ANALYSIS OF INTERSYMBOL INTERFERENCE IN FADING CHANNELS

A Dissertation

Presented to

the Faculty of the Department of Electrical Engineering

University of Houston

In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

by

Jorge Valerdi

December 1972

TO WHOM IT MAY CONCERN

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ABSTRACT

Tropospheric propagation paths are becoming very attractive in both military and commercial use due to their high data rate capability. This type of propagation has the characteristic of a frequency-selective fading channel. It is of interest in this dissertation to study how this selectivity affects the digital error probabilities, since the data rate capability of radio channels is limited by the symbol distortion and intersymbol interference.

Two procedures for the evaluation of the probability of error for binary receivers in the presence of intersymbol interference in random channels are studied. The first one is observed to be useful when the channel is slowly varying as compared with the signal transmission rate. The second procedure is showed to have an advantage over the slowly-varying method in that the selectivity of the medium can be easily incorporated.

For the slowly-varying channel, a unified analysis of the effects of fading and intersymbol interference due to filtering is presented for both the Rayleigh and the Rician fading channel. The matched filter operation is replaced by a Butterworth filter at the front end of the receiver. It is observed that when no diversity is used (L=1), the main degradation of the performance curves is due to fading, but when this degradation is totally or partially eliminated (perhaps using diversity) the effects of bandlimiting become an important factor for the design. When the probability of error is about 10^{-2}, the de-

gradation due to intersymbol interference in the presence of fading is approximately 2 dB for WT=0.5. If WT$\geq$1.5 there is essentially no degradation due to filtering. Ideal and realizable filters are compared, and it is observed that the relative degradation due to filtering is diminished about 0.5 dB when realizable filters are used.

The largest improvement with diversity is between the L=1 and the L=2 family of curves. For WT=1.0 this gain is 8 dB for a probability of error of 10^{-2} and 16 dB for a probability of error of 10^{-4}. It is also observed that if the probability of error is 10^{-4} the degradation due to filtering is of the order of 1.5 dB when L=2 and WT=0.5, which is small when compared with the 4 dB of degradation obtained for the non-fading case.

For the study of the distribution of errors in Rayleigh fading, an expression in terms of the gamma function is obtained. This qualitative analysis determines the range of SNR values responsible for the errors when fading is present as a function of the filter bandwidth and the order of diversity.

When the decision variable of a binary receiver can be formulated as a quadratic form, an expression for the probability density function is obtained. A general type of fading channel, which includes not only a random component but also a specular component, is considered. The average probability of error for the generalized quadratic form is obtained. It is proven that binary symmetric operation, in general, does not exist if the channel is characterized as a random filter. Thus the hypothesis that selective fading introduces non-symmetric operation is demonstrated.

Some results for the probability of error as a function of the intersymbol interference in DPSK are plotted. From the performance curves it is observed that there exists an irreducible probability of error. Values for this irreducible probability of error are given as a function of the filter bandwidth and the channel correlation bandwidth.

A generalized form for the moments is derived. Also, it is shown how the deterministic part of the moments can be evaluated independently of the channel characteristics from a general formula.

Finally, it is concluded that a great variety of problems can be solved by simply formulating the decision variable in a quadratic form of Gaussian random variables and using the expressions for the probability of error obtained in this dissertation. This expression offers not only the ability of analyzing a large class of receivers without performing many computations, but it also gives a unified way of studying the effects of intersymbol interference in random channels.

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A	- magnitude and phase of specular component (non-fading)
a,b,c	- real constants
α_k	- magnitude of the kth specular component
B	- channel frequency spread
B_c	- channel correlation bandwidth
D	- decision variable for Lth-order diversity
$D_i^2(\ )$	- SNR degradation due to filter bandlimiting
$d_k(t)$	- generalized quadratic form (real) of the kth branch
δ_{ij}	- Dirac delta
δ_k	- phase of the kth specular component
$\delta(t)$	- delta function
Δ	- det [I-sMQ]
ΔR	- det R
E	- signal energy
$\mathrm{erf}(\ )$	- error function
$\mathrm{erfc}(\ )$	- complementary error function
f	- frequency
F_c	- overall doppler spreading
${}_1F_1(a,b;z)$	- confluent hypergeometric function
${}_2F_1(a,b;c;z)$	- hypergeometric function
γ	- generalized SNR
γ_o	- average value of γ
$\gamma(\alpha,\nu)$	- incomplete gamma function

P_ε	- average probability of error
P_{ei}	- probability of error for the ith pattern
$P_r(S_k)$	- a priori probability for signaling waveform $S_k(t)$
Q	- a Hermitian matrix
ρ	- signal to noise ratio (dB)
R_{av}	- average signal-to-noise ratio
$R(\tau)$	- $R(0,\Delta t)$
$r(t)$	- received signal = $R_e\{u(t)e^{jw_o t}\}$
$R(\Delta f)$	- $R(\Delta f,0)$
$R(\Delta t)$	- $R(0,\Delta t)$
$R(\tau,\Delta t)$	- multipath autocovariance function
$\mathcal{R}(\Delta f;\Delta t)$	- time-frequency correlation function
s	- complex variable in the domain of the characteristic function
S	- column vector formed from a set fo complex random variables, the real and imaginary parts of which are normally distributed
$\sigma(\tau,f)$	- scattering function
S^T	- transpose of S
S^*	- conjugate of S
$S(t)$	- general signaling waveform
τ	- multipath delay
τ_k	- additional relative delay of the kth path
T	- pulse length
T_c	- channel coherence time
t_1	- average multipath delay

$\Gamma(n)$ - gamma function

$H_1(f)$ - spectrum of the lowpass equivalent of the information filter

$H_2(f)$ - spectrum of the lowpass equivalent of the reference filter

$h_1(\tau,t)$ - time-varying equivalent lowpass impulse response of the channel

$H_n(z)$ - hermite polynomial

$h(\tau;t)$ - impulse response of time-varying channel

$H(f;t)$ - transfer function of time-varying channel

I - unit matrix

$I_n(x)$ - modified Bessel function of order n

L - order of diversity

L_c - channel multipath spread

M - the complex covariance (or moment) matrix assumed nonsingular ($2L\times 2L$)

μ - cross correlation coefficient

m_u - variance of the random variable U

m_{uv} - covariance of the random variables U and V

N - number of bits in a pattern

N_o - spectral density of additive white, Gaussian noise

$n(t)$ - additive Gaussian noise process

$N(x)$ - probability of error conditional on the SNR$<x$

$\Phi_k(\ )$ - characteristic function of the kth quadratic form in complex Gaussian variables

$p(D)$ - probability density function of D

$u_k(t)$ - complex Gaussian variables (non-zero mean)

$u(t)$ - complex envelope of received signal

$v_k(t)$ - complex Gaussian variables (non-zero mean)

$\langle V \rangle$ - ensemble average of V

W - receiver filter bandwidth

W_s - signal bandwidth

$w(t,\tau,u)$ - deterministic response of the receiver filter to the ith pattern

CHAPTER I

INTRODUCTION

1.1 Physical System

When the atmosphere is used as a communication channel to transmit information, the transmitted signal can become distorted. This distortion can be in amplitude, frequency or phase, and if the information is in the form of a binary pulse, the distortion can also be in pulse duration. When the amplitude of the transmitted pulse changes, then the channel is attenuating the signal. If the channel changes the frequency of the transmitted carrier, it will produce a doppler effect on the signal. When the channel changes the phase of the original signal, the received carrier phase will be unstable. This will determine the stâbility or coherence of the channel. If the time duration of the transmitted pulse changes, then pulse smearing or multipath is present. If any of the above parameters change with time, then we have a time varying or random channel.

Tropospheric propagation paths are becoming very attractive to both military and commercial use due to the high data rate capability. This type of propagation has the characteristic of a frequency-selective fading channel. It is of interest in this dissertation to study how this selectivity affects the digital error probabilities, since the data rate capability of radio channels is limited by the symbol distortion and intersymbol interference.

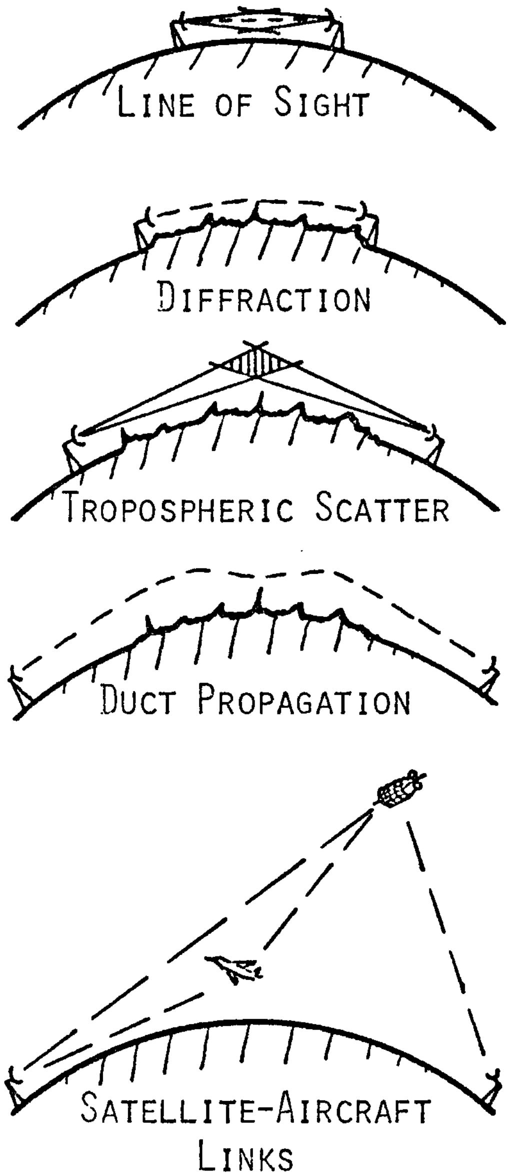
In order to characterize the behavior of the troposphere, one can represent the communication channel as a linear filter [1,2,3]. The random time-varying nature of the channel requires that the corresponding filter be a time-variant filter. Thus, the most general model of the channel model should include a random time-variant filter, whose output contains both slowly-varying and fast-varying random signal components, and a deterministic filter to take into account the fact that the slow- and fast-varying components can have an ensemble average*. This random time-varying filter can be reduced to more simplified channel models such as the time-flat frequency-flat, or flat-flat, fading channel;the frequency-flat time selective, or pure time selective, fading channel; and the frequency-selective time-flat, or pure frequency-selective, fading channel. The flat-flat fading model is one in which the received digital pulse differs from the transmitted pulse only in an amplitude change and a phase shift. The time-selective fading channel has the property that the amplitude and phase fluctuations observed in the channel response to a sinusoid are the same for any excitation frequency. The frequency-selective channel model is represented as a random but time-invariant linear filter. Further discussion

*Note the implication of the existence of an ergodic process. Consequently, stationarity is being assumed. See Chapter II for a justification of the above assumptions.

of the particular cases of time-variant or random channels is given in Section 2.3-C.

Previous analyses which pertain to the performance of digital communication systems of practical interest, e.g., correlation receivers, have not used such a general time-variant model for the channel. In fact, the great majority of the analyses performed on this class of channels include only non-coherent detection of signals. There are situations where the study of coherent (or partially coherent) detection in random channels can be of interest. For example, in tactical troposcatter links, coherent detection can be important when the transmission of scientific data has to be done at the highest speed possible under unpredictable environmental conditions (see Fig. I-1c). Also, if we attempt to send commands to a spacecraft which is under control of a omni-antenna and a correlation receiver on board the vehicle, we have to make use of a coherent receiver (see Fig. I-1e). Further, in planetary relay links, when telemetry data involves the transmission through a dense atmosphere, partially coherent detection can be used. As another example, in satellite down links, where transmission is through turbulent layers such as the troposphere, a coherent receiver could be used (see Fig. I-1e). Other situations of tropospheric propagation are depicted in Fig. I-1.

Then a knowledge of the performance of coherent receivers under the above conditions, some of which are already in operation, is of paramount importance. Most of these receivers



	Frequency Range	Propagation Range	Special Features
A. Line of Sight	70 MHz-20 GHz	$\leq$50 Km	-High reliability* -Critical adjustment -Low-cost
B. Diffraction	70 MHz-20 GHz	50-180Km	-Medium reliability* -Seasonal effects
C. Tropospheric Scatter	200 MHz-15 GHz	100-1000Km	-High reliability* -Critical adjustment -High power
D. Duct Propagation	~ 30 MHz	~ 2000Km	-Low reliability*
E. Satellite-Aircraft Links	100 MHz-100 GHz		-High reliability* -Critical adjustment -Power limitations

*High reliability → $P_\varepsilon \sim 10^{-7}$
Medium reliability → $P_\varepsilon \sim 10^{-5}$
Low reliability → $P_\varepsilon \sim 10^{-3}$
(at highest data rate possible)

Fig. I-1 Some Tropospheric Propagation Paths have been designed and found to be optimum, when the received signal is known (statistically speaking) and when the channel has been modeled as a special case of the above linear time-variant general model. Also information distortion due to receiver filtering has been completely neglected when studying communication systems in random channels. As shown in later chapters, the effects of filtering are negligible as compared with the degradation due to fading, but when the latter degradation is partially or totally eliminated, then the degradation due to bandlimiting at the receiver becomes important at high transmission rates.

1.2 Fading

A large number of references are available on the flat-flat, or slowly, fading channel. A review of some of the papers which have been useful in this work will be given in this section. In the areas of selective-fading and fast-fading, a complete coverage of the reported research is attempted, since these two topics have received relatively little attention because of their mathematical difficulties. Nevertheless, the efforts to analyze selective random channels have been very meaningful since selective-fading and fast-fading are very realistic situations in communication channels.

Error analyses for slowly-fading multipath and diversity channels have been carried out by Turin [4], Pierce [5], and Lindsey [6]. Turin [7] considers optimum diversity reception through dependent Rayleigh-fading paths with no explicit channel measurement involved. Pierce and Stein [8] consider the analysis for optimum predetection combining when the channel undergoes dependent Rayleigh-fading that is presumed to be measured perfectly. For independent fading, Bello and Nelin [derive an expression for the error rate when the measurements are noisy. Further, error analysis for single-channel and diversity reception has been extended by Lindsey [6] and Hahn [10] to M-ary orthogonal signaling when the channels fade independently. Pierce [11], on the other hand, shows that there exists an optimum number of diversity branches when the channel fading obeys the Rayleigh law.

For adaptive receivers, Proakis, Drouilhet and Price [12] give experimental results pertaining to the performance of coherent detection systems using decision-directed channel measurements. Nesenbergs [13] reports results of an adaptive fading model for slowly fading channels. Walker [14] examines the performance of a frequency differential system in the presence of noise and fading. Hancock and Lindsey [26] derive the a posteriori probabilities necessary for specifying different receiver modes for various forms of a priori channel knowledge. Price [28] evaluates the error probability for the adaptive multichannel reception of binary signals. Proakis [30] derives the probabilities of error for PSK signaling over an L-diversity branch, Rayleigh fading channel.

For fast fading, Price [15] considers the fast-fading scatter channel, while Harris [16] has examined semi-coherent Rake-type receivers. Voelcher [17] has considered the effect of nonzero fading bandwidths on differentially coherent PSK. Bello and Nelin [18] have examined matched-filter receivers which are subjected to fast fading. Pierce [11] computed error probabilities for a certain spread channel and Kennedy [19,20] has developed system performance limitations and error probabilities expressions for some dispersive fading channels. Chadwick [25] finds the error probability for both low and high fading bandwidths. Harper [31] studies the effect of correlated Rayleigh fading in dual diversity reception of 1- and 2-bit per baud DPSK.

For selective fading, Bello and Nelin [24] consider the effect of frequency-selective fading on the binary error probabilities of incoherent and differentially coherent matched filter receivers employing diversity combining. Frendberg and Stein [25] evaluate the performance of correlation systems in selective fading. Turin [26] considers receiving binary information through a noisy, selectively fading, discrete multipath channel. Bello [27] studies the Kathryn system under the effects of selective fading. Bello and Nelin [29] study the transmission in FDM-SSB over selectively fading media.

For arbitrary selectivity and rapidity, Bello [21] and Hingorani [22] derive a general expression for error probabilities in binary transmissions of signal over selectively fading diversity channels containing specular and fast-fading components. Stein (Chapter 9, [23]) sketches some of the steps to evaluate the performance of binary transmission in channels with arbitrary selectivity and rapidity. Bello [32] studies the distortion on the transmitted signal due to both time-selective and frequency-selective fading.

1.3 Deterministic Intersymbol Interference

When transmitting pulses in contiguous form, these pulses can suffer distortion that takes the form of a time overlap, known as intersymbol interference. This interference can be caused by fading or by receiver filters. The type of deterministic intersymbol interference caused by filters is of particular interest in this dissertation. Such a distortion can exist due to economy in frequency allocation, or simply when trying to eliminate noise by bandlimiting the received signal.

The effect of intersymbol interference on error rate is also particularly important when the signal-to-noise ratios are large, such as in coaxial cable PCM systems. This type of real transmission channel usually exhibits some form of time dispersion [36,38]. Thus this pulse distortion represents a kind of deterministic impairment, and if the characteristics of the channel and/or the filters are known, it is possible in theory to remove intersymbol interference.

It is important to observe that the probability of error under the above conditions, has been very rarely obtained in closed form. In fact, the majority of the investigators have been able to obtain bounds, and the exact computation remains an unsolved problem.

The research done to evaluate system performance under the influence of intersymbol interference should be reviewed. For other references and a historical note see Tufts [37]. The problem of designing systems to decrease probability of detection error due to intersymbol interference has received considerable attention [37-43, 45-47, 54-56, 58, 60-64]. The approaches used to attack this problem can be divided into two major groups. The first group is composed of those approaches which try to eliminate the intersymbol interference completely and to minimize the effects of the additive noise under this constraint, [41,46,48,58]. This general approach does not necessarily minimize the effects of the intersymbol interference and noise at the same time. The approaches included in the second group try to minimize the total effect, [38-40, 42, 43, 45, 50, 54, 55-64]. This group can again be divided into two subgroups: the approaches which restrict the receiver to be linear [40, 42, 54, 61, 37, 64] and the ones which do not have the linearity restriction, [38,39,43,45,49,50].

These approaches to the problem of obtaining digital systems with low error probability have resulted in a variety of systems. In general, these systems are of different complexities and have different error probabilities under identical operating conditions. To analyze and to compare these systems, one would like to be able to evaluate the probability of error for each system. Various authors have attempted to evaluate this probability, [37, 44, 45, 52, 57, 59]. They have been able to obtain bounds, but the exact computation remains unresolved.

1.4 Scope of this Study

In the last section, a brief summary of the research done in two major areas (fading and intersymbol interference) has been given in an attempt to report original work, some of it classical and some applied to special situations.

The scope of this dissertation is to evaluate the performance of bandlimited binary receivers in fading channels. Two procedures for the evaluation of the probability of error will be studied. First, in Chapter III, for the slowly-fading medium we calculate the steady error probability and average it over all possible signal patterns. The resulting conditional error probability is further averaged over the medium fluctuations. Second, in Chapter IV, an evaluation of the conditional probabilities that the decision variable $D>0$ and $D<0$ is made, given that a specific signal pattern has been transmitted. This conditional error probability is then averaged over all possible transmitted patterns in the L-data channels.

For slowly-fading channels, the average probability of error is evaluated, using a unified analysis of the effects of fading and intersymbol interference due to filtering for both the Rayleigh and Rician channel. It is shown how the fading degrades the signal, and when diversity is used to improve the system performance, the intersymbol interf-ence due to filtering becomes important.

A qualitative study of the fading effects is achieved by examining the range of the signal-to-noise rat responsible for the errors. The evaluation of the probab. or error conditional on the signal-to-noise ratio lying i: certain range is given.

For a fading channel with arbitrary rapidity and selectivity, the formulation of the detection problem has accomplished in a generalized form in the receiver structu The receiver decision variable has been modeled as a generalized Hermitian quadratic form D, where

$$D = \sum_{k=1}^{L} d_k \tag{1-1}$$

and

$$d_k = a\,|u_k|^2 + b\,|v_k| + c\, u_k v_k^* + c^* u_k^* v_k \tag{1-2}$$

where (u_k, v_k) and (u_m, v_m) are statistically independent an identically distributed pairs of Gaussian random variables This quadratic form will, for example, represent a receiver which employs crosscorretation if a=b=0, or a square law d tection if c=0.

Before obtaining the average probability of error effort was made to express the probability density funct the signal-to-noise ratio, which is of interest if one i cerned with the range of the values responsible for the rather than just a simple average probability of error.

Even though the mathematical details are computed for DPSK systems only, once the probability of error for the general quadratic form has been obtained, the problem of using the results obtained in another type of detection scheme becomes a matter of re-evaluating the moments of the quadratic form and perhaps changing the value of a,b, or c in (1-1). The final forms of the probability of error (4-12) and (4-13) are still valid.

Finally, an adaptive scheme called the Decision-Directed Method for combating the errors made due to the phase variations in fading channels is discussed, and it is shown that once more the problem can be formulated in a generalized quadratic form, allowing for the use of the results previously obtained for the probability of error.

It will be shown that, in general, the selectivity and/or the rapidity of the fading channel make the probability of error evaluation non-symmetric. The expression for the case of Rician fading with arbitrary selectivity and rapidity is obtained in a suitable form which does not require numeric integration or the assistance of tabulated functions. Nevertheless, the moments of the quadratic form will require the use of the error function.

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CHAPTER II

CHANNEL CHARACTERIZATION

2.1 Introduction

Many of the radio communication channels in use are subject to time and frequency spreading. Time spreading manifests itself most clearly when a narrow pulse at the channel input is converted into an output that is spread out over a significant period of time relative to the signal duration. Frequency spreading results when the bandwidth of the signal at the channel output is spread out over a wider bandwidth than that of the input signal. When a communication channel exhibits either time spreading or frequency spreading, it is called a selective channel.

Part of this chapter is tutorial in nature. It attempts to review the wealth of mathematically oriented literature on selective channels. Both the underlying physical mechanism and the engineering interpretations are discussed. First, the canonic forms for linear filters are discussed, and without proof, the relationship among these canonic forms is given. Then the two-frequency correlation function, the tap-gain correlation function, and the scattering function are introduced, with emphasis on their physical interpretation rather than their mathematical derivation. It is latter shown that these three quantities are related by the Fourier transform. Finally a short section on signal design is presented, showing the dependence of the signal duration on the order of diversity that can be used.

Several physical channels have been characterized by observing their time spread, L_c, and frequency spread, B. Typical examples are shown in Table II-1.

Channel	B (Hz)	L_c (μs)
Tropospheric Scatter	10	1
Ionospheric Scatter	10	100
Orbital Dipole	1000	100
Chaff Clouds	100	5
Moon Reflection	10	10000

Table II-1 Typical Values of Time and Frequency Spread for Several Channels.

Physical examples on the measurement of B and L_c in a random channel are given in Sect. 2.3; including their interpretation and the methods used for their measurement.

2.2 Canonic Forms for Linear Filters

A. Linear time-invariant filters

One way to characterize a linear time-invariant filter is in terms of its impulse response, h(t), which is a function of the elapsed time t only. When x(t) is the filter input, the output, z(t), is given by the superposition integral

$$z(t) = \int_0^\infty x(t-y)\, h(y)\, dy \tag{2-1}$$

The lower limit of this integral reflects the physical realizability conditions that h(t)=0 for t<0. Equation (2-1) can be interpreted physically by observing that the output at time t is the weighted sum of all previous inputs, where the weighting function h(t) is dependent only on the age of the previous inputs and not on the observation time t.

B. Linear time-variant filters [16]

The impulse response of a time-variant network is defined as $h(t,\tau)$, the output measured at time t in response to an impulse $\delta(t-\tau)$ applied at time $t=\tau$. This response differs from the commonly used impulse response in that is a function of both the absolute time, t, at which the response is being observed and the age of the applied impulse. The output of the system is measured as a function of the usual running time variable, t. The input signal is inserted at an absolute time $\xi=t-\tau$. The age variable τ represents the difference between the time of observation t and the signal insertion time ξ, and as the name implies, it represents the antiquity of the applied signal [4]. Since the network is linear, its response to any $x(t-\tau)$ can be found by superposing the responses of the system to a weighted series of impulses whose sum is equal to $x(t-\tau)$. Thus

$$z(t) = \int_{-\infty}^{\infty} h(t,\tau)\, x(t-\tau)\, d\tau \tag{2-2}$$

where $h(t,\tau)=0$ for $\tau<0$ and $x(t-\tau)=0$ for $t<\tau$.

C. Other forms for the time-variant impulse response [2]

If the channel impulse response is $h_1(t,\tau)$, the realizability condition is that the response be identically zero for $t<\tau$. This constraint involves both t and τ, and therefore is often inconvenient to use. Let us define $h_2(z,\tau)$ as the response measured at time $t=\tau+z$ to a unit impulse applied at time τ, where z is the elapsed time since the application of the impulse. The realizability condition is that the response be zero for $z<0$.

As another example, let $h_3(y,t)$ be the response measured at time t to a unit impulse applied at time t-y, y measuring the antiquity or age of the input. The realizability condition is zero response for $y<0$. The distinction between $h_2(z,\tau)$ and $h_3(y,t)$ where $z=t-\tau=y$, is the choice of time reference.

In short, the output of the network can be expressed in terms of $h_1(\cdot)$, $h_2(\cdot)$, or $h_3(\cdot)$ as follows:

$$z(t) = \int_{-\infty}^{t} h_1(t,\tau)\, x(\tau)\, d\tau \tag{2-3}$$

or

$$z(t) = \int_{0}^{\infty} h_2(z, t-z)\, x(t-z)\, dz \tag{2-4}$$

or

$$z(t) = \int_{0}^{\infty} h_3(y,t)\, x(t-y)\, dy \tag{2-5}$$

2.3 Characterization of Time and Frequency Spread Channels

Even though $h(\tau,t)$ can be measured in many cases [2], usually it will not be available. Therefore if we assume we have the unit impulse response, we are assuming more information than is likely to exist in physical situations.

A less ambitious but more practical goal is the channel characterization in terms of the two-frequency correlation function, the tap-gain correlation function, or the scattering function. Since $h(\tau,t)$ and $H(f,t)$ are statistically varying, one is interested in knowing the statistics of these functions rather than the actual functions themselves. It will be shown that the correlation functions and the scattering functions are related by the Fourier transform (see Sect. 4). At this point it is of greater interest to have clear understanding of the various correlation functions rather than to get involved with the detailed mathematics which describe these functions.

One general form of the channel model [7] can be visualized as three parallel channels. The first is random time-variant filter $h_F(\tau,t)$ which contains the fast fluctuations, and whose response to an input signal is a narrowband, zero mean, Gaussian process which can be nonstationary. The second is a slowly-varying filter, $h_S(\tau,t)$, whose response to an input signal will produce a random amplitude and phase that obey the Rayleigh and uniform probability distribution laws, respectively. Finally, a deterministic filter, A, is included whose output amplitude, α, and phase, δ, are fixed but can vary for different diversity channels. This characterization is shown in Fig. II-1.

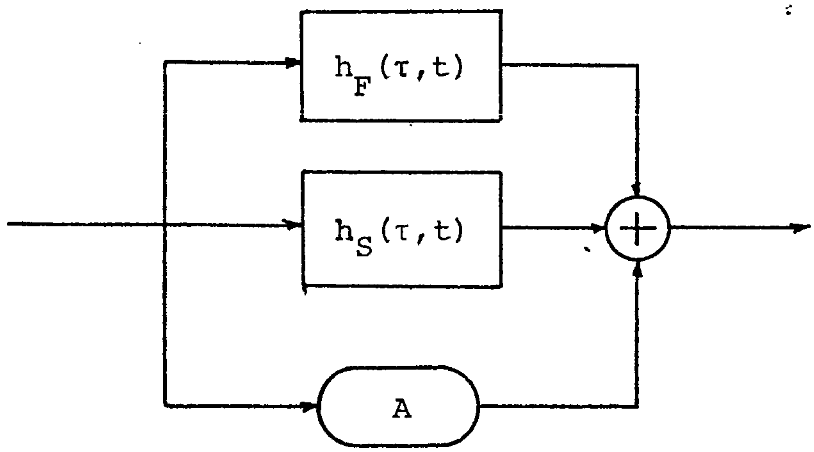

Fig. II-1 A General Form of the Channel Model.

