}\|_{2,\Omega}^2
\end{aligned}$$

$$\leq 2K(\|v-\hat{v}\|_{2,M}^2 + \|u-\hat{u}\|_{2,M}^2) + K\|u-\hat{u}\|_{2,\Omega}^2$$

From Lemma 3.0.8 for $p = 2$ and $\epsilon = \frac{d_{min}}{2K}$, we have

$$\|u-\hat{u}\|_{2,M}^2 \leq \frac{d_{min}}{2K}\|\nabla(u-\hat{u})\|_{2,\Omega}^2 + \tilde{C}_\epsilon\|u-\hat{u}\|_{2,\Omega}^2 \tag{5.0.4}$$

Using (5.0.4)

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(\|v-\hat{v}\|_{2,M}^2 + \|u-\hat{u}\|_{2,\Omega}^2\right) &\leq 2K\|v-\hat{v}\|_{2,M}^2 + K(1+2\tilde{C}_\epsilon)\|u-\hat{u}\|_{2,\Omega}^2 \\ &\leq C_{\epsilon,k}\left(\|v-\hat{v}\|_{2,M}^2 + \|u-\hat{u}\|_{2,\Omega}^2\right)\end{aligned}$$

Observe, $(u-\hat{u}), (v-\hat{v}) = 0$ at $t = 0$ and $\left(\|u-\hat{u}\|_{2,\Omega}^2 + \|v-\hat{v}\|_{2,M}^2\right) \geq 0$. Therefore, applying Gronwall's inequality, we get $\|v-\hat{v}\|_{2,M} = 0$ and $\|u-\hat{u}\|_{2,\Omega} = 0$. Thus, $v = \hat{v}$ and $u = \hat{u}$. Hence system (5.0.1) has a unique global solution. $\square$

Theorem 5.0.10. *Suppose F, G, and H are locally Lipschitz. Then there exists $T_{\max} > 0$ such that* (5.0.1) *has a unique maximal solution (u, v) with $T = T_{\max}$. Moreover, if $T_{\max} < \infty$ then*

$$\limsup_{t\to T_{\max}^-}\|u(\cdot,t)\|_{\infty,\Omega} + \limsup_{t\to T_{\max}^-}\|v(\cdot,t)\|_{\infty,M} = \infty$$

Proof. Recall that $u_0 \in W_p^{2-\frac{2}{p}}(\Omega)$, $v_0 \in W_p^{2-\frac{2}{p}}(M)$ with $p > n$ and u_0, v_0 satisfies the compatibility condition for $p > 3$. From Sobolev imbedding (see [13], [27]), u_0, v_0 are bounded functions. Therefore there exists $\tilde{r} > 0$ such that $\|u_0(\cdot)\|_{\infty,\Omega} \leq \tilde{r}$ and $\|v_0(\cdot)\|_{\infty,M} \leq \tilde{r}$.

For each $r > \tilde{r}$, we define cut off functions $\phi_r \in C_0^\infty(\mathbb{R}^k, [0,1])$ and $\psi_r \in C_0^\infty((\mathbb{R}^k \times \mathbb{R}^m), [0,1])$ such that $\phi_r(z) = 1$ for all $|z| \le r$, and $\phi_r(z) = 0$ for all $|z| > 2r$. Similarly $\psi_r(z,w) = 1$ when $|z| \le r$ and $|w| \le r$, and $\psi_r(z,w) = 0$ when $|z| > 2r$, and $|w| > 2r$. In addition, we define $H_r = H\phi_r, F_r = F\psi_r$ and $G_r = G\psi_r$. From construction, $H_r(z) = H(z), F_r(z,w) = F(z,w)$ and $G_r(z,w) = G(z,w)$ when $|z| \le r$ and $|w| \le r$. Also, there exists $M_r > 0$ such that H_r, G_r and F_r are Lipschitz functions with Lipschitz coefficient M_r. Moreover H_r, F_r and G_r are uniformly bounded functions with bounds depending on r.

Consider the "restricted" system

$$
\begin{aligned}
u_t &= D\Delta u + H_r(u) && x \in \Omega, \quad t > 0 \\
v_t &= \tilde{D}\Delta_M v + F_r(u,v) && x \in M, \quad t > 0 \\
D\frac{\partial u}{\partial \eta} &= G_r(u,v) && x \in M, \quad t > 0 \\
u &= u_0 && x \in \Omega, \quad t = 0 \\
v &= v_0 && x \in M, \quad t = 0
\end{aligned}
\tag{5.0.5}
$$

From Theorem 5.0.9, system (5.0.5) has a unique global solution (u_r, v_r), and there exists $T_r > 0$ (it is possible $T_r = \infty$, in which case we have global existence) such that

$$|(u_r(x,t), v_r(z,t))| \le r \quad \forall t \in [0, T_r], x \in \overline{\Omega}, z \in M$$

and for all $\tau > T_r$ there exists t such that $T_r < t < \tau$, $x \in \overline{\Omega}$ and $z \in M$ such that

$$|(u_r(t,x), v_r(t,z))| > r$$

Note that T_r is increasing with respect to r. Let $T_{\max} = \lim_{r\to\infty} T_r$. Now we define (u,v) as follows. Given $0 < t < T_{\max}$, there exists $r > 0$ such that $t < T_r \le T_{\max}$. For all $x \in \overline{\Omega}$, $u(x,t) = u_r(x,t)$, and for all $x \in M$, $v(x,t) = v_r(x,t)$. Furthermore (u,v) solves (5.0.1) with $T = T_{\max}$. Also, uniqueness of (u_r, v_r) implies uniqueness of (u,v). It remains to show that the solution of (5.0.1) is maximal and if $T_{\max} < \infty$ then

$$\limsup_{t\to T_{\max}^-} \|u(\cdot,t)\|_{\infty,\Omega} + \limsup_{t\to T_{\max}^-} \|v(\cdot,t)\|_{\infty,M} = \infty.$$

Let $T_{\max} < \infty$ and set,

$$\limsup_{t\to T_{\max}^-} \|u(\cdot,t)\|_{\infty,\Omega} + \limsup_{t\to T_{\max}^-} \|v(\cdot,t)\|_{\infty,M} = R.$$

If $R = \infty$ then (u,v) is a maximal solution. If $R < \infty$ there exists $L > 0$ such that

$$\|u\|_{\infty,\Omega\times(0,T_{\max})} + \|v\|_{\infty,M\times(0,T_{\max})} \le L.$$

As a result, $T_{2L} > T_{\max}$, contradicting the construction of T_{2L}. $\square$

Now we prove that under some extra assumptions, the solution to (5.0.1) is componentwise non-negative. Consider the system,

$$\begin{aligned}
u_t &= D\Delta u + H(u^+) && x \in \Omega, \quad 0 < t < T \\
v_t &= \tilde{D}\Delta_M v + F(u^+, v^+) && x \in M, \quad 0 < t < T \\
D\frac{\partial u}{\partial \eta} &= G(u^+, v^+) && x \in M, \quad 0 < t < T \\
u &= u_0 && x \in \Omega, \quad t = 0
\end{aligned} \tag{5.0.6}$$

$$v = v_0 \qquad\qquad x \in M, \quad t = 0$$

where $u^+ = \max(u(x,t), 0)$ and $u^- = -\min(u(x,t), 0)$.

Proposition 5.0.11. *Suppose F, G and H are locally Lipschitz, quasi-positive functions, and u_0, v_0 are componentwise non-negative functions. Then the unique solution (u, v) of* (5.0.6) *is componentwise non-negative.*

Proof. Note that $F(u^+, v^+)$, $G(u^+, v^+)$ and $H(u^+)$ are locally Lipschitz functions of u and v. Therefore from Theorem 5.0.10 there exists a unique maximal solution to (5.0.6) on $(0, T_{\max})$. Consider (5.0.6) componentwise. Multiply the v_{it} equation by v_i^- and the u_{j_t} equation by u_j^-,

$$v_i^- \frac{\partial v_i}{\partial t} = \tilde{d}_i v_i^- \Delta_M v_i + v_i^- F_i(u^+, v^+) \tag{5.0.7}$$

$$u_j^- \frac{\partial u_j}{\partial t} = d_j u_j^- \Delta u_j + u_j^- H_j(u^+) \tag{5.0.8}$$

Since $w^- \frac{dw}{dt} = \frac{-1}{2} \frac{d}{dt}(w^-)^2$,

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial t}(v_i^-)^2 + \frac{1}{2}\frac{\partial}{\partial t}(u_j^-)^2 = &-\tilde{d}_i v_i^- \Delta_M v_i - v_i^- F_i(u^+, v^+)\\ &- d_j u_j^- \Delta u_j - u_j^- H_j(u^+)\end{aligned}$$

Integrating (5.0.7) and (5.0.8) over M and Ω respectively, gives

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|v_i^-(\cdot,t)\|_{2,M}^2 + \frac{1}{2}\frac{d}{dt}\|u_j^-(\cdot,t)\|_{2,\Omega}^2 = &\ \tilde{d}_i \int_M \nabla v_i^- \cdot \nabla v_i - \int_M v_i^- F_i(u^+, v^+)\\ &+ d_j \int_\Omega \nabla u_j^- \cdot \nabla u_j - \int_\Omega u_j^- H_j(u^+)\end{aligned}$$

$$- \int_M u_j^- G_j(u^+, v^+)$$

Consequently,

$$\frac{1}{2}\frac{d}{dt}\|v_i^-(\cdot,t)\|_{2,M}^2 + \frac{1}{2}\frac{d}{dt}\|u_j^-(\cdot,t)\|_{2,\Omega}^2 + \tilde{d}_i \int_M |\nabla v_i^-|^2 + d_j \int_\Omega |\nabla u_j^-|^2$$
$$= - \int_\Omega u_j^- H_j(u^+) - \int_M u_j^- G_j(u^+, v^+) - \int_M v_i^- F_i(u^+, v^+)$$

