FIXED POINT THEOREMS FOR
SET VALUED FUNCTIONS

A Dissertation

Presented to

the Faculty of the Department of Mathematics

University of Houston

Houston, Texas

In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

by

Patrick Owen Wheatley

May, 1970

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ABSTRACT

This dissertation considers fixed point theorems for set valued functions, $F : X \to X$. The first part of this dissertation extends the class of those set valued functions which induce homomorphisms $h : H_* (X) \to H_* (X)$ that satisfy the Lefschetz Fixed Point Theorem. An example is given of a collection of set valued maps on a 2-cell which are fixed point free. Moreover, the class of spaces X for which F induces $h : H_* (X) \to H_* (X)$ is extended from polyhedra to ANR's and NR_δ's.

It is also shown that the inverse limit of spaces with the fixed point property for set valued maps has the fixed point property.

Finally, the notion of contractive set valued functions is introduced and investigated for fixed point theorems.

CONTENTS

CHAPTER 1

Introduction

For each $x \in X$, let $F(x)$ be a closed subset of Y. Then $F : X \to Y$ is called a set valued function on X to Y. For any $A \subset Y$, define $F^{-1}(A) = \{x \mid F(x) \cap A \neq \phi\}$. Then if for every closed subset K (open subset O) $F^{-1}(K)$ $(F^{-1}(O))$ is closed (open) in X, F is said to be upper semi-continuous, u.s.c. (lower semi-continuous, l.s.c.) respectively. A function $F : X \to Y$ that is both u.s.c. and l.s.c. is called a continuous set valued function, or simply a set valued map. Finally, let $F : X \to X$. Then x in X is a fixed point of F if $x \in F(x)$.

This dissertation investigates fixed point theorems for set valued functions, $F : X \to Y$. Making use of the Vietoris Theorem, Eilenberg and Montgomery[5] proved a fixed point theorem for u.s.c. $F : X \to X$ where for each $x \in X$, $F(x)$ is acyclic and X is a compact metric ANR. Later Barratt O'Neill[8] considered the case where $F(x)$ consists of one or n acyclic components and X a polyhedron. With the further restriction to set valued maps $F : X \to X$, O'Neill proved not only fixed point theorems, but also that F induces a nontrivial homomorphism, $h_* : H_*(X) \to H_*(X)$ such that h_* satisfies the Lefschetz Fixed Point Theorem.

The first part of this dissertation (Chapter 2) extends the class of set valued functions which have fixed points and induce homomorphisms on the homology groups. In particular, a set valued function based on recent papers of D. G. Bourgin[3a,3b] is shown to have many of the properties of a single valued map. An example is given which shows that in general, some restriction on the number of acyclic components of F(x) must be made; otherwise, F could be fixed point free. However, it is shown that with additional hypotheses on the map $F : X \to X$, F(x) can have 1, 2, ..., n acyclic components and still behave like the functions considered by O'Neill. In the final part of Chapter 2, the results are extended to compact ANR's and more generally to NR_δ's, spaces treated in an earlier paper by D. G. Bourgin[1].

In Chapter 3 it is shown that under certain conditions, the inverse limit of spaces having the fixed point property also has the fixed point property. This is not true in the case of single valued functions.

In the final chapter, the notion of contractive set valued functions is introduced. Here it is shown that contractive set valued functions indeed have a fixed point[4] although nothing can be said about the uniqueness of such points. Likewise the notion of a condensing function as introduced by I. N. Sadovskii[10] is extended to the set valued case to obtain a fixed point theorem.

Chapter 4 is concluded by two theorems comparing fixed point theorems for single valued maps versus set valued maps. In the first case, it is shown that there can be no contractive single valued map of a compact metric space onto itself. A simple example shows this to be otherwise for the set valued maps. In the second case, a theorem for unique fixed points is found to be true for both types of maps.

Unless stated otherwise, it is assumed throughout that:

(a) All spaces are compact metric.

(b) Only Cech homology over the field of rational numbers is used, i.e. $H_* (X) = H_* (X, Q)$.

(c) Invariably, u.s.c. and l.s.c. will be used for upper semi-continuous and lower semi-continuous respectively.

CHAPTER 2

Set Valued Functions that satisfy the Lefschetz Fixed Point Theorem

Definition 2.1. Given the set valued functions $\{F_i\}_{i=1}^{\infty}$, F on $X \to Y$, then $\{F_i\}_i \to F$ iff for every $\varepsilon > 0$, there is an $n(\varepsilon)$ such that for all $n \geqq n(\varepsilon)$, $F_n(x) \subset N_\varepsilon(F(x))$ for all x in X. The set $N_\varepsilon(F(x))$ is the open ε neighborhood about $F(x)$.

Proposition 2.2. A set valued function $F : X \to Y$ is u.s.c. iff for every open $N_\varepsilon(F(x))$, there is a $\delta > 0$ such that $F(N_\delta(x)) \subset N_\varepsilon(F(x))$.

Proposition 2.3. Let $F : X \to Y$ be a set valued function. Then F is u.s.c. iff $\Gamma(F)$ is closed in $X \times Y$ where $\Gamma(F)$ is the graph of F in $X \times Y$.

Definition 2.4a. [8]. Let A,B be chain groups with supports in X, Y respectively, and let $\varepsilon > 0$. A chain map $\phi: A \to B$ is accurate with respect to $F : X \to Y$ provided $|\phi(a)| \subset F(|a|)$ for each a in A.

Definition 2.4b. [8]. Let $\phi : A \to B$, $F : X \to Y$ as above. For each x, define $\varepsilon(x) = \{x' \mid d(x,x') \geqq \varepsilon\}$. Then $\phi : A \to B$

is <u>ε-accurate</u> with respect to F, provided ϕ is accurate with respect to $\varepsilon F \varepsilon : X \to Y$.

<u>Definition</u> 2.4c. [8]. A homomorphism $h: H_*(X) \to H_*(Y)$ is an <u>induced homomorphism</u> of a set valued function $F : X \to Y$, if for every $\varepsilon > 0$, there is a chain map $\phi : C_*(X) \to C_*(Y)$ such that ϕ is ε-accurate with respect to F and $\phi_* = h$. Moreover, $h_0 : H_0(X) \to H_0(Y)$ is a non-zero homomorphism.

<u>Theorem</u> 2.5. Let $F : X \to X$ be u.s.c. and let $\{F_i\}_{i=1}^{\infty} \to F$ where each F_i is u.s.c. and has a fixed point x_i in X. Then

(a) F has a fixed point.

Moreover, suppose each F_i induces a $h_i : H_*(X) \to H_*(X)$; then

(b) F induces an $h : H_*(X) \to H_*(X)$, provided $H_*(X)$ is finitely generated.

<u>Proof</u>

(a) The sequence $\{x_n\}_{n=1}^{\infty}$ contains a subsequence $\{x_{n_j}\}_{j=1}^{\infty}$ that converges to a point $\bar{x}$ in X.

Claim: The point $\bar{x} \in F(\bar{x})$. Suppose not. Then $d(\bar{x},F(\bar{x})) = \eta > 0$.

Let $\varepsilon = \eta/4$. Since F is u.s.c. there is a $\delta > 0$ such that $\delta < \varepsilon$ and $F(N_\delta(\bar{x})) \subset N_\varepsilon(F(\bar{x}))$. There is a j_0 such that for all $j \geq j_0$, $F_{n_j}(x) \subset N_\varepsilon(F(x))$ for all x and $x_{n_j} \in N_\delta(\bar{x})$.

Let $i_0 = n_{j_0}$. Then $x_{i_0} \in F_{i_0}(x_{i_0}) \subset N_\varepsilon(F(x_{i_0})) \subset N_{2\varepsilon}(F(\bar{x}))$.

Therefore $d(\bar{x}_{i_0}, F(\bar{x})) < 2\varepsilon < \eta/2$.

Since $d(\bar{x}, x_{i_0}) < \delta < \eta/4$ implies $d(\bar{x}, F(\bar{x})) < \eta$, there is a contradiction. Thus $\bar{x}$ is in $F(\bar{x})$.

(b) By hypothesis the set of homomorphisms of $H_*(X) \to H_*(X)$ is a finite dimensional vector space, L. Let $A(\varepsilon)$ be the collection of h in L that preserve the Kronecker index on $H_0(X) \to H_0(X)$ and are induced by an ε-accurate chain map $\phi: C(X) \to C(X)$.
Claim: The set $A(\varepsilon)$ is not empty. For let n_0 be such that for all $n \geq n_0$, $F_n(x) \subset N_{\varepsilon/4}(F(x))$ for all x. Now F_{n_0} induces a $h_{n_0} : H_*(X) \to H_*(X)$. Thus there is a $\phi: C_*(X) \to C_*(X)$ such that $|\phi(a)| \subset \frac{\varepsilon}{4} F_{n_0} \frac{\varepsilon}{4}(|a|) \subset \frac{\varepsilon}{2} F \frac{\varepsilon}{4} (|a|) \subset \varepsilon F \varepsilon (|a|)$

for all a in $C_*(X)$.

Now $\phi_* = h_{n_0} \Rightarrow h_{n_0} \varepsilon A(\varepsilon)$.

Claim: $\bigcap_{\varepsilon>0} A(\varepsilon) \neq \phi$, since $\eta < \varepsilon \Rightarrow A(\eta) \subset A(\varepsilon)$ and $A(\varepsilon)$ is a variety. (See Bourgin[2], p. 126) Let $h \in \bigcap_{\varepsilon>0} A(\varepsilon) \Rightarrow h : H(X) \to H(X)$ is an induced homomorphism of $F : X \to X$.

Lemma 2.6.[8]. Let X be a compact polyhedron, $F : X \to X$ an upper semi-continuous set valued function. If h is an induced homology homomorphism of F and the Lefschetz number $L(h) = \Sigma(-1)^q$ trace h_q is not zero, then F has a fixed point.

Theorem 2.7.[8]. Let F be a set valued self-map of a compact polyhedron X such that if $x \varepsilon X$, F(x) is homologically trivial or consists of n homologically trivial components. Then F has a nontrivial homomorphism h such that if $L(h) \neq 0$, F has a fixed point. If, further, X is homologically trivial, then F has a fixed point.

Corollary 2.8 Let X be a polyhedron: $\{F_i\} \to F$ on $X \to X$ where each $F_i : X \to X$ is a map having 1 or n_i acyclic components for each x, and F is u.s.c. Then F induces $h : H_*(X) \to H_*(X) \ni$ if $L(h) \neq 0$, then F has a fixed point.

Proof This follows from Theorem 2.7 and Theorem 2.5.

Theorem 2.9. [5]. Let X be a compact metric ANR. Suppose Suppose $F : X \to X$ is a u. s. c. set valued function such that for each x, $F(x)$ is acyclic. Then there is induced a nontrivial homomorphism h_* on $H_*(X) \to H_*(X)$ such that if $L(h_*) \neq 0$, then F has a fixed point.

As an application of what has been said, let X be a compact metric ANR and $F : X \to X$ a u.s.c. set valued function such that for every $\varepsilon > 0$, there is a neighborhood $N(x)$ about each x in X such that for all but a finite number of points in $N(x)$, $F(N(x))$ is contained in an open ball of radius ε. (This function is motivated by recent papers of D.G.Bourgin [3a,3b].) From Theorem 2.10 to the proof of Theorem 2.13 inclusive the spaces X,Y are assumed to be compact metric ANR's.

Theorem 2.10. Let $F : X \to X$ be defined as above. Then F induces a sequence of functions $\{F_n\}_{n=1}^{\infty} \to F$ on X to X and homomorphisms $\{h_n\}_{n=1}^{\infty} : H_*(X) \to H_*(X)$ such that $L(h_i) \neq 0$ implies F_i has a fixed point.

Proof a) It can be assumed that X is contained in some open subset U of a Hilbert cube where X is a retract of U;i.e., $r : U \to X$ is a retraction map. Moreover it can be assumed that there is an $\eta > 0$ such that X is covered by a family of open convex balls of radius η and contained in U. Let δ be the Lebesque number for this covering. Choose ε where $0 < \varepsilon < \frac{1}{2}\delta$. Then for each x, there is an $N(x)$ such that except for a finite number of points in $N(x)$, $F(N(x))$ is contained in an open convex ball of radius $< \varepsilon$. This is by the assumption of the theorem.

b) Let $\{N(x_i) \mid i = 1,\ldots, r\}$ be a finite cover of X and let $\{B_i \mid i = 1,\ldots, r\}$ be the corresponding set of open balls of radius $< \varepsilon$. Also let $Y = \{y_j \mid j = 1,\ldots,s\}$ denote the set of points $\ni F(y_j)$ cannot be contained in any of the B_i's. Suppose $y_1 \in N(x_1)$ say, then since F is u.s.c., $F(y_1) \cap \bar{B}_1 \neq \phi$.

c) Define $F_1 : X \to X$ by:

$\tilde{F}(x)$ = closed convex hull of $F(x)$ for all $x \in X - Y$.

$\tilde{F}(y)$ = closed convex hull of $F(y) \cap \bar{B}_j$ where y is in $N(x_j)$.

If y is in several such $N(x_j)$, choose the smallest such j. All this is possible since each $\bar{B}_j$ is contained in a convex open set of the cover of X.

Now let

$$F_1(x) = \begin{cases} r\,\tilde{F}(x) \\ \\ r\,\tilde{F}(y) & \text{respectively where} \end{cases}$$

$r : U \to X$ is the retraction map on U.

d) Claim: $F_1 : X \to X$ is u.s.c. and $F_1(x)$ is acyclic for all $x \in X$. That $F_1(x)$ is acyclic follows from the fact that $\tilde{F}(x)$ is. Now let $F_1(x) \subset V$ open. For $x \in X - Y$, $F(x) \subset F_1(x) \subset V$. Thus $\tilde{F}(x) \subset r^{-1}(V)$ and $\tilde{F}(x)$ convex implies $\exists$ an open convex set W about $\tilde{F}(x) \ni \tilde{F}(x) \subset W \subset r^{-1}(V)$. Since F is u.s.c. $\exists$ open $O(x)$ such that $F(O(x)) \subset W$ implies $\tilde{F}(O(x)) \subset W \subset r^{-1}(V)$ implies $F_1(O(x)) \subset V$.