Another characterization [8] of the channel can be obtained when very "slow" fluctuations are superimposed upon more "rapid" fluctuations. The latter will exhibit approximate satisti stationarity. This situation is often called quasi-stationari Then, time and frequency selective behavior of the channel can be regarded as wide-sense stationary for time and frequency intervals much greater than the durations and bandwidths, of the signal waveforms respectively. Moreover, the more "rapid" fluctuations often appear to be characterizable in terms of appropriately defined Gaussian statistics.

Since a Gaussian process can be completely described statistically if its correlation function is known, it follows that a fairly complete statistical description of many quasi-stationary radio channels should be achievable by measuring channel correlation function . These measurements have to be taken for time and frequency intervals small compared to the fluctuation intervals of the slow channel variations. Then a measure of the statistical behavior of these quasi-stationary channel correlation functions as caused by the slow channel variations has to be taken. The computations performed in this dissertation are for quasi-stationary error probabilities which reflect the "short-time" error rate behavior of the channel. The "long-time" error rate behavior could then be predicted by averaging the "short-time" error rate behavior over the "long-time" fading statistics of the channel.

At this point one can observe a discrepancy in terminology. While one author [7] calls slow fading amplitude variations which are constant over a bit period, another author [8] labels this phenomena as fast fading. The reason for this difference is that the latter author means fast variations (a few Hz for troposcatter) with respect to the long term diurnal (several hours) variations, but still slowly varying if compared with the signal bit rate.

It is then assumed that the transmission characteristics of a time- and frequency-selective channel are specified by the time-varying impulse response $h(t,\tau)$ and its Fourier transform $H(t,f)$, the time varying frequency response. It will be assumed that $h(t,\tau)$ and $H(t,f)$, considering f and τ as parameters, can be adequately modeled as sample functions of stationary ergodic random processes. Obviously, for channels such as chaff channels, where the scattering particles fall to the ground in a number of hours, the stationary assumption cannot be strictly valid. This is because the chaff clouds cannot be present at all times. However, over the period of a measurement it is perfectly reasonable to assume stationarity. It has been pointed out [17] that the assumption of ergodicity is no problem when one is constructing a random process model from a sample function. In fact, if the mathematical model is to have any engineering relevance at all, then some aspect of the model must represent (or approximate) something that could be measured experimentally.

A. Two-frequency correlation function

The two-frequency correlation function $\mathcal{R}(\Delta f,\Delta t)$ is defined as [5]

$$\mathcal{R}(\Delta f,\Delta t) = 2\langle H^*(f_0 - \frac{\Delta f}{2},t)H(f_0 + \frac{\Delta f}{2},t+\Delta t)\rangle \ . \qquad (2\text{-}6)$$

Using the complex lowpass representation, this equation becomes

$$\mathcal{R}(\Delta f,\Delta t) = \frac{1}{2}\langle H^*(-\frac{\Delta f}{2},t)H(\frac{\Delta f}{2},t+\Delta t)\rangle \ , \qquad (2\text{-}7)$$

which implies that the correlation function is not a function of the carrier frequency f_0 over a broad range of f . Expression (2-6) has also been called the frequency-time cross-variance [11], time-frequency correlation function [8], and spaced-time spaced-frequency correlation function [12]. The definition

(2-6) implies that there exists stationarity in both frequency and time since it does not depend on the time and frequency references. That is, if $h(t,\tau)$ is stationary, at least in the local sense, one is naturally considering short-term fading. Thus the correlation function (2-6) depends on t_1 and t_2 through the difference $\Delta t = t_1 - t_2$. The concept of stationarity in the frequency domain means that the correlation function (2-6) is a function of the frequency spacing only. Some particular cases of (2-7) are of interest.

i) $\mathcal{R}(\Delta f, \Delta t) = \mathcal{R}(0, \Delta t)$

This is simply the autocorrelation of the channel response to a sinusoid of frequency f_0, also known as time correlation function [8] and as the echo correlation function [12]. If the channel coherence time T_c is defined as the interval in Δt over which $\mathcal{R}(0,\Delta t)$ is essentially zero, it can be seen that T_c is a measure of the time over which coherent integration can be performed on the channel output and also the duration of the fades on the channel. Therefore as T_c approaches infinity the selective channel becomes a coherent channel.

ii) $\mathcal{R}(\Delta f, \Delta t) = \mathcal{R}(\Delta f, 0)$

This is the autocorrelation of the channel response to different frequencies, also known as frequency correlation function [8], spaced-frequency correlation function [12] and frequency crosscorrelation [10]. The channel coherence bandwidth F_c is defined as the frequency at which $\mathcal{R}(\Delta f, 0)$ has dropped significantly toward zero from $\mathcal{R}(0,0)$. This frequency correlation function is a rather important quantity in the design of frequency diversity systems since it determines how far apart in frequency the channels must be to achieve essentially uncorrelated signals.

Another important aspect of $\mathcal{R}(\Delta f,0)$ is that in all the reported measurements, the correlation actually measured has been on the envelopes simultaneously received for two separate tones. Since it has also been verified that each fading-tone envelope is Rayleigh distributed, which leads to the extrapolation that the Gaussian model is then correct, the measurement of envelope crosscorrelation can be regarded as a measurement of $|\mathcal{R}(\Delta f,0)|$. Unfortunately, this is the only case for which detailed measurements $\mathcal{R}(\Delta f,0)$ have been made.

B. Tap-gain correlation function

The tap-gain correlation function $R(\tau,\Delta t)$ is defined as [3]

$$R(\tau,\Delta t) = \frac{1}{2}\left\langle h^*(\tau,t)\, h(\tau,t+\Delta t)\right\rangle \qquad (2\text{-}10)$$

which has also been called multipath autocovariance profile [10] and the delay cross-power spectral density [8]. It can be interpreted as the correlation function of the return due to scatterers at path length of delay τ. The dependence on τ can imply a different shape of the autocovariance (or of the equivalent power density spectrum) for different τ.

The tap-gain correlation function presents another possibility, in addition to $(\Delta f,\Delta t)$, for statistical measurement of the channel. This is done by transmitting extremely short pulses to "sound" the medium, thus measuring $R(\tau,\Delta t)$ by correlation of values observed with successive pulses. This measurement has been accomplished at HF [13].

A particular case of (2-10) is when $R(\tau,\Delta t)=R(\tau,0)$. This function is proportional to the average returned power at relative delay τ, or in other words, it is the return due to scatterers for which the path length yields a delay τ. The above equation has been termed the multipath intensity profile [10], delay power density program [8], power impulse response [12] and delay spectrum [5].

The multipath smear, L_c, is the time duration in τ over which $R(\tau,0)$ is effectively nonzero. This interval is about twice the effective duration of the impulse response $h(\tau,t)$.

C. Scattering function

The scattering function, $\sigma(\tau,f)$, of a time and frequency selective channel is defined as [6]

$$\sigma(\tau,f) = \tilde{\sigma}(\tau,f)\left[\iint \tilde{\sigma}(\tau,f)\, d\tau\, df\right]^{-1} \qquad (2\text{-}9)$$

where $\tilde{\sigma}(\tau,f)$ is the average scatterer cross section. The scattering function is expressed in terms of square meters of effective radar cross section. That is, one watt per square meter of incident power flux will scatter toward the receiver σ watts per steradian of solid angle subtended. Thus (2-12) is a normalized density of cross section. The function $\sigma(\tau,f)$ unlike $R(\tau,\Delta t)$ and $\mathcal{R}(\Delta f,\Delta t)$, is necessarily real and non-negative. The scattering function has a much more direct physical interpretation than either $\mathcal{R}(\Delta f,\Delta t)$ or $R(\tau,\Delta t)$, and one can think of $\sigma(\tau,f)$ as being a measure of the power returned by scatterers with delay τ and doppler shift f. Thus, interpreted as doppler effects, the functional dependence on f describes the probability density function of velocity for all the scatterers with propagation delay τ. From the scattering function, we can determine L_c, the time duration in τ, and B, the frequency interval in f outside of which $\sigma(\tau,f)$ is effectively zero. The quantity B is commonly called the doppler spread and L_c the multipath spread of the channel.

A typical result obtained in a troposcatter link for $\sigma(\tau,f)$ is shown in Fig. II-2. From Fig. II-2 one can observe that for this particular example the doppler spread is 1 Hz and the multipath delay is approximately 0.9μs. Another characteristic of $\sigma(\tau,f)$ observed above is that it represents a unimodal two-dimensional function.

Some particular cases of the scattering function can be obtained, by letting $\sigma(\tau,f)$ be independent of τ or f as follows.

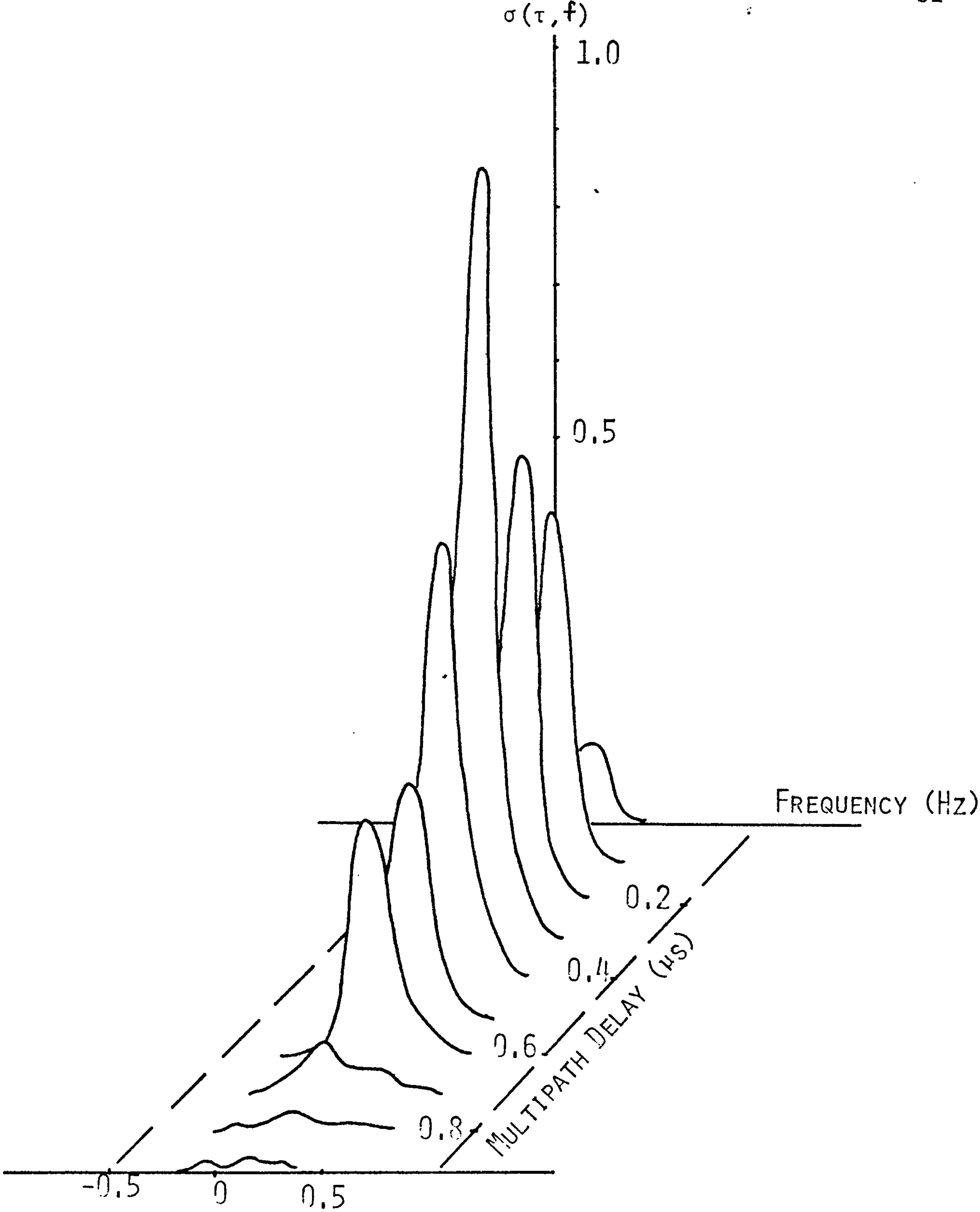

FIG. II-2 SCATTERING FUNCTION FOR A TROPOSCATTER LINK

i) $\sigma(\tau,f) = \sigma(f)\cdot\delta(\tau)$

When the scattering function is of the above form then the channel is said to be dispersive only in frequency [6], which is sometimes called frequency-flat fading or time-selective fading [14]. In this case the multipath spread appears to be zero. If the transmitted signal is a sine wave of frequency f_0, the signal scattered is detected at the receiver at a frequency f. The difference $f-f_0$ depends upon the scatterer velocity and is called the doppler shift. Since the scatterers have slightly different velocities and consequently a small difference in doppler shifts, the received signal is composed of a sum of sine waves with a distribution of frequencies as shown in Fig. II-3(b). The quantity $f_1=f_D-f_0$ is the average doppler shift and B is the distribution width or doppler spread.

The reason this case is also called time-selective is that the doppler spread can be thought of as a random amplitude modulation of a carrier f_0. In fact the rapidity of this random modulation is determined by 1/B if we interpret it as a modulation "bandwidth". This is illustrated in Fig. II-3(a).

ii) $\sigma(\tau,f) = \delta(f)\cdot\sigma(\tau)$

If the scattering function takes the above form, is said to be dispersive only in time [6] or often referred to as frequency selective or time-flat channel [14]. In this case the doppler spread of the scatterers appears to be zero. If one transmits an impulse and the received signal is stretched.

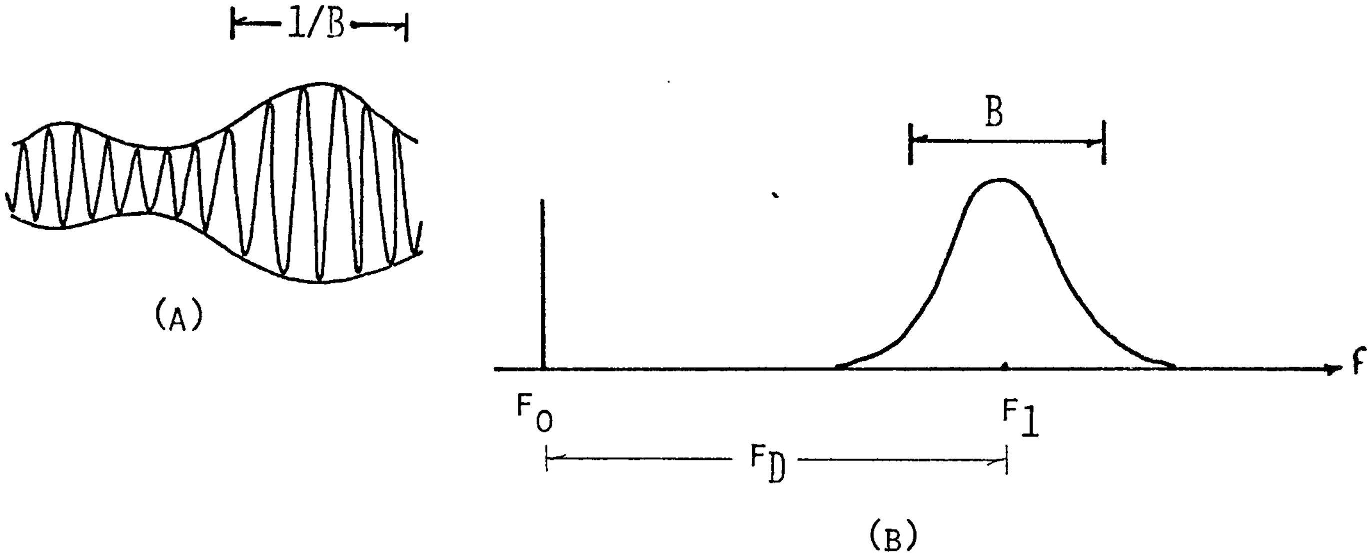

FIG. II-3 FREQUENCY DISPERSION

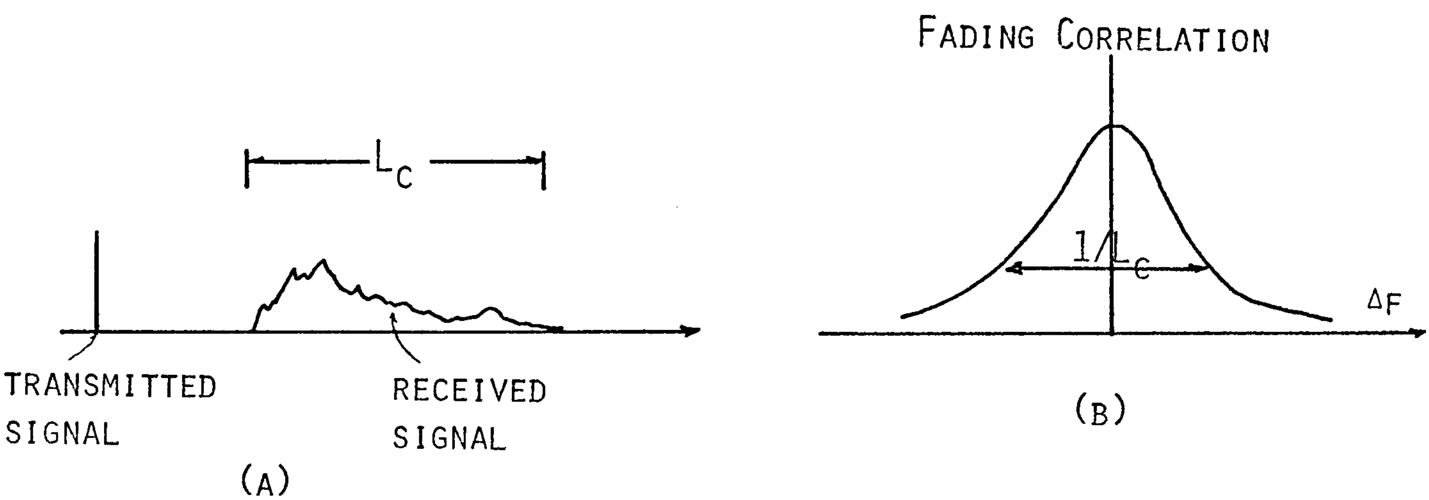

FIG. II-4 TIME DISPERSION

in time, then some multipath spread, L_c, exists. This is shown in Fig. II-4(a).

The reason this type of channel is called frequency selective can be seen from Fig. II-4(b). If two sine waves of frequencies f_1 and f_2, respectively, are transmitted simultaneously, each of these will have the behavior shown in Fig. II-3(b). Then by observing the envelopes of the two sine waves one can determine their correlation as a function of the frequency separation $\Delta f = f_1 - f_2$. Obviously, if $f_1 = f_2$ the received signals are identical and a maximum correlation will be obtained. At frequency differences much greater than the reciprocal of the multipath spread, $1/L_c$, the correlation approaches zero, or the two signals fade independently.

iii) $\sigma(\tau, f) = \delta(\tau)\,\delta(f)$

This type of channel is said to be nondispersive but fading [6], or a flat-flat fading channel [14]. In this case both the doppler spread and the multipath spread appear to be zero. This is the random-phase Rayleigh fading channel discussed extensively in the literature.

2.4 Relationships Among the Channel Characterizations

Because of the large number of mathematical steps involved, only a summary of the relationships among the expressions discussed in Section 2.3 will be attempted.

The correlation functions (2-7) and (2-8) are related by the Fourier transform as

$$R(\tau,\Delta t) = \int_{-\infty}^{\infty} \mathcal{R}(\Delta t,\Delta f)\, e^{j2\pi(\Delta f)\tau}\, d(\Delta f) \tag{2-10}$$

and

$$\mathcal{R}(\Delta t,\Delta f) = \int_{-\infty}^{\infty} R(\tau,\Delta t)\, e^{-j2\pi(\Delta f)\tau}\, d\tau \,. \tag{2-11}$$

The scattering function (2-9) is also related to these correlation functions by the Fourier transforms

$$\sigma(\tau,f) = \int_{-\infty}^{\infty} R(\tau,\Delta t)\, e^{-j2\pi f(\Delta t)}\, d(\Delta t) \tag{2-12}$$

and

$$\sigma(\tau,f) = \iint_{-\infty}^{\infty} \mathcal{R}(\Delta f,\Delta t)\, e^{-j2\pi\left(f(\Delta t)-\tau(\Delta f)\right)}\, d(\Delta t)\, d(\Delta f) \,. \tag{2-13}$$

Relationships among several of the channel parameters can be obtained if $\mathcal{R}(\Delta f,\Delta t)$ and $\sigma(\tau,f)$ are unimodal. Then

$T_c \approx 1/\text{doppler spreading} = 1/B$

$F_c \approx 1/\text{impulse response duration} = 1/L_c$.

Also if the scattering function is two-dimensional Gaussian with standard deviation L_c and B, then using (2-13) we can observe that $\mathcal{R}(\Delta f,\Delta t)$ is also Gaussian with approximate standard deviations 1/L and 1/B.

2.5 Signal Design in Troposcatter Channels

In general, the expressions for the probability of error depend on the average signal-to-noise ratio, order of diversity and fading parameters. The order of diversity L, although it can be chosen arbitrarily, is a function of the channel parameters defined in Section 2.4 [6], i.e.,

$$L = (B + W_s)(L_c + T) \tag{2-14}$$

Once the type of channel to be used is known, L_c and B are known, and by letting $W_s = 1/T$, the pulse length of the transmitted signal is determined from (2-20) by specifying the order of diversity L.

For example, for troposcatter channels one can see from Table II-1 that $B_c = 10\text{Hz}$ and $L_c = 10^{-6}\text{sec}$. For the pulse lengths of interest, the transmitted signal bandwidth is several order of magnitude greater than the channel frequency dispersion B. Hence

$$L = (W_s)(L_c + T) = 1/T\,(L_c + T) \tag{2-15}$$

so that

$$T = \frac{L_c}{L-1} \approx \frac{10^{-6}}{L-1} \text{ seconds. } (L>1) \tag{2-16}$$

One can be expected to find an optimum pulse width which minimizes the probability of error for a given signal-to-noise ratio and constraint on the rate of transmission. Some typical values of T are shown in Table II-2.

$T(\mu s)$	L
1	2
0.50	3
0.33	4
0.14	8

Table II-2 Typical Pulse Duration T as a Function of Diversity L.

In order to interpret the significance of the multipath spread L_c and the coherence time T_c, consider the use of phase-shift-keying for transmission of digital data. It was observed in Table II-2 that the duration of a symbol T can be of the same order or less than L_c depending on the order of diversity used. If no diversity is used then T should be much greater than L_c to avoid intersymbol interference due to multipath. On the other hand, T should be less than T_c to allow for coherent integration of the received signal. Obviously when $T_c<L_c$, these conditions cannot be met and phase-shift-keying is clearly a poor data transmission. When $T_c<L_c$ (or $B_cL_c>1$) the channel is called an overspread channel [12].

2.6 Shape of the Scattering Function

From Fig. II-2 and the parameters obtained in Sect. 2.5 can conclude that troposcatter channels are frequency selective. This can be seen by observing that $L_c \approx T$ and $W_s >> B$. The first statement indicates that multipath is of the same order of magnitude as the pulse duration. The second statement implies that the doppler effect is very small as compared with the signal bandwidth. Then the scattering function (2-19) is of the form

$$\sigma(\tau, f) = \sigma^2 \frac{B_c}{B_f} \sqrt{\pi}\, e^{-\left(\frac{\pi B_c \tau}{2}\right)^2 - \left(\frac{f}{B_c}\right)^2} \qquad (2\text{-}17)$$

where σ^2 is the average power received when a sinusoid of unity peak value is transmitted. B_f is the fading bandwidth defined as the frequency at which the fading power spectrum drops to e^{-1} of its maximum value, and B_c is the correlation bandwidth defined as the frequency separation at which the correlation drops to e^{-1} of its maximum value.

Then for the frequency-selective channel, (2-22) becomes

$$\sigma(\tau) = \sigma^2 B_c \sqrt{\pi}\, e^{-\left(\frac{\pi B_c \tau}{2}\right)^2} \qquad (2\text{-}18)$$

which implies that the frequency correlation function has also a Gaussian shape, namely

$$R(\Omega) = 2\sigma^2 e^{-\left(2 \frac{\Omega}{B_c}\right)^2} \quad . \qquad (2\text{-}19)$$

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CHAPTER III

PERFORMANCE OF RECEIVERS IN SLOWLY-VARYING NON-SELECTIVE FADING CHANNELS: THE CLASSICAL APPROACH

3.1 Introduction

As mentioned in the previous chapter, when the scattering function $\sigma(\tau,f)=\delta(\tau)\delta(f)$ the fading channel becomes non-dispersive in both time and frequency. Such a channel is also known as a slow, nonselective channel [2], or simply a flat-flat fading channel. This implies that the information sent through a non-dispersive channel will suffer no multipath spread and no doppler spread. In fact, if a pulse is transmitted, the output of the channel will be the same pulse multiplied by a complex quantity whose envelope is Rayleigh-distributed and whose phase is uniformly-distributed.

In this chapter, the performance for binary receivers will be evaluated when the intersymbol interference is introduced by the received filters. The channel will be considered to be a flat-flat fading channel. Most of the material in this chapter is based upon results obtained in [4].

3.2 Intersymbol Interference and Rayleigh Fading

A. Intersymbol interference caused by filtering

The concept of intersymbol interference has been explained in Chapter I. A study of the effects of the pulse spreading on the bit under detection should require the consideration of future and previous bits. If the filter at the receiver is an ideal filter one has to consider as many future as previous bits, since the response of the ideal filter is symmetrical and non-causal. This is illustrated in Fig. III-1, and Fig. III-2 shows the response of a Butterworth filter of order N. If the filter at the receiver is a realizable filter, say a Butterworth filter, followed by an integrate and dump operation, it can be proved [5] that instead of performing the integration between 0 and the pulse duration T, there is an optimum interval of integration (in the sense of minimizing the probability of error) between τ and $T+\tau$, where $0 \leq \tau < T$ is the time delay. Then, in this case we have to consider not only the previous bits but also the following one. This situation can be illustrated by considering the response of a 3rd order Butterworth filter to a signal pattern 0111 as shown in Fig. III-3. This situation will be further discussed when interpreting the results obtained.

