NONLINEAR INTEGRAL EQUATIONS AND PRODUCT INTEGRALS

A Dissertation

Presented to

the Faculty of the Department of Mathematics

University of Houston

In Partial Fulfillment

of the Requirements of the Degree

Doctor of Philosophy

by

Alvin John Kay

August, 1975

To Mary Ida

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ABSTRACT

Let S be a linearly ordered set, $\{G,+,0,||\ ||\}$ a complete normed abelian group, H the set of functions from G to G that take 0 to 0, OA and OM classes of functions from SXS to H that are order-additive and order-multiplicative respectively and satisfy a Lipschitz-type condition, E be J. S. Mac Nerney's reversible mapping from OA onto OM. Definitions. If each of K and M is a function from SXS to H, K is differentially equivalent to M means there is a function k from SXS to the real numbers such that for each $\{x,y,P\}$ in SXSXG $_x\sum^y k = 0$ and $||K(x,y)P - M(x,y)P|| \leqq k(x,y)||P||$. Φ and ψ are mappings such that if $\{V,W\}$ is in OAXOM, $\Phi(V)$ and $\psi(W)$ are the sets of all functions that are differentially equivalent to V and W-1 respectively. Theorem. $\psi[E] = \Phi$. This analysis is used to prove existence theorems for product integrals of the form $W(x,c)P = {}_x\Pi^c[1-M]^{-1}[1+K]P$ (where each of K and M is in $\Phi(OA)$) which we show solves a nonlinear integral equation of the form $f(x) = P + {}_x\int^c(Kf[R] + Mf[L])$.

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INTRODUCTION

In his 1963 paper [9], J. S. Mac Nerney obtained product integral solutions of linear integral equations of the form

$$f(x) = f(c) + \int_x^c Vf[R] \quad \text{and} \quad f(x) = f(c) + \int_c^x f[L]V,$$

where the integration is directed along intervals in some linearly ordered set S, with ordering O, the kernel function V for the integral equation is an O-additive function, and the Right and Left integrals are of the subdivision-refinement type.

In 1964, Professor Mac Nerney extended [9] into a nonlinear setting [10] by imbedding the normed ring in [9] into a normed near-ring and adding a Lipschitz-type condition to the kernel function V (see def. 1.6, this dissertation).

In 1966 in [3], B. W. Helton extended [9] by obtaining product integral solutions of linear integral equations of the form

$$(1) \qquad f(x) = f(c) + \int_x^c (Kf[R] + Mf[L]),$$

where each of K and M is differentially equivalent to an O-additive function (see def. 1.7, this dissertation).

This dissertation extends Professor Helton's paper [3] into the nonlinear setting developed by Professor Mac Nerney in [10] thereby also obtaining an extension of [10].

In [9] and [10] Professor Mac Nerney identified two classes, OA and OM, O-additive and O-multiplicative respectively, and established that there is a reversible function E from OA onto OM such that if V is in OA and E(V) = W, then for each {x,y} in SXS

$$W(x,y) = \prod_x^y [1+V] \quad \text{and} \quad V(x,y) = \sum_x^y [W-1]$$

(see def. 1.6 and thm. 1.2, this dissertation).

In [3] Professor Helton expanded the linear OA in [9] into a larger class $OA^0 \cdot OB^0$ (see def. 5.3, this dissertation) and the linear OM in [9] into a related larger class $OM^0 \cdot OB^0$ and showed that $OA^0 \cdot OB^0 = OM^0 \cdot OB^0$.

In Chapter II we expand the nonlinear OA in [10] into a larger class $\Phi(OA)$ and the nonlinear OM in [10] into a related larger class $\psi(OM)$ and show that $\psi[E] = \Phi$ (see def. 1.7 and thm. 2.2, this dissertation).

In Chapter III we use the analysis developed in Chapter II to prove existence theorems and establish identities for non-linear product integrals of the form

(2) $$W(x,c)f(c) = \prod_x^c [1-M]^{-1}[1+K]f(c),$$

where each of K and M is in $\Phi(OA)$. Linear product integrals of this form were introduced by Professor Helton in [3] as the solution to (1).

In Chapter IV we show that (2) solves (1) when they are both taken in the nonlinear setting of [10].

In Chapter V we show that as in [9] and [10] the theory of a seemingly more general equation

$$f(x) = P_1 + \int_x^c (Kf[R] + Mf[L]) + V(x,c)P_2$$

is subsumed in this treatment.

Finally in Chapter VI we show that the product integral solution Reneke gave for nonlinear Stieltjes-Volterra integral equations in [12] can be refined still further using the results presented here. This refinement was noticed by W. L. Gibson (July 1974, oral communication), and is reproduced here with his permission.

Chapters II through V constitute an elaboration of some of the author's results which will appear in the *Pacific Journal of Mathematics* under the title of "Nonlinear Integral Equations and Product Integrals".

CHAPTER I

PRELIMINARY DEFINITIONS AND THEOREMS

Let S denote a nondegenerate set with linear ordering O in the sense that O is a subset of SXS having the following properties:

(i) if each of $\{x,y\}$ and $\{y,z\}$ is in O, then $\{x,z\}$ is in O;

(ii) if $\{x,y\}$ is in SXS, then $\{x,y\}$ or $\{y,x\}$ is in O, and

(iii) if $\{x,y\}$ is in O and $\{y,x\}$ is in O, then y is x.

<u>Definition 1.1</u>: The statement that t is an O-subdivision of the member $\{x,y\}$ of SXS means that t is a finite sequence $\{t_j\}_0^n$ such that $\{t_0,t_n\}$ is $\{x,y\}$ and

(i) if $\{x,y\}$ is in O, then $\{t_{j-1},t_j\}$ is in O for each positive integer j not greater than n, and

(ii) if $\{y,x\}$ is in O, then $\{t_j,t_{j-1}\}$ is in O for each positive integer j not greater than n.

<u>Definition 1.2</u>: The statement that r is a refinement of the O-subdivision t of the member $\{x,y\}$ of SXS means that r is an O-subdivision of $\{x,y\}$ of which t is a subsequence.

<u>Definition 1.3 [9]</u>: OA^+ denotes the class of all functions α from SXS to the nonnegative real numbers such that α is O-additive in the sense that, for each $\{x,z\}$ in SXS, if $\{x,y,z\}$

is an 0-subdivision of $\{x,z\}$, then $\alpha(x,y) + \alpha(y,z) = \alpha(x,z)$. OM^+ denotes the class of all functions μ from SXS to the set of real numbers not less than one such that μ is 0-multiplicative in the sense that, for each $\{x,z\}$ in SXS, if $\{x,y,z\}$ is an 0-subdivision of $\{x,z\}$, then $\mu(x,y)\mu(y,z) = \mu(x,z)$.

Theorem 1.1 [9]: There is a reversible function E^+, from OA^+ onto OM^+, such that the following statements are equivalent:

(i) $\{\alpha,\mu\}$ is in E^+.

(ii) $\mu(x,y) = {}_x\Pi^y[1+\alpha] = \text{L.U.B. } \Pi_t[1+\alpha]$ for all 0-subdivisions t of $\{x,y\}$.

(iii) $\alpha(x,y) = {}_x\sum^y[\mu-1] = \text{G.L.B. } \sum_t[\mu-1]$ for all 0-subdivisions t of $\{x,y\}$.

Definition 1.4 [10]: $\{G,+,0,\|\ \|\}$ denotes a complete normed abelian group and H denotes the class of all functions from G to G to which $\{0,0\}$ belongs with identity function 1.

Definition 1.5 [10]: If g is a function from SXS to G and h is a function from SXS to H and $\{x,y\}$ is in SXS and P is in G, then

(i) the sum integral ${}_x\sum^y g$ denotes a member P_1 of G with the property that, for each positive number c, there is an

0-subdivision s of $\{x,y\}$ such that if $\{t_j\}_0^n$ is a refinement of s, then $||P_1 - \sum_t g|| < c$, where $\sum_t g$ denotes the sum (in G)

$$\sum_1^n g(t_{j-1},t_j) = g(t_0,t_1) + \ldots + g(t_{n-1},t_n), \text{ and}$$

(ii) the product integral ${}_x\Pi^y h[P]$ denotes a member P_2 of G with the property that, for each positive number c, there is an 0-subdivision s of $\{x,y\}$ such that if $\{t_j\}_0^n$ is a refinement of s, then $||P_2 - \Pi_t h[P]|| < c$, where $\Pi_t h[P]$ denotes the image of P under the product (functional composite)

$$\Pi_1^n h(t_{j-1},t_j) = h(t_0,t_1)\cdots h(t_{n-1},t_n).$$

Remark: We adopt the notational convention that $\Pi_1^n A_i$ always means the "left-to-right" continued product $A_1 \cdots A_n$ and that each of $\Pi_1^0 A_i$ and $\Pi_{n+1}^n A_i$ denotes 1.