Since F, G and H are quasi-positive and $\tilde{d}_i, d_j > 0$,

$$\frac{1}{2}\frac{d}{dt}\|v_i^-(\cdot,t)\|_{2,M}^2 + \frac{1}{2}\frac{d}{dt} \parallel u_j^-(\cdot,t) \parallel_{2,\Omega}^2 \leq 0$$

Therefore, the solution (u, v) is componentwise non-negative. □

Corollary 5.0.12. *Suppose F, G and H are locally Lipschitz, quasi-positive functions, and u_0, v_0 are componentwise non-negative functions. Then the unique solution (u, v) of (5.0.1) is componentwise non-negative.*

Proof. From Theorem 5.0.10 and Proposition 5.0.11, there exists a unique, componentwise non-negative and maximal solution, (u, v) to (5.0.6) on $(0, \hat{T})$. Infact (u, v) also solves (5.0.1). The only observation left to make is $\hat{T} = T_{\max}$. If $\hat{T} < T_{\max}$, then since (u, v) satisfies (5.0.1), it has to be bounded, which contradicts the fact that $(0, \hat{T})$ is the maximal interval of existence of (u, v) to (5.0.6). Therefore $\hat{T} = T_{\max}$. □

Chapter 6

Global Existence of Solutions

In this section we will establish the global existence of solutions of the reaction-diffusion system

$$\begin{aligned}
u_t &= d\Delta u + H(u) && (x,t) \in \Omega \times (0,T) \\
v_t &= \tilde{d}\Delta_M v + F(u,v) && (x,t) \in M \times (0,T) \\
d\frac{\partial u}{\partial \eta} &= G(u,v) && (x,t) \in M \times (0,T) \\
u &= u_0 && x \in \Omega, \quad t = 0 \\
v &= v_0 && x \in M, \quad t = 0
\end{aligned} \tag{6.0.1}$$

Throughout, we assume $F \in C^1(\mathbb{R} \times \mathbb{R}, \mathbb{R})$, $G \in C^1(\mathbb{R} \times \mathbb{R}, \mathbb{R})$, $H \in C^1(\mathbb{R}, \mathbb{R})$, $u_0 \in W_p^{2-\frac{2}{p}}(\Omega)$, $v_0 \in W_p^{2-\frac{2}{p}}(M)$ with $p > n$ and, for $p > 3$, u_0 and v_0 satisfy compatibility condition

$$d\frac{\partial u_0}{\partial \eta} = G(u_0, v_0) \quad \text{on } M.$$

Also, F, G and H are quasi-positive (as in definition 5.0.8) and u_0, v_0 are non-negative functions. Then from Theorem 5.0.10 there exists a unique non-negative solution of system (6.0.1) on $(\overline{\Omega} \times [0, T_{max}), M \times [0, T_{max}))$.

In addition to the assumptions stated above, we assume the following conditions:

(V1) There exists $\alpha, \beta > 0$ such that

$$F(\zeta,\nu)+G(\zeta,\nu) \leq \alpha(\zeta+\nu+1) \quad \text{and} \quad H(\zeta) \leq \beta(\zeta+1) \quad \text{for all} \quad \nu \geq 0, \zeta \geq 0$$

(V2) There exists $K_g > 0$ such that

$$G(\zeta,\nu) \leq K_g(\zeta+\nu+1) \quad \text{for all} \quad \nu \geq 0,\ \zeta \geq 0$$

(V3) There exists $l \in \mathbb{N}$ and $K_f > 0$ such that

$$F(\zeta,\nu) \leq K_f(\zeta+\nu+1)^l \quad \text{for all} \quad \nu \geq 0,\ \zeta \geq 0$$

Remark 6.0.13. We refer to (V2) in some sense as a linear "intermediate sums" condition mentioned by Morgan in [30], [31]. (V1) allows high-order nonlinearities in F but requires cancellation of high-order positive terms by G. (V3) implies F is polynomialy bounded above.

Remark 6.0.14. We will show that (V1) provides L_1 estimates for u on Ω and v on M, (V2) helps us obtain get better L_p estimates of u on $M \times (0, T_{max})$ from L_p estimates of v on $M \times (0, T_{max})$, and finally (V3) allows us to use L_q bounds to obtain sup norm bounds on u and v.

Theorem 6.0.15. *If* $(V1)-(V3)$ *are satisfied then the unique solution* (u,v) *of system* (6.0.1) *exists with* $T_{max}=\infty$. *i.e,* (6.0.1) *has a unique, non-negative global solution.*

Before proceeding to the proof of the Theorem 6.0.15, we obtain some preliminary estimates and set up a bootstrapping framework.

6.1 Bootstrapping Strategy

The following system will play a central role in duality arguments.

$$\begin{aligned} &\Psi_t = -\tilde{d}\Delta_M\Psi - \tilde{\vartheta} && (x,t)\in M\times(\tau,T) \\ &\Psi = 0 && x\in M,\quad t=T \end{aligned} \qquad (6.1.2\text{ a})$$

$$\begin{aligned} \varphi_t &= -d\Delta\varphi - \vartheta && (x,t)\in\Omega\times(\tau,T) \\ \kappa_1 d\frac{\partial\varphi}{\partial\eta} + \kappa_2\varphi &= \Psi && (x,t)\in M\times(\tau,T) \\ \varphi &= 0 && x\in\Omega,\quad t=T \end{aligned} \qquad (6.1.2\text{ b})$$

Here, $0<\tau<T<T_{max}$, $\tilde{\vartheta}\in L_p(M\times(\tau,T))$ and $\tilde{\vartheta}\geq 0$ with $\|\tilde{\vartheta}\|_{p,(M\times(\tau,T))}=1$, and $\vartheta\in L_p(\Omega\times(\tau,T))$ and $\vartheta\geq 0$ with $\|\vartheta\|_{p,(\Omega\times(\tau,T))}=1$. Also $d>0$, $\tilde{d}>0$, $\kappa_1\neq 0$ and $\kappa_2\in\mathbb{R}$. Lemmas 6.1.1 to 6.1.6 provide helpful estimates.

Lemma 6.1.1. *Let* $p>1$ *and suppose* $\tilde{\vartheta}\in L_p(M\times(\tau,T))$. *Then* (6.1.2*a*) *has a unique solution* $\Psi\in {W_p}^{2,1}(M\times(\tau,T))$ *and there exists* $\hat{C}_{p,T}>0$ *independent of* $\tilde{\vartheta}$

such that

$$\|\Psi\|^{(2)}_{p,M\times(\tau,T)} \le \hat{C}_{p,T}\|\tilde{\vartheta}\|_{p,M\times(\tau,T)}$$

Proof. See Theorem 3.1.2. □

Lemma 6.1.2. *Let $p > 1$ and suppose $\vartheta \in L_p(\Omega\times(\tau,T))$, $\tilde{\vartheta} \in L_p(M\times(\tau,T))$ and $\Psi \in W_p^{1-\frac{1}{p},\frac{1}{2}-\frac{1}{2p}}(M\times(\tau,T))$. Then (6.1.2b) has a unique solution $\varphi \in W_p^{2,1}(\Omega\times(\tau,T))$. Moreover, there exists $C_{p,T} > 0$ independent of ϑ and $\tilde{\vartheta}$ and dependent on $d, \tilde{d}, \kappa_1$ and κ_2 such that*

$$\|\varphi\|^{(2)}_{p,(\Omega\times(\tau,T))} \le C_{p,T}(\|\vartheta\|_{p,(\Omega\times(\tau,T))} + \|\tilde{\vartheta}\|_{p,M\times(\tau,T)})$$

Proof. The result follows from Theorem 9.1, Ladyzenskaya [27] and Lemma 6.1.1. □

Remark 6.1.3. If $p > n+2$ and $\kappa_1 \neq 0$, then $\nabla\varphi$ is Holder continuous in x and t. See Corollary after Theorem 9.1, chapter 4 of Ladyzenskaya [27].