For $x \in Y$, assume $x = y_1$ and $\tilde{F}(y_1)$ = closed convex hull of $F(y_1) \cap \bar{B}_1$. As above, there is $W \ni \tilde{F}(y_1) \subset W \subset r^{-1}(V)$. Now $\exists$ a neighborhood $N(F(y_1)) \ni N(F(y_1)) \cap \bar{B}_1 \subset W$. Take $O(y_1) \subset N(x_1) \ni F(O(y_1)) \subset N(F(y_1))$. Consider $O_1(y_1) = O(y_1) - \{y_1,\dots,y_s\}$. Then

$$F(O_1(y_1)) \subset W \Rightarrow \tilde{F}(O_1(y_1)) \subset W \subset r^{-1}(V)$$

Thus $F_1(O_1(y_1))$ is contained in V.

e) By Theorem 2.9 F_1 induces a homomorphism h_1 on $H_*(X)$ so that if $L(h_1) \neq 0$, F_1 has a fixed point.

f) As in the previous, let $\varepsilon_n = 1/2\ \varepsilon_{n-1}$ where $\varepsilon_1 = \varepsilon$.

This gives the sequence $\{F_n\}_{n=1}^{\infty} : X \to X$ and homomorphisms $\{h_n\}_{n=1}^{\infty} : H_*(X) \to H_*(X)$ such that $L(h_i) \neq 0$ implies F_i has a fixed point.

By definition $\{F_n\} \to F$.

Corollary 2.11. Let $F : X \to X$ be a u.s.c. set valued function such that for every $\varepsilon > 0$, there is a neighborhood N(x) about each $x \in X$ such that for all but a finite number of points in N(x), F(N(x)) is contained in a open ball of radius ε. Then F induces $h : H_*(X) \to H_*(X)$ that satisfies the Lefschetz Fixed Point Theorem.

Proof This follows from the preceeding theorem and Theorem 2.5.

In the following, it will be shown that the class of u.s.c. functions $F : X \to Y$ described above are homotopic to a single valued map in a certain sense.

Definition 2.12. Let $F : X \to Y$ be as above.
Then $H : X \times I \to Y$ is a homotopy if $H(x,0) = F(x)$ for all x, $H(x,1) = f(x)$, a single valued function on $X \to Y$, and for fixed t $H(x,t)$ is u.s.c. and satisfies the same conditions as F.

Theorem 2.13. Let $F : X \to Y$. Then $\exists\, F' : X \to Y$ such that $F'(x) = F(x)$ for all but a finite number of x and $F'(x) \subset F(x)$ otherwise. Then there is a homotopy $H : X \times I \to Y$ such that $H(x,0) = F'(x)$ and $H(x,1) = f(x)$, a single valued map on $X \to Y$.

Lemma 2.13a. $Y \subset U$ open is contained in a Banach space.

Proof Since $Y \subset U$ an open subset of a Hilbert cube, Q, and since Q is a metrisable space, it can be embedded in a Banach Space, B.

Lemma 2.13b. Let $Y \subset U \subset B$ and let α be a finite convex cover of $Y \subset U$. Then there is $F' : X \to Y$ such that $F'(x) = F(x)$ for all but a finite number of x and $F'(x) \subset F(x)$ otherwise.

Proof Since $Y \subset U$, there is a finite cover α of $F(X) \subset U$. Let δ be the Lebesgue number for this cover and $0 < \varepsilon < \frac{1}{2}\delta$. Then as in Theorem 2.10, one has $\{N(x_i) \mid i = 1,\ldots,r\}$ and $\{B_i \mid i = 1,\ldots,r\}$ the corresponding set of open balls of radius less than ε. Let $\alpha = \{B_i\}$. Let $X_0 = \{y_j \mid j = 1,\ldots,s\}$ denote the set of points such that $F(y_j)$ cannot be contained in any of the B_i's. Suppose $y_1 \in N(x_1)$ say, then since F is u.s.c., $F(y_1) \cap \bar{B}_1 \neq \phi$. If y_1 is in several B_j's, choose the smallest such j.

Define $F' : X \to Y$ by:

$$F'(x) = F(x) \text{ for } x \in X - X_0 ;$$

$$F'(x) = F(x) \cap \bar{B}_j \text{ for } x \in X_0 .$$

That F' is u.s.c. is proved in the same way as in Theorem 2.10.

Lemma 2.13c. F, F', $Y \subset U$, α as above. Then there is a single valued function $\phi : X \to U$ such that for each x, $\phi(x)$, $F'(x)$ are in the same $\bar{B}_i \in \alpha$.

Proof Let $P = \{p\}$ be a partition of the identity based on $\{N(x_i)\}$ such that $\sum_{P} p(x) = 1$ and each $p = 0$ outside some $N(x_i)$. For each p, suppose $p = 0$ on $X - N(x_i)$. Then let $y(p) \in B_i$ where $F'(N(x_i)) \subset \bar{B}_i$.

Define $\phi : X \to U$ by

$\phi(x) = \sum_P p(x)\, y(p)$. Since the $\bar{B}_i$'s are convex, the function ϕ satisfies the lemma.

<u>Lemma</u> 2.13d. Let F, F', X, U, α, ϕ be as above. Then there is a homotopy $H' : X \times I \to U \ni H'(x,0) = F'(x)$ and $H'(x,1) = \phi(x)$.

<u>Proof</u> For each x, $\phi(x)$ and $F'(x)$ are in the same convex ball, B_1 say. For each $y \in F'(x)$, the straight line segment $\{t\ \phi(x) + (1 - t)\, y \mid t \in [0,1]\} \subset \bar{B}_1$.

Define $H' : X \times I \to U$ by

$$H'(x,t) = \bigcup_{y \in F'(x)} \{t\ \phi(x) + (1 - t)\, y \mid t \in [0,1]\} .$$

<u>Proof of Theorem</u> 2.13. Let $r : U \to Y$ be the retraction map onto Y. Then define $H = rH'$, $f = r\,\phi$.

<u>Remark</u> The class of u.s.c. set valued functions considered from 2.10 to 2.13 inclusive, satisfy the Lefschetz Fixed Point Theorem and in the sense above, are homotopic to a single valued map. Later it will be shown that if F induces $h_* : H_*(X) \to H_*(Y)$

and if $H : X \times I \to Y$ as in 2.12, then $h_* = f_*$ where $f : X \to Y$ is a single valued map homotopic to F.

In the next part of this chapter a set valued map $F : X \to X$ is presented so that F(x) consists of 1, 2, or 3 points and F is fixed point free. Since X is a two cell, F cannot induce an $h_* : H_* (X) \to H_* (Y)$ although F can be shown to be homotopic to the identity.

EXAMPLE 2.14. Let X be the 2-cell, $\{(r,\theta) \mid 0 \leqq r \leqq 4,$ $0 \leqq \theta \leqq 2\pi$.

It is possible to define a continuous set valued map, $F : X \to X$ such that:

a) card $F(x) \in \{n, n + 1, n + 2\}$ for any fixed n and all $x \in X$;

b) F has no fixed point.

Here the case for n = 1 is treated in detail and the direction for treating general n will be indicated.

Definition of $F : X \to X$

1. On the lines $\{(r,\theta) \mid 0 \leqq r \leqq 4, \theta = \frac{\pi}{6}, \frac{5}{6}\pi, \frac{3}{2}\pi\}$,

$F(x) = \left\{(4, \frac{\pi}{2}), (4, \frac{7}{6}\pi), (4, -\frac{\pi}{6})\right\}$.

See Figure 1.

2. F will be defined on the top section (I) of X; i.e., on $\{(r,\theta) \mid 0 \leqq r \leqq 4, \frac{\pi}{6} \leqq \theta = \frac{5}{6}\pi\}$. It can be similarly defined on the other two sections of X.

See Figure 2.

a) $\{(r,\theta) \mid 0 \leqq r \leqq 1, \frac{\pi}{6} \leqq \theta \leqq \frac{5}{6}\pi\}$.

i) $F(r, \frac{3}{4}\pi) = \{(4, \frac{7}{6}\pi - \frac{r}{3}\pi), (4, \frac{\pi}{2} - \frac{\pi}{3}r), (4, -\frac{\pi}{6} + \frac{\pi}{3}r)\}$ for $0 \leqq r \leqq 1$.

See Figure 3.

ii) For $\frac{\pi}{4} \leqq \theta \leqq \frac{3}{4}\pi$, $F(r,\theta) = F(r, \frac{3}{4}\pi)$.

iii) For $\frac{3}{4}\pi \leqq \theta \leqq \frac{5}{6}\pi$,

$$F(r, \theta) = \begin{cases} (4, \frac{7}{6}\pi - 4r(\frac{5}{6}\pi - \theta)) \\ (4, \frac{\pi}{2} - 4r(\frac{5}{6}\pi - \theta)) \\ (4, -\frac{\pi}{6} + 4r(\frac{5}{6}\pi - \theta)). \end{cases}$$

iv) For $\frac{\pi}{6} \leqq \theta \leqq \frac{\pi}{4}$

$$F(r, \theta) = \begin{cases} (4, \frac{7}{6}\pi - 4r(\theta - \frac{\pi}{6})) \\ (4, \frac{\pi}{2} - 4r(\theta - \frac{\pi}{6})) \\ (4, -\frac{\pi}{6} + 4r(\theta - \frac{\pi}{6})). \end{cases}$$

b) $\{(r, \theta) \mid 1 \leqq r \leqq 4, \frac{3}{4}\pi \leqq \theta \leqq \frac{5}{6}\pi\}$

$F(r, \theta) = F(1, \theta)$.

c) $\{(r, \theta) \mid 1 \leqq r \leqq 2 \text{ and } \frac{\pi}{6} \leqq \theta \leqq \frac{3}{4}\pi\}$.

i) For $\frac{\pi}{4} \leqq \theta \leqq \frac{3}{4}\pi$

$$F(r, \theta) = \begin{cases} (4, \frac{5}{6}\pi - \frac{2}{3}(r - 1)(\frac{3}{4}\pi - \theta)) \\ \\ (4, \frac{\pi}{6} + \frac{2}{3}(r - 1)(\frac{3}{4}\pi - \theta)) \end{cases}$$

See Figure 4.

ii) For $\frac{\pi}{6} \leqq \theta \leqq \frac{\pi}{4}$

$$F(r, \theta) = \begin{cases} (4, \frac{7}{6}\pi - 4r(\theta - \frac{\pi}{6})) \\ (4, -\frac{\pi}{6} + 4r(\theta - \frac{\pi}{6})) \\ (4, \frac{\pi}{2} - 4(2 - r)(\theta - \frac{\pi}{6})). \end{cases}$$

See Figure 5.

d) $\{(r, \theta) \mid 2 \leqq r \leqq 3, \frac{\pi}{6} \leqq \theta \leqq \frac{3}{4}\pi\}$

i) For $\frac{\pi}{4} \leqq \theta \leqq \frac{3}{4}\pi$,

$$F(r, \theta) = \begin{cases} (4, \frac{5}{6}\pi - \frac{2}{3}(\frac{3}{4}\pi - \theta)) \\ \\ (4, \frac{\pi}{6} + \frac{2}{3}(3 - r)(\frac{3}{4}\pi - \theta) \end{cases}$$

ii) For $\frac{\pi}{6} \leqq \theta \leqq \frac{\pi}{4}$

$$F(r, \theta) = \begin{cases} (4, \frac{\pi}{2}) \\ (4, \frac{7}{6}\pi - 8(\theta - \frac{\pi}{6}) \\ (4, -\frac{\pi}{6} + 4(4 - r)(\theta - \frac{\pi}{6})) \end{cases}$$

See Figure 6.

e) $\{(r, \theta) \mid 3 \leqq r \leqq 4, \frac{\pi}{6} \leqq \theta \leqq \frac{3}{4}\pi\}$

i) For $\frac{\pi}{4} \leqq \theta \leqq \frac{3}{4}\pi$

$$F(r, \theta) = \begin{cases} (4, \frac{5}{6}\pi - \frac{2}{3}(\frac{3}{4}\pi - \theta)) \\ \\ (4, \frac{\pi}{6}) \end{cases}$$

ii) For $\frac{\pi}{6} \leqq \theta \leqq \frac{\pi}{4}$,

$$F(r, \theta) = \begin{cases} (4, \frac{\pi}{2}) \\ (4, \frac{7}{6}\pi - 8(\theta - \frac{\pi}{6})) \\ (4, -\frac{\pi}{6} + (4 - r)(\theta - \frac{\pi}{6})). \end{cases}$$

Continuity of $F : X \to X$

We make use of an equivalent definition of continuity (for compact, Hausdorff spaces) given by W. Strothers.[9]

Definition 2.15. [9] A point $\bar{y} \in Y$ is said to be in the cofinal limit (residual limit) of a sequence of sets $\{Y_d\}$, indexed by a directed set D, if whenever V is an open set containing $\bar{y}$ there is a cofinal subset (residual subset)

$A \subset C$ such that $V \cap Y_a \neq \phi$ for all $a \varepsilon A$.

Proposition 2.16. A set valued function $G : Y \to Y$ where Y is a compact Hausdorff space is continuous iff for every $\{y_n\} \to y \varepsilon Y$, $F(y) =$ cofinal limit $\{F(y_n)\} =$ residual limit $\{F(y_n)\}$.

Theorem 2.17. $F : X \to X$ is a continuous set valued map.