Definition 1.6 [10]: The class OA consists of all functions V from SXS to H such that

(i) V is 0-additive in the sense that, for each $\{x,z,P\}$ in SXSXG, if $\{x,y,z\}$ is an 0-subdivision of $\{x,z\}$, then

$$V(x,y)P + V(y,z)P = V(x,z)P, \text{ and}$$

(ii) there is a member α in OA^+ such that if $\{x,y\}$ is in SXS and $\{P,Q\}$ is in GXG, then

$$||V(x,y)P - V(x,y)Q|| \leq \alpha(x,y)||P-Q||.$$

The class OM consists of all functions W from SXS to H such that

(i) W is 0-multiplicative in the sense that, for each $\{x,z,P\}$ in SXSXG, if $\{x,y,z\}$ is an 0-subdivision of $\{x,z\}$, then

$$W(x,y)W(y,z)P = W(x,z)P, \text{ and}$$

(ii) there is a member μ of OM^+ such that if $\{x,y\}$ is in SXS and $\{P,Q\}$ is in GXG, then

$$||[W(x,y)-1]P - [W(x,y)-1]Q|| \leq [\mu(x,y)-1]||P-Q||.$$

<u>Lemma 1.1 [10]</u>: If $\{A_i\}_1^n$ is a sequence with values in H and $\{a_i\}_1^n$ is a numerical sequence such that, for each $\{P,Q\}$ in GXG and $i = 1, \ldots ,n$,

$$||(A_i-1)P - (A_i-1)Q|| \leq (a_i-1)||P-Q||,$$

then, for each $\{P,Q\}$ in GXG,

(i) $||(\prod_{i=1}^n A_i-1)P - (\prod_{i=1}^n A_i-1)Q|| \leq (\prod_{i=1}^n a_i-1)||P-Q||,$

and

(ii) $||(\prod_{i=1}^n A_i-1)P - \sum_{i=1}^n [A_i-1]P|| \leq (\prod_{i=1}^n a_i-1 - \sum_{i=1}^n [a_i-1])||P||.$

<u>Lemma 1.2 [10]</u>: If each of $\{A_i\}_1^n$ and $\{B_i\}_1^n$ is a sequence with values in H, and each of $\{a_i\}_1^n$ and $\{c_i\}_1^n$ is a numerical sequence such that for each $\{P,Q\}$ in GXG and $i=1, \ldots ,n$,

$$||A_iP - A_iQ|| \leqq a_i||P-Q|| \quad \text{and} \quad ||B_iP - A_iP|| \leqq c_i||P||,$$

then, for each P in G,

$$||(\prod_{i=1}^{n} B_i)P - (\prod_{i=1}^{n} A_i)P|| \leqq (\prod_{i=1}^{n} [a_i+c_i] - \prod_{i=1}^{n} a_i)||P||.$$

Theorem 1.2 [10]: There is a reversible function E from OA onto OM such that the following are equivalent:

(i) $\{V,W\}$ belongs to E.

(ii) W is in OM and V is the function defined by the condition that, for each $\{x,y,P\}$ in SXSXG, $V(x,y)P = {}_x\sum^y[W-1]P$.

(iii) V is in OA and W is the function defined by the condition that, for each $\{x,y,P\}$ in SXSXG, $W(x,y)P = {}_x\prod^y[1+V]P$.

(iv) $\{V,W\}$ is in OAXOM and there is a member $\{\alpha,\mu\}$ in E^+ such that for each $\{x,y,P\}$ in SXSXG,

$$||W(x,y)P - P - V(x,y)P|| \leqq [\mu(x,y) - 1 - \alpha(x,y)]||P||.$$

Definition 1.7: If each of K and M is a function from SXS to H, the statement that K is differentially equivalent to M means there is a function k from SXS to the real numbers such that $||K(x,y)P - M(x,y)P|| \leqq k(x,y)||P||$ and ${}_x\sum^y k = 0$ for each $\{x,y,P\}$ in SXSXG. Φ denotes a function from OA such that if V is in OA, $\Phi(V)$ is the set to which K belongs only in case K is differentially equivalent to V. ψ denotes a function from OM such that if W is in OM, $\psi(W)$ is the set to which K belongs only in case K is differentially equivalent to W-1.

Remark: We adopt the notational convention that $\Phi(OA)$ is the set to which K belongs only in case there is a V in OA such that K is in $\Phi(V)$.

Remark: The concept of differential equivalence between real valued functions was introduced in 1930 by A. Kolmogoroff in [7], where he proved the following theorem.

Theorem 1.3 [7]: If K is a function from SXS to the real numbers such that ${}_x\sum^y K$ exists for each $\{x,y\}$ in SXS, and $|K(x,y) - {}_x\sum^y K| = k(x,y)$, then ${}_a\sum^b k = 0$ for all $\{a,b\}$ in SXS.

Theorem 1.4 [3]: If K is a function from SXS to the real numbers such that ${}_x\Pi^y[1+K]$ exists for each $\{x,y\}$ in SXS, and $|1+K(x,y)-{}_x\Pi^y[1+K]| = k(x,y)$, then ${}_a\sum^b k = 0$ for each $\{a,b\}$ in SXS.

Remark: In [3] Professor Helton proved that both Theorems 1.3 and 1.4 hold in any finite dimensional normed ring N. This does not remain true in an infinite dimensional ring. W. D. L. Appling in [1] showed that if N is infinite dimensional, there is a function K from SXS to N for which Theorems 1.3 and 1.4 are false.

Definition 1.8: If K is a function from SXS into G, the statement that K is of bounded variation on each O-interval of S means if $\{x,y\}$ is in SXS, then there is a positive number b and an O-subdivision s of $\{x,y\}$ such that if $\{t_j\}_0^n$ is a refinement of s, then $\sum_1^n ||K(t_{j-1},t_j)|| < b$. If f is a function from S to G, the statement that f is of bounded variation on each O-interval of S means df (i.e., $df(a,b) = f(b) - f(a)$ for each $\{a,b\}$ in SXS) is of bounded variation on each O-interval of S.

Theorem 1.5 [3]: If $\{x,y\}$ is in SXS and each of K and M is a function from SXS to the real numbers that is of bounded variation on each O-interval of S, there is a number b such that if $\{t_j\}_0^n$ is an O-subdivision of $\{x,y\}$, then

$$\left|\prod_{j=1}^{n}[1+K(t_{j-1},t_j)] - \prod_{j=1}^{n}[1+M(t_{j-1},t_j)]\right|$$

$$\leq b\sum_{j=1}^{n}|K(t_{j-1},t_j) - M(t_{j-1},t_j)|.$$

Theorem 1.6 [10]: If $\{c,P\}$ is in SXG and W is in OM, then $W(\ ,c)P$ is of bounded variation on each O-interval of S.

CHAPTER II

$\psi[E] = \Phi$

In this chapter we prove two theorems that will be used in the proofs of later theorems. In the first theorem we prove that if K is in $\psi(OM)$, then the sum and product integrals of K exist, and in the second theorem we prove that if $\{V,W\}$ is in E, the collection of functions which are differentially equivalent to V is the same as the collection of functions which are differentially equivalent to W-1.

<u>Theorem 2.1</u>: If $\{V,W\}$ is in E and K is in $\psi(W)$, then

(1) $W(x,y)P = {}_x\Pi^y[1+V]P = {}_x\Pi^y[1+K]P$ for every $\{x,y,P\}$ in SXSXG, and

(2) $V(x,y)P = {}_x\sum^y[W-1]P = {}_x\sum^y KP$ for every $\{x,y,P\}$ in SXSXG.

<u>Proof</u>: (1) Let W be in OM and K in $\psi(W)$, k be a function from SXS to the real numbers such that for $\{x,y,P\}$ in SXSXG

$||K(x,y)P - [W(x,y)-1]P|| \leqq k(x,y)||P||$ and ${}_x\sum^y k = 0$, and

μ be a member of OM^+ such that for each $\{x,y\}$ in SXS and $\{P,Q\}$ in GXG

$||[W(x,y)-1]P - [W(x,y)-1]Q|| \leqq [\mu(x,y)-1]||P-Q||$.