The method that will be followed to compute the combined effects of intersymbol interference due to filtering and fading, is to evaluate the bit error probability for a given signal pattern assuming a steady signal. This result will then be averaged over all possible patterns, using the average method [8]. This probability of error will be further averaged over the ensemble of values of the multiplicative noise. This second averaging process is valid since the slow fading assumption implies that the multiplicative fading process varies so slowly compared with the signal keying rate that it can be regarded

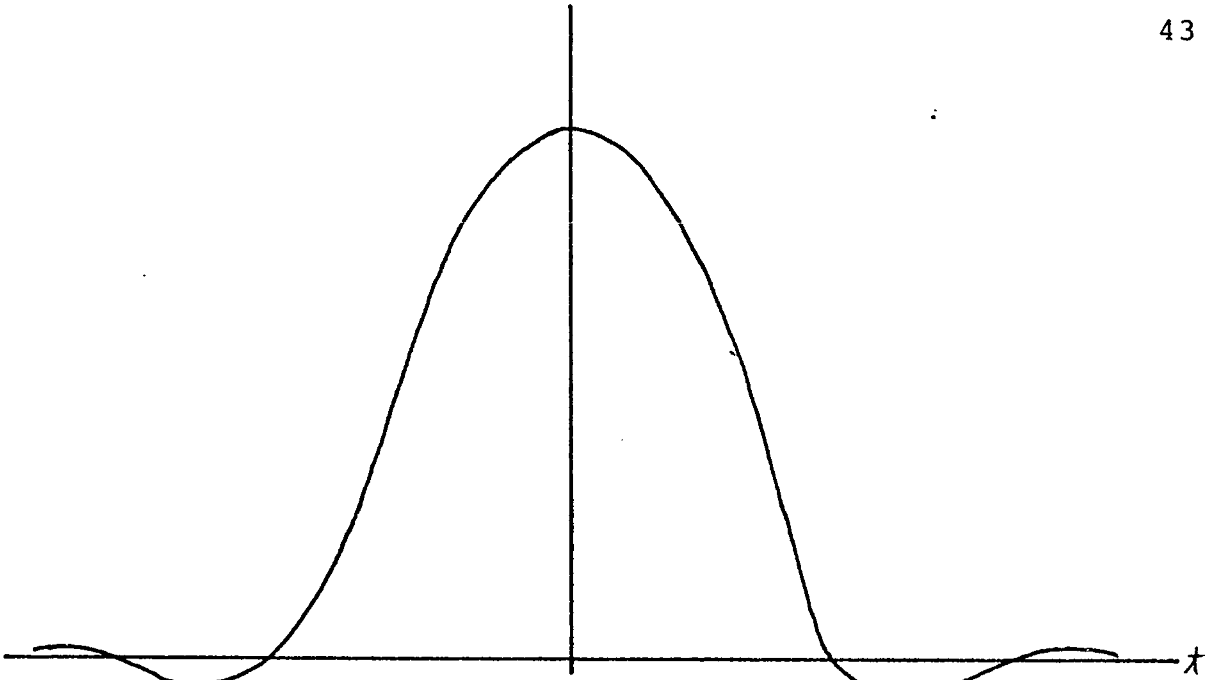

Fig. III-1 Response of an Ideal Filter

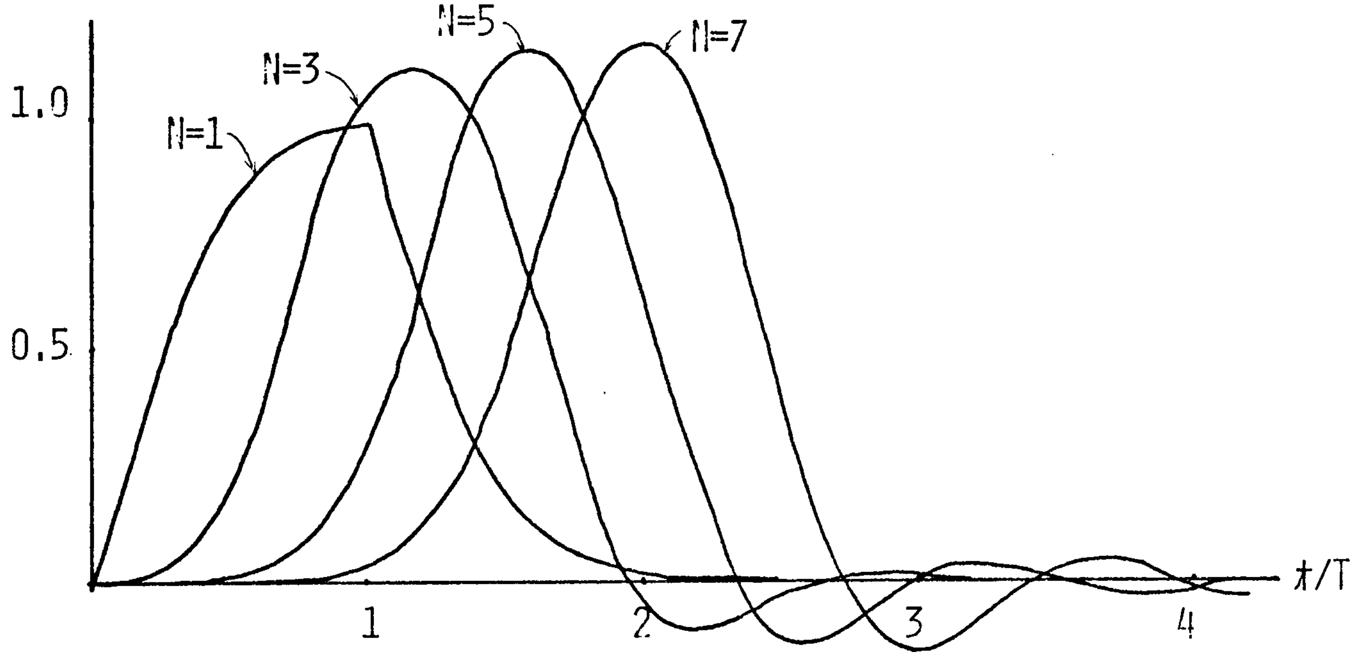

Fig. III-2 Response of a Butterworth Filter of Order N

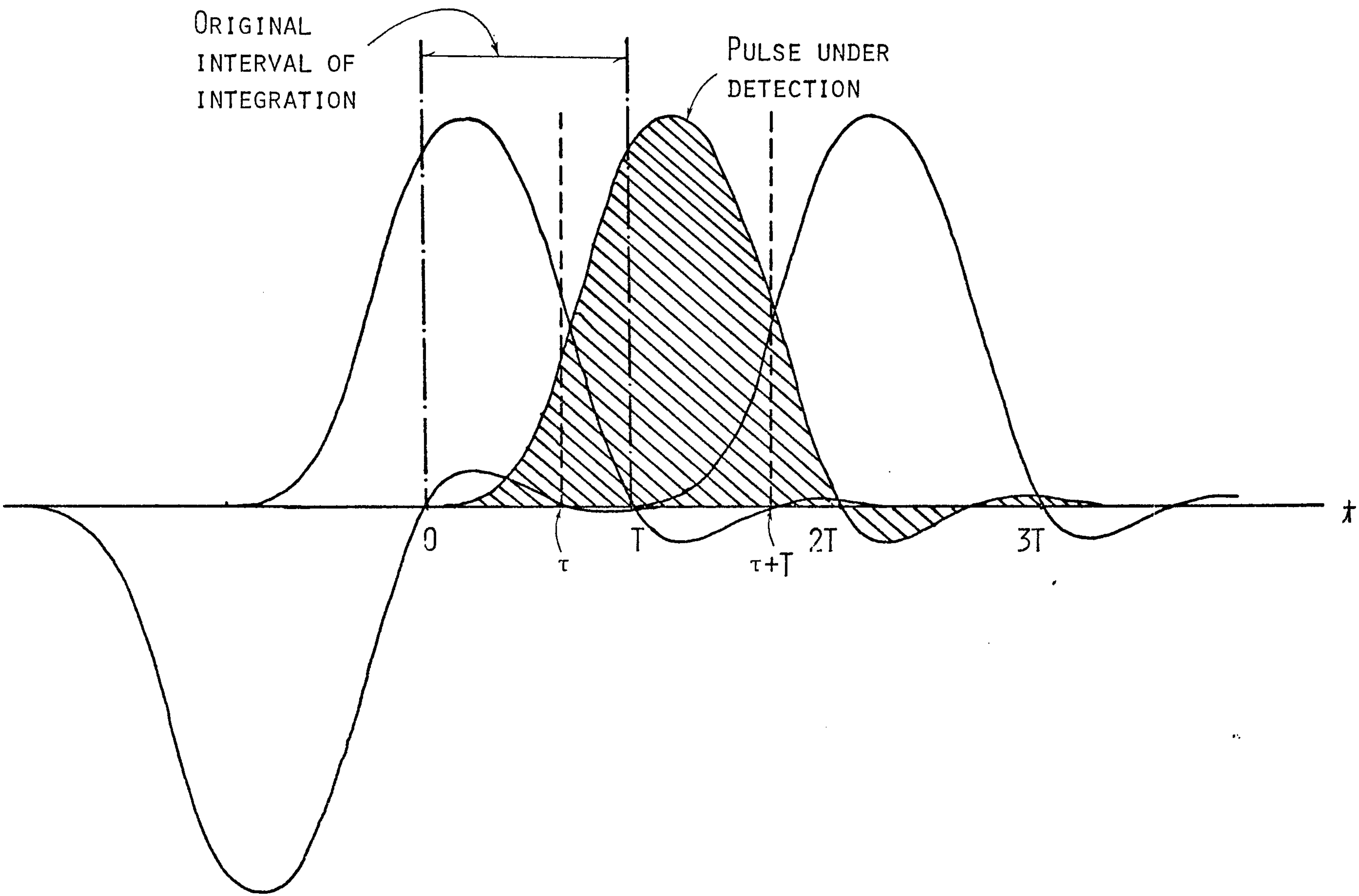

FIG. III-3 PULSE RESPONSE OF A 3RD ORDER BUTTERWORTH FILTER TO THE SIGNAL PATTERN 0111

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effectively constant during the course of each signal pulse. One last assumption that will be made at this point, and will be relaxed in the next section, is that the fading can be described completely as a multiplicative complex zero-mean Gaussian process.

For the ith bit pattern the probability of error can be expressed as [9]

$$P_{ei} = \frac{1}{2} \operatorname{erfc} \left[D_i^2(WT) \frac{E}{N_o} \right]^{1/2} \quad . \tag{3-1}$$

The quantity $D_i^2(WT)$, which is defined mathematically in Appendix C, is a multiplicative factor which reduces the signal energy-to-noise density ratio, E/N_o, to account for the intersymbol interference effects of preceding and future bits. The filter bandwidth is W and the bit duration is T.

To obtain the probability of error, P_ε, for any bit pattern n bits long, (3-1) must be averaged over all $N=2^n$ bit patterns giving

$$P_\varepsilon = \frac{1}{N} \sum_{i=1}^{N} P_{ei} \quad . \tag{3-2}$$

Thus (3-2) gives the average probability of error for detection of an NRZ signal in an additive noise channel with bandlimiting. The effect of bandlimiting for different types of filters (realizable [5] and ideal [9]) is included in the $D_i^2(WT)$ factor.

B. Fading and intersymbol interference

To obtain the overall average performance over fading channels, one can use the steady bit error probability and further average this result over the Rayleigh distribution for the envelope factor. Thus the overall average probability of error, P, for a slowly fading channel can be evaluated by averaging (3-2) over the signal-energy-to-noise density ratio, γ, [2]

$$P = \int_0^\infty P_\varepsilon \, p(\gamma) \, d\gamma \tag{3-3}$$

where $p(\gamma)=1/\gamma_0 \exp(-\gamma/\gamma_0)$ and γ_0 is the average value of γ. The density function $p(\gamma)$ results from a transformation of the Rayleigh density function of the envelope, μ, using the relationship $\gamma=\mu^2/2N$. Now, substituting (3-2) into (3-3) and interchanging the order of summation and integration one obtains

$$P = \frac{1}{N}\sum_{i=1}^{N}\int_0^\infty \frac{1}{2}\,\mathrm{erfc}\left[\sqrt{D_i^2(WT)\gamma}\right]\cdot\frac{1}{\gamma_0}\, e^{-\gamma/\gamma_0}\, d\gamma \quad . \tag{3-4}$$

This integral can be broken into two parts

$$P = \frac{1}{2N}\sum_{i=1}^{N}\left[\int_0^\infty \frac{1}{\gamma_0}\, e^{-\gamma/\gamma_0}\, d\gamma + \int_0^\infty \frac{1}{\gamma_0}\,\mathrm{erf}\left[\sqrt{D_i^2(WT)\gamma}\right] e^{-\gamma/\gamma_0}\, d\gamma\right] . \tag{3-5}$$

Solving the first integral, decomposing the second by integrating by parts, and observing that by using a Hermite polynomial

$$\frac{d^{n+1}}{dz^{n+1}} \operatorname{erf}(z) = (-1)^n \frac{2}{\sqrt{\pi}} H_n(z) e^{-z^2} ,$$

the overall probability of error becomes

$$P = \frac{1}{2N} \sum_{i=1}^{N} \left[1 - \operatorname{erf}\left(\frac{\sqrt{\gamma}}{2}\right) e^{-\gamma/\gamma_0} \Big|_0^{\infty} + \int_0^{\infty} \frac{e^{-\gamma\left(\frac{1}{\gamma_0} + D_i^2(WT)\right)}}{\sqrt{\pi\gamma}} d\gamma \right.$$

$$= \frac{1}{2N} \sum_{i=1}^{N} \left[1 - \frac{1}{\sqrt{1 + 1/[\gamma_0 D_i^2(WT)]}} \right] \qquad (3\text{-}6)$$

since $H_0(z) = 1$.

From (3-6) one can observe that there is no irreducible error, since the probability of error can be reduced to any desired level by simply increasing the SNR (signal-energy to-noise density ratio).

In Fig. III-4 the performance of a fading channel with ideal filtering is illustrated for four values of WT. Five bits on either side of the bit undergoing detection are considered. The effect of bits located further away is negligible. For comparison purposes the performance curves for a non-fading infinite-bandwidth channel and a non-fading finite bandwidth (WT=0.5) channel are also given.

In Fig. III-5 the curves shown are for the same conditions as in Figure III-4 except that the filter is a second-order Butterworth filter. The factors $D_i^2(WT)$ were obtained from Cheng [5]. Results were also obtained for third- and fourth-order Butterworth filters. In both cases the curves were within 0.2dB of the second-order case.

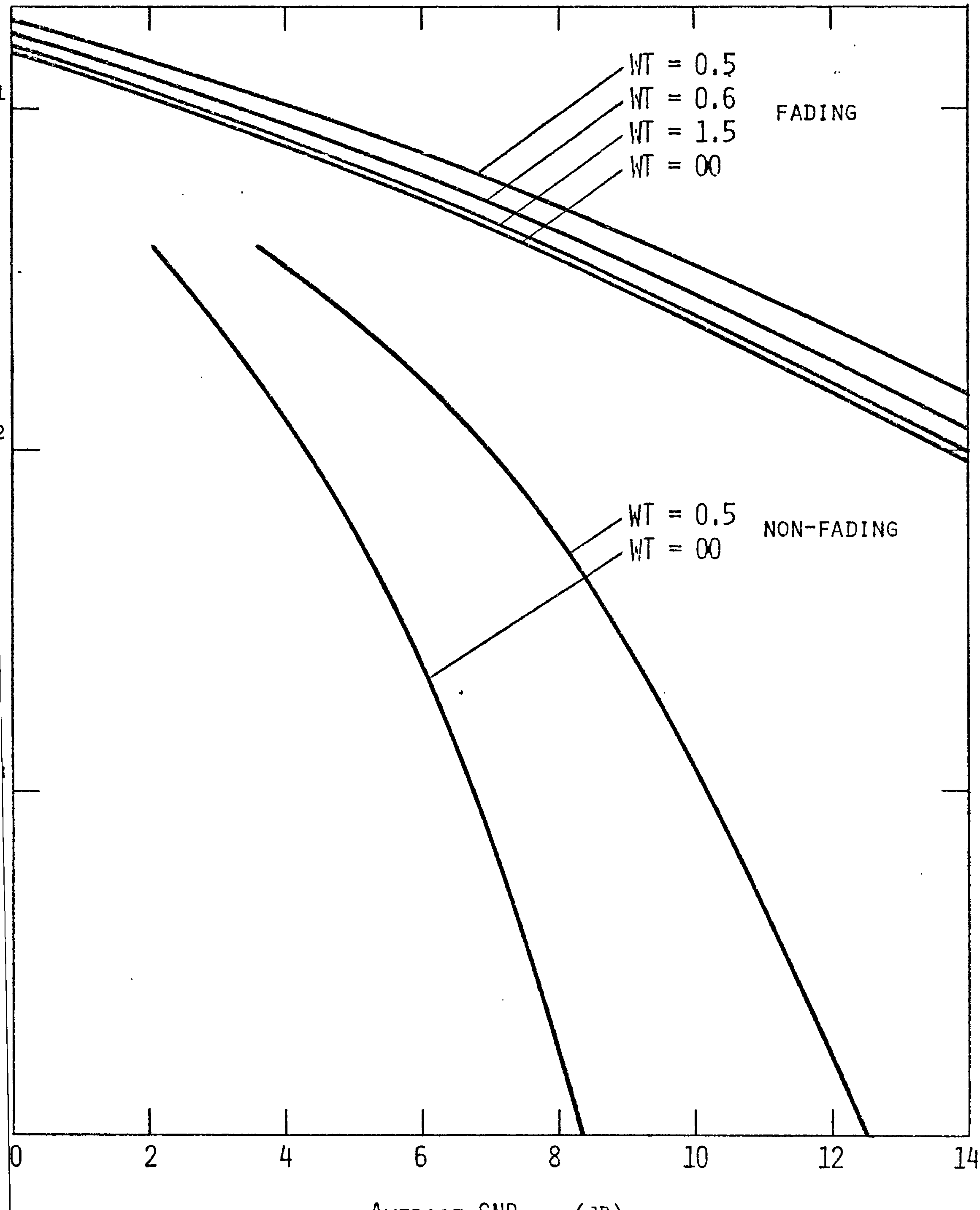

FIG. III-4 PERFORMANCE OF FADING CHANNEL WITH IDEAL FILTERING

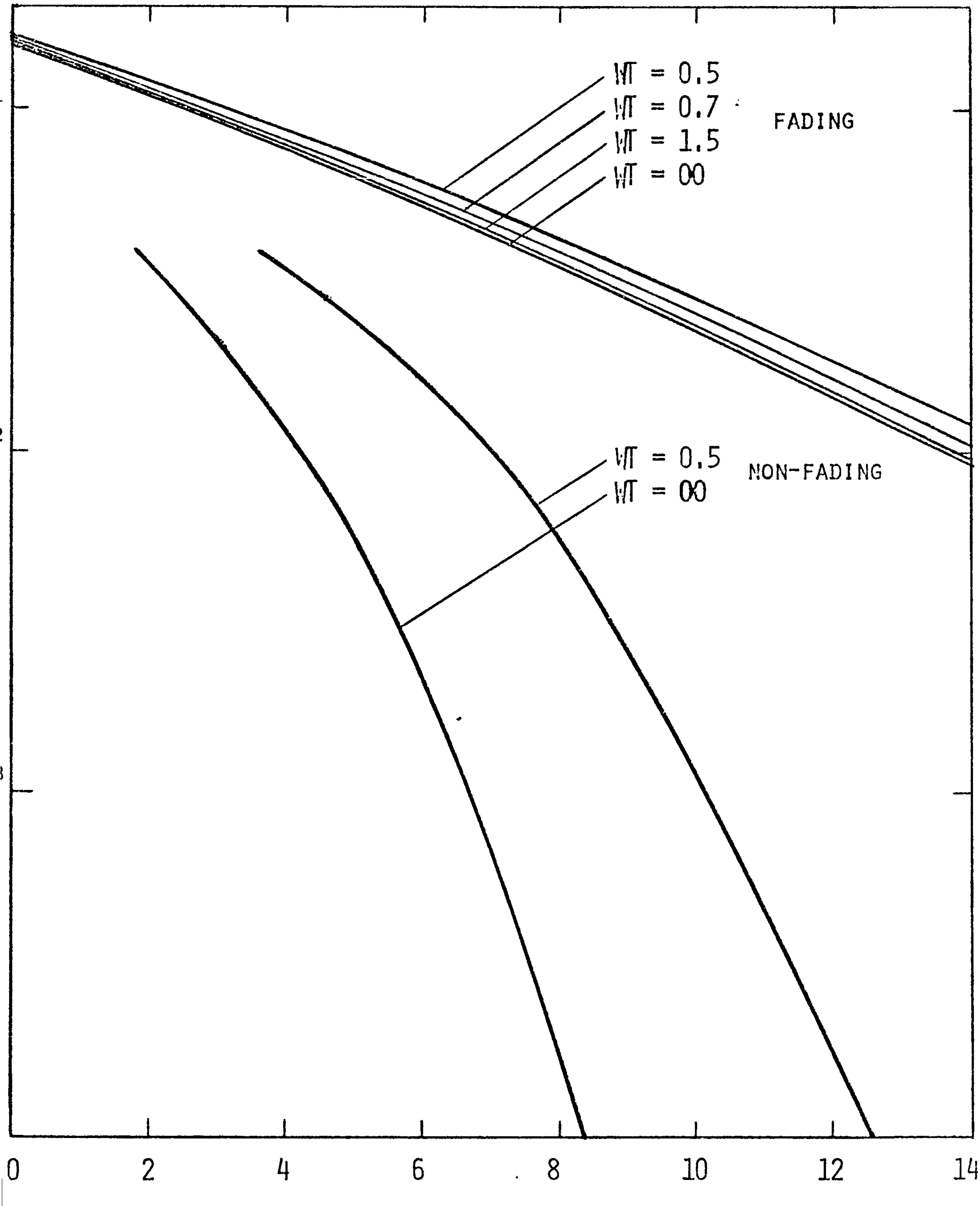

FIG. III-5 PERFORMANCE OF FADING CHANNEL WITH BUTTERWORTH FILTERING

Obviously from these two figures one can appreciate that the main source of degradation is, as expected, the channel fading. The degradation due to intersymbol interference in the presence of fading is less than approximately 2dB for BT=0.5. If BT>1.5 there is essentially no degradation due to filtering.

C. Linear diversity combining

When independently fading diversity branches are considered, the density function for γ can be determined by first evaluating the characteristic function and then transforming. If γ_0 is the same for all L diversity branches and the SNR ratios of the branches are linearly combined over the short-term fading [3], then

$$p(\gamma) = \frac{1}{(L-1)!} \cdot \frac{\gamma^{L-1}}{\gamma_0^{\,L}} \; e^{-\gamma/\gamma_0} \qquad (3\text{-}7)$$

This type of combining is known as maximal ratio combining.

By comparing (3-7) with the $p(\gamma)$ used in (3-3) one can see that there is an improvement in γ due to diversity. That is, $p(\gamma)$ in (3-3) is modified in such a way that it tends to be very small for small γ, rather than its exponential behavior in (3-3) for which $p(\gamma)$ actually peaks for small γ. Now, using (3-3) we have

$$P = \int_0^{\infty} \frac{1}{N} \sum_{i=1}^{N} \frac{1}{2}\left[1 - \mathrm{erf}\left(\frac{E}{N_0} D_i^2(WT)\right)^{1/2}\right] p(\gamma)\, d\gamma \quad . \tag{3-8}$$

Interchanging the order of integration and summation and using (3-7)

$$P = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{(L-1)!} \cdot \frac{1}{\gamma_0^L} \int_0^{\infty} \frac{1}{2}\left[1 - \mathrm{erf}\left(D_i^2(WT)\gamma\right)^{1/2}\right] \cdot \gamma^{L-1} \cdot e^{-\gamma/\gamma_0}\, d\gamma \quad . \tag{3-9}$$

With the change of variable $x^2=\gamma$

$$P = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{(L-1)!} \cdot \frac{1}{\gamma_0^L} \int_0^{\infty} \left[1 - \mathrm{erf}\left(D_i^2(WT)\, x^2\right)^{1/2}\right] x^{2L-1} \cdot e^{-x^2/\gamma_0}\, dx \tag{3-10}$$

and using Integral 6.286 [10]

$$P = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{(L-1)!} \cdot \frac{1}{\gamma_0^L} \cdot \frac{\Gamma\left(\frac{2L+1}{2}\right)}{2L\left[D_i^2(WT)\right]^{2L}} \cdot F\left(L, \frac{2L+1}{2}; L+1; \frac{-1}{\gamma_0 D_i^2(WT)}\right) \tag{3-11}$$

where the Gauss hypergeometric function is defined as [11]

$$F(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\,\Gamma(b+n)}{\Gamma(c+n)} \cdot \frac{z^n}{n!} \tag{3-12}$$

and $\Gamma(k+1)=k!$. Finally we obtain

$$P = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{\Gamma(L)} \cdot \frac{1}{\gamma_0^L} \cdot \frac{1}{2\sqrt{\pi}\left[D_i^2(WT)\right]} \cdot \sum_{m=0}^{\infty} \frac{\Gamma\left(L+m+\frac{1}{2}\right)}{(L+m)\,\Gamma(m+1)} \cdot \left(\frac{-1}{\gamma_0 D_i^2(WT)}\right)^m \quad . \tag{3-13}$$

The convergence of (3-13) is absolute if $D_i^2(BT)>-1/\gamma_0$, and L>0. The expression (3-13) can be easily evaluated since it converges rapidly (six terms are adequate for the infinite series to converge within one unit in 10^4).

Equation (3-13) is plotted in Fig. III-6, showing the performance of the diversity receiver for L=1,2,3 and 4, where L is the order of diversity. Curves are shown for WT=0.5 and WT=1.0 using a second-order Butterworth filter. Again, the performance of a non-fading infinite-bandwidth channel is shown for comparison. From this figure one can see the improvement obtained with diversity. The largest gain is observed between the L=1 and L=2 family of curves. For WT=1.0 this gain is 8dB for $P=10^{-2}$ and 16dB for $P=10^{-4}$. It is observed that for WT=1.0 the performance is better than the infinite bandwidth non-fading case if $L\geq 3$ and $\gamma_0<7$dB. Also, the lower the probability of error the higher the order of diversity necessary to give a performance equal to the non-fading infinite-bandwidth situation. Finally, if the probability of error is 10^{-4}, it is noticeable that the degradation due to filtering is about 1.5dB when M>2 and BT=0.5, which is small when compared with the 4dB of degradation obtained for the non-fading case of Fig. III-5.

D. Qualitative analysis of errors in Rayleigh fading

It has been observed that for large SNR the error rate is inversely proportional to the mean SNR [2]. On the other hand, when studying the error rates for steady signals,

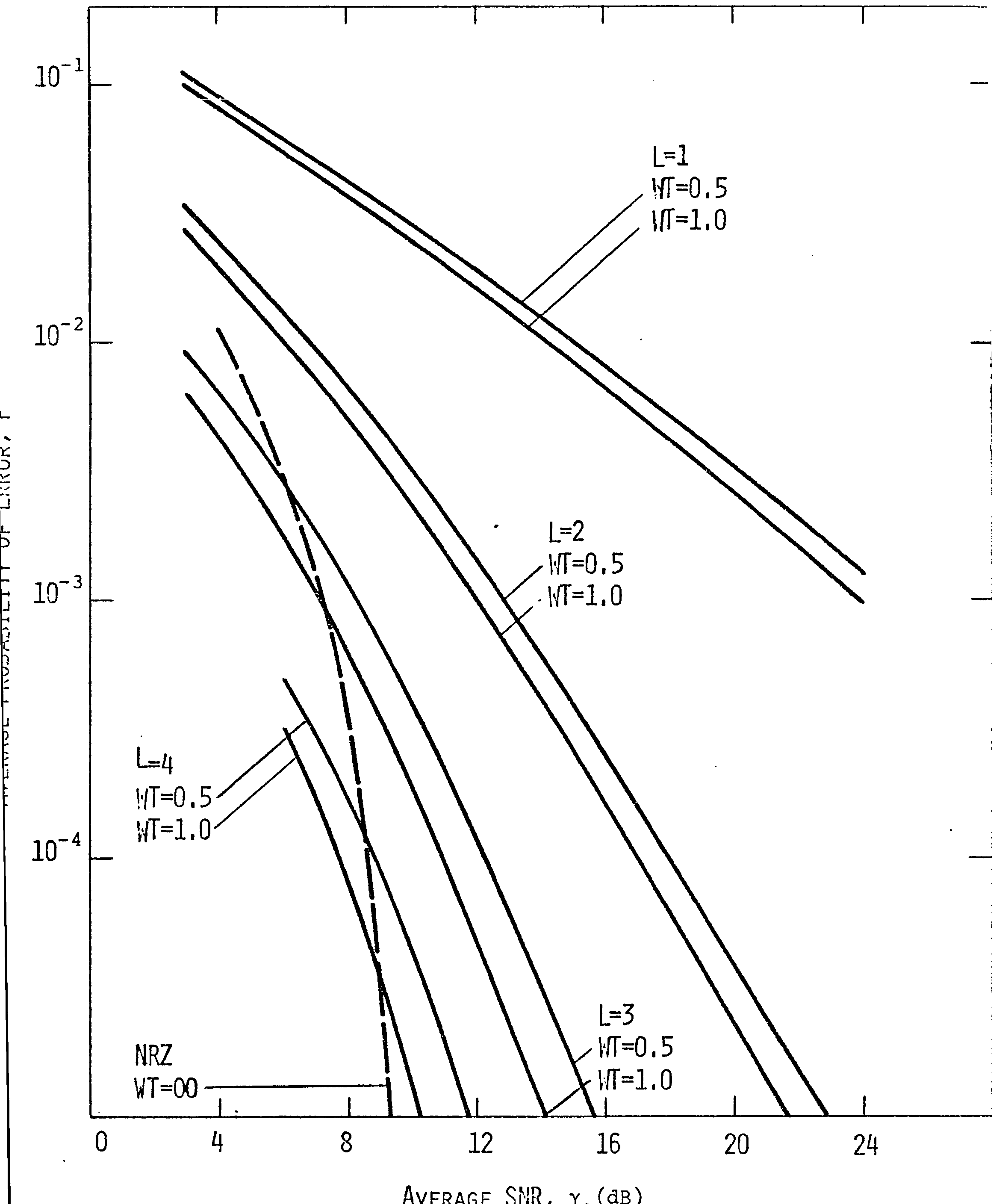

FIG. III-6 PERFORMANCE OF A FADING CHANNEL WITH DIVERSITY L the error rate has been observed to be asymptotic to an exponential function [1]. The behavior for large SNR shows that there exists a significant difference in performance that motivates the creation of extra equipment (such as diversity and adaptive receivers) to combat the fading. This is the dominant qualitative difference between the fading and the non-fading situations.