Lemma 6.1.4. *Suppose $l > 0$ is a non-integral number, $d > 0$, $\vartheta \in C^{l,\frac{l}{2}}(\overline{\Omega}\times[\tau,T])$, $\tilde{\vartheta} \in C^{l,\frac{l}{2}}(M\times[\tau,T])$, $\varphi(x,T) \in C^{2+l}(\overline{\Omega})$ and $\Psi \in C^{l+1,\frac{(l+1)}{2}}(M\times[\tau,T])$. Then (6.1.2b) has a unique solution in $C^{l+2,\frac{l}{2}+1}(\overline{\Omega}\times[\tau,T])$. Moreover there exists $c > 0$ independent of Ψ and ϑ such that*

$$|\varphi|^{(l+2)}_{\Omega\times[\tau,T]} \le c\left(|\vartheta|^{(l)}_{\Omega\times[\tau,T]} + |\Psi|^{(l+1)}_{M\times(\tau,T)}\right)$$

Proof. See Theorem 5.3 in chapter 4 of Ladyzenskaya [27]. □

Lemma 6.1.5. *Suppose* $1 < p < \infty$ *and* $\varphi \in W_p^{2,1}(\Omega \times (\tau, T))$. *If* $q \geq p$ *and* $2 - 2r - s - \left(\frac{1}{p} - \frac{1}{q}\right)(n+2) \geq 0$ *then there exists* $\tilde{K} > 0$ *depending on* Ω, r, s, n, p *such that*

$$\|D_t^r D_x^s \varphi\|_{q,\Omega\times(\tau,T)} \leq \tilde{K} \|\varphi\|_{p,\Omega\times(\tau,T)}^{(2)}$$

Proof. See Lemma 3.3 in chapter 2, Ladyzenskaya [27]. □

Lemma 6.1.6. *Suppose* $1 < p < \infty$ *and* $\varphi \in W_p{}^{2m,m}(\Omega \times (\tau, T))$. *Then for* $2r + s < 2m - \frac{2}{p}$, *there exist* $c > 0$ *independent of* φ *such that*

$$D_t^r D_x^s \varphi|_{t=\tau} \in W_p{}^{2m-2r-s-\frac{2}{p}}(\Omega) \text{ and } \|\varphi\|_{p,\Omega}^{(2m-2r-s-\frac{2}{p})} \leq c \|\varphi\|_{p,\Omega\times(\tau,T)}^{(2m)}$$

In addition, for $2r + s < 2m - \frac{1}{p}$

$$D_t^r D_x^s \varphi|_{M\times(\tau,T)} \in W_p{}^{2m-2r-s-\frac{1}{p},\ m-r-\frac{s}{2}-\frac{1}{2p}}(M \times (\tau, T))$$
$$\text{and } \|\varphi\|_{p,M\times(\tau,T)}^{(2m-2r-s-\frac{1}{p})} \leq c \|\varphi\|_{p,\Omega\times(\tau,T)}^{(2m)}$$

Proof. See Lemma 3.4 in chapter 2, Ladyzenskaya [27]. □

Now we state some results for (6.1.2b) with $\kappa_1 = 0$, which are also used in later arguments.

Lemma 6.1.7. *Let* $p > 1$, $\kappa_1 = 0$ *and suppose* $\vartheta \in L_p(\Omega \times (\tau, T))$ *and* $\Psi \in W_p^{2-\frac{1}{p},1-\frac{1}{2p}}(M \times (\tau, T))$. *Then* (6.1.2$b$) *has a unique solution* $\varphi \in W_p^{2,1}(\Omega \times (\tau, T))$. *Furthermore there exists* $C_T > 0$ *independent of* ϑ, φ_0 *and* Ψ *such that*

$$\|\varphi\|_{p,\Omega\times(\tau,T)}^{(2)} \leq C_T(\|\vartheta\|_{p,\Omega\times(\tau,T)} + \|\Psi\|_{p,M\times(\tau,T)}^{(2-\frac{1}{p},1-\frac{1}{2p})})$$

Proof. See Theorem 9.1 in chapter 4 of Ladyzenskaya [27]. □

Remark 6.1.8. If $p > \frac{n+2}{2}$, $\kappa_1 = 0$ and φ satisfies system (6.1.2b), then φ is a Holder continuous function in x and t. See Corollary after Theorem 9.1, chapter 4 of Ladyzenskaya [27].

Remark 6.1.9. There is more regularity on Ψ than is required in further arguments.

Remark 6.1.10. By Lemma 6.1.1, Lemma 6.1.2 and Lemma 6.1.6, we have $\varphi(\cdot,\tau) \in W^{2-\frac{2}{p}}(\Omega)$, $\Psi(\cdot,\tau) \in W^{2-\frac{2}{p}}(M)$ and there exists $c > 0$ independent of φ, Ψ such that

$$\|\varphi(\cdot,\tau)\|_{p,\Omega}^{(2-\frac{2}{p})} \leq c\|\varphi\|_{p,\Omega\times(\tau,T)}^{(2)} \text{ and } \|\Psi(\cdot,\tau)\|_{p,M}^{(2-\frac{2}{p})} \leq c\|\Psi\|_{p,M\times(\tau,T)}^{(2)}$$

respectively. Moreover, if $p > n$ there exists $c > 0$ independent of φ, Ψ such that

$$\|\varphi\|_{\infty,\Omega\times(\tau,T)} \leq c\|\varphi(\cdot,\tau)\|_{p,\Omega}^{(2-\frac{2}{p})} \text{ and } \|\Psi\|_{\infty,M\times(\tau,T)} \leq c\|\Psi(\cdot,\tau)\|_{p,M}^{(2-\frac{2}{p})}$$

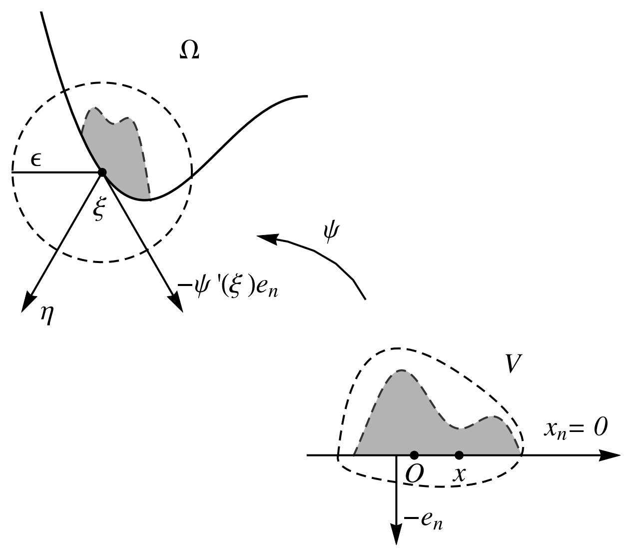
respectively.

Lemma 6.1.11. *Let $1 < p < n+2$ and $1 < q \leq \frac{(n+1)p}{n+2-p}$. There exists a constant $\hat{C} > 0$ depending on p, T, M and n such that if $\varphi \in W_p^{2,1}(\Omega \times (\tau,T))$, then*

$$\left\|\frac{\partial\varphi}{\partial\eta}\right\|_{q,M\times(\tau,T)} \leq \hat{C}\|\varphi\|_{p,\Omega\times(\tau,T)}^{(2)}$$

Proof. It suffices to consider the case when φ is smooth in $\overline{\Omega} \times [\tau,T]$ as such functions are dense in $W_p^{2,1}(\overline{\Omega} \times [\tau,T])$. M is a $C^{2+\mu}$, $n-1$ dimensional manifold ($\mu > 0$). Therefore, for every $\hat{\xi} \in M$ there exists $\epsilon_{\hat{\xi}} > 0$, an open set $V \subset \mathbb{R}^n$ containing 0, and a $C^{2+\mu}$ diffeomorphism $\psi : V \to B(\hat{\xi}, \epsilon_{\hat{\xi}})$ such that $\psi(\mathbf{0}) = \hat{\xi}$,

$\psi(\{x \in V : x_n > 0\}) = B(\hat{\xi}, \epsilon_{\hat{\xi}}) \cap \Omega$ and $\psi(\{x \in V : x_n = 0\}) = B(\hat{\xi}, \epsilon_{\hat{\xi}}) \cap M$.



Since ψ is a C^2 diffeomorphism, $(\psi^{-1})_n$, the nth component of ψ^{-1}, is differentiable in $B(\hat{\xi}, \epsilon_{\hat{\xi}})$, and by definition of ψ, $(\psi^{-1})_n(\xi) = 0$ if and only if $\xi \in B(\hat{\xi}, \epsilon_{\hat{\xi}}) \cap M$. Further, $\nabla(\psi^{-1})_n(\xi)$ is nonzero and orthogonal to $B(\hat{\xi}, \epsilon_{\hat{\xi}}) \cap M$ at each $\xi \in B(\hat{\xi}, \epsilon_{\hat{\xi}}) \cap M$. Without loss of generality, we assume the outward unit normal is given by

$$\eta(\xi) = \frac{\nabla(\psi^{-1})_n(\xi)}{|(\nabla\psi^{-1})_n(\xi)|} \quad \forall\ \xi \in B(\hat{\xi}, \epsilon_{\hat{\xi}}) \cap M$$

We know,

$$\frac{\partial\varphi}{\partial\eta}(\xi) = \nabla_\xi\varphi(\xi) \cdot \eta(\xi) \quad \forall\ \xi \in B(\hat{\xi}, \epsilon_{\hat{\xi}}) \cap M$$

Now in order to transform $\frac{\partial\varphi(\xi)}{\partial\eta}$ back to $\mathbb{R}^n$, pick $L > 0$, such that

$E = \underbrace{[-L, L] \times [-L, L] \times ... \times [-L, L]}_{(n-1) \text{ times}} \times [0, L] \subset V$, and define $\tilde{\varphi}$ such that

$$\tilde{\varphi}(x) = -\int_0^{x_n} \nabla_x\varphi(\psi(x', z))(\psi^{-1}(x', z))' \cdot \eta(\psi(x', z))dz \quad \forall\ x \in E$$

