STRESS BOUNDS FOR BARS IN TORSION

A Dissertation

Presented to

the Faculty of the Department of Mechanical Engineering

University of Houston

In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy in Mechanical Engineering

By

Shein-Liang Fu

December, 1972

ACKNOWLEDGMENT

The author would like to express his sincere appreciation to Dr. Lewis T. Wheeler, associate professor of Mechanical Engineering, for his advice and assistance as an advisor and chairman of the dissertation committee, throughout the course of research and manuscript preparation, and to Dr. R. Ray Nachlinger, Dr. Pat G. Hedgcoxe, Dr. Garret J. Etgen and Dr. William L. Morris for serving on the dissertation committee.

Thanks are also due to the National Science Foundation for the financial support provided by NSF grant GK-31070.

The author is also grateful to his parents, Mr. and Mrs. T. C. Fu, wife, Li-Yu and son, Daniel, for their encouragement, understanding and patience.

Lastly, but not the least, thanks goes to an American couple, Mr. and Mrs. Jim Shackelford, for their continuous concern and encouragement.

December, 1972.

STRESS BOUNDS FOR BARS IN TORSION

An Abstract of A Dissertation

Presented to

the Faculty of the Department of Mechanical Engineering

University of Houston

In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy in Mechanical Engineering

By

Shein-Liang Fu

December, 1972

ABSTRACT

This investigation is concerned with bounds on the maximum shear stress in bars subjected to twisting by applied end couples. The results, which are found within the framework of the Saint-Venant formulation, are applicable to bars of homogeneous, anisotropic material, having a simply connected cross section.

In the case of isotropic bars we arrive at an upper bound that evidently constitutes an improvement over those available in the literature. On the other hand, there appears to be nothing in the literature concerning stress bounds for bars of anisotropic materials.

The key idea involved in the derivation of the upper bound for the isotropic bars is the minimum principle for superharmonic functions. The stress bounds for anisotropic bars are found both in a manner analogous to the development to the isotropic case, and by directly applying an affine transformation to the results found in the isotropic case. Both methods are used for orthotropic and anisotropic bars.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

1. Introduction

In the Saint-Venant theory of torsion for homogeneous bars of elastic and isotropic material, bounds for the magnitude of the shear stress have been deduced by Colombo [1]* and Protter and Weinberger [2, p. 148]. Of these two results the latter is considerably simpler and easier to derive. Moreover, it is not, like Colombo's bound, restricted to bars of star-shaped cross section. However, Colombo's estimate has the advantage that it is optimal for the circular cross section, whereas it is not clear that there is any cross section for which Protter and Weinberger's result is exact.

In the present investigation, we derive an alternative upper bound, which is determined by only two geometric parameters: the radius of the largest disk contained in the cross section, and the minimum curvature of the boundary. It is established by methods closely related to those employed by Colombo. This new estimate, in addition to being exact for the circular cross section, is simpler than Colombo's, applies to cross sections not necessarily star shaped, and is sharper in certain cases of interest.

Two lower bounds are also derived for the maximum shear stress. Such lower bounds are useful, among other things for assessing the quality of the upper bounds. One is in essence found by applying the maximum principle for harmonic functions; the other is easily arrived at with the aid of Green's theorem for the plane. All of these results are considered and discussed in section 4.

The torsion problem for homogeneous bars of elastic and

---

*Numbers in brackets refer to the Bibliography at the end of the paper.

anisotropic material may be reduced to that of the isotropic case by an affine transformation. Thus, the upper and lower bounds for this case can be derived either in a manner analogous to the derivation in section 4, or by directly applying the affine transformation to the results of section 4. Both methods are exploited in section 6 and 7 to get results for orthotropic and anisotropic bars.

The torsion problem is formulated in section 2. Section 3 is in essence a compendium of results on subharmonic functions, drawn from the books of Radó [3] and Helms [4], as well as a paper [5] by Cimmino. A counterpart of this section is developed in section 6, which is aimed at establishing certain properties for elliptic operators that are needed to get stress bounds for anisotropic bars.

2. Formulation of the torsion problems

Let D designate the open plane domain occupied by a cross section of the bar, and let (x_1,x_2,x_3) stand for Cartesian coordinates relative to a frame that contains D in its (x_1,x_2)-plane. Throughout this investigation, D is assumed, unless otherwise stated, to be bounded and simply connected, so that its boundary ∂D is a simple closed curve. If the bar is homogeneous, and has at each point a plane of elastic symmetry normal to its axis, then the torsion problem may be reduced to finding a stress function Φ, that obeys*

$$\mathcal{L}\Phi = -2 \text{ on } D,\ \Phi = 0 \text{ on } \partial D. \tag{2.1}$$

Here $\mathcal{L}$ denotes the operator

$$\mathcal{L} \equiv a_{44}\frac{\partial^2}{\partial x_1^2} - 2a_{45}\frac{\partial^2}{\partial x_1 \partial x_2} + a_{55}\frac{\partial^2}{\partial x_2^2}, \tag{2.2}$$

where a_{44}, a_{45} and a_{55} are material constants. If G_{23} and G_{13} are the shear moduli for planes parallel to (x_2,x_3)- and (x_1,x_3)-planes, respectively, and $\mu_{3\alpha,\beta 3}$ $(\alpha,\beta=1,2)$ are the shear interaction coefficients that characterize the strain in (x_3,x_α)-plane resulting from shear stress in (x_β,x_3)-plane, then

$$a_{44} = \frac{1}{G_{23}},\ a_{55} = \frac{1}{G_{13}},\ a_{45} = \frac{\mu_{31,23}}{G_{23}} = \frac{\mu_{23,31}}{G_{13}}.$$

The shear stresses σ_{31}, σ_{32} are given in terms of Φ by

$$\sigma_{31} = \frac{m}{K}\frac{\partial\Phi}{\partial x_2},\ \sigma_{32} = -\frac{m}{K}\frac{\partial\Phi}{\partial x_1} \text{ on } D \tag{2.3}$$

where m is the applied twisting moment, and

$$K = 2\int_D \Phi\, dA \tag{2.4}$$

is the torsional rigidity of D.

---

*See, for example [6].

The problem may alternatively be formulated in terms of a function Ψ, that obeys a homogeneous partial differential equation. It is related to Φ through

$$\Psi = \Phi + \tfrac{1}{2}\left(\frac{x_1^{\,2}}{a_{44}} + \frac{x_2^{\,2}}{a_{55}} \right). \tag{2.5}$$

In view of (2.1),

$$L\Psi = 0 \text{ on } D,\ \Psi = \tfrac{1}{2}\left(\frac{x_1^{\,2}}{a_{44}} + \frac{x_2^{\,2}}{a_{55}} \right) \text{ on } \partial D. \tag{2.6}$$

If the material is orthotropic, then $a_{45} = 0$. Thus, if we write $G_{23} = \mu_2$, $G_{13} = \mu_1$, then equations (2.1), (2.5) and (2.6) reduce to

$$\mu_1 \frac{\partial^2\Phi}{\partial x_1^{\,2}} + \mu_2 \frac{\partial^2\Phi}{\partial x_2^{\,2}} = -2\mu_1\mu_2 \text{ on } D,\ \Phi = 0 \text{ on } \partial D, \tag{2.7}$$

$$\Psi = \Phi + \tfrac{1}{2}\left(\mu_2 x_1^{\,2} + \mu_1 x_2^{\,2}\right), \tag{2.8}$$

and

$$\mu_1 \frac{\partial^2\Psi}{\partial x_1^{\,2}} + \mu_2 \frac{\partial^2\Psi}{\partial x_2^{\,2}} = 0 \text{ on } D,\ \Psi = \tfrac{1}{2}\left(\mu_2 x_1^{\,2} + \mu_1 x_2^{\,2}\right) \text{ on } \partial D. \tag{2.9}$$

Equations (2.3) and (2.4) remain unchanged. For an isotropic bar, equations (2.7), (2.8) and (2.9) furnish

$$\Delta\phi = -2 \text{ on } D,\ \phi = 0 \text{ on } \partial D, \tag{2.10}$$

$$\psi = \phi + \tfrac{1}{2}\left(x_1^{\,2} + x_2^{\,2}\right), \tag{2.11}$$

and

$$\Delta\psi = 0 \text{ on } D, \ \psi = \tfrac{1}{2}\left(x_1^{\,2} + x_2^{\,2}\right) \text{ on } \partial D, \tag{2.12}$$

where Δ denotes the Laplacian operator on D, and $\phi = \Phi/\mu$, $\psi = \Psi/\mu$ and $\mu = \mu_1 = \mu_2$ is the shear modulus. For this case, (2.3) reduces to

$$\sigma_{31} = \frac{m}{k}\frac{\partial\phi}{\partial x_2}, \ \sigma_{32} = -\frac{m}{k}\frac{\partial\phi}{\partial x_1} \tag{2.13}$$

where

$$k = 2\int_D \phi \, dA \,. \tag{2.14}*$$

*If μ is the shear modulus, μk is the torsional rigidity.

3. Results on superharmonic functions

The purpose of this section is to bring together a number of results that have proved useful in investigations of the present kind. For convenience, let $C(A)$ be the class of functions defined and continuous on A, if A is a set, and let $C^m(A)$ be the set of functions m-times continuously differentiable on A. The symbol $\mathcal{S}(D)$ denotes the set of functions that on the domain D are superharmonic in sense of Radó [3]. Points in D will be identified with their position vectors, which we ordinarily denote by x, y, or z. If u is defined and continuous on the closure, $\bar{S}_\delta(x)$, of

$$S_\delta(x) = \{y \mid |x-y| < \delta\},$$

we put

$$M(u,x,\lambda) = \frac{1}{2\pi\lambda} \int_{C_\lambda(x)} u\, ds \quad (0 < \lambda \le \delta), \tag{3.1}$$

where $C_\delta(x) = \partial S_\delta(x)$, and write

$$\underline{\Delta} u(x) = 4 \liminf_{\lambda\to 0} \{\frac{1}{\lambda^2} [M(u,x,\lambda) - u(x)]\}. \tag{3.2}$$

Lemma 3.1 (Properties of superharmonic functions)

(A) $u \in \mathcal{S}(D) \cap C(\bar{D}) \Rightarrow \min_{\bar{D}} u = \min_{\partial D} u$;

(B) $u \in C(\bar{D})$, $\underline{\Delta} u \le 0$ on D $\Rightarrow u \in \mathcal{S}(D)$;

(C) $u \in C^2(D) \Rightarrow \underline{\Delta} u = \Delta u$ on D.

For the proof of Part (A), see p. 59 of [4]. Part (B) follows at once from a result on subharmonic functions that appears in 3.7 of Radó's monograph [3]. Finally, (C) is easily established by introducing polar coordinates in the obvious way and invoking L'Hospital's rule.

The next lemma requires the notion of the outward normal derivative, which we take to mean

$$\frac{\partial u}{\partial n}(x) = \lim_{\lambda \to 0^-} \frac{u(x + \lambda n(x)) - u(x)}{\lambda}, \tag{3.3}$$

where $n(x)$ stands for the unit outward normal vector at $x \in \partial D$.

Lemma 3.2 Assume:

(a) $u \in C(\bar{D})$, $u = 0$ on ∂D, $\frac{\partial u}{\partial n}$ exists on ∂D, and

$$\Delta u \leq -M \text{ on } D, \tag{3.4}$$

where $M > 0$ is a constant;

(b) $v, w \in C^2(D) \cap C(\bar{D})$, $v = w$ on ∂D, and

$$-M' \leq \Delta v \leq M'', \quad \Delta w = 0 \text{ on } D, \tag{3.5}$$

where M', M'' are positive constants.

Then,

$$\frac{\partial v}{\partial n} + \frac{M''}{M}\frac{\partial u}{\partial n} \leq \frac{\partial w}{\partial n} \leq \frac{\partial v}{\partial n} - \frac{M'}{M}\frac{\partial u}{\partial n} \text{ on } \partial D. \tag{3.6}$$

Proof. Consider the function

$$v' = w - v + \frac{M'}{M} u \quad \text{on } \overline{D}. \tag{3.7}$$

Since $u = 0$ and $v = w$ on ∂D,

$$v' = 0 \text{ on } \partial D. \tag{3.8}$$

Application of Part (C) of Lemma 3.1 yields

$$\underline{\Delta} v' = \Delta w - \Delta v + \frac{M'}{M} \underline{\Delta} u \text{ on } D.$$

Thus, by (3.4) and (3.5),

$$\underline{\Delta} v' \leq 0 \text{ on } D. \tag{3.9}$$

From (3.7) and the fact that $u, v, w \in C(\overline{D})$, it follows that $v' \in C(\overline{D})$. Hence, by (3.8), (3.9), and Parts (A) and (B) of Lemma 3.1,

$$v' \geq 0 \text{ on } \overline{D}. \tag{3.10}$$

The hypothesis of the present lemma ensures that $\frac{\partial v'}{\partial n}$ exists on ∂D and is given by

$$\frac{\partial v'}{\partial n} = \frac{\partial w}{\partial n} - \frac{\partial v}{\partial n} + \frac{M'}{M}\frac{\partial u}{\partial n} \quad \text{on } \partial D.$$

Thus, by (3.8) and (3.10),

$$\frac{\partial v'}{\partial n} \leq 0 \text{ on } \partial D \Rightarrow \frac{\partial v}{\partial n} - \frac{M'}{M}\frac{\partial u}{\partial n} \geq \frac{\partial w}{\partial n} \quad \text{on } \partial D.$$

An analogous argument applied to

$$v'' = v - w + \frac{M''}{M} u$$

yields

$$\frac{\partial w}{\partial n} \geq \frac{\partial v}{\partial n} + \frac{M''}{M} \frac{\partial u}{\partial n} \quad \text{on } \partial D.$$

The proof is now complete.

A result closely related to the foregoing lemma appears in a paper by Cimmino [5]. Application of the lemma entails making a suitable choice of the functions u and v. A particularly useful specification of u, which was deduced by Cimmino [5], is given in the following lemma. In this lemma, ∂D is assumed to have continuous curvature, the sign of which is determined in the usual way*. Also, we adopt the notation

$$\delta(x) = \min_{y \in \partial D} |x-y| \tag{3.11}$$

for the distance of a point $x \in D$ from ∂D.

Lemma 3.3 Let D be a plane domain whose boundary has continuous curvature, the minimum value of which is κ, and let ρ be the radius of a maximal disk contained in D. Let $\rho_0 = \rho$, if D is a disk. If D is not a disk and $\kappa > 0$, let $\rho_0 \in \left(\rho, \frac{1}{\kappa}\right)$. If $\kappa \leq 0$, let $\rho_0 > \rho$. Define f(s) for every $s \in [0, \rho_0]$ by

*Thus, if y is the center of curvature corresponding to $x \in \partial D$, then the curvature has the same sign as the inner product $(x-y) \cdot n(x)$, where n(x) refers to the unit normal outward to ∂D at x.

$$f(s) = \begin{cases} \dfrac{1}{\kappa^2\rho_o(2-\kappa\rho_o)}\left[\kappa s - \dfrac{\kappa^2 s^2}{2} + (1-\kappa\rho_o)^2 \ln(1-\kappa s)\right], \ \underline{\text{if}}\ \kappa \neq 0 \\ s\left(1 - \dfrac{s}{2\rho_o}\right)\ \underline{\text{if}}\ \kappa = 0, \end{cases} \tag{3.12*}$$

and define u for every $x \in \overline{D}$ by

$$u(x) = f(\delta(x)). \tag{3.13}$$

Then $u \in C(\overline{D})$,

$$u = 0 \ \underline{\text{on}}\ \partial D,\ \frac{\partial u}{\partial n} = -1\ \underline{\text{on}}\ \partial D,\ \underline{\Delta u} \leq \frac{-2}{\rho_o(2-\kappa\rho_o)}\ \underline{\text{on}}\ D. \tag{3.14}$$

Proof. It is easily verified that f has the properties

$$\begin{gathered} f \in C^2[0,\rho_o],\ f(0) = 0,\ f'(0) = 1,\ f' \geq 0 \text{ on } [0,\rho_o], \\ f'' - \frac{\kappa}{1-\kappa s} f' = \frac{-2}{\rho_o(2-\kappa\rho_o)} \text{ on } [0,\rho_o], \end{gathered} \tag{3.15}$$

and that these, together with (3.13) imply $u \in C(\overline{D})$ and

$$u = 0,\ \frac{\partial u}{\partial n} = -1 \text{ on } \partial D.$$

---

*The form of f(s) for $\kappa = 0$ is the limit as $\kappa \to 0$ of the form for $\kappa \neq 0$.

