STABILIZATION OF OPEN-LOOP UNSTABLE SYSTEMS BY THE COMBINE USE OF STATE-VARIABLE AND TRANSFER-FUNCTION TECHNIQUES

A Thesis

Presented to

the Faculty of the Department of Electrical Engineering

University of Houston

In Partial Fulfillment

of the Requirements for the Degree

Master of Science in Electrical Engineering

---

by

Hassan Hamidi-Hashemi

August 1973

ACKNOWLEDGMENT

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DEDICATION

TO: Farahnaz

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ABSTRACT

The inverted pendulum system, expressed mathematically by a third order transfer function with a zero at the origin, has many practical applications. Because of the complexity of the third order transfer function with zero at the origin, optimization is difficult. In this paper a combination of the classical and state variable approach is used to analyze, compensate, and design a working model of an inverted pendulum system. The state variable feedback technique is used to provide system stability. Performance criteria optimization of the above third order transfer function is investigated in classical methods for application to practical models.

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CHAPTER I

INTRODUCTION

In the early 1950's, a new method, called the state variable approach, was introduced for modeling and analyzing control systems. The motivations for development of this new method were[4]:

1. The general convenience of powerful vector space and its related theorems regarding the analytic structure of system response.

2. The necessity for a general and basic structure for theoretical investigations of indefinite and varied systems.

3. The acknowledgment of the classical approach limitations.

One of the limitations of classical approaches in control system design is synthesis. However, in the state variable approach, the availability of potentiometers and amplifiers makes the problem of design synthesis relatively straightforward.

Moreover, the advent of the general purpose digital computer suited for analysis of highly complex systems further enforces the use of the state variable approach. The digital computer provides high speed capabilities which minimizes the control theorist's complicated computation difficulties. Since the state variable approach uses the algebra of matrices extensively, complex systems may be characterized in simple and concise ways. Thus the complexity is overcome allowing a better intuitive insight into the functioning of a physical system.

Optimum control is the most modern and direct of all design methods. It utilizes exclusively the state variable, rather than transfer-function or system descriptions. It supplies the unique solution (which is truely optimal for given indices of performance) for linear or nonlinear and time variant or invariant systems. In addition, it incorporates as adequately as possible all the factors that add to the cost of performance. By using optimum control methods, engineers may design systems with performance characteristic that the classical methods may not allow.

However, although the state variable approach is useful in the analysis of many systems, the classical frequency response techniques and transfer function[3] description excel in certain cases, for example the steady state analysis of systems is best done with the classical approach. Also, system descriptions based on the desired closed loop transfer function are in some cases the only design information of practical value. There are many other examples and the classical theory must not be regarded as inferior in the hierarch of methods.

The following chapters of this paper is devoted to the investigation analysis, compensation, and optimization of a stabilized inverted pendulum system.

CHAPTER II

STRUCTURE OF INVERTED PENDULUM SYSTEM

The structure of the inverted pendulum system to be considered is presented in Figure 1. It consists of a cart with an inverted pendulum hinged on top of it and the cart acted upon by a driving force, u, and a viscous friction force, f. The control objective is to maintain the pendulum in a vertical position, or as nearly vertical as possible, with the assistance of the control force u. This artificial method of balancing the pendulum represents a very accurate dynamic model of a space missile[1] on take off or of a helicopter[7] in the air, as depicted in Figure 2. The missile[1] is balanced on top of the rocket engine thrust vector, T. The thrust vector can be given small horizontal components which have the same effect on the rocket as the force u has on the inverted pendulum system. The viscous friction of the wheel bearings of the cart is a suitable substitute for the air viscous friction on the missile. The only basic difference between the missile and the inverted pendulum systems is that the inverted pendulum is restricted to perform dynamics in only one vertical plane, i.e., a two dimensional system. However, the missile system can be considered as a two dimensional system by treating the pitch and yaw dynamics separately.

Similarly, the torque produced by the horizontal rotor system of the helicopter[7] causes lateral rotation of the air-

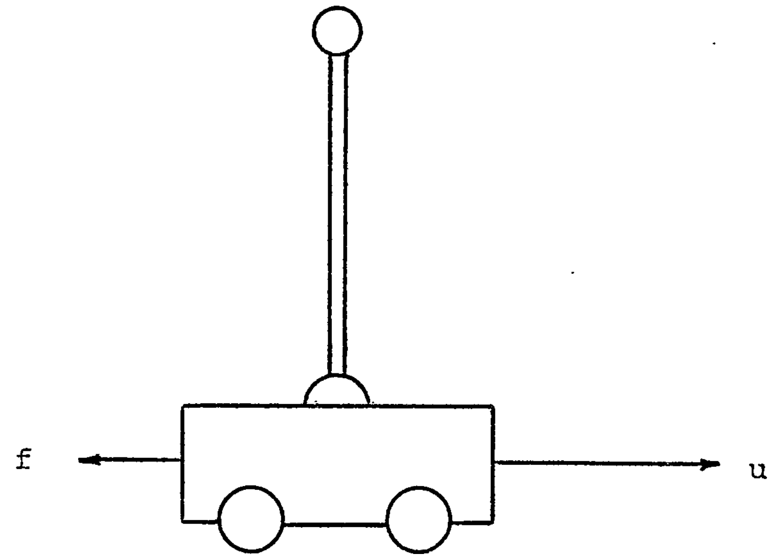


Figure 1. Structure of Inverted Pendulum System

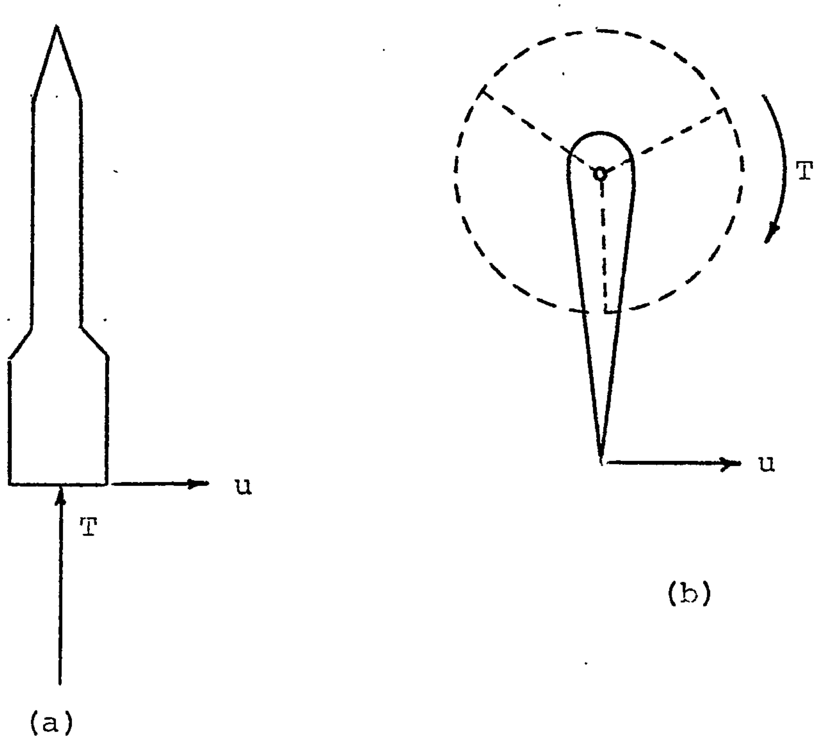


Figure 2. a) Missile on Take Off

b) Helicopter in Air

craft. Applying a horizontal component at the rear of the aircraft offsets the lateral rotation.

CHAPTER III

CONSTRUCTION OF MATHEMATICAL MODEL

The mathematical model of this inverted pendulum system can be simply obtained by Lagrange's equation:[2,6]

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = \Sigma F_{q_k} \tag{1}$$

where L, called Langrangian Function, is the difference between the total kinetic and potential energies

$$L = T-U \tag{2}$$

and q is the system variable. Subscript k denotes the different system variables and ΣF represents the sum of all forces acting upon the body of the system.

The realtionships between the system coordinates, shown in Figure 3, are:

$$X_2 = X_1 + \ell \sin\theta \tag{3}$$

$$Y_2 = \ell \cos\theta \; * \tag{4}$$

which result

$$\dot{X}_2 = \dot{X}_1 + \ell\dot{\theta}\cos\theta \tag{5}$$

$$\dot{Y}_2 = -\ell\dot{\theta}\sin\theta \tag{6}$$

---

*Assuming Y_1 is zero.

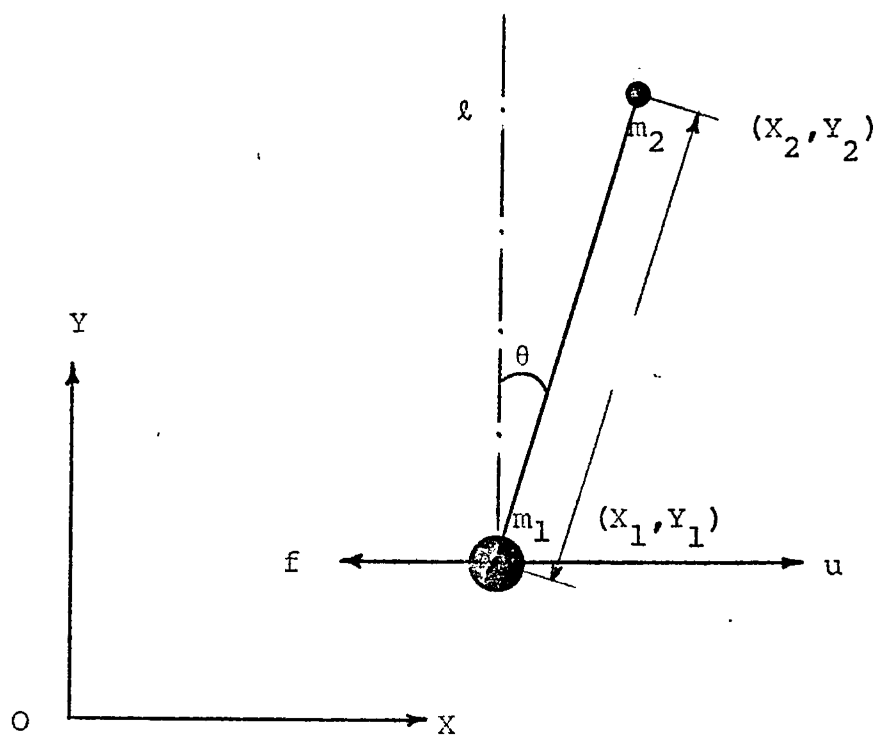


Figure 3. Coordinate System for the Inverted Pendulum

then,

$$T = \Sigma \frac{1}{2} m_i v_i^2 = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 + \frac{1}{2} m_2 \dot{y}_2^2 \tag{7}$$

$$U = \Sigma m_i g h_i = m_2 g y_2 \tag{8}$$

or

$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 (\dot{x}_1 + \ell\dot{\theta}\cos\theta)^2 + \frac{1}{2} m_2 \ell^2 \dot{\theta}^2 \sin^2\theta \tag{9}$$

$$U = m_2 g \ell \cos\theta \tag{10}$$

therefore,

$$L = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 (\dot{x}_1 + \ell\dot{\theta}\cos\theta)^2 + \frac{1}{2} m_2 \ell^2 \dot{\theta}^2 \sin^2\theta - m_2 g \ell \cos\theta \tag{11}$$

and

$$\frac{\partial L}{\partial x_1} = 0 \tag{12}$$

$$\frac{\partial L}{\partial \dot{x}_1} = m_1 \dot{x}_1 + m_2 (\dot{x}_1 + \ell\dot{\theta}\cos\theta) \tag{13}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) = m_1 \ddot{x}_1 + m_2 \ddot{x}_1 + m_2 \ell \ddot{\theta}\cos\theta - m_2 \ell \dot{\theta}^2 \sin\theta \tag{14}$$

$$\frac{\partial L}{\partial \theta} = -m_2 \ell \dot{x}_1 \dot{\theta} \sin\theta + m_2 g \ell \sin\theta \tag{15}$$

$$\frac{\partial L}{\partial \dot{\theta}} = m_2 \ell \dot{x}_1 \cos\theta + m_2 \ell^2 \dot{\theta} \tag{16}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = m_2 \ell \ddot{x}_1 \cos\theta - m_2 \ell \dot{x}_1 \dot{\theta}_1 \sin\theta + m_2 \ell^2 \ddot{\theta} \tag{17}$$

In addition,

$$\Sigma F = u - K\dot{x}_1 \qquad \text{for} \qquad x_1 \text{ variable} \tag{18}$$

$$\Sigma F = 0 \qquad \text{for} \qquad \theta \text{ variable} \tag{19}$$

where K is the viscous friction force constant. Consequently,

$$(m_1+m_2)\ddot{x}_1+m_2\ell\ddot{\theta}\cos\theta-m_2\ell\dot{\theta}^2\sin\theta = u-K\dot{x}_1 \tag{20}$$

$$m_2\ell\ddot{x}_1\cos\theta+m_2\ell^2\ddot{\theta}-m_2g\ell\sin\theta = 0 \tag{21}$$

Equation (20) and (21) form the basis for all the future investigation analysis and design of the system.