The strong degradation of the system performance introduced by fading is caused by the low signal levels during which the conditional probability of error is very close to 0.5. As was observed in (3-6), the probability of error approaches 0.5 when the average SNR approaches zero. Therefore, it is of interest to investigate the values of instantaneous SNR at which errors tend to occur. This will help to illustrate more qualitatively the fading effects by examining the ranges of γ responsible for the errors. Then using (3-10) the over all probability of error, $N(\gamma_1)$, conditional on the combiner output SNR being less than some value γ_1, can be expressed by

$$N(\gamma_1) = \frac{1}{2N} \sum_{i=1}^{N} \frac{1}{(L-1)!} \cdot \frac{1}{\gamma_0^L} \int_0^{\gamma_1} \left[1 - \mathrm{erf}\left(D_i^2 (WT)\gamma\right)^{1/2}\right] \gamma^{L-1} e^{-\gamma/\gamma_0} \, d\gamma \quad (3\text{-}14)$$

By expanding the error function in the infinite series

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \sum_{k=1}^{\infty} (-1)^{k+1} \cdot \frac{x^{2k-1}}{(2k-1)(k-1)!} \quad (3\text{-}15)$$

(3-14) becomes

$$N(\gamma_1) = \frac{1}{2N} \sum_{i=1}^{N} \frac{1}{(L-1)!} \cdot \frac{1}{\gamma_0^L} \cdot \left(\mathcal{I}_1 - \mathcal{I}_2 \right) \quad , \tag{3-16}$$

where

$$\mathcal{I}_1 = \int_0^{\gamma_1} \gamma^{L-1} e^{-\gamma/\gamma_0} \, d\gamma$$

$$= \frac{(L-1)!}{(1/\gamma_0)^{L-1}} - e^{-\gamma_1/\gamma_0} \sum_{k=0}^{\infty} \frac{(L-1)!}{k!} \cdot \frac{(\gamma_1)^k}{(1/\gamma_0)^{L-k}} \tag{3-17}$$

and

$$\mathcal{I}_2 = \frac{2}{\sqrt{\pi}} \sum_{k=1}^{\infty} \frac{(-1)^{k+1} \left[D_i^2(WT) \right]^{k-\frac{1}{2}}}{(2k-1)\,(k-1)!} \int_0^{\infty} \gamma^{\frac{2k-1}{2}+L-1} e^{-\gamma/\gamma_0} \, d\gamma \tag{3-18}$$

$$= \frac{2}{\sqrt{\pi}} \sum \frac{(-1)^{k+1} \left[D_i^2(WT) \right]^{k-\frac{1}{2}}}{(2k-1)\,(k-1)!} (\gamma_0)^{L+k-\frac{1}{2}} \gamma\left(L+k-\tfrac{1}{2}, \tfrac{\gamma_1}{\gamma_0}\right) \quad . \tag{3-19}$$

$\gamma(\alpha,x)$ is the incomplete gamma function:

$$\gamma(\alpha,x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{\alpha+n}}{n!\,(\alpha+n)} = \frac{x^{\alpha} e^{-x}}{\alpha} \, {}_1F_1(1, 1+\alpha; x) \quad . \tag{3-20}$$

Then using (3-19) and (3-18) in (3-16) one obtains

$$N(\gamma_1) = \frac{1}{2N}\sum_{i=1}^{N} \frac{1}{(L-1)!}\cdot\frac{1}{\gamma_0^{L}}\left\{\gamma_0^{L-1}(L-1)! - \frac{(L-1)!\,e^{-\gamma_1/\gamma_0}}{(1/\gamma_0)^{L-k}}\sum_{k=0}^{L-1}\frac{(\gamma_1)^k}{k!}\right.$$

$$\left. - \frac{2}{\sqrt{\pi}}\sum_{k_1=0}^{\infty}\frac{(-1)^{k_1+1}\left[D_i^2(WT)\right]^{k_1-\frac{1}{2}}}{(2k_1-1)(k_1-1)!}\gamma_0^{L+k_1-\frac{1}{2}}\,\gamma\left(L+k-\tfrac{1}{2},\frac{\gamma_1}{\gamma_0}\right)\right\} \tag{3-21}$$

$$= \frac{1}{2N}\sum_{i=1}^{N}\left\{\frac{1}{\gamma_0} - e^{-\gamma_1/\gamma_0}\sum_{k=0}^{L-1}\left(\frac{\gamma_1}{\gamma_0}\right)^k\cdot\frac{1}{k!}\right.$$

$$\left. - \frac{2}{(L-1)!\sqrt{\pi}}\sum_{k_1=0}^{\infty}\frac{(-1)^{k_1+1}\left[D_i^2(WT)\right]^{k_1-\frac{1}{2}}}{(2k_1-1)(k_1-1)!}\gamma_0^{k_1-\frac{1}{2}}\,\gamma\left(L+k-\tfrac{1}{2},\frac{\gamma_1}{\gamma_0}\right)\right\} \tag{3-22}$$

From (3-22) we can see that the conditional probability of error $N(\gamma_1)$ turned out to be a rather complicated expression. Nevertheless, we can observe that the ratio γ_1/γ_0 plays a very important role in determining $N(\gamma_1)$.

3.3 Intersymbol Interference and Fading in the Presence of a Specular Component

In assuming Rayleigh fading in the previous sections, we have specifically excluded the models involving a specular component, i.e., the multiplicative Gaussian process has been assumed to have a zero-mean. In this section the zero-mean restriction will be relaxed. Physically, it is possibly to have a channel that can have a major stable path with a number of additional weak paths. The stable path often appears to arise as a result of a stable specular reflection, giving rise to the term "specular component". This terminology is often used to represent any steady signal in a fading environment [12] whose envelope is stable but whose phase can be continually changing with respect to the phase reference at the transmitter.

Thus, the composite signal will appear to be composed of the sum of a steady signal and of a Rayleigh fading signal, the latter having the usual Gaussian quadrature components. Then for this situation, the fading SNR has a p.d.f. (p. 458, [3])

$$p(\gamma) = \frac{1}{\Gamma_R} e^{-\Gamma_0/\Gamma_R} e^{-\gamma/\Gamma_R} I_0\left(\frac{2\sqrt{\gamma\Gamma_0}}{\Gamma_R}\right) , \tag{3-23}$$

where Γ_R is the ratio of mean-square value of the fading component to the noise density, and Γ_0 is the ratio of mean-square value of specular component to the noise density. Equation (3-23) can be generalized for multichannel reception [7]

$$p(\gamma) = \frac{1}{2\sigma^2}\left(\frac{\gamma}{P}\right)^{\frac{L-1}{2}} e^{-\frac{\gamma+\Gamma_0}{2\sigma^2}} I_{L-1}\left(\frac{\sqrt{P\gamma}}{\sigma^2}\right) \tag{3-24}$$

where L is the diversity order. If α_i is the strength of the fixed component in the ith channel and $2\sigma^2$ is mean-squared value of the random or scatter component in the ith channel, then one can define

$$\begin{aligned} P &= \sum_{m=1}^{L} \alpha_m^2 \\ \eta_i^2 &= \alpha_i^2/2\sigma^2 \end{aligned} \tag{3-25}$$

The average error rate can be obtained by averaging (3-2) over all γ, i.e.

$$P = \frac{1}{2N} \sum_{i=1}^{N} \int_0^\infty \left[1 - \operatorname{erf}\left(D_i^2 (WT)\, \gamma \right)^{1/2} \right] \cdot$$

$$\frac{1}{2\sigma^2} \left(\frac{\gamma}{\rho} \right)^{\frac{L-1}{2}} \cdot e^{-\frac{\gamma+\rho}{2\sigma^2}}\, I_{L-1}\left(\frac{\sqrt{\rho\gamma}}{2\sigma^2} \right) d\gamma \tag{3-26}$$

where in (3-26) the order of integration and summation has been interchanged. Now, expanding the modified Bessel function into an infinite series

$$I_\nu(z) = \left(\frac{z}{2} \right)^\nu \sum_{k_2=0}^{\infty} \frac{(z/2)^{2k_2}}{k_2!\, \Gamma(\nu + k_2 + 1)} \qquad , \tag{3-27}$$

and substituting (3-27) into (3-26) one obtains

$$P = \frac{1}{2N} \sum_{i=1}^{N} \frac{e^{-\rho/2\sigma^2}}{2\sigma^2\, \rho^{L-\frac{1}{2}}} \int_0^\infty \left[1 - \operatorname{erf}\left(D_i^2 (WT)\, \gamma \right)^{1/2} \right] \gamma^{L-\frac{1}{2}}\, e^{-\gamma/2\sigma^2} \cdot$$

$$\left(\frac{\sqrt{\rho\gamma}}{2\sigma^2} \right)^{L-1} \sum_{k_2=0}^{\infty} \frac{\left(\sqrt{\rho\gamma}/2\sigma^2 \right)^{2k_2}}{k_2!\, \Gamma(L + k_2)}\, d\gamma \qquad . \tag{3-28}$$

Once more, interchanging the order of integration and summation

$$P = \frac{1}{2N} \sum_{i=1}^{N} \frac{e^{-\rho/2\sigma^2}}{(2\sigma^2)^L} \cdot \sum_{k_2=0}^{\infty} \left(\frac{\sqrt{\rho}}{2\sigma^2} \right)^{2k_2} \cdot \frac{1}{k_2!\, \Gamma(L + k_2)} \cdot J_1 \tag{3-29}$$

where by making the change of variable $x^2 = \gamma$

$$\mathcal{J}_1 = 2\int_0^\infty \left[1-\operatorname{erf}\left(\sqrt{D_i^2(WT)}\,x\right)\right] x^{2(L+k_2-\frac{1}{2})}\, e^{-x^2/2\sigma^2}\, dx. \tag{3-30}$$

Next, using Integral 6.286(1) in [10] one obtains

$$\mathcal{J}_1 = \frac{\Gamma(L+k_2+\frac{1}{2})}{\sqrt{\pi}\,(L+k_2)} \cdot \left(\frac{1}{D_i^2(WT)}\right)^{2(L+k_2)} \cdot F\left(L+k_2, L+k_2+\tfrac{1}{2}; L+k_2+1; \frac{-1/2\sigma^2}{D_i^2(WT)}\right). \tag{3-31}$$

Finally, by substituting (3-31) into (3-29) one obtains the desired expression for the overall error probability for Rician fading. We can now check (3-29) by letting $\alpha\to o$, $\eta^2\to o$ and observing that all the terms in the summation over k_2 vanish, except when $k_2=0$, then

$$P \longrightarrow \frac{1}{2N}\sum \left(\frac{1}{2\sigma^2}\right)^L \cdot \frac{1}{\Gamma(L)}\, \frac{\Gamma(L+1/2)}{\sqrt{\pi}\cdot L} \cdot \left(\frac{1}{D_i^2(WT)}\right)^{2L} F\left(L, L+\tfrac{1}{2}; L+1; \frac{-1/2\sigma^2}{D_i^2(WT)}\right). \tag{3-32}$$

Identifying $2\sigma^2$ with Γ_R and $\Gamma(L)$ with $(L-1)!$ one readily obtains (3-11) which is the expression for the Rayleigh fading case previously obtained in (3-13). Several plots of (3-29) are shown in Figs. III-7 and III-8, where we have defined

$$\gamma^2 = \frac{\alpha^2}{2\sigma^2}$$

as the ratio of the mean squared value of the specular component

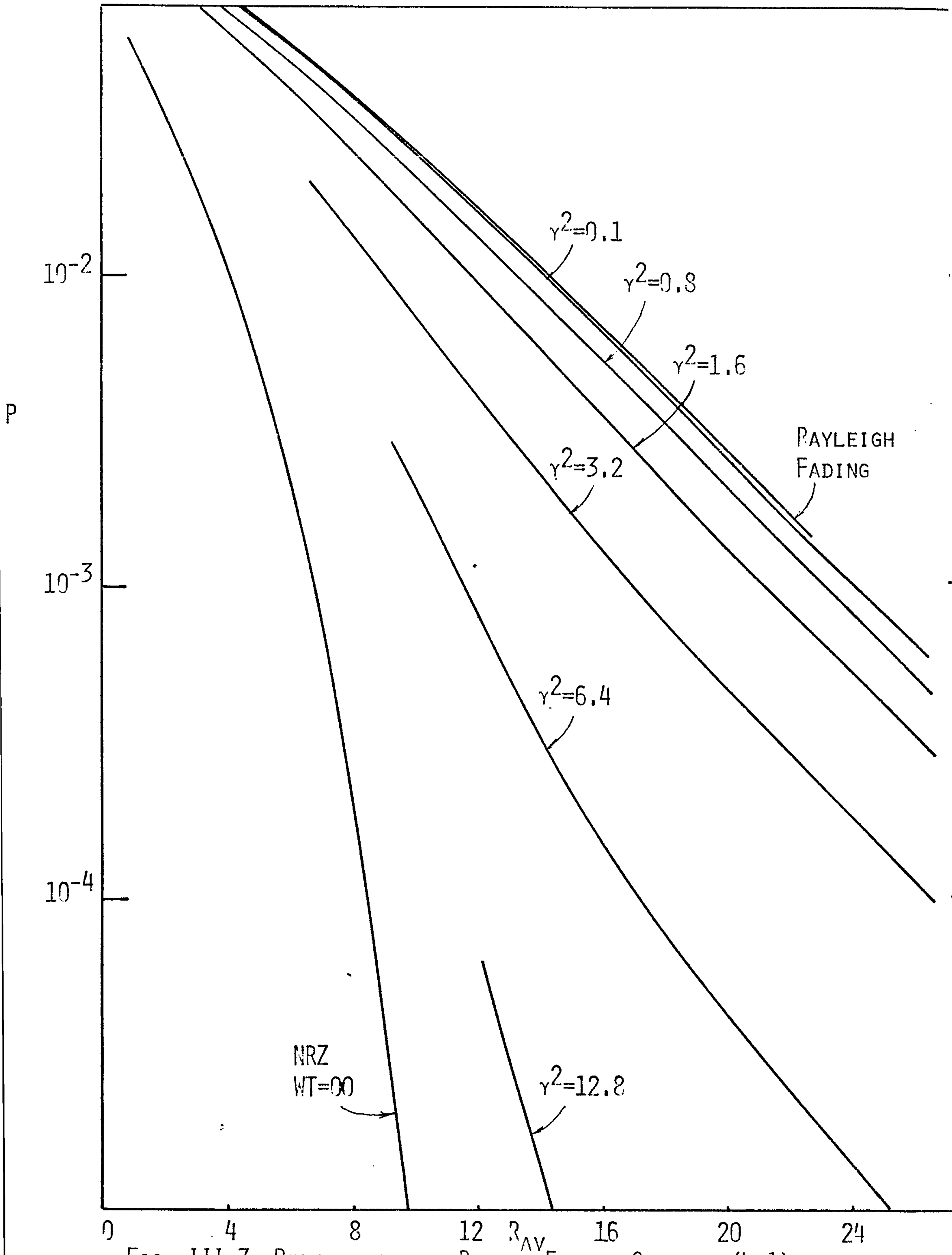

FIG. III-7 PERFORMANCE OF RICIAN FADING CHANNEL (L=1)

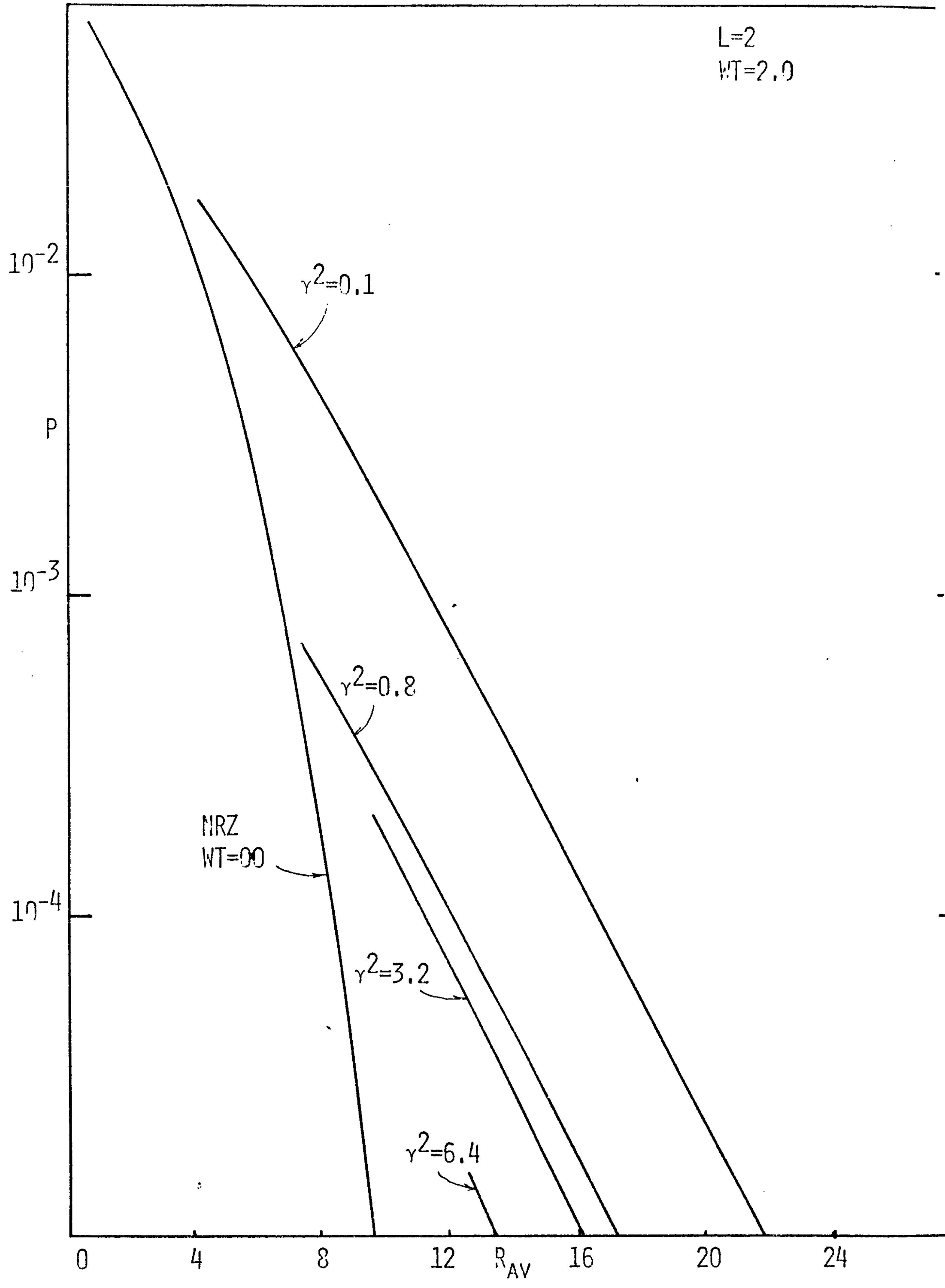

FIG. III-8 PERFORMANCE OF RICIAN FADING CHANNEL (L=2)

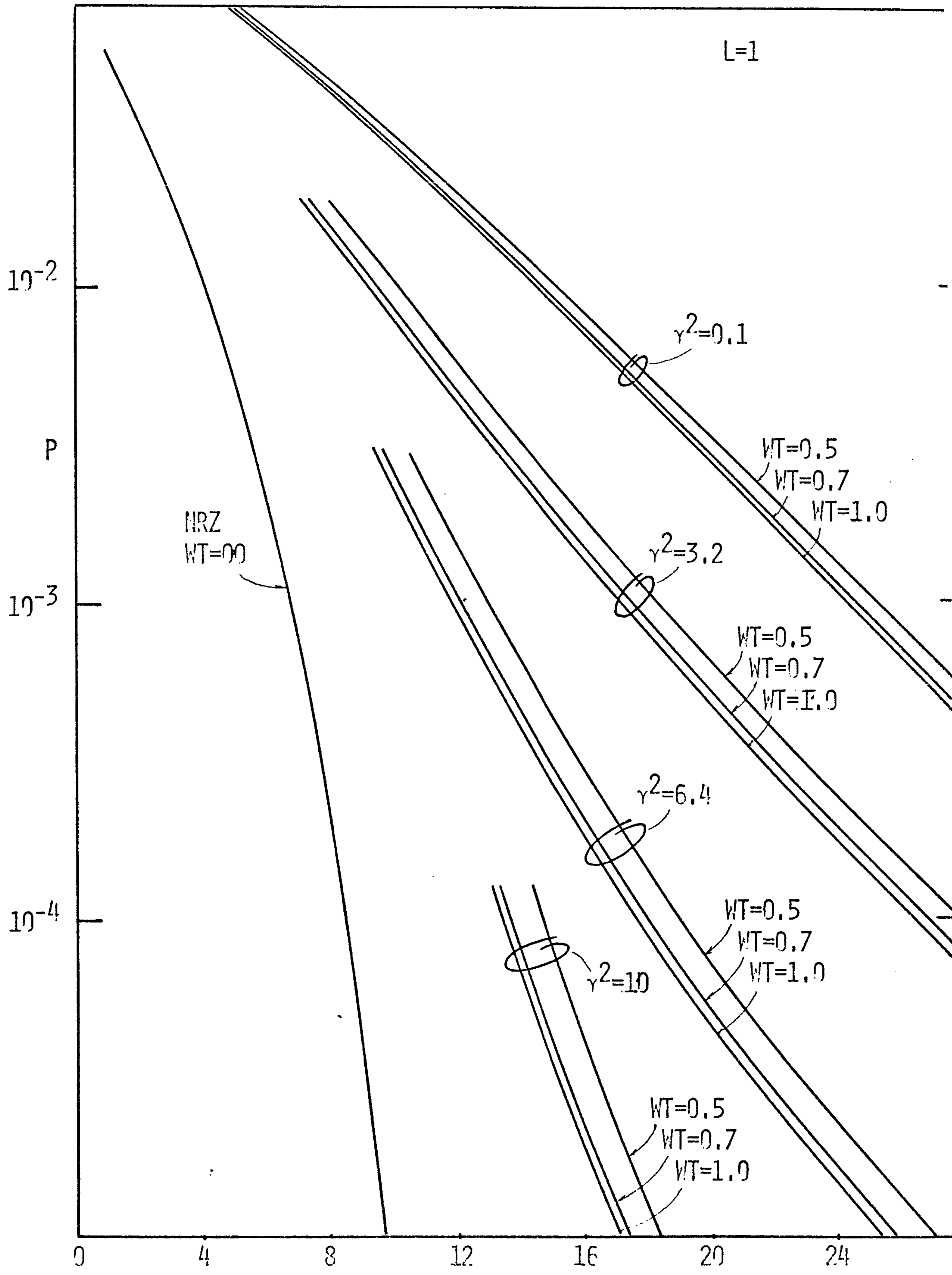
L=1
P
10^{-2}
10^{-3}
10^{-4}
γ^2=0.1
WT=0.5
WT=0.7
WT=1.0
γ^2=3.2
WT=0.5
WT=0.7
WT=1.0
γ^2=6.4
WT=0.5
WT=0.7
WT=1.0
γ^2=10
WT=0.5
WT=0.7
WT=1.0
NRZ
WT=00
0
4
8
12
16
20
24

Fig. III. [illegible]

to the mean squared value of the random component. Also the average SNR

$$R_{av} = \left(\frac{\alpha^2 + 2\sigma^2}{N_0}\right)E = (1+\gamma^2)\beta$$

is the ratio of the average energy received per channel to the noise power density. Note that for

$$\gamma^2 = \begin{cases} 0 \longrightarrow & \text{Rayleigh fading channel} \\ \infty \longrightarrow & \text{Non-fading channel} \end{cases}$$

We observe from Fig. III-5 and III-7 that for $\gamma^2 \simeq 0.1$ we have a Rayleigh fading channel, where the specular component is of no influence. We also observe from Figs. III-7 and III-8 that approximately the same performance is obtained when L=1 and $\gamma^2=6.4$ as when L=2 and $\gamma^2=0.1$. This shows the great improvement that one can obtain by using diversity, also that Rayleigh fading for dual diversity performs the same as singly diversity Rician fading with $\gamma^2=6.4$. Finally, from Figs. III-8 and III-9 we can appreciate that the parameter γ^2 has a definite effect on the system performance.

CHAPTER III

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1. Schwartz, M., Bennett, W. P., and Stein, S., "Communication Systems and Techniques", McGraw-Hill, 1966.

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CHAPTER IV

EVALUATION OF THE AVERAGE PROBABILITY OF ERROR FOR A FADING CHANNEL WITH ARBITRARY SELECTIVITY AND RAPIDITY

4.1 Introduction

In Appendix A, the characteristic function of a quadratic form, D, in complex Gaussian random variables is given and further expressed in a more tractable fashion for integration. In Appendix B, the operation of the kth diversity branch of a canonic receiver is described as a quadratic operation on two complex valued filter outputs. Some examples are given for specific types of receivers, all of which are special cases of the canonic receiver. In Chapter II several descriptions of fading channels were introduced, and it was concluded that the scattering function $\sigma(\tau,f)$ is the most general characterization of this type of channel. Any other characterization is either a particular case of the scattering function or is related to it by the Fourier transform.

In this chapter the evaluation of the probability of error for the canonic receiver of Fig. IV-1 will be obtained independently of the signaling scheme. Non-symmetric operation is assumed, and consequently different expressions will be obtained for the transmission of a "1" or a "0" through the fading channel. Equal a priori probability for the binary signal is assumed, but if this is not the case, then the total probability of error P_T will be just the weighted sum of the

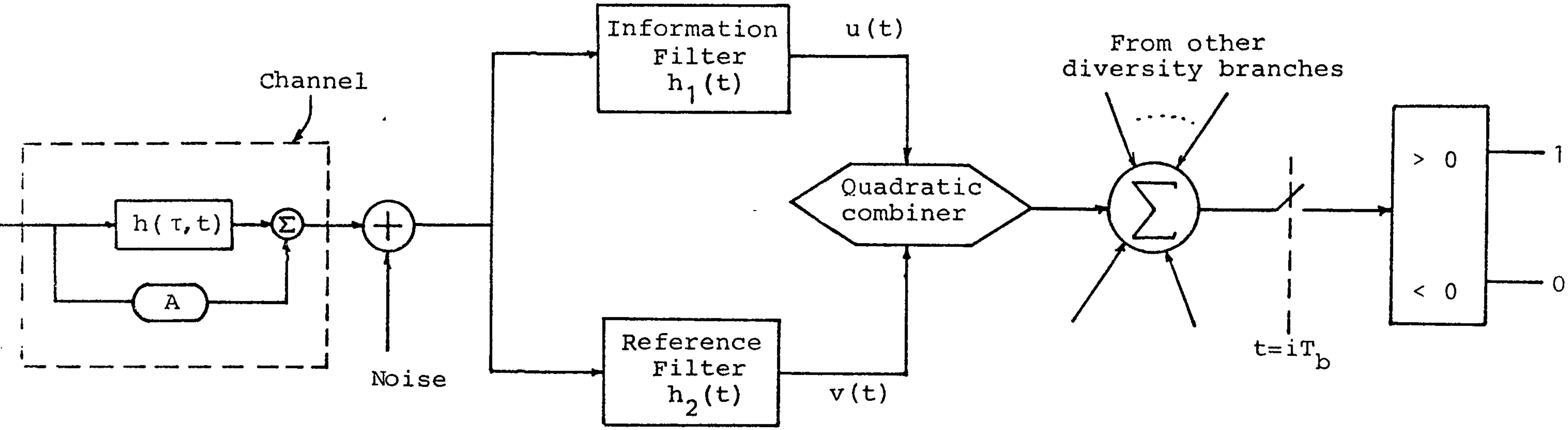

FIG. IV-1 KTH BRANCH FOR A CANONIC RECEIVER individual probabilities, that is

$$P_T = P(S_0) \cdot P_r\{D>0 \mid S=S_0\} + P(S_1) \cdot P_r\{D<0 \mid S=S_1\}$$

where $P(S_i)$ is the a priori probability of the signal S_i, and $P_r\{D>0 \mid S=S_i\}$ is the conditional probability that the quadratic form D is greater than zero, given that the transmitted signal is S_i. All the diversity channels are assumed to fade independently but with identical statistics, which will allow us to use the results of Appendix A. Finally, the type of fading channel does not affect the computation of the probability of error since its effects are included in the moment matrix which is reflected as a parameter in the final expression.