Consider now the last of (3.14), and assume first that $\kappa > 0$. If D is a disk, $\rho_o = \frac{1}{\kappa}$, and (3.12) gives

$$f(s) = s - \frac{\kappa s^2}{2}.$$

Thus, in this instance,

$$u(x) = \rho_o - r - \frac{1}{2\rho_o}(\rho_o - r)^2,$$

where r is the distance between x and the center of D. Therefore

$$\Delta u = \frac{-2}{\rho_o},$$

which confirms the last of (3.15) for D a disk. Suppose that D is not a disk. Choose $x° \in D$, and let $y° \in \partial D$ be such that $\delta(x°) = |x° - y°|$ (see Fig.1). Put $\eta = \frac{1}{\kappa}$, and let z be such that $C_\eta(z)$ is tangent to ∂D at y° and $n(y°)\cdot(y°-z) > 0$. Define δ' on $\overline{D}$ by

$$\delta'(x) = \min_{y \in C_\eta(z)} |y-x|$$

for every $x \in \overline{D}$. Let $\zeta > 0$ be such that $\delta(x) \leq \delta'(x) < \rho_o$ for every $x \in \overline{S}_\zeta(x°)$, and set

$$u'(x) = f(\delta'(x))$$

for all $x \in \overline{S}_\zeta(x°)$. By (3.15),

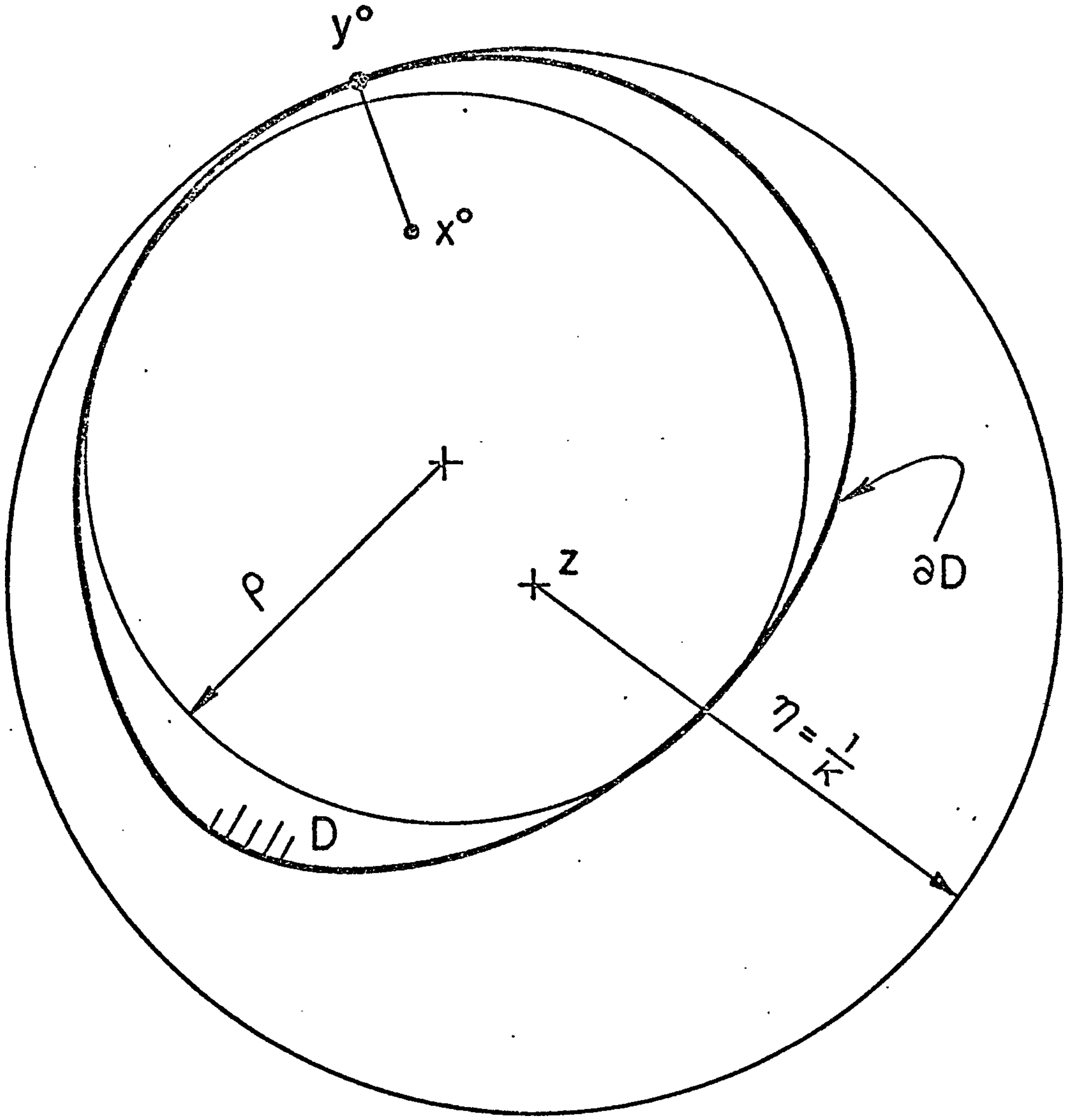

Figure 1. Cross Section with Positive Minimum Curvature

$$u' \in C^2(\overline{S}_\zeta(x^\circ)). \tag{3.16}$$

Since $\delta'(x^\circ) = \delta(x^\circ)$, and $\delta'(x) \geq \delta(x)$ for every $x \in S_\zeta(x^\circ)$, and because $f' \geq 0$ on $[0,\rho_o]$, there follows

$$u(x) \leq u'(x),\ u'(x^\circ) = u(x^\circ). \tag{3.17}$$

Accordingly,

$$M(u,x^\circ,\lambda) - u(x^\circ) \leq M(u',x^\circ,\lambda) - u'(x^\circ).$$

Hence, by (3.2), (3.16), and Part (C) of Lemma 3.1,

$$\underline{\Delta}u(x^\circ) \leq \underline{\Delta}u'(x^\circ). \tag{3.18}$$

By introducing polar coordinates centered at z, one may show that

$$\Delta u'(x^\circ) = f''(\delta'(x^\circ)) - \frac{\kappa}{1-\kappa\delta'(x^\circ)} f'(\delta'(x^\circ)),$$

whence (3.18) and the last of (3.15) imply

$$\underline{\Delta}u(x^\circ) \leq \frac{-2}{\rho_o(2-\kappa\rho_o)}\ .$$

To establish the last of (3.14) for $\kappa < 0$, take $\eta = -\frac{1}{\kappa}$, choose z such that $C_\eta(z)$ is tangent to ∂D at y° and $n(y^\circ)\cdot(y^\circ - z) < 0$,

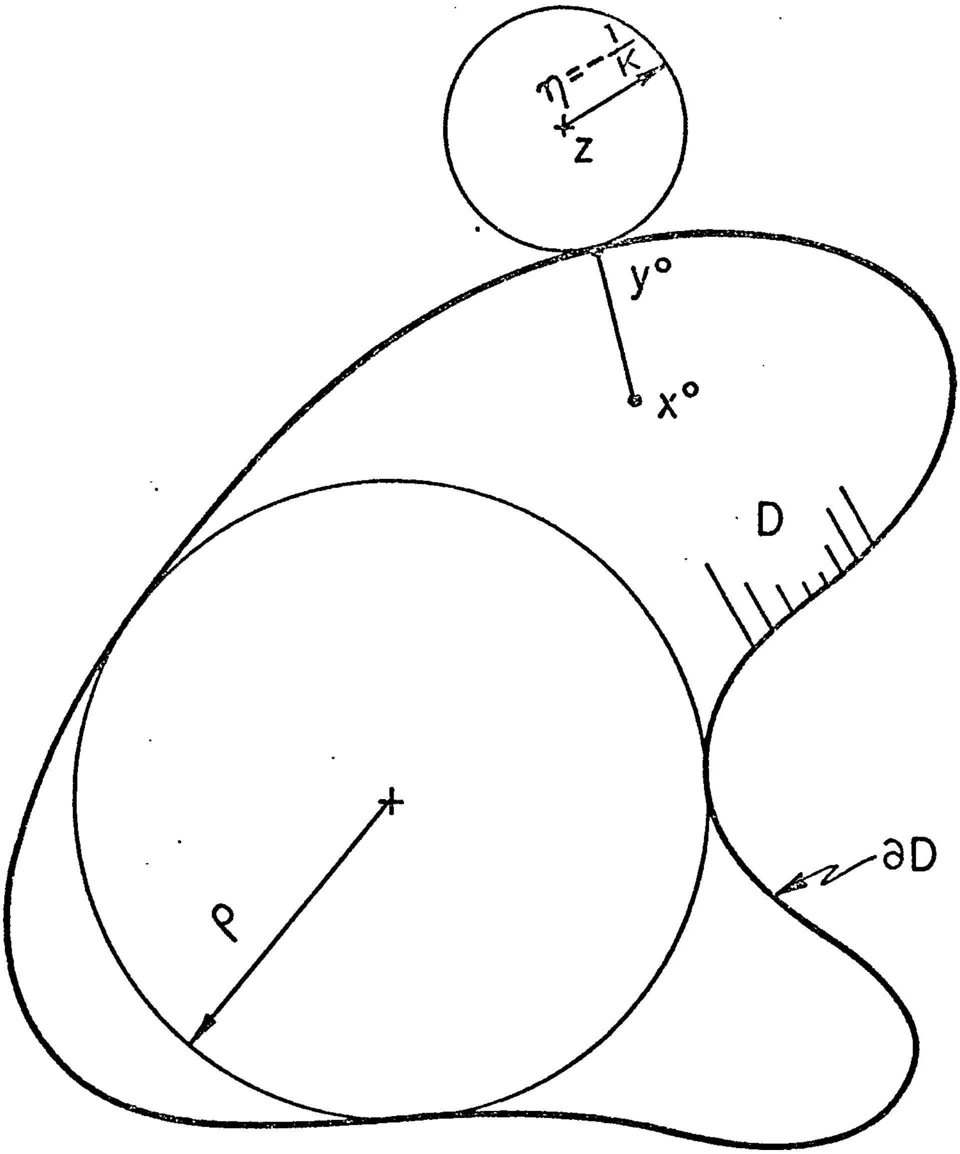

Figure 2. Cross Section with Negative Minimum Curvature

(see Fig.2). The remainder of the proof for $\kappa < 0$ is then entirely analogous to the foregoing proof for $\kappa > 0$.

Consider finally the case $\kappa = 0$. Fix $x^\circ \in D$, and let x_1, x_2 designate Cartesian coordinate in a frame with its origin on ∂D at a distance $\delta(x^\circ)$ from x°, its x_2-axis tangent to ∂D, and x° lying on the positive x_1-axis (see Fig.3). Let $\zeta > 0$ be such that $\delta(x) \leq x_1 < \rho_o$ for all $x \in S_\zeta(x^\circ)$, define

$$u'(x) = f(x_1),$$

and proceed as in the case $\kappa > 0$ to arrive at

$$\underline{\Delta}u(x^\circ) \leq \Delta u'(x^\circ).$$

By (3.12),

$$\Delta u'(x^\circ) = f''(x_1{}^\circ) = -\frac{1}{\rho_o}.$$

The proof is now complete.

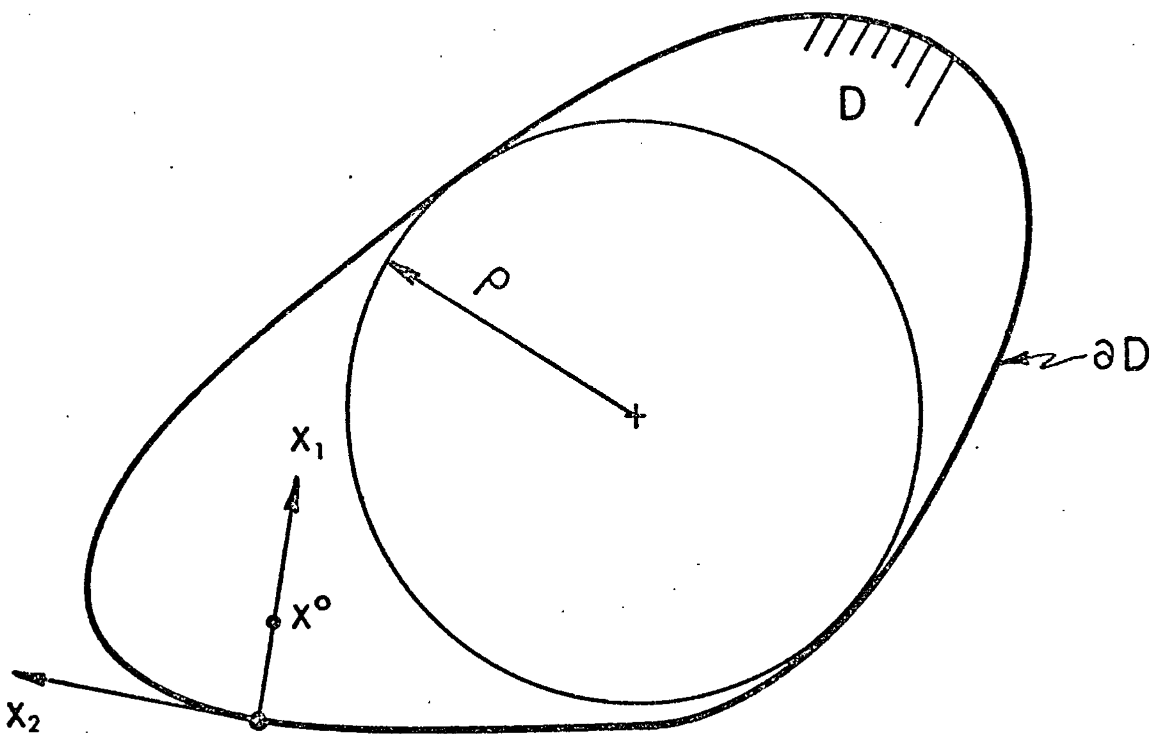

Figure 3. Cross Section with Zero Minimum Curvature

4. Stress bounds for homogeneous elastic bars

In view of (2.13), the maximum shear stress is given by

$$\sigma = \max_{\overline{D}} \sqrt{\sigma_{31}^{2} + \sigma_{32}^{2}} = \frac{m}{k} \max_{\overline{D}} |\nabla\phi|, \tag{4.1}$$

from which it is clear that an upper bound on σ may be found by establishing a lower bound for k and an upper bound on

$$\tau = \max_{\overline{D}} |\nabla\phi|. \tag{4.2}$$

On the other hand, lower bounds for σ can be obtained by deriving upper bounds on k and lower bounds on τ. The present investigation is concerned mainly with upper bounds on τ. The principal mathematical tools have been set forth in the preceding paragraph for establishing upper bounds on τ. We are now in a position to state

Theorem 4.1. Let D be bounded and simply connected, and assume ∂D has continuous curvature. Suppose that $\phi \in C^3(D) \cap C^1(\overline{D})$ and satisfies (2.10).

Then,

$$\max_{\overline{D}} |\nabla\phi| \le \rho(2-\kappa\rho), \tag{4.3}$$

where κ and ρ are as in Lemma 3.3.

Proof. By (2.10),

$$\Delta\ |\nabla\phi|^2 = 2 \sum_{\alpha=1}^{2} \sum_{\beta=1}^{2} \left(\frac{\partial^2\phi}{\partial x_\alpha \partial x_\beta}\right)^2 \geq 0.$$

Thus, Parts (A) and (B) of Lemma 3.1 yield

$$\max_{\overline{D}} |\nabla\phi|^2 = \max_{\partial D} |\nabla\phi|^2.$$

Consequently, and since Φ is constant on ∂D,

$$\max_{\overline{D}} |\nabla\phi| = \max_{\partial D} |\nabla\phi| = \max_{\partial D} \left|\frac{\partial\phi}{\partial n}\right|.$$

It therefore suffices to show that

$$\max_{\partial D} \left|\frac{\partial\phi}{\partial n}\right| \leq \rho(2-\kappa\rho) \tag{4.4}$$

in order to establish (4.3).

Let

$$v(x) = \tfrac{1}{2}(x_1^2 + x_2^2)$$

for all $x \in \overline{D}$ and put

$$w = \phi + v \text{ on } \overline{D}\cdot$$

Then,

$$\Delta v = 2,\ \Delta w = 0 \text{ on } D,\ w = v \text{ on } \partial D,$$

because of (2.10). Thus, by Lemma 3.2,

$$\frac{2}{M}\frac{\partial u}{\partial n} \leq \frac{\partial \phi}{\partial n} \leq -\frac{2}{M}\frac{\partial u}{\partial n} \quad \text{on } \partial D,$$

provided, of course, u and M conform to hypothesis (a) of the lemma. Consequently, Lemma 3.3 yields

$$-\rho_o(2-\kappa\rho_o) \leq \frac{\partial \phi}{\partial n} \leq \rho_o(2-\kappa\rho_o),$$

whence,

$$\left|\frac{\partial \phi}{\partial n}\right| \leq \rho_o(2-\kappa\rho_o) \text{ on } \partial D, \tag{4.5}$$

where $\rho_o = \rho$ if D is a disk, $\rho_o \in (\rho, \frac{1}{\kappa})$ if $\kappa > 0$, or $\rho_o > \rho$ if $\kappa \leq 0$. Since the left-hand member in (4.5) is independent of ρ_o if D is not a disk, (4.3) now follows and the proof is complete.