Proof As it is constructed, F satisfies the statement of Proposition 2.16.

Theorem 2.18. $F : X \to X$ has no fixed point.

Remark To define an $F : X \to X$ for general n, one need only divide X into $n + 2$ sections. Then F is defined on Section 1, making use of the two adjacent sections. Then one proceeds to Section 2, ... See Figure 7 where A, B replace $(4, \frac{7}{6}\pi)$, $(4, -\frac{\pi}{6})$ respectively.

Remark In [8] O'Neill mentions that there are a series of unpublished examples of self-maps F of a 2-cell that are without fixed points and such that the number of points in $F(x)$ occurs in a particular finite set of integers.

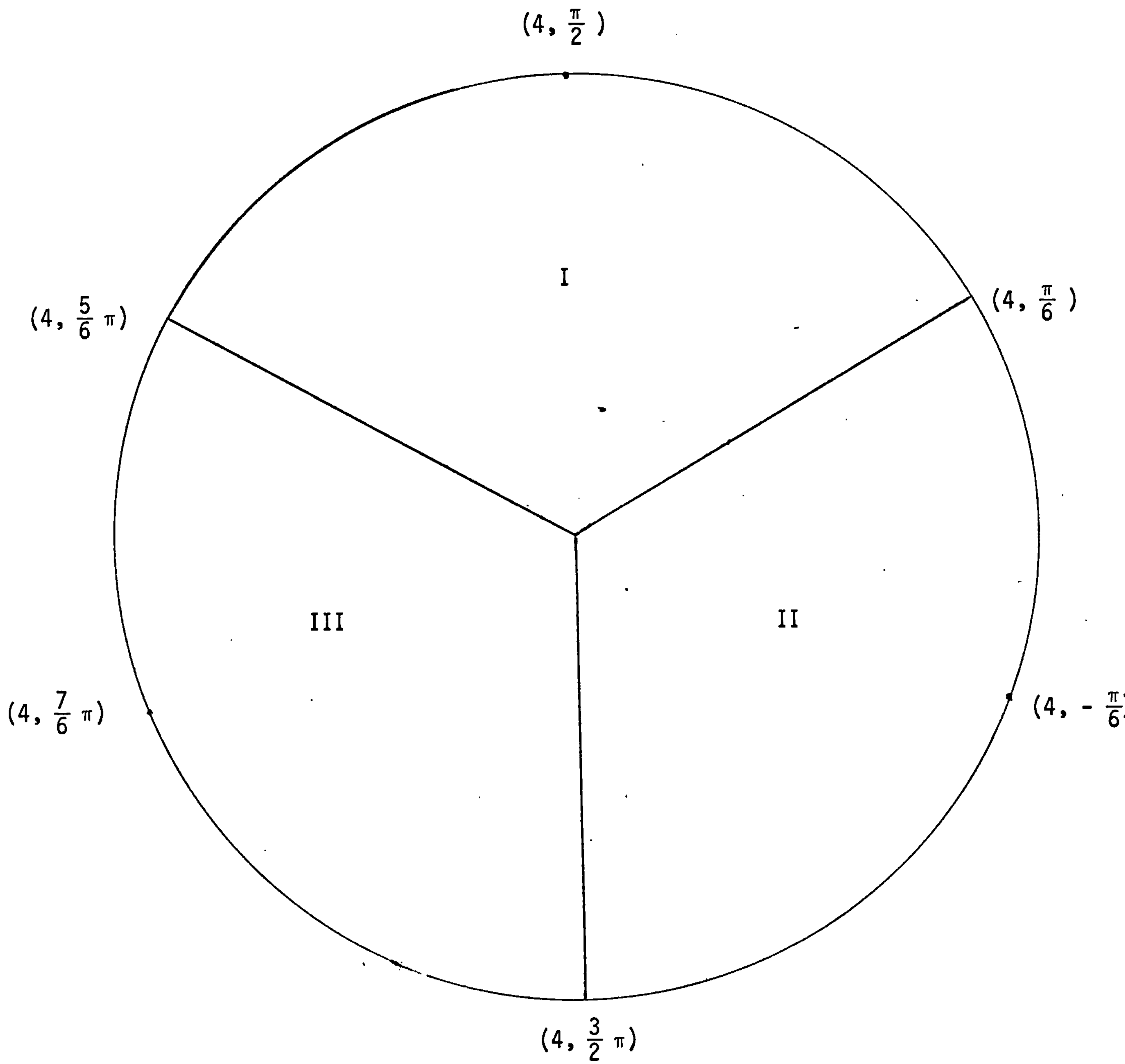

(4, $\frac{\pi}{2}$)
I
(4, $\frac{5}{6}\pi$)
(4, $\frac{\pi}{6}$)
III
II
(4, $\frac{7}{6}\pi$)
(4, - $\frac{\pi}{6}$)
(4, $\frac{3}{2}\pi$)