CHAPTER IV

LINEARIZATION OF MATHEMATICAL MODEL

Unfortunately, nature is inherently nonlinear; therefore, in constructing models for physical systems, one arrives more often than not at nonlinear equations[1] such as Equation 20 and 21 in Chapter III. The present mathematical methods are inadequate to handle any but the very simplest types of nonlinear differential equations.

Fortunately, the apparently hopeless problem posed by the model's nonlinearity can be solved by linearization. To linearize any model, some limitations and assumptions are essential. The choice of a suitable model embodying all the features of a physical system, critical to its performance, may be difficult. If an overly simplified model is used, the results obtained from it will not closely approximate the behavior of the physical system. If, on the other hand, an unnecessarily complicated model is used, it may be difficult or even impossible to analyze.

In the inverted pendulum system, the force u corrects the angle of deviation θ, as soon as the pendulum starts falling. Therefore, the angle θ, will in general be very small, so small in fact that

$$\theta<<1$$

then, with sufficient accuracy

$$\sin\theta = \theta - \frac{\theta^3}{6} \ldots . \simeq \theta \qquad (22)$$

$$\cos\theta = 1 - \frac{\theta^2}{6} + \ldots \simeq 1 \qquad (23)$$

In addition, the angular acceleration $\dot{\theta}$, can be assumed small. Consequently, $\dot{\theta}^2$ can be neglected. Then, the nonlinear equations (20) and (21) change into the linear equations:

$$m_2\ell^2\ddot{\theta} + m_2\ell\ddot{x} - m_2 g\ell\theta = 0 \qquad (24)$$

$$(m_1+m_2)\ddot{x} + K\dot{x} + m_2\ell\ddot{\theta} = u \qquad (25)$$

An infinite number of sets of state variables may be obtained from Equation 24 and 25; but only those which can be practically measured will be chosen for analyzing the system.

The selected state variables for this system are the angular position of the pendulum θ, the angular acceleration of the pendulum $\dot{\theta}$, and the velocity of the cart $\dot{x}_1$. Then, the state variable equations derived from the linear equations by simple manipulations are:

$$\dot{z}_1 = z_2 \qquad (26a)$$

$$\dot{z}_2 = \frac{g}{\ell}\left(1 + \frac{m_2}{m_1}\right)z_1 + \frac{K}{\ell m_1} z_3 - \frac{u}{m_1\ell} \qquad (26b)$$

$$\dot{z}_3 = -\frac{m_2 g}{m_1} z_1 - \frac{K}{m_1} z_3 + \frac{u}{m_1} \qquad (26c)$$

where,

$$z_1 = \theta \quad (27a)$$

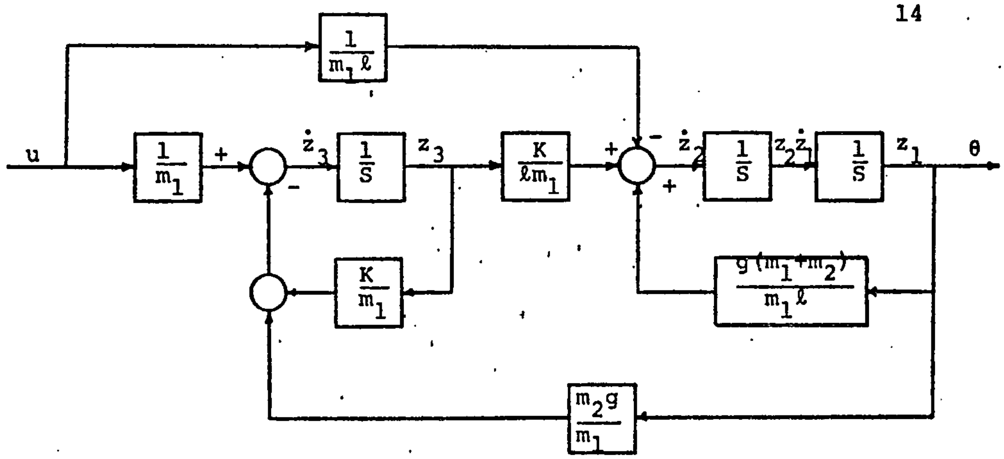
$$z_2 = \dot{\theta} \quad (27b)$$

$$z_3 = \dot{x}_1 \quad (27c)$$

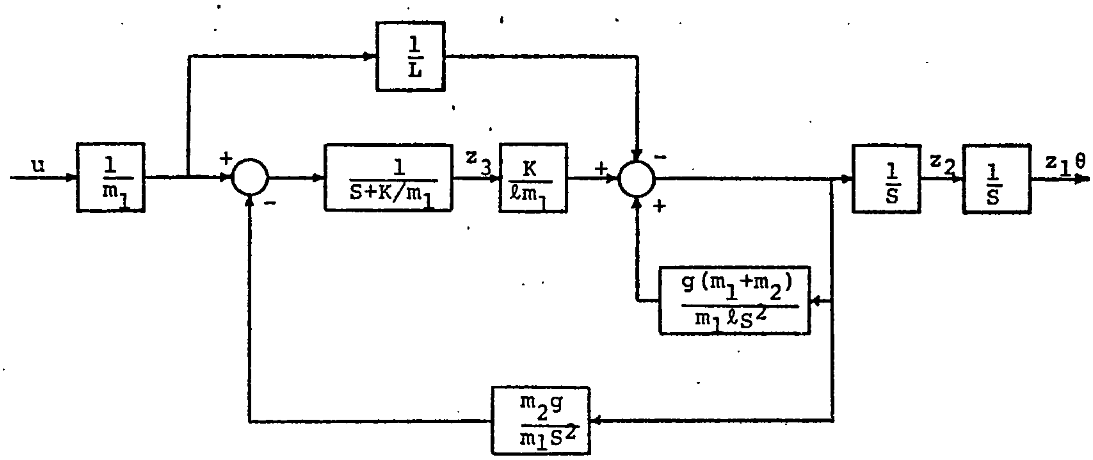
A complete procedure to obtain the simple block diagram[1,2] of the system is shown in Figure 4. The transfer function, from the block diagram,

$$\frac{\theta}{u} = \frac{-\frac{1}{m_1 \ell} S}{S^3 + (\frac{K}{m_1}) S^2 - \frac{g}{\ell}(1 + \frac{m_2}{m_1}) S - \frac{Kg}{m_1 \ell}}$$

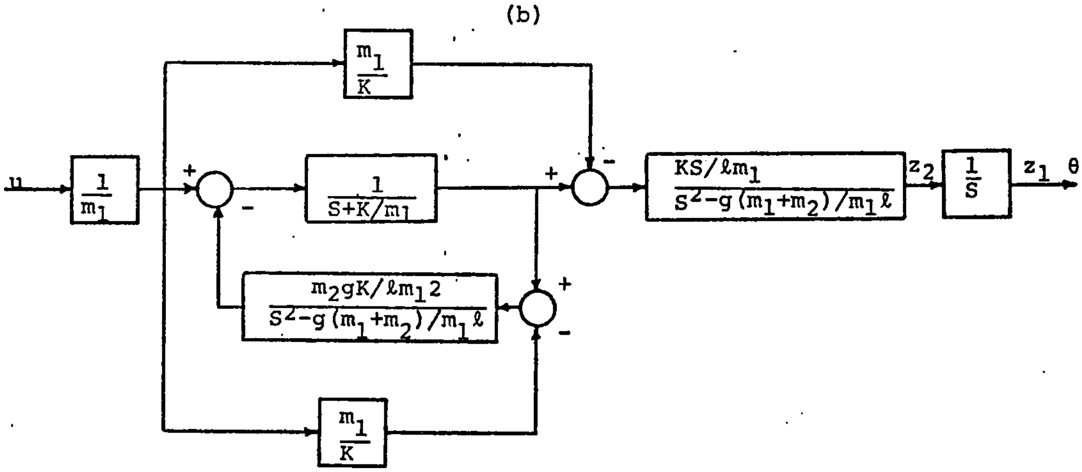
clearly indicates that the system is unstable, i.e., negative coefficients in the denominator.


u
1/m₁ℓ
1/m₁
ż₃
1/S
z₃
K/ℓm₁
ż₂
1/S
z₂ ż₁
1/S
z₁
θ
K/m₁
g(m₁+m₂)/m₁ℓ
m₂g/m₁

(a)


u
1/m₁
1/L
1/(S+K/m₁)
z₃
K/ℓm₁
1/S
z₂
1/S
z₁θ
g(m₁+m₂)/m₁ℓS²
m₂g/m₁S²

(b)


u
1/m₁
m₁/K
1/(S+K/m₁)
KS/ℓm₁ / S²-g(m₁+m₂)/m₁ℓ
z₂
1/S
z₁ θ
m₂gK/ℓm₁2 / S²-g(m₁+m₂)/m₁ℓ
m₁/K

(c)

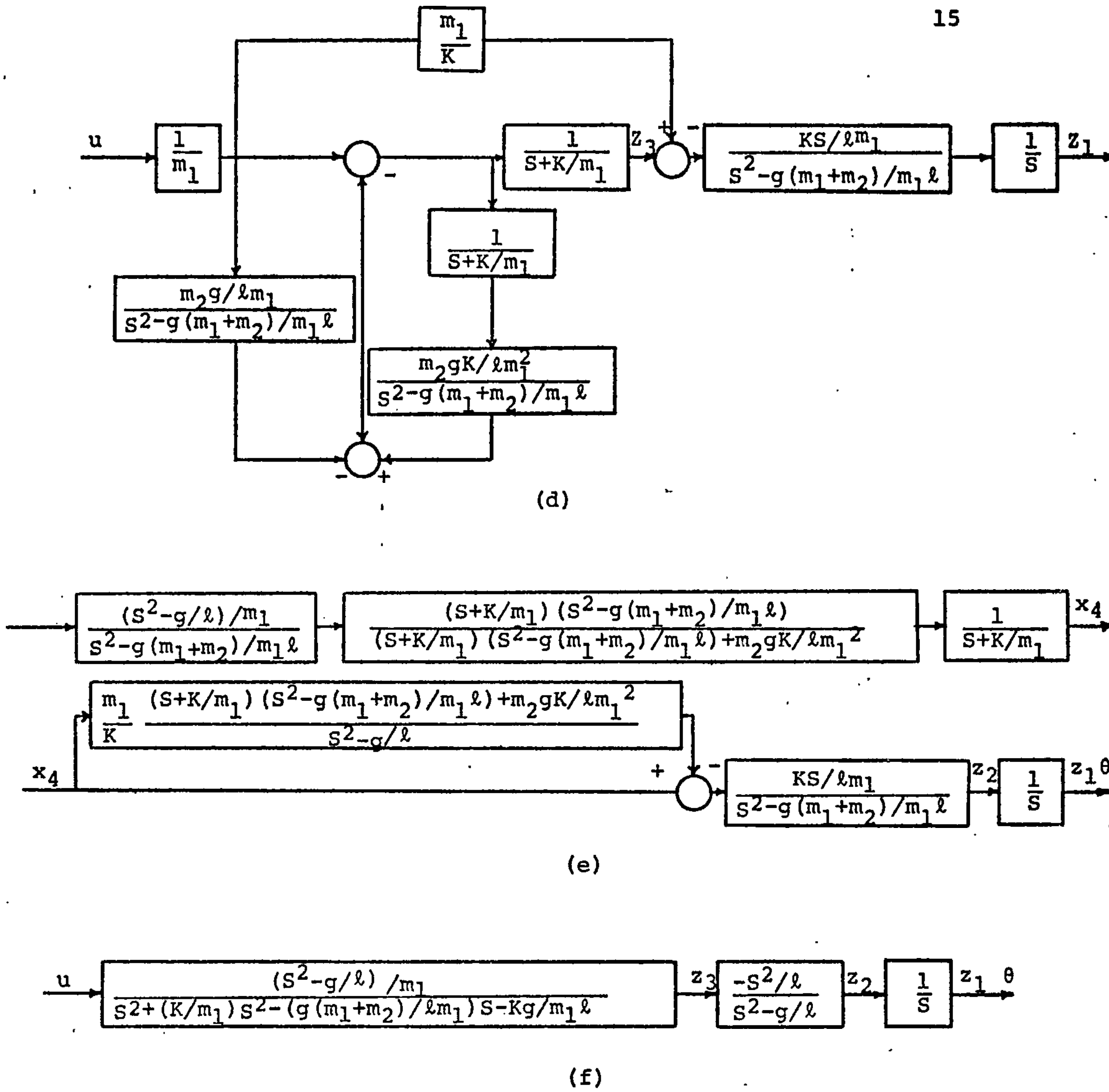


Figure 4. Simplification Procedure of the System Block Diagram.

CHAPTER V

FEEDBACK CONTROL MODEL

It is important in any feedback model to choose those variables which can be practically measured and fed into the system; a feedback model with no practical synthesis is of little use.