A different choice of the function v enables us to establish

Theorem 4.2 (Colombo's bound). Let D conform to the hypothesis of Lemma 3.3. Assume further that ∂D admits a representation

$$r = \alpha(\theta),\ 0 \leq \theta < 2\pi$$

in polar coordinates (r,θ), where

$$\alpha \in C^2(-\infty,\infty),\ \alpha > 0 \text{ on } [0,2\pi],$$

and α *is periodic of period* 2π. *Let*

$$\bar{\rho} = \min_{[0,2\pi]} \alpha,\ \hat{\rho} = \max_{[0,2\pi]} \alpha,\ \gamma = \max_{[0,2\pi]} |\alpha'|,\ \lambda = \max_{[0,2\pi]} \alpha''.$$

Finally, suppose $\phi \in C^3(D)\cap C'(\bar{D})$ *and obeys* (2.10). *Then*

$$2\max_{\bar{D}}|\nabla\phi| \leq \frac{\hat{\rho}^2 + 2\hat{\rho}\bar{\rho} - \bar{\rho}^2}{\bar{\rho}} + \frac{\gamma^2}{\bar{\rho}}(1 + \frac{\hat{\rho}^2}{\bar{\rho}^2}) + \frac{\rho}{2}(2-\kappa\rho)\frac{\lambda}{\bar{\rho}}(1 + \frac{\hat{\rho}^2}{\bar{\rho}^2}). \tag{4.6}$$

Proof. Since the scheme used to prove Theorem 4.1 may be used for the present theorem, we need only mention that here one chooses the function v as

$$v(r,\theta) = \tfrac{1}{2}\{r[\alpha(\theta) - \frac{\hat{\rho}^2}{\alpha(\theta)}] + \hat{\rho}^2\}, \tag{4.7}$$

and uses the fact that

$$\frac{\partial\phi}{\partial n} \leq 0 \text{ on } \partial D. \tag{4.8}$$

The inequality (4.8) follows from (2.10) and Part (A) of Lemma 3.2.

The bounds (4.3) and (4.6) are optimal for D a disk. Indeed, if D is a disk of radius ρ, and r stands for the distance of (x_1,x_2) from its center, then

$$\phi(x_1,x_2) = \tfrac{1}{2}(\rho^2 - r^2) \tag{4.9}$$

satisfies (2.10). Thus,

$$|\nabla\phi| = \left|\frac{\partial\phi}{\partial n}\right| = r.$$

Whence,

$$\max_{\overline{D}} |\nabla\phi| = \rho.$$

The right-hand members in (4.3) and (4.6) reduce to ρ, since[1]

$$\frac{1}{\kappa} = \rho = \overline{\rho} = \hat{\rho},\ \gamma = \lambda = 0.$$

We turn now to the task of finding lower bounds on τ. First, we introduce the following lemma.

Lemma 4.1. Assume D to be bounded and simply connected. Let $D_*\subset D$, and be simply connected such that the set $P = \partial D\cap\partial D_*$ is not empty, and let $x^\circ \in P$. Let $\phi \in C^2(D)\cap C^1(\overline{D})$ and obey (2.10), let $\phi_* \in C^2(D_*)\cap C^1(\overline{D_*})$ and satisfy

$$\Delta\phi_* = -2 \text{ on } D_*,\ \phi_* = 0 \text{ on } \partial D_*.$$

---

[1]Here we have taken $\alpha \equiv \rho$, so that the origin of the polar coordinates referred to in Theorem 4.2 is situated at the center of D.

Then,

$$|\nabla\phi(x^\circ)| \geq |\nabla\phi_*(x^\circ)|. \tag{4.10}$$

The proof of this lemma may be found on p. 148 of [2].

In the following theorem, we arrive at a simple lower bound on τ by taking D_* to be a disk.

Theorem 4.3 Let D be bounded and simply connected. Assume $\phi \in C^2(D)\cap C^1(\bar{D})$ and satisfies (2.10). Then,

$$\max_{\bar{D}} |\nabla\phi| \geq \rho, \tag{4.11}$$

where ρ is the radius of the largest disk contained in D.

Proof. Let D_* be a disk of radius ρ contained in D. Thus, if $x^\circ \in P = \partial D\cap\partial D_*$, Lemma 4.1 furnishes

$$|\nabla\phi(x^\circ)| \geq |\nabla\phi_*(x^\circ)|,$$

where ϕ_* is the stress function for the disk. By (4.9),

$$|\nabla\phi_*| = \rho \text{ on } \partial D_*.$$

Therefore,

$$\max_{\bar{D}} |\nabla\phi| \geq |\nabla\phi| \geq \rho,$$

and the proof is complete.

The key to finding the lower bound (4.11) was to choose a subdomain D_* for which the stress function is known, namely, a disk. As we shall see later on, sharper estimates may be obtained through other choices of D_*. Thus, in the case of the cross section bounded by the Booth's lemniscate, we will arrive at further lower bounds for τ by choosing for D_* a disk with a circular notch, a half disk, and an ellipse.

We turn now to an alternative way of getting lower bounds, which takes advantage of the well-known identity

$$\int_{\partial D} \frac{\partial \phi}{\partial n}\, ds = -2A, \tag{4.12}$$

where A is the area of D.

Theorem 4.4. Let D be bounded and simply connected. Assume $\phi \in C^3(D) \cap C'(\bar{D})$ and satisfies (2.10). Further, assume $\bar{D}$ to be a plane regular* region.

Then,

$$\max_{\bar{D}} |\nabla\phi| \geq \frac{2A}{C}, \tag{4.13}$$

where C is the length of ∂D.

Proof. The identity (4.12) is an elementary consequence of (2.10) and Green's theorem for the plane. From (4.12), there follows

---

*Cf. Kellogg [7], p. 99-100.

$$2A \le \max_{\partial D} \left|\frac{\partial \phi}{\partial n}\right| C.$$

Therefore, since ϕ is constant on ∂D and satisfies

$$\max_{\partial D} |\nabla\phi| = \max_{\bar{D}} |\nabla\phi|,$$

one has

$$\max_{\bar{D}} |\nabla\phi| \ge \frac{2A}{C}.$$

The proof is now complete.

Bounds for the torsional rigidity have been investigated extensively, and it is not our intention to go into the matter for the sake of finding new bounds. The brief summary offered here enables us to arrive at estimates on the shear stress σ by means of (4.1). Our main source of reference for the bounds on k is the recent survey article [8] by Payne.

Theorem 4.5. (The Saint-Venant isoperimetric inequality). Of all isotropic elastic beams of a given simply connected cross-sectional area, the circular beam has the highest torsional rigidity, i.e.

$$2\pi k \le A^2 \tag{4.14}$$

where A is the area of D.

This theorem was proved by Pólya [9].

Let ξ be any function Dirichlet integrable over D which vanishes on ∂D, let $\zeta_\alpha \in C'(\bar{D})$ $(\alpha = 1,2)$, and assume

$$\frac{\partial \zeta_1}{\partial x_1} + \frac{\partial \zeta_2}{\partial x_2} = -2 \text{ on } D.$$

Then,

$$\int_D (\zeta_1^2 + \zeta_2^2)\, dA \geq k \geq \frac{4(\int_D \xi\, dA)^2}{\int_D |\nabla \xi|^2\, dA}. \tag{4.15}$$

The choice $\zeta_\alpha = -x_\alpha$, leads to

$$k \leq I_o \tag{4.16}$$

where I_o is the polar moment of inertia about centroid. Another useful result is obtained from (4.15) by setting $\zeta_\alpha = -a_\alpha x_\alpha$, $a_1 + a_2 = 1$, and suitably choosing a_1 and a_2. This gives

$$k \leq \frac{4 I_1 I_2}{I_1 + I_2}, \tag{4.17}$$

where I_1, I_2 are moments of inertia about the two principal axes through the centroid.

For the lower bounds on k, Pólya and Szegő [10] have established

$$2k \geq \pi r^4 \tag{4.18}$$

where r is the maximum inner radius*. It is optimal for D a disk. A result due to Weinberger [11, p. 55] implies

$$2k \geq \pi\rho^4, \tag{4.19}$$

where ρ is the radius of the largest disk contained in D. Equality holds when D is a disk.

The bounds on the maximum shear stress, σ, are now easy to obtain. For instance, by (4.1), (4.3), (4.11), (4.17) and (4.18), it follows that

$$\frac{I_1+I_2}{4I_1I_2}\,\rho \leq \frac{\sigma}{m} \leq \frac{2}{\pi \dot{r}^4}\,\rho(2-\kappa\rho). \tag{4.20}$$

On the other hand, (4.1), (4.3), (4.13), (4.14) and (4.19) furnish

$$\frac{4\pi}{AC} \leq \frac{\sigma}{m} \leq \frac{2}{\pi\rho^3}\,(2-\kappa\rho). \tag{4.21}$$

Equality holds throughout (4.20) and (4.21) for D a disk.

The literature on torsion furnishes a substantial number of opportunities to compare the foregoing bounds with the exact value for τ. From the cross sections for which τ is known, we have selected two: the elliptic cross section and the cross section bounded by Booth's lemniscate. They were chosen mainly for the sake of simplicity, but the choice was also guided by the desire to consider both convex and non-convex cross sections. Let T_I stand for the upper bound in (4.3),

---

*For the definition of inner radius, see p. 2 of [10].

and T_{II} for the one in (4.6). The right-hand member of (4.11) will be denoted by t_I, and the lower bound in (4.13) by t_{II}.

For the ellipse represented by

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1 \qquad (0 < b < a).$$

One finds

$$\rho = b,\ \kappa = \frac{b}{a^2},\ \bar{\rho} = b,\ \hat{\rho} = a,\ \lambda = b(1 - \frac{b^2}{a^2}),$$

$$\gamma = a\ h(\epsilon),\ \text{where } \epsilon = \frac{a}{b}\ ,\ \text{and}$$

$$h(\epsilon) = [(\epsilon^2+1)(\epsilon^4-\epsilon^2+1)^{\frac{1}{2}} - (\epsilon^4+1)]^{\frac{1}{2}}/[(\epsilon^2+1) - (\epsilon^4-\epsilon^2+1)^{\frac{1}{2}}]^{3/2}.$$

Moreover, $A = \pi ab$, $C = 4a\ E(\theta)$, where $E(\theta)$ is the complete elliptic integral of second kind, and

$$\theta = \sin^{-1}\left(\frac{\sqrt{\epsilon^2-1}}{\epsilon}\right).$$

One can show that

$$C \le 2\pi\sqrt{\frac{a^2+b^2}{2}}\ \ \underline{\underline{\text{def}}}\ \ \bar{C}. \tag{4.22}$$

Thus, (4.3) and (4.6) give

$$T_I = \frac{a}{\epsilon}\ (2 - \frac{1}{\epsilon^2}), \tag{4.3'}$$

$$T_{II} = \frac{a}{\epsilon}\left[\epsilon^2+2\epsilon-1+\epsilon^2(1+\epsilon^2)h^2(\epsilon) + \frac{\epsilon^2}{2}\left(2-\frac{1}{\epsilon^2}\right)\left(1-\frac{1}{\epsilon^4}\right)\right], \qquad (4.6')$$

while (4.11) and (4.13) furnish

$$t_I = \frac{a}{\epsilon}, \qquad (4.11')$$

$$t_{II} = \frac{\pi a}{2\epsilon E(\theta)}. \qquad (4.13')$$

It follows from (4.13') and (4.22) that

$$t_{II} \geq \bar{t}_{II} = a\sqrt{\frac{2}{1+\epsilon^2}}. \qquad (4.23)$$

The exact value for τ is given [12, p. 121] by

$$\tau = \frac{2a\epsilon}{1+\epsilon^2}. \qquad (4.24)$$

Table 1 gives values of τ/a and its bounds as determined by (4.3'), (4.6'), (4.11'), (4.13') and (4.23). For the upper bounds, values of T_I are closer to the exact values than those of T_{II}. Moreover, the table suggests that T_I improves with increasing ϵ, while T_{II} deteriorates. Indeed, it is not difficult to verify that

$$\lim_{\epsilon\to\infty} (\tau - T_I) = 0,$$

whereas the limit corresponding to T_{II} tends to $-\infty$. The table also

Table 1. Upper and lower bounds for elliptic cross sections, a = 1.

ϵ	τ	lower bounds			upper bounds	
		(4.11')	(4.23)	(4.13')	(4.3')	(4.6')
1.1	0.995	0.910	0.951	0.951	1.067	1.208
1.2	0.984	0.834	0.906	0.908	1.088	1.428
1.6	0.899	0.625	0.751	0.957	1.006	2.524
2.0	0.800	0.500	0.633	0.649	0.875	4.466
2.4	0.714	0.417	0.545	0.563	0.761	8.220
3.0	0.600	0.333	0.448	0.470	0.630	20.45
3.6	0.516	0.278	0.379	0.402	0.534	47.10

tells us that t_{II} is sharper than t_I. By (4.11') and (4.23),

$$t_I^2 - \bar{t}_{II}^2 = \frac{a^2}{\epsilon^2(1+\epsilon^2)}(1-\epsilon^2) \le 0.$$

Thus, by (4.23),

$$t_I \le \bar{t}_{II} \le t_{II}.$$

In order to assess the upper and lower bounds T_I, $\bar{t}_{II}$, we define the relative error,

$$\omega = 2\left(\frac{T_I - \bar{t}_{II}}{T_I + \bar{t}_{II}}\right). \tag{4.25}$$

Hence, (4.25), (4.3'), (4.23) furnish

$$\omega = \left[\frac{(2\epsilon^2-1)\sqrt{1+\epsilon^2} - \sqrt{2}\,\epsilon^3}{(2\epsilon^2-1)\sqrt{1+\epsilon^2} + \sqrt{2}\,\epsilon^3}\right]. \tag{4.26}$$

Thus,

$$\omega(\epsilon) = 2\,\frac{f_1(\epsilon) - f_2(\epsilon)}{f_1(\epsilon) + f_2(\epsilon)},$$

where $f_1 = (2\epsilon^2-1)\sqrt{1+\epsilon^2}$, $f_2(\epsilon) = \sqrt{2}\,\epsilon^3$.

Consequently,

$$\omega'(\epsilon) = 4\,\frac{f_2(\epsilon)f_1'(\epsilon) - f_1(\epsilon)f_2'(\epsilon)}{[f_1(\epsilon) + f_2(\epsilon)]^2}$$

and

$$f_1'(\epsilon) = \frac{3\epsilon(2\epsilon^2+1)}{\sqrt{1+\epsilon^2}}, \quad f_2'(\epsilon) = 3\sqrt{2}\,\epsilon^2.$$

Finally, we obtain

$$\omega'(\epsilon) = \frac{12\sqrt{2}\,\epsilon^2}{\sqrt{1+\epsilon^2}\,[f_1(\epsilon)+f_2(\epsilon)]^2} \geq 0. \tag{4.27}$$

It follows from (4.27) that $\omega(\epsilon)$ is a non-decreasing function of ϵ. Obviously, when $\epsilon\to 1$, $\omega\to 0$. Moreover,

$$\lim_{\epsilon\to\infty} \omega(\epsilon) = 2\left(\frac{2-\sqrt{2}}{2+\sqrt{2}}\right) = 0.344 = 34.4\%$$

which shows that the relative error is not more than 34.4%.