By Lemma 1.2 and Theorem 1.5 we have that, for each $\{x,y,P\}$ in SXSXG, there is a number b and an O-subdivision s of $\{x,y\}$ such that if t is a refinement of s, then

$$||\Pi_t[1+K]P - \Pi_t WP|| \leqq \{\Pi_t[\mu+k] - \Pi_t\mu\}||P|| \leqq b\sum_t k||P||.$$

Since ${}_x\sum^y k = 0$ the proof of (1) is complete.

(2) For each O-subdivision t of $\{x,y\}$ in SXS

$$||V(x,y)P - \sum_t KP|| \leqq \sum_t||[W-1]P - KP|| + ||\sum_t[W-1]P - V(x,y)P||$$
$$\leqq \{\sum_t k + \sum_t[\mu-1] - \alpha(x,y)\}||P||.$$

Since ${}_x\sum^y k + {}_x\sum^y[\mu-1] - \alpha(x,y) = 0$ the proof is complete.

Remark: The proof of the following theorem is similar to the proof of Theorem 3.4 [3, p. 301] of which this theorem is an extension.

Theorem 2.2: $\psi[E] = \Phi$.

Proof: Part I. Let V be in OA and E(V) = W and K be in $\psi(W)$; there is a μ in OM^+ such that for each $\{x,y\}$ in SXS and $\{P,Q\}$ in GXG

$||[W(x,y)-1]P - [W(x,y)-1]Q|| \leqq [\mu(x,y)-1]||P-Q||$; and there is a function k from SXS to the real numbers such that ${}_x\sum^y k = 0$

and $||[1+K(x,y)]P - W(x,y)P|| \leqq k(x,y)||P||$. By Theorem 1.2

$$\{[\mu(x,y)-1] - {}_x\sum^y[\mu-1]\}||P|| \geqq ||[W(x,y)-1]P - {}_x\sum^y[W-1]P||$$

$$\geqq ||K(x,y)P - {}_x\sum^y[W-1]P|| - ||[1+K(x,y)]P - W(x,y)P||; \text{ hence}$$

$$||K(x,y)P - V(x,y)P|| \leqq \{[\mu(x,y)-1] - {}_x\sum^y[\mu-1] + k(x,y)\}||P||$$

so K is in $\Phi(V)$.

Part II. Let K be in $\Phi(V)$; there is an α in OA^+ such that if $\{x,y\}$ is in SXS and $\{P,Q\}$ is in GXG, then $||V(x,y)P - V(x,y)Q|| \leqq \alpha(x,y)||P-Q||$ and there is a function h from SXS to the real numbers such that $||V(x,y)P - K(x,y)P|| \leqq h(x,y)||P||$ and ${}_x\sum^y h = 0$. By Theorem 1.2, $||[1+K(x,y)]P - W(x,y)P||$

$$\leqq ||[1+V(x,y)]P - {}_x\Pi^y[1+V]P|| + ||V(x,y)P - K(x,y)P||$$

$\leqq \{{}_x\Pi^y[1+\alpha] - \alpha(x,y)-1 + h(x,y)\}||P||$; therefore K is in $\psi(E(V))$.

CHAPTER III

EXISTENCE THEOREMS

In this chapter we will prove that if each of K and M is in $\Phi(OA)$ and $[1-M(x,y)]^{-1}$ exists and is bounded sufficiently there is a member V of OA such that $[1-M]^{-1}[1+K] - 1$ is in $\Phi(V)$; hence, $W(x,y)P = {}_x\Pi^y[1-M]^{-1}[1+K]P$ exists for every $\{x,y,P\}$ in SXSXG. This extends existence theorems proven by J. S. Mac Nerney [9] [10], B. W. Helton [4], J. V. Herod [6], and J. C. Helton [5]. Also we give an extension of a theorem of D. L. Lovelady's [8] which provides some new identities for W.

<u>Theorem 3.0</u>: If β is in OA^+ and $\beta(x,y) < 1$ for each $\{x,y\}$ in SXS, then

(1) $\quad \alpha(x,y) = {}_x\Sigma^y([1-\beta]^{-1} - 1)$ exists for each $\{x,y\}$ in SXS and α is in OA^+;

(2) $\quad \mu(x,y) = {}_x\Pi^y[1-\beta]^{-1}$ exists for each $\{x,y\}$ in SXS and μ is in OM^+; and

(3) $\quad \{\alpha,\mu\}$ is in E^+.

<u>Proof</u>: Let $\varepsilon = [1-\beta]^{-1} - 1$; if $\{r,s,t\}$ is an 0-subdivision of $\{r,t\}$ in SXS,

$$\varepsilon(r,t) \geqq \varepsilon(r,s) + \varepsilon(s,t) \geqq 0.$$

Hence, $\alpha(x,y) = {}_x\sum^y \varepsilon = \text{G.L.B. } \sum_t \varepsilon \geqq 0$ for all 0-subdivisions t of $\{x,y\}$ in SXS. ε is in $\Phi(\alpha)$ and from Theorem 2.2 ε is in $\psi(E(\alpha))$. Hence, from Theorem 2.1

$$\mu(x,y) = {}_x\Pi^y[1+\varepsilon] = {}_x\Pi^y[1+\alpha] \text{ for all } \{x,y\} \text{ in SXS, and}$$

the proof is complete.

<u>Lemma 3.1 [11]</u>: If T is in H and $0 < t < 1$ and

$||TP-TQ|| \leqq t||P-Q||$ for all $\{P,Q\}$ in GXG, then

$(1-T)^{-1}$ is in H, $(1-T)^{-1} = 1 + T(1-T)^{-1}$, and for each

such $\{P,Q\}$ $||(1-T)^{-1}P - (1-T)^{-1}Q|| \leqq (1-t)^{-1}||P-Q||$.

<u>Remark</u>: These and closely related inequalities are used in the sequel, usually without explicit reference.

<u>Theorem 3.1</u>: If V_1 is in OA and α_1 is in OA^+ such that for $\{x,y\}$ in SXS and $\{P,Q\}$ in GXG, $\alpha_1(x,y) < 1$, and

$$||V_1(x,y)P - V_1(x,y)Q|| \leqq \alpha_1(x,y)||P-Q||, \text{ then}$$

(1) $V(x,y)P = {}_x\sum^y\{[1-V_1]^{-1} - 1\}P$ exists for each $\{x,y,P\}$ in SXSXG and V is in OA;

(2) $W(x,y)P = {}_x\Pi^y[1-V_1]^{-1}P$ exists for each $\{x,y,P\}$ in SXSXG and W is in OM; and

(3) $\{V,W\}$ is in E.

Proof: (1) Note that $[1-V_1]^{-1} - 1 = V_1[1-V_1]^{-1}$

and if $\{x,y,P\}$ is in SXSXG and $\{x,s,t,y\}$ is an O-subdivision of $\{x,y\}$, then

$$||[1-V_1(x,y)]^{-1}P - [1-V_1(s,t)]^{-1}P||$$

$$= ||V_1(x,y)[1-V_1(x,y)]^{-1}P - V_1(s,t)[1-V_1(s,t)]^{-1}P$$

$$\pm V_1(x,y)[1-V_1(s,t)]^{-1}P||$$

$$\leq \alpha_1(x,y)||[1-V_1(x,y)]^{-1}P - [1-V_1(s,t)]^{-1}P||$$

$$+ [\alpha_1(x,y) - \alpha_1(s,t)][1-\alpha_1(s,t)]^{-1}||P||$$

$$\leq \{[1-\alpha_1(x,y)]^{-1} - [1-\alpha_1(s,t)]^{-1}\}||P||.$$

For each $\{x,y,P\}$ in SXSXG and 0-subdivision $\{t_j\}_0^n$ of $\{x,y\}$

$$||V_1(x,y)[1-V_1(x,y)]^{-1}P - \sum_t V_1[1-V_1]^{-1}P||$$

$$= ||(\sum_t V_1)[1-V_1(x,y)]^{-1}P - \sum_t V_1[1-V_1]^{-1}P||$$

$$\leq \sum_1^n \alpha_1(t_{j-1},t_j)\{[1-\alpha_1(x,y)]^{-1} - [1-\alpha_1(t_{j-1},t_j)]^{-1}\}||P||$$

$$= \{[1-\alpha_1(x,y)]^{-1} - 1 - \sum_t([1-\alpha_1]^{-1} - 1)\}||P||.$$

It follows that if s is a refinement of t

$$||\sum_s([1-V_1]^{-1} - 1)P - \sum_t([1-V_1]^{-1} - 1)P||$$

$$\leq \{\sum_s([1-\alpha_1]^{-1} - 1) - \sum_t([1-\alpha_1]^{-1} - 1)\}||P||.$$

Hence, by the completeness of $\{G,+,0,||\ ||\}$ and Theorem 3.0 $V(x,y)P = {}_x\sum^y\{[1-V_1]^{-1} - 1\}P$ exists. For each $\{x,y\}$ in SXS and $\{P,Q\}$ in GXG $||V(x,y)P - V(x,y)Q|| \leq \alpha(x,y)||P-Q||$ where α is defined as in Theorem 3.0. Therefore V is in OA and, with ε as in the proof of Theorem 3.0, considerations of $\varepsilon-\alpha$ may be seen to show that $[1-V_1]^{-1} - 1$ is in $\Phi(V)$.