We know $\varphi \in W_p^{2,1}(\Omega \times (\tau, T))$. Therefore from Lemma 6.1.5, there exists $0 < \alpha < L$ and $K_{\hat{\xi}} > 0$, depending on Ω, n, p such that

$$\int_{S_\alpha} \left| \frac{\partial \tilde{\varphi}(x', \alpha, t)}{\partial x_n} \right|^q < K_{\hat{\xi}} \|\varphi\|_{p,\Omega\times(\tau,T)}^{(2)} \quad \forall\ 1 < q \leq \frac{(n+2)p}{n+2-p} \tag{6.1.2}$$

where $S_\alpha = E|_{x_n=\alpha} \times (\tau, T)$ and $S_{x_n} = E|_{0\leq x_n \leq \alpha} \times (\tau, T)$. Using the fundamental theorem of calculus,

$$\int_{E\times(\tau,T)} \left| \frac{\partial \tilde{\varphi}(x', 0, t)}{\partial x_n} \right|^q \leq \int_{S_\alpha} \left| \frac{\partial \tilde{\varphi}(x', \alpha, t)}{\partial x_n} \right|^q + q \int_{S_{x_n}} \left| \frac{\partial \tilde{\varphi}(x', s, t)}{\partial x_n} \right|^{q-1} \cdot \left| \frac{\partial^2 \tilde{\varphi}(x', s, t)}{\partial x_n^2} \right|$$

Using (6.1.2),

$$\int_{E\times(\tau,T)} \left| \frac{\partial \tilde{\varphi}(x', 0, t)}{\partial x_n} \right|^q \leq K_{\hat{\xi}} \|\varphi\|_{p,\Omega\times(\tau,T)}^{(2)} + q \int_{S_{x_n}} \left| \frac{\partial \tilde{\varphi}(x', s, t)}{\partial x_n} \right|^{q-1} \cdot \left| \frac{\partial^2 \tilde{\varphi}(x', s, t)}{\partial x_n^2} \right|$$

Applying Hölder inequality,

$$\begin{aligned}\int_{E\times(\tau,T)} \left| \frac{\partial \tilde{\varphi}(x', 0, t)}{\partial x_n} \right|^q \leq q \left(\int_{S_{x_n}} \left| \frac{\partial \tilde{\varphi}(x', s, t)}{\partial x_n} \right|^{\frac{(q-1)p}{p-1}} \right)^{\frac{p-1}{p}} \left(\int_{S_{x_n}} \left| \frac{\partial^2 \tilde{\varphi}(x', s, t)}{\partial x_n^2} \right|^p \right)^{\frac{1}{p}} \\ + K_{\hat{\xi}} \|\varphi\|_{p,\Omega\times(\tau,T)}^{(2)}\end{aligned}$$

Recall $\frac{\partial^2 \tilde{\varphi}}{\partial x_n^2} \in L_p(S_{x_n})$. So using Lemma 6.1.5 we have

$$\int_{E\times(\tau,T)} \left| \frac{\partial \tilde{\varphi}(x', 0, t)}{\partial x_n} \right|^q \leq \hat{K} \|\varphi\|_{p,\Omega\times(\tau,T)}^{(2)} \tag{6.1.3}$$

Now, M is a compact manifold. Therefore there exists set $A = \{P_1, ..., P_N\} \subset M$ such that $M \subset \cup_{1\leq i\leq N} B(P_i, \epsilon_{P_i})$. Let V_i, $\hat{K}_i$ and α_i be the open sets and constants respectively obtained above when $\hat{\xi} = P_i$. Then,

$$\begin{aligned}
\int_\tau^T \int_M \left|\frac{\partial \varphi}{\partial \eta}\right|^q E\Sigma &\leq \sum_{P_i \in A} \int_\tau^T \int_{B(P_i,\epsilon)} \left|\frac{\partial \varphi}{\partial \eta}\right|^q E\Sigma \\
&\leq C \sum_{P_i \in A} \int_\tau^T \int_{V_i|_{x_n=0}} \left|\frac{\partial \tilde{\varphi}}{\partial x_n}\right|^q \\
&\leq C \sum_{P_i \in A} \tilde{K}_i \|\varphi\|^{(2)}_{p,\Omega\times(\tau,T)}
\end{aligned}$$

Therefore, for some $\hat{C} > 0$, depending only upon p, τ, T, M and n, we get

$$\left\|\frac{\partial \varphi}{\partial \eta}\right\|_{q,M\times(\tau,T)} \leq \hat{C}\|\varphi\|^{(2)}_{p,\Omega\times(\tau,T)}$$

□

The following Lemma plays a key role in bootstrapping L_p estimates of solutions to (6.0.1).

Lemma 6.1.12. *Suppose* $(V1)$ *holds and* $0 < T_{\max} < \infty$. *Then* $u \in L_1(M \times (0, T_{\max}))$, $u \in L_1(\Omega \times (0, T_{\max}))$ *and* $v \in L_1(M \times (0, T_{\max}))$.

Proof. Consider the system

$$\begin{aligned}
\varphi_t &= -d\Delta\varphi & (x,t) &\in \Omega \times (0,T) \\
d\frac{\partial \varphi}{\partial \eta} &= \alpha\varphi + 1 & (x,t) &\in M \times (0,T) \qquad (6.1.4)\\
\varphi &= 0 & x &\in \Omega, \quad t = T
\end{aligned}$$

where α is given in (V1), $d > 0$, $0 < \mu < 1$, M is a $C^{2+\mu}$ manifold, and non-negative $\varphi_T \in C^{2+\Upsilon}(\overline{\Omega})$ for some $0 < \Upsilon < 1$. From Lemma 6.1.4, $\varphi \in C^{2+\Upsilon,\frac{\Upsilon}{2}+1}(\overline{\Omega} \times [0,T])$

and therefore by standard sequential argument $\varphi \in C^{\Upsilon+2,\frac{\Upsilon}{2}+1}(M \times [0,T])$. Now having enough regularity for φ on $M \times [0,T]$, consider

$$\Delta_M \varphi = -\frac{1}{\sqrt{det\ g}} \partial_j (g^{ij} \sqrt{det\ g}\ \partial_i \varphi)$$

and further let $\tilde{\vartheta} = -\varphi_t - \tilde{d}\Delta_M \varphi$. Note that although Δ_M (Laplace Beltrami operator) is defined in local coordinates, infact this expression is independent of the choice of local coordinates (see Rosenberg [39], Ex 19), and as a result we obtain the following system on $M \times (0,T)$.

$$\begin{aligned} \varphi_t &= -\tilde{d}\Delta_M \varphi - \tilde{\vartheta} && (x,t) \in M \times (0,T) \\ \varphi &= \tilde{\varphi} && x \in M, \quad t = T \end{aligned}$$

Now, multiplying v with $\tilde{\vartheta}$ and integrating over $M \times (0,T)$,

$$\begin{aligned} \int_0^T \int_M v\tilde{\vartheta} &= \int_0^T \int_\Omega u(-\varphi_t - d\Delta\varphi) + \int_0^T \int_M v(-\varphi_t - \tilde{d}\Delta_M \varphi) \\ &= \int_0^T \int_\Omega \varphi(u_t - d\Delta u) + \int_0^T \int_M \varphi(v_t - \tilde{d}\Delta_M v) - d\int_0^T \int_M u\frac{\partial \varphi}{\partial \eta} \\ &\quad + d\int_0^T \int_M \frac{\partial u}{\partial \eta}\varphi + \int_\Omega u(x,0)\varphi(x,0) + \int_M v(x,0)\varphi(x,0) \end{aligned}$$

Using $d\frac{\partial \varphi}{\partial \eta} = \alpha\varphi + 1$

$$\begin{aligned} \int_0^T \int_M u \le \int_0^T \int_\Omega \varphi H(u) + \int_0^T \int_M (F(u,v) + G(u,v))\varphi \\ + \int_\Omega u(x,0)\varphi(x,0) + \int_M v(x,0)\varphi(x,0) - \int_0^T \int_M v\tilde{\vartheta} \end{aligned}$$

Using (V1),

$$\int_0^T \int_M u \leq \int_0^T \int_\Omega \beta\varphi(u+1) + \int_0^T \int_M \alpha(v+1)\varphi \tag{6.1.5}$$
$$+ \int_\Omega u(x,0)\varphi(x,0) + \int_M v(x,0)\varphi(x,0) - \int_0^T \int_M v\tilde{\vartheta}$$

Now, integrating the u equation over Ω and the v equation over M,

$$\begin{aligned}\frac{d}{dt}\left(\int_\Omega u + \int_M v\right) &= d\int_\Omega \Delta u + \int_\Omega H(u) + \tilde{d}\int_M \Delta v + \int_M F(u,v)\\ &\leq \beta\int_\Omega (u+1) + \int_M (G(u,v) + F(u,v))\\ &\leq \beta\int_\Omega (u+1) + \alpha\int_M (u+v+1)\end{aligned} \tag{6.1.6}$$