All the state variables of the inverted pendulum system can be simply measured by different techniques which will be discussed in Chapter VII. The feedback control model of the system is presented in Figure 5. The closed loop transfer function, from the feedback model, is:

$$\frac{\theta}{u} = \frac{-C/m_1 \ell \; S}{S^3 + \left(\frac{K}{m_1} - \frac{CH_2}{m_1 \ell} + \frac{CH_3}{m_1}\right) S^2 + \left(-\frac{g}{\ell} - \frac{m_2 g}{m_1 \ell} - \frac{CH_1}{m_1 \ell}\right) S + \left(-\frac{Kg}{m_1 \ell} - \frac{CgH_3}{m_1 \ell}\right)} \tag{29}$$

obviously, by choosing different forms for the state variable feedback compensators, any desired closed loop transfer function can be obtained. To maintain the order of the system as low as possible, the compensators H_1, H_2, H_3, and C are chosen as constants. However, it is still necessary to choose the numerical values of the chosen closed loop transfer function coefficients which is to be synthesized. To obtain optimum values for C, H_1, H_2 and H_3, an investigation analysis for a general third order transfer function with a zero at the origin is essential.

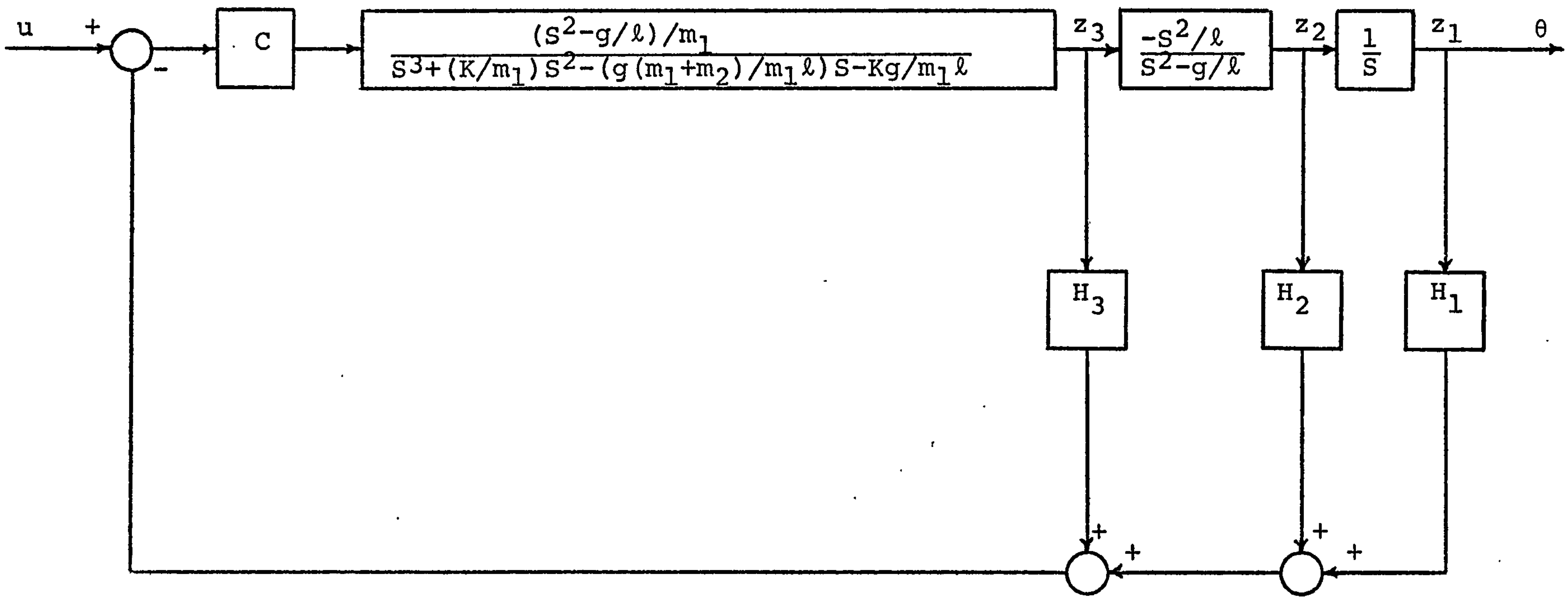


Figure 5. Feedback Control Model of the System

CHAPTER VI

OPTIMIZATION* OF THIRD ORDER TRANSFER FUNCTION WITH A ZERO AT THE ORIGIN

The third order transfer function with a zero at the origin is expressed in general by the equation:

$$T = \frac{S}{S^3+aS^2+bS+c} \tag{30}$$

The integral square value[5], ISV, of the transfer function is:

$$ISV = \frac{1}{2}\frac{\begin{vmatrix} 0 & -1 & 0 \\ 1 & b & 0 \\ 0 & a & c \end{vmatrix}}{\begin{vmatrix} a & c & 0 \\ 1 & b & 0 \\ 0 & a & c \end{vmatrix}} = \frac{1}{2(ab-c)} \tag{31}$$

The transfer function can be changed into a new form:

$$T = \frac{S}{(S+z)(S^2+2\xi\omega_n S+\omega_n^2)} \tag{32}$$

where,

$$a = 2\xi\omega_n + z \tag{33a}$$

$$b = \omega_n^2 + 2\xi\omega_n z \tag{33b}$$

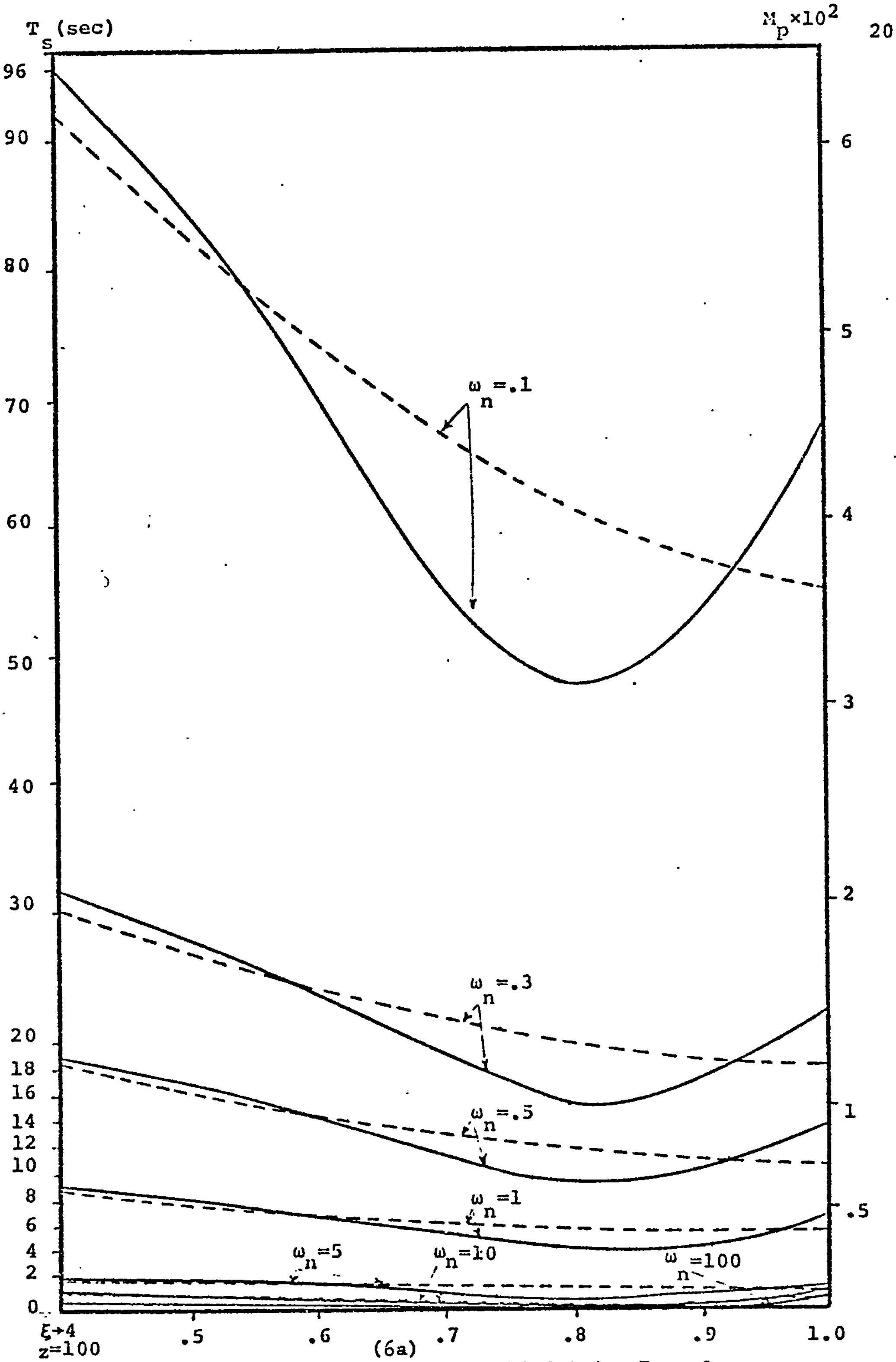
$$c = z\omega_n^2 \tag{33c}$$

*With respect to maximum peak and settling time.

then,

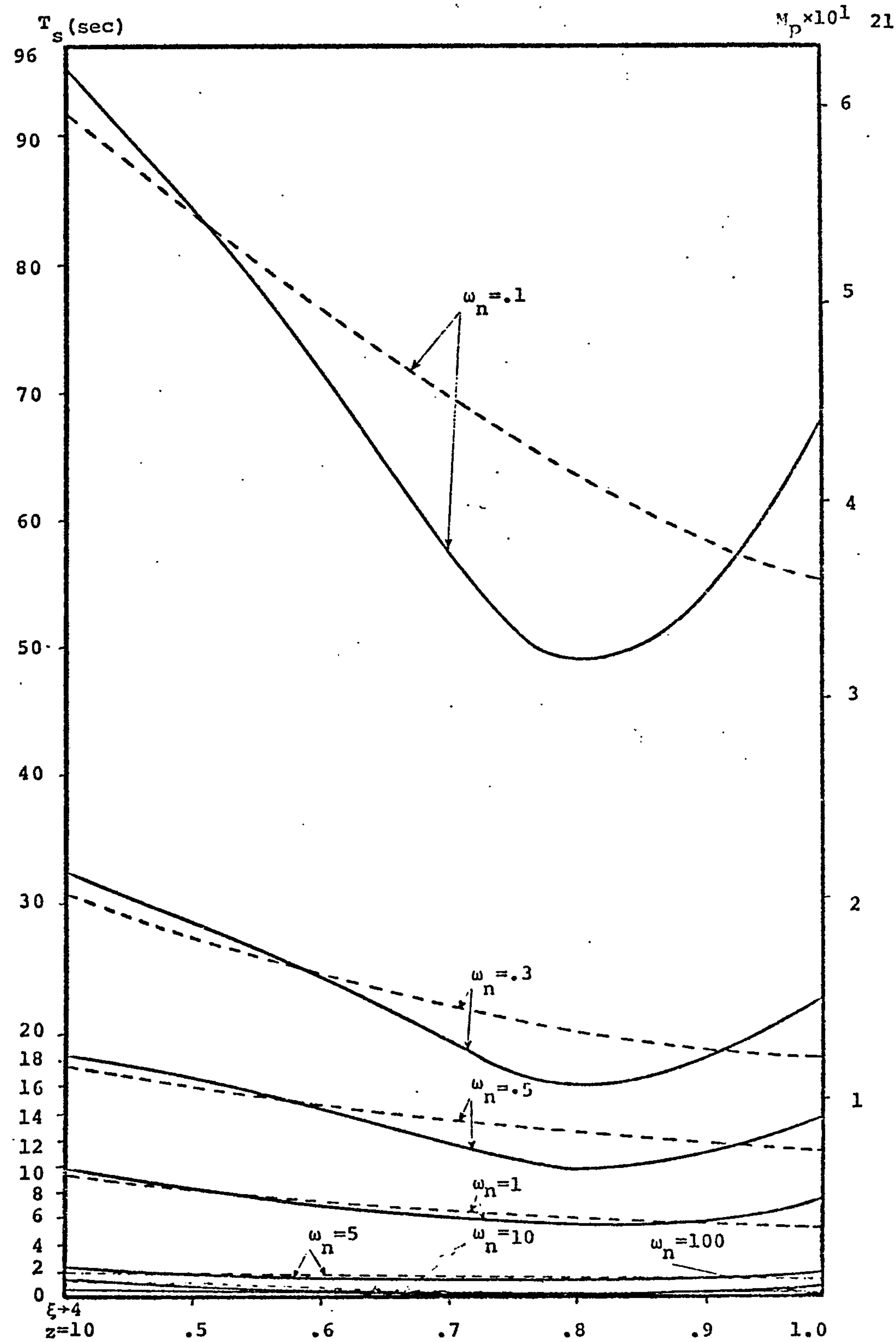
$$ISV = \frac{1}{4\xi\omega_n(z^2+2\xi\omega_n z+\omega_n^2)} \tag{34}$$

Equation (34) clearly shows that ISV is a decreasing monotonic function for all positive values of ξ, ω_n, and z. Therefore, absolute optimization can not be approached. In addition, the indices of performance are not available for this type of transfer function in order to optimize the system by the Riccati equation. However, by using computer programming, a set of curves relating reasonable performance criteria such as the maximum peak M_p, and the setting time T_s of the unit step response versus ξ, ω_n, and z can be obtained. Then knowing the required M_p, and T_s, one can select the desired closed loop transfer function from the curves shown in Figure 6. The curves can be used to intelligently design third order systems with a zero at the origin.