4.2 Probability Density Function of the Hermitian Quadratic Form

To calculate the probability density function p(D), we will take the Fourier transform of the characteristic function, $\Phi(j\nu)$, of the hermitian quadratic form, D. This characteristic function has been derived in Appendix A. Thus

$$p(D) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi(j\nu)\, e^{-j\nu D}\, d\nu \qquad (4-1)$$

or using (A-13)

$$p(D) = \frac{(\nu_1 \nu_2)^L}{2\pi} e^{-A_1} \int [illegible] \qquad (4-2)$$

,

where

$$\int_{-\infty}^{\infty} f_1(\nu)\,d\nu = \int_{-\infty}^{\infty} \frac{e^{-j\nu D}}{(\nu+j\nu_1)^{L}(\nu-j\nu_2)} \cdot e^{j\frac{A_2}{\nu+j\nu_1} - j\frac{A_3}{\nu-j\nu_2}}\,d\nu\,. \tag{4-3}$$

To evaluate (4-2) we shall expand the second exponential term in (4-3) in an infinite series of the form

$$e^{j\frac{A}{\nu+j\nu_1}} = \sum_{k_1=0}^{\infty} \frac{1}{k_1!}\,(j)^{k_1}\,(A)^{k_1}\left(\frac{1}{\nu+j\nu_1}\right)^{k_1}. \tag{4-4}$$

Equation (4-3) then becomes

$$\int_{-\infty}^{\infty} f_1(\nu)\,d\nu = \sum_{k_1=0}^{\infty} \frac{(jA_2)^{k_1}}{k_1!} \sum_{k_2=0}^{\infty} \frac{(-jA_3)^{k_2}}{k_2!} \cdot \int_{-\infty}^{\infty} f_2(\nu)\,d\nu, \tag{4-5}$$

where

$$\int_{-\infty}^{\infty} f_2(\nu)\,d\nu = \frac{(j)^{-(L+k_1)}}{(-j)^{L+k_2}} \cdot \int_{-\infty}^{\infty} \frac{e^{-j\nu D}}{(\nu_2+j\nu)^{L+k_1}(\nu_1-j\nu)^{L+k_2}}\,d\nu\,. \tag{4-6}$$

It is important to observe that use of the series expansion in (4-2) is permissible since the series itself is convergent and each one of its terms is finite for any real value of ν. If ν takes values from $-\infty$ to $+\infty$ on the real axis, it will never be equal to $+j\nu_2$ or $-j\nu_1$, which are the two values of ν that can make the series expansion invalid. See Fig. IV-1.

Now, evaluating (4-6) using Integral 3.384(9) in [1] one obtains

$$\int_{-\infty}^{\infty} f_1(\nu)\, d\nu = \begin{cases} \dfrac{-2\pi/R}{(\nu_1+\nu_2)^{L+\frac{k_1+k_2}{2}}} \cdot \dfrac{D^{L+\frac{k_1+k_2}{2}-1}}{\Gamma(L+k_1)} \cdot e^{\left(\frac{\nu_1-\nu_2}{2}\right)D} \cdot W_{\frac{k_2-k_1}{2},\,\frac{1}{2}-L-\frac{k_1+k_2}{2}}\left[(\nu_1+\nu_2)D\right] & D>0 \\[2ex] \dfrac{2\pi/R}{(\nu_1+\nu_2)^{L+\frac{k_1+k_2}{2}}} \cdot \dfrac{(-D)^{L+\frac{k_1+k_2}{2}-1}}{\Gamma(L+k_1)} \cdot e^{\left(\frac{\nu_2-\nu_1}{2}\right)D} \cdot W_{\frac{k_1-k_2}{2},\,\frac{1}{2}-L-\frac{k_1+k_2}{2}}\left[-(\nu_1+\nu_2)D\right] & D<0 \end{cases} \tag{4-7}$$

where $\Gamma(n+1)=n!$, and $W_{\lambda,\xi}(z)$ is the Wittaker function [2]. The probability density function for a generalized quadratic form is found by substituting (4-7) in (4-5) and this result used back into (4-2) to obtain

$$f(D) = \begin{cases} \displaystyle\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} \frac{(A_2)^{k_1}}{k_1!}\frac{(A_3)^{k_2}}{k_2!}\frac{(-1)^{1-L-k_2}}{j^{2L}}\frac{(\nu_1\nu_2)^L/R}{(\nu_1+\nu_2)^{L+\frac{k_1+k_2}{2}}} \cdot \frac{e^{-A_1}}{\Gamma(L+k_2)} \cdot D^{L+\frac{k_1+k_2}{2}-1} \cdot e^{\left(\frac{\nu_1-\nu_2}{2}\right)D}\, W_{\frac{k_2-k_1}{2},\,\frac{1}{2}-L-\frac{k_1+k_2}{2}}\left[(\nu_1+\nu_2)D\right] & D>0 \\[2ex] \text{[illegible]} \end{cases} \tag{4-8}$$

4.3 Average Probability of Error

Let p_o denote the probability that the receiver decides that a "1" was received given that a "0" was sent, and p_1 the probability that a "0" is decided given that a "1" was sent. Then an error will occur if a "1" was sent and the sign of the quadratic form D is negative¶. Conversely, if a "0" was sent and D>0 then the receiver will be in error. Thus

$$
\begin{aligned}
p_o &= P_r\left\{ D>0 \,\middle|\, s(t)=s_0(t),\ 0\le t\le T \right\} = \int_0^{\infty} p(D)\,dD \\
p_1 &= P_r\left\{ D<0 \,\middle|\, s(t)=s_1(t),\ 0\le t\le T \right\} = \int_{-\infty}^{0} p(D)\,dD \quad .
\end{aligned}
\tag{4-9}
$$

These probabilities of error will be evaluated, and it will be shown that, in general, $p_o \neq p_1$ indicating that the channel is not always symmetric.

For p_o we use (4-8) in (4-9) and interchanging the order of integration and summation giving

$$
P_\varepsilon\left\{ D>0 \right\} = \int_0^{\infty} p(D)\,dD
$$

$$
= \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \frac{(A_1)^{k_1}}{k_1!} \cdot \frac{(A_2)^{k_2}}{k_2!} \frac{(-1)^{j-L-k_1}}{j^{2L}} \frac{(D_1 D_2)^{L}}{(D_1+D_2)^{L+\frac{k_1+k_2}{2}}} \cdot \frac{e^{-A_1}}{\Gamma(L+k_2)} \cdot \int_0^{\infty} f_3(D)\,dD \;, \tag{4-10}
$$

¶For some examples on how the sign of the quadratic form determines the presence of a "0" or a "1", see Appendix B.

where

$$f_3(D) = e^{-\frac{\nu_2+\nu_1}{2}D}\, D^{L+\frac{k_1+k_2}{2}-1}\, W_{\frac{k_2-k_1}{2},\, \frac{1}{2}-L-\frac{k_1+k_2}{2}}\left[(\nu_1+\nu_2)D\right] . \tag{4-11}$$

Using Integral 7.621(3) from [1] in (4-10), one readily obtains

$$P_\varepsilon\{D>0\} = \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} \frac{(A_2)^{k_1}}{k_1!}\cdot\frac{(A_3)^{k_2}}{k_2!}\cdot\frac{(\nu_1\nu_2)^L/\nu_2}{(\nu_1+\nu_2)^{2L-1+k_1+k_2}}\cdot\frac{e^{-A_1}}{\Gamma(L+k_1)}\cdot\frac{(-1)^{-1-k_2}}{j^{2L}}\cdot$$

$$\frac{\Gamma(2L+k_1+k_2)}{\Gamma(L+k_2+k_1)}\ {}_2F_1\left(1,\, 1-L-k_2;\, L+k_1+1;\, -\frac{\nu_2}{\nu_1}\right) , \tag{4-12}$$

where ${}_2F_1(a,b;c;z)$ is the Gauss hypergeometric function [3]. In a similar manner we can compute $P_\varepsilon\{D<0\}$ obtaining

$$P_\varepsilon\{D<0\} = \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} \frac{(A_2)^{k_1}}{k_1!}\cdot\frac{(A_3)^{k_2}}{k_2!}\cdot\frac{(\nu_1\nu_2)^L/\nu_1}{(\nu_1+\nu_2)^{2L-1+k_1+k_2}}\cdot\frac{e^{-A_1}}{\Gamma(L+k_1)}\cdot\frac{(-1)^{1-L-k_2}}{j^{2L}}\cdot$$

$$\frac{\Gamma(2L+k_1+k_2)}{\Gamma(L+k_2+k_1)}\ {}_2F_1\left(1,\, 1-L-k_1;\, L+k_2+1;\, -\frac{\nu_1}{\nu_2}\right) . \tag{4-13}$$

Even though (4-12) and (4-13) show that to evaluate the average probability of error one has to compute two infinite series, the computer runs show that the summations over k_1 and k_2

converge rather quickly. In fact, typical values for the number of iterations necessary to achieve a convergence of one unit in 10^7 are $k_1=7$ and $k_2=4$, for L=1. The hypergeometric series is known to be composed of a finite number of terms if either a or b in ${}_2F_1(a,b;c;z)$ is negative or zero, leading to absolute convergence. This convergence can also be seen from the condition of convergence of the hypergeometric series when $|z|=1$, i.e., the series is absolutely convergent throughout the entire unit circle if

$$\mathrm{Re}\{1-2L-k_1\}<0 \quad . \tag{4-14}$$

Some results are shown in Fig. IV-2 where we have plotted (4-11) versus average SNR, ρ, using the definitions

$$\xi = \frac{\alpha_i^2}{2\sigma^2} \tag{4-15}$$

and

$$\rho = \frac{E\ \alpha_i^2}{2N_o} \quad . \tag{4-16}$$

The parameter ξ can be interpreted as the amount of power received via the specular component, α_i^2, as compared to the power received when a sinisoid of unity peak value is transmitted.

The main observation that can be made from these figures is that there exists an irreducible probability of error if the random channel is selective, i.e., if we let the SNR approach infinity, the probability of error approaches a finite number. To illustrate the dependence of performance on bandlimiting, families of curves are plotted in Fig. IV-2 for WT=0.5

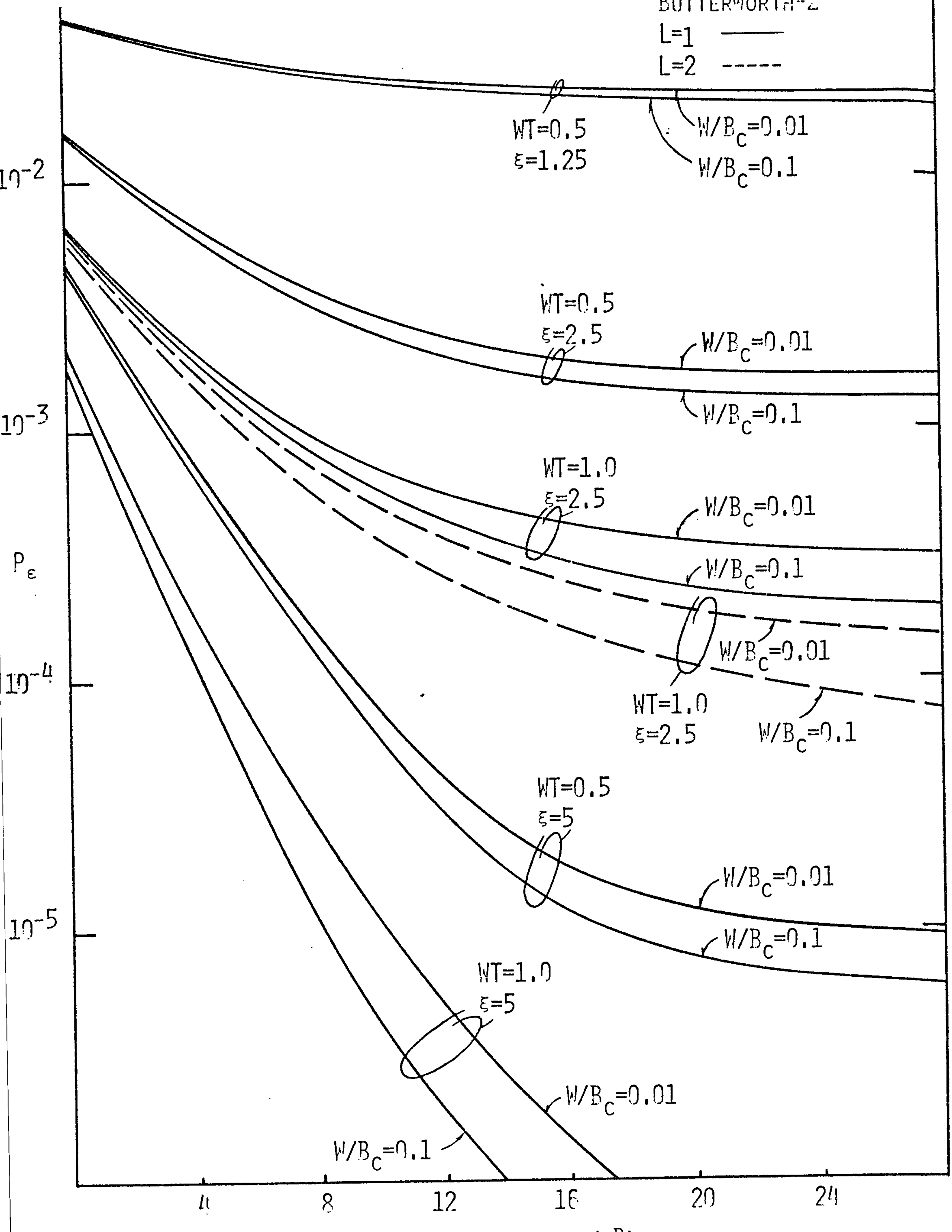

FIG. IV-2 PERFORMANCE OF A FREQUENCY SELECTIVE CHANNEL and WT=1.0. The reason for chosing these two values for WT is that WT<0.5 is of no practical interest, and for WT>1.0 the performance curves are very close to the WT=1.0 curve. In fact when $\xi=5$, L=1, and $W/B_c=0.01$, the difference between the curves for WT=1.0 and WT=1.5 is about 0.2 dB. We can also see that for $W<B_c$ the receiver performance is strongly dependent on the product WT. For example, when $\xi=2.5$, L=1,and $W/B_c=0.1$, the irreducible probability of error is 1.25×10^{-3} for WT=0.5 and 1.75×10^{-4} for WT=1.0.

Another parameter which greatly affects the performance curves of Fig IV-2 is ξ. We can observe that doubling ξ from $\xi=2.5$ to $\xi=5.0$ gives better improvement than to double the diversity order from L=1 to L=2. This shows that the specular, or non-fading, component is of great importance in determining the system performance. Note that the irreducible probability of error is not eliminated when increasing ξ but rather decreased to a smaller number.

4.4 Rayleigh Fading

The results obtained for the generalized case of Rician fading can be reduced to a purely Rayleigh fading situation. This is done by letting $A_1=A_2=A_3=0$ defined in (A-14), which implies that the specular component of the fading channel is zero, and observing that all the terms in both summations in (4-12) and (4-13) vanish except when $k_1=k_2=0$. Then

$$P_\varepsilon = \frac{(\nu_1\nu_2)^L}{(\nu_1+\nu_2)^{2L-1}} \cdot \frac{\Gamma(2L)}{\Gamma(L)\Gamma(L+1)} \cdot \begin{cases} \frac{1}{\nu_2}\,{}_2F_1\left(1;1-L;L+1;-\frac{\nu_1}{\nu_2}\right) & D>0 \\ \\ \frac{1}{\nu_1}\,{}_2F_1\left(1,1-L;L+1;-\frac{\nu_2}{\nu_1}\right) & D<0 \end{cases} \tag{4-17}$$

The same argument is used here as in Section 4.3 on the hypergeometric infinite series, i.e., when $L \geq 2$ the hypergeometric series

$${}_2F_1(a,b;c;z) = 1 + \frac{a \cdot b}{c \cdot 1} z + \frac{a(a+1)\,b(b+1)}{c(c+1)\cdot 1 \cdot 2} z^2 + \cdots \tag{4-18}$$

terminates since $b=1-L$ is a negative integer. Then it is valid to assume that when $a=1$ and b is a negative integer, the series (4-16) will have a finite number of terms. Also note that when $L=1$ (no diversity) the hypergeometric series becomes

$${}_2F_1(1,0;1;z)=1 \quad . \tag{4-19}$$

Then (4-12) and (4-13) can be expressed as

$$P_\varepsilon = \begin{cases} \frac{\nu_1}{\nu_1+\nu_2} & D>0 \\ \\ \frac{\nu_2}{\nu_1+\nu_2} & D<0 \end{cases} , \tag{4-20}$$

which is the well known result previously obtained in the literature [4,5,6]. Therefore, by reducing (4-12) and (4-13) to (4-20) we have specialized our general result to a rather simple expression which has been widely used when selective Rayleigh-fading channels are under study.

It is interesting to note that when comparing these results with those of systems employing other types of combining and/or detection schemes, the expression for the probability of error can be identical, but the expression for the moments is expected to be different. This situation arises from the fact that the receiver operation for linear combining involves the comparasion of a quadratic form with a zero threshold, but the constants a,b, and c multiplying the quadratic form (1-2) can be different.

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Chapter IV

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CHAPTER V

THE MOMENT MATRIX M

5.1 Introduction

In the development of Appendix A it was shown that the characteristic function of the quadratic form (A-2) is a function, among other things, of the moment matrix M. Further manipulation of (A-2) proved to give (A-13)which is a function of ν_1, ν_2, A_1, A_2, and A_3 given in (A-10) and (A-14). These parameters are a function of the elements of the moment matrix, called "moments", multiplied by some constants. Then once the moments are known, the characteristic function (A-13) and consequently the probability of error (4-17), are known.

In this chapter the explicit computation of the moments (A-4) will be carried out for the particular case of DPSK detection; however, the formulation of the moments will be given for other types of detection schemes. The channel characterizations will include the flat-flat fading, the time-flat fading, and the frequency-flat fading channels, in combination with the matched filter and the Butterworth filter receivers. Complex envelope notation will be used because of the resulting simplification it allows when dealing with narrowband processes. It will be shown that the moments can be expressed in a general form which is a function of the filter bandwidth W, the channel correlation bandwidth B_c, and the pulse duration T. Also it is illustrated that the deterministic part of the moments can be evaluated independently of the channel characteristics, which act on the moments as a weighting factor.

5.2 General Form of the Moments m_u, m_v, and m_{uv}

Let the received signal be

$$r(t) = \mathrm{Re}\left\{y(t)\, e^{j2\pi f_0 t}\right\} \tag{5-1}$$

where

$$y(t) = A \cdot s(t-t_1) + \int h(t-\tau, t)\, s(\tau)\, d\tau + n(t) \tag{5-2}$$

is the complex envelope of $r(t)$, and

$$A = \langle h(t-\tau, t)\rangle = \alpha\, e^{j\delta} \tag{5-3}$$

is the complex factor describing the relative amplitude and phase of the specular component. Let $h(\tau, t)$ be the stationary complex zero-mean Gaussian process describing the time-varying equivalent lowpass impulse response of the medium introduced in Chapter II, and t_1 be a delay relative to some nominal mean time. If the information filter in Fig. IV-1 has an impulse response of $h_1(t)$ then its output given that (5-2) is the input is

$$\mu(t) = \int h_1(t-x)\, y(x)\, dx + \int h_1(t-x)\, n(x)\, dx \tag{5-4}$$

where the mean value of this output signal is

$$\langle \mu(t) \rangle = A \int h_1(t-x)\, s(x-t_1)\, dx \,. \tag{5-5}$$

The output for the reference filter with impulse response $h_2(t)$ is

$$\begin{aligned} v(t) &= \int h_2(t-x)\, y(x)\, dx + \int h_2(t-x)\, n(x)\, dx \\ &= A \cdot \int h_2(t-x)\, s(x-t_1)\, dx + \int h_2(t-x)\, n(x)\, dx + \\ &\qquad \iint h_2(t-x)\, h(\sigma; x)\, s(x-\sigma)\, d\sigma\, dx \,. \end{aligned} \tag{5-6}$$

If v(t) and u(t) are the inputs to the kth quadratic combiner, the output will be

$$d_k = a|\mu_k|^2 + b|v_k|^2 + c\,\mu_k^* v_k + c^* \mu_k v_k^* \tag{5-7}$$

We shall now evaluate the quantities defined in (A-4). First, the moment of the information filter is

$$m_u = \left\langle \left| U_k(t) \right|^2 \right\rangle \tag{5-8}$$

where we define the random part of $u_k(t)$ as

$$U_k(t) = u_k(t) - \left\langle u_k(t) \right\rangle \tag{5-9}$$

Then using (5-4) and (5-5) into (5-9), and the latter into (5-8) we obtain

$$m_u = \left\langle \iiiint h_1^*(t-x_1)\, h_2(t-x_2)\, h^*(\tau_1, x_1)\, h(\tau_2, x_2)\, s^*(x_1-\tau_1)\, s(x_2-\tau_2)\; d\tau_1\, dx_1\, d\tau_2\, dx_2 \right\rangle . \tag{5-10}$$

Since the only random quantities are $h(\cdot)$ and $h^*(\cdot)$, the order of the ensemble averaging operation and the integration can be interchanged. At this point the multipath autocovariance defined in (2-11) can be identified as

$$R_T(\tau_1; x_2-x_1)\, \delta(\tau_1-\tau_2) = \left\langle h^*(\tau_1; x_1)\, h(\tau_2; x_2) \right\rangle . \tag{5-11}$$

Substituting (5-11) into (5-10) gives

$$m_u = \iiiint h_1^*(t-x_1)\, h_1(t-x_2)\; R_T(\tau_1; x_2-x_1)\, \delta(\tau_1-\tau_2)\, s^*(x_1-\tau)\, s(x_2-\tau)\; d\tau_1\, dx_1\, dx_2\, d\tau_2 \,, \tag{5-12}$$

which can be further simplified by using the sifting property of the delta function to yield

$$m_\mu = \iiiint h_1^*(t-x_1)\, h_1(t-x_2)\, R_T(\tau, x_2-x_1) \\ s^*(x_1-\tau)\, s(x_2-\tau)\, d\tau\, dx_1\, dx_2 \quad . \tag{5-13}$$

The next step is to express (5-13) in terms of the scattering function, $\sigma(\tau,f)$, by using the relationship (2-12) this gives

$$\begin{aligned} m_\mu &= \iiiint h_1^*(t-x_1)\, h_1(t-x_2)\, \sigma(\tau,\mu)\, e^{-j2\pi\mu(x_2-x_1)} \\ &\qquad s^*(x_1-\tau)\, s(x_2-\tau)\, d\tau\, dx_1\, dx_2\, d\mu \\ &= \iiiint \left[h_1^*(t-x_1)\, s^*(x_1-\tau)\, e^{j2\pi\mu x_1}\right] \\ &\qquad \left[h_1(t-x_2)\, s(x_2-\tau)\, e^{-j2\pi\mu x_2}\right] \sigma(\tau,\mu)\, dx_1\, dx_2\, d\tau\, d\mu \; . \end{aligned} \tag{5-14}$$

Now if we let

$$\begin{aligned} w_\mu(t,\tau,\mu) &= e^{j2\pi\mu\tau} \int h_1(t-x)\, s(x-\tau)\, e^{-j2\pi\mu x}\, dx \\ &= \int h_1(t-\tau-x)\, s(x)\, e^{-j2\pi\mu x}\, dx \end{aligned} \tag{5-15}$$

and

$$w_\mu^*(t,\tau,\mu) = e^{-j2\pi\mu\tau} \int h_1^*(t-x)\, s(x-\tau)\, e^{j2\pi\mu x}\, dx \; , \tag{5-16}$$

we can write (5-14) as

$$m_\mu = \iint \left[w_\mu(t,\tau,\mu)\, w_\mu^*(t,\tau,\mu) \right] \sigma(\tau,\mu)\, d\tau\, d\mu \tag{5-17}$$

or

$$m_\mu = \iint \left| w_\mu(t,\tau,\mu) \right|^2 \sigma(\tau,\mu)\, d\tau\, d\mu . \tag{5-18}$$

In an analogous fashion we shall define m_v as

$$m_v = \iint \left| w_v(t,\tau,\mu) \right|^2 \sigma(\tau,\mu)\, d\tau\, d\mu \tag{5-19}$$

with

$$w_v(t,\tau,\mu) = e^{j2\pi\mu\tau} \int h_2^*(t-x)\, s(x-\tau)\, e^{-j2\pi\mu x}\, dx . \tag{5-20}$$

Similarly

$$m_{\mu v} = \iiiint \left[h_1^*(t-x_1)\, s(x_1-\tau)\, e^{j2\pi\mu x_1} \right] \left[h_2(t-x_1)\, s(x_1-\tau)\, e^{-j2\pi\mu x_2} \right] \sigma(\tau,\mu)\, dx_1\, dx_2\, d\tau\, d\mu$$

$$= \iint \left[w_\mu(t,\tau,\mu)\, w_v^*(t,\tau,\mu) \right] \sigma(\tau,\mu)\, d\tau\, d\mu . \tag{5-21}$$

It can be seen that $w(t,\tau,u)$ is a deterministic quantity which can be evaluated independently of the channel state. The channel behavior will be reflected in the moments by weighting the w's which could be identified as convolution integrals of the signal and filters impulse response.

We shall now use (5-14), (5-19), and (5-21) to evaluate the moments m_u, m_v and m_{uv} for the flat-flat fading, the time-flat fading and the frequency-flat fading, in combination with different type of receiver filters, namely, the matched filter and the Butterworth filter. Five cases will be presented giving the explicit form of the scattering function and the moments. The first three will be using the matched filter at the front end of the receiver to show how the general formulation for the scattering function (2-12) and the moment (5-18), (5-19) and (5-21) can be reduced to particular cases in the literature. Butterworth filtering is then used to determine the influence of the filter bandwidth on the moments. Special interest is given to the time-flat, frequency-selective fading channel, which is developed in detail in Appendix D.