Note from Remarks 6.1.3 and 6.1.10, φ is Hölder continuous on $\overline{\Omega} \times [0,T]$, and using the regularity of the initial data u_0 and v_0, all the integrands of each term are integrable. Integrating (6.1.6) over $(0,T)$ and using (6.1.5), gives

$$\int_\Omega u(x,T) + \int_M v(x,T) \leq \tilde{\beta}\int_0^T \int_\Omega u + \tilde{\alpha}\int_0^T \int_M v + \tilde{L}(T) \tag{6.1.7}$$

where

$$\tilde{L}(T) = \alpha|M|T + \beta|\Omega|T + \alpha\beta\|\varphi\|_{1,\Omega\times(0,T)} + \alpha^2\|\varphi\|_{1,M\times(0,T)} + \alpha\|u(x,0)\|_{1,\Omega}\cdot\|\varphi(x,0)\|_{\infty,\Omega}$$
$$+ \|v(x,0)\|_{1,M} + \alpha\|v(x,0)\|_{1,M}\cdot\|\varphi(x,0)\|_{\infty,M} + \|u(x,0)\|_{1,\Omega}$$

$$\tilde{\alpha} = \alpha^2\|\varphi\|_{\infty,M\times(0,T)} + \alpha + \alpha\|\tilde{\theta}\|_{M\times(0,T)} \quad \text{and} \quad \tilde{\beta} = \beta + \alpha\beta\|\varphi\|_{\infty,\Omega\times(0,T)}$$

Now using Gronwall's inequality, for all $0 \le t < T < T_{max}$

$$\begin{aligned}\int_\Omega u(x,t) + \int_M v(\zeta,t) &\le \tilde{L}(T) + (\tilde{\alpha}(T) + \tilde{\beta}(T)) \int_0^T \tilde{L}(s) \exp\left(\int_s^T \tilde{\alpha}(r) + \tilde{\beta}(r) dr\right) ds \\ &\le C_T\end{aligned}$$

Therefore, (u,v) being nonnegative implies $\int_\Omega u < C_T$ and $\int_M v \le C_T$. Substituting the L_1 estimate of u on Ω and v on M in (6.1.5), yields

$$\begin{aligned}\int_0^T \int_M u \le{}& \beta \left(\|\varphi\|_{\infty,\Omega\times(0,T)} \|u\|_{1,\Omega\times(0,T)} + |\Omega| T \|\varphi\|_{\infty,\Omega\times(0,T)} \right) \\ &+ \alpha \left(\|\varphi\|_{\infty,M\times(0,T)} \|v\|_{1,M\times(0,T)} + |M| T \|\varphi\|_{\infty,M\times(0,T)} \right) \\ &+ \|u(\cdot,0)\|_{1,\Omega} \|\varphi(\cdot,0)\|_{\infty,\Omega} + \|v(\cdot,0)\|_{1,M} \|\varphi(\cdot,0)\|_{\infty,M} + \|v\|_{1,M\times(0,T)} \|\tilde{\vartheta}\|_{\infty,M}\end{aligned}$$

Hence $u \in L_1(M \times (0,T))$. □

Lemma 6.1.13. *Suppose* $(V1)$ *and* $(V2)$ *hold and if* $q > 1$ *such that* $v \in L_q(M \times (0,T))$. *Then* $u \in L_q(M \times (0,T))$ *and* $u \in L_q(\Omega \times (0,T))$.

Proof. Multiplying the u_t equation by u^{q-1}, we get

$$\begin{aligned}\int_0^t \int_\Omega u^{q-1} u_t &= d \int_0^t \int_\Omega u^{q-1} \Delta u + \int_0^t \int_\Omega u^{q-1} H(u) \\ &= d \int_0^t \int_M u^{q-1} \frac{\partial u}{\partial \eta} - d \int_0^t \int_\Omega (q-1) u^{q-2} |\nabla u|^2 + \int_0^t \int_\Omega u^{q-1} H(u)\end{aligned}$$

Using $(V2)$

$$\int_\Omega \frac{u^q}{q} + d \int_0^t \int_\Omega \frac{4(q-1)}{q^2} |\nabla u^{\frac{q}{2}}|^2 \le K_g \int_0^t \int_M u^{q-1}(u+v+1) + \beta \int_0^t \int_\Omega (u+1) u^{q-1}$$

$$+\int_\Omega \frac{u_0^q}{q} \tag{6.1.8}$$

Also for $1 < q \leq \infty$, for all $\epsilon > 0$ there exists $C_\epsilon > 0$ such that,

$$\int_0^t \int_M u^q \leq C_\epsilon \int_0^t \int_\Omega u^q + \epsilon \int_0^t \int_\Omega |\nabla u^{\frac{q}{2}}|^2 \tag{6.1.9}$$

Applying Young's inequality in (6.1.8) and using (6.1.9) for some $\epsilon > 0$, gives

$$\frac{1}{q}\frac{d}{dt}\int_0^t \int_\Omega u^q \leq \tilde{K}_1 \int_0^t \int_\Omega u^q + \tilde{K}_2 \tag{6.1.10}$$

for some $\tilde{K}_1, \tilde{K}_2 > 0$ depending on t. Therefore, from Gronwall's Inequailty

$$u \in L_q(\Omega \times (0,t))$$

and there exists $L_1 > 0$ such that

$$\int_0^t \int_\Omega u^q \leq L_1 \left(e^{\tilde{k}_1 q t + 1}\right)$$

From (6.1.8),

$$\begin{aligned} \epsilon \int_0^t \int_\Omega |\nabla u^{\frac{q}{2}}|^2 \quad &\leq \quad 2\epsilon \int_0^t \int_M u^q + \epsilon \int_0^t \int_M v^q + \epsilon \left(\int_0^t \int_M u^q\right)^{\frac{q-1}{q}} (t|M|)^{\frac{1}{q}} + \\ &\epsilon \int_\Omega \frac{u_0^q}{q} + \beta\epsilon \int_0^t \int_\Omega (u^q + u^{q-1}) \end{aligned} \tag{6.1.11}$$

Using (6.1.9) and (6.1.11) we have,

$$\begin{aligned}\int_0^t\int_M u^q &\leq C_\epsilon\int_0^t\int_\Omega u^q+\epsilon\int_\Omega\frac{u_0^q}{q}+\beta\epsilon\int_0^t\int_\Omega(u^q+u^{q-1})\\ &\quad 2\epsilon\int_0^t\int_M u^q+\epsilon\int_0^t\int_M v^q+\epsilon\left(\int_0^t\int_M u^q\right)^{\frac{q-1}{q}}(t|M|)^{\frac{1}{q}}\end{aligned}$$

Now choosing ϵ such that $1-2\epsilon-\epsilon(t|M|)^{\frac{1}{q}}>0$ and using the estimate above for u on $(\Omega\times(0,t))$, we have $u\in L_q(M\times(0,T))$. $\square$

Lemma 6.1.14. *Suppose* $(V1)$ *and* $(V2)$ *hold. If* $q\geq 1$ *such that* $u\in L_q(\Omega\times(\tau,T))$ *and* $u,v\in L_q(M\times(\tau,T))$*. Then* $u\in L_p(\Omega\times(\tau,T))$ *and* $u,v\in L_p(M\times(\tau,T))$ *for all* $p>1$*, and there exists* $C_{p,T}>0$ *such that*

$$\|u\|_{p,\Omega\times(\tau,T)}+\|v\|_{p,M\times(\tau,T)}\leq C_{p,T}\left(\|u\|_{q,M\times(\tau,T)}+\|u\|_{q,\Omega\times(\tau,T)}+\|v\|_{q,M\times(\tau,T)}\right)$$

Proof. First we show there exists $r>1$ such that $u\in L_{rq}(\Omega\times(\tau,T))$ and $v\in L_{rq}(M\times(\tau,T))$. Consider the system (6.1.2a) and (6.1.2b) with $\kappa_1=0$, $\kappa_2=1$, and $\tilde{\vartheta}\geq 0$ and $\tilde{\vartheta}\in L_p(M\times(\tau,T))$ with $\|\tilde{\vartheta}\,\|_{p,(M\times(\tau,T))}=1$, and $\vartheta\geq 0$ and $\vartheta\in L_p(\Omega\times(\tau,T))$ with $\|\vartheta\,\|_{p,(\Omega\times(\tau,T))}=1$. Multiplying u with ϑ and v with $\tilde{\vartheta}$ and integrating over $\Omega\times(\tau,T)$ and $M\times(\tau,T)$ respectively, gives

$$\begin{aligned}\int_\tau^T\int_\Omega u\vartheta+\int_\tau^T\int_M v\tilde{\vartheta}&=\int_\tau^T\int_\Omega u(-\varphi_t-d\Delta\varphi)+\int_\tau^T\int_M v(-\Psi_t-\tilde{d}\Delta_M\Psi)\\ &=\int_\tau^T\int_\Omega\varphi(u_t-d\Delta u)+\int_\tau^T\int_M\Psi(v_t-\tilde{d}\Delta_M v)\\ &\quad-d\int_\tau^T\int_M u\frac{\partial\varphi}{\partial\eta}+d\int_\tau^T\int_M\frac{\partial u}{\partial\eta}\varphi+\int_\Omega u(x,\tau)\varphi(x,\tau)\\ &\quad+\int_M v(x,\tau)\Psi(x,\tau)-\int_M v(x,T)\Psi(x,T)-\int_\Omega u(x,T)\varphi(x,T)\end{aligned}$$

Since $\Psi(x,T)=0$ and $\varphi(x,T)=0$,

$$\begin{aligned}\int_\tau^T\int_\Omega u\vartheta+\int_\tau^T\int_M v\tilde{\vartheta}\le \int_\tau^T\int_\Omega \varphi H(u)+\int_\tau^T\int_M (F(u,v)+G(u,v))\Psi\\ -d\int_\tau^T\int_M u\frac{\partial\varphi}{\partial\eta}+\int_\Omega u(x,\tau)\varphi(x,\tau)\\ +\int_M v(x,\tau)\Psi(x,\tau)\end{aligned}$$

Using $(V1)$,

$$\begin{aligned}\int_\tau^T\int_\Omega u\vartheta+\int_\tau^T\int_M v\tilde{\vartheta}\le \int_\tau^T\int_\Omega \beta\varphi(u+1)+\int_\tau^T\int_M \alpha(u+v+1)\Psi\\ -d\int_\tau^T\int_M u\frac{\partial\varphi}{\partial\eta}+\int_\Omega u(x,\tau)\varphi(x,\tau)\\ +\int_M v(x,\tau)\Psi(x,\tau)\end{aligned} \tag{6.1.12}$$

Now we break the argument in two cases.