Theorem 3.2: If $\{\alpha_j\}_1^n$ is a sequence each value of which is in OA^+, then $\mu(x,y) = {}_x\Pi^y\{\Pi_1^n[1+\alpha_j]\}$ exists and μ is in OM^+. Furthermore $\Pi_1^n[1+\alpha_j] - 1$ is in $\psi(\mu)$.

Proof: Let $\gamma = \Pi_1^n[1+\alpha_j]$ and $\{x,z\}$ be in SXS. If $\{x,y,z\}$ is an 0-subdivision of $\{x,z\}$, then

$$1 \leqq \gamma(x,z) \leqq \gamma(x,y)\gamma(y,z) \leqq \mathrm{Exp}[\textstyle\sum_1^n \alpha_j(x,y)].$$

Hence, $\mu(x,y) = {}_x\Pi^y\gamma = \text{L.U.B. } \Pi_t\gamma$ for all 0-subdivisions t of $\{x,y\}$ in SXS. By Theorem 1.4 $\gamma-1$ is in $\psi(\mu)$.

Remark: The following theorem is an extension of D. L. Lovelady's Theorem 6[8, p.425].

Theorem 3.3: If $\{V_i\}_1^n$ is a sequence with values in OA and $\{K_i\}_1^n$ is a sequence such that for each positive integer i not greater than n, K_i is in $\Phi(V_i)$, then

$$W(x,y)P = {}_x\Pi^y\{\Pi_1^n[1+V_i]\}P = {}_x\Pi^y\{\Pi_1^n[1+K_i]\}P \quad \text{for each}$$

$\{x,y,P\}$ in SXSXG and W is in OM.

Furthermore $\Pi_1^n[1+K_i] - 1$ is in $\psi(W)$.

Proof: It will suffice to show that if each of V_1 and V_2 is in OA and K_1 is in $\Phi(V_1)$ and K_2 is in $\Phi(V_2)$, then

$W(x,y)P = {}_x\Pi^y[1+V_1][1+V_2]P = {}_x\Pi^y[1+K_1][1+K_2]P$ for each $\{x,y,P\}$ in SXSXG and $[1+K_1][1+K_2] - 1$ is in $\psi(W)$.

Let $\{\alpha_i\}_1^2$ and $\{h_i\}_1^2$ be sequences with values in OA^+ and $[0,\infty)^{SXS}$ respectively such that for $i = 1, 2$, $\{x,y\}$ in SXS, and $\{P,Q\}$ in GXG, ${}_x\sum^y k_i = 0$,

$$||K_i(x,y)P - V_i(x,y)P|| \leqq k_i(x,y)||P||, \text{ and}$$

$$||V_i(x,y)P - V_i(x,y)Q|| \leqq \alpha_i(x,y)||P-Q||.$$

Let $\{x,y\}$ be in SXS; then for each 0-subdivision $\{t_j\}_0^n$ of $\{x,y\}$

$$||[1+V_1(x,y)][1+V_2(x,y)]P - P - \textstyle\sum_t([1+V_1][1+V_2] - 1)P||$$

$$= ||(\textstyle\sum_t V_1)[1+V_2(x,y)]P - \sum_t V_1[1+V_2]P||$$

$$\leqq \textstyle\sum_1^n \alpha_1(t_{j-1},t_j)||[1+V_2(x,y)]P - [1+V_2(t_{j-1},t_j)]P||$$

$$\leqq \textstyle\sum_1^n \alpha_1(t_{j-1},t_j)\{\alpha_2(x,y) - \alpha_2(t_{j-1},t_j)\}||P||$$

$$= |[1+\alpha_1(x,y)][1+\alpha_2(x,y)] - 1 - \textstyle\sum_t\{[1+\alpha_1][1+\alpha_2] - 1\}|\ ||P||.$$

It follows that if s is a refinement of t

$$||\sum_s([1+V_1][1+V_2] - 1)P - \sum_t([1+V_1][1+V_2] - 1)P||$$

$$\leqq \{\sum_s([1+\alpha_1][1+\alpha_2] - 1) - \sum_t([1+\alpha_1][1+\alpha_2] - 1)\}||P||.$$

Hence, by the completeness of $\{G,+,0,||\ ||\}$ and Theorem 3.2,

$V(x,y)P = {}_x\sum^y([1+V_1][1+V_2] - 1)P$ exists. For each $\{x,y\}$ in SXS and $\{P,Q\}$ in GXG

$||V(x,y)P = V(x,y)Q|| \leqq \alpha(x,y)||P-Q||$ where $\alpha(x,y) = {}_x\sum^y([1+\alpha_1][1+\alpha_2] - 1)$. Therefore V is in OA and considerations of $[1+\alpha_1][1+\alpha_2] - 1 - \alpha$ may be seen to show that $[1+V_1][1+V_2] - 1$ is in $\Phi(V)$. Let $W = E(V)$; from Theorems 2.1 and 2.2, $[1+V_1][1+V_2] - 1$ is in $\psi(W)$.

For each $\{x,y,P\}$ in SXSXG and positive integer n

$${}_x\Pi^y(\Pi_1^n[1+K_i])P = {}_x\Pi^y[1+\tilde{V}][1+K_n]P$$

where $\tilde{W}(x,y)P = {}_x\Pi^y(\Pi_1^{n-1}[1+K_i])P$ and $\tilde{V} = E^{-1}(\tilde{W})$.

Hence, considerations of ${}_x\Pi^y[1+\tilde{V}][1+K_n]P$ and the assertion we established in the first part of this argument can be seen to establish the induction for this theorem.

Theorem 3.4: If each of V_1 and V_2 is in OA, and each of α_1 and α_2 is in OA^+ such that for each {x,y} in SXS and {P,Q} in GXG, $\alpha_2(x,y) < 1$, and for i = 1, 2

$$||V_i(x,y)P - V_i(x,y)Q|| \leqq \alpha_i(x,y)||P-Q||, \text{ then}$$

(1) $V(x,y)P = {}_x\sum^y([1-V_2]^{-1}[1+V_1] - 1)P$ exists for each {x,y,P} in SXSXG, and V is in OA;

(2) $W(x,y)P = {}_x\Pi^y[1-V_2]^{-1}[1+V_1]P$ exists for each {x,y,P} in SXSXG, and W is in OM; and

(3) {V,W} is in E.

Proof: This theorem is a corollary to Theorems 3.1 and 3.3.

Remark: The proof in the Pacific Journal of Mathematics paper yet to appear was independent of Theorem 3.3.

Theorem 3.5: Suppose

(1) each of V_1 and V_2 is in OA, and each of α_1 and α_2 is in OA^+ such that for each {x,y} in SXS and {P,Q} in GXG and for i = 1, 2

$$||V_i(x,y)P - V_i(x,y)Q|| \leqq \alpha_i(x,y)||P-Q||;$$

(2) K is in $\Phi(V_1)$ and M is in $\Phi(V_2)$ and each of h and k is a function from SXS to the real numbers such that for each $\{x,y,P\}$ in $SXSXG$, ${}_x\sum^y k = 0$, ${}_x\sum^y h = 0$,

$$||K(x,y)P - V_1(x,y)P|| \leqq k(x,y)||P|| \text{ and}$$

$$||M(x,y)P - V_2(x,y)P|| \leqq h(x,y)||P||;$$

(3) there is a number $a < 1$ such that for each $\{x,y\}$ in SXS $\alpha_2(x,y) + h(x,y) \leqq a$; and

(4) $\beta = [1-\alpha_2]^{-1}[1+\alpha_1]$ and $\gamma = [1-\alpha_2-h]^{-1}[1+\alpha_1+k]$.