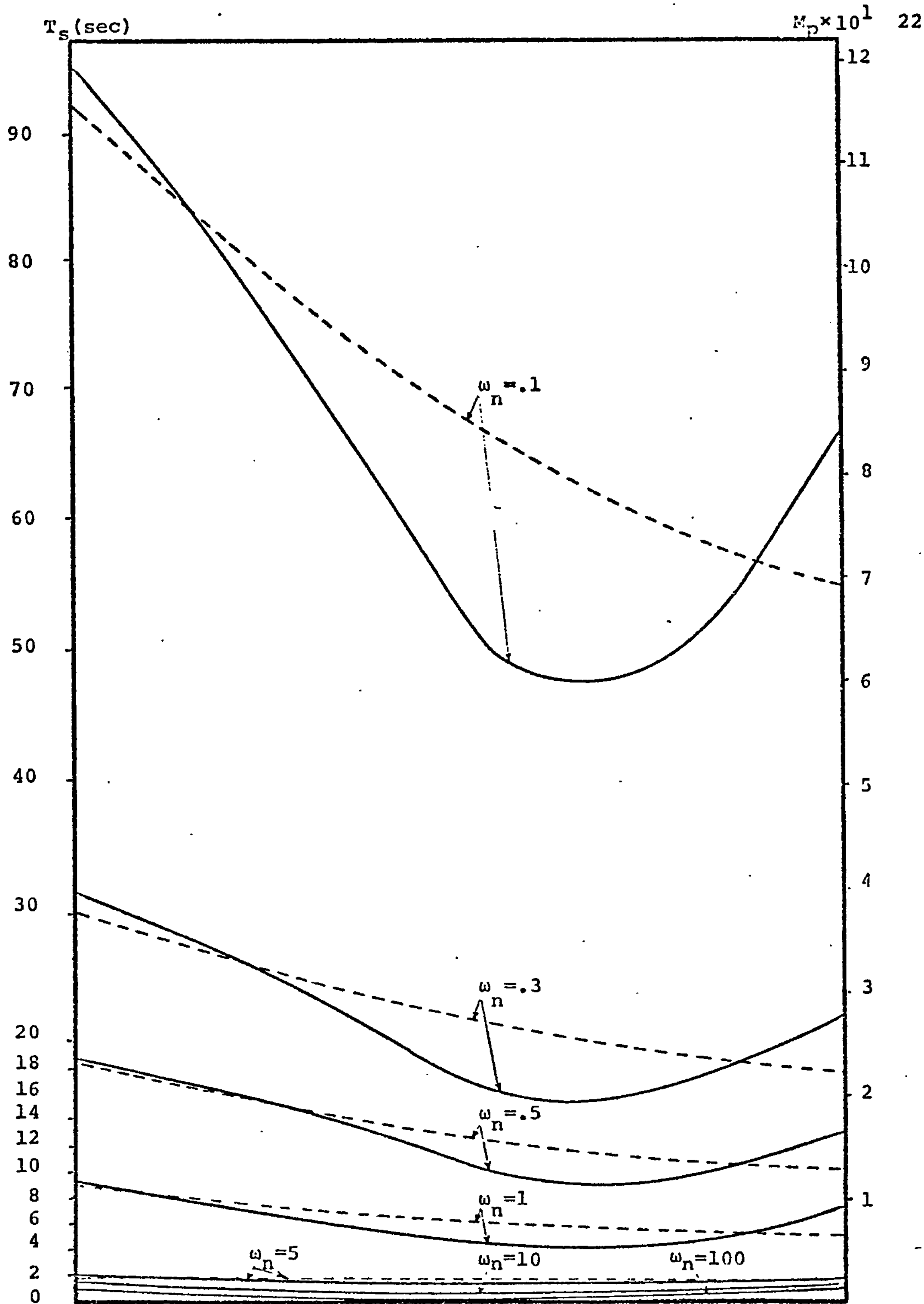
(6a)

M_p and T_s of Unit Step Response of Third Order Transfer Function with a Zero at the Origin vs. ξ&ω_n for z=100



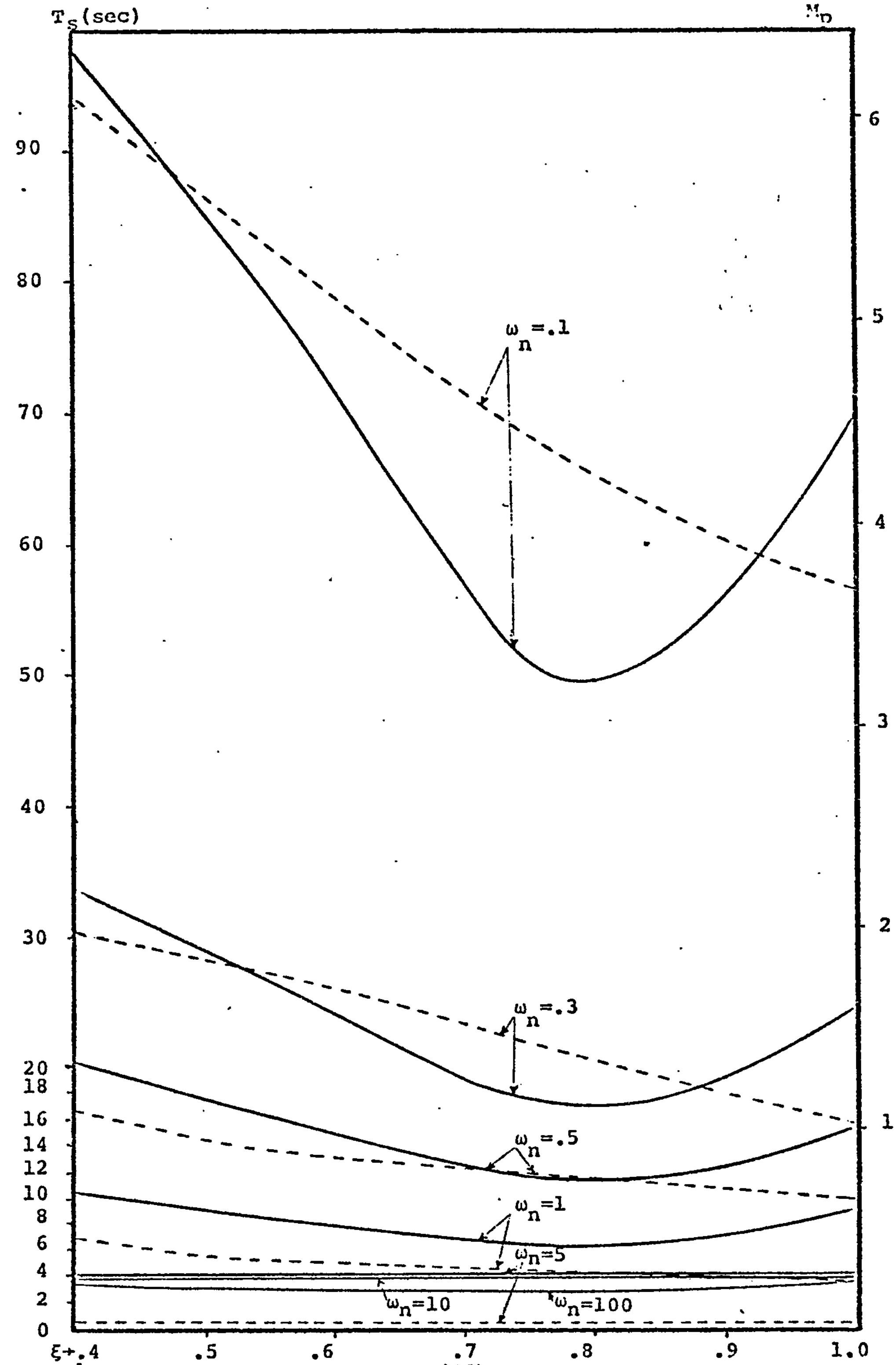
(6b)

M_p and T_s of Unit Step Response of Third Order Transfer Function with a zero at the Origin vs. ξ & ω_n for z=10



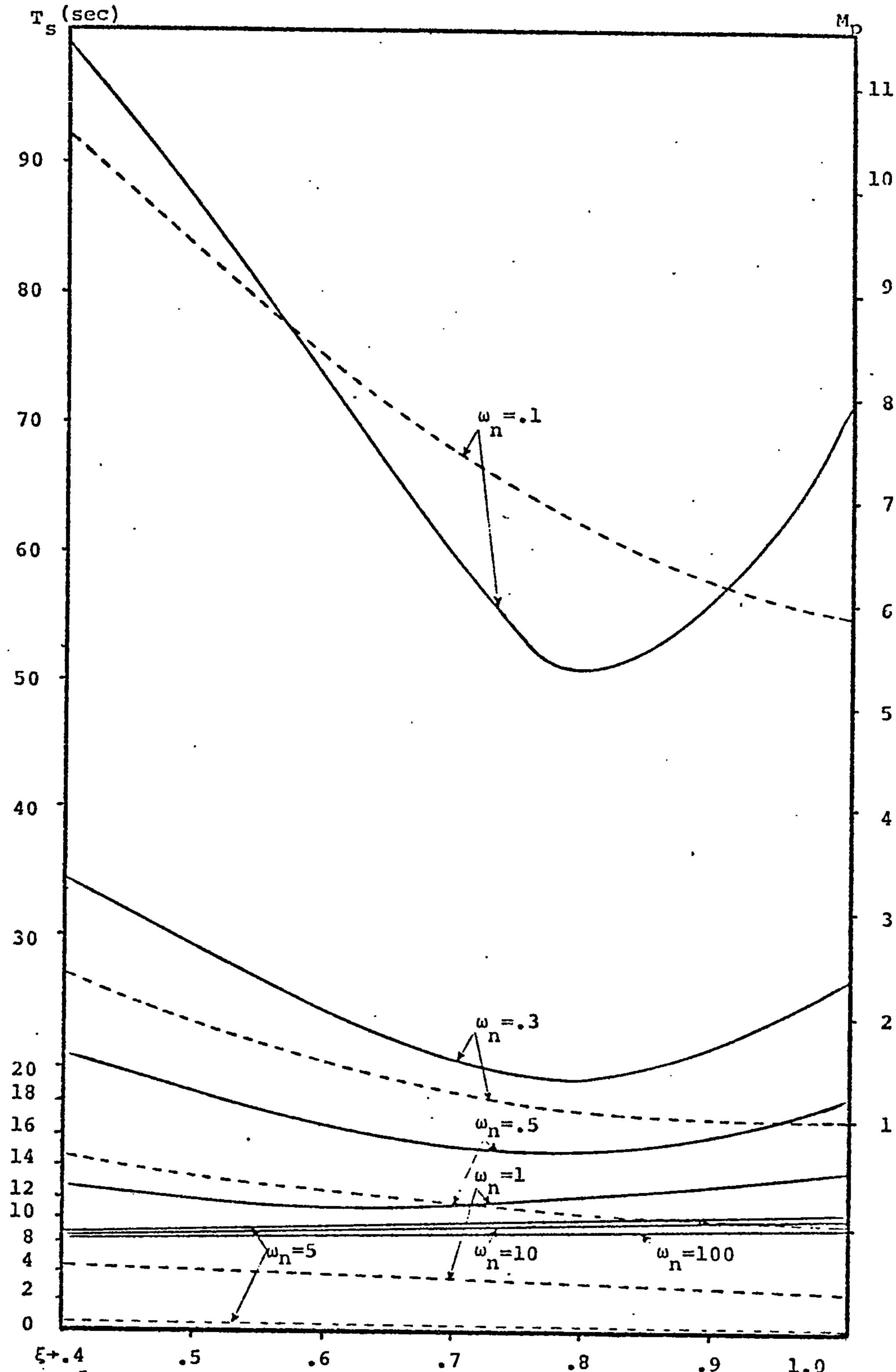
(6c)

M_p and T_s of Unit Step Response of Third Order Transfer Function with a Zero at the Origin vs. ξ&ω_n for z=5