Next, consider the cross section bounded by the Booth's lemniscate which is represented parametrically through Θ by

$$x_1 = b\,\frac{(a+1)\cos\Theta}{1+a^2+2a\cos 2\Theta}, \quad x_2 = b\,\frac{(a-1)\sin\Theta}{1+a^2+2a\cos 2\Theta}, \quad (0 \leq \Theta < 2\pi). \tag{4.28}$$

where $a < 1$, $b < 0$ (see Fig.4). An involved caluclation results in

$$\rho = \begin{cases} \frac{1}{2}\,\dfrac{b}{a-1}, & \text{for } 1 < a \leq 3, \\[2ex] \dfrac{b}{a+1}, & \text{for } a > 3, \end{cases}$$

$$\bar{\rho} = \frac{b}{a+1}, \quad \hat{\rho} = \frac{b}{a-1},$$

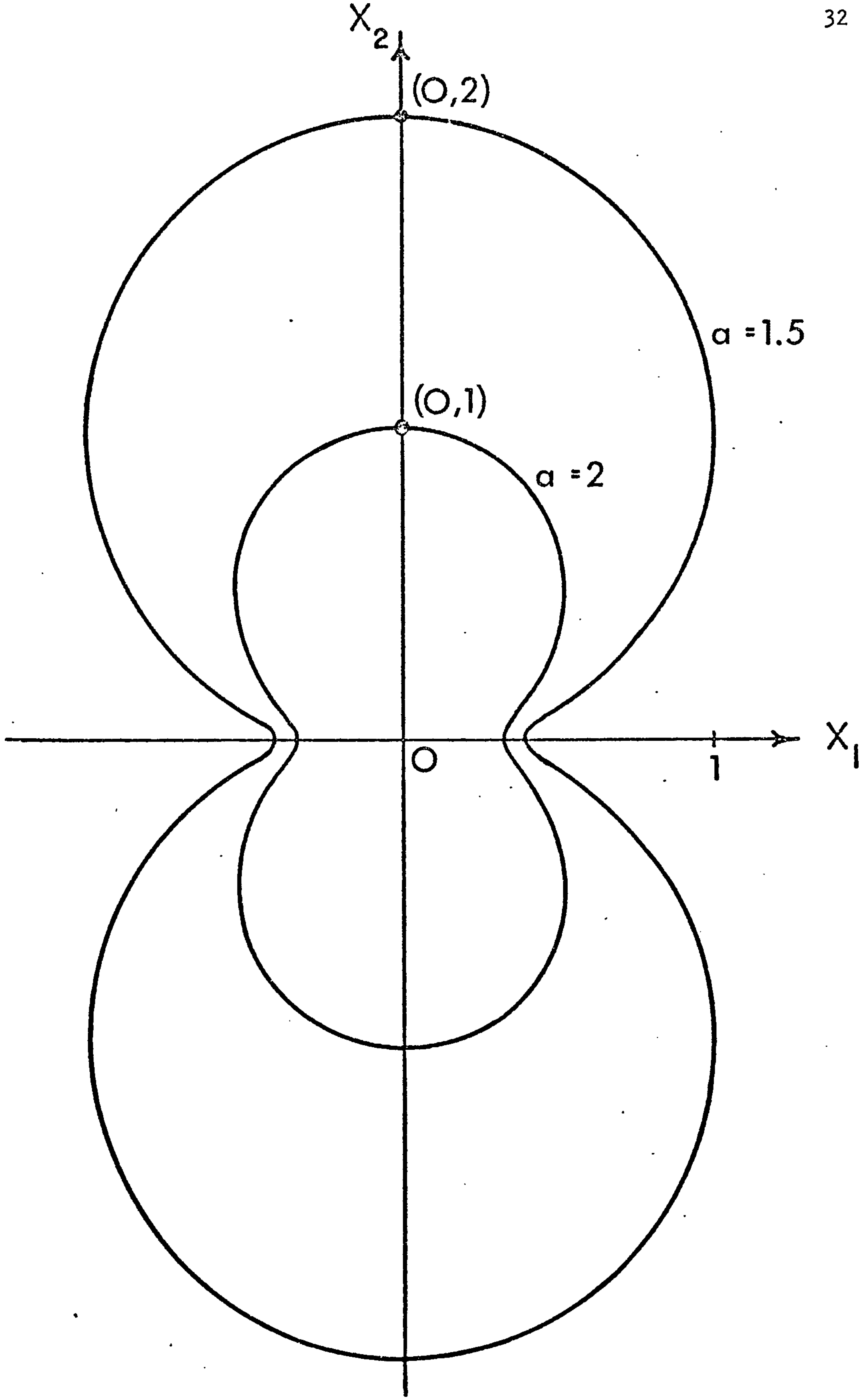

Figure 4. Cross Section Bounded by the Booth's Lemniscate

$$\kappa = \frac{(a+1)(1-6a+a^2)}{b(a-1)^2}, \quad \lambda = \frac{4ab}{(a-1)^2(a+1)},$$

and

$$\gamma = -\frac{b}{2}\left(\frac{a+1}{a-1}\right)\frac{P'(g)}{P^{\frac{1}{2}}(g)Q(g)},$$

where $P(g) = 1 + a^2 + 2ag$, $P'(g) = -4a(1-g^2)^{\frac{1}{2}}$, and $Q(g) = (1+a)^2 + 2a(1-g)$.
To get g, let

$$U = \frac{1+a^2}{2a^2}, \quad V = \frac{a^4+1+2a(a^2+1)}{2a^2}, \quad W = \frac{1+4a+a^2}{2a},$$

and

$$F = \frac{1}{3}(3V - U^2), \quad G = \frac{1}{27}(2U^3 - 9UV + 27W),$$

$$X = [-\frac{G}{2} + \left(\frac{G^2}{4} + \frac{F^3}{27}\right)^{\frac{1}{2}}]^{1/3}, \quad Y = [-\frac{G}{2} - \left(\frac{G^2}{4} + \frac{F^3}{27}\right)^{\frac{1}{2}}]^{1/3}.$$

Then,

$$g = X + Y - \frac{U}{3}.$$

Thus, (4.3) and (4.6) yield

$$T_I = \begin{cases} \dfrac{b}{2(a-1)}\left[2 - \dfrac{(a+1)(1-6a+a^2)}{(a-1)^3}\right], & \text{for } 1 < a \le 3, \\[2ex] \dfrac{b}{a+1}\left[2 - \dfrac{1-6a+a^2}{(a-1)^2}\right], & \text{for } a > 3. \end{cases} \tag{4.3''}$$

$$T_{II} = b \begin{cases} \dfrac{3a-1}{(a-1)^2} - \dfrac{1}{a+1} + \{\gamma^2(a+1) + \dfrac{a}{(a-1)^3}\left[2- \dfrac{(a+1)(1-6a+a^2)}{(a-1)^2}\right]\left[1 + \left(\dfrac{a+1}{a-1}\right)^2\right]\}, \\ \qquad \text{for } 1 < a \le 3; \\[2ex] \dfrac{3a-1}{(a-1)^2} - \dfrac{1}{a+1} + \{\gamma^2(a+1) + \dfrac{2a}{(a+1)(a-1)^2}\left[2 - \dfrac{1-6a+a^2}{(a-1)^2}\right]\left[1 + \left(\dfrac{a+1}{a-1}\right)^2\right]\}, \\ \qquad \text{for } a > 3. \end{cases} \tag{4.6''}$$

On the other hand, (4.11) leads to

$$t_I = b \begin{cases} \dfrac{1}{2}\,\dfrac{1}{a-1}, & \text{for } 1 < a \le 3; \\[2ex] \dfrac{1}{a+1}, & \text{for } a > 3. \end{cases} \tag{4.11''}$$

As pointed out earlier, one might expect to get lower bounds sharper than t_I by imbedding in D non-circular cross sections for which the stress function is known. For instance, we may imbed the largest disk with a circular notch such that the bottom of the notch touches ∂D at x° (see Fig.5). If the radius of the notch is η, then the radius of the disk is $\bar{\rho} + \frac{1}{2}\eta$. It is clear from [13, p. 303] that

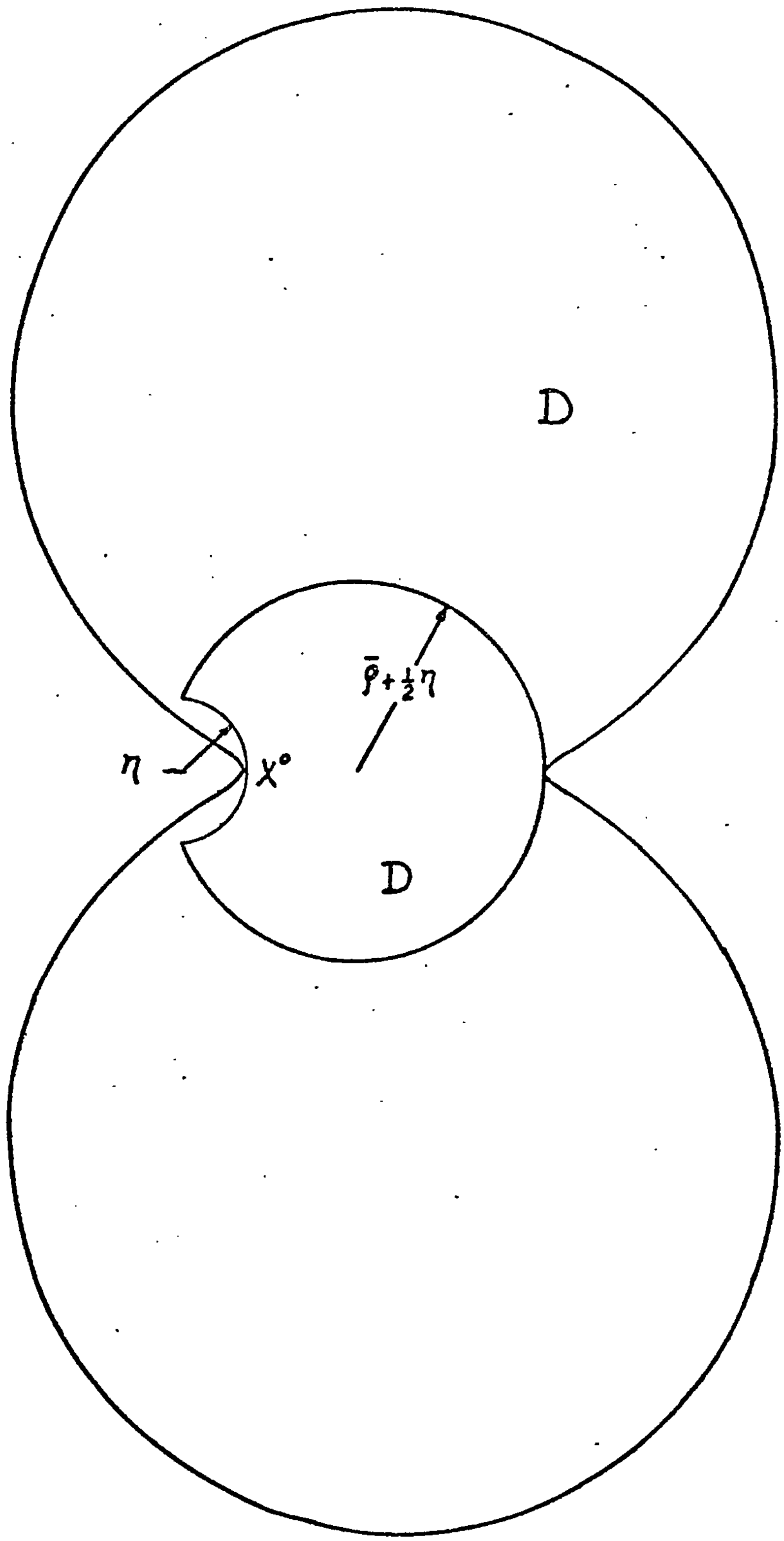

Figure 5. Booth's Lemniscate Imbedded with A Notched Disk

$$\max_{\bar{D}_*} |\nabla\phi_*| = 2(\bar{\rho} + \tfrac{1}{2}\eta) - \eta = 2\bar{\rho}.$$

Consequently, and by Lemma 4.1,

$$\max_{\bar{D}} |\nabla\phi| \geq \frac{2b}{a+1} . \tag{4.29}$$

The largest half disk contained in D with the center located at x° may also be employed as the subdomain D_* (see Fig.6). Since,[1]

$$\max_{\bar{D}_*} |\nabla\Phi_*| = 0.849 \times (2\bar{\rho}) = 1.698\bar{\rho},$$

then,

$$\max_{\bar{D}} |\nabla\phi| \geq \frac{1.698b}{a+1} . \tag{4.30}$$

Furthermore, we also can imbed in D an ellipse such that the point x° is at one end of the minor axis (see Fig.7). In this case, $\frac{b}{a-1}$ and $\frac{b}{a+1}$ will be its length of semi-major and semi-minor axes, respectively. This furnishes

$$\max_{\bar{D}} |\nabla\phi| \geq \frac{b(a+1)}{a^2+1} . \tag{4.31}$$

Finally, the exact value for τ is given [14, p. 592] by

---

[1]For example, see p. 314 of [13].

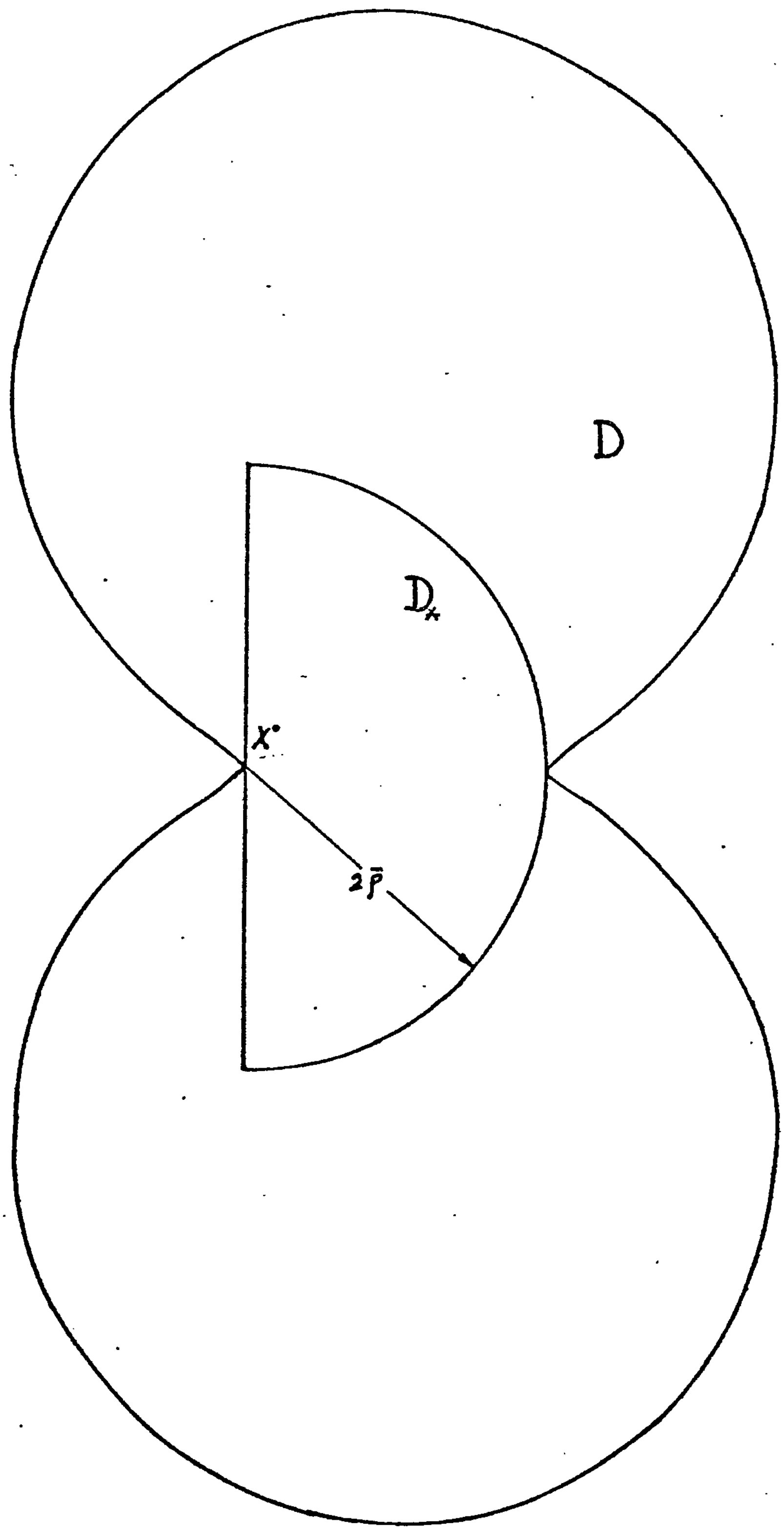

Figure 6. Booth's Lemniscate Imbedded with A Half Disk

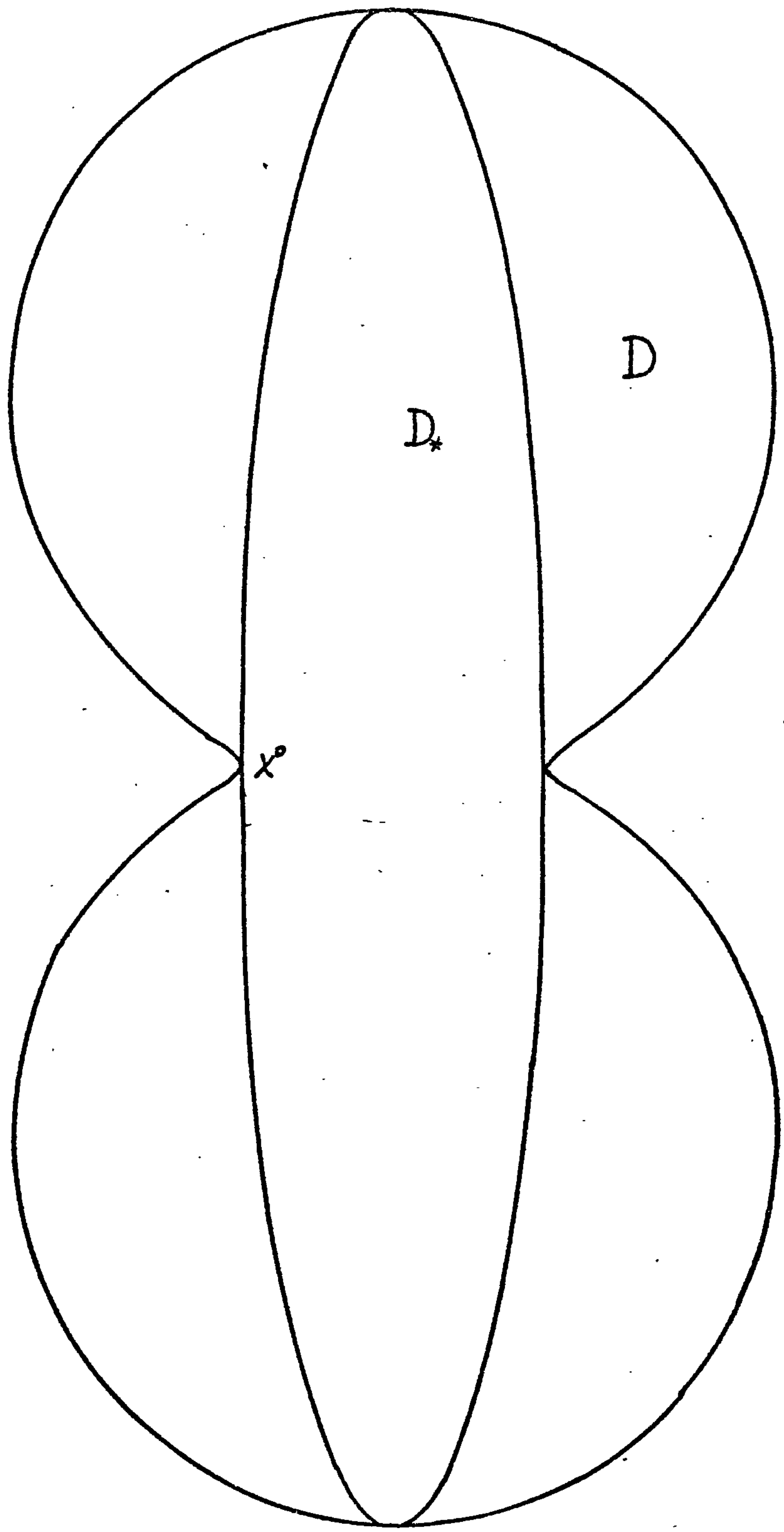

Figure 7. Booth's Lemniscate Imbedded with An Ellipse

$$\tau = \frac{b(a^2+1)}{(a+1)(a-1)^2} . \tag{4.32}$$

Computations based on (4.32), (4.3''), (4.6''), (4.11''), (4.29), (4.30) and (4.31) are shown in Table 2. Concerning the upper bounds, we observe from the table that the sharper is T_I. On the other hand, lower bounds calculated from (4.29) - (4.31) are better than those values obtained from (4.11'') for some values of a: (4.29) supercedes (4.11'') for a in [1.667, 3.6]; (4.30) is sharper for a in [1.83, 4.6]; (4.31) is closer if $a > \sqrt{3}$.