Figure 1

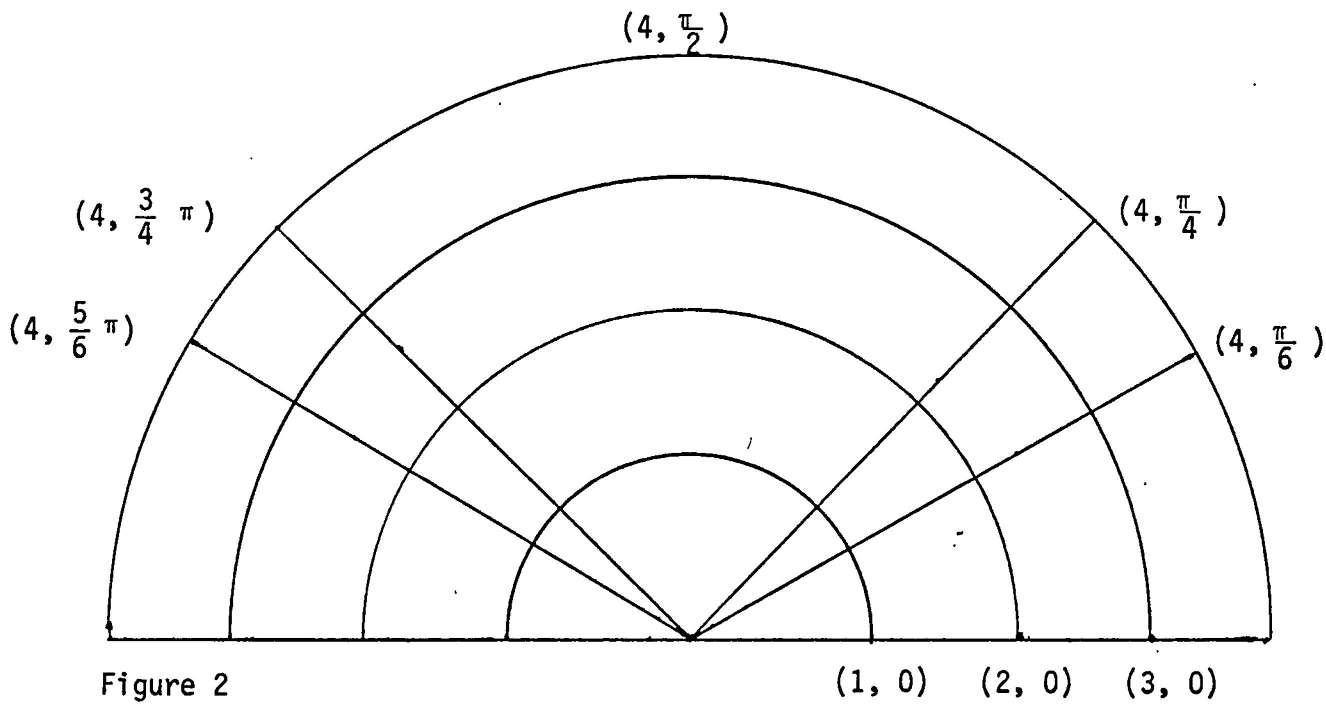


Figure 2

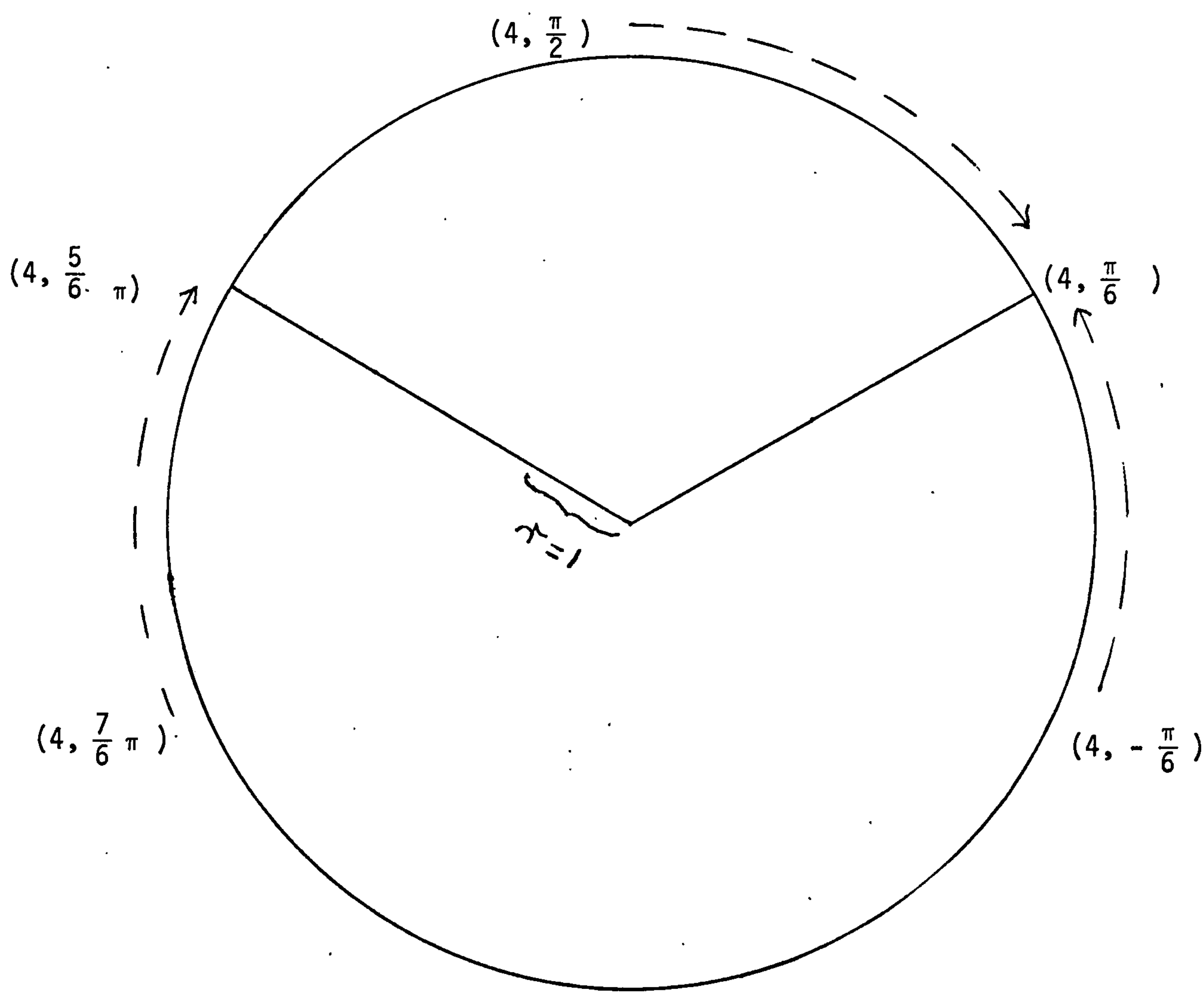


Figure 3

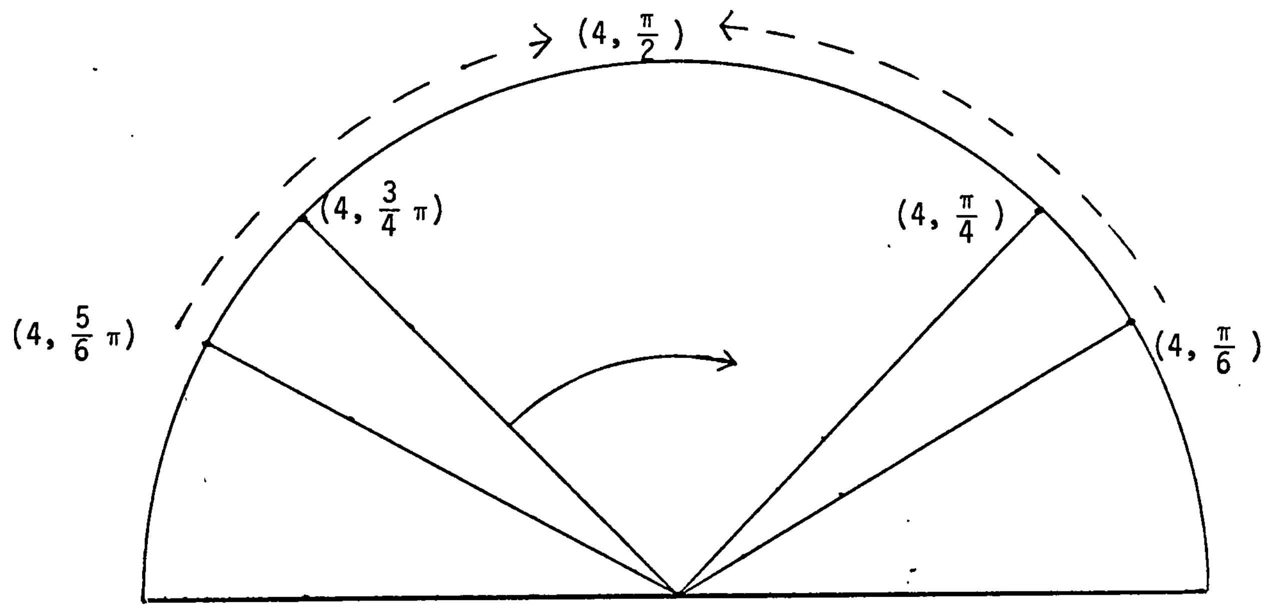


Figure 4

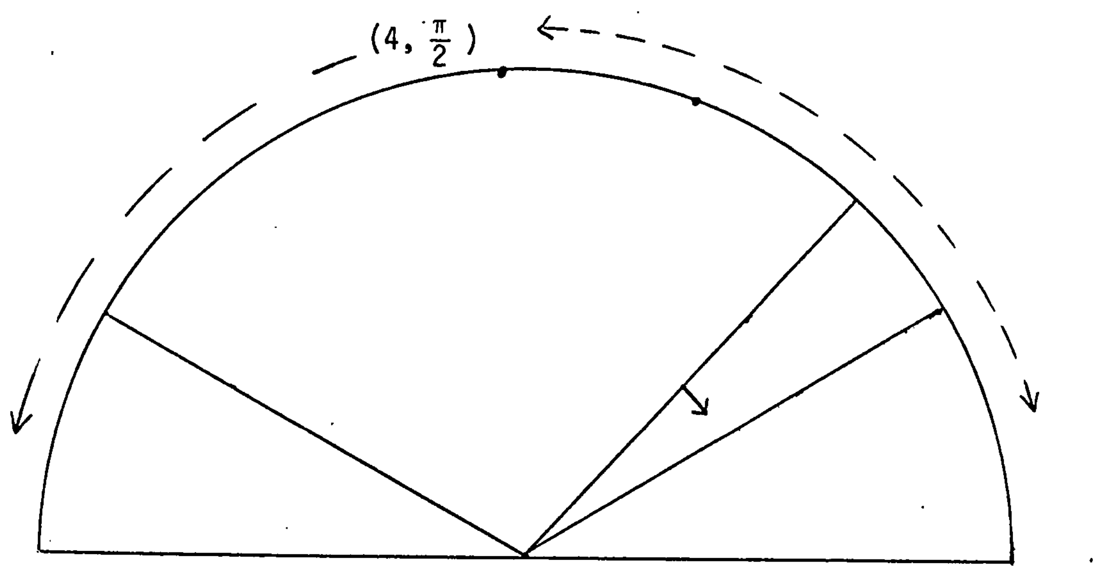


Figure 5

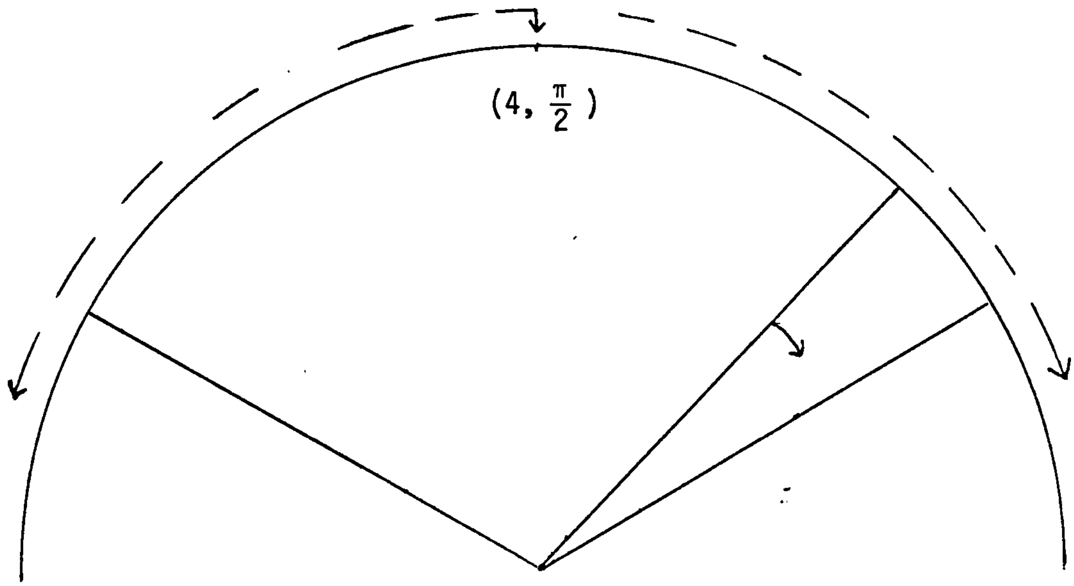


Figure 6

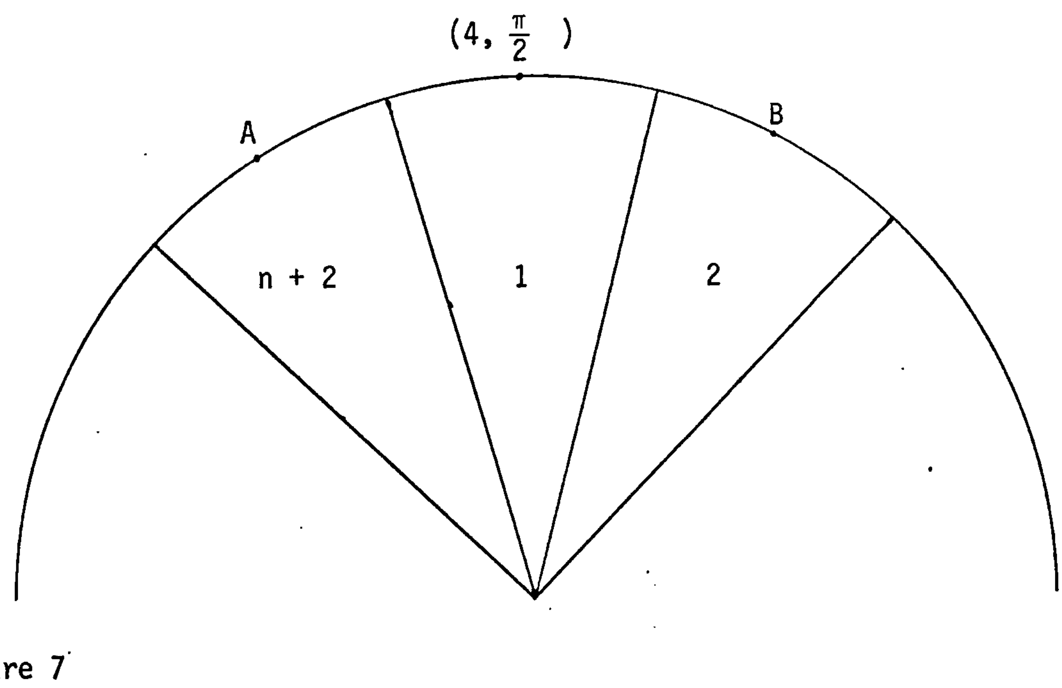


Figure 7

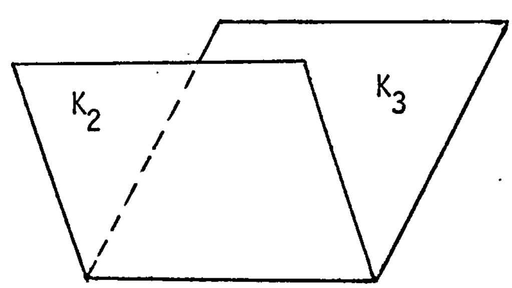
In the previous example it was shown that if a set valued map $F : X \to X$ is such that $F(x)$ has more than 1 or n acyclic components, then F need not have a fixed point. Moreover, this function cannot induce a nontrivial homomorphism on the homology groups.

However, it is possible that under certain conditions $F(x)$, for each x, may have 1, 2, ..., n acyclic components and still induce a nontrivial homomorphism on the homology groups.

Definition 2.19. Let $F : X \to X$ be a set valued map. Let $K_j = \{x : x \in X \text{ and } F(x) \text{ has 1 or } j \text{ acyclic components}\}$.

Theorem 2.20. Suppose K_1, K_m, K_n are nonempty polyhedra in X and $X = K_m \cup K_n$. Then $F : X \to X$ induces a nontrivial homomorphism $h : H_*(X) \to H_*(X)$ such that if $L(h) \neq 0$, then F has a fixed point.

EXAMPLE. Consider the case $X = K_2 \cup K_3$.



$K_1 = K_2 \cap K_3$

Lemma 2.20a. Consider the diagram

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[d, "\alpha"'] & V \arrow[r, "\psi"] \arrow[d, dashed, "\overline{\Psi}"] & B \arrow[d, "\beta"] \\ C \arrow[r, "\phi_1"] & W \arrow[r, "\psi_1"] & D \end{tikzcd}$$

(where $\operatorname{Im}\phi \subset \ker\psi$ and $\operatorname{Im}\phi_1 \subset \ker\psi_1$).

of vector spaces over the rationals and linear transformations.

Then $\exists\ \overline{\Psi} : V \to W$ such that both squares commute iff $\alpha : \ker\phi \to \ker\phi_1$ and $\beta : \operatorname{Im}\psi \to \operatorname{Im}\psi_1$

Proof of lemma. Straightforward.

Proof of the theorem

Let $h_*(m) : H_*(K_m) \to H_*(X)$ and $h_*(n) : H_*(K_n) \to H_*(X)$ be the nontrivial homomorphisms induced by F on K_m, K_n respectively. Thus for every $\varepsilon > 0$, $\exists\ \phi_\#(j) : C_*(K_j) \to C_*(X)$ ε-accurate with respect to F, and $\phi_*(j) = h_*(j)$ for $j = m,n$. Consider

$$\begin{tikzcd} 0 \arrow[r] & C_*(K_1) \arrow[r, "i_\#"] & C_*(K_m) \oplus C_*(K_n) \arrow[r, "\psi_\#"] \arrow[d, dashed, "\alpha"] & C_*(X) \arrow[r] & 0 \arrow[d] \\ & 0 \arrow[r] & C_*(X) \arrow[r, "1"] & C_*(X) \arrow[r] & 0 \end{tikzcd}$$

where $i_\#(m) : C_*(K_1) \to C_*(K_m)$ is the inclusion homomorphism

Now let $i_{\#} : C_*(K_1) \to C_*(K_m) \oplus C_*(K_n)$ be defined by

$i_{\#}(c_*(K_1)) = (i_{\#}(m)\ c_*(K_1), -\ i_{\#}(n)\ c_*(K_1))$.

Also $\psi_{\#}(c_*(K_m), c_*(K_n)) = c_*(K_m) + c_*(K_n)$.

Thus $\ker \psi_{\#} = \operatorname{Im} i_{\#}$.

Define $\alpha : C_*(K_m) \oplus C_*(K_n) \to C_*(X)$ by

$\alpha(c_*(K_m), c_*(K_n)) = n\phi_{\#}(m)\ c_*(K_m) + m\phi_{\#}(n)\ c_*(K_n)$.

Then $\alpha : \ker\ \psi_{\#} \to 0$. For instance,

Let $v_0 \in C_0(K_1)$. Then since $\phi_{\#}(m)\ i_{\#}(m)\ v_0 = mv_0$, by combining terms it follows that

$$\begin{aligned}
&\alpha(i_{\#}(m)v_0, -\ i_{\#}(n)v_0) \\
&\quad = n\phi_{\#}(m)\ i_{\#}(m)v_0 - m\phi_{\#}(n)\ i_{\#}(n)v_0 \\
&\quad = nm\bar{v}_0 - mn\bar{v}_0 = 0.
\end{aligned}$$

By the lemma $\exists$

$\gamma : C_*(X) \to C_*(X)$, which is ε-accurate with respect to F since $\phi_{\#}(n)$, $\phi_{\#}(m)$ are, and $\gamma_0 = nh_0(m) + mh_0(n)$ implies γ_0 is nontrivial.

<u>Theorem</u> 2.21. Let $F : X \to X$ and let $K_1, K_2, \ldots, K_n$ be polyhedra where $X = K_2 \cup K_3 \cup \ldots \cup K_n$ and in particular $K_1 \neq \phi$. Then F induces a nontrivial homomorphism.

Lemma 2.21a. Suppose $X = K_p \cup K_q \cup K_n$ where $1 < p < q < n$ and nonempty. Then $\exists\ \alpha : C_*(K_p \cup K_q) \oplus C_*(K_n) \to C_*(X)$ in the diagram that maps : $\ker \psi_\# \to 0 \in C_*(X)$.

$$\begin{tikzcd}
0 \arrow[r] & C_*(K_1) \arrow[r, "i'_\#"] & C_*(K_p \cup K_q) \oplus C_*(K_n) \arrow[r, "\psi'_\#"] \arrow[d, dashed, "\alpha'"] & C_*(X) \arrow[r] \arrow[d] & 0 \\
& 0 \arrow[r] & C_*(X) \arrow[r, equal] & C_*(X) \arrow[r] & 0
\end{tikzcd}$$

Proof of lemma From the previous theorem there is the diagram:

$$\begin{tikzcd}
0 \arrow[r] & C_*(K_1) \arrow[r, "i_\#"] & C_*(K_p) \oplus C_*(K_q) \arrow[r, "\psi_\#"] \arrow[d, "\alpha"] & C_*(K_p \cup K_q) \arrow[r] \arrow[d, "\beta"] & 0 \\
& 0 \arrow[r] & C_*(X) \arrow[r, equal] & C_*(X) \arrow[r] & 0
\end{tikzcd}$$

where $\beta : C_*(K_p \cup K_q) \to C_*(X)$ exists from the first lemma.