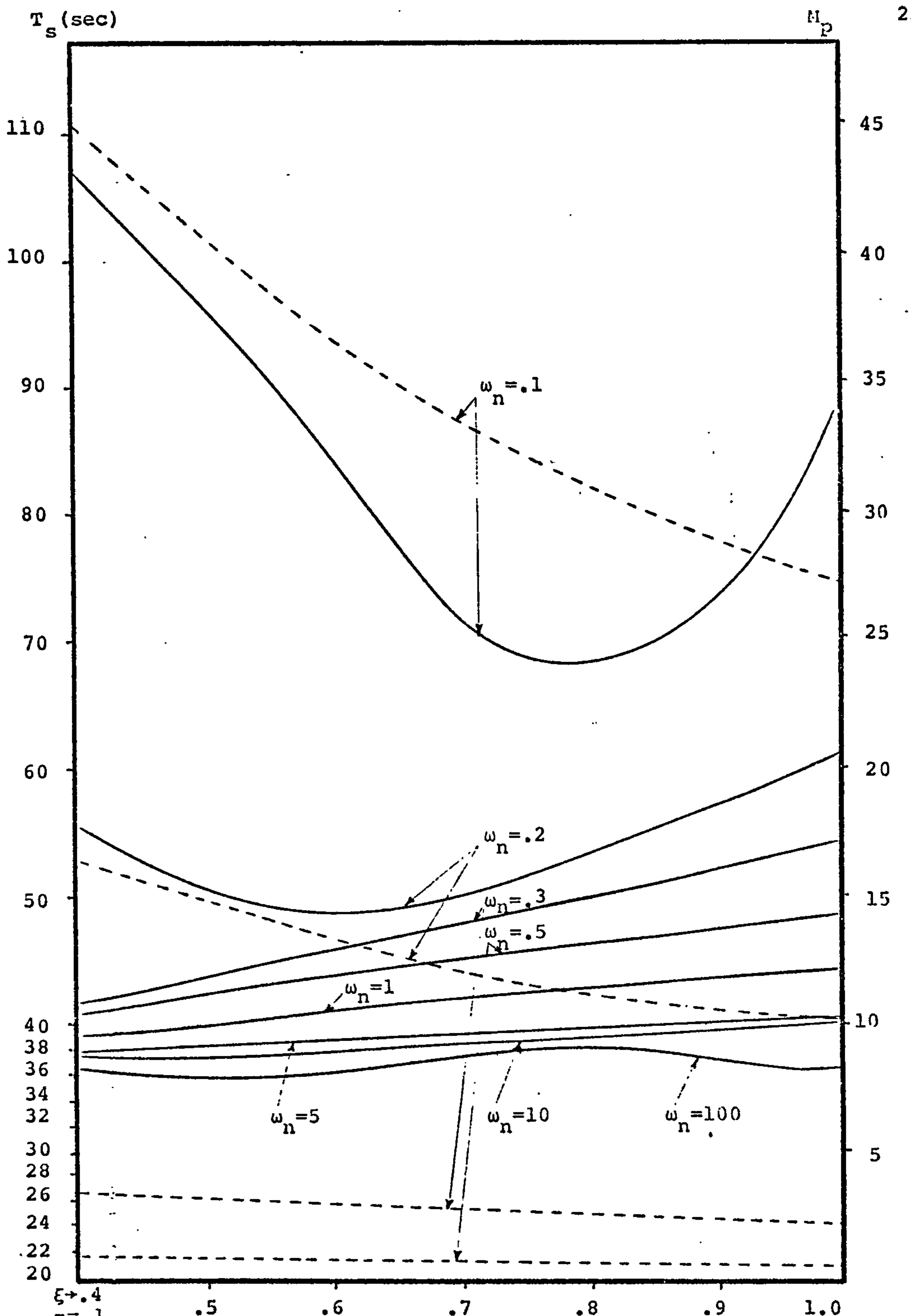


(6d)

M_p and T_s of Unit Step Response of Third Order Transfer Function with a Zero at the Origin vs. ξ&ω_n for z=1



(6e)
M_p and T_s of Unit Step Response of Third Order Transfer Function with a Zero at the origin vs. ξ&ω_n for z=.5



(6f)

Figure 6. M_p and T_s of Unit Step Response of Third Order Transfer Function with a zero at the Origin vs. ξ&ω_n for Constant z

CHAPTER VII

DESIGN OF THE FEEDBACK MODEL

A small maximum peak, M_p, and short settling time, T_s, are essential for the inverted pendulum system being compensated. The maximum peak must be limited to values for which the linear approximation is valid; the settling time should be small enough to keep the maximum cart displacement to a few inches. A suitable set of values is:

$$M_p \leq .2 \quad \text{(rad)}$$

$$T_s \leq 1 \quad \text{(sec)}$$

An infinite number of sets of values for z, ω_n, and ξ can be obtained from the curves of Figure 6 which will meet the above requirements. However, it is desired to meet the requirement with the minimum gains setting needed for the three state variables. From the curves the values of z, ω_n, and ξ are found to be:

$$\omega_n = 10$$

$$\xi = .6$$

$$z = 5$$

Then, the transfer function is:

$$T = \frac{S}{(S+5)(S^2+12S+100)} \tag{35}$$

Comparison of Equations 29 and 35 will result in:

$$\frac{K}{m_1} - \frac{CH_2}{m_1 \ell} + \frac{CH_3}{m_1} = 17 \tag{36a}$$

$$\frac{-g}{\ell} - \frac{m_2 g}{m_1 \ell} - \frac{CH_1}{m_1 \ell} = 160 \tag{36b}$$

$$\frac{-Kg}{m_1 \ell} - \frac{gCH_3}{m_1 \ell} = 500 \tag{36c}$$

The parameters of the system to be built are chosen as:

$m_1 = 2$ lbm

$m_2 = .25$ lbm

$\ell = 8.$ in

$K = .013$ lbf sec/ft

Therefore,

$H_1 = 214.2935$

$H_2 = 27.7789$

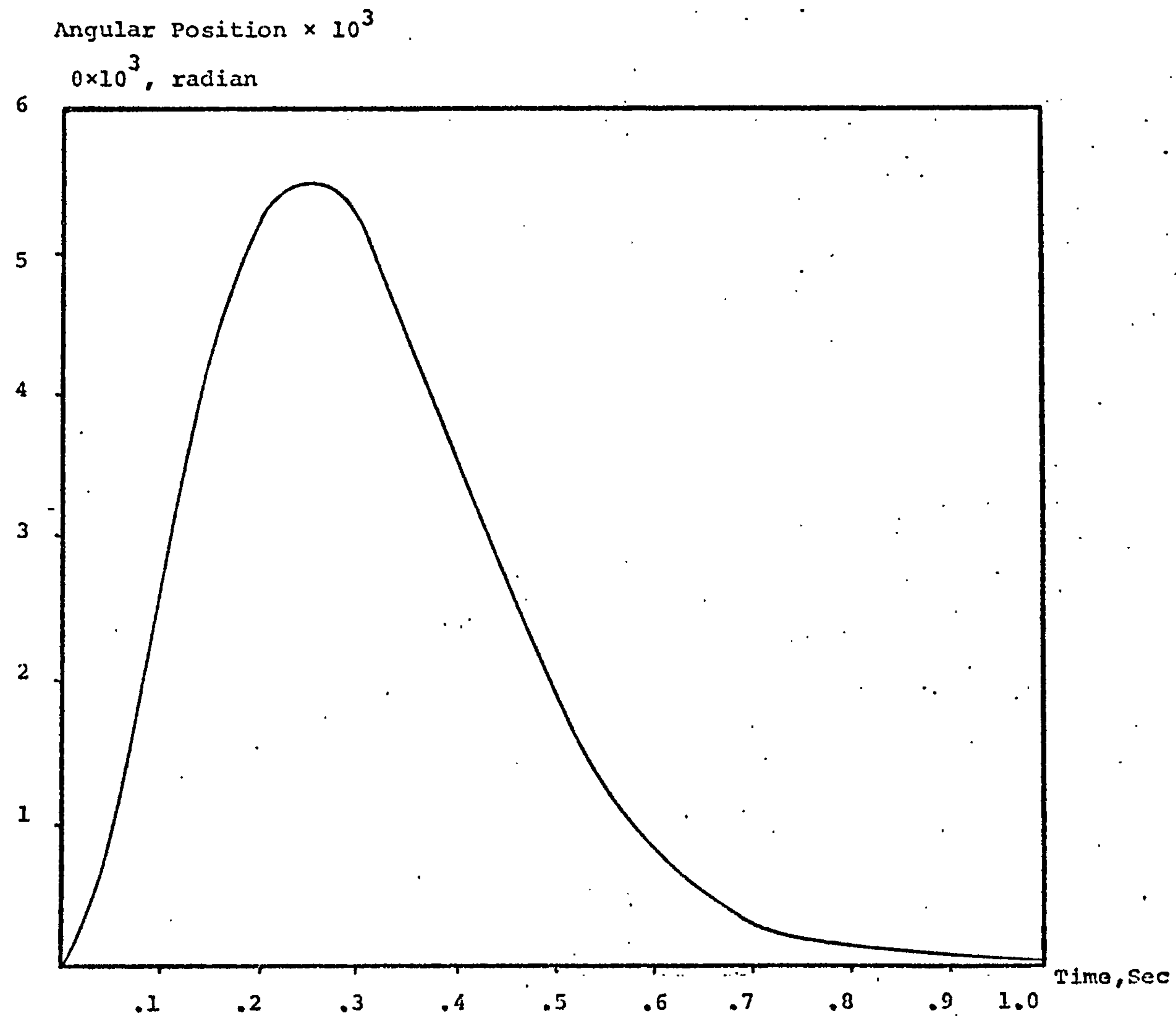
$H_3 = 15.8549$

$C = -.0414$

The unit-step and impulse responses of the closed loop transfer function having these values of feedback constants are shown in Figures 7a and 7b respectively. Since the design is based on approximated linear state variable equations, it is necessary to observe the response of the nonlinear model to insure that the system is stable. The unit-step response of the nonlinear model is presented in Figure 8. The results obtained from the responses of the linear and nonlinear models, shown in Table 1, are so close, that the difference is insignificant. In addition, the zero input response of the system for an initial angular position of .2 radians is shown in Figure 9. The computer runs indicate that the closed loop transfer function chosen would provide the proper system behavior.

The system components are represented by the additional blocks shown in Figure 10. The force u is obtained from a pair of driving wheels driven through a set of gears by a D.C. servomotor. The cart velocity is measured by a tachometer geared to the driving wheels. The pendulum angular position is measured by a potentiometer and the angular velocity is measured by a tachometer. The corresponding voltages from the tachometers and the potentiometer are amplified and summed and then used to drive the servomotor.

The system as described was built. Photographs of the system are shown in Figure 11. The electronic circuit built for the system, presented in Figure 12, consists of four low power operational amplifiers connected to a high power amplifier which drives the motor. The constant voltage drop of the zener


Angular Position × 10^3
θ×10^3, radian
6
5
4
3
2
1
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.0
Time,Sec

(7a)