The foregoing examples raise the question whether (4.3) is in general sharper than (4.6). Although we have unfortunately not succeeded in finding the answer, we have been able to establish that for a significant class of cross sections, the answer is affirmative.

Theorem 4.7 Assume that the hypothesis of Theorem 4.2 holds, and let T_I, T_{II} stand for the respective right-hand members of (4.3), (4.6). If

$$\frac{\lambda}{\rho} \geq 2, \tag{4.33}$$

then $T_I < T_{II}$. If $K(\theta)$ $(0 \leq \theta \leq 2\pi)$ denotes the curvature associated with the angle θ, $K(\bar{\theta})=\kappa$, and

$$\kappa \leq \frac{-1}{\alpha(\bar{\theta})} , \tag{4.34}$$

then (4.33) is true.

Table 2. Upper and lower bounds for cross section bounded by Booth's lemniscate, b = 1.

a	τ	lower bounds				upper bounds	
		(4.11'')	(4.29)	(4.30)	(4.31)	(4.3'')	(4.6'')
1.1	105.24	5.0				0.231×10^5	0.112×10^8
1.2	27.73	2.5				0.164×10^4	0.601×10^7
1.6	3.803	0.833				59.5	0.931×10^4
2.0	1.667	0.500	0.667	0.566	0.600	6.25	129.13
2.4	1.015	0.354	0.588	0.499	0.571	2.40	22.42
3.0	0.625	0.250	0.500	0.425	0.400	1.0	4.871
3.6	0.449	0.217	0.435	0.369	0.385	0.681	2.232
4.0	0.378	0.200		0.340	0.333	0.556	1.525
4.6	0.305	0.178		0.303	0.278	0.432	0.978
5.0	0.271	0.167			0.250	0.375	0.772
5.5	0.237	0.154			0.222	0.321	0.602

Proof. By (4.3), (4.6),

$$2(T_{II} - T_I) = \frac{1}{\bar\rho}(\hat\rho + \bar\rho) + 2\hat\rho + \frac{\gamma^2}{\bar\rho}\left(1 + \frac{\hat\rho^2}{\bar\rho^2}\right)$$

$$+ \frac{\rho}{2}(2-\kappa\rho)\frac{\lambda}{\bar\rho}\left(1 + \frac{\hat\rho^2}{\bar\rho^2}\right) - 2\rho(2-\kappa\rho).$$

Thus, since $\bar\rho \le \rho \le \hat\rho$ and $\kappa\rho \le 1$,

$$2(T_{II} - T_I) \ge \frac{1}{\bar\rho}(\hat\rho+\bar\rho)(\hat\rho-\bar\rho) + \frac{2\gamma^2}{\bar\rho} + \rho[2 + (2-\kappa\rho)\left(\frac{\lambda}{\bar\rho} - 2\right)]. \qquad (4.35)$$

Since $\kappa\rho \le 1$, (4.33) ensures that the last term in (4.35) is positive, it now follows that $T_{II} > T_I$.

With a view toward establishing that (4.34) implies (4.33), recall that

$$\kappa = K(\bar\theta) = \frac{1 + 2\left(\frac{r'}{r}\right)^2 - \frac{r''}{r}}{r[1 + \left(\frac{r'}{r}\right)^2]^{3/2}},$$

where

$$r = \alpha(\bar\theta),\ r' = \alpha'(\bar\theta),\ r'' = \alpha''(\bar\theta).$$

Thus, by (4.34)

$$\frac{\lambda}{\bar\rho} \ge \frac{r''}{r} = -r\kappa[1 + \left(\frac{r'}{r}\right)^2]^{3/2} + 1 + 2\left(\frac{r'}{r}\right)^2 \ge -r\kappa + 1 \ge 2,$$

which completes the proof.

The main importance of the conclusion that $T_I < T_{II}$ for $\kappa \leq \dfrac{-1}{\alpha(\bar{\theta})}$ is its implication that (4.3) is sharper than (4.6) for a class of cross-sections with negative curvature.

5. Some results on elliptic operators

This section, which is a counterpart of section 3, is aimed at establishing certain properties for elliptic operators that are needed to get bounds for anisotropic bars.

We denote by L the differential operator

$$L \equiv \sum_{\alpha,\beta=1}^{2} C_{\alpha\beta} \frac{\partial^2}{\partial x_\alpha \partial x_\beta}, \tag{5.1}$$

where $C_{\alpha\beta} = C_{\beta\alpha}$ are constants. We assume henceforth that L is elliptic, so that $C_{\alpha\beta}$ are elements of a positive definite matrix. Since $C_{\alpha\beta}$ are constants, L is uniformly elliptic. Thus, there is a non-singular linear coordinate transformation such that L becomes the Laplacian on the transformed domain D'. Let

$$x_1' = \xi_1 x_1 + \xi_2 x_2, \; x_2' = \eta_1 x_1 + \eta_2 x_2 \tag{5.2}$$

be such a transformation. Here ξ_α, η_α (α=1,2) are constants. Let J be the matrix associated with (5.2).

Let $N_\delta(x^\circ)$ denote the inverse image of $S_\delta(y^\circ)$ under the mapping (5.2) where $y^\circ = Jx^\circ$. If a function u is defined and continuous on $\bar{N}_\delta(x^\circ)$, we put

$$M'(u,x^\circ,\lambda) = \frac{1}{\ell} \int_{E_\lambda(x^\circ)} u \, ds, \; (0 < \lambda \le \delta) \tag{5.3}$$

where $E_\lambda(x^\circ) = \partial N_\lambda(x^\circ)$, and ℓ is the length of $E_\lambda(x^\circ)$, and write

$$\underline{L}u(x) = 4 \liminf_{\lambda\to 0}\left\{\frac{1}{\lambda^2}\,[M'(u,x,\lambda) - u(x)]\right\}. \tag{5.4}$$

We will call a function u super c-harmonic* on D if:

(a) $M'(u,x,\lambda)$ is defined and $u(x) \geq M'(u,x,\lambda)$ on D;

(b) u is not identical $+\infty$ on D;

(c) $u > -\infty$ on D;

(d) u is lower semi-continuous on D.

The class of super c-harmonic functions on D will be designated by $S'(D)$.

Lemma 5.1. (Properties of super c-harmonic functions)

(A) $u \in S'(D)\cap C(\bar{D}) \Rightarrow \min_{\bar{D}} u = \min_{\partial D} u$;

(B) $u \in C(\bar{D})$, $\underline{L}u \leq 0$ <u>on</u> D $\Rightarrow u \in S'(D)$;

(C) $u \in C^2(D) \Rightarrow \underline{L}u = Lu$ <u>on</u> D.

The truth of this lemma is easily established with the aid of Lemma 3.1 and the affine transformation (5.2).

Lemma 5.2. <u>Assume</u>:

(a) $u \in C(\bar{D})$, $u = 0$ <u>on</u> ∂D, $\frac{\partial u}{\partial n}$ <u>exists</u> <u>on</u> ∂D, <u>and</u>

$$\underline{L}u \leq -M \text{ on } D, \tag{5.5}$$

---

*We adopt the terminology of Morrey [15] which would label a solution of Lu = 0 "c-harmonic."

where $M > 0$ *is a constant;*

(b) $v, w \in C^2(D) \cap C(\bar{D})$, $v = w$ *on* ∂D *and*

$$-M' \le Lv \le M'', \quad Lw = 0 \text{ on } D, \tag{5.6}$$

where M', M'' *are positive constants.*

Then

$$\frac{\partial v}{\partial n} + \frac{M''}{M}\frac{\partial u}{\partial n} \le \frac{\partial w}{\partial n} \le \frac{\partial v}{\partial n} - \frac{M'}{M}\frac{\partial u}{\partial n} \text{ on } \partial D. \tag{5.7}$$

The proof of this lemma is sufficiently close to the proof of Lemma 3.2 that it may safely be omitted.

Let Φ be a function such that

$$L\Phi = -2 \text{ on } D, \; \Phi = 0 \text{ on } \partial D. \tag{5.8}$$

By (5.2), the above equation may be reduced to

$$\Delta'\Phi = -2G \text{ on } D', \; \Phi = 0 \text{ on } \partial D'. \tag{5.9}$$

Here $\Delta' \equiv \frac{\partial^2}{\partial x_1'^2} + \frac{\partial^2}{\partial x_2'^2}$, and G is a constant. Moreover, if we write $\tilde{\Phi} = \Phi/G$, then (5.9) gives way to

$$\Delta'\tilde{\Phi} = -2 \text{ on } D', \; \tilde{\Phi} = 0 \text{ on } \partial D'. \tag{5.10}$$

If A is a square matrix, we write $\|A\|$ for the square root of the largest eigenvalue of $A^T A$; it is easily verified that $\|\cdot\|$ is a norm.

Lemma 5.3. Let $\Phi \in C^3(D) \cap C^1(\bar{D})$ and satisfy (5.8). Then

$$G\|H^{-1}\|^{-1} \max_{\bar{D}'} |\nabla' \tilde{\Phi}| \leq \max_{\bar{D}} |\nabla \Phi| \leq G\|H\| \max_{\bar{D}'} |\nabla \tilde{\Phi}|. \tag{5.11}$$

where $H = J^T$.

Proof. By (5.2), one has

$$GH\nabla' \tilde{\Phi} = \nabla \Phi. \tag{5.12}$$

Take the Euclidean vector norm on both sides of (5.12) to get

$$G\|H\| \; |\nabla' \tilde{\Phi}| \geq |\nabla \Phi|. \tag{5.13}$$

By (5.12),

$$G\nabla' \tilde{\Phi} = H^{-1} \nabla \Phi.$$

Accordingly,

$$G\|H^{-1}\|^{-1} \; |\nabla' \tilde{\Phi}| \leq |\nabla \Phi|. \tag{5.14}$$

The proof is now complete.

The significance of (5.11) is that the bounds on $\max_{\bar{D}} |\nabla \Phi|$ for anisotropic materials may be obtained from the bounds for isotropic bars.

Lemma 5.4. Let D be simply connected, and suppose that $\Phi \in C^3(D) \cap C^1(\bar{D})$ and obeys (5.8). Assume further, that the coefficients in (5.8) satisfy the condition

$$c_{12}^2 \le c_{11}c_{22}. \tag{5.15}$$

Then,

$$\max_{\bar{D}} |\nabla\Phi| = \max_{\partial D} \left|\frac{\partial\Phi}{\partial n}\right| . \tag{5.16}$$

Proof.

$$L\ |\nabla\Phi|^2 = 2\left\{[c_{11}\left(\frac{\partial^2\Phi}{\partial x_1^2}\right)^2 + 2c_{12}\left(\frac{\partial^2\Phi}{\partial x_1^2}\right)\left(\frac{\partial^2\Phi}{\partial x_1\partial x_2}\right) + c_{22}\left(\frac{\partial^2\Phi}{\partial x_1\partial x_2}\right)^2]\right.$$

$$\left. + [c_{22}\left(\frac{\partial^2\Phi}{\partial x_2^2}\right)^2 + 2c_{12}\left(\frac{\partial^2\Phi}{\partial x_2^2}\right)\left(\frac{\partial^2\Phi}{\partial x_1\partial x_2}\right) + c_{11}\left(\frac{\partial^2\Phi}{\partial x_1\partial x_2}\right)^2]\right\}.$$

The condition (5.15) implies

$$L\ |\nabla\Phi|^2 \ge 0.$$

Application of Lemma 5.1 gives

$$\max_{\bar{D}} |\nabla\Phi|^2 = \max_{\partial D}|\nabla\Phi|^2$$

Consequently, and since Φ is constant on ∂D,

$$\max_{\bar{D}} |\nabla\Phi| = \max_{\partial D} |\nabla\Phi|,$$

which completes the proof.

Lemma 5.5. Let D be bounded and simply connected, and let $D_* \subset D$ be simply connected such that the set $P = \partial D \cap \partial D_*$ is not empty, and let $x^\circ \in P$. If $\Phi \in C^2(D) \cap C^1(\bar{D})$ and obeys (5.8), $\Phi_* \in C^2(D_*) \cap C^1(\bar{D}_*)$ and satisfies

$$L\Phi_* = -2 \text{ on } D_*, \quad \Phi_* = 0 \text{ on } \partial D_*.$$

Then

$$|\nabla\Phi(x^\circ)| \geq |\nabla\Phi_*(x^\circ)|. \tag{5.17}$$

Proof. By hypothesis and Lemma 5.1,

$$\Phi \geq 0 \text{ on } D, \quad \Phi_* \geq 0 \text{ on } D_*.$$

Define $\bar{\Phi} = \Phi - \Phi_*$ on $\bar{D}_*$. Clearly $\bar{\Phi} \geq 0$ on ∂D_*, $\bar{\Phi}(x^\circ) = 0$, and

$$L\bar{\Phi} = 0 \text{ on } D_*.$$

Thus, and by Lemma 5.1,

$$\frac{\partial \Phi}{\partial n}(x^\circ) \leq \frac{\partial \Phi_*}{\partial n}(x^\circ).$$

Since $\frac{\partial \Phi_*}{\partial n} \leq 0$ on ∂D_* and Φ, Φ_* are constant on ∂D and ∂D_*, the desired conclusion (5.17) now follows.

6. Stress bounds for homogeneous orthotropic elastic bars

If the material is orthotropic, the operator L reduces to

$$L_1 \equiv \mu_1 \frac{\partial^2}{\partial x_1^2} + \mu_2 \frac{\partial^2}{\partial x_2^2} . \tag{6.1}$$

Clearly, L_1 is uniformly elliptic on D. The affine transformation (5.2) may take the form

$$x_1' = \sqrt{\frac{\mu_2}{\mu}}\; x_1, \; x_2' = \sqrt{\frac{\mu_1}{\mu}}\; x_2, \tag{6.2}$$

where $\mu = \frac{\mu_1+\mu_2}{2}$.

The maximum shear stress is given by

$$\sigma = \frac{m}{K} \max_{\bar{D}} |\nabla\Phi| . \tag{6.3}$$

As pointed out earlier, bounds on σ may be arrived at by getting bounds on K and the quantity

$$\tau = \max_{\bar{D}} |\nabla\Phi| . \tag{6.4}$$

We will consider the bounds on τ first.

By selecting suitable functions u and v in Lemma 5.2, we may obtain an upper bound on τ. The next lemma is concerned with a function u that confirms to hypothesis (a) of Lemma 5.2.

Lemma 6.1. Let D be a plane domain whose boundary has continuous curvature, the minimum value of which is κ, and let ρ be the radius of the largest disk contained in D. Let $\rho_o = \rho$ if D is a disk. If D is not a disk and $\kappa > 0$, let $\rho_o \in (\rho, \frac{1}{\kappa})$. If $\kappa \le 0$, let $\rho_o > \rho$. Define f(s) for every $s \in [0,\rho_o]$ by

$$f(s) = \begin{cases} \dfrac{1}{\kappa[(1-\kappa\rho_o)^{\nu+1}-1]}\left\{\dfrac{(1-\kappa\rho_o)^{\nu+1}}{\nu-1}\left[\left(\dfrac{1}{1-\kappa s}\right)^{\nu-1}-1\right]+\tfrac{1}{2}[(1-\kappa s)^2-1]\right\}, & \text{if } \kappa \neq 0 \\ s(1-\dfrac{s}{2\rho_o}), & \text{if } \kappa = 0 \end{cases} \tag{6.5*}$$

where $\nu = \mu_2/\mu_1$ and assume $\mu_2 > \mu_1 > 0$. Define a function u for every $x \in \bar{D}$ by

$$u(x) = f(\delta(x)). \tag{6.6}$$

Then $u \in C(\bar{D})$, $u = 0$ on ∂D, $\frac{\partial u}{\partial n} = -1$ on ∂D, and

$$L_1 u \le \frac{-\mu_1(\nu+1)\kappa}{1-(1-\kappa\rho_o)^{\nu+1}}. \tag{6.7}$$

Proof. It is easy to verify that f has the properties

$$\begin{gathered} f \in C^2[0,\rho_o],\ f(0) = 0,\ f'(0) = 1,\ f' \ge 0 \text{ on } [0,\rho_o], \\ f'' - \frac{\kappa}{1-\kappa s}\nu f' = \frac{-(\nu+1)\kappa}{1-(1-\kappa\rho_o)^{\nu+1}} \text{ on } [0,\rho_o]. \end{gathered} \tag{6.8}$$

---

*The form of f(s) for $\kappa = 0$ is the limit as $\kappa \to 0$ of the form for $\kappa \neq 0$.