Let $j : C(K_1) \to C(K_p \cup K_q)$ be the inclusion homomorphism and consider the commutative diagrams:

$$\begin{tikzcd}
& C_*(K_1) \arrow[dl, "\bar{s}"'] \arrow[d, "j"] & \\
C_*(K_p) \oplus C_*(K_q) \arrow[r, "\psi"] \arrow[d, "\alpha"] & C_*(K_p \cup K_q) \arrow[r] \arrow[d, "\beta"] & 0 \\
C_*(X) \arrow[r, equal] & C_*(X) &
\end{tikzcd}$$

where $\bar{s}(C(K_1)) = (0, i_2(C(K_1))$ and $j = \psi\,\bar{s}$.

Let v_0 be a 0-chain in $C_*(K_1)$. Then $\beta\ \psi\ \bar{s}(v_0) = \alpha\ \ \bar{s}(v_0) =$

$\alpha(0, i_2(v_0)) = p\phi(q)\ i_2(v_0) = pq(\bar{v}_0)$.

Since K_1 is a closed polyhedron in X, and since each $\phi(j)$ is allowable, $\partial\phi(q) = \phi(q)\partial$. Thus if $e_1 = (v^0 v^1)$ is an elementary 1-chain in $C_*(K_1)$, $\phi(q)\ e_1 = \phi(q)(v^1 - v^0) = q(\bar{v}^1 - \bar{v}^0)$.

Thus $\phi(q)\ e\ \ = q\ \bar{e}_1$ in $C_*(X)$. Similarly for each elementary n-chain.

In like manner, $\beta j(e_0) = pq(\bar{e}_0)$ in $C_0(X)$, $\beta j(e_n) = pq\ \bar{e}_n \in C_n(X)$.

Now define: $\alpha' : C_*(K_p \cup K_q) \oplus C_*(K_n) \to C_*(X)$ by

$$\alpha' : (c_*(K_p \cup K_q), c_*(K_n)) \to n\beta(c_*(K_p \cup K_q)) + pq\phi(n)[c_*(K_n)]$$

Let $e_0 \in C_0(K_1)$. Then

$\alpha'(j(e_0), - i_2(e_0)) = npq\bar{e}_0 - pqn\bar{e}_0 = 0$

Thus $\alpha : \ker \psi_\# \to 0$ by previous discussion.

<u>Proof</u> of Theorem 2.21. This follows by induction on Lemma 2.21a.

<u>Theorem</u> 2.22. Let $X = K_1 \cup K_2 \cup \ldots \cup K_n$ where K_n is not empty, and $F : X \to X$ the set valued map that defines the K_i. Then F induces a nontrivial homomorphism, $h : H(X) \to H(X)$ such that if $L(h) \neq 0$, then F has a fixed point.

<u>Proof</u> This theorem follows from Theorem 2.21 and Lemma 2.6.

Extension of Spaces on which $F : X \to Y$ Induce Homomorphisms and Satisfy the Lefschetz Fixed Point Theorem

Definition 2.23. [1]. Let $\{X^i, p_j^{\,i}\}^-$ be an inverse system of finite polyhedra ordered by inverse inclusion in some parallelotope P and $r(\rho) : U \to X^\rho$ is the retraction map.
Then $X = \underleftarrow{L}\{X^i, p_j^{\,i}\}^- = \bigcap X^i$ is an NR_δ where there are maps $t^i : X^i \to X$ for each i as well as inclusion maps $j^i : X \to X^i$.
Thus $i < j$ iff $X^j \subset X^i$ and
$r^i_{\,j} = r(j)r(i)$.

Lemma 2.24. Let $X = \bigcap_i X^i$ be an NR_δ as in the definition.

Let $F : X \to X$ satisfy Theorem 2.7. Then for each i, there is a set valued map $F^i : X^i \to X^i$ that induces h^i on $H_*(X^i) \ni L(h^i) \neq 0 \Rightarrow F^i$ has a fixed point, x_i.

Proof F^i is defined from

$$\begin{tikzcd} X^i \arrow[r, dashed, "F^i"] \arrow[d, "t^i"'] & X^i \\ X \arrow[r, "F"] & X \arrow[u, "j^i"'] \end{tikzcd}$$

where $F^i = j^i\, F\, t^i$ so that each F^i satisfies Theorem 2.7 and so induces $h^i : H_*(X^i) \to H_*(X^i)$.

<u>Theorem</u> 2.25. Let $X = \bigcap_i X^i$, F^i, h^i, F be as above. Then:

(a) If $t^\nu r^\rho_\nu \simeq t^\rho$, there is induced $h : H_*(X) \to H_*(X)$ such that $Lh = Lh^\rho$ for $\rho > \rho_0$.

(b) If moreover, for every covering α of X, there is $\rho(\alpha)$ such that for all $\rho > \rho(\alpha)$, $(x, t^\rho(x)) \in \alpha \times \alpha$, then $L(h) \neq 0$ implies F has a fixed point.

<u>Lemma</u> 2.25a. If $X \xrightarrow{H} Y \xrightarrow{G} Z$ where H, G, are u.s.c. set valued functions with induced homomorphisms h_H, h_G respectively, then $h_G h_H$ is an induced homomorphism of GH.

<u>Lemma</u> 2.25b. Let $H : Z \times I \to Y$ where Z, Y are compact polyhedra, and $H(z, t)$ consists of 1 or n acyclic components for all $z \in Z$ and t in I. Then $H_{0*} = H_{1*} : H_*(Z) \to H(Y)$ where $H_0(Z) = H(Z,0)$, and $H_1(Z) = H(Z, 1)$.

<u>Proof of Lemma</u> 2.25b. H induces $k_* \neq 0$ such that $k_* : H(Z \times I) \to H(Y)$. Let $g_i : Z \to Z \times I$, $i = 0,1$ be defined by $g_i(z) = (z,i)$ $i = 0,1$. Then $H g_i = H_i$. Now $g_{0*} = g_{1*}$ and thus $H_{0*} = k_* g_{0*} = k_* g_{1*} = H_{1*}$. [6]

<u>Corollary</u> 2.25c. If $G \simeq K : Z \to W$ by the homotopy $H : Z \times I \to W$ in the lemma, then the induced homomorphisms

$$k_* = g_* : H_*(Z) \to H_*(W).$$

Proof of Theorem 2.25.

To show there is an $h : H(X) \to H(X)$, it must be shown that the diagram

$$\begin{array}{ccc} H(X^\rho) & \xrightarrow{h^\rho} & H(X^\rho) \\ {\scriptstyle p_\mu{}^\rho}\downarrow & & \downarrow{\scriptstyle p_\mu{}^\rho} \\ H(X^\mu) & \xrightarrow{h^\mu} & H(X^\mu) \end{array}$$ commutes for $\mu < \rho$.

Now $t^\rho\, r^\mu{}_\rho \simeq t^\mu$ implies $F^\mu \simeq p_\mu{}^\rho\, F^\rho\, r^\mu{}_\rho$ implies $F^\mu p_\mu{}^\rho \simeq p_\mu{}^\rho\, F^\rho$ in the sense of Lemma 2.25b. By Corollary 2.25c, $h^\mu\, p_{\mu *}{}^\rho = p_{\mu *}{}^\rho\, h^\rho$. Thus the diagram commutes to give

$$h : H(X) = \underset{\leftarrow}{L} H(X^\rho).$$

That $Lh = Lh^\rho$ for all $\rho > \rho_0$ would be proven in exactly the same manner as is done in [1] by making use of essential cycles.

The proof of (b) will follow a proof in Bourgin [1]. Assume $\bar{x} \notin F(\bar{x})$. Then there are open neighborhoods U, V of $\bar{x}$, $F(\bar{x})$ respectively such that $U \cap V = \phi$ and $F(U) \subset V$. Choose a cover α of X such that $St(St(\bar{x},\alpha),\alpha) \subset U$. Select ρ so that $\bar{x}$, $t^\rho\, \bar{x}\ \varepsilon\ \alpha \times \alpha$.

(Here $\alpha \times \alpha$ or α^2 will denote the set of $A_i \times A_i$ where A_i's are in α.)

For some $\nu > \rho$, x_ν and $\bar{x} \in a_i$, say, of α. Then $\bar{x}$, $t^\nu x_\nu \in \alpha^2$ implies $t^\nu x_\nu \in U$. Thus $V \supset Ft^\nu(x_\nu)$. But $x_\nu \in F^\nu(x_\nu) \subset V$ and $x_\nu \in U$. Contradiction.

Thus $\bar{x} \in F(\bar{x})$.

Remark In this section it is possible that X is not a metric space even though each X^ρ is.

Theorem 2.26. Let P be a finite polyhedron; Y a compact metric ANR. Let $F : P \to Y$ satisfy the statement of Theorem 2.7. Then F induces a homomorphism $h : H_*(P) \to H_*(Y)$.

Theorem 2.27. Let $X = P \times Q$ where P is a finite polyhedron, and Q is a Hilbert cube. Let $F : X \to X$. Then F induces a homomorphism

(a) $h : H\ (X) \to H\ (X)$ such that

(b) $L(h) \neq 0$ implies F has a fixed point.

Lemma 2.27a. Let F, P, Q, X be as in the theorem. Then there is a family of u.s.c. functions $\{F_i\} : X \to X$ such that $\{F_i\}_{i=1} \to F$.

Proof Let $Q = Q_n \times R_n$ where $Q_n = \prod_{i=1}^{n} [-\frac{1}{i}, \frac{1}{i}]$ and

$R_n = \prod_{i=n+1}^{\infty} [-\frac{1}{i}, \frac{1}{i}]$. Let R_n^0 be the zero of R_n. Let

$P_i = P \times Q_i \times R_i^0$ and $\pi_i : X \to P_i$ the projection onto P_i.

Let $\varepsilon > 0$ be given. Then since F is u.s.c., there is a $\delta > 0$ so that $F(N_\delta(x)) \subset \varepsilon/4\ F(x)$. The compactness of X implies there is a δ_0 so that $F(N_{\delta_0}(x)) \subset \varepsilon/4\ F(x)$. Since $\lim_{i\to\infty} d(\pi_i(x), x) \to 0$ uniformly, there is an i_0 so that for all $i \geqq i_0$, $\pi_i\ F\ \pi_i\ (x) \subset \varepsilon\ F(x)$.

Let $F_i = \pi_i\ F\pi_i$. Thus, $\{F_i\} \to F$.

Lemma 2.27b. Let F, X, P, Q, F_i be as given above. Let $F_i^1 = F\pi_i$

Then $\{F_i^1\}$ induces $\{h_i\} : H(X) \to H(X)$ and also $h : H(X) \to H(X)$.

Proof Let $j_i : P_i \to X$ be an inclusion map. Then $Fj_i : P_i \to X$ induces an $s_{i*} : H_*(P_i) \to H_*(X)$. Let $h_{i*} = s_{i*}\pi_{i*} : H_*(X) \to H_*(X)$ where h_{i*} is induced from the map $Fj_i\pi_i : X \to X$. Since $\{Fj_i\pi_i\} \to F$ as in the previous lemma, a previous result implies F induces $h : H(X) \to H(X)$

Proof of theorem

(a) follows from the lemmas.

(b) $\pi_{i*}\ hj_{i*} : H_*(P_i) \to H_*(P_i)$.

Since $\pi_{i*}j_{i*} = 1_{i*} : H_*(P_i) \to H\ (P\ )$, and since

$\pi_{i*} : H_*(X) \to H_*(P_i)$ is an isomorphism where $j_{i*} = \pi_{i*}^{-1}$.

Thus with Trace h = trace $\pi_{i*}hj_{i*}$, $Lh = L\pi_{i*}hj_{i*}$. But

$L(h) \neq 0$ implies $\pi_i F j_i$ has a fixed point x_i for each i.

Thus $x_i \in \pi_i F j_i(x_i) = \pi_i F \pi_i j_i(x_i)$. Since $\{\pi_i F \pi_i\} \to F$ by a previous lemma, F has a fixed point.

<u>Theorem</u> 2.28. Let X be a compact metric ANR and $F : X \to X$ as before.

Then F induces

(a) $h : H(X) \to H(X)$ such that if

(b) $L(h) \neq 0$, F has a fixed point.

<u>Proof</u>

(a) Let $Z = P \times Q$ such that

$$\begin{tikzcd} X \arrow[rr, "1"] \arrow[dr, "j"'] & & X \\ & Z \arrow[ur, "r"'] & \end{tikzcd}$$

commutes. Then $Fr : Z \to X$ is such that for all x, $Fr(x)$ consists of 1 or n acyclic components and thus induces $t_* : H(Z) \to H(X)$. Let $h = t_*j_* : H_*(X) \to H_*(X)$.

(b) Let A, B be n x m, m x n matrices respectively. Then Trace AB = Trace BA shows that Trace t_*j_* = Trace j_*t_*. Now $L(h) \neq 0 \Rightarrow L(j_*t_*) \neq 0$ where $j_*t_* : H(Z) \to H(Z)$. From the previous $iFr : Z \to Z$ has a fixed point x. But $x \in iFr(x) \Rightarrow x \in X$ and $x \in F(x)$.

Corollary 2.29. Let X be an ANR, $\{F_i\} \to F : X \to X$ where each $F_i : X \to X$ is a continuous set valued map such that $F_i(x)$ consists of 1 or n acyclic components. Then if F is u.s.c., F induces an

(a) $h : H(X) \to H(X)$ such that

(b) $L(h) \neq 0 \Rightarrow$ F has a fixed point.

Corollary 2.30. Let $\{X^\rho, p_\nu{}^\rho\}$ be an inverse system of ANR's ordered by inverse inclusion. Let $X = \bigcap_\rho X^\rho$ be an NR_δ. Then the results of Theorems 2.23 - 2.25 still follow.

CHAPTER 3

Inverse Limits and the Fixed Point Property for Set Valued Maps

In this chapter, it is shown that the fixed point property is invariant for inverse limits if the bonding maps are surjections. This is not true for the single valued case.

Definition 3.1.