A. Case Study I: Time-flat and frequency-flat fading with matched filter receiver

For this case the scattering function becomes

$$\sigma(\tau, f) = K \cdot \delta(\tau - \tau_0)\, \delta(f - f_0) \tag{5-22}$$

where τ_o can be interpreted as a path delay and f_0 a center frequency of operation. From (5-15)

$$\omega(t',0,0) = \int h(t'-x)\, s(x)\, dx \quad . \tag{5-23}$$

Since a matched filter operation is performed,

$$h(t_o - t) = S^*(t) \ . \tag{5-24}$$

Substituting (5-24) and (5-23) into (5-14) we have

$$\begin{aligned} m_\mu &= K \iint h_1^*(t-x_1)\, h_1(t-x_2)\, S^*(x_1)\, S(x_2)\; dx_1\, dx_2 \\ &= K \iint S_1(x_1)\, S_1^*(x_2)\, S^*(x_1)\, S(x_2)\; dx_1\, dx_2 \quad . \end{aligned} \tag{5-25}$$

If we define the energy of the binary pulse as

$$E = \frac{1}{2} \int_0^T |s(t)|^2\, dt \tag{5-26}$$

and the correlation coefficient between mark and space as

$$\beta = \frac{1}{2E} \int_0^T S_1^*(t)\, S_0(t)\, dt \tag{5-27}$$

then

$$m_\mu = K\int_0^T s_1(x_1)\, s^*(x_1)\, dx_1 \int_0^T s_1^*(x_2)\, s(x_2)\, dx_2 = 4K\,E^2 \quad . \tag{5-28}$$

Similarly,

$$m_\nu = K\int_0^T s_2(x_1)\, s^*(x_1)\, dx_1 \int_0^T s_2^*(x_2)\, s(x_2)\, dx_2 = 4KE^2|\beta|^2 \tag{5-29}$$

$$m_{\mu\nu} = K\int_0^T s_1(x_1)\, s^*(x_1)\, dx_1 \int_0^T s_0^*(x_2)\, s(x_2)\, dx_2 = 4KE^2\beta \, . \tag{5-30}$$

These equations can be identified with (19) of Bello [1].

B. Case Study II: Time-selective and frequency flat fading with matched-filter receiver

For this case the scattering function is

$$\sigma(\eta, f) = \delta(\eta)\, \sigma(\mu) \quad . \tag{5-31}$$

Using (5-24) and (5-31) in (5-13) we obtain

$$m_\mu = \iiint h_1^*(t-x_1)\, h_1(t-x_2)\; R_T(x_2-x_1)\; s^*(x_1)\, s(x_2)\; dx_1\, dx_2 \quad . \tag{5-32}$$

Making the change of variable

$$\begin{aligned} x &= x_2 \\ X &= x_2 - x_1 \end{aligned} \tag{5-33}$$

we can express (5-32) as

$$m_\mu = \iint \left[h_1^*(t-x+X)\, s^*(x-X)\right]\left[h(t-x)\, s(x)\right] R_T(X)\, dx\, dX$$

$$= \iint S_1(x+X)\, s(x)\, S_1^*(x)\, s(x-X)\, R_T(X)\, dx\, dX \tag{5-34}$$

which can be identified with (15) in Bello [2].

C. Case Study III: Time-flat and frequency-selective fading with matched filter receiver

For this case the scattering function is

$$\sigma(\tau, f) = \delta(f)\, \sigma(\tau), \tag{5-35}$$

Using (5-35) and (5-24) in (5-14) we have

$$m_\mu = \iiint \left[h_1^*(t-x_1)\, h_1(t-x_2)\right] \sigma(\tau)\, s^*(x_1-\tau)\, s(x_2-\tau)\, d\tau\, dx_1\, dx_2$$

$$= \iiint S(x_1)\, S^*(x_2)\, s^*(x_1-\tau)\, s(x_2-\tau)\, \sigma(\tau)\, d\tau\, dx_1\, dx_2 \tag{5-36}$$

$$= \int \sigma(\tau) \int S(x_1)\, s^*(x_1-\tau)\, dx_1 \int S(x_2)\, s(x_2-\tau)\, dx_2\, d\tau$$

This equation is equivalent to (32) in Bello [1].

D. Case Study IV: Time-flat and frequency-flat fading with Butterworth filtering

For this case, we are using the same scattering function as in (5-22), but the matched filter has been replaced by a Butterworth filter. Also the explicit form of the moments will not be given because, as will be shown in Case Study V, they can be obtained as a particular case of Case Study V. Before calculating the deterministic part of the moments, $w(\cdot)$, the response of the filter to an input signal pattern is discussed.

When the input signal to a linear filter is a bit pattern, the filter will cause intersymbol interference depending on the value of its time constant. Interference results because the binary information is sent in contiguous form and there will be some symbol overlapping due to the filter response. Since we are considering causal systems, only the previous bits will induce interference into the bit under detection, if the sampling time, t_s, is between 0 and T. This effect can be illustrated as in Fig. V-1, where 3 previous bits are considered (a future bit is shown for generality, since it will be considered later in the computation of the moments) to cause the most significant part in the intersymbol interference.

The computation of $w(t',0,0)$ will have to be done for all the 2^4=16 patterns since, in general, the effect of the channel on a 0 will not be the same as the effect on a 1. Nevertheless, there will be symmetry conditions which will simplify the computation. Let abcd denote the pattern of transmitted bits and $r_{abcd}(t)$ the corresponding received waveform. If p_{abcd} is the probability that the quadratic form is negative given that the transmitted

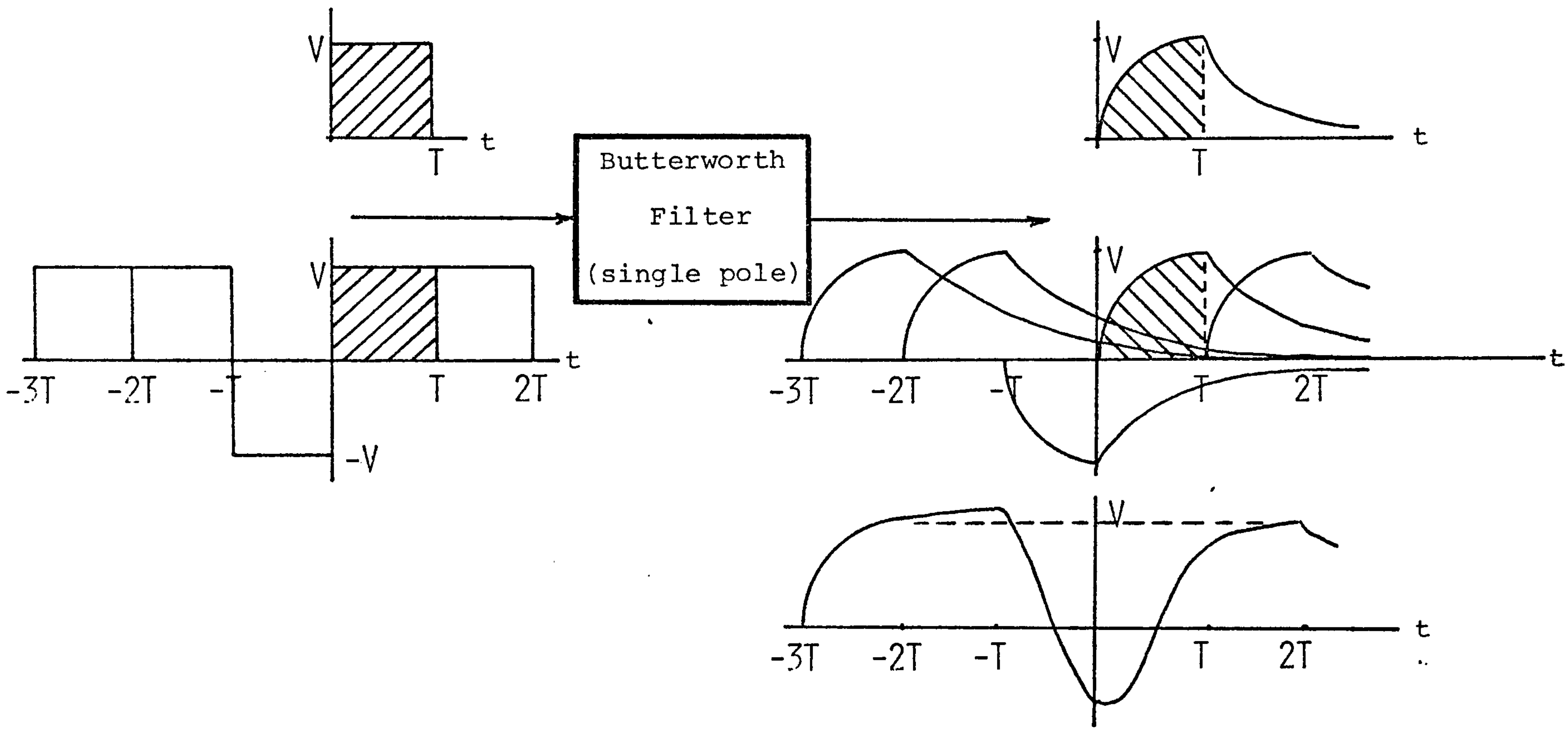

FIG. V-1 PULSE AND PATTERN RESPONSE OF A BUTTERWORTH FILTER pattern is abcd, where a,b,c and d=1 or 0, then the symmetry conditions are [1]

$$\rho_{abcd} = \rho_{\overline{abcd}} \tag{5-36}$$

and

$$\rho_{abcd} = \rho_{dcba} \tag{5-37}$$

where the bar over the symbol pattern means binary complement. Eq. (5-36) implies that a change in polarity of the transmitted signal does not change the detector output (signal only). In (5-37) we are saying that if $s_{dcba}(t) = s_{abcd}(-t)$, then, replacing $s(t)$ by $s(-t)$ in U_k and V_k is statistically equivalent to interchanging U_k and V_k, which does not change the detector output statistics. Then the signal patterns to be considered are

$$\begin{array}{ll} S_4 = 0100 & S_6 = 0110 \\ S_5 = 0101 & S_7 = 0111 \\ S_{12} = 1100 & S_{15} = 1111 \end{array} \tag{5-38}$$

We now proceed to obtain the expression of $w(t',0,0)$ for the patterns of interest. A general formula that can be used to obtain $w(t',0,0)$ for any signal pattern. This general formulation can be easily checked by induction.

Thus

$$
\begin{aligned}
\omega(t',0,0) = 2V\, e^{-at'} \Big\{ & \big[\mu_o(n+k)\big]\, e^{at'} && \forall t' \\
& + \big[\mu_o(n) - \mu_o(n+1)\big]\, e^{aT} && (T \leq t' \leq 2T) \\
& + \big[\mu_o(n-1) - \mu_o(n)\big] \cdot 1 && (0 \leq t' \leq 2T) \\
& + \big[\mu_o(n-2) - \mu_o(n-1)\big] \cdot e^{-aT} && (-T \leq t' \leq 2T) \\
& + \big[\mu_o(n-3) - \mu_o(n-2)\big] \cdot e^{-2aT} && (-2T \leq t' \leq 2T) \\
& \quad \vdots \\
& \big[\mu_o(n+N) - \mu_o(n-N-1)\big] \cdot e^{-(N-1)aT} \Big\}
\end{aligned}
\tag{5-39}
$$

where

$$
\mu_o(k) = \begin{cases} \pm 1 & \text{depending on the sign of the kth bit} \\ 0 & \text{if not considered, or if the term is not within the interval of interest} \end{cases},
$$

$k=n$ when $nT \leq t' \leq (n+1)T$,

N=number of bits considered previous to the bit under detection,

M=number of future bits ,

$V=\sqrt{2E/T} \quad 0 \leq t \leq T$.

Equation (5-39) can also be expressed as

$$w(t',0,0) = 2V e^{-at'} \left\{ \left[\mu_0(n+k)\, e^{at'} + \sum_{k=-N}^{M} \left[\mu_0(n+k-1) - \mu_0(n+k) \right] e^{k a T} \right\} \right. \tag{5-40}$$

for $kT \le t' \le (M+1)T$.

Since the filter is realizable, only the previous bits will cause interference, but the channel behavior will cause the future and the previous bits interfere with the present bit. Then (5-40) considers N previous bits and M future bits as well as the present bit which is the first term in (5-40).

We now proceed to use (5-40) to obtain the w(t',0,0) for the signal patterns (5-38). To use (5-39) one has to consider (N+M+1) intervals T seconds wide, and then get the terms of interest from the summation. Thus, for signal pattern S_4 with M=1 and N=2

$$S_4 = 0 \quad 1 \quad 0 \quad 0 \Rightarrow \begin{cases} \mu_0(n+1) = -1 \ , & k = 1 \\ \mu_0(n) = 1 \ , & k = 0 \\ \mu_0(n-1) = -1 \ , & k = -1 \\ \mu_0(n-2) = -1 \ , & k = -2 \end{cases} \tag{5-41}$$

The deterministic part of the moment for signal pattern S_4 is

$$W_4(t',0,0) = 2Ve^{-at'} \begin{cases} -e^{at'} + e^{-2aT} & (-2T \leq t' \leq -T) \\ e^{at'} + e^{-2aT} - 2e^{-aT} & (-T \leq t' \leq 0) \\ -e^{at'} + e^{-2aT} - 2e^{-aT} + 2 & (0 \leq t' \leq T) \\ -e^{at'} + e^{-2aT} - 2e^{-aT} + 2 & (T \leq t' \leq 2T) \end{cases} \tag{5-42}$$

Similarly, for signal patterns S_5, S_6, S_7, S_{12}, and S_{15} we obtain

$$W_5(t',0,0) = 2Ve^{-at'} \begin{cases} -e^{at'} + e^{-2aT} & (-2T \leq t' \leq -T) \\ e^{at'} + e^{-2aT} - e^{-aT} & (-T \leq t' \leq 0) \\ -e^{at'} + e^{-2aT} - e^{-aT} + 2 & (0 \leq t' \leq T) \\ e^{at'} + e^{-2aT} - e^{-aT} + 2 - 2e^{aT} & (T \leq t' \leq 2T) \end{cases} \tag{5-43}$$

$$W_6(t',0,0) = 2Ve^{-at'} \begin{cases} -e^{at'} + e^{-2aT} & (-2T \leq t' \leq -T) \\ e^{at'} + e^{-2aT} - 2e^{-aT} & (-T \leq t' \leq T) \\ -e^{at'} + e^{-2aT} - 2e^{-aT} + 2e^{aT} & (T \leq t' \leq 2T) \end{cases} \tag{5-44}$$

$$W_7(t',0,0) = 2Ve^{-at'} \begin{cases} -e^{at'} + e^{-2aT} & (-2T \leq t' \leq -T) \\ e^{at'} + e^{-2aT} - 2e^{-aT} & (-T \leq t' \leq 2T) \end{cases} \tag{5-45}$$

$$W_{12}(t',0,0) = 2Ve^{-at'} \begin{cases} e^{at'} + e^{-2aT} & (-2T \leq t' \leq 0) \\ -e^{at'} + e^{-2aT} + 2 & (0 \leq t' \leq 2T) \end{cases} \tag{5-46}$$

$$W_{15}(t',0,0) = 2Ve^{-at'} \begin{cases} e^{at'} - e^{-2aT} & (-2T \leq t' \leq 2T) \end{cases} \tag{5-47}$$

Once the filter responses are computed, we proceed to evaluate the moments (5-18), (5-19) and (5-21). First, the squares and the cross term of the w(t',0,0) will be computed.

For the flat-flat situation, one can anticipate several results. First, the filter responses w(t',0,0) will be independent of τ and u, the variables of integration to obtain the moments. Second, the intersymbol interference present is due to the effects of filtering at the receiver only. Since the flat-fading will be a particular case of the next section, we proceed without explicitly showing the moments of this section.

E. Case Study V: Time-flat and frequency-selective fading with Butterworth filtering

For this case, we let μ=0 in (5-15) to obtain

$$w(t,\tau,0) = \int h_1(t-\tau-x)\, s(x)\, dx \tag{5-48}$$

Now, if $h_1(t)$ is the unit impulse response of a first-order Butterworth filter, then

$$\begin{aligned} h_1(t) &= 4\pi W e^{-2\pi W t} \\ &= 2a\, e^{-at} \qquad (a = 2\pi W)\,. \end{aligned} \tag{5-49}$$

At this point it is convenient to refer back to Case Study I and note the similarity between (5-23) and (5-48). Then, if we let

$$t' = t - \tau \tag{5-50}$$

then

$$\omega(t',0,0) = \omega(t,\tau,0) \quad . \tag{5-51}$$

From (5-51) one can see that it is possible to use the results obtained for w(t',0,0) in Case Study IV by just making the appropriate change of variables.

To illustrate the change of variable suggested in (5-50) and also to compare it with the ideal case (the matched filter), a comparasion will be shown where the pattern S_5=0101 is arbitrarily used. First we show the matched filter response in Fig. V-2. The convolution of the bit pattern and the impulse response of the filter $h_1(t)$ is shown in Fig. V-3.

We can observe that in Fig. V-3 intersymbol interference is present due to the bits previous to the bit under detection. On the other hand, when matched filtering is used, only the adjacent bit can affect the filter response, and if sampling at t=T is performed then no intersymbol interference is present. Fig. V-3 also shows that when the change of variable $t'=T+\tau$ is performed, the response in the t' plane is shifted and folded to give the response in the τ plane.

We proceed to the evaluation of the moments (5-18), (5-19) and (5-21) for the range $|\tau|\leq T$. The reason for restricting the computation of the moments (and consequently the $w(\cdot)$'s) to the interval $|\tau|\leq T$ is that this is in fact the range of interest for the channel pulse spreading. As shown in Chapter 2 the signal was designed in such a way that the channel multipath spread was less than the signal duration. The question is how much is this channel selectivity is going to affect

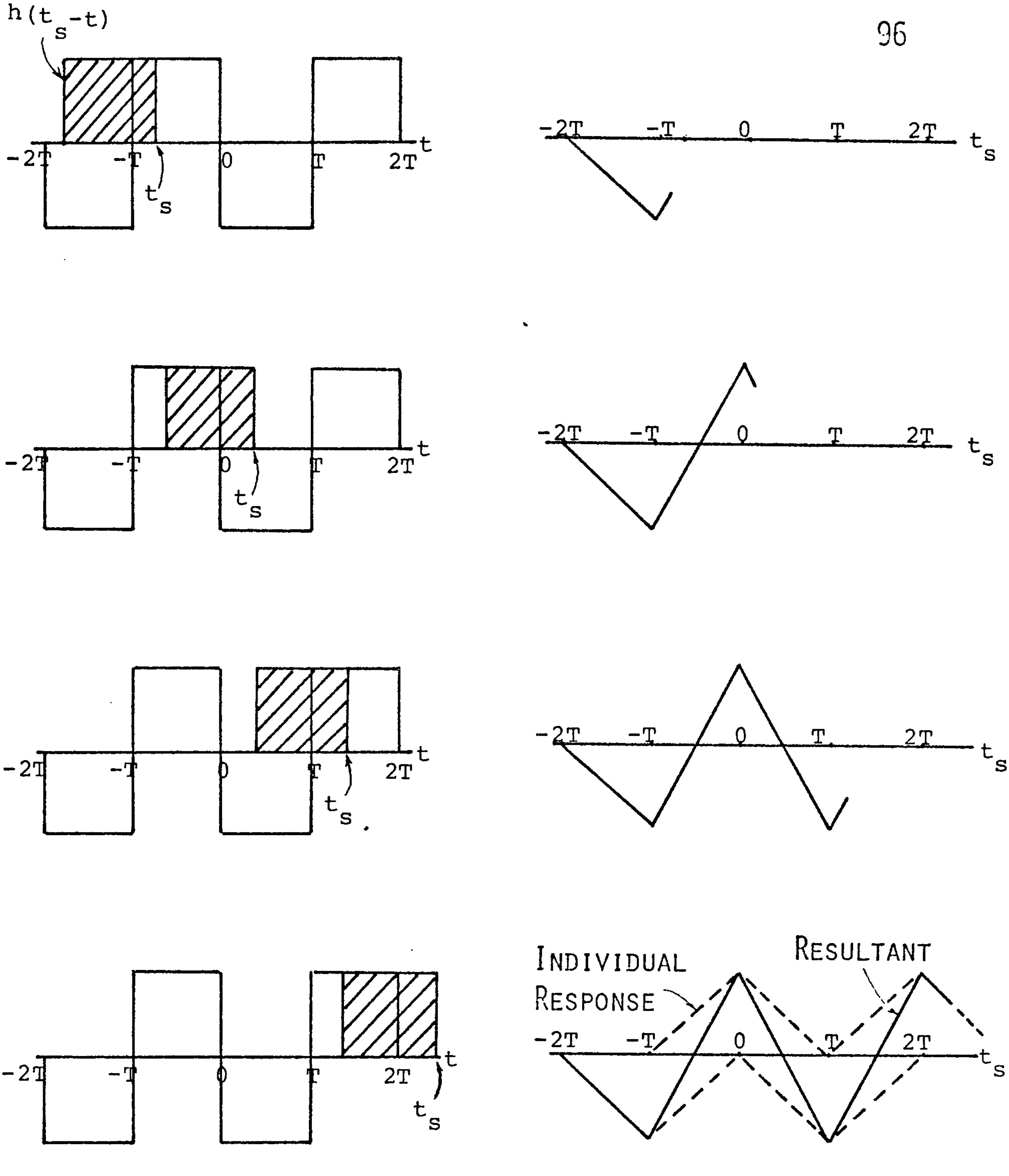

FIG. V-2 MATCHED FILTER RESPONSE FOR SIGNAL PATTERN S_5

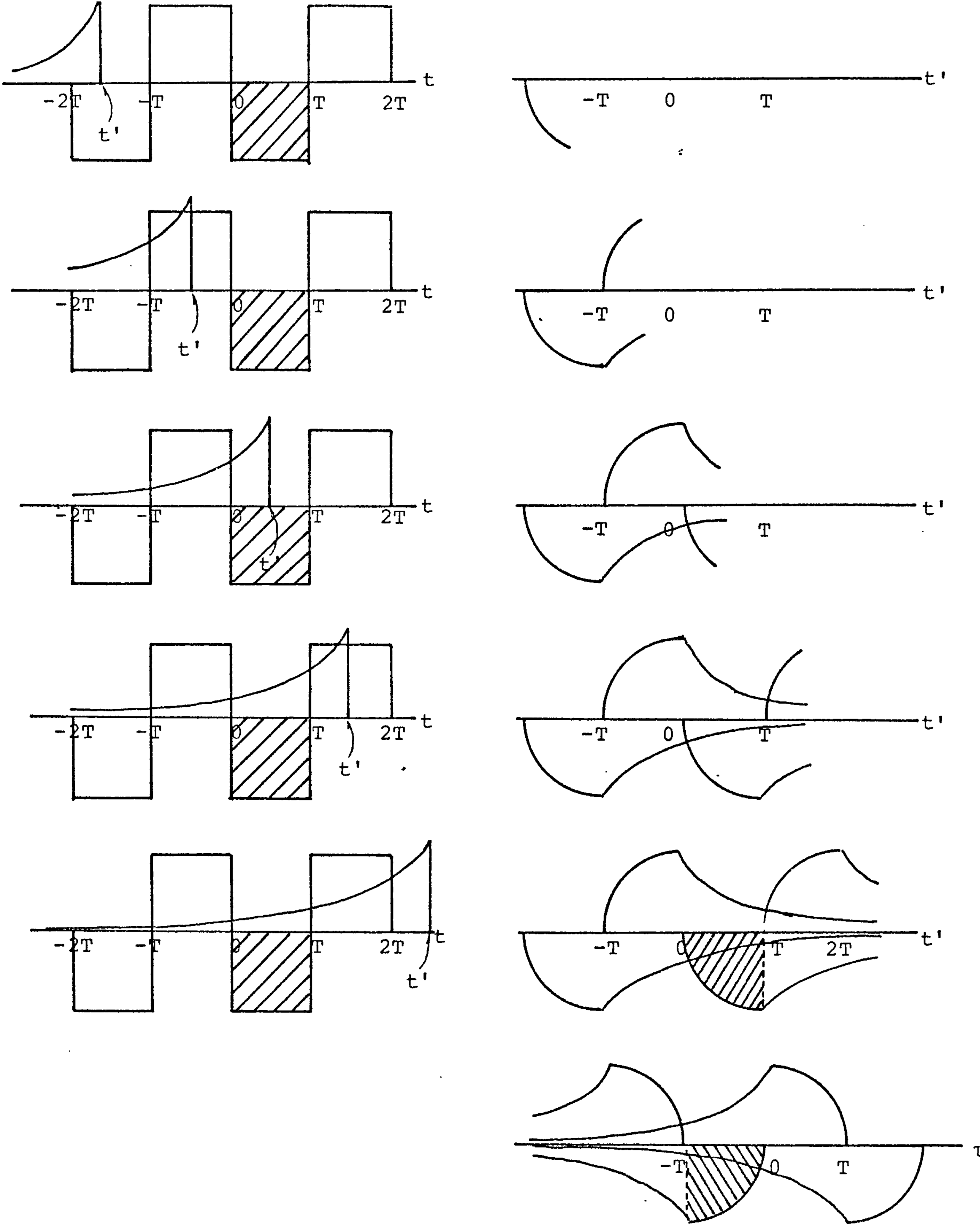

FIG. V-3 GRAPHICAL CONVOLUTION OF BIT PATTERN S_5 AND THE IMPULSE RESPONSE OF THE BUTTERWORTH FILTER IN THE t' AND τ DOMAIN.

the signal detection eventhough the multipath is within the tolerance range. The detailed evaluation of the moments (5-18) (5-19) and (5-21) is given in Appendix D. A general term for these moments was observed in (D-72) to be

$$m_{r,i} = 4V^2\sigma^2 \left\{ 2 + e^{\left(\frac{4W}{B_c}\right)^2} \left[C_{1,i} + C_{2,i}\, \mathrm{erf}\left(\frac{4W}{B_c}\right) \right] \right. .$$

$$\left. + e^{\left(\frac{2W}{B_c}\right)^2} \left[C_{3,i} - C_{4,i}\, \mathrm{erf}\left(\frac{2W}{B_c}\right) \right] \right\} \qquad (5\text{-}52)$$

where r=u,v or uv for the ith pattern. $C_{m,i}$ is the mth constant for the ith pattern, W is the filter bandwidth and B_c is the correlation bandwidth for the fading channel. The quantity σ^2 is the average power that would be received when a sinusoid of unity peak value is transmitted.

Some reductions can be made in (5-52) for the slow-fading situation. This is done by letting $B_c \to \infty$, and the generalized term (5-52) becomes

$$m_{r,i} \longrightarrow 4V^2\sigma^2 \left\{ 2 + C_{1,i} + C_{2,i} \right\}$$

$$= \frac{8E\sigma^2}{T} \left\{ \text{constant} \right\}. \qquad (5\text{-}52)$$

The above simplification can be justified from the frequency correlation function (2-19). If (2-19) is essentially constant over the interval of interest, then

$$R(\Omega) = 2\sigma^2 e^{-\frac{4\Omega^2}{B_c^2}} = 2\sigma^2 \qquad (5\text{-}53)$$

only if $B_c \to \infty$. Expression (5-53) is the frequency correlation function for the flat-flat fading case.

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CHAPTER VI

CONCLUSIONS AND RECOMMENDATIONS

6.1 Conclusions

Two procedures for the evaluation of the probability of error for binary receivers in the presence of intersymbol interference in random channels have been studied. The first one in Chapter III, was observed to be useful when the channel is slowly-varying as compared with the signal transmission rate. The second procedure in Chapter IV, accounted for the selectivity of the medium.

For the slowly-varying channel, a unified analysis of the effects of fading and intersymbol intersymbol interference due to filtering was presented. The matched filter operation was replaced by a more realistic filter at the front end of the receiver. It was observed that when no diversity is used (L=1), the main degradation is due to fading, but when this degradation is totally or partially eliminated (perhaps using diversity) the effects of bandlimiting become an important factor for design. When the probability of error is about 10^{-2}, the degradation due to intersymbol interference in the presence of fading is approximately 2 dB for WT=0.5. If WT$\geqslant$1.5, there is essentially no degradation due to filtering. Ideal and realizable filters were compared, and it was concluded that degradation due to filtering was diminished about 0.5 dB when the realizable filters are used.

The largest improvement obtained with diversity was between the L=1 and L=2 family of curves. For WT=1.0 this gain is 8 dB for a probability of error of 10^{-2}, and 16 dB for a probability of error of 10^{-4}. It was observed that for WT=1.0 the performance is better than the infinite-bandwidth non-fading case if $L \geq 3$ and $\gamma_o < 7$ dB. Also the lower the probability of error, the higher the order of diversity necessary to maintain the same performance that the non-fading infinite-bandwidth situation. Finally, if the probability of error is 10^{-4} the degradation due to filtering is of the order of 1.5 dB when L=2 and WT=0.5, which is small when compared with the 4 dB of degradation obtained for the non-fading case.