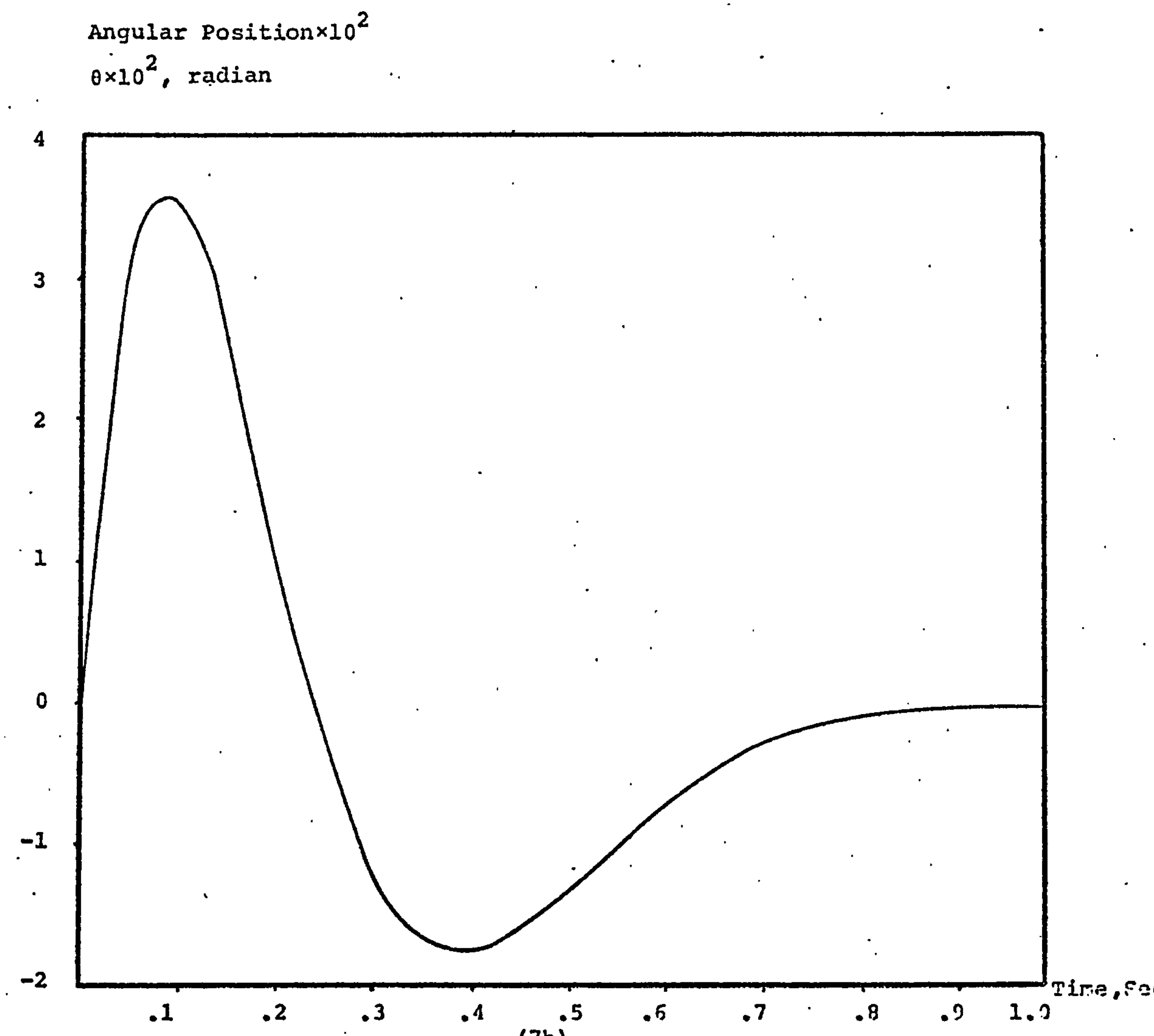
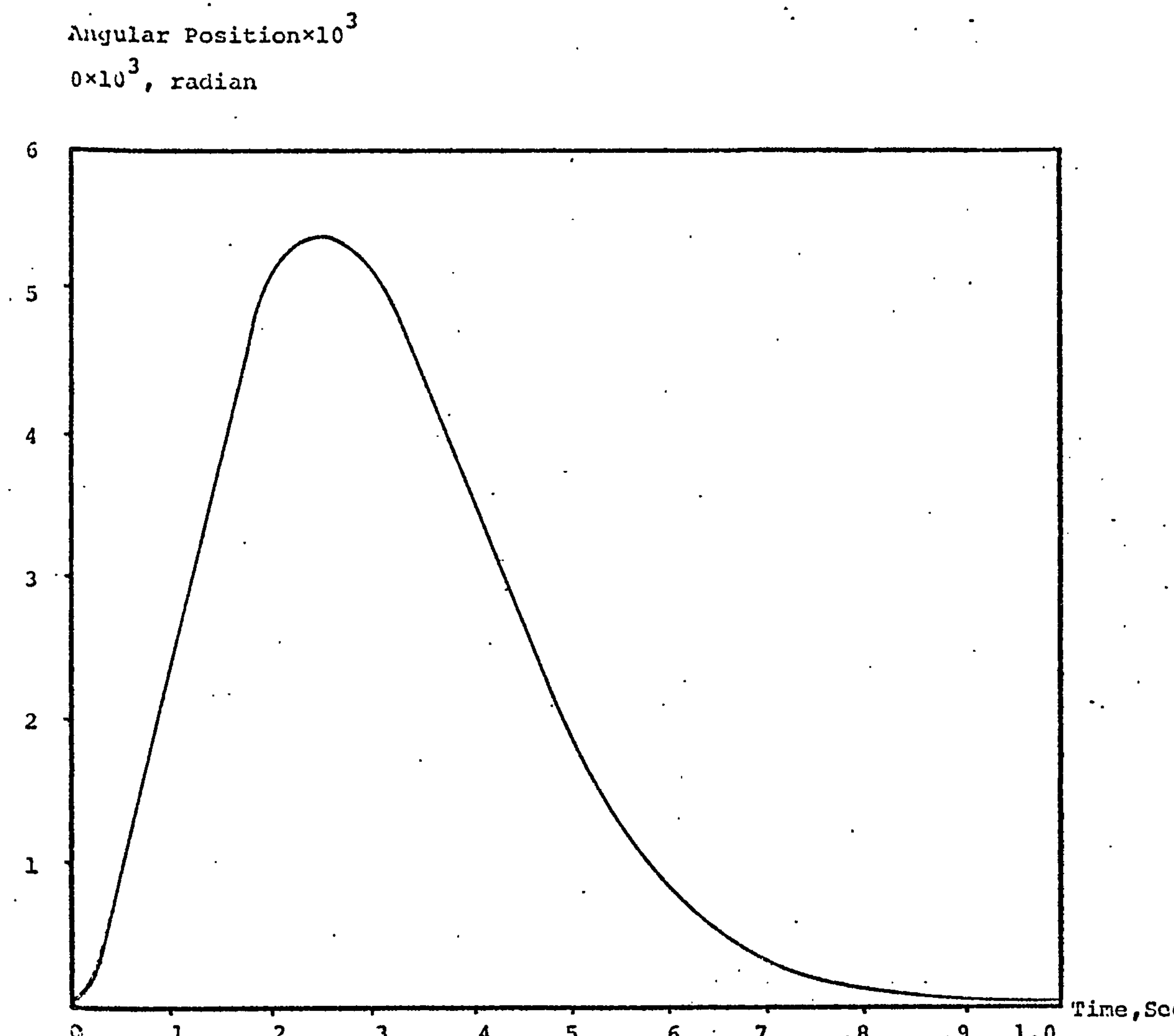
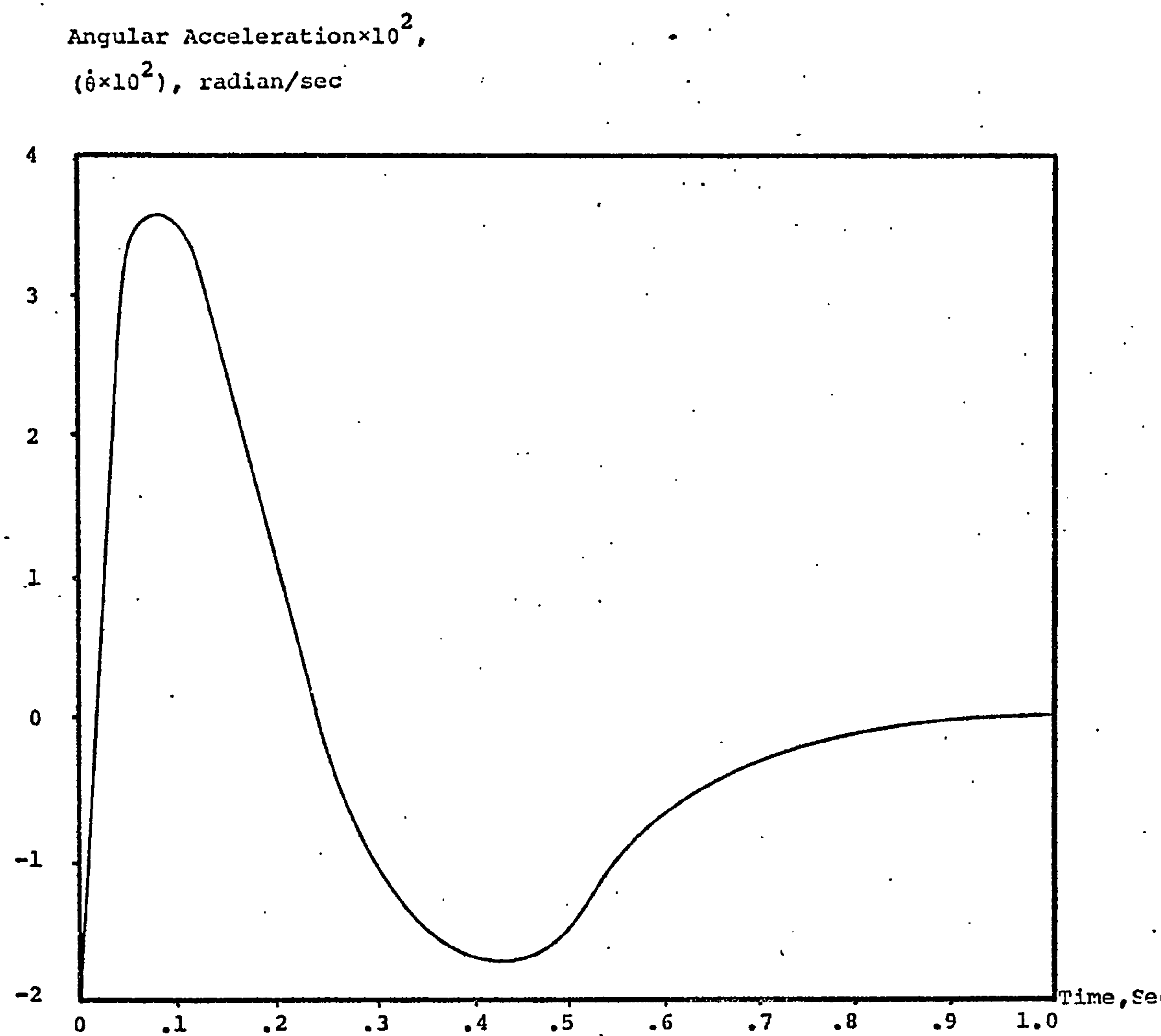


Figure 7. Response of the Closed Linear System (a) Unit Step Response
(b) Impulse Response


Angular Position×10^3
θ×10^3, radian
6
5
4
3
2
1
0
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.0
Time,Sec

(8a)


Angular Acceleration×10², (θ̇×10²), radian/sec
4
3
2
1
0
-1
-2
0
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.0
Time, Sec

(8h)

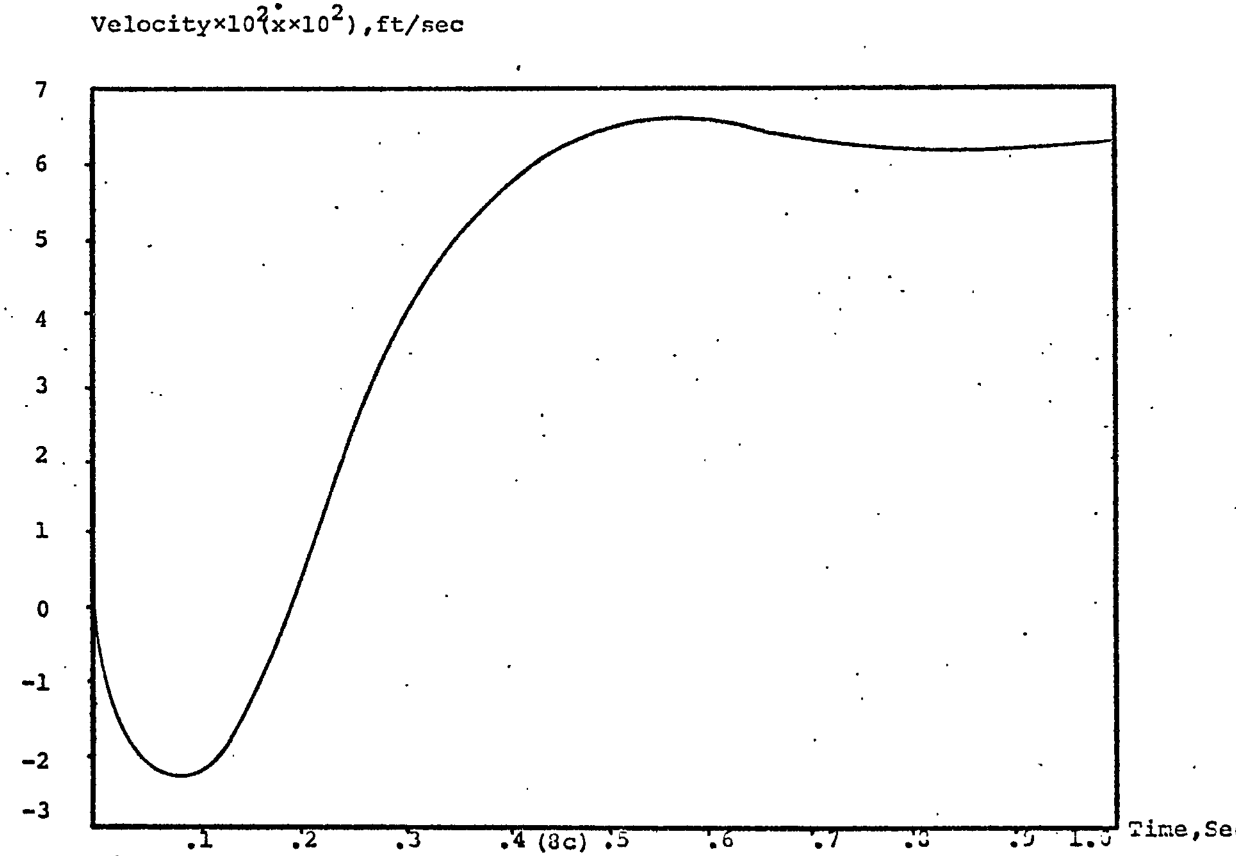


Figure 8. Unit Step Response of the Closed-Loop Nonlinear System
(a) Angular Position Response
(b) Angular Acceleration Response
(c) Velocity Response