(a) Let $\mathrm{cov}^f(X)$ be the cofinal family of finite open covers of X.

(b) Let $\alpha, \beta \in \mathrm{cov}^f(X)$. Then $\alpha > \beta$ means α refines β and $\alpha \overset{*}{>} \beta$ means star refines β in the sense that for every $A \in \alpha$, $St(A,\alpha) = \bigcup A'$ (where $A' \in \alpha$ and $A' \cap A \neq \phi) \subset B$ for some $B \in \beta$.

(c) If $\alpha \in \mathrm{cov}^f(X)$, $\bar{\alpha} = \{\bar{A} \mid A \in \alpha\}$.

Theorem 3.2. Let $X = \underleftarrow{L}\{X^\lambda : p_\mu^\lambda ; D\}^-$ be an inverse limit where

(a) X, $\{X^\lambda\}_{\lambda \in D}$ are compact metric spaces;

(b) $p_\mu^\lambda : X^\lambda \to X^\mu$ are surjections;

(c) Each X^λ has the fixed point property for set valued maps.

Then if $F : X \to X$ is a set valued map, F has a fixed point.

Outline of the Proof

Step 1. Assume F has no fixed point. Then there is an ε, δ such that $\mathcal{N} = \{N_\varepsilon(x_i)\}_{i=1}^{n_0}$ such that $N_\varepsilon(x_i) \cap N_\delta F(x_i) = \phi$ and $FN_\varepsilon(x_i) \subset N_\delta F(x_i)$. Let $\mathcal{N}$, ε, δ be denoted as $\mathcal{N}_0$, ε_0, δ_0 respectively. Define $\varepsilon_n < \varepsilon/2^n$, etc., and take refinements $\mathcal{N}_n > \mathcal{N}_{n-1} > \ldots > \mathcal{N}_0$ of diameters $< \varepsilon_n$.

Step 2. A sequence $\{F_n\}_0^\infty : X \to X$ is defined such that for each n, $F_n = \underleftarrow{L}\{F_n^\lambda \mid D_n\}^-$ where D_n is cofinal in D_{n-1} and $D_0 = D$. Moreover, $F_n^\lambda : X^\lambda \to X^\lambda$ is a set valued map and has a fixed point x_n^λ. Let $x_n = \underleftarrow{L}\{x_n^\lambda \mid p_n^\lambda ; D_n\}$. Then $x_n \in F_n(x_n)$.

Step 3. For each $x \in X$, $F_0(x) \supset F_1(x) \supset \ldots \supset F_n(x) \supset \ldots$ Thus by the finite intersection property, $\check{F}(x) = \bigcap_{n=0}^{\infty} F_n(x) \neq \phi$ can be defined.

Step 4. Let $\{x_{n_j}\}_{j=1}^\infty \subset \{x_n\}$. Let $\bar{x}$ be the limit point of $\{x_{n_j}\}$. Then $\bar{x} \in F(\bar{x})$.

Step 5. Show $\bar{x} \in F(\bar{x})$. This contradicts the assumption in Step 1.

Proof

Step 1. Assume for each x, $x \notin F(x)$. Then $\exists\ \varepsilon_0, \delta_0 > 0$, $\mathcal{N}_0 = \{N_{\varepsilon_0}(x_i)\}_{i=1}^{n_0} \in \mathrm{cov}^f(X)$ such that $N_{\varepsilon_0}(x_i) \cap N_{\delta_0}F(x_i)$ is empty and $F(N_{\varepsilon_0}(x_i)) \subset N_{\delta_0}F(x_i)$, $i = 1, \ldots, n_0$, and for each x' in $N_{\varepsilon_0}(x_i)$, $F(x_i) \subset N_{\delta_{0/2}}(F(x'))$. These follow from the fact that F is both u.s.c. and l.s.c. Now find covers $\gamma_0, \beta_0 \in \mathrm{cov}^f(X)$ such that $\gamma_0 \overset{*}{>} \beta_0 > \overline{\beta}_0 > \mathcal{N}_0$. Then $\exists\ \mu_0 \in D$, $\alpha_0 \in \mathrm{cov}^f(X^{\mu_0})$ such that $p_{\mu_0}^{-1}(\alpha_0) > \gamma_0$ [6].

Define $r_{\mu_0} : X^{\mu_0} \to X$ by $r_{\mu_0}(x') = \overline{St(p_{\mu_0}^{-1}(x'); p_{\mu_0}^{-1}(\alpha_0))}$.

Then r_{μ_0} is a set valued map.

(a) $r_{\mu_0}(x')$ is closed for each x'.

(b) Let U be open in X. Then $r_{\mu_0}^{-1}(U) = \{z \mid r_{\mu_0}(z) \cap U \neq \phi\}$. Suppose $z \in \{A_1, \ldots, A_r\} \subset \alpha_0$. Then $\exists\ V \subset A_1 \cap \ldots \cap A_r$ such that for any $w \in V$, $p_{\mu_0}^{-1}(w) \cap U \neq \phi$. This will follow from the fact that $p^{\mu_0} : X \to X^{\mu_0}$ is a closed map.

(c) Let C be closed in X. Then $r_{\mu_0}^{-1}(C) = \{z \mid r_{\mu_0}(z) \cap C \neq \phi\}$.

Let $\bar{z}$ be a limit point of $r_{\mu_0}^{-1}(C)$. Then $\{z_n\} \to \bar{z}$ such that

$r_{\mu_0}(z_n) \cap C \neq \phi$. As in (b) above, $\bar{z}$ can be assumed to be in

$V \subset A_1 \cap \ldots \cap A_r$. Then $\exists\, N_0$ such that $n \geq N_0 \Rightarrow z_n \in V$ and

thus $r_{\mu_0}(z_n) = r_{\mu_0}(\bar{z})$. Therefore $r_{\mu_0}^{-1}(C)$ is closed.

<u>Define</u> $F_0^{\mu_0} = \pi_{\mu_0} Fr_{\mu_0} : X^{\mu_0} \to X^{\mu_0}$. $F_0^{\mu_0}$ is continuous and has

a fixed point $x_0^{\mu_0}$.

(a) Let $D_0 \subset D$ where $D_0 = \{\lambda \mid \lambda > \mu\}$ and is cofinal in

D. Then define $F_0^{\lambda} : X^{\lambda} \to X^{\lambda}$ by

$F_0^{\lambda} = \pi_{\lambda}\ Fr_{\mu_0} p_{\mu_0}{}^{\lambda} : X^{\lambda} \to X^{\lambda}$ which has a fixed point, x_0^{λ}

(b) $\{F_0^{\lambda}\}_{\lambda \in D_0} : \{X^{\lambda} \mid p_{\mu}{}^{\lambda} ; D_0\}^{-} \to \{X^{\lambda}; D_0\}^{-}$ induces

$F_0 : \underleftarrow{L}\{X^{\lambda} ; D_0\} \to \underleftarrow{L}\{X^{\lambda} ; D_0\}$. Commutativity easily follows as

is obvious from the diagram:

$$
\begin{tikzcd}
X^{\mu} \arrow[r, "F_0^{\mu}"] \arrow[d, "p_{\lambda}^{\mu}"'] & X^{\mu} \arrow[d, "p_{\lambda}^{\mu}"] \\
X^{\lambda} \arrow[r, "F_0^{\lambda}"] \arrow[d, "p_{\mu_0}^{\lambda}"'] & X^{\lambda} \arrow[d] \\
X^{\mu_0} \arrow[r, "F_0^{\mu_0}"] \arrow[d, "r_{\mu_0}"'] & X^{\mu_0} \\
X \arrow[r, "F"] & X \arrow[u, "p_{\mu_0}"'] \arrow[uuu, bend right=60, "p\mu"']
\end{tikzcd}
$$

Since $X \simeq \underset{\leftarrow}{L} \{X^{\lambda} ; D_0\}^{-}$ one can assume $F_0 : X \to X$ with fixed point $x_0 = \{x_0^{\lambda}\}$.

<u>Step</u> <u>2</u>. Let $\mathcal{N}_1 > p_{\mu}^{-1}(\alpha_0)$ where diam $\mathcal{N}_1 < \varepsilon/2 = \varepsilon_1$ and $N_{\varepsilon_1}(x') \in \mathcal{N}_1 \Rightarrow FN_{\varepsilon_1}(x') \subset N_{\delta_1}F(x')$, for each x" in $N_{\varepsilon_1}(x')$, $F(x') \subset N_{\delta_{1/2}}(F(x''))$, and $N_{\varepsilon_1}(x') \cap N_{\delta_1}F(x') = \phi$.

Again $\exists\ \gamma_1, \beta_1$

$\gamma_1 \overset{*}{>} \beta_1 > \bar{\beta}_1 > \mathcal{N}_1$. Also $\exists\ \mu_1 \in D_0$ and $\alpha_1 \in \mathrm{cov}^f(X^{\mu_1})$ such that $p_{\mu_1}^{-1}(\alpha_1) > \gamma_1$. Moreover, $r_{\mu_1} : X^{\mu_1} \to X$ and $F_1^{\mu_1} : X^{\mu_1} \to X^{\mu_1}$ with fixed point $x_1^{\mu_1}$. Let $D_1 = \{\lambda \in D_0 \mid \lambda > \mu_1\}$ cofinal in D.

Thus there is induced $F_1 : X \to X$ with fixed point x_1.

<u>Claim</u> $F_1(x) \subset F_0(x)$ for all $x \in X$.

Let $\lambda \in D_1$. Now $F_1(x) = \underleftarrow{L}\{F_1^{\lambda} p_\lambda(x) \mid D_1\}^-$.

Since $p_{\mu_0} = p_{\mu_0}^{\mu_1} \; p_{\mu_1}$,

$$p_{\mu_0}^{-1}(\alpha_0) = p_{\mu_1}^{-1} \, p_{\mu_0}^{\mu_1 \, -1}(\alpha_0) < \beta_1 < p_{\mu_1}^{-1}(\alpha_1).$$

$$\begin{tikzcd}
X^{\lambda} \arrow[d, "p_{\mu_1}^{\lambda}"] \\
X^{\mu_1} \arrow[d, "p_{\mu_0}^{\mu_1}"] \arrow[dd, bend right=60, "r_{\mu_1}"'] \\
X^{\mu_0} \arrow[d, "r_{\mu_0}"] \\
X \arrow[uuu, bend right=60, "p^{\lambda}"']
\end{tikzcd}$$

Then it can be shown that $r_{\mu_1} p_{\mu_1}^{\lambda}(x_\lambda) \subset r_{\mu_0} p_{\mu_0}^{\lambda}(x_\lambda)$ where $x_\lambda = p_\lambda(x)$.

$$\text{Now } r_{\mu_1} p_{\mu_1}{}^{\lambda}(x_\lambda) = \overline{St(p_{\mu_1}^{-1} p_{\mu_1}{}^{\lambda}(x_\lambda),\ p_{\mu_1}^{-1}(\alpha_1))}$$

$$= \overline{St(p_{\mu_1}^{-1}(p_{\mu_1}(x)),\ p_{\mu_1}^{-1}(\alpha_1))}$$

$$\subset \overline{St(p_{\mu_0}^{-1} p_{\mu_0}(x),\ p_{\mu_0}^{-1}(\alpha_0))} = r_{\mu_0}(p_{\mu_0}(x))$$

$$= r_{\mu_0} p_{\mu_0}{}^{\lambda}(x_\lambda), \text{ since}$$

$$p_{\mu_1}^{-1}(x_{\mu_1}) \subset p_{\mu_1}^{-1}\, p_{\mu_0}^{\mu_1\,-1}\ p_{\mu_0}^{\mu_1}(x_{\mu_1}) = p_{\mu_0}^{-1}\, p_{\mu_0}(x)$$

Then it follows that $F_1{}^{\lambda}(x_\lambda) \subset F_0{}^{\lambda}(x_\lambda)$ for all $\lambda \in D_1$.

This gives $F_1(x) \subset F_0(x)$; and in particular $x_1 \in F_1(x_1) \subset F_0(x_1)$.

<u>In general</u>, let $\eta_{n+1} > p_{\mu_n}^{-1}(\alpha_n)$ and $\gamma_{n+1} \overset{*}{>} \beta_{n+1} > \bar{\beta}_{n+1} > \eta_{n+1}$.

As before, an $F_{n+1} : X \to X$ is induced with fixed point x_{n+1}.

Since for every x, $F_0(x) \supset F_1(x) \supset \ldots \supset F_{n+1}(x)$,

$x_{n+1} \in F_i(x_{n+1})$, $i \leqq n+1$.

<u>Step 3</u>. For each $x \in X$, define $\check{F}(x) = \bigcap_{n=0}^{\infty} F_n(x)$ which is non-empty by the finite intersection property.

<u>Step 4</u>. There is $\bar{x} \ni \bar{x} \in F(\bar{x})$. Now $\{x_n\}$ contains a subsequence $\{x_{n_j}\} \to \bar{x}$. Let $d(\bar{x}, \check{F}(\bar{x})) = \delta'$. Let $\eta = \delta'/8$. There is an n_0 such that for all $n \geqq n_0$, $F_n(\bar{x}) \subset N_\eta(\check{F}(\bar{x}))$. Assume $\{x_{n_j}\} \subset N_\eta(\bar{x})$. Then for $n_{j(n)} \geqq n$, $F_n(x_{n_{j(n)}}) \subset N_\eta \check{F}(\bar{x})$. Then

$x_{n_{j(n)}} \in F_{n_{j(n)}}(x_{n_{j(n)}}) \subset F_n(x_{n_{j(n)}})$. This implies

$d(\bar{x}, \check{F}(\bar{x})) \leqq d(\bar{x}, x_{n_{j(n)}}) + 0 + \eta$ $\not\to$ therefore $\bar{x} \in \check{F}(\bar{x})$.