For the study of the distribution of errors in Rayleigh fading a rather complicated expression was obtained. This qualitative analysis determines the range of SNR values responsible for the errors when fading is present, as a function of the filter bandwidth and the order of diversity.

The Rayleigh fading channel was generalized for the case of Rician fading. Performance curves were obtained showing the dependence of the probability of error with the filter bandwidth W, the pulse duration T, and the ratio of the specular-to-random components γ^2. We observed that for $\gamma^2=0.1$ we have a Rayleigh fading channel, and the specular component is of no influence. We also concluded that approximately the same performance was obtained when L=1 and $\gamma^2=6.4$ as when L=2 and $\gamma^2=0.1$. This shows the great improvement that one can obtain by using diversity and that Rayleigh fading for dual diversity performs the same as single diversity for Rician fading with γ^2=6.4.

For the type of receivers whose decision variable can be formulated in a quadratic form, an expression for the probability density function was obtained. A general type of channel which includes not only a random component but also a specular component, was considered. The average probability of error for the generalized quadratic form was obtained. It was proved that binary symmetric operation, in general, does not exist if the channel is characterized as a random filter. Thus the hypothesis that selective fading introduces non-symmetric operation, has been proved. Some results for the probability of error as a function of the intersymbol interference in DPSK were plotted. From the curves provided it was observed that in selective channels there exists an irreducible probability of error. Values for this irreducible probability of error are given in Fig IV-2, as a function of the filter bandwidth W and the channel correlation bandwidth B_c.

A generalized form for the moments was derived by induction. It was shown that the deterministic part of the moments can be evaluated independently of the channel characteristics using a general formula.

Finally, it is concluded that a great variety of problems can be solved by simply formulating the decision variable in a quadratic form, and using the expression for the probability of error obtained in this dissertation. This expression offers not only the ability of analyzing a large class of receivers without performing many computations, but it also gives a unified way of studying the effects of intersymbol interference in random channels.

6.2 Recommendations for Further Study

When estimating the phase of a signal in fading channels, decision-directed methods have the disadvantage that wrong decisions are never corrected. In a coded system, decision-direction does not suffer from this shortcoming.[1] If the decoder is observing the correct code word (e.g. correct path in convolution codes) all "decisions" used in forming the phase are correct. It is of interest to further study this coded decision-directed detection scheme.

Another extension of this work would be to consider the channel characterization to be in terms of a multimodal scattering function. Even though troposcatter channels are very unlikely to have this characteristic, other type of channels can have such a representation, such as acoustic underwater communication channels, ionospheric channels, etc.

In the area of adaptive successive detectors, Kalman filters followed by a likelihood ratio computer have been suggested [2] for non-selective fading channels. It would be of interest to study the actual construction and analysis of the sub-optimum receiver that will realize the theoretical design. Also a study of the limitation on the selectivity of the medium in the analysis would be of great value.

Further, the design of the optimum receiver for troposcatter channels including time variant channel equalization is of great difficulty. There are some reports on this subject using the Viterbi algorithm and/or dynamic programming [3] that have attempted to formulate the problem. An extension could be an attempt to design such an equalizer that will take into consideration the time variant behavior of the channel.

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APPENDIX A

CHARACTERISTIC FUNCTION OF A REAL HERMITIAN QUADRATIC FORM

In this appendix the characteristic function used in Chapter 4 will be derived. It will be shown that when the moment matrix is of second order, the expression obtained in matrix form for this characteristic function can be expressed in an algebraic form which simplifies manipulation.

Turin [1] has evaluated the characteristic function $\Phi_k(s)$ of a general Hermitian quadratic form $d_k(t)$ in complex Gaussian variables. The quadratic form can be expressed as

$$\begin{aligned} d_k(t) &= a|\mu_k|^2 + b|v_k|^2 + c\mu_k^* v_k + c^*\mu_k v_k^* \\ &= (\mu_k \quad v_k)^* \begin{pmatrix} a & c \\ c^* & b \end{pmatrix} \begin{pmatrix} \mu_k \\ v_k \end{pmatrix} = V^{T*}QV \quad , \end{aligned} \tag{A-1}$$

where μ_k and v_k are Gaussian random variables (which in general can have non-zero means), a,b and c are non-random quantities, the asterisk denotes the complex conjugate, T means transpose, and the subscript k implies that we are considering the kth branch of a multireceiver †. Then, $\Phi_k(s)$ is given by

$$\Phi_k(s) = \frac{\exp\left\{-S_k^{T*} M^{-1}\left[I-(I-sMQ)^{-1}\right] S_k\right\}}{\det\left[I-sMQ\right]} \tag{A-2}$$

---

†The term multireceiver is used here in the sense of having L branches and M receivers, where L>M, since the type of diversi used could be time, frequency, space, angle, polarizarion or a combination.

where

$$S_k = \begin{pmatrix} \langle \mu_k \rangle \\ \langle v_k \rangle \end{pmatrix} \qquad M = \begin{pmatrix} m_\mu & m_{\mu v} \\ m_{\mu v}^* & m_v \end{pmatrix} \qquad Q = \begin{pmatrix} a & c \\ c^* & b \end{pmatrix} ,$$

and I is the unity matrix. The symbol $\langle \ \rangle$ implies expected value, which will be interchangeably used with a bar over the variable of interest. We can observe that S_k is the vector of the means, Q is a Hermitian matrix, and M is the complex covariance matrix (assumed to be non-singular) whose elements are the moments

$$\begin{aligned} m_\mu &= \left\langle \left| \mu_k - \langle \mu_k \rangle \right|^2 \right\rangle \\ m_v &= \left\langle \left| v_k - \langle v_k \rangle \right|^2 \right\rangle \qquad \text{(A-3)} \\ m_{\mu v} &= \left\langle \left(\mu_k - \langle \mu_k \rangle \right)^* \left(v_k - \langle v_k \rangle \right) \right\rangle \end{aligned}$$

which are independent of k.

If we define the Hermitian quadratic form D as

$$D = \sum_{k=1}^{L} d_k \qquad \text{(A-4)}$$

then the characteristic function Φ (s) can be defined as the product of the L individual $\Phi_k(s)$ functions of the form of (A-2), that is,

$$\Phi(s) = \prod_{k=1}^{L} \Phi_k(s) \quad .$$

It appears to be very involved to try to evaluate the exponent of (A-2) L times. Instead of working with matrix algebra, the exponent of (A-2) will be expressed in a more tractable algebraic fashion. Consider only the exponent of (A-2). By performing the matrix operations, it can be shown that [2]

$$S_k^{T*} M^{-1}\left[I-(I-sMQ)^{-1}\right]S_k = \frac{s^2\Delta_R S_k^{T*}M^{-1}S_k - s\,S_k^{T*}QS_k}{1-s(r_u+r_v)+s^2\Delta_R} \tag{A-5}$$

In this equation the exponent of (A-2) has been reduced to an algebraic form in s. The next step is to express the right-hand side of (A-5) in a partial fraction expansion. First let

$$\begin{aligned} A_k &= \Delta_R\left(S_k^{T*}M^{-1}S_k\right), \\ B_k &= S_k^{T*}QS_k, \\ \alpha &= \sqrt{\left(\frac{r_u+r_v}{2}\right)^2+\Delta_R} - \frac{r_u+r_v}{2}, \\ \beta &= \sqrt{\left(\frac{r_u+r_v}{2}\right)^2+\Delta_R} + \frac{r_u+r_v}{2}, \end{aligned} \tag{A-6}$$

and

$$s = j\nu .$$

Then

$$\frac{-s^2\Delta_R\left(S_k^{T*}M^{-1}S_k\right)+s\left(S_k^{T*}QS_k\right)}{1-s(r_u+r_v)+s^2\Delta_R} = \frac{\nu^2A_k+j\nu B_k}{(1-j\nu\beta)(1+j\nu\alpha)} . \tag{A-7}$$

Now, if we define

$$\alpha_1 = \sum_{k=1}^{L} A_k = \sum_{k=1}^{L} \left(|c|^2 - ab\right)\left(|\bar{\mu}_k|^2 m_v + |\bar{v}_k|^2 m_\mu - \bar{\mu}_k^* \bar{v}_k m_{\mu v} - \bar{\mu}_k \bar{v}_k^* m_{\mu v}\right) \tag{A-8}$$

and

$$\alpha_2 = \sum_{k=1}^{L} B_k = \sum_{k=1}^{L} a|\bar{\mu}_k|^2 + b|\bar{v}_k|^2 + c\,\bar{\mu}_k^* \bar{v}_k + c^* \bar{\mu}_k \bar{v}_k^* ,$$

then after some algebraic manipulation

$$\frac{\nu^2 A_k + j\nu B_k}{(1-j\nu\beta)(1+j\nu\alpha)} = \frac{\nu_1 \nu_2 (j\nu\alpha_2 - \nu^2\alpha_1)}{(\nu+j\nu_1)(\nu-j\nu_2)} , \tag{A-9}$$

where

$$\nu_1 = \sqrt{\frac{r_u + r_v}{2\Delta_R} + \frac{1}{\Delta_R}} - \left(\frac{r_u + r_v}{2\Delta_R}\right) , \tag{A-10a}$$

$$\nu_2 = \sqrt{\frac{r_u + r_v}{2\Delta_R} + \frac{1}{\Delta_R}} + \frac{r_u + r_v}{2\Delta_R} , \tag{A-10b}$$

$$R = \begin{pmatrix} r_u & r_{uv} \\ r_{vu} & r_v \end{pmatrix} = MQ = \begin{pmatrix} a\,m_u + c^* m_{uv} & c\,m_u + b\,m_{uv} \\ a\,m_{uv}^* + c^* m_v & c\,m_{uv}^* + b\,m_v \end{pmatrix} , \tag{A-11}$$

and

$$\Delta_R = \det R = \left(|c|^2 - ab\right)\left(|m_{uv}|^2 - m_u m_v\right) .$$

Using (A-9) in (A-7) and substituting the result into (A-2) gives

$$\Phi(j\nu) = \frac{(\nu_1\nu_2)^L}{(\nu+j\nu_1)^L(\nu-j\nu_2)^L}\, e^{\frac{\nu_1\nu_2(j\nu\alpha_2-\nu^2\alpha_1)}{(\nu+j\nu_1)(\nu-j\nu_2)}} \quad . \tag{A-12}$$

A final step will be to expand the exponent in a partial fraction expansion giving

$$\Phi(j\nu) = \frac{(\nu_1\nu_2)^L}{(\nu+j\nu_1)^L(\nu-j\nu_2)}\, e^{-A_1+j\frac{A_2}{\nu+j\nu_1}-j\frac{A_3}{\nu-j\nu_2}} \tag{A-13}$$

where

$$A_1 = \alpha_1\nu_1\nu_2 \quad ,$$

$$A_2 = \frac{\nu_1^2\nu_2(\nu_1\alpha_1+\alpha_2)}{(\nu_1+\nu_2)} \quad , \tag{A-14}$$

and

$$A_3 = \frac{\nu_1\nu_2^2(\nu_2\alpha_1-\alpha_2)}{(\nu_1+\nu_2)} \quad .$$

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APPENDIX B

EXAMPLES OF QUADRATIC FORMS IN BINARY RECEIVERS

In this appendix several examples of binary receivers which can be mathematically represented as a particular case of the quadratic form $d_k(t)$, will be discussed. It will be shown that by letting $a=b=0$ and $c\neq 0$ in (1-2) we are representing a class of receivers which can include a correlation operation. If we let $c=0$ and $a\neq b\neq 0$ a class of noncoherent recei ers can be represented, which includes a squaring operation or envelope detection. Also, it will be illustrated that if u_k and v_k in (1-2) are random quantities, then complex valued Gaussian process can be used to characterize fading channels. The kth branch of an L-diversity canonic binary receiver is shown in Fig. IV-1.

1. Pilot-Tone Phase-Shift-Keying (PT-PSK)

Suppose that at the receiver, one has available two signals. One of them could be the information bearing signal $r_1(t)$, and the other the reference signal $r_2(t)$, which in this case is a pilot tone. The multiplication of the information signal with the reference signal and selection of the low frequency components becomes $r_1(t)\cdot r_2(t)$ and a real part extraction operation. Then if $r_1(t)$ and $r_2(t)$ are expressed in the complex notation

$$r_1(t) = Re\{u_1(t)e^{j\omega_0 t}\} = x_1(t)\cos\omega_0 t - y_1(t)\sin\omega_0 t$$

and (B-1)

$$r_2(t) = Re\{u_2(t)e^{j\omega_0 t}\} = x_2(t)\cos\omega_0 t - y_2(t)\sin\omega_0 t$$

we have that

$$2\left(r_1(t)\cdot r_2(t)\right) = u_1 u_2^* + u_1^* u_2 \tag{B-2}$$

Then in a PSK detection, the multiplication of two signals $r_1(t)$ and $r_2(t)$ and low frequency extraction is equivalent to having the sum of the product of two complex quantities, u_1 and u_2, which is given by the quadratic combiner.

Also, it can be proved [2] that

$$r_1(t)\cdot r_2(t) = Re\{u_1 u_2^*\} = R_1 R_2 \cos(\theta_1 - \theta_2) \tag{B-3}$$

where we are representing $u_i = Re^{j\theta_i}$. Therefore in a synchronous detector, the decision is based on the algebraic sign of $\cos(\theta_1-\theta_2)$ and since (B-3) is equivalent to (B-2), the decision can be made based on the sign of the quadratic form

$$d = u_1 u_2^* + u_1^* u_2 \tag{B-4}$$

which is of the form of the generalized Hermitian quadratic form (A-1) when a=b=0 and c=c*=1. Note that in this case, the pilot tone filter (the reference filter) will have a narrower bandwidth than the information filter. This difference will be of importance when determining the moments of the random variables.

Several authors have used (B-4) to represent the decision variable for PT-PSK analysis. Hingorani [3] formulated the output of the information and reference filter as

and
$$\mu = \alpha g_k + n_k \quad . \qquad \qquad v = g_k + m_k \quad , \tag{B-5}$$

where $\alpha = I_1 + jI_2$ is the transmitted signal and $I_i = \pm 1$, making the decision variable

$$d_k = Re\{\mu_k v_k^*\} \quad . \tag{B-6}$$

Bello [5] performs a pilot tone extraction by obtaining (g+h)*, where the channel impulse response to a single path has a gain of (g+h) affecting the pilot tone and the transmitted waveform. In [6] the same author showed that a received signal

$$w(t) = \left[g_k^*(t) + \xi_k^*(t)\right]\left[h_k(t)\,S(t)\,e^{j\theta} + n_k(t)\,e^{-j2\pi Ft}\right] \quad , \tag{B-7}$$

where S(t) is the transmitted signal and $n_k(t)$ is the additive noise, can be represented in the quadratic form

$$d_k = \frac{1}{2}(t_1 - jt_2) Z_k h_k^* + \frac{1}{2}(t_1 + jt_2) Z_k^* h_k + t_3 |Z_k|^2 \tag{B-8}$$

This expression is the same as (A-1) if we let a=0, $b=t_3$, and $c=t_1+jt_2$. Proakis [7] also obtains a quadratic form for PT-PSK by letting a pilot tone estimate, $\hat{g}_k$, and a sampled received signal, x_k, form the decision variable

$$d = \sum_{k=1}^{L} x_k \hat{g}_k^* \tag{B-9}$$

which resembles (A-1) if a=b=0 and c=c*=1/2.

2. Differential Phase-Shift-Keying (DPSK)

The same type of decision scheme given in (B-4) is used here except that the output of the reference filter is not a pilot tone but the previous signal. Note that the information filter and the reference filter are the same. The same argument that was used in the previous section to obtain the quadratic form applies here except that $r_2(t)$ in (B-1) represents the received signal of the previous bit. Therefore by letting a=b=0 and c=c*≠0, the quadratic form in (A-1) represents a DPSK detection scheme. A block diagram of a DPSK receiver is shown in Fig. B-1.

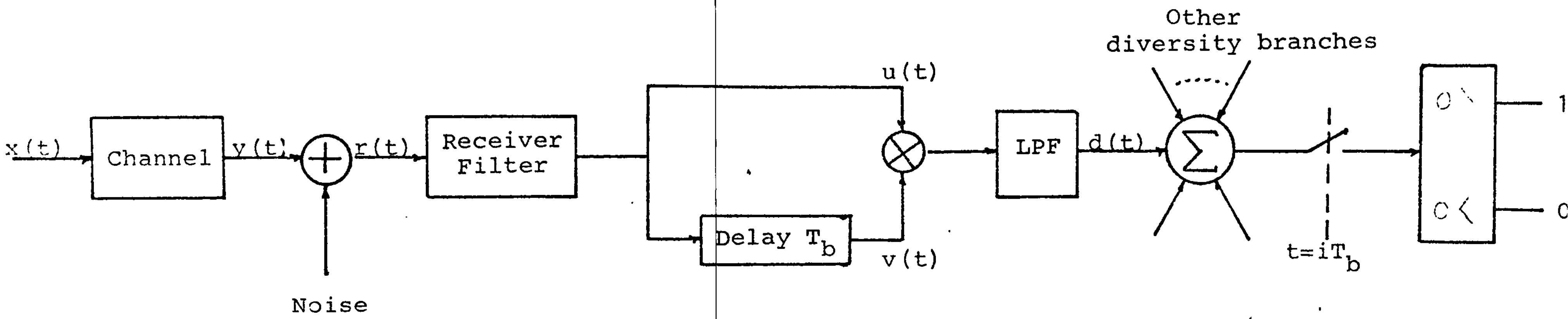

FIG. B-1 KTH BRANCH FOR A DPSK RECEIVER

3. Frequency-Shift-Keying

At the receiver there is one of two signals which was transmitted at frequencies f_1 or f_2. The receiver structure will be composed of two bandpass filters an envelope detector and a decision stage. This requires that the information filter and the reference filter in Fig. B-1 be two bandpass filters centered at f_1 and f_2. The quadratic combining will be a squaring operation at the output of the envelope detectors, also known as square-law envelope detectors. Then if the bandpass filter outputs are

$$r_1(t) = Re\left\{u_1(t)\, e^{j\omega_1 t}\right\} = x_1(t)\cos\omega_1 t - y_1(t)\sin\omega_1 t$$

and (B-10)

$$r_2(t) = Re\left\{u_2(t)\, e^{j\omega_2 t}\right\} = x_2(t)\cos\omega_2 t - y_2(t)\sin\omega_2 t$$

the output of the envelope detectors will be

$$R_1(t) = \sqrt{x_1^2(t) + y_1^2(t)} = |u_1(t)|$$

$$R_2(t) = \sqrt{x_2^2(t) + y_2^2(t)} = |u_2(t)| \qquad \text{(B-11)}$$

After the squaring operation we obtain $|u_1(t)|^2$ and $|u_2(t)|^2$ and the binary decision is made by testing which of these is larger at the sampling instant. The reason for the squaring operation, is to facilitate the mathematical analysis. Actually, this analysis results in no loss of generality for the zero threshold case.

Equivalently, one can have at the input of the threshold detector the function

$$d = |u_1(t)|^2 + |u_2(t)|^2 \quad . \tag{B-12}$$

where the test would be on the sign of d rather than a comparison between $u_1(t)$ and $u_2(t)$. Then by comparing (B-12) with (A-1), one can see that they are equal if a=b=1 and c=c*=0, meaning that the canonic receiver can be used for the noncoherent reception of binary signals.

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APPENDIX C

THE SNR DEGRADATION $D_i^2(WT)$

The quantity $D_i^2(WT)$ used in Chapter III, is defined in [1] and [2] as

$$D_i^2(WT) = \frac{\left[J(WT,0) + \sum_{n=1}^{5} \left(\frac{A_n + A_{-n}}{A}\right) J(WT,n) \right]}{J(WT,0)} \tag{C-1}$$

where

$$J(WT,n) = \frac{2}{\pi} \int_0^{\pi} \frac{\sin^2 x}{x} \cos 2nx \, dx \quad , \tag{C-2}$$

and the nth bit of information can be represented by

$$a_n(t) = \begin{cases} A_n & NT \leq t \leq (N+1)T \\ 0 & \text{elsewhere} \end{cases} \tag{C-3}$$

where

$$A_n = A \text{ or } -A \quad . \tag{C-4}$$

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APPENDIX D

EVALUATION OF THE MOMENTS m_u, m_v, AND m_{uv}

To evaluate the moments (5-18), (5-19) and (5-21) we first have to compute $w_u(t,\tau,0)$, $w_v(t,\tau,0)$ and the cross product $w_u(t,\tau,0)w_v(t,\tau,0)$ for the range $\tau \le T$. Then, using the transformation (5-50) in (5-42)-(5-47), the deterministic outputs of the information filter are

$$w_{u,6}(t,\tau,0) = 2V \begin{cases} -1-2e^{-2aT+a\tau}+e^{-3aT+a\tau}+2e^{a\tau} & (-T\le\tau\le 0) \\ 1-2e^{-2aT+a\tau}+e^{-3aT+a\tau} & (0\le\tau\le T) \end{cases} \tag{D-1}$$

$$w_{u,7}(t,\tau,0) = 2V \begin{cases} 1-2e^{-2aT+a\tau}+e^{-3aT+a\tau} & (-T\le\tau\le 0) \\ 1-2e^{-2aT+a\tau}+e^{-3aT+a\tau} & (0\le\tau\le T) \end{cases} \tag{D-2}$$

$$w_{u,15}(t,\tau,0) = 2V \begin{cases} 1-e^{-3aT+a\tau} & (-T\le\tau\le 0) \\ 1-e^{-3aT+a\tau} & (0\le\tau\le T) \end{cases} \tag{D-3}$$

$$w_{u,5}(t,\tau,0) = 2V \begin{cases} 1+e^{a\tau}\left(2e^{-aT}-2-2e^{-2aT}+e^{-3aT}\right) & (-T\le\tau\le 0) \\ 1+e^{a\tau}\left(2e^{-aT}-2e^{-2aT}+e^{-3aT}\right) & (0\le\tau\le T) \end{cases} \tag{D-4}$$

$$w_{u,4}(t,\tau,0) = 2V\left\{-1+e^{a\tau}\left(2e^{-aT}-2e^{-2aT}+e^{-3aT}\right)\right\} \tag{D-5}$$

$$w_{u,12}(t,\tau,0) = 2V\left\{1+e^{a\tau}\left(2e^{-aT}+e^{-3aT}\right)\right\} \tag{D-6}$$

To obtain $|w_u(t,\tau,0)|^2$ we just square (5-65)-(5-70), since all the w( )'s are real quantities. Then

$$|w_{u,6}(t,\tau,0)|^2 \approx 4V^2 \begin{cases} 1+e^{2a\tau}(4-8e^{-2aT})+e^{a\tau}(4e^{-2aT}-4) & (-T\le\tau\le 0) \\ 1+e^{2a\tau}(4e^{-4aT})+e^{a\tau}(2e^{-3aT}-4e^{-2aT}) & (0\le\tau\le T) \end{cases} \tag{D-7}$$

$$|w_{u,7}(t,\tau,0)|^2 \approx 4V^2 \left\{ 1+e^{2a\tau}(4e^{-4aT})+e^{a\tau}(2e^{-3aT}-4e^{-2aT}) \right. \tag{D-8}$$

$$|w_{u,13}(t,\tau,0)|^2 \approx 4V^2 \left\{ 1+e^{a\tau}(-2e^{-3aT}) \right. \tag{D-9}$$

$$|w_{u,5}(t,\tau,0)|^2 \approx 4V^2 \begin{cases} 1+e^{2a\tau}(4e^{-2aT}+4)+e^{a\tau}(4e^{-aT}-4) \\ 1+e^{2a\tau}(4e^{-2aT}+8e^{-4aT}-8e^{-3aT}) \\ \quad +e^{a\tau}(2e^{-aT}-4e^{-2aT}+2e^{-3aT}) \end{cases} \tag{D-10}$$

$$|w_{u,4}(t,\tau,0)|^2 \approx 4V^2 \left\{ 1+e^{2a\tau}(8e^{-4aT}+4e^{-2aT}-8e^{-3aT}) \right. \\ \quad + e^{a\tau}(4e^{-2aT}-2e^{-3aT}-4e^{-aT}) \tag{D-11}$$

$$|w_{u,12}(t,\tau,0)|^2 \approx 4V^2 \left\{ 1+e^{2a\tau}(4e^{-2aT}+4e^{-4aT}) \right. \\ \quad + e^{a\tau}(2e^{-3aT}+4e^{-aT}) \,. \tag{D-12}$$

We now proceed to compute the deterministic output for the reference filter $w_v(t,\tau,0)$. If we let $\mu=0$ and $s(t)=s(t-T)$ in (5-20), we obtain

$$w_v(t,\tau,0) = \int h(t-x)\, s(x-\tau-T)\, dx \tag{D-13}$$

The reason for the difference between (5-20) and (D-13) i

that in DPSK reception the reference filter output is del

by one baud, as shown in Fig. B-2. Rearranging (D-13) we h

$$w_v(t,\tau,0) = \int h(t-T-\tau-x)\, s(x)\, dx$$

and comparing (D-14) with (5-23) we can observe that

$$w_v(t,\tau,0) = w(t',0,0)$$

if $t'=t-T-\tau$. Therefore, once more we can use the results obtained for $w(t',0,0)$ by using the above change of variable. Thus, when using (5-42)-(5-47)

$$w_{v,5}(t,\tau,0) = 2V \begin{cases} -1 + e^{a\tau}\left(2 - 2e^{-aT} + e^{-2aT}\right) \\ 1 + e^{a\tau}\left(-2e^{-aT} + e^{-2aT}\right) \end{cases} , \qquad \text{(D-}$$

$$w_{v,6}(t,\tau,0) = 2V \left\{ 1 + e^{a\tau}\left(-2e^{-aT} + e^{-2aT}\right) \right. , \qquad \text{(D-}$$

$$w_{v,7}(t,\tau,0) = 2V \left\{ 1 + e^{a\tau}\left(-2e^{-aT} + e^{-2aT}\right) \right. ,$$

$$w_{v,15}(t,\tau,0) = 2V \left\{ 1 + e^{a\tau}\left(-e^{-2aT}\right) \right. ,$$

$$w_{v,4}(t,\tau,0) = 2V \begin{cases} -1 + e^{a\tau}\left(e^{-2aT} - 2e^{-aT} + 2\right) \\ 1 + e^{a\tau}\left(e^{-2aT} - 2e^{-aT}\right) \end{cases}$$

and

$$w_{v,12}(t,\tau,0) = 2V \begin{cases} 1 + e^{a\tau}\left(2 + e^{-2aT}\right) \\ 1 + e^{a\tau}\left(e^{-2aT}\right) \end{cases} ,$$

where we have set $t=t_s=T$.

The evaluation of $|w_v(t,\tau,0)|^2$ using (D-16)-(D-21) is

$$|w_{v,6}(t,\tau,0)|^2 = 4V^2\left\{1 + e^{2a\tau}\left(4e^{-2aT} + e^{-4aT} - 4e^{-3aT}\right) + e^{a\tau}\left(2e^{-2aT} - 4e^{-aT}\right)\right\} \tag{D-22}$$

$$|w_{v,7}(t,\tau,0)|^2 = |w_{v,6}(t,\tau,0)|^2 \tag{D-23}$$

$$|w_{v,13}(t,\tau,0)|^2 = 4V^2\left\{1 + e^{2a\tau}\left(e^{-4aT}\right) + e^{a\tau}\left(-2e^{-2aT}\right)\right\} \tag{D-24}$$

$$|w_{v,4}(t,\tau,0)|^2 = 4V^2\begin{cases} 1 + e^{2a\tau}\left(8e^{-2aT} + 4 - 8e^{-aT}\right) + e^{a\tau}\left(4e^{-aT} - 2e^{-2aT} - 4\right) \\ 1 + e^{2a\tau}\left(e^{-4aT} + e^{-2aT} - 4e^{-3aT}\right) + e^{a\tau}\left(2e^{-2aT} - 4e^{-aT}\right) \end{cases} \tag{D-25}$$

$$|w_{v,5}(t,\tau,0)|^2 = 4V^2\begin{cases} 1 + e^{2a\tau}\left(4 + 8e^{-2aT} - 8e^{-aT}\right) + e^{a\tau}\left(4e^{-aT} - 4 - 2e^{-2aT}\right) \\ 1 + e^{2a\tau}\left(4e^{-2aT} + e^{-4aT} - 4e^{-3aT}\right) + e^{a\tau}\left(2e^{-2aT} - 4e^{-aT}\right) \end{cases} \tag{D-26}$$

$$|w_{v,12}(t,\tau,0)|^2 = 4V^2\begin{cases} 1 + e^{2a\tau}\left(4 + 4e^{-2aT}\right) + e^{a\tau}\left(4 + 2e^{-2aT}\right) \\ 1 + e^{2a\tau}\left(e^{-4aT}\right) + e^{a\tau}\left(2e^{-2aT}\right) \end{cases} \tag{D-27}$$

For the computation of the cross products $w_u(\ )w_v(\ )$ the subindex "-" in the $w(\ )$'s implies that $-T \le \tau \le 0$ and subindex "+" that $0 \le \tau \le T$.