	Results of Unit-Step Response for θ	
	Non Linear Model	Linear Model
Maximum Peak Radian	5.39×10^{-3}	5.45×10^{-3}
Settling Time Sec	8.70×10^{-1}	9.70×10^{-1}
Rise Time Sec	2.40×10^{-1}	2.40×10^{-1}

Table 1

Comparison of the Results of the Nonlinear and Linear Models

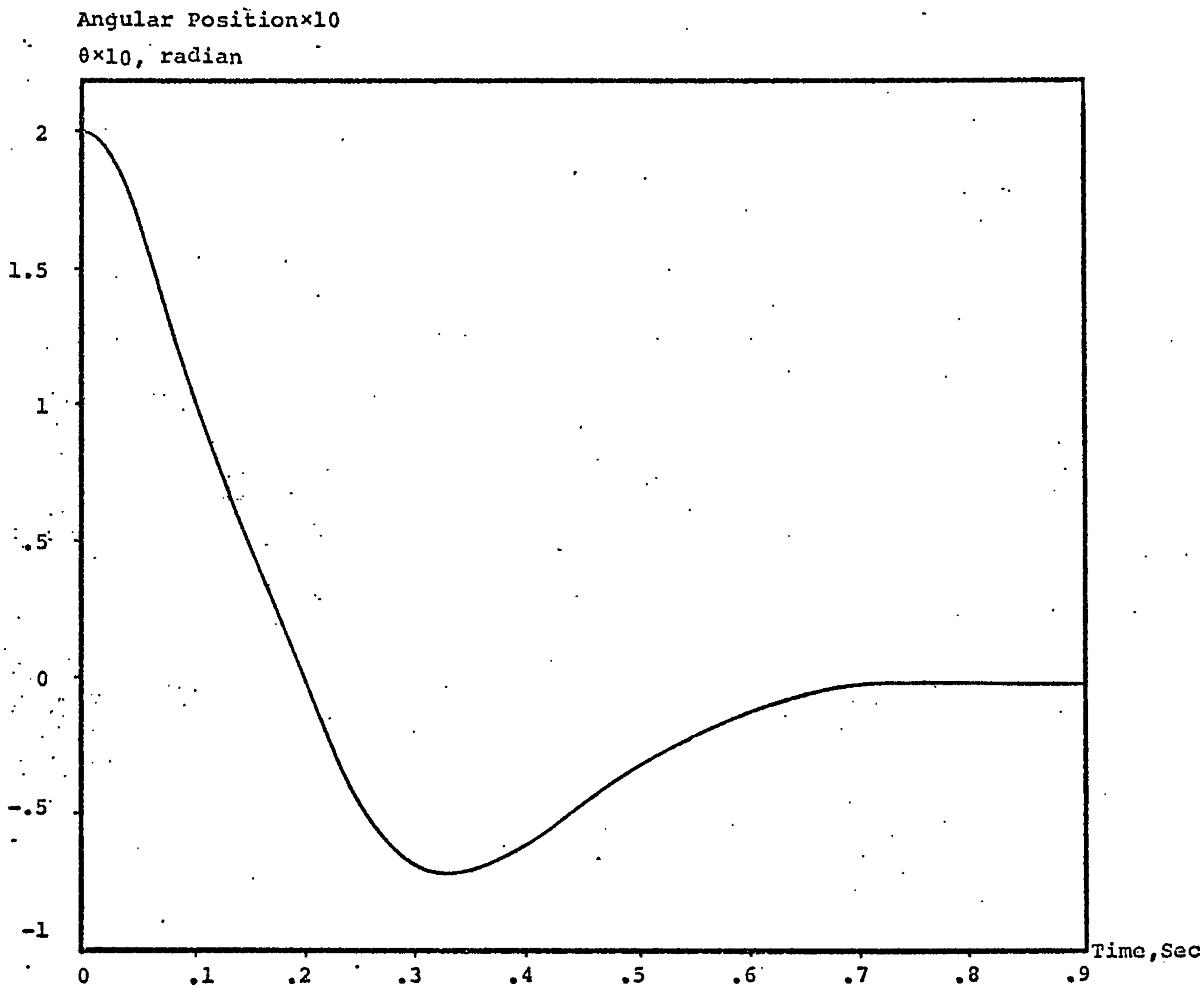


Figure 9. Zero Input Response of the Angular Position For an Initial Angular Position of .2 Radians

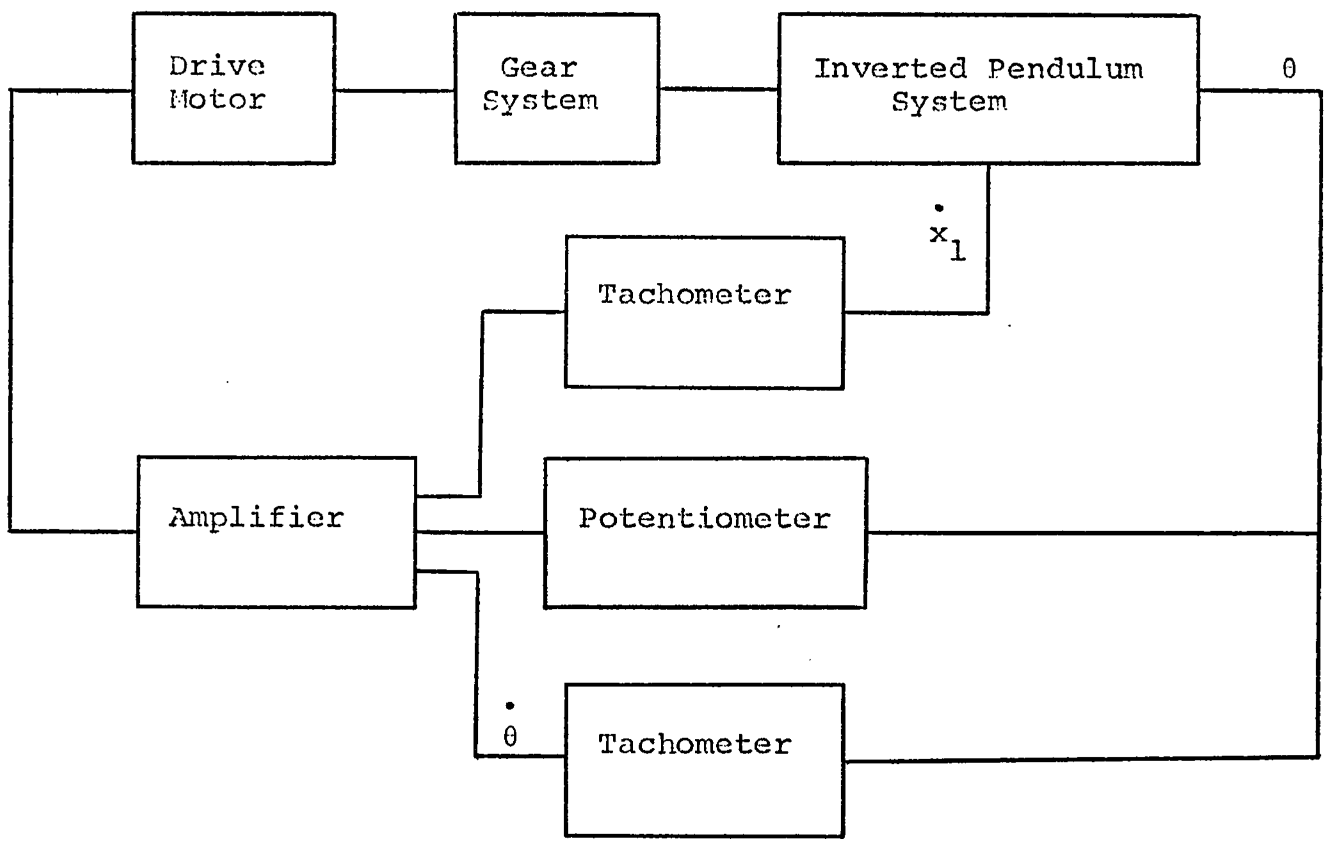
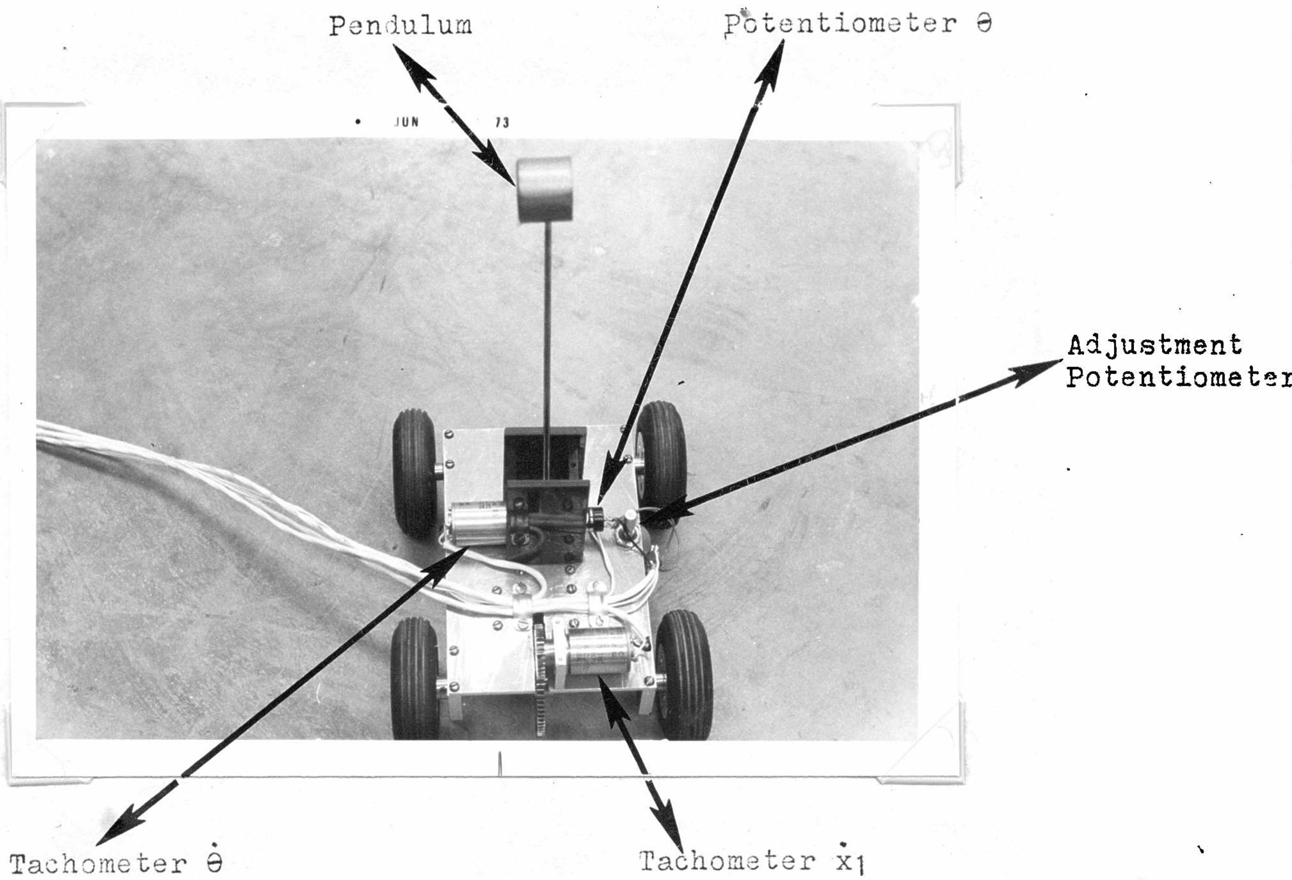


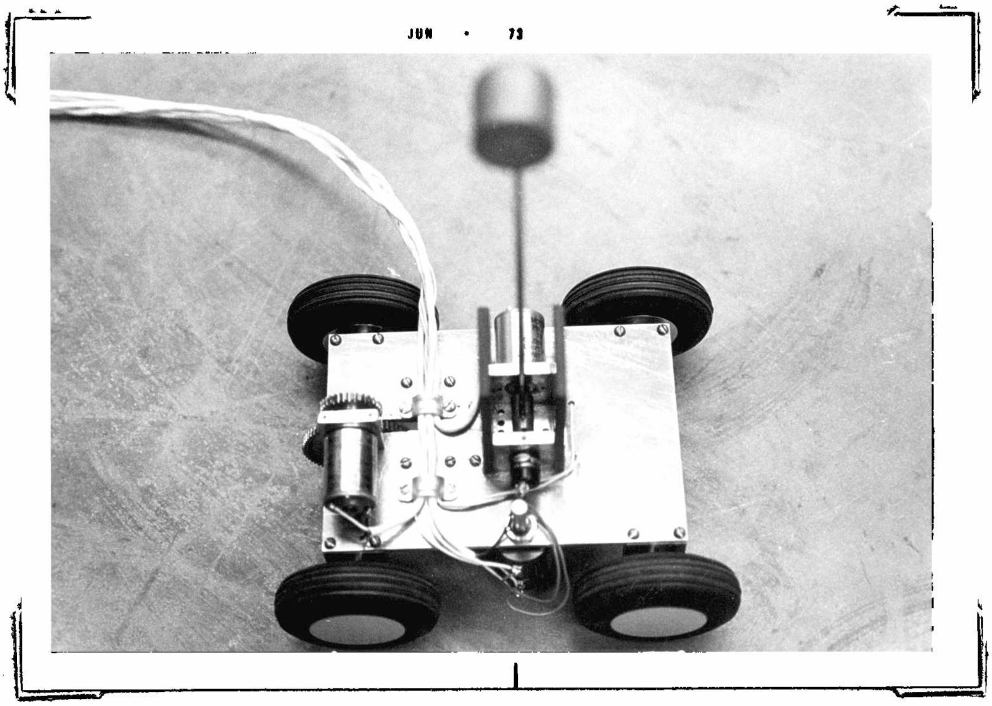
Figure 10. A Practical Model for the System