<u>Step 5</u>. $\bar{x} \in F(\bar{x})$. By assumption $\bar{x} \notin F(\bar{x})$ and this implies $\exists$ $\lambda \ni p_\lambda(\bar{x}) \notin p_\lambda F(\bar{x})$. Let $\bar{x}_\lambda = p_\lambda(\bar{x}) \in U_\lambda$ and $p_\lambda F(\bar{x}) \subset V_\lambda$ $\ni$ $U_\lambda \cap V_\lambda = \phi$. Now $\exists$ $n \ni p_\lambda N_{\delta_n}(F(\bar{x})) \subset V_\lambda$. Let $\nu > \mu_{n+1}, \lambda$.

Then $\bar{x}_\nu \in p_\lambda^{\nu\ -1}(U_\lambda)$, $p_\nu(F(\bar{x})) \subset p_\lambda^{\nu\ -1}(V_\lambda) \ni p_\lambda^{\nu\ -1}(U_\lambda) \cap p_\lambda^{\nu\ -1}(V_\lambda) = \phi$.

$$\begin{tikzcd}
 & X^\lambda & \\
X^\nu \arrow[ur, "p_\lambda^\nu"] \arrow[dr, "p^\nu_{\mu_{n+1}}"'] & & X \arrow[ul, "p^\lambda"'] \arrow[dl, "p^{\mu_{n+1}}"] \\
 & X^{\mu_{n+1}} &
\end{tikzcd}$$

Now $r_{\mu_{n+1}} p^\nu_{\mu_{n+1}} (\bar{x}_\nu) \subset N_{\varepsilon_{n+1}}(x') \in \mathcal{V}_{\varepsilon_{n+1}}$.

Then $Fr_{\mu_{n+1}} p^\nu_{\mu_{n+1}} (\bar{x}_\nu) \subset N_{\delta_{n+1}} F(x') \subset N_{\delta_n} F(\bar{x})$ where

$\delta_n > 2\delta_{n+1}$. This implies $F^\nu_{n+1}(\bar{x}_\nu) \subset p_\nu N_{\delta_n} F(\bar{x}) \subset p_\lambda^{\nu\ -1}(V_\lambda)$

where $p_\lambda^\nu p_\nu = p_\lambda$.

But in Step 4, it was shown that $\bar{x} \in \check{F}(\bar{x})$ which shows that $\bar{x} \in F_n(\bar{x})$ for all n. Thus $\bar{x}_\nu \in F^\nu_{n+1}(\bar{x}_\nu) \subset p_\nu N_{\delta_n} F(\bar{x}) \subset p_\lambda^{\nu\ -1}(V_\lambda)$.

This contradicts the assumption in Step 1. Thus $\bar{x} \in F(\bar{x})$.

CHAPTER 4

Contractive and Non-Expansive Set Valued Functions

Let (X, d) be a metric space with metric d. (X need not be compact.).

Definition 4.1. Let $D(A, B)$ be the Hausdorff metric defined on all closed and bounded subsets of X.

Theorem 4.2. Suppose (X, d) is a complete metric space and $F : X \to X$ where each $F(x)$ is compact. Then if for every $(a, b) \in X \times X$, $D(F(a), F(b)) \leqq k\, d(a, b)$ where $0 < k < 1$, there is a $\xi \in X$ such that $\xi \in F(\xi)$.

Proof Suppose F has no fixed point. Then for any x, let

$$d(F(x), x) = \min_{y \in F(x)} d(y, x) = \varepsilon > 0.$$

Now $F(x)$ compact implies $\exists\, x_1 \in F(x)$ such that $d(x_1, x) = \varepsilon$. By assumption $D(F(x_1), F(x)) \leqq k\ \varepsilon$.

The compactness of $F(x_1)$ implies there is $x_2 \in F(x_1)$ so that

$d(x_2, x_1) \leqq D(F(x_1), F(x)) \leqq k\ \varepsilon$.

Now $D(F(x_2), F(x_1)) \leqq k^2 \varepsilon$. In general one finds

$x_{n+1} \in F(x_n)$ so that

$$d(x_{n+1}, x_n) \leqq D(F(x_n), F(x_{n-1})) \leqq k^n \varepsilon$$

and

$$D(F(x_{n+1}), F(x_n)) \leqq k\, d(x_{n+1}, x_n).$$

Since $\{x_n\}_0$ is a Cauchy sequence, $\{x_n\} \to \xi$ say.

Claim $\xi \in F(\xi)$. Suppose $d(F(\xi), \xi) = \min\limits_{z \in F(\xi)} d(z, \xi) = \varepsilon > 0$.

Then $\exists\ \xi' \in F(\xi)$ where $d(\xi, \xi') = \varepsilon$.

For each j there is y_j in $F(\xi)$ so that

$$d(x_{j+1}, y_{j+1}) \leqq D(F(x_j), F(\xi)) \leqq k\, d(x_j, \xi)$$

Since $d(x_j, \xi) \to 0$ as $j \to \infty$, implies $\lim\limits_{j\to\infty} d(x_j, F(\xi)) = 0$

Thus $\xi = \xi' \in F(\xi)$.

Proposition 4.3. Let $F : X \to X$ where $D(F(x), F(y)) < d(x, y)$ for all $x \neq y$. Then this implies that F is a set valued map.

<u>Proof</u> That F is u.s.c. is obvious. That F is l.s.c., assume that $\exists x$, $z \in F(x)$ and a sequence $\{y_n\} \to x$ where $d(y_n, x) < \frac{1}{n}$ and for some open set V of z, $F(y_n) \subset X - V$. However this contradicts the assumption that

$D(F(y_n), F(x)) < d(y_n, x) \to 0$ as $n \to \infty$.

<u>Theorem</u> 4.4. Let (X, d) be a metric space and $F : X \to X$ such that

(a) $D(F(x), F(y)) < d(x, y)$ for all $x \neq y$;

(b) $\exists \; x \ni x_i \in F^i(x)$ and

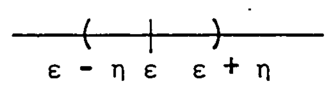
$\{x_i\} \supset \{x_{i_r}\} \to \xi$. Also $\{x_i\}$ are such that

$d(x_{i+1}, x_i) \leqq D(F(x_i), F(x_{i-1}))$. Then $\xi \in F(\xi)$.

<u>Proof of Theorem</u>. The following lemmas are needed.

<u>Lemma</u> 4.4a. Define $\sigma : X \times X \to$ reals by $\sigma(x, y) = D(F(x), F(y))$. Then σ is continuous on $X \times X$.

<u>Proof of lemma</u>. Suppose $\sigma(x, y) = D(F(x), F(y)) = \varepsilon$. Let $S_\eta(\varepsilon)$ be an open η interval about ε; ie.



<u>Claim</u> There is an open set $U \times V$ containing (x, y) such that $\sigma(U \times V) \subset S_\eta(\varepsilon)$. Let $\delta < \eta/8$ and $N_\delta F(x)$, $N_\delta F(y)$ be open δ-neighborhoods about $F(x)$, $F(y)$. Since F is u.s.c., there is a U_1, V_1 open about x, $y \ni F(U_1) \subset N_\delta F(x)$ etc. Let $\{S_\delta(x_i)\}$, $\{S_\delta(y_j)\}$ be a finite open cover of $F(x)$, $F(y)$ respectively.

Then since F is l.s.c. $U_2 = \bigcap_i F^{-1}S_\delta(x_i)$, $V_2 = \bigcap_j F^{-1}S_\delta(y_j)$ are open neighborhoods of x, y respectively. Let $U = U_1 \cap U_2$ and $V = V_1 \cap V_2$. Then for any $u \in U$, $\sigma(u, x) =$ $D(F(u), F(x)) < 2\delta$.

Similarly for any $v \in V$, $\sigma(y, v) < 2\delta$. To show this, let $u \in U_1 \cap U_2$. Thus $u \in U_1$ implies $F(u) \subset N_\delta(F(x))$. Also $u \in U_2 = \bigcap F^{-1}S_\delta(x_i)$ implies $F(x) \subset N_{2\delta}F(u)$ since $u' \in F(u) \Rightarrow$ $u' \in S_\delta(x_1)$ say, and $S_\delta(x_1) \subset S_{2\delta}(u')$. Here as elsewhere $S_{2\delta}(u')$ means the sphere of radius 2δ about u'.

For any $(u, v) \in U \times V$,

$$\sigma(u, v) = D(F(u), F(v)) \leqq D(F(u), F(x)) + D(F(x), F(y)) + D(F(y), F(v))$$

$$\leqq 2\delta + \varepsilon + 2\delta < \varepsilon + \eta.$$

Also $\sigma(x, y) \leqq \sigma(x, u) + \sigma(u, v) + \sigma(v, y) \Rightarrow \varepsilon - \eta < \sigma(u,v)$.
Thus $\varepsilon - \eta < \sigma(u, v) < \varepsilon + \eta$ for all $(u, v) \in U \times V$.

Lemma 4.4b. $F : X \to Y$ a continuous set valued map implies that if $\{x_n\} \to x$, then $F(x)$ is a cofinal and residual limit of $\{F(x_n)\}$.

Proof of Theorem Assume $d(\xi, F(\xi)) = \delta$. Let $\eta < \frac{\delta}{4}$ and consider $W = S_\eta(\xi) \times N_\eta(F(\xi))$. From the previous lemma there is a $U \times V$ containing $\xi \times F(\xi)$ and $U \times V \subset W$ such that $\sigma(u, v) \leqq \alpha\, d(u, v)$, $0 < \alpha < 1$ for all $(u, v) \in U \times V$.
From the second lemma, it can be assumed that $\{x_{n_i}\}_{i=1}^{\infty} \subset U$, and $\{F(x_{n_i})\}_{i=1}^{\infty} \subset V$. Thus

$$D(F(x_{n_1}), F(x_{n_1+1})) \leqq \alpha d(x_{n_1}, x_{n_1+1}).$$

By assumption $d(x_{n_1+1}, x_{n_1+2}) \leqq D(F(x_{n_1}), F(x_{n_1+1}))$

Similarly $d(x_{n_2-1}, x_{n_2}) \leqq D(F(x_{n_2-2}), F(x_{n_2-1})) < d(x_{n_2-2}, x_{n_2-1})$.

$$D(F(x_{n_2}), F(x_{n_2+1})) \leqq \alpha\, d(x_{n_2}, x_{n_2+1}) < \alpha^2 d(x_{n_1}, x_{n_1+1})$$

This gives $d(x_{n_2+1}, x_{n_2+2}) < \alpha^2 d(x_{n_1}, x_{n_1+1})$ and

$d(x_{n_3}, x_{n_3+1}) < \alpha^2 d(x_{n_1}, x_{n_1+1})$.

In general $d(x_{n_r}, x_{n_r+1}) < \alpha^{r-1} d(x_{n_2}, x_{n_1+1}) \to 0$ as $r \to \infty$.

This contradicts the assumption that $\{x_{n_i}\} \subset U$ and

$\{F(x_{n_i})\} \subset V$ where $U \times V \subset S_\eta(\xi) \times N_\eta(F(\xi))$.

As a corollary to what has been said, let $X = \underset{\leftarrow}{L} \{X^\lambda, p_\mu{}^\lambda, D\}^-$

be the inverse limit of compact metric spaces, X^λ, with metric d^λ . Suppose there is a u.s.c. set valued function $F : X \to X$ such that for each $\lambda \in D$, the function

$F^\lambda = p^\lambda F p^{\lambda\ -1} : X^\lambda \to X^\lambda$ satisfies for each

$x^\lambda \times y^\lambda \in X^\lambda \times X^\lambda - \Delta$

$D^\lambda(F^\lambda(x^\lambda), F^\lambda(y^\lambda)) < d^\lambda(x^\lambda, y^\lambda)$ where D^λ is the Hausdorff metric induced by d^λ.

Theorem 4.5. Let $F : X \to X = \underset{\leftarrow}{L} \{X^\lambda\}^-$ be as stated above. Then F has a fixed point.

Proof By the previous theorem for each λ, F^λ has a nonempty set of fixed points $Y^\lambda \subset X^\lambda$. Since each Y^λ is closed and so compact, and since $\{Y^\lambda, p_\mu{}^\lambda \mid Y^\lambda\}^-$ is an inverse system $Y = \underleftarrow{L}\{Y^\lambda\}$ is not empty.

Claim For $y \in Y$, $y \in F(y)$. Suppose not. Then for some $\mu \in D$, $y^\mu \notin p_\mu F(y)$. Let U^μ, V^μ be disjoint open neighborhoods of y^μ, $p_\mu F(y)$ respectively. Also let U, V be disjoint open neighborhoods of y, $F(y)$ respectively such that $F(U) \subset V$ and $p_\mu(U) \subset U^\mu$, $p_\mu(V) \subset V^\mu$.

By[6] $\exists$ $\sigma > \mu$, $\alpha(\sigma) \in \text{cov}^f(X^\sigma)$ such that $p_\sigma{}^{-1}(\alpha(\sigma))$ refines $\alpha = \{U, V, X - \{y\} \cup \{F(y)\}\}$.
Let $y^\sigma = p_\sigma(y) \in A \in \alpha(\sigma)$. Then
$p_\sigma{}^{-1}(A) \subset U \Rightarrow p_\sigma{}^{-1}(y^\sigma) \subset U$. Then
$p_\mu F\, p_\sigma{}^{-1}(y^\sigma) \subset p_\mu V \subset V^\mu$.

Also $p_\mu{}^{\sigma\,-1}\, p_\mu F\, p_\sigma{}^{-1}(y^\sigma) \subset p_\mu{}^{\sigma\,-1}(V^\mu)$.