Then

$$\omega_{u,6}(t,\tau,0)_-\,\omega_{v,6}(t,\tau,0)_- = 4V^2\Big\{-1+e^{2a\tau}\left(5e^{-2aT}-2e^{-aT}\right) + e^{a\tau}\left(2-e^{-2aT}\right) \tag{D-29}$$

$$\omega_{u,6}(t,\tau,0)_+\,\omega_{v,6}(t,\tau,0)_+ = 4V^2\Big\{1+e^{2a\tau}\left(4e^{-3aT}-4e^{-4aT}\right) + e^{a\tau}\left(e^{-3aT}-e^{-2aT}-2e^{-aT}\right) \tag{D-30}$$

$$\omega_{u,7}(t,\tau,0)_-\,\omega_{v,7}(t,\tau,0)_- = 4V^2\Big\{1+e^{a\tau}\left(-e^{-aT}-e^{-2aT}\right) \tag{D-31}$$

$$\omega_{u,7}(t,\tau,0)_+\,\omega_{v,7}(t,\tau,0)_+ = 4V^2\Big\{1+e^{2a\tau}\left(4e^{-3aT}-4e^{-4aT}\right) + e^{a\tau}\left(e^{-3aT}-e^{-aT}-e^{-2aT}\right) \tag{D-32}$$

$$\omega_{u,15}(t,\tau,0)_-\,\omega_{\ ,15}(t,\tau,0)_- = 4V^2\Big\{1+e^{a\tau}\left(e^{-2aT}\right) \tag{D-33}$$

$$\omega_{u,15}(t,\tau,0)_+\,\omega_{\ ,15}(t,\tau,0)_+ = 4V^2\Big\{1+e^{a\tau}\left(-e^{-2aT}-e^{-3aT}\right) \tag{D-34}$$

$$\omega_{u,4}(t,\tau,0)_-\,\omega_{\ ,4}(t,\tau,0)_- = 4V^2\Big\{1+e^{2a\tau}\left(-4e^{-2aT}\right)+e^{a\tau}\left(4e^{-aT}-2-e^{-2aT}\right) \tag{D-35}$$

$$\omega_{u,4}(t,\tau,0)_+\,\omega_{\ ,4}(t,\tau,0)_+ = 4V^2\Big\{-1+e^{2a\tau}\left(-4e^{-2aT}+6e^{-3aT}-4e^{-4aT}\right) + e^{a\tau}\left(4e^{-aT}-3e^{-2aT}+e^{-3aT}\right) \tag{D-36}$$

$$\omega_{u,5}(t,\tau,0)_-\,\omega_{\ ,5}(t,\tau,0)_- = 4V^2\Big\{-1+e^{2a\tau}\left(8e^{-aT}-4-10e^{-2aT}\right) + e^{a\tau}\left(-4e^{-aT}+4+3e^{-2aT}\right) \tag{D-37}$$

$$\omega_{u,5}(t,\tau,0)_+\,\omega_{v,5}(t,\tau,0)_+ = 4V^2\Big\{1+e^{2a\tau}\left(-4e^{-2aT}+6e^{-3aT}-4e^{-4aT}\right) + e^{a\tau}\left(-e^{-2aT}+e^{-3aT}\right) \tag{D-38}$$

$$\omega_{u,12}(t,\tau,0)_-\,\omega_{v,12}(t,\tau,0)_- = 4V^2\Big\{1+e^{2a\tau}\left(4e^{-aT}\right) + e^{2a\tau}\left(2+2e^{-aT}+e^{-2aT}\right) \tag{D-39}$$

and

$$\omega_{u,12}(t,\tau,0)_+\,\omega_{v,12}(t,\tau,0)_+ = 4V^2\Big\{1+e^{2a\tau}\left(e^{-3aT}\right) + e^{a\tau}\left(2e^{-aT}+e^{-2aT}+e^{-3aT}\right). \tag{D-40}$$

Using (2-23) and (D-7)-(D-12) one can compute the moments m_u given in (5-18) as

$$m_u = \int_{-\infty}^{\infty} |\omega_u(t,\tau,0)|^2 \sigma(\tau)\, d\tau = m_{u-} + m_{u+}, \tag{D-41}$$

where

$$m_{u-} = \int_{-\infty}^{0} |\omega_u(t,\tau,0)|^2 \sigma(\tau)\, d\tau$$

and (D-42)

$$m_{u+} = \int_{0}^{\infty} |\omega_u(t,\tau,0)|^2 \sigma(\tau)\, d\tau$$

If we let $C_{m,i}$ be the mth constant for the moment of pattern i, we obtain

$$m_{u,6-} = 4V^2\sigma^2 \left\{ 1 + C_{1,6}\, e^{\left(\frac{4W}{B_c}\right)^2} \left[1 - \operatorname{erf}\left(\frac{4W}{B_c}\right)\right] + C_{2,6}\, e^{\left(\frac{2W}{B_c}\right)^2} \left[1 - \operatorname{erf}\left(\frac{2W}{B_c}\right)\right] \right\} \tag{D-43}$$

and

$$m_{u,6+} = 4V^2\sigma^2 \left\{ 1 + C_{3,6}\, e^{\left(\frac{4W}{B_c}\right)^2} \left[1 + \operatorname{erf}\left(\frac{4W}{B_c}\right)\right] + C_{4,6}\, e^{\left(\frac{2W}{B_c}\right)} \left[1 + \operatorname{erf}\left(\frac{4W}{B_c}\right)\right] \right\} \tag{D-44}$$

where

$$C_{1,6} = 4 - 8e^{-2aT},$$

$$C_{2,6} = 4e^{-2aT} - 4,$$

$$C_{3,6} = 4e^{-4aT}, \tag{D-45}$$

and

$$C_{4,6} = 2e^{-3aT} - 4e^{-2aT}.$$

The rest of the moments are

$$m_{\mu,7\,-,+} = 4V^2\sigma^2 \left\{ 2 + 2\,C_1^7\, e^{\left(\frac{4W}{Bc}\right)^2} + 2\,C_2^7\, e^{\left(\frac{2W}{Bc}\right)^2} \right\} \tag{D-46}$$

$$C_{1,7} = 4e^{-4aT}$$

$$C_{2,7} = 2e^{-3aT} - 4e^{-2aT}$$

$$m_{\mu,15\,-,+} = 4V^2\sigma^2 \left\{ 2 + 2\,C_{1,15}\, e^{\left(\frac{2W}{Bc}\right)^2} \right\} \tag{D-47}$$

$$C_{1,15} = -2e^{-3aT}$$

$$m_{\mu,5\,-} = 4V^2\sigma^2 \left\{ 1 + C_{1,5}\, e^{\left(\frac{4W}{Bc}\right)^2}\left[1 - \operatorname{erf}\left(\frac{4W}{Bc}\right)\right] + C_{2,5}\, e^{\left(\frac{2W}{Bc}\right)^2}\left[1 - \operatorname{erf}\left(\frac{2W}{Bc}\right)\right] \right\} \tag{D-48}$$

$$m_{\mu,5\,+} = 4V^2\sigma^2 \left\{ 1 + C_{3,5}\, e^{\left(\frac{4W}{Bc}\right)^2}\left[1 + \operatorname{erf}\left(\frac{4W}{Bc}\right)\right] + C_{4,5}\, e^{\left(\frac{2W}{Bc}\right)^2}\left[1 + \operatorname{erf}\left(\frac{2W}{Bc}\right)\right] \right\} \tag{D-49}$$

$$C_{1,5} = 4e^{-2aT} + 4$$

$$C_{2,5} = 4e^{-aT} - 4$$

$$C_{3,5} = 4e^{-2aT} + 8e^{-4aT} - 8e^{-3aT}$$

$$C_{4,5} = 2e^{-aT} - 4e^{-2aT} + 2e^{-3aT}$$

and

$$m_{\mu,4\,-,+} = 4V^2\sigma^2 \left\{ 2 + 2\,C_{1,4}\, e^{\left(\frac{4W}{Bc}\right)^2} + 2\,C_{2,4}\, e^{\left(\frac{2W}{Bc}\right)^2} \right\} \tag{D-50}$$

$$C_{1,4} = 8e^{-4aT} + 4e^{-2aT} - 8e^{-3aT}$$

$$C_{2,4} = 4e^{-2aT} - 2e^{-3aT} - 4e^{-aT}.$$

Similarly, the moments m_V can be computed using (5-19). Then

$$m_{V,6\,-,+} = 4V^2\sigma^2 \left\{ 2 + 2\,C_{3,6}\, e^{\left(\frac{4W}{Bc}\right)^2} + 2\,C_{4,6}\, e^{\left(\frac{2W}{Bc}\right)^2} \right\} \tag{D-51}$$

$$C_{3,6} = 4e^{-2aT} + e^{-4aT} - 4e^{-3aT}$$

$$C_{4,6} = 2e^{-2aT} - 4e^{-aT}$$

$$m_{V,7\,-,+} = m_{V,6\,-,+} \tag{D-52}$$

$$m_{v,15-,+} = 4V^2\sigma^2\left\{2 + 2\,C_{2,15}\,e^{\left(\frac{4W}{B_c}\right)^2}\right\} \qquad \text{(D-53)}$$

$$C_{2,15} = e^{-4aT} - 2e^{-2aT}$$

$$m_{v,4-} = 4V^2\sigma^2\left\{1 + C_{3,4}\,e^{\left(\frac{4W}{B_c}\right)^2}\left[1 - \mathrm{erf}\left(\frac{4W}{B_c}\right)\right] + C_{4,4}\,e^{\left(\frac{2W}{B_c}\right)^2}\left[1 - \mathrm{erf}\left(\frac{2W}{B_c}\right)\right]\right\} \qquad \text{(D-54)}$$

$$m_{v,4+} = 4V^2\sigma^2\left\{1 + C_{5,4}\,e^{\left(\frac{4W}{B_c}\right)^2}\left[1 + \mathrm{erf}\left(\frac{4W}{B_c}\right)\right] + C_{6,4}\,e^{\left(\frac{2W}{B_c}\right)^2}\left[1 + \mathrm{erf}\left(\frac{2W}{B_c}\right)\right]\right\} \qquad \text{(D-55)}$$

$$C_{3,4} = 8e^{-2aT} + 4 - 8e^{-aT}$$

$$C_{4,4} = 4e^{-aT} - 2e^{-2aT} - 4$$

$$C_{5,4} = e^{-4aT} + 4e^{-2aT} - 4e^{-3aT}$$

$$C_{6,4} = 2e^{-2aT} - 4e^{-aT}$$

$$m_{v,5-} = 4V^2\sigma^2\left\{1 + C_{5,5}\,e^{\left(\frac{4W}{B_c}\right)^2}\left[1 - \mathrm{erf}\left(\frac{4W}{B_c}\right)\right] + C_{6,5}\,e^{\left(\frac{2W}{B_c}\right)^2}\left[1 - \mathrm{erf}\left(\frac{2W}{B_c}\right)\right]\right\} \qquad \text{(D-56)}$$

$$m_{v,5+} = 4V^2\sigma^2\left\{1 + C_{7,5}\,e^{\left(\frac{4W}{B_c}\right)^2}\left[1 + \mathrm{erf}\left(\frac{4W}{B_c}\right)\right] + C_{8,5}\,e^{\left(\frac{2W}{B_c}\right)^2}\left[1 + \mathrm{erf}\left(\frac{2W}{B_c}\right)\right]\right\} \qquad \text{(D-57)}$$

$$C_{5,5} = 4 + 8e^{-2aT} - 8e^{-aT}$$

$$C_{6,5} = 4e^{-aT} - 4 - 2e^{-2aT}$$

$$C_{7,5} = 4e^{-2aT} + e^{-4aT} - 4e^{-3aT}$$

$$C_{8,5} = 2e^{-2aT} - 4e^{-aT}$$

and

$$m_{v,12-} = 4V^2\sigma^2\left\{1 + C_{3,12}\,e^{\left(\frac{4W}{B_c}\right)^2}\left[1 - \mathrm{erf}\left(\frac{4W}{B_c}\right)\right] + C_{4,12}\,e^{\left(\frac{2W}{B_c}\right)^2}\left[1 - \mathrm{erf}\left(\frac{2W}{B_c}\right)\right]\right\} \qquad \text{(D-5}$$

$$m_{v,12+} = 4V^2\sigma^2\left\{1 + C_{5,12}\,e^{\left(\frac{4W}{B_c}\right)^2}\left[1 + \mathrm{erf}\left(\frac{4W}{B_c}\right)\right] + C_{6,12}\,e^{\left(\frac{2W}{B_c}\right)^2}\left[1 + \mathrm{erf}\left(\frac{4W}{B_c}\right)\right]\right\} \qquad \text{(D-5}$$

$$C_{3,12} = 4+4e^{-2aT}$$
$$C_{4,12} = 4+2e^{-2aT}$$
$$C_{5,12} = e^{-4aT}$$
$$C_{6,12} = 2e^{-2aT} .$$

Now, using the cross products (D-29)-(D-40) to compute the moments m_{uv} given by (5-21) we obtain

$$m_{\mu v,6-} = 4V^2\sigma^2\left\{-1 + C_7^6 E_1\left(\frac{4W}{\beta c}\right) + C_8^6 E_1\left(\frac{2W}{\beta c}\right)\right\} \quad \text{(D-60)}$$

$$m_{\mu v,6+} = 4V^2\sigma^2\left\{1 + C_9^6 E_2\left(\frac{4W}{\beta c}\right) + C_{10}^6 E_2\left(\frac{2W}{\beta c}\right)\right\} \quad \text{(D-61)}$$

$$C_7^6 = 5e^{-2aT} - 2e^{-aT}$$
$$C_8^6 = 2 - e^{-2aT}$$
$$C_9^6 = 4e^{-3aT} - 4e^{-4aT}$$
$$C_{10}^6 = e^{-3aT} - e^{-2aT} - 2e^{-aT}$$
$$E_1(x) = e^{-x^2}\left[1 - \operatorname{erf}(x)\right]$$
$$E_2(x) = e^{-x^2}\left[1 + \operatorname{erf}(x)\right]$$

$$m_{\mu v,7-} = 4V^2\sigma^2\left\{1 + C_1^7 E_1\left(\frac{4W}{\beta c}\right) + C_2^7 E_1\left(\frac{2W}{\beta c}\right)\right\} \quad \text{(D-62)}$$

$$m_{\mu v,7+} = 4V^2\sigma^2\left\{1 + C_3^7 E_2\left(\frac{4W}{\beta c}\right) + C_4^7 E_2\left(\frac{2W}{\beta c}\right)\right\} \quad \text{(D-63)}$$

$$C_1^7 = 0$$
$$C_2^7 = -e^{-aT} - e^{-2aT}$$
$$C_3^7 = 4e^{-3aT} - 4e^{-4aT}$$
$$C_4 = e^{-3aT} - e^{-aT} - e^{-2aT}$$

$$m_{\mu v,15-} = 4V^2\sigma^2\left\{1 + C_{3,15} E_1\left(\frac{4W}{\beta c}\right) + C_{4,15} E_1\left(\frac{2W}{\beta c}\right)\right\} \quad \text{(D-64)}$$

$$m_{\mu,v,15+} = 4V^2\sigma^2\left\{1 + C_{5,15} E_2\left(\frac{4W}{\beta c}\right) + C_{6,15} E_2\left(\frac{2W}{\beta c}\right)\right\} \quad \text{(D-65)}$$

$$C_{3,15} = 0$$

$$C_{4,15} = e^{-2aT}$$

$$C_{5,15} = 0$$

$$C_{6,15} = -e^{-2aT} - e^{-3aT}$$

$$m_{\mu v,4-} = 4V^2\sigma^2\left\{1 + C_{7,4}E_1\left(\frac{4W}{Bc}\right) + C_{8,4}E_1\left(\frac{2W}{Bc}\right)\right\} \qquad \text{(D-66)}$$

$$m_{\mu v,4+} = 4V^2\sigma^2\left\{-1 + C_{9,4}E_2\left(\frac{4W}{Bc}\right) + C_{10,4}E_2\left(\frac{2W}{Bc}\right)\right\} \qquad \text{(D-67)}$$

$$C_{7,4} = -4e^{-2aT}$$

$$C_{8,4} = 4e^{-aT} - e^{-2aT}$$

$$C_{9,4} = 6e^{-3aT} - 4e^{-2aT} - 4e^{-4aT}$$

$$C_{10,4} = 4e^{-aT} + e^{-3aT} - 3e^{-2aT}$$

$$m_{\mu v,5-} = 4V^2\sigma^2\left\{-1 + C_{9,5}E_1\left(\frac{4W}{Bc}\right) + C_{10,5}E_1\left(\frac{2W}{Bc}\right)\right\} \qquad \text{(D-68)}$$

$$m_{\mu v,5+} = 4V^2\sigma^2\left\{1 + C_{11,5}E_2\left(\frac{4W}{Bc}\right) + C_{12,5}E_2\left(\frac{2W}{Bc}\right)\right\} \qquad \text{(D-69)}$$

$$C_{9,5} = 8e^{-aT} - 4 - 10e^{-2aT}$$

$$C_{10,5} = 3e^{-2aT} + 4 - 4e^{-aT}$$

$$C_{11,5} = 6e^{-3aT} - 4e^{-2aT} - 4e^{-4aT}$$

$$C_{12,5} = e^{-3aT} - e^{-2aT}$$

and

$$m_{\mu v,12-} = 4V^2\sigma^2\left\{1 + C_{7,12}E_1\left(\frac{4W}{Bc}\right) + C_{8,12}E_1\left(\frac{2W}{Bc}\right)\right\} \qquad \text{(D-70)}$$

$$m_{\mu v,12+} = 4V^2\sigma^2\left\{1 + C_{9,12}E_2\left(\frac{4W}{Bc}\right) + C_{10,12}E_2\left(\frac{2W}{Bc}\right)\right\} \qquad \text{(D-71)}$$

$$C_{7,12} = 4e^{-aT}$$

$$C_{8,12} = 2 + 2e^{-aT} + e^{-2aT}$$

$$C_{9,12} = e^{-3aT}$$

$$C_{10,12} = 2e^{-aT} + e^{-2aT} + e^{-3aT}.$$

A general term for the moments (D-43)-(D-71) is observed to be

$$m_{r,i} = 4V^2\sigma^2 \left\{ 2 + e^{\left(\frac{4W}{B_c}\right)^2}\left[C_{1,i} + C_{2,i}\, \mathrm{erf}\left(\frac{4W}{B_c}\right)\right] + e^{\left(\frac{2W}{B_c}\right)^2}\left[C_{3,i} + C_{4,i}\, \mathrm{erf}\left(\frac{2W}{B_c}\right)\right] \right\} \tag{D-72}$$

where r=u,v or uv for the ith pattern. $C_{m,i}$ is the mth constant for the ith pattern, W is the filter bandwidth and B_C is the correlation bandwidth for the fading channel. The quantity σ^2 is equal to the average power that would be received when a sinusoid of unity peak value is transmitted.

From the general term (D-72) we can derive the moments for any signal pattern. Then the moments for the signal patterns S_4, S_5, S_6, S_7, S_{12} and S_{15} are completely specified if the constants $C_{m,i}$ are known. Thus

$$\begin{aligned} C_{1,4} &= 8e^{-2aT} - 8e^{-3aT} + 8e^{-4aT} \\ C_{2,4} &= 0 \\ C_{3,4} &= -4e^{-aT} + 4e^{-2aT} - 2e^{-3aT} \qquad (r=u) \\ C_{4,4} &= 0 \end{aligned} \tag{D-73}$$

$$\begin{aligned} C_{1,4} &= 4 - 8e^{-aT} + 12e^{-2aT} - 4e^{-3aT} + e^{-4aT} \\ C_{2,4} &= -4 + 8e^{-aT} - 4e^{-2aT} - 4e^{-3aT} + e^{-4aT} \\ & \qquad\qquad (r=v) \\ C_{3,4} &= -4 \\ C_{4,4} &= 4 - 8e^{-aT} + 4e^{-2aT} \end{aligned} \tag{D-74}$$

$$\begin{aligned} C_{1,5} &= 4 + 8e^{-2aT} - 8e^{-3aT} + 8e^{-4aT} \\ C_{2,5} &= -4 - 8e^{-3aT} + 8e^{-4aT} \\ C_{3,5} &= -4 + 6e^{-aT} - 4e^{-2aT} + 2e^{-3aT} \\ C_{4,5} &= 4 - 2e^{-aT} - 4e^{-2aT} + 2e^{-3aT} \end{aligned} \qquad (r = \mu) \qquad \text{(D-75)}$$

$$\begin{aligned} C_{1,5} &= 4 - 8e^{-aT} + 12e^{-2aT} - 4e^{-3aT} + e^{-4aT} \\ C_{2,5} &= -4 + 8e^{-aT} - 4e^{-2aT} - 4e^{-3aT} + e^{-4aT} \\ C_{3,5} &= -4 \\ C_{4,5} &= 4 - 8e^{-aT} + 4e^{-2aT} \end{aligned} \qquad (r = \nu) \qquad \text{(D-76)}$$

,

$$\begin{aligned} C_{1,6} &= 4 - 8e^{-2aT} + 4e^{-4aT} \\ C_{2,6} &= -4 + 8e^{-2aT} + 4e^{-4aT} \\ C_{3,6} &= -4 + 2e^{-3aT} \\ C_{4,6} &= 4 - 8e^{-2aT} + 2e^{-3aT} \end{aligned} \qquad (r = \mu) \qquad \text{(D-77)}$$

$$\begin{aligned} C_{1,6} &= 4e^{-2aT} + e^{-4aT} - 4e^{-3aT} \\ C_{2,6} &= 0 \\ C_{3,6} &= 2e^{-3aT} - 4e^{-aT} \\ C_{4,6} &= 0 \end{aligned} \qquad (r = \nu) \qquad \text{(D-78)}$$

,

$$\begin{aligned} C_{1,7} &= 4e^{-4aT} \\ C_{2,7} &= 0 \\ C_{3,7} &= 2e^{-3aT} - 4e^{-2aT} \\ C_{4,7} &= 0 \end{aligned} \qquad (r = \mu) \qquad \text{(D-79)}$$

$$\begin{aligned} C_{1,7} &= 4e^{-2aT} + e^{-4aT} - 4e^{-3aT} \\ C_{2,7} &= 0 \\ C_{3,7} &= 2e^{-3aT} - 4e^{-aT} \\ C_{4,7} &= 0 \end{aligned} \qquad (r = \nu) \qquad \text{(D-80)}$$

,

$$\begin{aligned} C_{1,12} &= 8e^{-2aT} + 8e^{-4aT} \\ C_{2,12} &= 0 \\ C_{3,12} &= 8e^{-aT} + 4e^{-3aT} \\ C_{4,12} &= 0 \end{aligned} \qquad (r = \mu) \qquad \text{(D-81)}$$

$$\begin{aligned} C_{1,12} &= 4 + 4e^{-2aT} + e^{-4aT} \\ C_{2,12} &= -4 - 4e^{-2aT} + e^{-4aT} \\ C_{3,12} &= 4 + 4e^{-2aT} \\ C_{4,12} &= -4 \end{aligned} \qquad (r = \nu) \qquad \text{(D-82)}$$

,

$$\begin{aligned} C_{1,15} &= -2e^{-3aT} \\ C_{2,15} &= 0 \\ C_{3,15} &= 0 \\ C_{4,15} &= 0 \end{aligned} \qquad (r = \mu) \qquad \text{(D-83)}$$

$$\begin{aligned} C_{1,15} &= -2e^{-2aT} + e^{-4aT} \\ C_{2,15} &= 0 \\ C_{3,15} &= 0 \\ C_{4,15} &= 0 \end{aligned} \qquad (r = \nu) \qquad \text{(D-84)}$$

,

and the task is completed.

APPENDIX E

DECISION-DIRECTED PHASE ESTIMATION

An adaptive detection scheme over a channel with a time-varying phase is described and analyzed here. Channel phase measurements based on the biphase-modulated data signal is accomplished by using past channel symbol decisions to effectively remove the modulation. Results in the form of a quadratic form are given, to allow once more for the use of the generalized expressions for the probability of error obtained in Chapter IV.

Decision-directed phase estimation is of interest in adaptive signal detection and unsupervised pattern recognition because of its computational simplicity and relatively easy implementation. The decision-directed procedure involves assuming a reference, obtaining a correlation with a successive observation and, upon decision, updating the reference by a linear average of the most recently classified observation with the previous reference for that class. When this procedure is used in phase-reversal signaling systems, the receiver is actually performing a channel measurement of the phase. Thus, the receiver is forming the dot product between the incoming signal V_i and the phase reference vector V_r. The decision as to which symbol was transmitted is determined by the sign of the successive dot products. That is,

$$V_{r+1} = \omega V_r \pm V_i \tag{E-1}$$

where V_{r+1} is the new reference, and ω is a weighting factor. Thus by forming $V_r \cdot V_i$ and comparing the sign of this product with the sign of $V_{r-1} \cdot V_{i-1}$ we have a decision-directed measurement.

If we consider a receiver of the form of Fig. B-2, the decision variable can be formulated in the following manner. Let

$$V_{r,k} = \mu_2 = \left(R_2\, e^{j\Theta_{k,2}}\right) a_k \tag{E-2}$$

and

$$V_{i,k} = \mu_1 = \left(R_1\, e^{j\Theta_{k,1}}\right) a_k \tag{E-3}$$

be the reference and the incoming vector respectively. The a_k's are either +1 or -1, $\Theta_{k,1}$ is the angle of the incoming information, and $\Theta_{k,2}$ is the estimate of the channel phase in interval k based on the received data and the modulation decisions from the past p intervals, i.e.,

$$\Theta_{k,2} = \hat{\Theta}_k = \text{angle of } V_{r,k} = \text{angle of} \sum_{\ell=k-p}^{k-1} \hat{a}_\ell V_{i,\ell} \tag{E-4}$$

Thus, the dot product is

$$\begin{aligned} d_k &= \frac{1}{2} \sum_{\ell=k-p}^{k-1} \hat{a}_k \hat{a}_\ell \left(V_{i,\ell} V_{i,k}^* + V_{i,\ell}^* V_{i,k}\right) \\ &= \frac{1}{2} \sum_{\ell=k-p}^{k-1} \hat{a}_k a_k \hat{a}_\ell a_\ell \left(V_{i,\ell} V_{i,k}^* + V_{i,\ell}^* V_{i,k}\right) \end{aligned} \tag{E-5}$$

where we have premultiplied the decision variable by the a_m's to remove the modulation.

If we assume that one of two messages was sent, and all the decisions $\hat{a}_\ell$ are correct, the dot product d_k tends to be positive. Then the receiver will be in error if $d_k<0$, that is,

$$P_\varepsilon = P_r\left\{d_k \leq 0\right\}$$

$$= \frac{1}{2} P_r\left\{\sum_{\ell=k-p}^{k-1} \hat{a}_k a_k \hat{a}_\ell a_\ell \left(V_{i,\ell} V_{i,k}^* + V_{i,\ell}^* V_{i,k}\right)\right\} . \qquad \text{(E-6)}$$

The error probability is therefore the probability that a quadratic form in complex Gaussian random variables is less than zero. This formulation allows us to use the results obtained in Chapter IV.