Case 1: Suppose $q=1$. Then $u\in L_1(\Omega\times(\tau,T))$ and $u,v\in L_1(M\times(\tau,T))$. Let $\epsilon>0$ and set $p=n+2+\epsilon$. Set $p'=\frac{n+2+\epsilon}{n+1+\epsilon}$ (conjugate of p). Remarks 6.1.3 and 6.1.10, and Lemma 6.1.12 imply all of the integrals on the right hand side of (6.1.12) are finite. Application of Hölder's inequality in (6.1.12), yields $v\in L_{p'}(M\times(0,T))$, and there exists $C_{p,T}>0$ such that

$$\|u\|_{p',\Omega\times(\tau,T)}+\|v\|_{p',M\times(\tau,T)}\le C_{p,T}(\|u\|_{1,\Omega\times(\tau,T)}+\|v\|_{1,M\times(\tau,T)}+\|u\|_{1,M\times(\tau,T)})$$

Therefore, Lemma 6.1.13 implies $u\in L_{p'}(M\times(0,T))$. So for this case $r=\frac{n+2+\epsilon}{n+1+\epsilon}$.

Case 2: Suppose $q>1$ such that $u\in L_q(\Omega\times(\tau,T))$ and $u,v\in L_q(M\times(\tau,T))$. Recall $p>1$, $0\le\tilde{\vartheta}\in L_p(M\times(\tau,T))$ with $\|\tilde{\vartheta}\|_{p,(M\times(\tau,T))}=1$ and $0\le\vartheta\in$

$L_p(\Omega \times (\tau, T))$ with $\|\vartheta\|_{p,(\Omega\times(\tau,T))} = 1$. Also $p' = \frac{p}{p-1}$, $q' = \frac{q}{q-1}$. Applying Hölder inequality in 6.1.12 and further using Remark 6.1, Lemma 6.1.11, yields $v \in L_{p'}(M \times (\tau, T)), u \in L_{p'}(\Omega \times (\tau, T))$ that is there exists $\tilde{C}_{p,T}$ such that

$$\|u\|_{p',\Omega\times(\tau,T)} + \|v\|_{p',M\times(\tau,T)} \leq C_{p,T}(\|u\|_{q,\Omega\times(\tau,T)} + \|v\|_{q,M\times(\tau,T)} + \|u\|_{q,M\times(\tau,T)})$$

provided $p' \leq \frac{(n+2)q}{n+1}$. So, in this case $r = \frac{(n+2)q}{n+1}$.

Now, by repeating the above argument for rq instead of q, we get $v \in L_{r^2q}(M \times (\tau, T))$, $u \in L_{r^2q}(\Omega \times (\tau, T))$, and continuing in this manner, we get $v \in L_{r^mq}(M \times (\tau, T))$, $u \in L_{r^2q}(\Omega \times (\tau, T))$, for all $m > 1$. As $r > 1$, $\lim_{m\to\infty} r^m q \to \infty$, and as a result, $v \in L_p(M \times (\tau, T))$ for all $p > 1$. Hence from Lemma 6.1.13, $u \in L_p(M \times (\tau, T))$ and $u \in L_p(\Omega \times (\tau, T))$ for all $p > 1$, and there exists $C_{p,T} > 0$ such that

$$\|u\|_{p,\Omega\times(\tau,T)} + \|v\|_{p,M\times(\tau,T)} \leq C_{p,T}\left(\|u\|_{q,M\times(\tau,T)} + \|u\|_{q,\Omega\times(\tau,T)} + \|v\|_{q,M\times(\tau,T)}\right)$$

□

Theorem 6.1.15. *Suppose conditions* $(V1)-(V3)$ *are satisfied. Then* (6.0.1) *has a componentwise non-negative global solution.*

Proof. From Theorem 5.0.10, we already have local existence and uniqueness for (6.0.1). If $T_{\max} = \infty$, then we are done. So, by way of contradiction assume $T_{\max} < \infty$. From Lemma 6.2.1, we have L_p estimates for our solution for all $p \geq 1$, on $\Omega \times (0, T_{\max})$ and $M \times (0, T_{\max})$. We know from $(V2)$ and $(V3)$ that F and G

are polynomially bounded above. So, consider the system

$$\begin{aligned} U_t &= d\Delta U + \beta(u+1) && (x,t) \in \Omega \times (0, T_{max}) \\ V_t &= \tilde{d}\Delta_M V + K_f(u+v+1)^l && (x,t) \in M \times (0, T_{max}) \\ d\frac{\partial U}{\partial \eta} &= K_g(u+v+1) && (x,t) \in M \times (0, T_{max}) \\ U &= u_0 && x \in \Omega, \quad t = 0 \\ V &= v_0 && x \in M, \quad t = 0 \end{aligned} \tag{6.1.13}$$

Note that $u \leq U$ and $v \leq V$ for all $t \geq 0$. For all $q \geq 1$, $K_f(u+v+1)^l$ and $K_g(u+v+1)$ lie in the $L_q(M \times (0, T_{max}))$. Now using Theorem 4.1.5, the solution of (6.2.16) is bounded for finite time. Therefore, by the Maximum Principle [37], the solution of (6.0.1) is bounded for finite time. This contradicts the Theorem 5.0.10. Therefore, $T_{max} = \infty$. □

6.2 Uniform Estimates

Lemma 6.2.1. *Suppose there exists $\tau \geq 0$, $q \geq 1$ such that $u \in L_q(\Omega \times (\tau, \tau+2))$ and $u, v \in L_q(M \times (\tau, \tau+2))$. If $p' = \frac{(n+3)q}{n+2}$, there exists $C > 0$ independent of τ such that*

$$\begin{aligned} &\|u\|_{p',\Omega\times(\tau+1,\tau+2)} + \|v\|_{p',M\times(\tau+1,\tau+2)} \\ &\qquad \leq C\left(\|u\|_{q,M\times(\tau,\tau+2)} + \|u\|_{q,\Omega\times(\tau,\tau+2)} + \|v\|_{q,M\times(\tau,\tau+2)}\right) \end{aligned}$$

Proof. In order to show that there exists $r > 1$ such that $u \in L_{rq}(\Omega \times (\tau, \tau +$

2)) and $v \in L_{rq}(M \times (\tau, \tau+2))$, consider the system (6.1.2a) with $T = \tau + 2$, $\kappa_1 = 0$ and $\kappa_2 = 1$. $0 \leq \tilde{\vartheta} \in L_p(M \times (\tau, \tau+2))$ with $\|\tilde{\vartheta}\|_{p,(M\times(\tau,\tau+2))} = 1$ and $0 \leq \vartheta \in L_p(\Omega \times (\tau, \tau+2))$ with $\|\vartheta\|_{p,(\Omega\times(\tau,\tau+2))} = 1$. Define a cut off function $\psi \in {C_0}^{\infty}(\mathbb{R}, [0,1])$ such that $\psi(t) = 1$ for all $t \geq \tau + 1$ and $\psi(t) = 0$ for all $t \leq \tau$. In addition, we define $w(x,t) = \psi(t)\Psi(x,t)$ and $z(x,t) = \psi(t)\varphi(x,t)$. From construction, $w(x,t) = \Psi(x,t)$ and $z(x,t) = \varphi(x,t)$ for all $(x,t) \in M \times (\tau+1, \tau+2)$ and $(x,t) \in \Omega \times (\tau+1, \tau+2)$ respectively . Also w, z satisfies the following system

$$
\begin{aligned}
z_t &= -d\Delta z - \psi(t)\vartheta + \psi'(t)\varphi(x,t) && (x,t) \in \Omega \times (\tau, \tau+2)\\
w_t &= -\tilde{d}\Delta_M w - \psi(t)\tilde{\vartheta} + \psi'(t)\Psi(t) && (x,t) \in M \times (\tau, \tau+2)\\
w &= z && (x,t) \in M \times (\tau, \tau+2)\\
z &= 0 && x \in \Omega, \quad t = \tau+2\\
w &= 0 && x \in M, \quad t = \tau+2
\end{aligned}
\tag{6.2.14}
$$