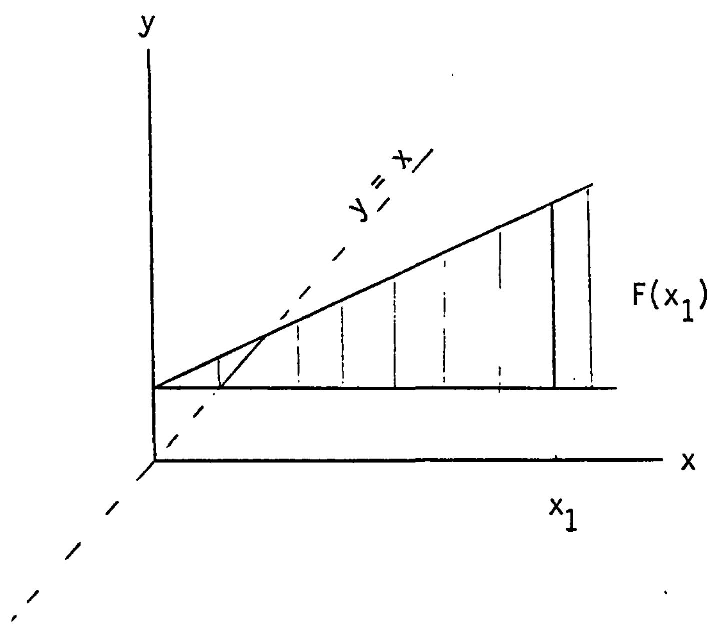
Now $y^\sigma \in p_\sigma F\, p_\sigma{}^{-1}(y^\sigma) \subset p_\mu{}^{\sigma\,-1}\, p_\mu F\, p_\sigma{}^{-1}(y^\sigma) \subset p_\mu{}^{\sigma-1}(V^\mu)$.
But $y^\sigma \in p_\mu{}^{\sigma\,-1}(U^\mu)$. $\rightarrow\!\leftarrow$

Therefore $y \in F(y)$.

<u>Remark</u> In the case of contractive set valued maps, nothing can be said about the uniqueness of the fixed point of the map $F : X \to X$.

<u>Example</u> Let X be the positive reals, R^+.

Let $F : R^+ \to R^+$ be defined by $F(x) = [\frac{1}{4}, \frac{1}{4} + \frac{1}{2}x]$. The F is a contractive, set valued map where $D(F(x_1), F(x_2)) = \frac{1}{2} | x_1 - x_2 |$. Then every point on the interval $[\frac{1}{4}, \frac{1}{2}]$ is a fixed point of F.



In[10] B. N. Sadovskii introduced the notion of a condensing operator from a Banach space B to a Banach space Y. This notion will be applied to set valued functions.

Definition 4.6. Let $A \subset B$ and A bounded in B.

$Q(A) = \{ \varepsilon \mid A \text{ has a finite } \varepsilon\text{-net}\}$.

Then $\quad \chi(A) = \inf Q(A)$.

Definition 4.7. A set valued function $F : B \to Y$ is a condensing operator if for every bounded subset $A \subset B$, $\chi(F(A)) \leqq \chi(A)$ and if $\chi(A) > 0$, then $\chi(F(A)) < \chi(A)$.

Proposition 4.8. Let $F : B \to Y$ be a condensing operator. Then $F(x)$ is compact.

Theorem 4.9. Let T be a closed and bounded convex subset of B. Let $F : T \to T$ be a set valued condensing map such that for each $x \in T$, $F(x)$ is a convex subset of T. (In the case that B is a separable Banach space $F(x)$ may consist of 1 or n acyclic components.) Then F has a fixed point.

Lemma 4.9a. If $\overline{co(A)}$ is the closed convex hull of the set A, then $\chi(\overline{co(A)}) = \chi(A)$.

Proof of the lemma. This is proven in[10]. There it is shown that $Q(A) \subset Q(\overline{co(A)})$.

Lemma 4.9b. Under the conditions of the theorem there is a nonempty compact subset $K \subseteq T$ which satisfies $F(K) = K$.

Proof of Lemma For any $x \varepsilon T$, let $M = \bigcup_{n=0}^{\infty} F^n(x)$ which is bounded since T is. Let $M_1 = F(M)$ and since $M - M_1$ is x or is the empty set, it follows that $\chi(M_1) = \chi(M) = 0$. Thus $\bar{M}$, $\bar{M}_1$ are compact. Now let $K = \{y \mid y$ is a limit point of $M\}$.

Claim: $K = F(K)$.

(a) Let $y \varepsilon K$. Then there is a sequence $\{x_{n_k}\}_{k=1}^{\infty}$ such that $\lim x_{n_k} = y$. Let $z_{n_k-1} \varepsilon F^{n_k-1}(x)$ where $x_{n_k} \varepsilon F(z_{n_k-1})$. Let $\{z_{n_k'-1}\}$ be a convergent subsequence of $\{z_{n_k-1}\}$ where $\lim z_{n_k'-1} = w$. Since F is u.s.c., this implies $y \varepsilon F(w)$ or $y \varepsilon F(K)$.

(b) Let $y \varepsilon F(K)$. Then this implies there is a sequence such that $\lim x_{n_k} = z$ and $y \varepsilon F(z)$. But since $\bar{M}$ is compact and Hausdorff, $F(z)$ = cofinal limit $(F(x_{n_k}))$ = residual limit $F(x_{n_k})$. Thus $y \varepsilon K$.

<u>Lemma</u> 4.9c. There is a compact, convex subset $X \subset T$ such that $X = \overline{co\ f(X)}$.

<u>Proof of Lemma</u> Let $\mathfrak{F}$ be a collection of subsets X' of T such that X' is closed and convex, $F(X') \subset X'$ and $K \subset X'$. Let $\mathfrak{F}$ be ordered by inverse inclusion. By Zorn's Lemma, $\mathfrak{F}$ has a smallest element, X.

Now $F(X) \subset X$ and since X is minimal $\overline{co\ F(X)} = X$.

<u>Proof of Theorem</u> 4.9. By the lemmas $\chi(X) = \chi(\overline{co\ (F(X))}) = \chi(F\ (X))$, it follows that X is a compact convex subset of T so that $F : X \to X$ has a fixed point by a Theorem of Kakutani[5] or in the second case by Theorem 2.28. In the latter case X is a compact ANR.

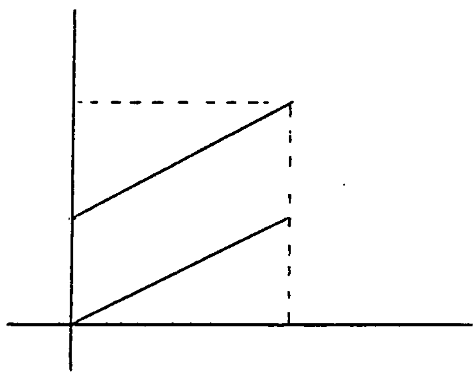
Another difference between single-valued and set valued maps of the simplest type can be brought out in the following theorem.

<u>Theorem</u> 4.10. Let $f : X \to X$ be a single-valued map of a compact metric space, X onto itself, such that for each

$x, x' \in X \ni d(x, x') < \varepsilon$, then $d(f(x), f(x')) \leqq d(x, x')$. Then there can be no pair of points $x_1, x_2 \ni d(f(x_1), f(x_2)) < d(x_1, x_2)$.

Before proving this theorem a counterexample for the set valued case can be given where the Hausdorff metric D replaces d. Thus let $X = [0, 1]$ and $F : X \twoheadrightarrow X$ defined by

$F(x) = \{\frac{1}{2}x, \frac{1}{2} + \frac{1}{2}x\}$. Then
$D(F(x), F(x')) < \frac{1}{2} | x - x' |$ and the graph of F, $\Gamma(F)$ is given by



Proof of Theorem

(a) Suppose there is x_1, x_2 such that $d(f(x_1), f(x_2)) < d(x_1, x_2)$. Then the continuous function
$\rho(x', y') = d(f(x'), f(y'))/d(x', y')$ defined on $X \times X - \Delta$ implies $\exists\ \varepsilon_1 > 0$ such that

$d(f(x'), f(y')) \leqq k\, d(x', y')$ for all

$(x', y') \in \overline{S_{\varepsilon_1}(x_1)} \times \overline{S_{\varepsilon_1}(x_2)}$ where

$0 < k < 1$ and $\overline{S_{\varepsilon_1}(x_i)}$ is a closed ball of radius ε_1 about

x_i, $i = 1,2$.

(b) Choose an $\eta < 1/8 \min \{d(x_1, x_2), \varepsilon, \varepsilon_1\}$.

Let $\{S_\eta(x_i)\}_{i=1}^N$ be an open irreducible cover for X. Also

since diameters do not increase,

$f^r(S_\eta(x_i)) \subset S_\eta(f^r(x_i))$, $i = 1, \ldots, N$ for all r, and

$\{f^r(\overline{S_\eta(x_i)})\}_{i=1}^N$ is a closed cover of X for all r.

(c) <u>Case 1</u>. Suppose $x_i = f(x_i)$, $i = 1,2$.

Let $d(\bar{x}_1, \bar{y}_1) = \min \{d(x', y') \mid (x', y') \in \overline{S_\eta(x_1)} \times \overline{S_\eta(x_2)}\}$

and $d(f(\bar{x}_1), f(\bar{y}_1)) \leqq k\, d(\bar{x}_1, \bar{y}_1)$.

Futhermore $d(f^r(\bar{x}_1), f^r(\bar{y}_1)) \leqq k^r\, d(\bar{x}_1, \bar{y}_1)$ and

$(x_1, x_2) \in f^r(\overline{S_\eta(x_1)}) \times f^r(\overline{S_\eta(x_2)}) \subset \overline{S_\eta(x_1)} \times \overline{S_\eta(x_2)}$. Thus

$$d(x_1, x_2) \leqq d(x_1, f^r(\bar{x}_1)) + d(f^r(\bar{x}_1), f^r(\bar{y}_1)) + d(f^r(\bar{y}_1), x_2)$$

$$\leqq 2\,\eta + k^r\, d(\bar{x}_1, \bar{y}_1)$$

$$< 3\,\eta \quad \text{for large } r$$

$$< d(x_1, x_2). \quad \cancel{\times}$$

<u>Case 2</u>. In general since $\{f^r(\overline{S_\eta(x_i)})\}_{i=1}^{N}$ is a closed cover of X for $r = 1,2,\ldots,$. there is a subsequence $\{r_n\}_{n=1}^{\infty} \subset \{r\}_{r=1}^{\infty}$ such that for a fixed x_{i_0} and all n,

$$x_1 \in f^{r_n}(\overline{S_\eta(x_{i_0})}) \subset S_{\varepsilon_1}(x_1).$$

Similarly $\exists$ a subsequence $\{r_{n_j}\}_{j=1}^{\infty} \subset \{r_n\}_{n=1}^{\infty}$

$$x_1 \times x_2 \in f^{r_{n_j}}(\overline{S_\eta(x_{i_0})}) \times f^{r_{n_j}}(\overline{S_\eta(x_{i_1})}) \subset S_{\varepsilon_1}(x_1) \times S_{\varepsilon_1}(x_2)$$

where $i_1 \neq i_0$ and both are fixed for all j.

Let $(\bar{x}_1, \bar{y}_1) \in f^{r_{n_1}}(\overline{S_\eta(x_{i_0})}) \times f^{r_{n_1}}(\overline{S_\eta(x_{i_1})})$ where $d(\bar{x}_1, \bar{y}_1) =$ min distance between these two compact sets.

Then $d(f(\bar{x}_1), f(\bar{y}_1)) \leqq k\, d(\bar{x}_1, \bar{y}_1)$.

$$d(f^2(\bar{x}_1), f^2(\bar{y}_1)) \leqq k\, d(\bar{x}_1, \bar{y}_1)$$

Let $m_1 = r_{n_2} - r_{n_1} + 1$. Then

$$d(f^m(\bar{x}_1), f^m(\bar{y}_1)) \leqq k^2\, d(\bar{x}_1, \bar{y}_1).$$

Thus as $j \to \infty$, let $m_{j+1} = r_{n_{j+2}} - r_{n_{j+1}} + 1$, and by assumption

$$d(f^{m_j}(\bar{x}_1), f^{m_j}(\bar{y}_1)) \leqq k^{j+1}\, d(\bar{x}_1, \bar{y}_1) \to 0$$

The proof is completed as for Case 1.

On the other hand, there is a type of theorem that is common to both single-valued and set valued functions. The proof for the set valued case is given, and the corresponding one for the single-valued function is as easily proved.

Theorem 4.11. Suppose $F : X \to X$ is a u.s.c. set valued function with a unique fixed point $\bar{x}$. Then for every $\varepsilon > 0$, there is a $\delta > 0$ such that $d(x, \bar{x}) > \varepsilon$ implies $d(F(x), x) > \delta$.

<u>Proof</u> Suppose for some $\varepsilon > 0$, there is a sequence

$\{x_n\}_{n=1}^{\infty}$ so that $d(x,x_n) > \varepsilon$ but $d(x_n, F(x_n)) \leqq \frac{1}{n}$.

Then the sequence $\{x_n\}$ contains a subsequence $\{x_{n_j}\}$ converging to a $\bar{y}$, say.

<u>Claim</u> $\bar{y} \varepsilon F(\bar{y})$. Otherwise let $d(\bar{y}, F(\bar{y})) = \eta > 0$. Take $\eta/2$ neighborhoods U, V about $\bar{y}$, $F(\bar{y})$ respectively such that $F(U)$ V. However, by hypothesis $\lim_{n\to\infty} d(x_n, F(x_n)) = 0$. <u>Contradiction</u>. Thus $\bar{y} \varepsilon F(\bar{y})$.

However, since $\bar{x}$ is unique, $\bar{y} = \bar{x}$; but $\bar{y}$ is outside an ε-neighborhood of $\bar{x}$. This leads to another contradiction and so implies the theorem.

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