Conclusion:

(1) $||[1-V_2(x,y)]^{-1}[1+V_1(x,y)]P - [1-V_2(x,y)]^{-1}[1+V_1(x,y)]Q||$

$\leqq \beta(x,y)||P-Q||$ for every $\{x,y\}$ in SXS and $\{P,Q\}$ in GXG;

(2) $||[1-M(x,y)]^{-1}[1+K(x,y)]P - [1-V_2(x,y)]^{-1}[1+V_1(x,y)]P||$

$\leqq [\gamma(x,y) - \beta(x,y)]||P||$ for every $\{x,y,P\}$ in $SXSXG$;

and

(3) ${}_x\Pi^y[1-M]^{-1}[1+K]P = {}_x\Pi^y[1-V_2]^{-1}[1+V_1]P$ for every $\{x,y,P\}$

in $SXSXG$.

Proof: Let $\{x,y\}$ be in SXS, $\{P,Q\}$ be in GXG and

$A = [1-V_2(x,y)]^{-1}[1+V_1(x,y)]$. First note that

$$A = 1 + V_1(x,y) + V_2(x,y)A.$$

$$||AP - AQ|| \leqq [1+\alpha_1(x,y)]||P-Q|| + \alpha_2(x,y)||AP - AQ||$$

and assertion (1) follows. Let $B = [1-M(x,y)]^{-1}[1+K(x,y)]$;

$$\begin{aligned}
||BP - AP|| &= ||[1+K(x,y)+M(x,y)B]P - [1+V_1(x,y)+V_2(x,y)A]P \\
&\qquad \pm V_2(x,y)BP|| \\
&\leqq k(x,y)||P|| + h(x,y)||BP|| + \alpha_2(x,y)||BP - AP|| \\
&\leqq k(x,y)||P|| + h(x,y)||AP|| + [h(x,y) + \alpha_2(x,y)]||BP - AP|| \\
&\leqq k(x,y)||P|| + h(x,y)[1+\alpha_1(x,y)][1-\alpha_2(x,y)]^{-1}||P|| \\
&\qquad + [h(x,y) + \alpha_2(x,y)]||BP - AP||
\end{aligned}$$

which, except for minor algebraic manipulation, establishes (2).

For each 0-subdivision t of $\{x,y\}$ it follows from Lemma 1.2 that

$$||\Pi_t[1-M]^{-1}[1+K]P - \Pi_t[1-V_2]^{-1}[1+V_1]P|| \leqq (\Pi_t\gamma - \Pi_t\beta)||P||.$$

By Theorem 1.5 and hypothesis (3) of this theorem, there is a number b such that

$$\Pi_t\gamma - \Pi_t\beta \leqq b\textstyle\sum_t(\gamma-\beta) \leqq b^2\sum_t k + b^3[1+\alpha_1(x,y)]\sum_t h.$$

Since ${}_x\sum^y k = 0$ and ${}_x\sum^y h = 0$ the proof is complete.

Theorem 3.6: If each of V_1 and V_2 is in OA, and each of α_1 and α_2 is in OA^+ such that for each $\{x,y\}$ in SXS and $\{P,Q\}$ in GXG, $\alpha_2(x,y) < 1$, and for $i = 1, 2$

$$||V_i(x,y)P - V_i(x,y)Q|| \leqq \alpha_i(x,y)||P||,$$

(H) ${}_x\sum^y\alpha_1\alpha_2 = 0$ and ${}_x\sum^y\alpha_2^2 = 0$, then

(C) for each $\{x,y,P\}$ in SXSXG

$${}_x\Pi^y[1-V_2]^{-1}[1+V_1]P = {}_x\Pi^y[1+V_1+V_2]P.$$

Proof: $||[1-V_2(x,y)]^{-1}[1+V_1(x,y)]P - P - V_1(x,y)P - V_2(x,y)P||$

$\leqq ||V_2(x,y)[1-V_2(x,y)]^{-1}[1+V_1(x,y)]P - V_2(x,y)P||$

$\leqq \alpha_2(x,y)||V_1(x,y)P + V_2(x,y)[1-V_2(x,y)]^{-1}[1+V_1(x,y)]P||$

$\leqq \alpha_2(x,y)\alpha_1(x,y)||P|| + \alpha_2(x,y)^2[1-\alpha_2(x,y)]^{-1}[1+\alpha_1(x,y)]||P||.$

Since ${}_x\sum^y(\alpha_2\alpha_1 + \alpha_2^2[1-\alpha_2]^{-1}[1+\alpha_1]) = 0$, $[1-V_2]^{-1}[1+V_1]$ is differentially equivalent to $1 + V_1 + V_2$. Hence this theorem is a corollary to Theorems 3.3 and 3.4.

Remark: The inequalities in the proof of the preceding theorem are sharp in the sense that if each of V_1 and V_2 is in OA^+, then (H) and (C) are equivalent.

CHAPTER IV

THE INTEGRAL EQUATIONS

Definition 4.1 [2]: R and L each denotes a function from SXS into S such that $R(x,y) = y$ and $L(x,y) = x$ for each $\{x,y\}$ in SXS.

Remark: This notation due to W. L. Gibson in [2] provides a more precise notation for left and right integral process than that used before. Hence

$$(RL)\int_x^y (Kf + Mf) \quad \text{becomes} \quad \int_x^y (Kf[R] + Mf[L]).$$

Definition 4.2[10]: $F(c,P)$ denotes the class of all functions f from S to G such that $f(c) = P$ and there is a member β of OA^+ such that $||f(y) - f(x)|| \leqq \beta(x,y)$ for each $\{x,y\}$ in SXS (i.e., f is of bounded variation on each O-interval of S).

Remark: The construction of the proof of the next lemma is similar to that of Lemma 2.2 [10, p. 629].

Lemma 4.1: Suppose

(1) each of V_1 and V_2 is in OA and α_1 and α_2 is in OA^+ such that for each $\{x,y\}$ in SXS and $\{P,Q\}$ in GXG and $i = 1, 2$

$$||V_i(x,y)P - V_i(x,y)Q|| \leqq \alpha_i(x,y)||P-Q||;$$

(2) f is in F(c,P); and

(3) for each {x,y} in SXS

$$C(x,y) = {}_x\int^y (V_1 f[R] + V_2 f[L]) - V_1(x,y)f(y) - V_2(x,y)f(x).$$

Conclusion: For each {x,y} in SXS ${}_x\sum^y ||C|| = 0$.

<u>Proof</u>: Let β be in OA^+ such that $||df|| \leqq \beta$, {x,y} be in SXS such that {x,y,c} is an O-subdivision of {x,c} where c is in S, and $\{t_i\}_0^n$ be an O-subdivision of {x,y}; then

$$||\textstyle\sum_t (V_1 f[R] + V_2 f[L]) - V_1(x,y)f(y) - V_2(x,y)f(x)||$$

$$\leqq ||\textstyle\sum_t V_2 f[L] - \sum_t V_2 f(x)|| + ||\sum_t V_1 f[R] - \sum_t V_1 f(y)||$$

$$\leqq \textstyle\sum_t ||V_2 f[L] - V_2 f(x)|| + \sum_t ||V_1 f[R] - V_1 f(y)||$$

$$\leqq \textstyle\sum_t \alpha_2 \beta[x,\ ][L] + \sum_t \alpha_1 \beta[\ ,y][R]$$

$$= \alpha_2(x,y)\beta(x,c) - \textstyle\sum_t \alpha_2 \beta[\ ,c][L]$$

$$+ \textstyle\sum_t \alpha_1 \beta[\ ,c][R] - \alpha_1(x,y)\beta(y,c).$$

Let $h(x,y) = \alpha_2(x,y)\beta(x,c) - {}_x\sum^y \alpha_2 \beta[\ ,c][L]$

$$+ {}_x\textstyle\sum^y \alpha_1 \beta[\ ,c][R] - \alpha_1(x,y)\beta(y,c).$$

Since $_x\sum^y \alpha_2 \beta[\ ,c][L]$ and $_x\sum^y \alpha_1 \beta[\ ,c][R]$ exist for every $\{x,y\}$ in SXS (as in [10, p.629]) and each is real valued, then by Theorem 1.3 $_a\sum^b h = 0$ for all $\{a,b\}$ in SXS and the proof is complete.