From these, together with (6.6), it follows that

$$u \in C(\bar{D}) \text{ and } u = 0,\ \frac{\partial u}{\partial n} = -1 \text{ on } \partial D.$$

The last of (6.7) remains to be proven. We consider first the case for $\kappa > 0$ and D not a disk. Pick $x^\circ \in D$, and let $y^\circ \in \partial D$ be such that $\delta(x^\circ) = |x^\circ - y^\circ|$ (see Fig.1). Put $\eta = \frac{1}{\kappa}$, and let z be such that the circle $C_\eta(z)$ with radius η centered at z is tangent to ∂D at y° and $n(y^\circ)\cdot(y^\circ - z) > 0$. Define δ' on $\bar{D}$ by

$$\delta'(x) = \min_{y \in C_\eta(z)} |y-x|$$

for every $x \in \bar{D}$. Let $\zeta > 0$ be such that $\delta(x) \le \delta'(x) < \rho_o$ for every $x \in \bar{N}_\zeta(x^\circ)$, and set

$$u'(x) = f(\delta'(x)) \tag{6.9}$$

for all $x \in \bar{N}_\zeta(x^\circ)$. By (6.8),

$$u' \in C^2(\bar{N}_\zeta(x^\circ)). \tag{6.10}$$

Since $\delta'(x^\circ) = \delta(x^\circ)$, and $\delta'(x) \ge \delta(x)$ for every $x \in \bar{N}_\zeta(x^\circ)$, and because $f' \ge 0$ on $[0,\rho_o]$; there follows

$$u(x) \le u'(x),\ u'(x^\circ) = u(x^\circ). \tag{6.11}$$

Accordingly,

$$M'(u,x^\circ,\lambda) - u(x^\circ) \le M'(u',x^\circ,\lambda) - u'(x^\circ), \quad (0<\lambda\le\zeta)$$

Hence, by (5.4), (6.9), and Part (C) of Lemma 5.1,

$$\underline{L}_1\, u(x^\circ) \le L_1\, u'(x^\circ). \tag{6.12}$$

For convenience, let the origin of the coordinate system be centered at z. One finds

$$L_1 u'(x^\circ) = \frac{\mu_1 x_1^{\circ 2} + \mu_2 x_2^{\circ 2}}{r^2}\left[\frac{\partial^2 u'}{\partial r^2} + \frac{1}{r}\,\frac{\mu_1 x_2^{\circ 2} + \mu_2 x_1^{\circ 2}}{\mu_1 x_1^{\circ 2} + \mu_2 x_2^{\circ 2}}\,\frac{\partial u'}{\partial r}\right], \tag{6.13}$$

where r is the distance of the point x° from z. Now fix the coordinate system such that $x_1^\circ = r$, $x_2^\circ = 0$. Therefore, and in view of (6.9), (6.13) reduces to

$$L_1\, u'(x^\circ) = \mu_1[f''(\delta'(x^\circ)) - \frac{\kappa\nu}{1-\kappa\delta'(x^\circ)}\, f'(\delta'(x^\circ))],$$

whence (6.12) and the last of (6.8) imply

$$\underline{L}_1\, u(x^\circ) \le \frac{-\mu_1(\nu+1)\kappa}{1-(1-\kappa\rho_0)^{\nu+1}}\,.$$

This confirms the last of (6.7) for $\kappa > 0$ and D not a disk.

The proofs for the remaining two cases, D a disk and $\kappa \le 0$, are easily constructed along the lines of the proofs of their counterparts in Lemma 3.3, and may safely be omitted.

We are now able to establish

Theorem 6.1. Let D be bounded and simply connected, and assume ∂D has continuous curvature. Suppose that $\Phi \in C^3(D) \cap C^1(\bar{D})$ and obeys (2.7).

Then,

$$\max_{\bar{D}} |\nabla\Phi| \le \frac{2\nu\mu_1}{\kappa(\nu+1)} [1 - (1-\kappa\rho)^{\nu+1}], \tag{6.14}$$

where κ and ρ are as in Lemma 6.1.

The proof of the theorem is strictly analogous to that of Theorem 4.1. Here we may choose the function

$$v(x) = \tfrac{1}{2}(\mu_2 x_1^2 + \mu_1 x_2^2),$$

and employ Lemma 6.1, Lemma 5.2 and Lemma 5.4.

If we put $\nu = 1$ in (6.14), then it reduces to (4.3). Indeed,

$$\max_{\bar{D}} |\nabla\Phi| \le \frac{\mu_1}{\kappa} (2\kappa\rho - \kappa^2\rho^2) = \mu_1\rho(2-\kappa\rho).$$

It is easy to show that for a circular cross section with radius ρ, the exact value of τ is

$$\tau = \frac{2\mu_1\nu}{\nu+1}\rho. \tag{6.15}$$

Thus, in view of (6.15), the bound in (6.14) is optimal when D is a disk.

The foregoing upper bound for τ was derived in a manner analogous to the derivation of (4.3). The affine transformation that relates L_1 to Δ plays a significant role in this procedure. One might expect that the affine transformation could be applied directly in connection with (4.3) to get a bound. This conjecture 's confirmed by

Theorem 6.2. Let D be bounded and simply connected, and assume ∂D has continuous curvature. Suppose that $\Phi \in C^3(D) \cap C^1(\bar{D})$ and obeys (2.7). Then

$$\max_{\bar{D}} |\nabla\Phi| \leq \sqrt{\mu\mu_2}\, \rho'(2-\kappa'\rho'), \tag{6.16}$$

where ρ' is the radius of the largest disk contained in D' and κ' is the minimum curvature of $\partial D'$.

Proof. Recall

$$\max_{\bar{D}} |\nabla\Phi| \leq G\|H\| \max_{\bar{D'}} |\nabla'\tilde{\Phi}|. \tag{6.17}$$

By (6.2), it is not difficult to show that

$$G = \mu \text{ and } \|H\| = \sqrt{\frac{\mu_2}{\mu}} . \tag{6.18}$$

Thus, the desired conclusion follows immediately from (6.17), (6.18) and (4.3), and the proof is now complete.

Equality holds in (6.16) if D is a disk and the material is isotropic.

By taking D_* to be a disk in Lemma 5.5, we may arrive at a simple lower bound.

Theorem 6.3. Let D be bounded and simply connected. Assume $\Phi \in C^2(D) \cap C^1(\bar{D})$ and satisfies (2.7). Then

$$\max_{\bar{D}} |\nabla\Phi| \geq \frac{2\nu\mu_1}{\nu+1}\rho, \tag{6.19}$$

where ρ is the radius of the largest disk contained in D.

Proof. Let D_* be a disk of radius ρ contained in D. Thus, if $x^\circ \in P = \partial D \cap \partial D_*$, Lemma 5.5 furnishes

$$|\nabla\Phi(x^\circ)| \geq |\nabla\Phi_*(x^\circ)|,$$

where Φ_* is the stress function for the disk. By (6.15),

$$|\nabla\Phi_*| = \frac{2\nu\mu_1}{\nu+1}\rho \text{ on } \partial D_*.$$

Therefore,

$$\max_{\bar{D}} |\nabla\Phi| \geq \frac{2\nu\mu_1}{\nu+1}\rho,$$

and the proof is complete.

Clearly, equality holds in (6.19) for D a disk.

Theorem 6.4. Let D be bounded and simply connected. Assume $\Phi \in C^2(D) \cap C^1(\bar{D})$ and satisfies (2.7).

Then

$$\max_{\bar{D}} |\nabla\Phi| \geq \sqrt{\mu\mu_1}\,\rho', \tag{6.20}$$

where ρ' is the radius of the largest disk contained in D'. Further, if $\bar{D}$ is a plane regular region, and $\Phi \in C^3(D) \cap C^1(\bar{D})$, then

$$\max_{\bar{D}} |\nabla\Phi| \geq 2\sqrt{\mu\mu_1}\,\frac{A'}{C'}, \tag{6.21}$$

where A' is the area of D' and C' is the length of $\partial D'$.

Both (6.20) and (6.21) are elementary consequence of (5.11), (6.2), (4.11) and (4.13).

So far, we have been concerned exclusively with bounds on τ. In order to get complete bounds on the maximum shear stress σ, we turn now to the task of finding bounds on the torsional rigidity.

Research aimed at bounds on the torsional rigidity for orthotropic bars has not been so extensive as that for isotropic media. In the sequel, we will deduce bounds for the orthotropic case from bounds appropriate to the isotropic case by means of the affine transformation (6.2).

In the next lemma, we designate by k' the torsional rigidity, corresponding to $\mu = 1$, for the transformed domain D' under (6.2). It is not difficult to verify that K and k' are related by

$$K = \frac{\mu^2}{\sqrt{\mu_1\mu_2}}\,k'. \tag{6.22}$$

A counterpart of the Saint-Venant isoperimetric inequality for the orthotropic case is furnished by

Theorem 6.5. Of all homogeneous orthotropic elastic beams of a given simply connected cross-sectional area A, the elliptic beam with the form

$$\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} = 1, \tag{6.23}$$

where

$$a_1^2 = \sqrt{\frac{\mu_1}{\mu_2}}\,\frac{A}{\pi}\,,\quad a_2^2 = \sqrt{\frac{\mu_2}{\mu_1}}\,\frac{A}{\pi}\,,$$

has the greatest torsional rigidity, i.e.,

$$2\pi K \le \sqrt{\mu_1\mu_2}\,A^2. \tag{6.24}$$

Proof. (6.24) is a straightforward consequence of (6.22), (4.14) and (6.2). For the derivation of (6.23), let D' be a disk of the form

$$x_1'^2 + x_2'^2 = a^2.$$

By (6.2), one has

$$\frac{x_1^2}{\frac{\mu}{\mu_2}a^2} + \frac{x_2^2}{\frac{\mu}{\mu_1}a^2} = 1 \text{ and } a^2 = \frac{\sqrt{\mu_1\mu_2}}{\mu}\,\frac{A}{\pi}\,.$$

Thus, the desired result follows, and the proof is complete.

Further application of (6.22) to the previous results (4.16) - (4.19) leads to

$$K \leq \frac{\mu^2}{\sqrt{\mu_1\mu_2}} I_o' , \tag{6.25}$$

$$K \leq \frac{\mu^2}{\sqrt{\mu_1\mu_2}} \frac{4I_1'I_2'}{(I_1'+I_2')} , \tag{6.26}$$

$$K \geq \frac{\mu^2}{\sqrt{\mu_1\mu_2}} \frac{\pi \dot{r}'^4}{2} , \tag{6.27}$$

and

$$K \geq \frac{\mu^2}{\sqrt{\mu_1\mu_2}} \frac{\pi \rho'^4}{2} , \tag{6.28}$$

where all quantities refer to the transformed domain D' which is obtained under (6.2).

A result of Weinberger [11] is extended by

Lemma 6.2. <u>Let</u> D <u>be the union of two disjoint domains</u> D_1 <u>and</u> D_2, <u>and let</u> K, K_1 <u>and</u> K_2 <u>be the torsional rigidities, respectively.</u>

<u>Then</u>

$$K \geq K_1 + K_2. \tag{6.29}$$

Proof. By (6.22) and a result from [11],

$$K = \frac{\mu^2}{\sqrt{\mu_1\mu_2}} k' \geq \frac{\mu^2}{\sqrt{\mu_1\mu_2}} (k_1' + k_2') = K_1 + K_2 ,$$

where k', k_1' and k_2' are the torsional rigidities corresponding to unit shear modulus, for D', D_1' and D_2', respectively, and the proof is complete.

The importance of this lemma is its implication that a lower bound on K may be obtained by imbedding a subdomain in D for which the torsional rigidity is known. For instance, by taking D_1 to be the largest disk contained in D, we arrive at

Theorem 6.6. Let D be bounded and simply connected. Then,

$$K \geq \frac{\mu_1 \mu_2}{\mu} \frac{\pi \rho^4}{2}, \tag{6.30}$$

where ρ is the radius of the largest disk contained in D.

Proof. Let D_1 be a disk of radius ρ contained in D. It is easy to verify that

$$K_1 = \frac{\mu_1 \mu_2}{\mu} \frac{\pi \rho^4}{2}.$$

Thus, Lemma 6.2 furnishes

$$K \geq \frac{\mu_1 \mu_2}{\mu} \frac{\pi \rho^4}{2},$$

and the proof is now complete.

Now the bounds on the maximum shear stress, σ, follow immediately. For example, by (6.3), (6.14), (6.19), (6.24) and (6.30),

$$\frac{4\pi\sqrt{\nu}}{\nu+1} \frac{\rho}{A^2} \leq \frac{\sigma}{m} \leq \frac{2}{\kappa \pi \rho^4} [1 - (1-\kappa\rho)^{\nu+1}]. \tag{6.31}$$

On the other hand, (6.3), (6.16), (6.20), (6.25) and (6.28) furnish

$$\frac{\mu_1}{\mu}\sqrt{\frac{\mu_2}{\mu}}\,\frac{\rho'}{I'_o} \leq \frac{\sigma}{m} \leq \frac{\mu_2}{\mu}\sqrt{\frac{\mu_1}{\mu}}\,\frac{2(2-\kappa'\rho')}{\pi\rho'^3}\,. \tag{6.32}$$

Equality holds throughout (6.31) and (6.32) for D an isotropic disk.

In order to demonstrate the quality of the bounds on τ, let us consider an elliptic cross section of the form

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1 \quad (0 < b < a). \tag{6.33}$$

One can easily show that

$$\rho = b,\ \kappa = \frac{b}{a^2}\,;$$

$$\rho' = \sqrt{\frac{\mu_1}{\mu}}\ b,\ \kappa' = \frac{\sqrt{\mu\mu_1}}{\mu_2}\,\frac{b}{a^2}\,;$$

$$A' = \sqrt{\frac{\mu_1\mu_2}{\mu}}\ ab,\ C' = 4\sqrt{\frac{\mu_2}{\mu}}\ a\,E(\theta'),$$

where $E(\theta')$ is the complete elliptic integral of second kind, and

$$\theta' = \sin^{-1}\sqrt{\frac{\mu_2 a^2-\mu_1 b^2}{\mu_2 a^2}}\,.$$

Thus, (6.14) and (6.16) give

$$\frac{\tau}{\mu_1 a} \leq \frac{2\nu\epsilon}{\nu+1}\left[1 - \left(1 - \frac{1}{\epsilon^2}\right)^{\nu+1}\right] \tag{6.34}$$

and

$$\frac{\tau}{\mu_1 a} \le \frac{\sqrt{\nu}}{\epsilon} \left(2 - \frac{1}{\nu\epsilon^2}\right), \tag{6.35}$$

while (6.19), (6.20) and (6.21) supply

$$\frac{\tau}{\mu_1 a} \ge \frac{2\nu}{(\nu+1)\epsilon}, \tag{6.36}$$

$$\frac{\tau}{\mu_1 a} \ge \frac{1}{\epsilon}, \tag{6.37}$$

and

$$\frac{\tau}{\mu_1 a} \ge \frac{\pi}{2\epsilon E(\theta')}. \tag{6.38}$$

Here, of course, $\epsilon = a/b$. Furthermore, it is not difficult to verify that the exact value of τ is given by

$$\frac{\tau}{\mu_1 a} = \frac{2\nu\epsilon}{\nu\epsilon^2+1}. \tag{6.39}$$

Table 3 gives a typical example of values of $\tau/\mu_1 a$ and various bounds for the case $\nu=2$. For $\nu=1.1$ and $\nu=3$, the results are plotted in Figures 8 and 9, respectively. In these figures, we denote the upper bounds in (6.34) and (6.35) by T_I and T_{II}, likewise, t_I, t_{II} and t_{III} stand for the lower bounds in (6.36) - (6.38). From these, we observe that for upper bounds, T_I is sharper than T_{II} for small values of ϵ,

Table 3. Upper and lower bounds for elliptic cross sections for $\nu = 2$.