Multiplying u with $\psi(t)\vartheta$ and v with $\psi(t)\tilde{\vartheta}$ and integrating over $\Omega \times (\tau, \tau+2)$ and $M \times (\tau, \tau+2)$,

$$
\begin{aligned}
&\int_\tau^{\tau+2}\int_\Omega u\psi\vartheta + \int_\tau^{\tau+2}\int_M v\psi\tilde{\vartheta}\\
&= \int_\tau^{\tau+2}\int_\Omega u(-z_t - d\Delta z + \psi'\varphi) + \int_\tau^{\tau+2}\int_M v(-w_t - \tilde{d}\Delta_M w + \psi'\Psi)\\
&= \int_\tau^{\tau+2}\int_\Omega z(u_t - d\Delta u) + u\psi'\varphi + \int_\tau^{\tau+2}\int_M w(v_t - \tilde{d}\Delta_M v) + v\psi'\Psi\\
&\quad -d\int_\tau^{\tau+2}\int_M u\frac{\partial z}{\partial \eta} + d\int_\tau^{\tau+2}\int_M \frac{\partial u}{\partial \eta} z + \int_\Omega u(x,\tau)z(x,\tau) - \int_\Omega u(x,\tau+2)z(x,\tau+2)\\
&\qquad + \int_M v(x,\tau)w(x,\tau) - \int_M v(x,\tau+2)w(x,\tau+2)
\end{aligned}
$$

Since $w(x,\tau+2)=0$, $z(x,\tau+2)=0$, $w(x,\tau)=0$ and $z(x,\tau)=0$.

$$\begin{aligned}\int_{\tau+1}^{\tau+2}\int_\Omega u\vartheta+\int_{\tau+1}^{\tau+2}\int_M v\tilde{\vartheta}\le \int_\tau^{\tau+2}\int_M (F(u,v)+G(u,v))w+\int_\tau^{\tau+2}\int_\Omega zH(u)\\ +\int_\tau^{\tau+2}\int_M v\psi'\Psi+\int_\tau^{\tau+2}\int_\Omega u\psi'\varphi\\ -d\int_\tau^{\tau+2}\int_M u\frac{\partial z}{\partial\eta}\end{aligned}$$

Using $(V1)$,

$$\begin{aligned}\int_{\tau+1}^{\tau+2}\int_\Omega u\vartheta+\int_{\tau+1}^{\tau+2}\int_M v\tilde{\vartheta}\le \int_\tau^{\tau+2}\int_M \alpha(u+v+1)w+\int_\tau^{\tau+2}\int_\Omega \beta(u+1)z\\ +\int_\tau^{\tau+2}\int_M v\psi'\Psi+\int_\tau^{\tau+2}\int_\Omega u\psi'\varphi\\ -d\int_\tau^{\tau+2}\int_M u\frac{\partial z}{\partial\eta}\end{aligned}\tag{6.2.15}$$

Now we break the argument in two cases.

Case 1: Suppose $q=1$. $u\in L_1(\Omega\times(\tau,\tau+2))$ and $u,v\in L_1(M\times(\tau,\tau+2))$ Set $p=n+3$ then $p'=\frac{n+3}{n+2}$ (conjugate of p). Remark 6.1.3, 6.1.10 and Lemma 6.1.12 implies all the integrals are finite. Application of Hölder inequality in 6.2.15, yields $v\in L_{p'}(M\times(\tau+1,\tau+2))$ that is, there exists C such that

$$\begin{aligned}&\|u\|_{p',\Omega\times(\tau+1,\tau+2)}+\|v\|_{p',M\times(\tau+1,\tau+2)}\\ &\qquad\le C(\|u\|_{1,\Omega\times(\tau,\tau+2)}+\|v\|_{1,M\times(\tau,\tau+2)}+\|u\|_{1,M\times(\tau,\tau+1)})\end{aligned}$$

and therefore, Lemma 6.1.13 implies $u\in L_{p'}(M\times(\tau+1,\tau+2))$. So, for this case $r=\frac{n+3}{n+2}$.

Case 2: Suppose $q > 1$. $u \in L_q(\Omega \times (\tau, \tau+2))$ and $u, v \in L_q(M \times (\tau, \tau+2))$ Let $p' = \frac{p}{p-1}$, $\tilde{\vartheta} \geq 0 \in L_p(M \times (\tau, \tau+2))$ with $\|\tilde{\vartheta}\|_{p,(M\times(\tau,\tau+2))} = 1$ and $\vartheta \geq 0 \in L_p(\Omega \times (\tau, \tau+2))$ with $\|\vartheta\|_{p,(\Omega\times(\tau,\tau+2))} = 1$. Applying Hölder inequality in 6.2.15 and further using Remark 6.1, Lemma 6.1.11, yields $v \in L_{p'}(M \times (\tau+1, \tau+2)), u \in L_{p'}(\Omega \times (\tau+1, \tau+2))$ that is there exists $\tilde{C}$ such that

$$\begin{aligned} &\|u\|_{p',\Omega\times(\tau+1,\tau+2)} + \|v\|_{p',M\times(\tau+1,\tau+2)} \\ &\qquad \leq \tilde{C}(\|u\|_{q,\Omega\times(\tau,\tau+2)} + \|v\|_{q,M\times(\tau,\tau+2)} + \|u\|_{q,M\times(\tau,\tau+2)}) \end{aligned}$$

since $p' \leq \frac{(n+2)q}{n+1}$. The result follows. □

Theorem 6.2.2. *Suppose F, G, and H are C^1, and satisfy $(V1)-(V3)$. If u_0, v_0 are component-wise bounded and nonnegative then $T_{max} = \infty$. Furthermore, if there exists $K_1 > 0$ independent of $\tau \geq 0$ such that*

$$\int_\tau^{\tau+1}\int_\Omega u + \int_\tau^{\tau+1}\int_M v + \int_\tau^{\tau+1}\int_M u \leq K_1$$

Then the solution is uniformly bounded in sup-norm.

Proof. From Theorem 6.1.15, we already have global existence of the solution to (6.0.1). Since our solution is uniformly bounded for $0 < t < 1$, repeated application of Lemma 6.2.1 and our hypothesis implies that if $p > 1$ and $\tau \geq 0$ then there exists $C_p \geq 0$ independent of τ such that

$$\|u\|_{p,\Omega\times(\tau+1,\tau+2)}, \|u\|_{p,M\times(\tau+1,\tau+2)}, \|v\|_{p,M\times(\tau+1,\tau+2)} \leq C_p$$

are bounded. We know from $(V2)$ and $(V3)$, F and G are polynomially bounded. So, consider the system

$$\begin{aligned}
\tilde{u}_t &= d\Delta\tilde{u} + \beta(u+1) && (x,t) \in \Omega \times (\tau, \tau+2)\\
\tilde{v}_t &= \tilde{d}\Delta_M\tilde{v} + K_f(u+v+1)^l && (x,t) \in M \times (\tau, \tau+2)\\
d\frac{\partial\tilde{u}}{\partial\eta} &= K_g(u+v+1) && (x,t) \in M \times (\tau, \tau+2)\\
\tilde{u} &= u_0 && x \in \Omega, \quad t=\tau\\
\tilde{v} &= v_0 && x \in M, \quad t=\tau
\end{aligned} \tag{6.2.16}$$

For some p sufficiently large depending on l, $K_f(u+v+1)^l$ and $K_g(u+v+1)$ has a uniform (independent of time) L_p bounds. Now we define a cut off function $\psi \in {C_0}^\infty(\mathbb{R}, [0,1])$ such that $\psi(t) = 1$ for all $t \geq \tau+1$ and $\psi(t) = 0$ for all $t \leq \tau$. In addition, we define $\hat{v}(x,t) = \psi(t)\tilde{v}(x,t)$ and $\hat{u}(x,t) = \psi(t)\tilde{u}(x,t)$. From construction, $\hat{v}(x,t) = \tilde{v}(x,t)$ and $\hat{u}(x,t) = \tilde{u}(x,t)$ for all $(x,t) \in M \times (\tau+1, \tau+2)$ and $(x,t) \in \Omega \times (\tau+1, \tau+2)$ respectively . Also $\hat{u}, \hat{v}$ satisfies the system

$$\begin{aligned}
\hat{u}_t &= d\Delta\hat{u} + \psi'(t)\tilde{u}(x,t) + \psi\beta(u+1) && (x,t) \in \Omega \times (\tau, \tau+2)\\
\hat{v}_t &= \tilde{d}\Delta_M\hat{v} + \psi(t)K_f(u+v+1)^l + \psi'(t)\tilde{v}(x,t) && (x,t) \in M \times (\tau, \tau+2)\\
d\frac{\partial\hat{u}}{\partial\eta} &= \psi K_g(u+v+1) && (x,t) \in M \times (\tau, \tau+2)\\
\hat{u} &= 0 && x \in \Omega, \quad t=\tau\\
\hat{v} &= 0 && x \in M, \quad t=\tau
\end{aligned}$$

Note that $u \leq \hat{u}$ and $v \leq \hat{v}$ for all $t \geq 0$. Now using Theorem 4.1.5, solution of system (6.2.16) is bounded. Therefore, by the Maximum Principle [37] solution of (6.0.1) is uniformly bounded (see [17]). □

Corollary 6.2.3. *Suppose* $F + G \leq 0$*. Then there exists* $K_1 > 0$ *such that*

$$\int_\tau^{\tau+1}\int_\Omega u + \int_\tau^{\tau+1}\int_M v + \int_\tau^{\tau+1}\int_M u \leq K_1$$

for all $\tau \geq 0$*, and the solution to* (6.0.1) *is sup norm bounded.*

Chapter 7

Examples

During bacterial cytokinesis, a proteinaceous contractile, called the Z ring assembles in the cell middle. The Z ring tethers to the membrane and contracts, when triggered, to form two identical daughter cells. Positioning the Z ring at midcell involves two independent processes, referred to as Min system inhibition and nucleoid occlusion ([40], [41] Sun and Margolin 2001). The Min system involves proteins MinC, MinD, and MinE ([38] Raskin and de Boer 1999). MinC inhibits Z ring assembly while the action of MinD and MinE serve to exclude MinC from the midcell region. This promotes the assembly of the Z ring at the midcell. Thomas Pollard in [36] gives an overview of progress in understanding the mechanism of cytokinesis in fission yeast using fluorescence microscopy of proteins tagged with fluorescent proteins to establish the temporal and spatial pathway for the assembly and constriction of the contractile ring. Zhang, Morgan, and Lindahl [44] considered the Min subsystem involving 6 chemical reactions and 5 components, under specific rates and parameters and performed a numerical investigation using a finite volume method on a one dimensional mathematical model.