Lemma 4.2: Suppose

(1) each of V_1 and V_2 is in OA, and each of α_1 and α_2 is in OA^+ such that for each $\{x,y\}$ in SXS and $\{P,Q\}$ in GXG, $\alpha_2(x,y) < 1$, and for $i = 1, 2$

$$||V_i(x,y)P - V_i(x,y)Q|| \leqq \alpha_i(x,y)||P-Q||;$$

(2) C is a function from SXS to G such that for each $\{x,y\}$ in SXS $_x\sum^y ||C|| = 0$; and

(3) $A(x,y)P = [1-V_2(x,y)]^{-1}([1+V_1(x,y)]P + C(x,y))$ for each $\{x,y,P\}$ in SXSXG.

Conclusion: $_x\Pi^y [1-V_2]^{-1}[1+V_1]P = {}_x\Pi^y AP$ for every $\{x,y,P\}$ in SXSXG.

Proof: This lemma is a corollary to Theorems 3.3 and 3.4.

<u>Theorem 4.1</u>: Suppose

(1) each of V_1 and V_2 is in OA,

(2) K is in $\Phi(V_1)$ and M is in $\Phi(V_2)$,

(3) f is a function from S to G that is bounded on each 0-interval of S, and

(4) for each $\{x,y\}$ in SXS, ${}_x\int^y (V_1 f[R] + V_2 f[L])$ exists.

Conclusion: For each $\{x,y\}$ in SXS

$$ {}_x\int^y (Kf[R] + Mf[L]) = {}_x\int^y (V_1 f[R] + V_2 f[L]). $$

<u>Proof</u>: Let each of h and k be a function from SXS to the real numbers such that for each $\{x,y,P\}$ in SXSXG, ${}_x\sum^y k = 0$,

$$ ||K(x,y)P - V_1(x,y)P|| \leqq k(x,y)||P||, \quad {}_x\sum^y h = 0, \text{ and} $$

$$ ||M(x,y)P - V_2(x,y)P|| \leqq h(x,y)||P||. $$

Pick $\{x,y\}$ in SXS and a number b such that if $\{x,z,y\}$ is an 0-subdivision of $\{x,y\}$ then $||f(z)|| \leqq b$. Let $\{t_i\}_0^n$ be an 0-subdivision of $\{x,y\}$; then

$$ ||\textstyle\sum_1^n [K(t_{i-1},t_i)f(t_i) + M(t_{i-1},t_i)f(t_{i-1})] $$

$$ - \textstyle\sum_1^n [V_1(t_{i-1},t_i)f(t_i) + V_2(t_{i-1},t_i)f(t_{i-1})]|| $$

$$ \leqq \textstyle\sum_1^n ||K(t_{i-1},t_i)f(t_i) - V_1(t_{i-1},t_i)f(t_i)|| $$

$$+ \sum_1^n ||M(t_{i-1},t_i)f(t_{i-1}) - V_2(t_{i-1},t_i)f(t_{i-1})||$$

$$\leq \sum_1^n k(t_{i-1},t_i)||f(t_i)|| + \sum_1^n h(t_{i-1},t_i)||f(t_{i-1})||$$

$$\leq b[\sum_1^n k(t_{i-1},t_i) + \sum_1^n h(t_{i-1},t_i)].$$

Remark: The construction of the proof of the next theorem is similar to that of Theorem 5.1 [3, p.310].

Theorem 4.2: Suppose

(1) each of V_1 and V_2 is in OA and each of α_1 and α_2 is in OA^+ such that for each $\{x,y\}$ in SXS, $\{Q_1,Q_2\}$ in GXG, and $i = 1, 2$

$$||V_i(x,y)Q_1 - V_i(x,y)Q_2|| \leq \alpha_i(x,y)||Q_1-Q_2||;$$

(2) K is in $\Phi(V_1)$ and M is in $\Phi(V_2)$ and each of h and k is a function from SXS to the real numbers such that for each $\{x,y,Q\}$ in SXSXG, ${}_x\sum^y k = 0$, ${}_x\sum^y h = 0$,

$$||V_1(x,y)Q - K(x,y)Q|| \leq k(x,y)||Q||, \text{ and}$$

$$||V_2(x,y)Q - M(x,y)Q|| \leq h(x,y)||Q||;$$

(3) there is a number $a < 1$ such that $\alpha_2(x,y) + h(x,y) \leq a$ for all $\{x,y\}$ in SXS; and

(4) $\{c,P\}$ is in SXG.

Conclusion: The following statements are equivalent:

(1) f is in F(c,P) and $f(x) = P + {}_x\int^c (Kf[R] + Mf[L])$

for each x in S;

(2) $f(x) = {}_x\Pi^c [1-M]^{-1}[1+K]P$ for each x in S; and

(3) if for each $\{a,b,Q\}$ in SXSXG,

$$V(a,b)Q = {}_a\sum^b ([1-M]^{-1}[1+K] - 1)Q, \text{ then}$$

$$f(x) = {}_x\Pi^c [1+V]P \quad \text{for each } x \text{ in } S.$$

<u>Proof</u>: (1 → 2): If $\{x,y,c\}$ is an O-subdivision of $\{x,c\}$, then by Theorem 4.1

$$f(x) = P + {}_x\int^c (Kf[R] + Mf[L]) = P + {}_x\int^c (V_1f[R] + V_2f[L])$$

$$= f(y) + {}_x\int^y (V_1f[R] + V_2f[L]).$$

Hence if $\{t_j\}_0^n$ is an O-subdivision of $\{x,c\}$ and j is an integer in [1,n], then

$$f(t_{j-1}) - f(t_j) = {}_{t_{j-1}}\int^{t_j} (V_1f[R] + V_2f[L]) \quad \text{and}$$

$$f(t_{j-1}) = f(t_j) + V_1(t_{j-1},t_j) + V_2(t_{j-1},t_j)f(t_{j-1})$$
$$+ C(t_{j-1},t_j)$$

where $C(t_{j-1},t_j) = {}_{t_{j-1}}\int^{t_j}(V_1f[R] + V_2f[L]) - V_1(t_{j-1},t_j)f(t_j)$
$$- V_2(t_{j-1},t_j)f(t_{j-1}).$$

$$[1-V_2(t_{j-1},t_j)]f(t_{j-1}) = [1+V_1(t_{j-1},t_j)]f(t_j) + C(t_{j-1},t_j).$$

$$f(t_{j-1}) = [1-V_2(t_{j-1},t_j)]^{-1}\{[1+V_1(t_{j-1},t_j)]f(t_j) + C(t_{j-1},t_j)\}.$$

Let $A(x,y)Q = [1-V_2(x,y)]^{-1}\{[1+V_1(x,y)]Q + C(x,y)\}$ for each $\{x,y,Q\}$ in SXSXG.

By iteration $j = n, n-1, n-2, \ldots, 1$, in order, we obtain

$$f(t_0) = \Pi_1^n A(t_{j-1},t_j)f(t_n).$$

Using our Lemmas 4.1 and 4.2 and Theorem 3.5

$$f(x) = {}_x\Pi^c[1-V_2]^{-1}[1+V_1]P = {}_x\Pi^c[1-M]^{-1}[1+K]P.$$

(2 → 1): If $\{x,c\}$ is in SXS and $\{t_i\}_0^n$ is an O-subdivision of $\{x,c\}$ and i is an integer in $[1,n]$, then from Theorem 3.5

$$f(t_{i-1}) = {}_{t_{i-1}}\Pi^{t_i}[1-M]^{-1}[1+K]f(t_i)$$

$$= {}_{t_{i-1}}\Pi^{t_i}[1-V_2]^{-1}[1+V_1]f(t_i)$$

$$= [1-V_2(t_{i-1},t_i)]^{-1}[1+V_1(t_{i-1},t_i)]f(t_i) + D(t_{i-1},t_i)f(t_i)$$

where $D(t_{i-1},t_i) = {}_{t_{i-1}}\Pi^{t_i}[1-V_2]^{-1}[1+V_1]$

$$- [1-V_2(t_{i-1},t_i)]^{-1}[1+V_1(t_{i-1},t_i)].$$

$$[1-V_2(t_{i-1},t_i)][f(t_{i-1})-D(t_{i-1},t_i)f(t_i)] = [1+V_1(t_{i-1},t_i)]f(t_i);$$