ϵ	$\tau/\mu_1 a$	upper bounds		lower bounds		
		(6.34)	(6.35)	(6.36)	(6.37)	(6.38)
1.1	1.2865	1.459	2.04	1.2121	0.9090	1.0938
1.2	1.2371	1.5544	1.9478	1.1111	0.8333	1.0314
1.5	1.0909	1.6571	1.6761	0.8889	0.6667	0.8778
2.0	0.8889	1.5417	1.3258	0.6667	0.50	0.6984
2.5	0.7407	1.3577	1.0861	0.5333	0.40	0.5777
3.0	0.6316	1.1907	0.9166	0.4444	0.3333	0.4914
3.6	0.5349	1.0276	0.7705	0.3704	0.2778	0.4190
4.0	0.4849	0.9388	0.6961	0.3333	0.25	0.3771
4.6	0.4248	0.8291	0.6076	0.2899	0.2174	0.3307
5.0	0.3922	0.7684	0.56	0.2667	0.20	0.3055
5.5	0.3577	0.7035	0.51	0.2424	0.1818	0.2788
6.0	0.3288	0.6483	0.4681	0.2222	0.1667	0.2563
6.6	0.2996	0.5923	0.4261	0.2020	0.1515	0.2339
7.0	0.2828	0.5596	0.402	0.1905	0.1429	0.2208

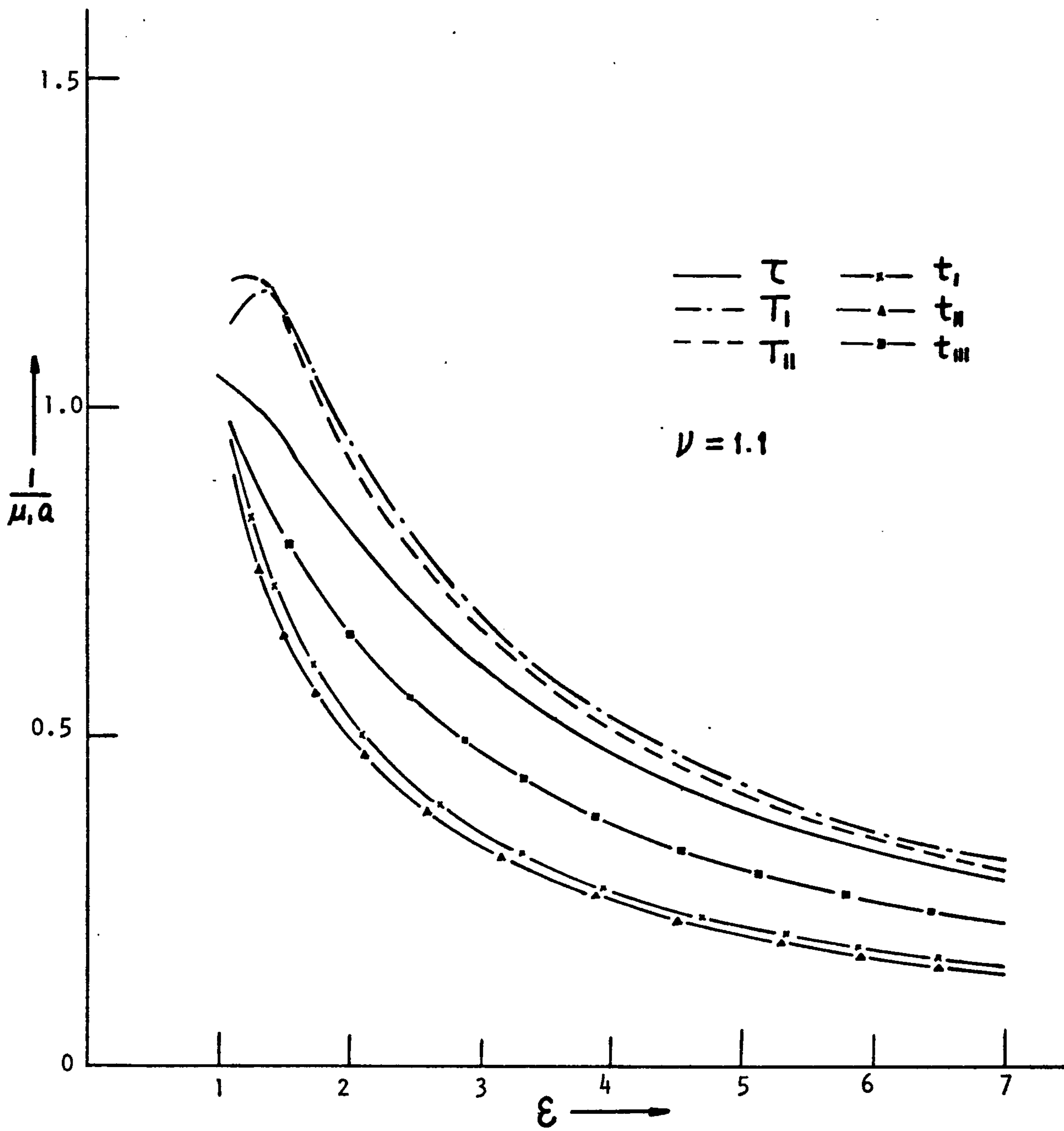

Figure 8. Upper and Lower Bounds for Various Elliptic Cross Sections ($\nu = 1.1$)

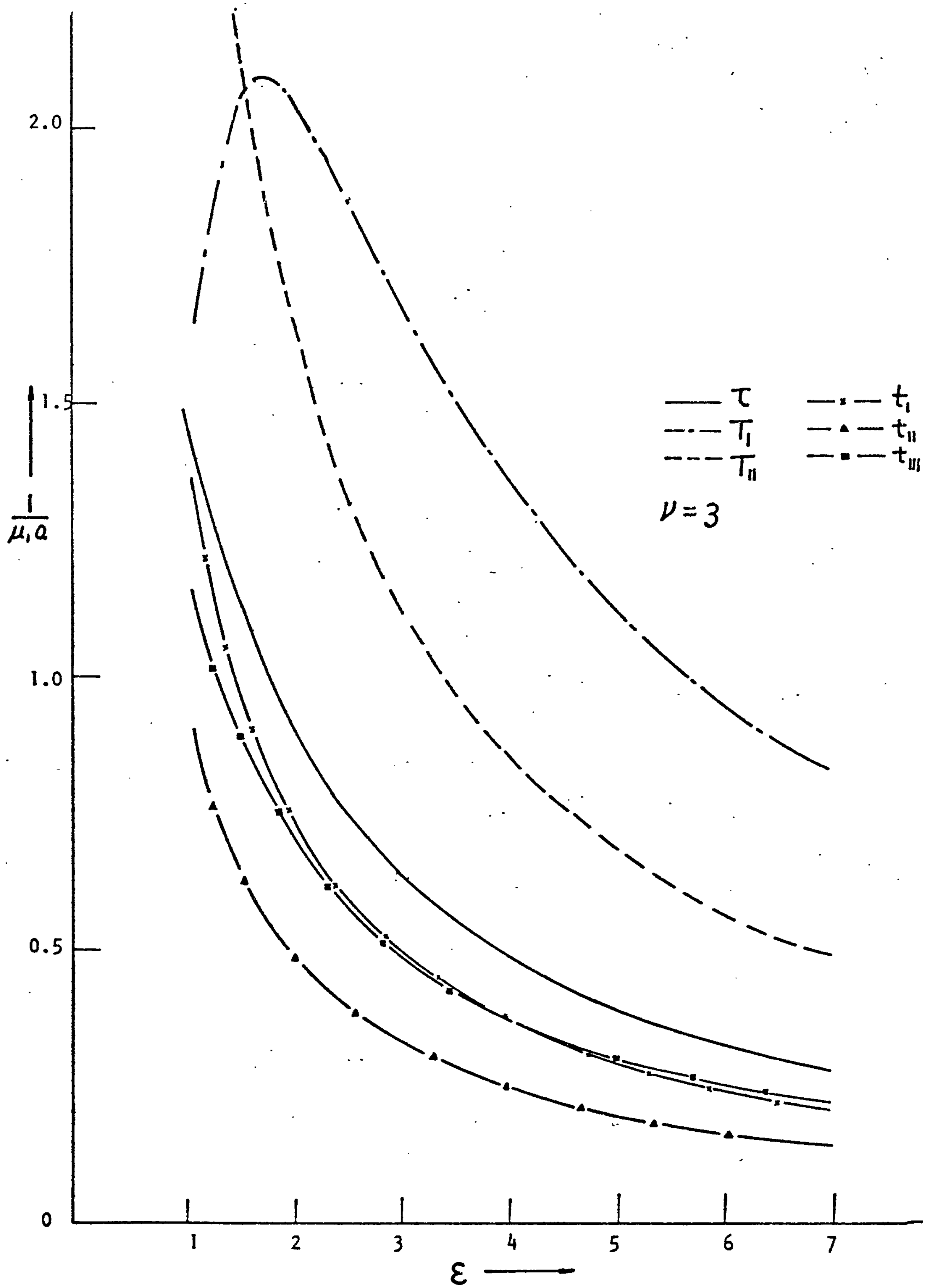

Figure 9. Upper and Lower Bounds for Various Elliptic Cross Sections ($\nu = 3$)

say $\epsilon < 1.5$. For $\epsilon > 1.5$, T_{II} is closer to the exact value. As for the lower bounds, the best one is afforded by t_{III} for large values of ϵ, while for small ϵ, t_I is sharper than t_{III}. In all cases, t_{II} is the crudest of the lower bounds. These conclusions are borne out by the plots in Figures 10 and 11.

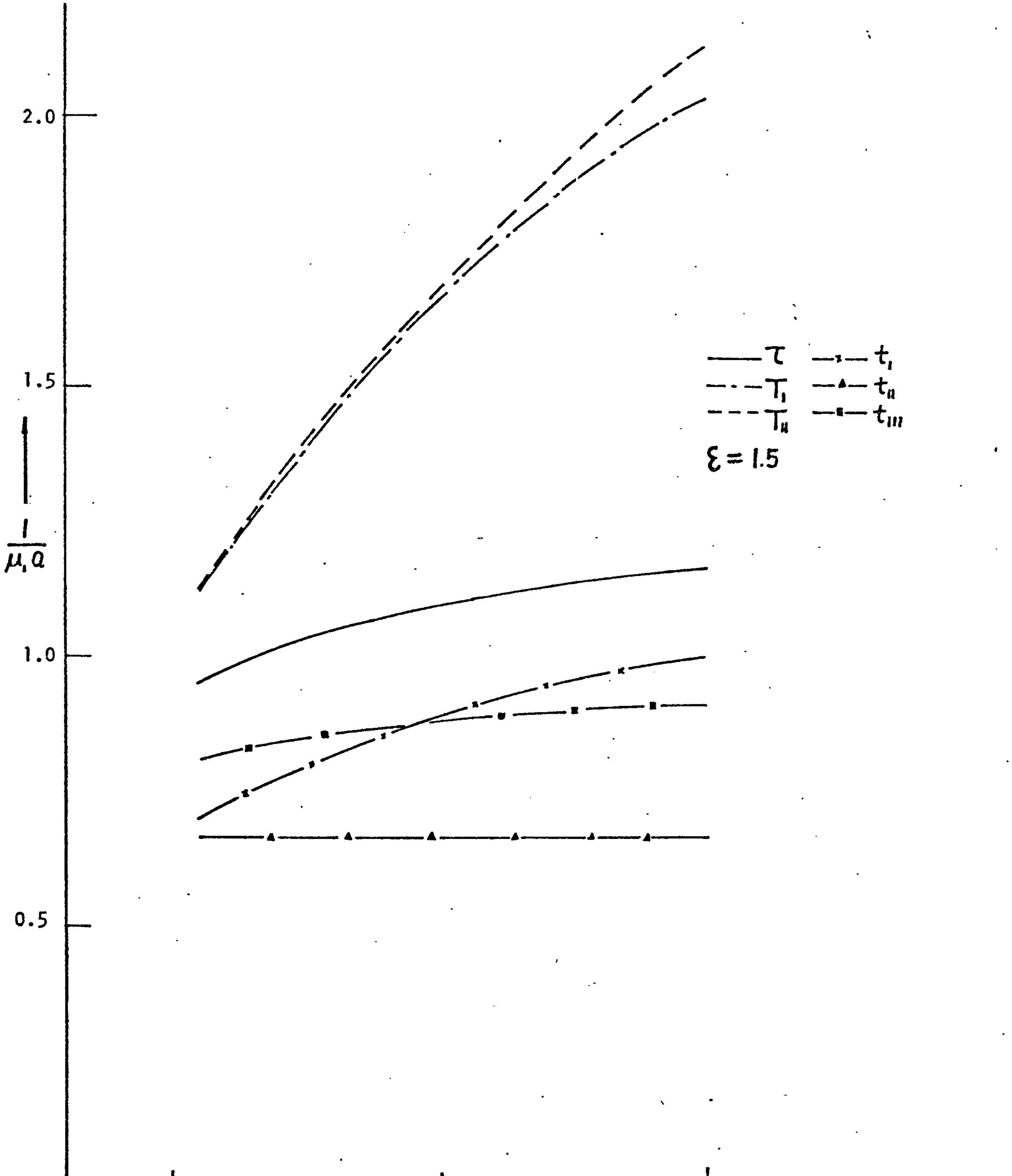

Figure 10. Upper and Lower Bounds for Various Values of ν (Elliptic Cross Section with $\epsilon = 1.5$)

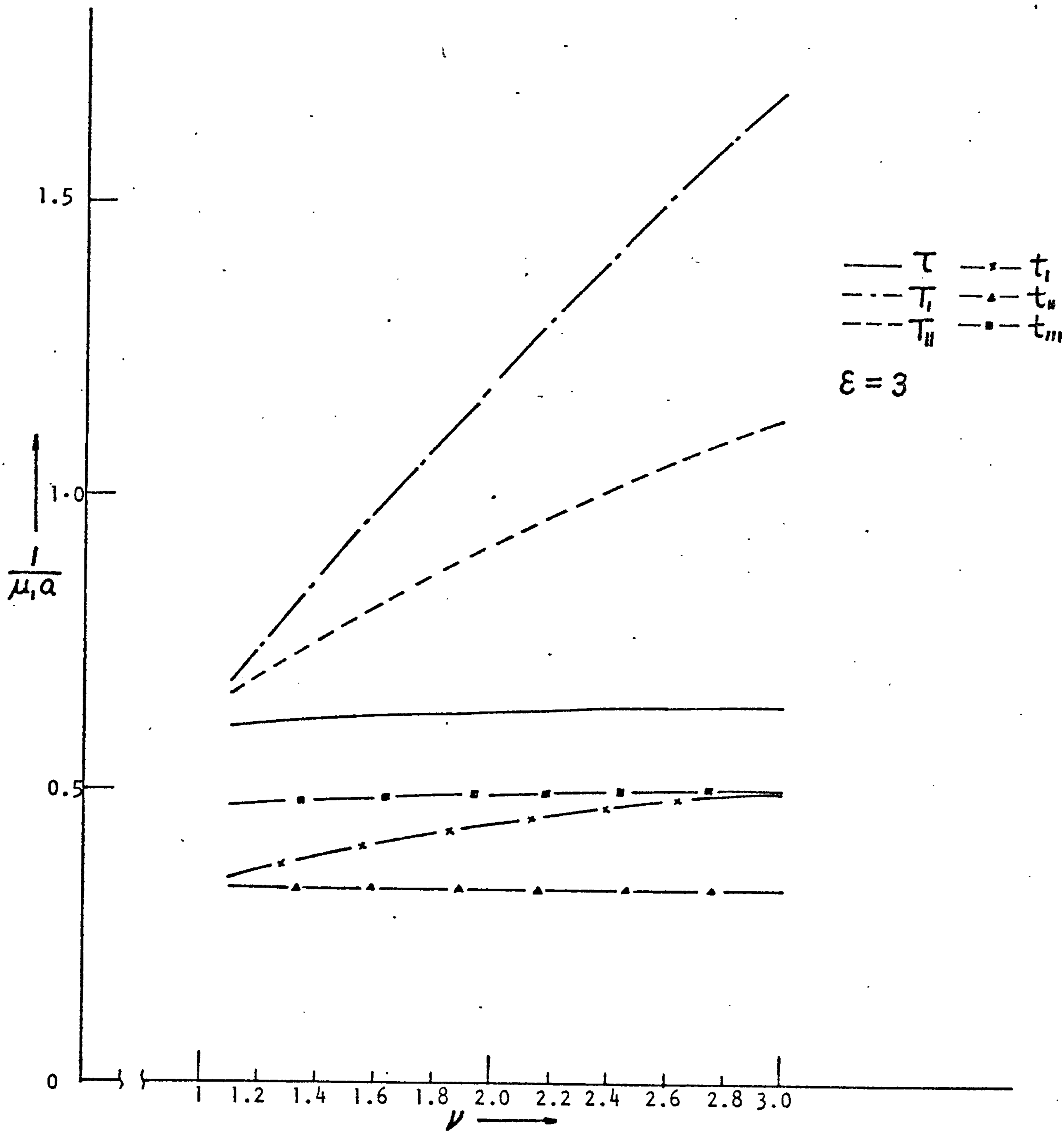

Figure 11. Upper and Lower Bounds for Various Values of ν (Elliptic Cross Section with ϵ = 3)

7. Stress bounds for homogeneous anisotropic elastic bars

The operator L, in view of (2.1), reduces to

$$L \equiv a_{44} \frac{\partial^2}{\partial x_1^2} - 2a_{45} \frac{\partial^2}{\partial x_1 \partial x_2} + a_{55} \frac{\partial^2}{\partial x_2^2} \text{ on } D. \tag{7.1}$$

The eigenvalues of the constant coefficient matrix

$$A = \begin{bmatrix} a_{44} & -a_{45} \\ -a_{45} & a_{55} \end{bmatrix} \tag{7.2}$$

in (6.1) are easily found to be

$$\alpha_2 = \tfrac{1}{2}[a_{44}+a_{55}+\sqrt{(a_{55}-a_{44})^2+4a_{45}^2}],$$

and (7.3)

$$\alpha_1 = \tfrac{1}{2}[a_{44}+a_{55}-\sqrt{(a_{55}-a_{44})^2+4a_{45}^2}].$$

The ellipticity of L on D requires that

$$a_{44}a_{55} \geq a_{45}^2. \tag{7.4}$$

Accordingly, A confirms to the condition (5.15) in Lemma 5.4. An affine transformation that takes this operator into the Laplacian is given by

$$x_1' = \frac{a_{55}-\alpha_1}{\lambda_1} x_1 + \frac{a_{45}}{\lambda_1} x_2 , \quad x_2' = \frac{a_{45}}{\lambda_2} x_1 + \frac{a_{44}-\alpha_2}{\lambda_2} x_2 , \tag{7.5}$$

where

$$\lambda_1 = \sqrt{\frac{\bar{\alpha}}{\alpha_2}[(a_{55}-\alpha_1)^2+a_{45}^2]} , \ \lambda_2 = \sqrt{\frac{\bar{\alpha}}{\alpha_1}[(a_{44}-\alpha_2)^2+a_{45}^2]} , \ \bar{\alpha} = \frac{\alpha_1+\alpha_2}{2}$$

We turn now to the task of getting bounds on

$$\tau = \max_{\bar{D}} |\nabla\Phi| . \tag{7.6}$$

As in the orthotropic case, two upper bounds will be derived by employing Lemmas 5.2 and 5.3. On the other hand, we will appeal to Lemmas 5.3 and 5.5 to arrive at the lower bounds. The following lemma ensures the existence of a function that confirms to hypothesis (a) of Lemma 5.2.