Note this Z ring is also referred to as the FtsZ ring. Table 7.1 shows the assumed chemical reactions. In the multidimensional setting, the concentration

Table 7.1: Reactions and Reaction Rates

Chemicals	Reactions	Reaction Rates
Min D	$D_{cyt}^{ADP} \xrightarrow{k_{exc}} D_{cyt}^{ATP}$	$R_{exc} = k_{exc}[D_{cyc}^{ADP}]$
Min D	$D_{cyt}^{ATP} \xrightarrow{k_{Dcyt} D_{mem}^{ATP}}$	$R_{Dcyt} = k_{Dcyt}[D_{cyc}^{ATP}]$
	$D_{cyt}^{ATP} \xrightarrow{k_{Dmem}[D_{mem}^{ATP}]} D_{mem}^{ATP}$	$R_{Dmem} = k_{Dmem}[D_{mem}^{ATP}][D_{cyc}^{ATP}]$
Min E	$E + D_{mem}^{ATP} \xrightarrow{k_{Ecyt}} E : D_{mem}^{ATP}$	$R_{Ecyt} = k_{Ecyt}[E[D_{mem}^{ATP}]$
	$E + D_{mem}^{ATP} \xrightarrow{k_{Emem}[E:D_{mem}^{ATP}]^2} E : D_{mem}^{ATP}$	$R_{Emem} = k_{Emem}[D_{mem}^{ATP}][E][E : D_{mem}^{ATP}]^2$
Min E	$E : D_{mem}^{ATP} \xrightarrow{k_{exp}} E + D_{cyt}^{ADP}$	$R_{exp} = k_{exp}[E : D_{mem}^{ATP}]$

density satisfy the reaction-diffusion system given by

$$
\begin{aligned}
\frac{\partial [D_{cyt}^{ATP}]}{\partial t} &= \sigma_{Dcyt} \Delta [D_{cyt}^{ATP}] + R_{exc} && x \in \Omega, \quad t > 0 \\
\frac{\partial [D_{cyt}^{ATP}]}{\partial \eta} &= -R_{Dmem} - R_{Dcyt} && x \in M, \quad t > 0 \\
\frac{\partial [D_{cyt}^{ADP}]}{\partial t} &= \sigma_{ADcyt} \Delta [D_{cyt}^{ATP}] - R_{exc} && x \in \Omega, \quad t > 0 \\
\frac{\partial [D_{cyt}^{ADP}]}{\partial \eta} &= R_{exp} && x \in M, \quad t > 0 \\
\frac{\partial [E_{cyt}]}{\partial t} &= \sigma_{Ecyt} \Delta [E_{cyt}] && x \in \Omega, \quad t > 0 \\
\frac{\partial [E_{cyt}]}{\partial \eta} &= R_{exp} - R_{Ecyt} - R_{Emem} && x \in M, \quad t > 0 \\
\frac{\partial [D_{mem}^{ATP}]}{\partial t} &= \sigma_{Dmem} \Delta_M [D_{mem}^{ATP}] \\
&\qquad + R_{Dcyt} + R_{Dmem} - R_{Ecyt} - R_{Emem} && x \in M, \quad t > 0
\end{aligned}
$$

$$\frac{\partial [E:D_{mem}^{ATP}]}{\partial t} = \sigma_{E:Dmem}\Delta_M [E:D_{mem}^{ATP}] - R_{exp} + R_{Ecyt} + R_{Emem} \qquad x \in M, \quad t > 0$$

The system is special case of (1.0.1), with

$$D = \begin{pmatrix} \sigma_{Dmem} & 0 \\ 0 & \sigma_{EDmem} \end{pmatrix} \text{ and } \tilde{D} = \begin{pmatrix} \sigma_{Dcyt} & 0 & 0 \\ 0 & \sigma_{ADyct} & 0 \\ 0 & 0 & \sigma_{Ecyt} \end{pmatrix},$$

$$\vec{u} = \begin{pmatrix} D_{cyt}^{ATP} \\ D_{cyt}^{ADP} \\ E_{cyt} \end{pmatrix}, \ \vec{v} = \begin{pmatrix} D_{mem}^{ATP} \\ E:D_{mem}^{ATP} \end{pmatrix}, \ \vec{F} = \begin{pmatrix} R_{Dcyt} + R_{Dmem} - R_{Ecyt} - R_{Emem} \\ -R_{exp} + R_{Ecyt} + R_{Emem} \end{pmatrix},$$

$$\vec{G} = \begin{pmatrix} -R_{Dcyt} - R_{Dmem} \\ R_{exp} \\ R_{exp} - R_{Ecyt} - R_{Emem} \end{pmatrix} \text{ and } \vec{H} = \begin{pmatrix} R_{exc} \\ -R_{exc} \\ 0 \end{pmatrix}.$$

Here

$$R_{exc} = k_{exc}[D_{cyt}^{ADP}]$$

$$R_{Dcyt} = k_{Dcyt}[D_{cyt}^{ATP}]$$

$$R_{Dmem} = k_{Dmem}[D_{mem}^{ATP}][D_{cyt}^{ATP}]$$

$$R_{exp} = k_{exp}[E:D_{mem}^{ATP}]$$

$$R_{Ecyt} = k_{Ecyt}[E_{cyt}][D_{mem}^{ATP}]$$

$$R_{Emem} = k_{Emem}[D_{mem}^{ATP}][E_{cyt}][E:D_{mem}^{ATP}]^2$$

represent reaction rates. This is a five component model with $(u,v) = (u_1,u_2,u_3,v_1,v_2)$, where $u_1 = D_{cyt}^{ATP}, u_2 = D_{cyt}^{ADP}$ and $u_3 = E_{cyt}$, and $v_1 = D_{mem}^{ATP}$ and $v_2 = E:D_{mem}^{ATP}$.

Our local existence result holds for any number of finite components, therefore this system has a unique maximal solution. Regarding global existence, we have in general analyzed the two component model. It is interesting to note in this example, that if we take two specific component at a time, we are able to obtain global existence. For that purpose we apply our results to (u_1, v_1), (u_3, v_2), and (u_2, v_2).

Notice that reaction vector fields associated to (u_1, v_1), (u_3, v_2), and (u_2, v_2) on M are (G_1, F_1), (G_3, F_2) and (G_2, F_2) respectively. Where $G_1 = -R_{Dcyt} - R_{Dmem}, G_2 = R_{exp}$ and $G_3 = R_{exp} - R_{Ecyt} - R_{Emem}$, and $F_1 = R_{Dcyt} + R_{Dmem} - R_{Ecyt} - R_{Emem}$, $F_2 = -R_{exp} + R_{Ecyt} + R_{Emem}$. And H is a reaction vector field associated to u on Ω. Where $H_1 = R_{exc}, H_2 = -R_{exc}$ and $H_3 = 0$. It is easy to see $G_1 + F_1 \leq 0$, $G_3 + F_2 \leq 0$ and G_i are linearly bounded for all $i = 1, 2, 3$, and $H_1 \leq k_{exc} u_2$, $H_2, H_3 \leq 0$. So, for each of the pair (u_1, v_1) and (u_3, v_2), conditions (V1), (V2), and (V3), (hypothesis of global existence, Theorem 6.1.15) are satisfied. As a result $\int_\Omega u_1 + \int_M v_1$ and $\int_\Omega u_3 + \int_M v_2$ is conserved and u_1, v_1, u_3 and v_2 exists for all time $t > 0$. Now, since $H_2 \leq 0$ and $\int_M v_2 < \infty$, therefore from [19], $\int_\Omega u_2 < \infty$.

We also discussed another example briefly in chapter 1. Consider the system

$$\begin{aligned}
u_t &= \Delta u && x \in \Omega, \quad 0 < t < T \\
v_t &= \Delta_M v + u^a v^b && x \in M, \quad 0 < t < T \\
\frac{\partial u}{\partial \eta} &= -u^a v^b && x \in M, \quad 0 < t < T \\
u &= u_0 && x \in \Omega, \quad t = 0
\end{aligned}$$

$$v = v_0 \qquad\qquad x \in M, \quad t = 0$$

Here $G(u,v) \leq 0$, $F + G \leq 0$, and (V1), (V2), and (V3) are clearly satisfied. Therefore, conditions for global existence are satisfied.

Remark 7.0.4. If we consider the similar system with $G(u,v) = u^a v^b$ and $F(u,v) = -u^a v^b$, then the results [19] apply in the non-manifold setting, but in our setting it is still an open question.

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