$$f(t_{i-1}) - f(t_i) = V_1(t_{i-1},t_i)f(t_i) + V_2(t_{i-1},t_i)[f(t_{i-1})$$

$$- D(t_{i-1},t_i)f(t_i)] + D(t_{i-1},t_i)f(t_i);$$

$$f(x) = f(c) + \int_x^c \{V_1 f[R]+V_2(f[L]+Df[R])\} + \int_x^c Df[R];$$

but $\int_x^c V_2(f[L]+Df[R]) + \int_x^c Df[R] = \int_x^c V_2 f[L]$ because

$$||\sum_1^n V_2(t_{i-1},t_i)[f(t_{i-1}) - D(t_{i-1},t_i)f(t_i)]$$

$$+ \sum_1^n D(t_{i-1},t_i)f(t_i) - \sum_1^n V_2(t_{i-1},t_i)f(t_{i-1})||$$

$$\leq \sum_1^n \alpha_2(t_{i-1},t_i)||D(t_{i-1},t_i)f(t_i)|| + \sum_1^n||D(t_{i-1},t_i)f(t_i)||$$

$$\leq \{1+\alpha_2(x,c)\}\sum_1^n||D(t_{i-1},t_i)f(t_i)||$$

$$\leq \{1+\alpha_2(x,c)\}\sum_1^n d(t_{i-1},t_i)||f(t_i)||$$

where $d(a,b) = {}_a\Pi^b[1-\alpha_2]^{-1}[1+\alpha_1] - [1-\alpha_2(a,b)]^{-1}[1+\alpha_1(a,b)]$

for each $\{a,b\}$ in $S X S$. The preceding inequality follows from the proof of Theorem 3.4 and it follows from Theorem 3.0 and Theorem 1.4 that ${}_a\sum^b d = 0$ for each $\{a,b\}$ in $S X S$. Hence from Theorem 4.1

$$f(x) = P + {}_x\int^c (V_1 f[R] + V_2 f[L])$$
$$= P + {}_x\int^c (Kf[R] + Mf[L]).$$

It follows from Theorems 3.4 and 3.5 that (3) is equivalent to (2) and the proof is complete.

Remark: A question that arises from the preceding theorem is: "Under what conditions is it true that $V = V_1 + V_2$?" Theorem 3.6 provides an answer to this question. Also from the foregoing argument it is evident that each of the following statements is equivalent to those in the conclusion of the preceding theorem:

(4) f is in $F(c,P)$ and $f(x) = P + {}_x\int^c (V_1 f[R] + V_2 f[L])$

for each x in S; and

(5) $f(x) = {}_x\Pi^c [1-V_2]^{-1}[1+V_1]P$ for each x in S.

Theorem 4.3: Suppose

(1) each of V_1 and V_2 is in OA and each of α_1 and α_2 is in OA^+ such that for each $\{x,y\}$ in SXS, $\{Q_1,Q_2\}$ in GXG, and i = 1, 2

$$||V_i(x,y)Q_1 - V_i(x,y)Q_2|| \leqq \alpha_i(x,y)||Q_1-Q_2|| \quad \text{and}$$

(2) K is in $\Phi(V_1)$ and M is in $\Phi(V_2)$ and each of h and k is a function from SXS to the real numbers such that for each $\{x,y,Q\}$ in SXSXG, ${}_x\sum^y k = 0$, ${}_x\sum^y h = 0$,

$$||V_1(x,y)Q - K(x,y)Q|| \leqq k(x,y)||Q||, \text{ and}$$

$$||V_2(x,y)Q - M(x,y)Q|| \leqq h(x,y)||Q||;$$

(3) there is a number $a < 1$ such that $\alpha_1(x,y) + k(x,y) \leqq a$ for all $\{x,y\}$ in SXS;

(4) $K'(y,x)Q = K(x,y)Q$ and $M'(y,x)Q = M(x,y)Q$ for each $\{x,y,Q\}$ in SXSXG; and

(5) $\{c,P\}$ is in SXG.

Conclusion: The following statements are equivalent:

(1) f is in F(c,P) and $f(x) = P + {}_c\int^x (Kf[R] + Mf[L])$ for each x in S; and

(2) $f(x) = {}_x\Pi^c[1-K']^{-1}[1+M']P$ for each x in S.

Proof: Since $\int_c^x (Kf[R] + Mf[L]) = \int_x^c (M'f[R] + K'f[L])$

this theorem is a corollary to Theorem 4.2.

CHAPTER V

A SEEMINGLY MORE GENERAL INTEGRAL EQUATION

In [10, pp.632-633] Professor Mac Nerney showed that the theory he developed in solving an integral equation of the form

$f(x) = P + {}_x\int^c Vf[R]$ could be used to solve a seemingly more general equation of the form

$$f(x) = P_1 + {}_c\int^x V_1 f[R] + V_2(x,c)P_2.$$

We repeat that procedure here by using the theory developed in the preceding chapters to solve an equation of the form

$$f(x) = P_1 + {}_c\int^x (Kf[R] + Mf[L]) + V(x,c)P_2,$$

and the solution of this equation in the purely linear case is shown to include the solutions Helton obtained in Theorems 5.1-5.4 [3, pp.310-314].

Definition 5.1 [10]: $\{GXG,+,\{0,0\},\|\ \|\}$ denotes the group with addition and norm defined by $\{P_1,P_2\} + \{Q_1,Q_2\} = \{P_1+Q_1,P_2+Q_2\}$, $\|\{P_1,P_2\}\| = \|P_1\| + \|P_2\|$. For this complete normed group, let OA" and OM" be the functional classes corresponding, respectively, to the classes OA and OM for the group $\{G,+,0,\|\ \|\}$, and let E" be the corresponding mapping from OA" onto OM" as given by the apparent analogue of Theorem 1.2.

Definition 5.2: Let Φ'' and ψ'' be the mappings corresponding to the mappings Φ and ψ in Definition 1.7.

If each of K and M is in $\Phi(OA)$ and V is in OA, then there are members K" and M" of $\Phi''(OA'')$ determined by the condition that, for $\{x,y\}$ in SXS and P in GXG,

$$K''(x,y)P = \{K(x,y)P_1, 0\} \quad \text{and}$$
$$M''(x,y)P = \{M(x,y)P_1 + V(x,y)P_2, 0\};$$

moreover, if c is in S and f_1 is in $F(c,P_1)$ and f_2 is in $F(c,P_2)$ then

$$_c\int^x (K''f[R] + M''f[L]) = \{_c\int^x (Kf_1[R] + Mf_1[L]) + {}_c\int^x Vf_2[L], 0\}$$

for x in S, where f is the function $\{f_1, f_2\}$ from S to GXG; hence, with these identifications, we see that the condition that, for each x in S,

$$f_1(x) = P_1 + {}_c\int^x (Kf_1[R] + Mf_1[L]) + V(c,x)P_2 \text{ and } f_2(x) = P_2$$

is equivalent to the condition that for each x in S,

$$f(x) = P + {}_c\int^x (K''f[R] + M''f[L]).$$

These considerations show that the following theorem is a reinterpretation of Theorem 4.3. We will not state the corresponding reinterpretation of Theorem 4.2.

<u>Theorem 5</u>: Assume the hypothesis of Theorem 4.3 with K and M as defined there. Let P be in GXG, V be in OA and each of K" and M" be in $\Phi''(OA'')$ such that

$$K''(x,y)Q = \{K(y,x)Q_1, 0\} \text{ and}$$
$$M''(x,y)Q = \{M(y,x)Q_1 + V(y,x)Q_2, 0\}$$

for each {x,y} in SXS and Q in GXG. If f is a function from S to G, the following are equivalent:

(1) $\{f(x), P_2\} = {}_x\Pi^c[1-K'']^{-1}[1+M'']P$ for each x in S, and

(2) f is in $F(c,P_1)$ such that for each x in S

$$f(x) = P_1 + \int_c^x (Kf[R] + Mf[L]) + V(x,c)P_2.$$

<u>Definition 5.3 [3]</u>: $\{N,+,0,\cdot,1,|\ |\}$ denotes a complete normed ring. Let K be a function from SXS to N; OA^0 denotes the set to which K belongs only in case for each {a,b} in SXS and O-subdivision {a,x,y,b} of {a,b}, ${}_x\sum^y K$ exists and ${}_x\sum^y k = 0$, where $k(x,y) = |K(x,y) - {}_x\sum^y K|$. OM^0 denotes the set to which K belongs only in case for each {a,b} in

SXS and 0-subdivision $\{a,x,y,b\}$ of $\{a,b\}$, ${}_x\Pi^y[1+K]$ exists and ${}_x\sum^y k = 0$, where $k(x,y) = |1 + K(x,y) - {}_x\Pi^y[1+K]|$. OB^0 denotes the set to which K belongs only in case for each $\{a,b\}$ in SXS there is a number b such that for each 0-subdivision $\{t_j\}_0^n$ of $\{a,b\}$, $\sum_1^n |K(t_{j-1},t_j)| < b$.