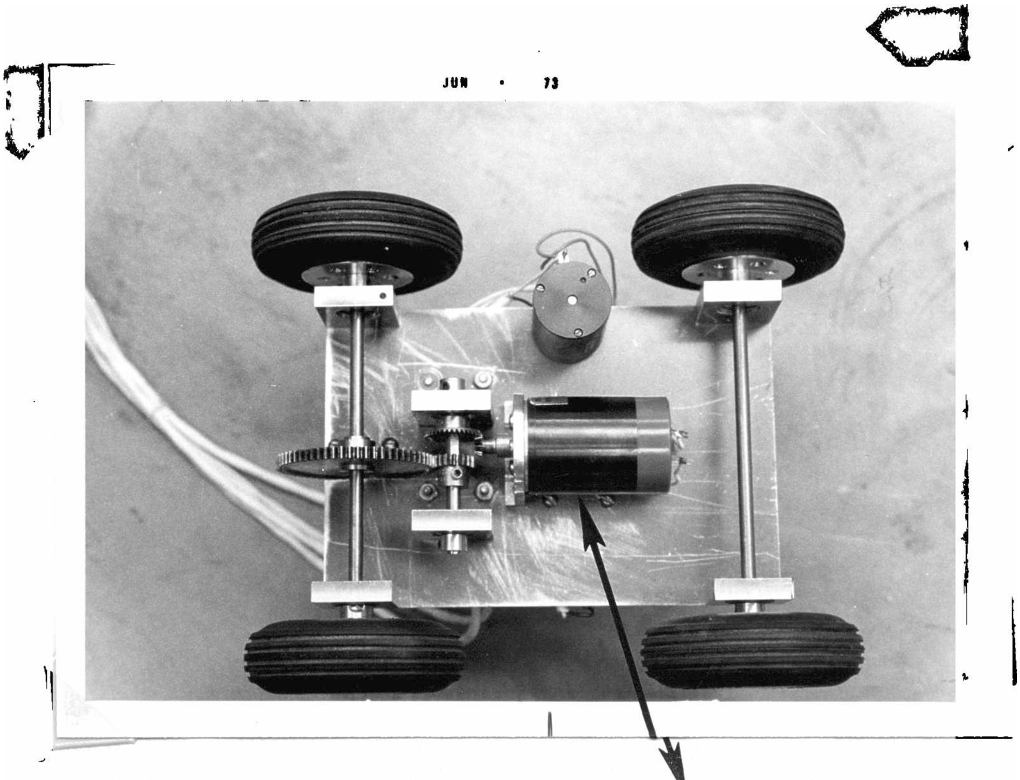
(11a)

Figure 11. Photographs of the Inverted Pendulum

a) Front view
b) Side view
c) Bottom view



(11b)



(11c)

diodes obtained from the voltage supplies of the high power amplifier, provides the supply voltages of the low power operational amplifiers. The angular position voltage adjustment potentiometer (mounted on the cart) is to provide zero voltage when the pendulum is in the vertical position. The relay at the output of the high power amplifier is for motor protection and limits the voltage across the motor. In addition, two power resistors in series with the power supplies limit the current into the circuit.

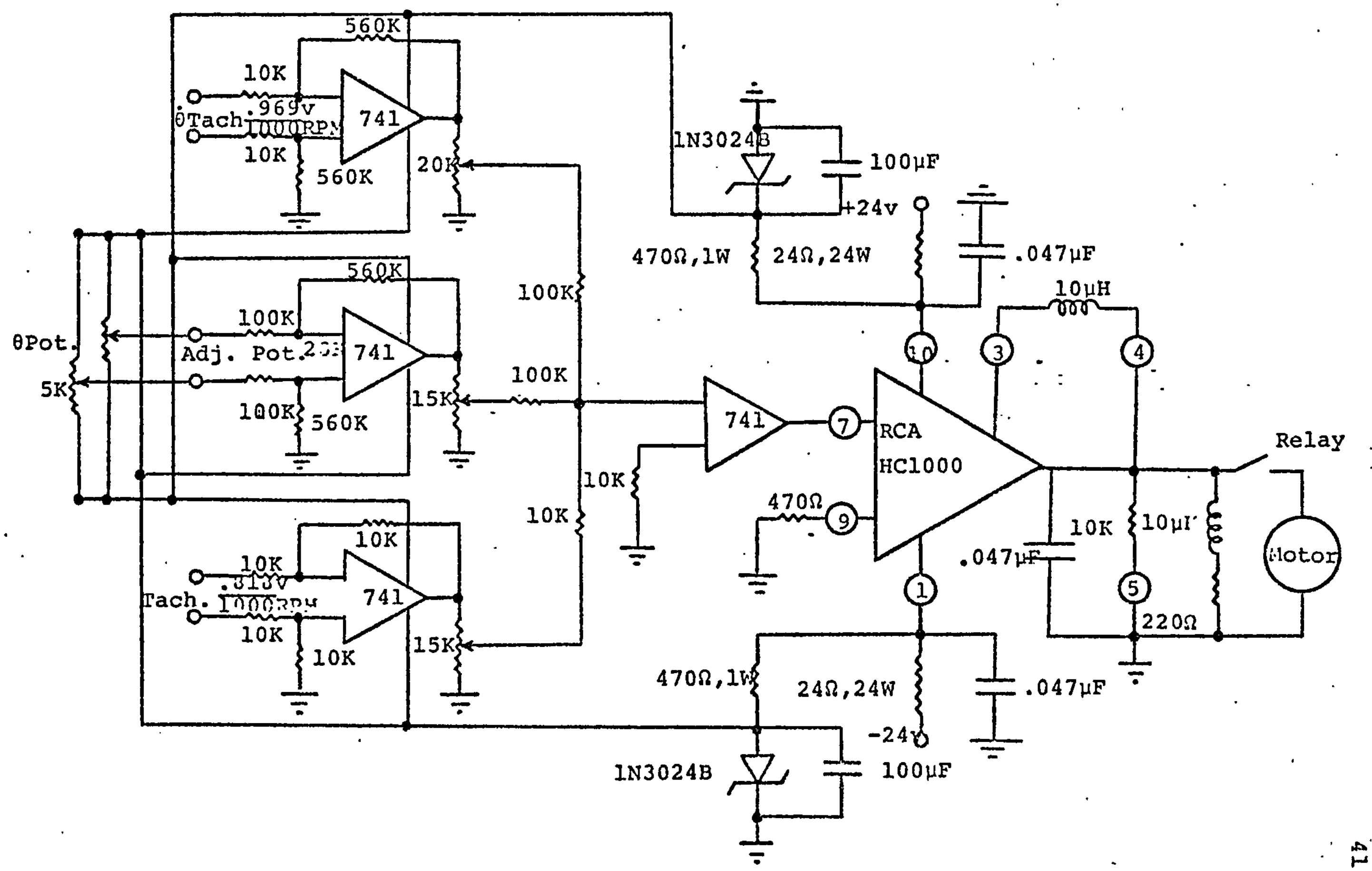


Figure 12. Electronic Circuit of the Inverted Pendulum System

CHAPTER VIII

EXPERIMENT AND RESULTS

The value of the theoretical feedback gains were obtained by adjusting the potentiometers at the output of the 741 operational amplifiers. With these gains, the system performed as predicted. No additional adjustments were necessary to maintain the pendulum in a vertical position. In general the system performance was excellent.

Figure 13 shows the angular position response of the pendulum for an initial angular offset of .2 radians. The measured settling time of the system compared very well with the predicted value. The difference was less than 15%.

System stability was maintained in the presence of external noise in the form of lateral forces applied to the pendulum mass as well as lateral forces applied to the cart. Some gear chatter was experienced, but this chatter produced insignificant effect on the system.

As expected, complete instability resulted when any one of the feedback variable was eliminated.

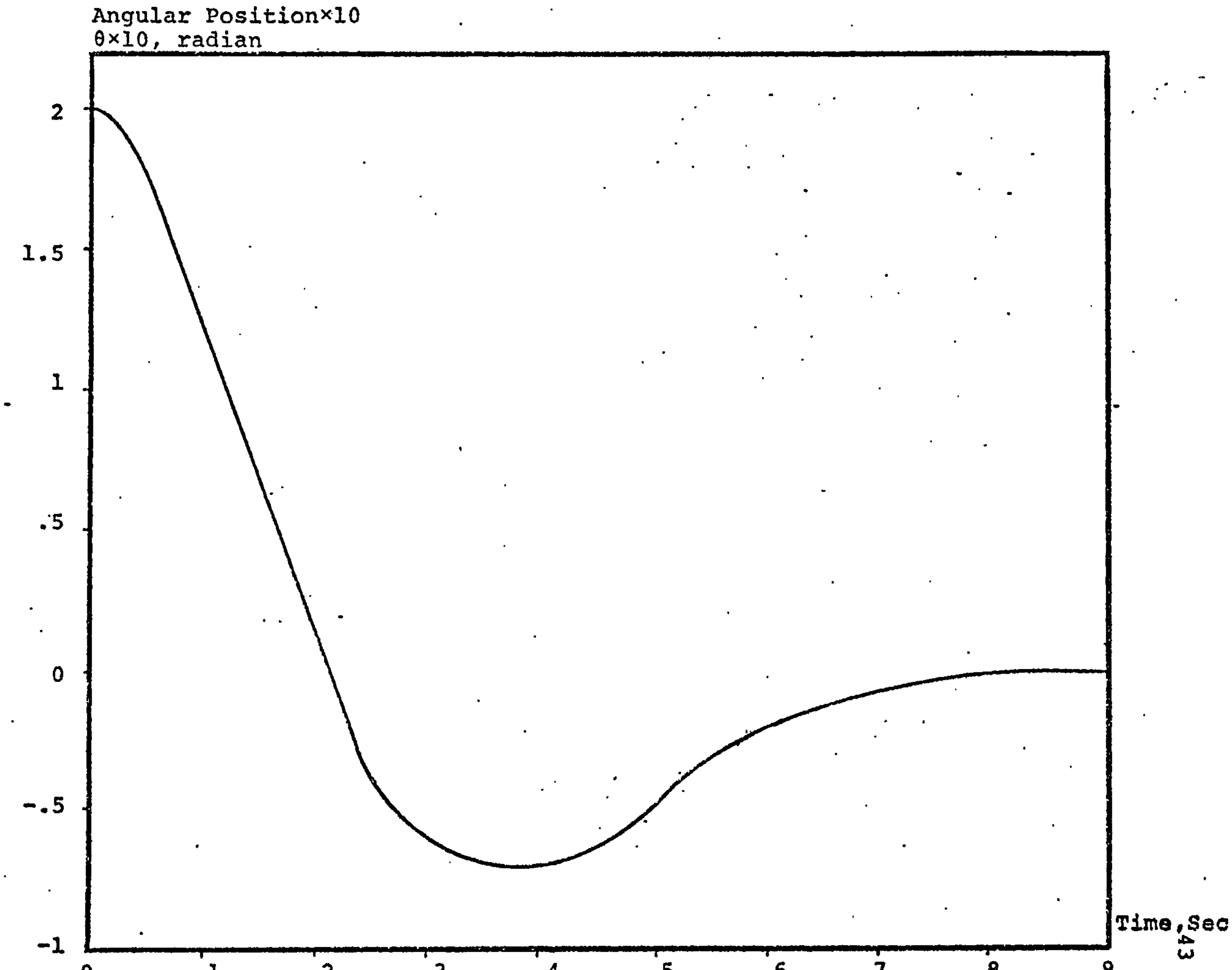


Figure 13. Zero Input Response of the Angular Position for an Initial Angular Position of .2 Radians

CHAPTER IX

CONCLUSIONS

In this study of inverted pendulum system several conclusions can be made.

By the use of both classical and state variable approach the best of both techniques can be used. For this case the state variable technique allowed a straight forward identification of the variables to be measured and the classical closed loop transfer function representation provided the stability and time response criterias.

In addition, the normalized third order system curves of Figure 6 provided the information for optimizing the system with respect to some measure of costs.

Also, the simple linearization of the system provided a sufficiently accurate description from which system design can be accomplished.

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