Lemma 7.1. Let D be a plane domain whose boundary has continuous curvature, the minimum value of which is κ, and let ρ be the radius of the largest disk contained in D. Let $\rho_o = \rho$ if D is a disk. If D is not a disk and $\kappa > 0$, let $\rho_o \in (\rho, \frac{1}{\kappa})$. If $\kappa \leq 0$, let $\rho_o > \rho$. Define $f(s)$ for every $s \in [0,\rho_o]$ by

$$f(s) = \begin{cases} \dfrac{1}{\kappa[(1-\kappa\rho_o)^{\beta+1}-1]} \left\{ \dfrac{(1-\kappa\rho_o)^{\beta+1}}{\beta-1} \left[\left(\dfrac{1}{1-\kappa s} \right)^{\beta-1} -1 \right] + \tfrac{1}{2}[(1-\kappa s)^2-1] \right\}, & \text{if } \kappa \neq 0 \\ s\left(1 - \dfrac{s}{2\rho_o}\right), & \text{if } \kappa = 0 \end{cases} \tag{7.7}^*$$

<u>where</u> $\beta = \alpha_2/\alpha_1$ <u>and</u> <u>assume</u> $a_{55} > a_{44} > 0$. <u>Define</u> <u>a</u> <u>function</u> u <u>for</u> <u>every</u> $x \in \bar{D}$ by

$$u(x) = f(\delta(x)). \tag{7.8}$$

<u>Then</u> $u \in C(\bar{D})$, $u = 0$ <u>on</u> ∂D, $\dfrac{\partial u}{\partial n} = -1$ <u>on</u> ∂D, <u>and</u>

$$L\,u \leq \frac{-\alpha_1(\beta+1)\kappa}{1-(1-\kappa\rho_o)^{\beta+1}} . \tag{7.9}$$

The proof is entirely analogous to that of Lemma 5.1. In this proof, one should note that, by placing the origin of the coordinate system at z (see Fig.1), it follows

$$Lu'(x^\circ) = \frac{a_{44}x_1^{\circ 2}-2a_{45}x_1^\circ x_2^\circ+a_{55}x_2^{\circ 2}}{r^2}\left[\frac{\partial^2 u'}{\partial r^2} + \frac{1}{r}\,\frac{a_{44}x_2^{\circ 2}+2a_{45}x_1^\circ x_2^\circ+a_{55}x_1^{\circ 2}}{a_{44}x_1^{\circ 2}-2a_{45}x_1^\circ x_2^\circ+a_{55}x_2^{\circ 2}}\,\frac{\partial u'}{\partial r}\right], \tag{7.10}$$

where r is the distance of x° from z. Thus, if we choose the

---

*The form of (7.7) is similar to that in (6.5), except that ν is replaced by β.

coordinate system

$$x_1^\circ = r\cos\theta,\ x_2^\circ = r\sin\theta$$

such that

$$\tan 2\theta = \frac{2a_{45}}{a_{55}-a_{44}},$$

then (7.10) reduces to

$$Lu'(x^\circ) = \alpha_1[f''(\delta'(x^\circ)) - \frac{\kappa}{1-\kappa\delta'(x^\circ)}\beta f'(\delta'(x^\circ))].$$

Now the conclusion can be easily reached by means of the procedure used to prove Lemma 5.1.

The foregoing lemma, (7.4), and Lemmas 5.2 and 5.4 furnish the upper bound in

Theorem 7.1. Let D be bounded and simply connected, and assume ∂D has continuous curvature. Suppose that $\Phi \in C^3(D)\cap C'(\bar{D})$ and obeys (2.1).

Then

$$\max_{\bar{D}} |\nabla\Phi| \le \frac{2}{\alpha_1(\beta+1)\kappa}[1 - (1-\kappa\rho)^{\beta+1}]. \tag{7.11}$$

The proof of this theorem is strictly analogous to that of Theorem 4.1. Here one picks

$$v(x) = \tfrac{1}{2}\left(\frac{x_1^2}{a_{44}} + \frac{x_2^2}{a_{55}}\right), \tag{7.12}$$

and employs Lemma 6.1, (7.4), and Lemmas 5.2 and 5.4.

If $a_{45} = 0$ in (7.11), it reduces to (6.14). Thus, $\alpha_1 = \alpha_{44} = \frac{1}{\mu_2}$, $\beta = \frac{a_{55}}{a_{44}} = \frac{\mu_2}{\mu_1} = \nu$, then

$$\max_{\bar{D}} |\nabla\Phi| \le \frac{2\nu\mu_1}{\kappa(\nu+1)} [1 - (1-\kappa\rho)^{\nu+1}].$$

It is not difficult to verify that for D a disk with radius ρ, τ is given by

$$\tau = \frac{2\rho}{a_{44}+a_{55}}. \tag{7.13}$$

Hence, the right-hand member of (7.11) reduces to the exact value when D is a disk.

We turn now to another upper bound which is in essence found by direct application of the affine transformation to (4.3).

Theorem 7.2. Let D be bounded and simply connected, and assume ∂D has continuous curvature. Suppose that $\Phi \in C^3(D) \cap C^1(\bar{D})$ and obeys (2.1). Then

$$\max_{\bar{D}} |\nabla\Phi| \le \frac{1}{\alpha_1}\sqrt{\frac{\bar{\alpha}}{\alpha_2}}\, \rho'(2-\kappa'\rho'), \tag{7.14}$$

where ρ' is the radius of the largest disk contained in D' and κ' is the minimum curvature of $\partial D'$.

Proof. Recall

$$\max_{\bar{D}} |\nabla\Phi| \le G\|H\| \max_{\bar{D}'} |\nabla'\tilde{\Phi}|. \tag{7.15}$$

For this case, by (7.5), one has

$$H = \begin{bmatrix} \dfrac{a_{55}-\alpha_1}{\lambda_1} & \dfrac{a_{45}}{\lambda_2} \\ \\ \dfrac{a_{45}}{\lambda_1} & \dfrac{a_{44}-\alpha_2}{\lambda_2} \end{bmatrix}$$

and $G = \bar{\alpha}/(\alpha_1\alpha_2)$. It is easily found that

$$\|H\| = \sqrt{\frac{\alpha_2}{\bar{\alpha}}} .$$

Thus, the desired conclusion follows from (7.15) and (4.3), and the proof is now complete.

The inequality (7.14) reduces to (6.16), if we let $a_{45} = 0$. Equality holds in (7.14) if D is a disk with the isotropic material.

A simple lower bound will be arrived at in the next theorem by picking D_* to be a disk.

Theorem 7.3. <u>Let</u> D <u>be bounded and simply connected</u>. <u>Assume</u> $\Phi \in C^2(D)\cap C^1(\bar{D})$ <u>and satisfies</u> (2.1). <u>Then</u>

$$\max_{\bar{D}} |\nabla\Phi| \ge \frac{2\rho}{a_{44}+a_{55}}, \tag{7.16}$$

<u>where</u> ρ <u>is the radius of the largest disk contained in</u> D.

The proof, which is closely analogous to the proof of Theorem 6.3, is easily constructed with the aid of Lemma 5.5.

Direct application of the affine transformation to the results in (4.11) and (4.13) is the key idea involved in

Theorem 7.4. Let D be bounded and simply connected. Assume $\Phi \in C^2(D) \cap C'(\bar{D})$ and satisfies (2.1).

Then

$$\max_{\bar{D}} |\nabla\Phi| \geq \frac{1}{\alpha_2}\sqrt{\frac{\bar{\alpha}}{\alpha_1}}\ \rho', \tag{7.17}$$

where ρ' is the radius of the largest disk contained in D'. Further, if $\bar{D}$ is a plane regular region, and $\Phi \in C^3(D) \cap C'(\bar{D})$, then

$$\max_{\bar{D}} |\nabla\Phi| \geq \frac{1}{\alpha_2}\sqrt{\frac{\bar{\alpha}}{\alpha_1}}\ \frac{2A'}{C'}, \tag{7.18}$$

where A' is the area of D' and C' is the length of $\partial D'$.

Proof. Lemma 5.3 gives

$$G\left\|H^{-1}\right\|^{-1} \max_{D'} |\nabla'\tilde{\Phi}| \leq \max_{\bar{D}} |\nabla\Phi|. \tag{7.19}$$

By (7.5), one has

$$G = \frac{\bar{\alpha}}{\alpha_1\alpha_2}, \quad H^{-1} = \frac{\lambda_1\lambda_2}{(a_{55}-\alpha_1)(a_{44}-\alpha_2)-a_{45}^2}\begin{bmatrix} \dfrac{a_{44}-\alpha_2}{\lambda_2} & -\dfrac{a_{45}}{\lambda_1} \\[2ex] -\dfrac{a_{45}}{\lambda_2} & \dfrac{a_{55}-\alpha_1}{\lambda_1} \end{bmatrix},$$

and hence

$$\| H^{-1} \| = \sqrt{\frac{\bar{\alpha}}{\alpha_1}} \ .$$

Thus, (7.17) follows from (7.19) and (4.11). On the other hand, (7.18) is implied by (7.19) and (4.13). The proof is now complete.

With a view toward getting bounds on the maximum shear stress σ, we now take up the problem of determining bounds on the torsional rigidity. The procedure followed here is closely related to the one used in the preceding section for orthotropic bars.

Let us denote by k' the torsional rigidity of the transformed domain, corresponding to $\mu = 1$. By (2.4) and (7.5), the relation between K and k' can be established by

$$K = \frac{\bar{\alpha}^2}{(\alpha_1 \alpha_2)^{3/2}} k' . \qquad (7.20)$$

Let $c = \det J$, where J is the matrix associated with the transformation (7.5). A generalization of the Saint-Venant isoperimetric inequality is given by

Theorem 7.5. Of all homogeneous anisotropic elastic beams of a given simply connected cross-sectional area A, the elliptic beam with the form

$$B_1 x_1^2 + B_2 x_1 x_2 + B_3 x_2^2 = A, \qquad (7.21)$$

where

$$B_1 = \frac{\pi}{c}\left[\left(\frac{a_{55}-\alpha_1}{\lambda_1}\right)^2 + \left(\frac{a_{45}}{\lambda_2}\right)^2\right],$$

$$B_2 = \frac{2\pi a_{45}}{c}\left[\frac{(a_{55}-\alpha_1)}{\lambda_1^2} + \frac{(a_{44}-\alpha_2)}{\lambda_2^2}\right],$$

$$B_3 = \frac{\pi}{c}\left[\left(\frac{a_{45}}{\lambda_1}\right)^2 + \left(\frac{a_{44}-\alpha_2}{\lambda_2}\right)^2\right],$$

<u>has the greatest torsional rigidity, i.e.</u>,

$$2\pi K \leq \frac{1}{\sqrt{\alpha_1\alpha_2}} A^2. \tag{7.22}$$

The proof is entirely analogous to that of Theorem 6.5, and may safely be omitted.

Further application of (7.20) to the previous results (4.16) - (4.19) of the isotropic case also furnishes

$$K \leq \frac{\bar{\alpha}^2}{(\alpha_1\alpha_2)^{3/2}} I_o', \tag{7.23}$$

$$K \leq \frac{\bar{\alpha}^2}{(\alpha_1\alpha_2)^{3/2}} \frac{4 I_1' I_2'}{I_1' + I_2'}, \tag{7.24}$$

$$K \geq \frac{\bar{\alpha}^2}{(\alpha_1\alpha_2)^{3/2}} \frac{\pi r'^4}{2}, \tag{7.25}$$

and

$$K \geq \frac{\bar{\alpha}^2}{(\alpha_1\alpha_2)^{3/2}} \frac{\pi\rho'^4}{2}, \tag{7.26}$$

where all quantities refer to D' which is obtained under (7.5).

By using (7.20) and a result of Weinberger [11], one may establish

Lemma 7.2. Let D be the union of two disjoint domains D_1 and D_2, and let K, K_1 and K_2 be their torsional rigidities.

Then

$$K \geq K_1 + K_2. \tag{7.27}$$

A proof may be constructed along the lines of the arguments used to establish Lemma 6.2.

We may imbed any subdomain D_1 with known torsional rigidity K_1 in D. Thus, (7.27) affords us a simple lower bound on K. If D_1 is the largest disk with radius ρ, we arrive at

$$K \geq \frac{\pi\rho^4}{a_{44}+a_{55}}. \qquad (7.28)^*$$

Now the information that we need to get complete bounds on the maximum shear stress is at hand. For instance, by (7.11), (7.16), (7.22) and (7.28), one obtains

---

*The exact value of the torsional rigidity for a disk can be found, for example, in [6, p. 197].

$$\frac{4\pi\rho\ \sqrt{\alpha_1\alpha_2}}{(a_{44}+a_{55})A^2} \leq \frac{\sigma}{m} \leq \left(\frac{a_{44}+a_{55}}{\pi\rho^4}\right) \frac{2}{\alpha_1(\beta+1)\kappa} [1 - (1-\kappa\rho)^{\beta+1}]. \tag{7.29}$$

BIBLIOGRAPHY

[1] Colombo, G., Maggiorazioni delle Componenti di Stress nel Problema di De Saint-Venant, Rend. Seminar. Mat. Univ. Padova, **24**, 70-83 (1955).

[2] Protter, M. H. and Weinberger, H. F., Maximum Principles in Differential Equations, Prentice-Hall, Inc. (1967).

[3] Radó, T., Subharmonic Functions, Ergebnisse der Mathematik und Ihrer Grenzgebeite, Berlin, 1937 (reprinted by Chelsea, New York, 1949).

[4] Helms, L. L., Introduction to Potential Theory, Wiley and Sons, Inc., (1969).

[5] Cimmino, G., Formole di Maggiorazione nel Problema di Dirichlet per le Funzioni Armoniche, Rend. Sem. Mat. Un. Padova, **3**, 46-66 (1932).

[6] Lekhnitskii, S. G., Theory of Elasticity of An Anisotropic Elastic Body, Holden-Day, Inc., San Francisco, (1963).

[7] Kellogg, O. D., Foundations of Potential Theory, New York, (1953).

[8] Payne, L. E., Isoperimetric Inequalities and Their Applications, S.I.A.M. Review, **9**, 453-488 (1967).

[9] Pólya, G., Torsional rigidity, principal frequency, electrostatic capacity and symmetrization, Quart. Appl. Math., 6, 267-277, (1948).

[10] Pólya, G. and Szegö, G., Isoperimetric Inequalities in Mathematical Physics, Ann. of Math. Studies, no. 27, Princeton University Press, Princeton, (1951).

[11] Weinberger, H. F., Upper and lower bounds for torsional rigidity, J. Math. and Phys., 32, 54-62, (1953).

[12] Sokolnikoff, I. S., Mathematical Theory of Elasticity, McGraw-Hill Co. (1956).

[13] Timoshenko, S. P. and Goodier, J. N., Theory of Elasticity, McGraw-Hill Co., 3rd ed., (1970).

[14] Muskhelishvili, N. I., Some Basic Problems of the Mathematical Theory of Elasticity, P. Noordhoff Ltd., Groningen, (1963).

[15] Morrey, C. B., Second order elliptic systems of differential equations, Ann. of Math. Studies, no. 33, Princeton University Press, Princeton, 101-159, (1954).