Remark: The next corollary shows that in the purely linear case Theorem 5 includes the solutions Helton obtained in his Theorems 5.1-5.4 [3, pp. 310-314].

Corollary: Suppose

(1) each of K and M is a function from SXS to N that is in the common part of OA^0 and OB^0;

(2) there exists a number $a < 1$ such that for each $\{x,y\}$ in SXS $|K(x,y) - {}_x\sum^y K| + {}_x\sum^y |K| \leqq a$ and $M'(x,y) = M(y,x)$ and $K'(x,y) = K(y,x)$; and

(3) c is in S and each of f and h is a function from S to N such that $f(c) = h(c)$ and dh is in OB^0.

Conclusions:

(1) The following two statements are equivalent:

(a) df is in OB^0 and

$f(x) = h(x) + {}_c\int^x (f[R]K + f[L]M)$ for each x in S; and

(b) $f(x) = f(c)\, {}_c\Pi^x[1+M][1-K]^{-1} + {}_c\int^x (dh)\, {}_t\Pi^x[1+M][1-K]^{-1}[R]$

for each x in S.

(2) The following two statements are equivalent:

(a) df is in OB^0 and

$$f(x) = h(x) + {}_c\int^x (Kf[R] + Mf[L]) \quad \text{for each } x \text{ in } S; \text{ and}$$

(b) $f(x) = {}_x\Pi^c[1-K']^{-1}[1+M']f(c)$

$$+ {}_c\int^x {}_x\Pi^t[1-K']^{-1}[1+M'][R](dh)$$

for each x in S.

(3) The following two statements are equivalent:

(a) df is in OB^0 and

$$f(x) = h(x) + {}_c\int^x (Kf[R] + f[L]M) \quad \text{for each } x \text{ in } S; \text{ and}$$

(b) $f(x) = {}_x\Pi^c[1-K']^{-1}f(c)\, {}_c\Pi^x[1+M]$

$$+ {}_c\int^x {}_x\Pi^t[1-K']^{-1}[R](dh)\, {}_t\Pi^x[1+M][R]$$

for each x in S.

(4) The following two statements are equivalent:

(a) df is in OB^0 and

$$f(x) = h(x) + {}_c\int^x (f[R]K + Mf[L]) \quad \text{for each } x \text{ in } S; \text{ and}$$

$$\text{(b)}\quad f(x) = {}_x\Pi^c[1+M']f(c)\,{}_c\Pi^x[1-K]^{-1}$$

$$+ {}_c\int^x {}_x\Pi^t[1+M'][R](dh)\,{}_t\Pi^x[1-K]^{-1}[R]$$

for each x in S.

Proof: (1) For each $\{x,y\}$ in $S\times S$ and Q in $N\times N$

$K''(x,y)Q = \{Q_1K(y,x),\ 0\}$ and

$M''(x,y)Q = \{Q_1M(y,x) - dh(x,y)Q_2,\ 0\}$.

Let P be in $N\times N$ such that $P_1 = h(c)$ and $P_2 = 1$;

$$h(x) + {}_c\int^x(f[R]K + f[L]M)$$

$$= P_1 + {}_x\int^c(K''f[R] + M''f[L]) + (-dh)(c,x)P_2$$

and for each 0-subdivision $\{t_j\}_0^n$ of $\{x,c\}$

$$\Pi_1^n[1-K''(t_{j-1},t_j)]^{-1}[1+M''(t_{j-1},t_j)]P$$

$$= \{f(c)\prod_{j=1}^{n}[1+M(t_{n-j+1},t_{n-j})][1-K(t_{n-j+1},t_{n-j})]^{-1}$$

$$+ \sum_{j=1}^{n} dh(t_{n-j},t_{n-j+1})\sum_{q=1}^{n-j}[1+M(t_{n-q+1},t_{n-q})]$$

$$\cdot[1-K(t_{n-q+1},t_{n-q})]^{-1},\ P_2\}.$$

(2) For each $\{x,y\}$ in $S\times S$ and Q in $N\times N$,

$K''(x,y)Q = \{K'(x,y)Q_1,\ 0\}$ and

$M''(x,y)Q = \{M'(x,y)Q_1 - dh(x,y)Q_2,\ 0\}$.

(3) For each $\{x,y\}$ in $S \times S$ and Q in $N \times N$,

$K''(x,y)Q = \{K'(x,y)Q_1, 0\}$ and

$M''(x,y)Q = \{Q_1 M(y,x) - dh(x,y)Q_2, 0\}$.

(4) For each $\{x,y\}$ in $S \times S$ and Q in $N \times N$,

$K''(x,y)Q = \{Q_1 K(y,x), 0\}$ and

$M''(x,y)Q = \{M'(x,y)Q_1 - dh(x,y)Q_2, 0\}$.

CHAPTER VI

A REFINEMENT OF A PRODUCT INTEGRAL SOLUTION OF A STIELTJES-VOLTERRA INTEGRAL EQUATION

In this chapter we show that the product integral solution Reneke gave for nonlinear Stieltjes-Volterra integral equations in [12] can be refined still further using the results contained in this dissertation. This refinement was noticed by W. L. Gibson as indicated in the introduction.

Definition 6.1 [12]: Let S be a number interval $[a,b]$ with O denoting the usual $\leq$ ordering of S and $\{G_1, 0, +, N_1\}$ be a complete normed abelian group with norm N_1. Let M denote the class of all functions from G_1 to G_1 to which $\{0,0\}$ belongs with identity function 1. Let C be the class of all functions F from SXS into M such that

(1) $F(t,t) = 1$ for all t in S,

(2) $F[\ ,t]P$ is quasi-continuous for each t in S and P in G_1 (i.e., if s is an increasing or decreasing sequence with final set in S then the limit of $F[s,t]P$ exists), and

(3) there is a nondecreasing function k on S, called a super function for F, such that

$N_1([F(t,x) - F(t,y)]P - [F(t,x) - F(t,y)]Q) \leq |k(x) - k(y)|N_1(P-Q)$ for all $\{t,x,y\}$ in SXSXS and $\{P,Q\}$ in G_1XG_1.

Definition 6.2 [12]: Let G be the class of all functions from S to G_1 which are quasi-continuous on S, + denote functional addition, 0 denote the member of G of which the only value is 0 in G_1, and $\|\ \|$ be the supremum norm for G. It should be noted that $\{G,+,0,\|\ \|\}$ is the complete normed abelian group with respect to which the class OA from Chapter I is to be identified.

Theorem 6.1 [12]: Suppose F is a member of C. The formulas (with f in G)

$$[V(x,y)f](u) = \int_y^x dF[u,\ ]f[L], \text{ for the } \{u,x,y\} \text{ in } SXSXS,$$

define a function V from SXS to H which belongs to OA and, for each member f of G and each c in S, the following statements are equivalent:

(1) h is a member of G such that if t is in S then

$$h(t) = f(t) + \int_c^t dF[t,\ ]h[L]; \text{ and}$$

(2) h is a function on S such that

$$h(t) = [{}_t\Pi^c(1+V)f](t) \text{ for each t in S.}$$

Theorem 6.2: Suppose F is a member of C with super function k, and V is a member of OA as described in Reneke's Theorem 6.1. The formulas (with f in G)

$$[K(x,y)f](u) = [F(u,x)-F(u,y)]f(y), \text{ for } \{u,x,y\} \text{ in } SXSXS,$$

define a function K from SXS to H such that if $\{f,g\}$ is in GXG then

$$||K(x,y)f - K(x,y)g|| \leqq |k(x)-k(y)|\ ||f-g|| \text{ for } \{x,y\}$$

in SXS,

and such that if K belongs to $\Phi(V)$ then, for each member f of G and each c in S, the following statements are equivalent:

(1) h is a member of G such that if t is in S then

$$h(t) = f(t) + \int_c^t dF[t,\ ]h[L]; \text{ and}$$

(2) h is a function on S such that

$$h(t) = [{}_t\Pi^c(1+K)f](t) \text{ for each } t \text{ in } S.$$

<u>Remarks</u>: This is a direct consequence of Theorem 3.3 (for the case n = 1) or of Theorem 2.1. To show that K belongs to $\Phi(V)$, one would need a function k_2 on SXS such that if f is in G and $\{x,y\}$ is in SXS then

$$||K(x,y)f - V(x,y)f|| \leqq k_2(x,y)||f|| \text{ and } {}_x\Sigma^y k_2 = 0.$$

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